Applications of hyperbolic geometry in physics

Scott Randall Rippy

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APPLICATIONS OF HYPERBOLIC GEOMETRY IN PHYSICS

A Project

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Scott Randall Rippy

June 1996
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5/16/96 Date
ABSTRACT

Question: How are the fundamental properties of hyperbolic geometry applied in physics?

In this treatment of the subject we include:

1) analogies from three dimensions, leading the reader into the realm of three space dimensions and one time dimension,

2) an introduction to Minkowski spacetime along with the significance of "distance" in this metric,

3) how we change from one frame of reference at rest to another in motion,

4) an algebraic journey from one form of the Lorentz transformation matrix to another,

5) a study of the Lorentz Group, and

6) how the Lorentz transformation goes beyond Minkowski spacetime into other applications, such as Maxwell's Equations.
ACKNOWLEDGMENTS

I would like to thank Dr. John Sarli for his unending patience in teaching, and for his guiding hand. I thank Dr. James Okon for his assistance in the physical production of this project.

Most of all, I dedicate my most profound feelings of gratitude to my wife Jamie, who has "steadied the ark" with a most worthy and capable hand.
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1. Minkowski Spacetime and Hyperbolic Geometry

A The Original Minkowski Spacetime, 1908.

Picture a cone in $\mathbb{R}^3$ (see figure 1) whose axis is the $z$-axis, and the equation of which is

$$x^2 + y^2 = z^2.$$

Another form of this equation is

$$x^2 + y^2 - z^2 = 0.$$

![Figure 1](image)

Consider two vectors in $\mathbb{R}^3$, $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. In three-dimensional hyperbolic geometry we have the bilinear form

$$b(\mathbf{x}, \mathbf{y}) = x_1y_1 + x_2y_2 - x_3y_3.$$

This is the inner product for hyperbolic geometry. Note that for any vector $\mathbf{v} = (x, y, z)$,

$$b(\mathbf{v}, \mathbf{v}) = x^2 + y^2 - z^2.$$
(remember the second equation above and see #3 below). Using this inner product we define a non-zero vector $v$ to be:

1. spacelike if $b(v, v) > 0$ (such as $(2, 3, 0)$),
2. timelike if $b(v, v) < 0$ (such as $(0, 0, 7)$),
3. lightlike if $b(v, v) = 0$ (such as $(3, 4, 5)$).

A spacelike vector is found exterior to the cone described in the first paragraph above (see Figure 1), a timelike vector is found inside the cone, and a lightlike vector is actually an element of the cone (the cone is often referred to as the "light cone").

We note that if $b(v, v) = -1$, then $v$ is a unit timelike vector inside the light cone, such as $(0, 0, 1)$ or $(1, 1, \sqrt{3})$. Then $x^2 + y^2 - z^2 = -1$ if and only if $v$ is a unit timelike vector. If we think of the ordered triple $(x, y, z)$ as a point in $\mathbb{R}^3$, the set of all such points (where $x^2 + y^2 - z^2 = -1$) forms a hyperboloid of two sheets, interior to the cone described above, and separated by the plane $z = 0$. We may use this model of hyperbolic geometry to help ourselves "visualize" Minkowski spacetime.

Adding the fourth dimension of time, we now have points with coordinates $(x, y, z, t)$. Analogous to the above three-dimensional model of hyperbolic geometry, we now obtain the equation

$$x^2 + y^2 + z^2 - c^2t^2 = -1$$

Minkowski writes this equation as $c^2t^2 - x^2 - y^2 - z^2 = 1$. We will see why the introduction of $c$ to the equation is so important later (77). The set of all solutions to this equation is the set of all unit timelike vectors in four variables, forming "two surfaces separated by $t = 0$, on the analogy of a hyperboloid of two sheets"
(Minkowski 77). The equation
\[ x^2 + y^2 + z^2 - c^2t^2 = 0 \]
represents the asymptotic light cone in four dimensions (analogous to the cone in Figure 1) which contains \( x^2 + y^2 + z^2 - c^2t^2 = -1 \) in its interior.

B The World Postulate.

Minkowski defines a world-point to be a point of space at a point of time, denoted by the coordinates \((x, y, z, t)\) where \(x, y, \) and \(z\) are rectangular coordinates in space and \(t\) denotes time. "The multiplicity of all thinkable \(x, y, z, t\) systems of values we will christen the world" (Minkowski 76). As a body travels through space and time, we obtain, as an image of infinitely connected world-points, a curve in the world called a world-line (Minkowski 76).

With this definition of a line in Minkowski spacetime, let us consider various situations:

1. A body "at rest" would have \(x, y, \) and \(z\) coordinates fixed with respect to the origin, creating a world-line parallel to the \(t\)-axis.

2. A body in uniform motion with respect to the origin would form a straight world-line at an angle to the \(t\)-axis.

3. A body in varying motion with respect to the origin would form a "world-line in some form of a curve" (Minkowski 80).

It is useful to be able to move the zero point of space and time to a specific world-point. When this is done, the interior of the upper half of the light cone (where \( t > 0 \)) becomes the future, while the lower half (where \( t < 0 \)) becomes the past.
The importance of $c$ in the equation $x^2 + y^2 + z^2 - c^2 t^2 = 0$ should now be discussed. Consider Figure 3, which is a cross-section of the Minkowski spacetime light cone; the plane including the $x$ and $t$ axes (where $y = 0$ and $z = 0$).

Since the three spatial axes measure distance, it is necessary to think of the time axis as measuring distance also. We can easily see that since

$$\text{distance} = (\text{rate})(\text{time}),$$

$ct$ represents distance. Any vector satisfying

$$x^2 + y^2 + z^2 - c^2 t^2 = 0$$
would then represent a body travelling at the speed of light. Thus, we preserve the
gometry and meaning of the "light cone". If in Figure 3 we let \( O c = 186,284 \) miles
and \( t = 1 \) second, the light cone can only have a slope of one if we stipulate that
\( OA = c \). Now we can see that any body which travels less than 186,284 miles in 1
second (a speed less than that of light) would be represented by a world-line with a
slope greater than one, and would be timelike. Any body travelling at a speed greater
than that of light would have a world-line with a slope less than one, and would be
spacelike.

Roger Penrose, in his article "The Geometry of the Universe", simplifies the
four-dimensional picture by choosing the speed of light to be equal to unity (\( c = 1 \)).
"This means that units of length and time become convertible into one another
according to the scheme:

\[
1 \text{ second} = 299,792,458 \text{ metres} = 186,284 \text{ miles} \\
1 \text{ year} = 1 \text{ light year}.
\]

Then the light cone of the origin is the locus

\[ t^2 = x^2 + y^2 + z^2 \] (Penrose110).

The surface of this cone also has a slope of one.

