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A study of finite and linear elasticity

Fen Rui Johnson

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A STUDY OF FINITE AND LINEAR ELASTICITY

A Project
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Mathematics

by
Fen Rui Johnson

June 1996
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Approved by:

Dr. John Sarli, Chair, Mathematics
Dr. Javier Torner, Physics
Dr. Chetan Prakash, Mathematics
Dr. J. Paul Vicknair, Department Chair, Mathematics

Date 5/6/96
ABSTRACT

The purpose of this paper is to study the basic concepts of finite and linear elasticity such as stress, strain and displacement, and to find out the relationship between them. It is important to know that linear elasticity is the special case of finite elasticity. Although linear elasticity was developed from finite elasticity, they have very different characters. This can be seen from the examples presented in the paper.
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CHAPTER 1
FINITE ELASTICITY

Finite elasticity is a theory of elastic materials capable of undergoing large deformation. This is a non-linear theory. To study the basic concepts of finite elasticity and to learn the difference between finite and linear elasticity, we need to look at the following definitions.

A body $B$ is a closed, connected set in $\mathbb{R}^3$ with piecewise smooth boundary. Points in $B$ are called material points and are denoted by $p$.

A deformation of $B$ is a smooth one to one map

$$u: B \rightarrow \mathbb{R}^3$$

$$u(p) = x$$

Thus, $u$ is a vector function.

The displacement is

$$u(p) - p = x - p$$

The point $u(p)$ is the place occupied by $p$ in the deformation $u$. $F = \nabla u$ is the deformation gradient. $\nabla u$ is a tensor field with components $(\nabla u)_{ij} = \partial u_i / \partial x_j$. Thus, we can think of $\nabla u$ as the Jacobian matrix, i.e., the classical derivative of $u$. In what follows we assume

$$\det \nabla u > 0$$

We let $F^T$ denote the transpose of $F$.

A mapping $x \rightarrow \Phi(x)$ with domain $u(B)$ is called a spatial field.

In finite elasticity surface forces can be modeled as smooth functions $t$ which associate a vector in $\mathbb{R}^3$ with a given point $x$ in $u(B)$ and a given unit vector $n$.

i.e.

$$x \rightarrow t(n, x)$$

$t(n, x)$ represents the force per unit area at $x$ on any oriented surface through $x$ with positive unit normal $n$. The reason the surface force is defined this way is because we treat
x as a point in a deformed body which the surface goes through. Surface forces arise when there is a physical contact with another body. Body forces may be modeled as continuous functions which associate a vector in \( \mathbb{R}^3 \) with each point \( x \in u(B) \).

\[ x \rightarrow b(x) \]

where \( b(x) \) is the force exerted at \( x \) per unit volume.

The laws of force and moments lead to a basic law of continuum mechanics known as Cauchy’s Theorem.

**Cauchy’s Theorem**

There exists a smooth, symmetric spatial tensor field \( T \) such that

\[ t(n, x) = T(x) \cdot n \]

for every unit vector \( n \) and all \( x \in u(B) \). Further

\[ \text{div} \ T + b = 0 \]

where \( \text{div} \ T \) is the vector field with components \( (\text{div} T)_i = \partial T_{ij} / \partial x_j \) and the usual summation is assumed.

Note: \( T \) is called the Cauchy stress which is usually just called stress. Also, the equation

\[ t(n, x) = T(x)n \]

is called the constitutive equation, \( b \) is the body force.

The main example we wish to work out in this chapter is about simple shears of a homogeneous, isotropic cube. We say a material is isotropic if the properties relating to its behavior under stress are the same in any direction at each point.

Let \( B \) be a homogeneous, isotropic body in the shape of a cube. Consider the deformation \( x = u(p) \) defined by

\[ x_1 = p_1 + \gamma p_2 \]
\[ x_2 = p_2 \]
\[ x_3 = p_3 \]

where \( \gamma = \tan \theta \)
is called the shearing strain, which will be precisely defined in chapter 3.

\[ \begin{bmatrix} e_2 \\ B \\ e_1 \end{bmatrix} \xrightarrow{u} \begin{bmatrix} e_2 \\ \theta \\ e_1 \end{bmatrix} \]

\( B \) is a homogeneous body, i.e., \( B \) is a body in which the corresponding stress \( T \) is constant. Also, \( T \) satisfies the equation of equilibrium \( \text{div} \ T = 0 \), for it is known from the general theory that a homogeneous body can be deformed without body force, so \( b = 0 \).