"The future light cone . . . [represents] the history of a spherical light pulse
travelling outwards from an initial flash at the center at \( t = 0 \), and the past light cone . . . is the history of a spherical light pulse converging inward" (110) (refer to Figure 2). The generators of the light cone are the world-lines of the individual photons of
the light flash. This also holds true "for any other type of 'massless' particle since
all such particles travel with the speed of light" (110). Since all massive particles, to
man's knowledge, travel with a speed less than that of light, all world-lines of massive
particles emitted in the same explosive event would lie in the interior of the future light cone (Penrose 110).

Previously, we described the movement of a body in varying motion with respect to the origin as a world-line in some form of a curve (see p.3). Penrose describes the same as "a curve - the world-line of the particle - whose tangent vectors . . . everywhere lie inside the local light cone" (111).

C Distance in Minkowski Spacetime.

Recall the inner product for hyperbolic geometry from page one. Take a world-point on some world-line of a massive particle as the vertex of the light cone (call it A) and some other world-point on the same world-line (call it B). Then let \( A = (0, 0, 0, 0) \) and \( B = (x, y, z, t) \). We note that \( b(A, B) = x^2 + y^2 + z^2 - t^2 \), and that this value is negative. Penrose defines the distance from A to B as

\[
AB^2 = t^2 - x^2 - y^2 - z^2 \quad \text{(Penrose 111),}
\]

which we can rewrite as

\[
AB = \sqrt{-b(A, B)},
\]

making all such distances real. "The significance of this 'distance' \( [AB] \) is that it measures the time-interval experienced, between the events \([A]\) and \([B]\), by a clock whose world-line is the straight path \([AB]\)" (Penrose 111-112). Note that if B is on the light cone, \( AB = 0 \), implying that if one were to ride a photon, one would experience no passage of time.

For a third world-point C on a different world-line containing B, where \( C = (x', y', z', t') \) and C is interior to the light cone of A,

\[
BC^2 = (t - t')^2 - (x - x')^2 - (y - y')^2 - (z - z')^2 \quad \text{(Penrose 112).}
\]
BC "measures the time-interval experienced by a clock in free motion from" B to C (Penrose 112).

Let us now use this definition of Minkowski spacetime distance to discover a phenomenon inexplicable through Newtonian mechanics and Euclidean geometry (here we begin to answer the question of why we use the metric of hyperbolic geometry in theoretical physics). Take three world-points:

\[ \mathbf{R} = (0, 0, 0, 0), \mathbf{S} = (5, 0, 0, 6), \mathbf{T} = (0, 0, 0, 14) \]

in the scenario that the earth travels from \( \mathbf{R} \) to \( \mathbf{T} \) in fourteen years and an astronaut travels from \( \mathbf{R} \) to \( \mathbf{S} \) in six earth years, and then from \( \mathbf{S} \) to \( \mathbf{T} \) in eight earth years (see Figure 4).

![Figure 4](image-url)
By our distance formula, we calculate that

\[
RT = \sqrt{14^2} = 14,
\]

\[
RS = \sqrt{6^2 - 5^2} \approx 3.3,
\]

\[
ST = \sqrt{8^2 - 5^2} \approx 6.2.
\]

Remembering that distance in Minkowski spacetime translates to the time-interval experienced by the body in motion, our calculations mean that those who stayed on earth aged the normal 14 years (being as a body at rest), while the astronaut aged approximately 9.5 years (being accelerated to a speed closer to that of light). Thus, we have a strange reversal of the Euclidean triangle inequality. In Minkowski spacetime,

\[
RT > RS + ST \text{ (Penrose112-113).}
\]

This upholds evidence of such a phenomenon found in "lifetimes of cosmic ray particles created in the upper atmosphere, accurate time measurements made in airplanes, [and] the behavior of particles in high energy accelerators" (112).

It is interesting to note that in special relativity, velocities are often represented as Minkowski unit vectors. We remember from hyperbolic geometry in three dimensions that the set of all timelike unit vectors creates the unit hyperbolic plane, analogous to the unit sphere in spherical geometry. In four dimensions, the set of all unit timelike vectors, representing the space of velocities in special relativity, forms the upper half of what Penrose refers to as the "Minkowski unit sphere" (113).
2. Changing Frames of Reference

A Transformation Equations.

In the Galilean transformation equations from one frame of reference to another, where one is moving with constant velocity \((V)\) with respect to the other, time is constant. These equations are

\[
\begin{align*}
x' &= x - Vt, \\
y' &= y, \\
z' &= z, \\
t' &= t \quad \text{(Kittel 346)}.
\end{align*}
\]

However, experiment upon experiment (as mentioned in the last paragraph of the previous chapter) has forced us to concede that measurement of time changes, depending upon the relative velocity of one's frame of reference.

For example, a radioactive \(\pi^+\) meson decays into a \(\mu^+\) meson and a neutrino (sic). The \(\pi^+\) meson in a frame in which it is at rest has a mean life before decaying of about \(2.5 \times 10^{-8}\) sec (Kittel 358). However, at an accelerated speed of \(c(1 - (5 \times 10^{-5}))\) their mean life has been recorded at \(2.5 \times 10^{-6}\) sec, or 100 times the proper lifetime of \(\pi^+\) mesons at rest (358). The clock with respect to the particle-in-motion's frame of reference advances more slowly than a clock at rest in the scientist's frame of reference.

Other experiments have been performed with accurate time measurements in airplanes, showing that clocks in frames of reference travelling at velocities greater than those in other frames of reference record a slower passage of time (Penrose 112).
Physicists were then presented with a puzzle: if time is not constant, what is (if anything)? In the following quote we find the answer.

The null result of the Michelson-Morley experiment to detect the drift of the earth through the ether can only be understood by making a revolutionary change in our thinking . . . . The speed of light is independent of the light source or receiver. (Kittel 345) 

The speed of light is our constant. Much of modern physics regarding the theories of relativity is based upon this premise.

Let us define a frame of reference $S$ to be at rest, and another, $S'$, to be moving in uniform motion at velocity $V$ along the $x$-axis with respect to $S$. Let a world-point in $S$ be represented by $(x, y, z, t)$ and the same point observed in $S'$ be represented by $(x', y', z', t')$. Taking the speed of light $c$ as constant, and allowing $S$ and $S'$ to coincide at $t = t' = 0$, we emit a pulse of light at the world-point $(0, 0, 0, 0)$. Because of our premise that light speed is constant regardless of the frame of reference, an observer in $S$ sees the path of the light pulse as a sphere with equation

$$x^2 + y^2 + z^2 = c^2 t^2,$$

and an observer in $S'$ sees it as a sphere with equation

$$x'^2 + y'^2 + z'^2 = c^2 t'^2 \quad \text{(Kittel 346)}.$$

Remember, the Galilean transformation equations connect "measurements in the two frames according to the equations

$$x' = x - Vt, \quad y' = y, \quad z' = z, \quad t' = t' \quad \text{" (Kittel 346).}$$

By substitution of these into the equation of the sphere in $S'$, we obtain

$$(x^2 - 2xVt + V^2 t^2) + y^2 + z^2 = c^2 t^2 \quad \text{(Kittel 346). \ (* )}$$
Since the transformation does not give us the expected
\[ x^2 + y^2 + z^2 = c^2 t^2, \]
we know the Galilean transformation is in error. "If the principle of the constancy of
the speed of light is valid, there should exist some transformation which reduces to the
Galilean as \( \frac{v}{c} \to 0 \) and which transforms \( x'^2 + y'^2 + z'^2 = c^2 t'^2 \) into \( x^2 + y^2 + z^2 = c^2 t^2 \).”
(Kittel 348). Let \( t' = t + kx \). Then (*) becomes
\[
(x^2 - 2xVt + V^2t^2) + y^2 + z^2 = (c^2 t^2 + 2c^2 ktx + c^2 k^2 x^2). (**)
\]
If we then let \(-2xVt = 2c^2 ktx\), we see that \( k = \frac{-V}{c} \). Then for our transformation
equation \( t' = t + kx \), we obtain \( t' = t + \frac{-Vx}{c^2} \). Now (**) can be written as
\[
(x^2 - 2xVt + V^2t^2) + y^2 + z^2 = [c^2 t^2 + 2c^2 (\frac{-Vx}{c^2})tx + c^2 (\frac{-Vx}{c^2})^2 x^2],
\]
which simplifies to
\[
x^2(1 - \frac{V^2}{c^2}) + y^2 + z^2 = c^2 t^2(1 - \frac{V^2}{c^2}).
\]
While this is closer to the desired \( x^2 + y^2 + z^2 = c^2 t^2 \), to make the correct transformation
we develop the following equations:
\[
x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}; \\
y' = y; \\
z' = z; \\
t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} \quad (Kittel 348).
\]
These are the Lorentz transformation equations. We can see that by using these
substitutions, the before described spherical path of the wave front of a pulse of
light \((x^2 + y^2 + z^2 = c^2t^2)\) has the same form of equation regardless of the frame of reference. We say that it is invariant under a Lorentz transformation (Kittel 349).

Now let us return to our initial example in this chapter, that of the \(\pi^+\) meson, but at a different velocity. If a beam of such particles is produced with a velocity \(V \approx 0.9c\), then a particle whose at rest position is the origin, and which has an at rest lifetime of \(2.5 \times 10^{-8}\) sec has a lifetime in our new accelerated frame of reference of

\[
\tau' = \frac{(2.5 \times 10^{-8}) - (0.9c\tau)(0)}{\sqrt{1 - (0.9c)^2}}
\]

\[
= \frac{2.5 \times 10^{-8}}{\sqrt{1 - 0.81}}
\]

\[
\approx 5.7 \times 10^{-8}\text{ sec},
\]

which is indicative of actual experiments carried out by R. P. Durbin, H. H. Loar, and W. W. Havens, Jr., Phys. Rev. 88, 179 (1952) . . . . It has been said that almost every high-energy physicist tests special relativity every day” (Kittel 358-359).

B Introduction to SO(3,1).

We will now show that distance in Minkowski spacetime is preserved by a Lorentz transformation. Recall that ”the Minkowskian distance \(OQ\) of a point \(Q\), with coordinates \((t, x, y, z)\) from the origin is given by

\[
OQ^2 = [c^2t^2 - x^2 - y^2 - z^2] \quad \text{(Penrose 111)}.
\]

(Note that \(OQ^2 = -b(O, Q)\).) Moreover, ”if \(R\) has coordinates \((t', x', y', z')\), then the Minkowski distance \(RQ\) is given by

\[
RQ^2 = c^2(t - t')^2 - (x - x')^2 - (y - y')^2 - (z - z')^2,
\]
(Penrose 112). We will show this distance to be invariant under a Lorentz transformation in the general case, which will also prove the hyperbolic inner product $b(Q, Q)$ to be invariant.

Since under this Lorentz transformation $y' = y$ and $z' = z$, we may simplify our situation by working in only one space dimension, $x$. Let $Q = (t_1, x_1)$ and $R = (t_2, x_2)$. With the transformation equations, we obtain $Q' = (t'_1, x'_1)$ and $R' = (t'_2, x'_2)$ by

$$
t'_1 = \frac{t_1 - \frac{V}{c^2} x_1}{\sqrt{1 - \frac{V^2}{c^2}}},
$$
$$
x'_1 = \frac{x_1 - \frac{V}{c} t_1}{\sqrt{1 - \frac{V^2}{c^2}}},
$$
$$
t'_2 = \frac{t_2 - \frac{V}{c^2} x_2}{\sqrt{1 - \frac{V^2}{c^2}}},
$$
$$
x'_2 = \frac{x_2 - \frac{V}{c} t_2}{\sqrt{1 - \frac{V^2}{c^2}}}.
$$

Then

$$
(R'Q')^2 = c^2(t'_1 - t'_2)^2 - (x'_1 - x'_2)^2
= c^2\left[\frac{(t_1 - \frac{V}{c^2} x_1) - (t_2 - \frac{V}{c^2} x_2)}{\sqrt{1 - \frac{V^2}{c^2}}}ight]^2
- \left[\frac{(x_1 - \frac{V}{c} t_1) - (x_2 - \frac{V}{c} t_2)}{\sqrt{1 - \frac{V^2}{c^2}}}\right]^2
= c^2\left[\frac{(t_1 - t_2) - \frac{V}{c^2} (x_1 - x_2)}{\sqrt{1 - \frac{V^2}{c^2}}}ight]^2
- \left[\frac{(x_1 - x_2) - \frac{V}{c} (t_1 - t_2)}{\sqrt{1 - \frac{V^2}{c^2}}}\right]^2
= c^2(t_1 - t_2)^2 - 2V(t_1 - t_2)(x_1 - x_2) + \frac{V^2}{c^2}(x_1 - x_2)^2
- \frac{1}{1 - \frac{V^2}{c^2}}
\left[\frac{1}{1 - \frac{V^2}{c^2}}(- (x_1 - x_2)^2 + 2V(t_1 - t_2)(x_1 - x_2) - V^2(t_1 - t_2)^2)
\right]
= \frac{(c^2 - V^2)(t_1 - t_2)^2 + (\frac{V^2}{c^2} - 1)(x_1 - x_2)^2}{1 - \frac{V^2}{c^2}}
= \frac{c^2(1 - \frac{V^2}{c^2})(t_1 - t_2)^2 - (1 - \frac{V^2}{c^2})(x_1 - x_2)^2}{1 - \frac{V^2}{c^2}}
$$

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Thus, Minkowski distance and the hyperbolic inner product are both invariant under the Lorentz transformation equations. One obvious consequence of this result is that each hyperbolic plane is invariant under the Lorentz transformation, for each is formed by vectors having the same Minkowski length. These ideas help to prepare us to discuss SO(3,1), a group of transformations in which Minkowski length is preserved. We will discuss this in more detail on page 23.
3. The Lorentz Group

Now that we have seen the necessity of the Lorentz transformation in physics, we will turn to more of a mathematical study of what is referred to as the Lorentz Group. Since in the transformation equations for a body travelling along the $x$-axis the $y$ and $z$ values remain unchanged, we again simplify our discussion by restricting ourselves to only the $t$ and $x$ axes. We can write the transformation in matrix form as

$$ T = \begin{pmatrix}
\frac{1}{\sqrt{1-\frac{V^2}{c^2}}} & \frac{-V}{\sqrt{1-\frac{V^2}{c^2}}} \\
\frac{-V}{\sqrt{1-\frac{V^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}
\end{pmatrix} $$

where for $u = \begin{pmatrix} t \\ x \end{pmatrix}$, $Tu = u'$. We should take note here that

$$ \det T = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \cdot \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} - \frac{-V}{\sqrt{1-\frac{V^2}{c^2}}} \cdot \frac{-V}{\sqrt{1-\frac{V^2}{c^2}}} $$

$$ = \frac{1 - \frac{V^2}{c^2}}{1 - \frac{V^2}{c^2}} $$

$$ = 1 $$

The fact that $\det T = 1$ will be important in identifying Lorentz transformation matrices later.