We calculate the deformation gradient

\[
F = \begin{bmatrix}
1 & \gamma & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

so

\[
F^T = \begin{bmatrix}
1 & 0 & 0 \\
\gamma & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

From these we calculate a quantity known as the \textit{left} Cauchy-Green strain tensor \( B = FF^T \)

\[
B = \begin{bmatrix}
1 & \gamma & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
\gamma & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 + \gamma^2 & \gamma & 0 \\
\gamma & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

From \( B \) we have \( B^{-1} \).
\begin{align*}
B^{-1} &= \begin{bmatrix} 1 & -\gamma & 0 \\
-\gamma & 1 + \gamma^2 & 0 \\
0 & 0 & 1 \end{bmatrix}
\end{align*}

and we calculate

\begin{align*}
B^2 &= \begin{bmatrix}
1 + 3\gamma^2 + \gamma^4 & 2\gamma + \gamma^3 & 0 \\
2\gamma + \gamma^3 & 1 + \gamma^2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\end{align*}

\begin{align*}
\text{tr}(B) &= A_{11} + A_{22} + A_{33} = 3 + \gamma^2 \\
\text{tr}(B^2) &= 3 + 4\gamma^2 + \gamma^4.
\end{align*}

From tensor analysis we know we can express $T$ in terms of three principal invariants:

\begin{align*}
l_1(B) &= \text{tr}(B) = 3 + \gamma^2 \\
l_2(B) &= \frac{1}{2} \left[ \left( \text{tr}(B) \right)^2 - \text{tr}(B^2) \right] = 3 + \gamma^2 \\
l_3(B) &= \text{det} B = 1
\end{align*}

hence the list of principal invariants of $B$ is

\begin{align*}
L_B = [l_1(B), l_2(B), l_3(B)] = (3 + \gamma^2, 3 + \gamma^2, 1).
\end{align*}

Therefore by the constitutive equation for an isotropic material, we have

\begin{align*}
T &= \beta_0 I + \beta_1 B + \beta_2 B^{-1}
\end{align*}

where $\beta_1$'s are scalar functions of $L_B$.

This is equivalent to

\begin{align*}
\begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{12} & T_{22} & T_{23} \\
T_{13} & T_{23} & T_{33}
\end{bmatrix}
&= \beta_0 \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix}
+ \beta_1 \begin{bmatrix} 1 + \gamma^2 & \gamma & 0 \\
\gamma & 1 & 0 \\
0 & 0 & 1 \end{bmatrix}
+ \beta_2 \begin{bmatrix} 1 & -\gamma & 0 \\
-\gamma & 1 + \gamma^2 & 0 \\
0 & 0 & 1 \end{bmatrix}
\end{align*}
Hence $T_{13}=T_{23}=0$ and 

$$T_{12} = \beta_1 \gamma + \beta_2 (-\gamma)$$

so 

$$\frac{T_{12}}{\gamma} = \mu$$

where 

$$\mu = \beta_1 - \beta_2$$

is a quantity called the \textit{generalized shear modulus}. 

In linear elasticity theory the normal stresses $T_{11}$, $T_{22}$, and $T_{33}$ in simple shear are zero, which we will learn in chapter 4. Here 

$$T_{11} = \gamma^2 \beta_1 + \tau$$  
$$T_{22} = \gamma^2 \beta_2 + \tau$$  
$$T_{33} = \tau$$

where $\tau$ is \textit{shear stress} (which will be defined in Chapter 2) and $\tau = \beta_0 + \beta_1 + \beta_2$. 

This shows there are relations between shear stress and normal stress in finite elasticity. In order to produce a pure shear the normal stresses need to be zero. By the equations we have above if $\beta_1 = \beta_2 = 0$, then the normal stresses will vanish. Hence $\tau = \beta_0$.