A Another Form of the Lorentz Transformation.

It would be useful to know some characteristics of the general form of a Lorentz transformation matrix. Let $L$ represent this general form, such that

$$ L = \begin{pmatrix} p & q \\ r & s \end{pmatrix} $$
Let \( u = (t, x) \). Let
\[
W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
such that
\[
\delta(u, u) = uWu^t = (t, x) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = t^2 - x^2.
\]
Since the inner product of hyperbolic geometry is invariant under a Lorentz transformation, it must be true that \( W \) is invariant under \( L \).

Thus, we have the following four results:

1. \( p^2 - q^2 = 1 \)
2. \( r^2 - s^2 = -1 \)
3. \( pr - qs = 0 \)
4. Since \( \det(W) = -1 \), then
\[
\det \left( \begin{array}{cc} p^2 - q^2 & pr - qs \\ pr - qs & r^2 - s^2 \end{array} \right) = -1.
\]
When we figure the determinant, we obtain
\[
\det \left( \begin{array}{cc} p^2 - q^2 & pr - qs \\ pr - qs & r^2 - s^2 \end{array} \right) = p^2r^2 - p^2s^2 - q^2r^2 + q^2s^2 - p^2r^2 + 2prqs - q^2s^2
\]
So, we may say that

\[(ps - qr)^2 = 1\]
\[
ps - qr = 1
\]

(this is simply \(\det \mathbf{L}\)). Thus, \(\det \mathbf{L} = 1\) is our fourth result. (But we already know this from our earlier discussion of the determinant of a Lorentz transformation matrix.)

We should mention here that

\[
\mathbf{TWT} = (c^2 - V^2) \begin{pmatrix}
    c^2(1 - V^2) & V(c^2 - 1) \\
    V(c^2 - 1) & -(c^4 - V^2)
\end{pmatrix},
\]

which is only equal to \(\mathbf{W}\) if \(c = 1\).

In Baez' chapter on Lie Groups he defines "the Lorentz transform mixing up the \(t\) and \(x\) coordinates" as the matrix

\[
\mathbf{M} = \begin{pmatrix}
    \cosh \theta & -\sinh \theta \\
    -\sinh \theta & \cosh \theta
\end{pmatrix}
\]

(Baez 164).

Here, \(\theta\) "is a convenient quantity called the rapidity, defined so that \(\tanh \theta = V\)" (Baez 10). However, this is still under the stipulation that \(c\) is unity. If you recall, in physics we use \(c\) as unity on the time axis. Mathematically, there should be a way we can move directly from \(\mathbf{T}\) to \(\mathbf{M}\), without the requirement that \(c = 1\).

If we let \(\mathbf{L} = \mathbf{M}\), we find (as we should expect) that all four of the requirements we found between \(p, q, r,\) and \(s\) are satisfied:

1. \(p^2 - q^2 = \cosh^2 \theta - (\sinh \theta)^2 = 1\)
2. \(r^2 - s^2 = (-\sinh \theta)^2 - \cosh^2 \theta = -1\)
3. \( pr - qs = -\cosh \theta \sinh \theta - (-\sinh \theta \cosh \theta) = 0 \)

4. \( ps - qr = \cosh^2 \theta - (-\sinh \theta)^2 = 1 \)

Even though we have shown that \( T \) and \( M \) both satisfy these conditions of \( L \) (for \( c = 1 \)), we must overcome the difficulty that \( M \) is symmetric, and \( T \) is not. We see that

\[
MWM^t = \begin{pmatrix} \cosh \theta & -\sinh \theta \\ -\sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh \theta & -\sinh \theta \\ -\sinh \theta & \cosh \theta \end{pmatrix} \\
= \begin{pmatrix} \cosh^2 \theta - \sinh^2 \theta & -\cosh \theta \sinh \theta + \cosh \theta \sinh \theta \\ -\cosh \theta \sinh \theta + \cosh \theta \cosh \theta & \sinh^2 \theta - \cosh^2 \theta \end{pmatrix} \\
= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
= W,
\]

whereas we have already seen that

\[
TWT = (c^2 - V^2) \begin{pmatrix} c^2(1 - V^2) & V(c^2 - 1) \\ V(c^2 - 1) & -(c^4 - V^2) \end{pmatrix},
\]

which is only equal to \( W \) if \( c = 1 \).

For the moment, let us suppose that \( T = M \).

Then \( \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} = \cosh \theta \)

and \( \frac{-V}{\sqrt{1 - \frac{V^2}{c^2}}} = -V \cosh \theta = -\sinh \theta \).

Then \( V = \frac{-\sinh \theta}{-\cosh \theta} = \tanh \theta \),

(which we already knew from Baez).
However, we also have the equation
\[
\frac{-V}{\sqrt{1 - \frac{V^2}{c^2}}} = -\frac{V}{c^2} \cosh \theta = -\sinh \theta,
\]
which would then force the result
\[
\frac{V}{c^2} = \tanh \theta.
\]

We will make a revised definition for the moment: \( \frac{V}{c} = \tanh \theta \). With this substitution into all the entries of \( T \) we obtain the following results:

1. \[
\frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{1}{\sqrt{1 - \tanh^2 \theta}} = \frac{1}{\sqrt{1 - \frac{\sinh^2 \theta}{\cosh^2 \theta}}} = \frac{1}{\cosh \theta} = \cosh \theta,
\]
2. \[
\frac{-V}{\sqrt{1 - \frac{V^2}{c^2}}} = -V \cosh \theta = -c(\frac{V}{c}) \cosh \theta = -c \tanh \theta \cosh \theta = -c \sinh \theta,
\]
3. \[
\frac{-\frac{V}{c}}{\sqrt{1 - \frac{V^2}{c^2}}} = -(\frac{V}{c^2}) \cosh \theta = -(\frac{1}{c})(\frac{V}{c}) \cosh \theta = -(\frac{1}{c}) \sinh \theta.
\]

Thus, \( T \) changes form to
\[
T_h = \begin{pmatrix}
\cosh \theta & -c \sinh \theta \\
-(\frac{1}{c}) \sinh \theta & \cosh \theta
\end{pmatrix},
\]
which is very close to \( M \). We must now look for some inner product matrix \( W_h \) such that
\[
T_h W_h (T_h)^t = W_h.
\]