This implies 

$$\mu = 0.$$ 

Thus if 

$$\mu \neq 0$$

then it is impossible to produce a simple (pure) shear by applying shear stresses alone because according to the equations we have, the normal stresses exist. This will not be the case in linear elasticity where there is no relationship between normal stresses and shear stresses.

This example is from “Topics in Finite Elasticity” by Morton E. Gurtin.
After this brief introduction of finite elasticity, I will focus on the linear elasticity in the next four chapters, then conclude the project with another example of finite elasticity.
CHAPTER 2
STRESS IN LINEAR ELASTICITY

In a body that is not deformed, all parts of the body are in mechanical equilibrium. This means the resultant force on the body is zero. When a deformation occurs the body will try to go back to its original state of equilibrium. Forces therefore arise which tend to return the body to equilibrium. These forces are called \textit{internal forces}. The internal forces depend on the external forces, the body forces and the surface forces. Body forces are the forces that are connected with the mass of the body and are distributed throughout the volume of a body; they are not the result of the contact of two bodies. Forces such as gravitational, magnetic, and inertia forces are body forces. They are measured in terms of force per unit volume. On the other hand surface forces arise from the physical contact between two bodies.

We can picture an imaginary surface within a body acting on an adjacent surface.

![Figure 2.1 External surface forces and internal forces](image)

Figure 2.1 External surface forces and internal forces.
The internal forces which occur when a body is deformed are called internal stress. This means the internal forces became internal stresses under deformation. If no deformation occurs, there are no internal stresses. The average force per unit area is

\[ P_{\text{AVE}} = \frac{\Delta F}{\Delta A} \]

From this we have the definition of stress, which is the limit value of average forces per unit area as the area of \( \Delta A \to 0 \).

\[ \text{i.e. } P = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}. \]

Here the stress on the area \( dA \) is a vector and has the same direction as the force vector \( dF \).

One thing we should be clear on is that stress is not a vector unless a specific plane is given. This means we can only represent the stress as a stress vector on a specific plane. When several stress act on the same plane, at this point we treat stress as a vector and can use rule for vector addition. Therefore we cannot talk about the stress on a point since through a point we can draw infinitely many planes and that will give us infinitely many vectors on different planes. Thus, stress does not behave as a vector because a vector quantity associates only a scalar (its component) with each direction in space whereas stress does not.

When a stress vector is decomposed, we call its components stress components. Let's look at an example
B is a point. Through B we draw plane 1 and plane 2 and we can see the stress on these two lines are different so the stress does not behave as a vector.

Although we cannot determine stress at a point, we can determine state of stress at a point. To determine the state of stress at a point we determine the stress components with respect to the orthogonal coordinates. The above example shows that we can determine the state of stress at point B because the description we set allows us to determine the stress on every plane that passes point B. Therefore, if we are given the stress on two planes (here the stress is a vector), then we will know the stress vector on any plane that goes through the point.

We conclude that in order to completely define a stress vector we must specify its magnitude, direction and the plane which it acts on.
In general, we use the term stress to represent stress vector, stress component, or stress tensor, as will be clear from context.

There are two ways to express the components of a stress vector; the \( x, y, z \) components and the normal and tangential (shearing) components. In the case of normal and shearing components of both external and internal stresses we let \( \sigma \) represent the normal stress which is the component of stress perpendicular to the plane on which it acts. Let \( \tau \) be the shear stress which is the component of stress that lies in the plane.

\[
\sigma
\]

\[
P^2 = \sigma^2 + \tau^2
\]

Fig 2.3 Normal and Shearing Stress Components

If the state of stress in a body is the same at all points, this body is said to be in a uniform state of stress.