First we will examine how we travel from \( M \) to \( T_h \). Let \( G \) be a matrix such that
\[
GMG^{-1} = T_h.
\]
Let
\[
G = \begin{pmatrix} 0 & g_1 \\ g_2 & 0 \end{pmatrix}.
\]
Then
\[
\begin{align*}
\text{GMG}^{-1} &= \begin{pmatrix} 0 & g_1 \\ g_2 & 0 \end{pmatrix} \begin{pmatrix} \cosh \theta & -\sinh \theta \\ -\sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} 0 & g_1 \\ g_2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -g_1 \sinh \theta & g_1 \cosh \theta \\ g_2 \cosh \theta & -g_2 \sinh \theta \end{pmatrix} \begin{pmatrix} 0 & g_1 \\ g_2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} g_1 g_2 \cosh \theta & -g_1^2 \sinh \theta \\ -g_2^2 \sinh \theta & g_1 g_2 \cosh \theta \end{pmatrix}
\end{align*}
\]

Therefore,
\[
\begin{pmatrix} g_1 g_2 \cosh \theta & -g_1^2 \sinh \theta \\ -g_2^2 \sinh \theta & g_1 g_2 \cosh \theta \end{pmatrix} = \begin{pmatrix} \cosh \theta & -c \sinh \theta \\ -(\frac{1}{c}) \sinh \theta & \cosh \theta \end{pmatrix}.
\]

By this equality we learn that
\[
\begin{align*}
g_1 &= \sqrt{c}, \\
g_1 &= \frac{1}{g_2},
\end{align*}
\]
and so
\[
g_2 = \frac{1}{\sqrt{c}}.
\]

Thus,
\[
G = \begin{pmatrix} 0 & \sqrt{c} \\ \frac{1}{\sqrt{c}} & 0 \end{pmatrix}.
\]

Now we use our claims that \(T_h W_h (T_h)^t = W_h\) and \(T_h = \text{GMG}^{-1}\). By substitution,
\[
\begin{align*}
W_h &= (\text{GMG}^{-1}) W_h (\text{GMG}^{-1})^t, \\
W_h &= \text{GMG}^{-1} W_h G^t M^t (G^{-1})^t.
\end{align*}
\]

Since \(G = G^{-1}\) and \(G^t = (G^{-1})^t\), we write
\[
W_h = \text{GMG}^{-1} W_h (G^{-1})^t M^t G^t.
\]
Now we need to recall that \( W = MWM^t \). Then we can use substitution to say that

\[
G(W)G^t = G(MWM^t)G^t = GM(G^{-1}G)W(G^t(G^{-1})^t)M'G'.
\]

Hence, we can make two observations: first that

\[
W_h = [GMG^{-1}]W_h[(G^{-1})^tM'G^t]
\]

and second that

\[
G(W)G^t = [GMG^{-1}]GWG^t[(G^{-1})^tM'G^t].
\]

These then force us to conclude that

\[
W_h = G(W)G^t.
\]

Thus,

\[
W_h = \begin{pmatrix}
0 & \sqrt{c} \\
\frac{1}{\sqrt{c}} & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
0 & \frac{1}{\sqrt{c}} \\
\sqrt{c} & 0
\end{pmatrix}

= \begin{pmatrix}
0 & -\sqrt{c} \\
\frac{1}{\sqrt{c}} & 0
\end{pmatrix}
\begin{pmatrix}
0 & \frac{1}{\sqrt{c}} \\
\sqrt{c} & 0
\end{pmatrix}

= \begin{pmatrix}
-c & 0 \\
0 & \frac{1}{c}
\end{pmatrix}.
\]

In order to test \( W_h \), we recall that we should obtain the result \( W_h = T_h W_h(T_h)^t \).

\[
T_h W_h(T_h)^t = \begin{pmatrix}
cosh \theta & -c \sinh \theta \\
-(\frac{1}{c}) \sinh \theta & \cosh \theta
\end{pmatrix}
\begin{pmatrix}
-c & 0 \\
0 & \frac{1}{c}
\end{pmatrix}
\begin{pmatrix}
cosh \theta & -(\frac{1}{c}) \sinh \theta \\
-c \sinh \theta & \cosh \theta
\end{pmatrix}

= \begin{pmatrix}
-c \cosh \theta & -\sinh \theta \\
\sinh \theta & \frac{1}{c} \cosh \theta
\end{pmatrix}
\begin{pmatrix}
cosh \theta & -(\frac{1}{c}) \sinh \theta \\
-c \sinh \theta & \cosh \theta
\end{pmatrix}
\]

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Since for \( \frac{V}{c} = \tanh \theta \) we know that \( T = T_h \), then \( W_h \) is an equivalent bilinear form (inner product matrix) for the non-symmetric form of the Lorentz transformation matrix \( T_h \) that was derived from \( T \). So, we have found a connection: \( T \) and \( T_h \) both have the bilinear form \( W_h \), while \( M \) has the regular hyperbolic geometry bilinear form \( W \), and \( GMG^{-1} = T_h \).

From this point forward we will restrict our study of the Lorentz Group to the form of symmetric matrices with hyperbolic trigonometric entries (like \( M \))(much of the preceding has been taken from instruction by Sarli).

B The Special Orthogonal Groups.

In two-dimensional Euclidean geometry, the set of all rotations about the origin is a group called \( \text{SO}(2) \), the Special Orthogonal group, consisting of all \( \text{rot} \theta \) where

\[
\text{rot} \theta = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\]

Note that \( \det[\text{rot} \theta] = 1 \).

Likewise, in Minkowski spacetime, if we return to three space dimensions and one time dimension, we have a general form of the Lorentz transformation which performs
rotations involving the \( t \) and \( x \) axes, and where \( y' = y \) and \( z' = z \):

\[
L_{t,x} = \begin{pmatrix}
\cosh \theta & -\sinh \theta & 0 & 0 \\
-\sinh \theta & \cosh \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

SO(3,1) is the Special Orthogonal group of all such 4 \( \times \) 4 matrices with a det = 1 and which preserve the Minkowski spacetime inner product. All types of these rotations make up the Lorentz Group. "It contains the spatial rotations in an obvious way, but also contains the Lorentz transformations that mix up space and time coordinates" (Baez 163). (We should note here that O(3,1) is the Orthogonal group of all such 4 \( \times \) 4 matrices with det = \( \pm 1 \), comprised of all reflections and rotations about the origin.)