The concept of state of stress at a point leads to the transformation of stress equations.
\[ F = P \times \text{Area} \]

Let \( Z = 1 \) be the uniform depth of a solid.

\[ F_x(b/2) - F_x(b/2) + F_y(a/2) - F_y(a/2) - \tau_{yx} a b - (\tau_{xy} b) a = 0 \]

By simplying, we have

\[ \tau_{yx} ab = \tau_{xy} ab \]

Since \( ab \neq 0 \)

So

\[ \tau_{yx} = \tau_{xy} \]

This means the stress component on the x plane acting in y direction is equal to the stress component on the y plane acting in x direction. For any two perpendicular planes we will have this result. Now let’s see how the transformation equations are developed.

Let \( \Sigma F_x = 0 \)

\[ F_{x1} - F_{x2} - \tau_{yx} A \sin \alpha = 0 \]

\[ P_x A - \sigma_x A \cos \alpha - \tau_{yx} A \sin \alpha = 0 \]

\[ P_x = \sigma_x \cos \alpha + \tau_{yx} \sin \alpha . \quad (1) \]

Similarly

\[ P_y = \sigma_y \sin \alpha + \tau_{yx} \cos \alpha . \quad (2) \]
By projecting $P_x$ and $P_y$ in the $x'$ direction, we get

$$\sigma_{x'} = P_x \cos \alpha + P_y \sin \alpha.$$  

By substituting (1) and (2) we get

$$\sigma_{x'} = \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha + 2 \tau_{xy} \sin \alpha \cos \alpha. \quad (3)$$

Similarly, $\tau_{x'y'} = P_y \cos \alpha - P_x \sin \alpha$

$$= (\sigma_y - \sigma_x) \sin \alpha \cos \alpha + \tau_{xy} \left( \cos^2 \alpha - \sin^2 \alpha \right). \quad (4)$$

By substituting $(\alpha + \pi/2)$ for $\alpha$ in Equation (3)

We have

$$\sigma_{y'} = \sigma_x \cos^2 (\alpha + \pi/2) + \sigma_y \sin^2 (\alpha + \pi/2) + 2 \tau_{xy} \sin(\alpha + \pi/2) \cos(\alpha + \pi/2)$$

$$= \sigma_x \sin^2 \alpha + \sigma_y \cos^2 \alpha - 2 \tau_{xy} \sin \alpha \cos \alpha. \quad (5)$$

Equations (3), (4) and (5) are the transformation of stress equations. The sum of the normal stresses on two perpendicular planes is independent of $\alpha$ since from equations (3) and (5)

$$\tau_{x'} + \tau_{y'} = \tau_x + \tau_y = \text{constant}.$$ 

Write equations (3), (4), and (5) in terms of $2\alpha$ using trigonometric identities.

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha \quad \sin^2 \alpha = \frac{1}{2} (1 - \cos 2\alpha)$$

$$\cos^2 \alpha = \frac{1}{2} (1 + \cos 2\alpha).$$

We have

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha + \tau_{xy} \sin 2\alpha \quad (6)$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha - \tau_{xy} \sin 2\alpha \quad (7)$$
\[ \tau_{xy'} = \frac{\sigma_y - \sigma_x}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha. \]  

(8)

The planes on which the shear stress vanishes are called the principle planes.

The normal stresses on these planes are called the principle stresses. To determine the maximum and minimum normal stress on a plane, we have

\[ \frac{d\sigma_x'}{d\alpha} = -\left(\alpha_x - \alpha_y\right) \sin 2\alpha + 2\tau_{xy} \cos 2\alpha = 0 \]  

(9)

or

\[ \tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \]  

(10)

since

\[ 1 + \tan^2 2\alpha = \sec^2 2\alpha \]

so we have

\[ \cos 2\alpha = \pm \frac{\sigma_x - \sigma_y}{\sqrt{4\tau_{xy}^2 + (\sigma_x - \sigma_y)^2}} \]

\[ \sin 2\alpha = \pm \frac{2\tau_{xy}}{\sqrt{4\tau_{xy}^2 + (\sigma_x - \sigma_y)^2}} \]

Substituting the above two expressions into equation (1), we have

\[ \sigma_{\text{MAX}} = \sigma_1 = \sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \sqrt{\frac{4\tau_{xy}^2 + (\sigma_x - \sigma_y)^2}{4\tau_{xy}^2 + (\sigma_x - \sigma_y)^2}} + \frac{4\tau_{xy}^2}{2 \sqrt{4\tau_{xy}^2 + (\sigma_x - \sigma_y)^2}} \]
Similarly $\sigma_{\text{min}} = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$.