We now show that the inner product for Minkowski spacetime is invariant in these four dimensional transformations. Consider a vector \( u = (t, x, y, z) \). Then

\[
u' = \begin{pmatrix}
\cosh \theta & -\sinh \theta & 0 & 0 \\
-\sinh \theta & \cosh \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
t \\
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
t \cosh \theta - x \sinh \theta \\
-t \sinh \theta + x \cosh \theta \\
y \\
z
\end{pmatrix} \begin{pmatrix}
t' \\
x' \\
y' \\
z'
\end{pmatrix}.
\]

Now

\[
b(u', u') = (t \cosh \theta - x \sinh \theta)^2 - (-t \sinh \theta + x \cosh \theta)^2 - y^2 - z^2
\]

\[
= t^2 \cosh^2 \theta - xt \sinh \theta \cosh \theta + x^2 \sinh^2 \theta
\]

\[
- t^2 \sinh^2 \theta + xt \sinh \theta \cosh \theta - x^2 \cosh^2 \theta - y^2 - z^2
\]

\[
= t^2 (\cosh^2 \theta - \sinh^2 \theta) - x^2 (\cosh^2 \theta - \sinh^2 \theta) - y^2 - z^2
\]

\[
= t^2 - x^2 - y^2 - z^2
\]

\[
= b(u, u).
\]
Hence, $L_{t,x}$ preserves the inner product (and $\det L_{t,x} = 1$), so it is an element of $SO(3, 1)$.

There are of course other members of this group representing other rotations:

$$L_{t,y} = \begin{pmatrix} \cosh \theta & 0 & -\sinh \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh \theta & 0 & \cosh \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{which mixes axes } t \text{ and } y,$$

$$L_{t,z} = \begin{pmatrix} \cosh \theta & 0 & 0 & -\sinh \theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh \theta & 0 & 0 & \cosh \theta \end{pmatrix} \quad \text{which mixes axes } t \text{ and } z,$$

$$L_{x,y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{which mixes axes } x \text{ and } y,$$

$$L_{y,z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix} \quad \text{which mixes axes } y \text{ and } z,$$

$$L_{x,z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & 0 & -\sin \theta \\ 0 & 0 & 1 & 0 \\ 0 & \sin \theta & 0 & \cos \theta \end{pmatrix} \quad \text{which mixes axes } x \text{ and } z.$$

These six matrices are canonical transformations from which we can understand the dimension of the Lorentz Group as a Lie group. These matrices generate the Lorentz
Group as $\theta$ varies. In fact, each of the above generators forms a one-parameter subgroup as $\theta$ varies. By differentiating with respect to $\theta$ and evaluating at $\theta = 0$, one obtains an element of the tangent space to the Lorentz Group at the identity. We are taking the view here that SO(3, 1) is a manifold. The fact that in addition it is also a Lie group tells us that this tangent space has the structure of a Lie algebra of dimension six. One should note here that physicists regard these Lie algebra matrices themselves as the (infinitesimal) generators of the Lorentz Group (Sarli). The derivative of $L_{t,x}$ with respect to $\theta$ is

$$\frac{dL_{t,x}}{d\theta} = \begin{pmatrix} \sinh \theta & -\cosh \theta & 0 & 0 \\ -\cosh \theta & \sinh \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

When this is evaluated at $\theta = 0$, we obtain

$$\frac{dL_{t,x}}{d\theta} [0] = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To simplify our notation, we will say that

$$D_{t,x} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The other five resultant matrices of this form are
The elements of the Lie algebra \( so(3,1) \), which has the operation \([AB] = AB - BA\). For example,

\[
[D_{t,y}, D_{x,y}] = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
- \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
- \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= \mathbf{D}_{t,z}.
\]

The table of operation is shown below.

<table>
<thead>
<tr>
<th>[A, B]</th>
<th>D_{t,z}</th>
<th>D_{t,y}</th>
<th>D_{t,z}</th>
<th>D_{x,y}</th>
<th>D_{y,z}</th>
<th>D_{x,z}</th>
</tr>
</thead>
<tbody>
<tr>
<td>D_{t,z}</td>
<td>0</td>
<td>-D_{x,y}</td>
<td>D_{x,z}</td>
<td>-D_{t,y}</td>
<td>0</td>
<td>D_{t,z}</td>
</tr>
<tr>
<td>D_{t,y}</td>
<td>D_{x,y}</td>
<td>0</td>
<td>-D_{y,z}</td>
<td>D_{t,x}</td>
<td>-D_{t,z}</td>
<td>0</td>
</tr>
<tr>
<td>D_{t,z}</td>
<td>-D_{x,z}</td>
<td>D_{y,z}</td>
<td>0</td>
<td>0</td>
<td>D_{t,y}</td>
<td>-D_{t,x}</td>
</tr>
<tr>
<td>D_{x,y}</td>
<td>D_{t,y}</td>
<td>-D_{t,x}</td>
<td>0</td>
<td>0</td>
<td>-D_{x,z}</td>
<td>D_{y,z}</td>
</tr>
<tr>
<td>D_{y,z}</td>
<td>0</td>
<td>D_{t,x}</td>
<td>-D_{t,y}</td>
<td>D_{x,z}</td>
<td>0</td>
<td>D_{x,y}</td>
</tr>
<tr>
<td>D_{x,z}</td>
<td>-D_{t,z}</td>
<td>0</td>
<td>D_{t,x}</td>
<td>-D_{y,z}</td>
<td>-D_{x,y}</td>
<td>0</td>
</tr>
</tbody>
</table>

The table is anti-symmetric (symmetric, but with opposite signs) and the quadrants show definite patterns, which teach us about combinations of rotations in Minkowski spacetime:

1. When both matrices are strictly spatial (like \([D_{x,y}, D_{x,z}]\)) or both involve time (like \([D_{t,y}, D_{t,z}]\)), the result is strictly spatial.

2. When one matrix is strictly spatial and the other involves time (like \([D_{x,y}, D_{t,y}]\)), the result involves time.

3. A matrix operated with itself results in the zero matrix (the identity rotation).

4. When all four axes are involved in the operation (like \([D_{x,y}, D_{t,z}]\)), the result is the zero matrix.
4. Maxwell’s Equations

One of my goals in researching hyperbolic geometry was to find applications. It turns out that Maxwell’s equations (involving the elements of the electromagnetic field) are also invariant under a Lorentz transformation (we will use \( L_{t,x} \)). In fact, these equations were actually the inspiration for general relativity (Baez 7). In Maxwell’s equations we will use the following notation:

1. \( \vec{B} \) will represent the magnetic vector field, which is defined in \( \mathbb{R}^3 \) and is a function of time \( t \).
2. \( \vec{E} \) will represent the electric vector field, which is defined in \( \mathbb{R}^3 \) and is a function of time \( t \).
3. \( \vec{j} \) will represent the electric current density, a vector field which is defined in \( \mathbb{R}^3 \) and is time-dependent.
4. \( \rho \) will represent the electric charge density, which is a time-dependent function on space, and which also depends upon \( \vec{j} \).

As with the symmetric form of the Lorentz transformation matrices, for Maxwell’s equations we must arrange our units such that \( c = 1 \). The equations are:

\[
\begin{align*}
\nabla \cdot \vec{B} &= 0, \\
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0, \\
\n\nabla \cdot \vec{E} &= \rho, \\
\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= \vec{j}.
\end{align*}
\]
To show invariance of these equations under a Lorentz transformation we must list the results of the transformation on each element of these equations. Just as for

\[ u = (t, x, y, z), \]

\[
\begin{pmatrix}
\cosh \theta & -\sinh \theta & 0 & 0 \\
-\sinh \theta & \cosh \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
t \\
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
t \cosh \theta - x \sinh \theta \\
-t \sinh \theta + x \cosh \theta \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
t' \\
x' \\
y' \\
z'
\end{pmatrix},
\]

we find that

\[
\rho' = \rho \cosh \theta - j_x \sinh \theta,
\]

\[
j'_x = -\rho \sinh \theta + j_x \cosh \theta,
\]

\[
j'_y = j_y,
\]

\[
j'_z = j_z.
\]

The electric and magnetic fields, however, render more complicated transformations:

\[
E_x' = E_x,
\]

\[
E_y' = E_y \cosh \theta - B_z \sinh \theta,
\]

\[
E_z' = B_y \sinh \theta + E_z \cosh \theta,
\]

\[
B_x' = B_x,
\]

\[
B_y' = B_y \cosh \theta + E_z \sinh \theta,
\]

\[
B_z' = -E_y \sinh \theta + B_z \cosh \theta.
\]
We are given that $\mathbf{V} \cdot \mathbf{S} = 0$. We expect to find that $\mathbf{V} \cdot \mathbf{B'} = 0$.

$$\nabla \cdot \mathbf{B'} = \frac{\partial B'_x}{\partial x'} + \frac{\partial B'_y}{\partial y'} + \frac{\partial B'_z}{\partial z'}$$

$$= \frac{\partial B_x}{\partial x'} + \frac{\partial (B_y \cosh \theta + E_z \sinh \theta)}{\partial y'} + \frac{\partial (-E_y \sinh \theta + B_z \cosh \theta)}{\partial z'}.$$  

We can see at this point that since $y' = y$ and $z' = z$, the equation may be rewritten as

$$\nabla \cdot \mathbf{B'} = \frac{\partial B_x}{\partial x'} + \frac{\partial (B_y \cosh \theta + E_z \sinh \theta)}{\partial y'} + \frac{\partial (-E_y \sinh \theta + B_z \cosh \theta)}{\partial z'}.$$  

Our problem is with $\frac{\partial \mathbf{B}}{\partial x'}$. We must develop a chain rule to accommodate $x'$. Remember that

$$\begin{pmatrix} \cosh \theta & -\sinh \theta \\ -\sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t' \\ x' \end{pmatrix}$$

so that

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}.$$  

Now let $w(t', x') = w(t(t', x'), x(t', x'))$. Then

$$\frac{\partial w}{\partial t'} = \frac{\partial w}{\partial t} \frac{\partial t}{\partial t'} + \frac{\partial w}{\partial x} \frac{\partial x}{\partial t'},$$

and

$$\frac{\partial w}{\partial x'} = \frac{\partial w}{\partial t} \frac{\partial t}{\partial x'} + \frac{\partial w}{\partial x} \frac{\partial x}{\partial x'}.$$  

We find that

$$\frac{\partial t}{\partial t'} = \cosh \theta,$$

$$\frac{\partial x}{\partial t'} = \sinh \theta,$$

$$\frac{\partial t}{\partial x'} = \sinh \theta,$$

and $\frac{\partial x}{\partial x'} = \cosh \theta$.  

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Therefore,

\[ \frac{\partial w}{\partial t'} = \frac{\partial w}{\partial t} \cosh \theta + \frac{\partial w}{\partial x} \sinh \theta \]

and

\[ \frac{\partial w}{\partial x'} = \frac{\partial w}{\partial t} \sinh \theta + \frac{\partial w}{\partial x} \cosh \theta. \]

Now we are ready.

\[
\nabla \cdot \vec{B}' = \frac{\partial B_z}{\partial x'} + \frac{\partial (B_y \cosh \theta + E_z \sinh \theta)}{\partial y} + \frac{\partial (-E_y \sinh \theta + B_z \cosh \theta)}{\partial z} \\
= \frac{\partial B_z}{\partial t} \sinh \theta + \frac{\partial B_z}{\partial x} \cosh \theta + \frac{\partial B_y}{\partial y} \cosh \theta \\
+ \frac{\partial E_z}{\partial y} \sinh \theta - \frac{\partial E_z}{\partial z} \sinh \theta + \frac{B_z}{\partial z} \cosh \theta \\
= \cosh \theta \left( \frac{\partial B_z}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{B_z}{\partial z} \right) + \sinh \theta \left[ \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \frac{\partial B_z}{\partial t} \right]
\]

We know that

\[
\left( \frac{\partial B_z}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{B_z}{\partial z} \right) = \nabla \cdot \vec{B} = 0.
\]

We also know that

\[
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0,
\]

and that \( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial x} \) and \( \frac{\partial B_z}{\partial t} \) are simply the \( x \) components of those vectors, therefore

\[
\left[ \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \frac{\partial B_z}{\partial t} \right] = 0.
\]

Hence, \( \nabla \cdot \vec{B}' = 0. \)
For Maxwell’s second equation, we expect that $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$. Let us take each component separately. The $x$ component is

$$
\left( \frac{\partial E'_x}{\partial y'} - \frac{\partial E'_y}{\partial x'} \right) + \frac{\partial B'_z}{\partial t'} = \left( \frac{\partial (B_y \sinh \theta + E_z \cosh \theta)}{\partial y} - \frac{\partial (E_y \cosh \theta - B_z \sinh \theta)}{\partial z} \right) \\
+ \frac{\partial B_z}{\partial t'} \\
= \left( \frac{\partial B_y}{\partial y} \sinh \theta + \frac{E_z}{\partial y} \cosh \theta - \frac{E_y}{\partial y} \cosh \theta + \frac{B_z}{\partial y} \sinh \theta \right) \\
+ \frac{\partial B_y}{\partial t} \cosh \theta + \frac{\partial B_z}{\partial t} \sinh \theta \\
= \sinh \theta \left( \frac{\partial B_y}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{B_z}{\partial z} \right) \\
+ \cosh \theta \left[ \left( \frac{E_z}{\partial y} - \frac{E_y}{\partial z} \right) + \frac{\partial B_z}{\partial t} \right]
$$

As in Maxwell’s first equation,

$$
\left( \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{B_z}{\partial z} \right) = \nabla \cdot \vec{B} = 0,
$$

and

$$
\left[ \left( \frac{E_z}{\partial y} - \frac{E_y}{\partial z} \right) + \frac{\partial B_z}{\partial t} \right] = 0
$$

(the $x$ component of $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t}$). Therefore, the $x$ component of $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t}$ is zero.

The $y$ component is

$$
\left( \frac{\partial E'_y}{\partial z'} - \frac{\partial E'_z}{\partial x'} \right) + \frac{\partial B'_x}{\partial t'} = \left( \frac{\partial E_z}{\partial z} - \frac{\partial (B_y \sinh \theta + E_z \cosh \theta)}{\partial x} \right) \\
+ \frac{\partial (B_y \cosh \theta + E_z \sinh \theta)}{\partial t'}
$$
\[
\frac{\partial E_z}{\partial z} - \left( \frac{\partial B_x}{\partial t} \sinh \theta + \frac{\partial B_y}{\partial x} \cosh \theta \right) \sinh \theta \\
- \left( \frac{E_z}{\partial t} \sinh \theta + \frac{E_z}{\partial x} \cosh \theta \right) \cosh \theta \\
+ \left( \frac{\partial B_y}{\partial t} \cosh \theta + \frac{\partial B_y}{\partial x} \sinh \theta \right) \cosh \theta \\
+ \left( \frac{E_z}{\partial t} \cosh \theta + \frac{E_z}{\partial x} \sinh \theta \right) \sinh \theta
\]

\[
\frac{\partial E_z}{\partial z} - \frac{\partial B_y}{\partial t} \sinh^2 \theta - \frac{\partial B_y}{\partial x} \cosh \theta \sinh \theta \\
- \frac{E_z}{\partial t} \sinh \theta \cosh \theta - \frac{E_z}{\partial x} \cosh^2 \theta + \frac{\partial B_y}{\partial t} \cosh^2 \theta \\
+ \frac{\partial B_y}{\partial x} \sinh \theta \cosh \theta + \frac{E_z}{\partial t} \cosh \theta \sinh \theta + \frac{E_z}{\partial x} \sinh^2 \theta
\]

\[
\frac{\partial E_z}{\partial z} - \frac{E_z}{\partial x} \left( \cosh^2 \theta - \sinh^2 \theta \right) + \frac{\partial B_y}{\partial t} \left( \cosh^2 \theta - \sinh^2 \theta \right)
\]

\[
= \left( \frac{\partial E_z}{\partial z} - \frac{E_z}{\partial x} \right) + \frac{\partial B_y}{\partial t}
\]

\[
= 0
\]

The \( z \) component of \( \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \) is

\[
\left( \frac{\partial E_y}{\partial x'} - \frac{\partial E_x}{\partial y'} \right) + \frac{\partial B_z}{\partial t'} = \left( \frac{\partial (E_y \cosh \theta - B_z \sinh \theta)}{\partial x'} - \frac{\partial E_x}{\partial y'} \right) \\
+ \frac{\partial (-E_y \sinh \theta + B_z \cosh \theta)}{\partial t'}
\]

\[
= \left( \frac{E_y}{\partial t} \sinh \theta + \frac{E_y}{\partial x} \cosh \theta \right) \cosh \theta \\
- \left( \frac{\partial B_z}{\partial t} \sinh \theta + \frac{\partial B_z}{\partial x} \cosh \theta \right) \sinh \theta - \frac{\partial E_z}{\partial y} \\
+ \left( \frac{\partial B_z}{\partial t} \cosh \theta + \frac{\partial B_z}{\partial x} \sinh \theta \right) \cosh \theta \\
- \left( \frac{E_y}{\partial t} \cosh \theta + \frac{E_y}{\partial x} \sinh \theta \right) \sinh \theta
\]
Since all components of \( \nabla \times \vec{E} \) are zero, it follows that
\[
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0.
\]

For Maxwell’s third equation, we would expect that \( \nabla \cdot \vec{E} = \rho’ \).
\[
\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}
= \frac{\partial E_x}{\partial x} + \frac{\partial}{\partial y} \left( E_y \cosh \theta - B_z \sinh \theta \right) + \frac{\partial}{\partial z} \left( B_y \sinh \theta + E_z \cosh \theta \right)
\]
\[
= \frac{\partial E_x}{\partial t} \sinh \theta + \frac{\partial E_x}{\partial x} \cosh \theta + \frac{\partial E_y}{\partial y} \cosh \theta - \frac{\partial B_z}{\partial y} \sinh \theta
+ \frac{\partial B_y}{\partial z} \sinh \theta + \frac{\partial E_z}{\partial z} \cosh \theta
\]
\[
= \cosh \theta \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) - \sinh \theta \left[ \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) - \frac{\partial E_x}{\partial t} \right]
\]
Similar to the case of Maxwell’s first equation, we see that
\[
\left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = \nabla \cdot \vec{E} = \rho.
\]
Also,
\[
\left[ \left( \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) - \frac{\partial E_x}{\partial t} \right] = j_z.
\]
(the $x$ component of $\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t}$). Therefore

$$\nabla \cdot \vec{E} = \rho \cosh \theta - j_z \sinh \theta$$

$$= \rho'.$$

And now for the fourth and final equation. We expect that $\nabla \times \vec{B}' - \frac{\partial \vec{E}}{\partial t'} = \vec{j}'$. As in the case of the second equation, we will figure each component's calculations separately.

The $x$ component is

$$\left( \frac{\partial B'_x}{\partial y'} - \frac{\partial B'_y}{\partial z'} \right) - \frac{E'_z}{\partial t'} = \frac{-\partial E_y}{\partial y} \sinh \theta + \frac{\partial B_z}{\partial y} \cosh \theta - \frac{\partial B_y}{\partial z} \cosh \theta$$

$$- \frac{\partial E_z}{\partial z} \sinh \theta - \frac{\partial E_z}{\partial t} \cosh \theta - \frac{\partial E_z}{\partial x} \sinh \theta$$

$$= -\sinh \theta \left( \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right)$$

$$+ \cosh \theta \left[ \left( \frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial z} \right) - \frac{\partial E_z}{\partial t} \right]$$

$$= -\rho \sinh \theta + j_z \cosh \theta$$

$$= j'_x.$$
The $z$ component of $\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t'}$ is

$$\left( \frac{\partial B_y'}{\partial x'} - \frac{\partial B_z'}{\partial y'} \right) - \frac{E_z'}{\partial t'} = \frac{\partial (B_y \cosh \theta + E_z \sinh \theta)}{\partial x'} - \frac{\partial B_z}{\partial y} - \frac{\partial (E_z \cosh \theta + B_y \sinh \theta)}{\partial t'}$$

$$= \frac{\partial B_y}{\partial t} \sinh \theta \cosh \theta + \frac{\partial B_y}{\partial x} \cosh^2 \theta + \frac{\partial E_z}{\partial t} \sinh^2 \theta$$

$$+ \frac{\partial E_z}{\partial x} \sinh \theta \cosh \theta - \frac{\partial B_z}{\partial y} - \frac{\partial E_z}{\partial t} \cosh^2 \theta$$

$$= \frac{\partial B_y}{\partial x} \left( \cosh^2 \theta - \sinh^2 \theta \right) - \frac{\partial B_z}{\partial y} - \frac{\partial E_z}{\partial t} \left( \cosh^2 \theta - \sinh^2 \theta \right)$$

$$= \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_z}{\partial y} \right) - \frac{\partial E_z}{\partial t}$$

$$= j_z$$

Thus,

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t'} = \vec{j'}.$$

Therefore, we have shown that Maxwell's equations are invariant under a Lorentz transformation.
BIBLIOGRAPHY