By taking the second derivative of equation (6) and evaluating it at two roots of equation (10) we know that $\sigma_1$ and $\sigma_2$ are the maximum and minimum values of $\sigma$.

To determine the particular values of $2\alpha$ (by equation (10)) corresponding to either $\sigma_1$ and $\sigma_2$, it is necessary to consider the signs of the numerator and denominator of $2\tau_{xy}/(\sigma_x - \sigma_y)$

The value of $2\alpha$ corresponding to the director of $\sigma_1$ is

- $0 < 2\alpha < \pi/2$ if $\tau_{xy} > 0$ and $(\sigma_x - \sigma_y) > 0$
- $\pi/2 < 2\alpha < \pi$ if $\tau_{xy} > 0$ and $(\sigma_x - \sigma_y) < 0$
- $\pi < 2\alpha < (3/2)\pi$ if $\tau_{xy} < 0$ and $(\sigma_x - \sigma_y) < 0$
- $3/2\pi < 2\alpha < 2\pi$ if $\tau_{xy} < 0$ and $(\sigma_x - \sigma_y) > 0$.

To determine the planes of maximum shear stress, we differentiate the expression for $\tau_{x'y'}$ and set this equal to zero

$$\frac{d\tau_{x'y'}}{d\alpha} = \left(\sigma_y - \sigma_x\right)\cos 2\alpha - 2\tau_{xy}\sin 2\alpha = 0$$

$$\tan 2\alpha = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \quad (11)$$

Since equations (5) and (11) are negative reciprocals to each other, so the planes of maximum shear stress are $45^0$ from the principal planes. From the equations (5) and (11) we develop the equation for maximum shear stress on the $x'y'$ coordinates.
\[ \tau_{xy'} = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}. \]

We already know how to use formulas to find principal stresses and maximum shear stresses. Here we use another method to do it without using the formulas. That is, Mohr's Circle.

Mohr's Circle enables to find the principal stresses and maximum shear graphically. To do this we need to rewrite the following formula

\[ \sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha + \tau_{xy} \sin 2\alpha \]

\[ \tau_{xy'} = \frac{\sigma_y - \sigma_x}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha \]

When we square and add them together we have the equation of a circle.

\[ \left(\sigma - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \]

Where \( \sigma_{x'} = \sigma \quad \tau_{xy'} = \tau \)

The center is \( \left(\frac{\sigma_x + \sigma_y}{2}, 0\right) \) on the \( \sigma \)-axis.

The radius is \( \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \).
Mohr’s Circle represents the relationship between the normal stress and the shear stress on any given plane. We can find the magnitude and direction of the normal and the shear stresses for a point on the circle regardless of the coordinate system used.

When we locate the point on Mohr’s Circle which represents the stresses on a plane inclined at an angle $\alpha$ with respect to the plane of $\sigma_i$ (assume $\sigma_i$ is given), we just need to substitute $\sigma_x = \sigma_1$, $\sigma_y = \sigma_2$, and $\tau_{xy} = 0$ into the transformation of stress equations (1), (2), (3) we have

$$\sigma_{x'} = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\alpha$$

$$\sigma_{y'} = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2\alpha$$

$$\tau_{x'y'} = \frac{\sigma_2 - \sigma_1}{2} \sin 2\alpha.$$
From these formulas we can find the normal and shear stress on $x'y'$ plane. In general, every point on the Mohr's circle corresponds to a plane in the body under stress and these stresses are given by the point.

The state of stress at a point also leads us to the equations governing the variation of the stress components in space. Now we come to the most important part of the chapter—that is, the development of the differential equations of equilibrium. The differential equations of equilibrium result from the expressions relating the space derivatives of the various components of stress. It is very important to study the differential equations of equilibrium since the stress is different from point to point in a stressed body.

If the stress at A is $\sigma_x$, with the rate of change $\partial\sigma/\partial x$ of $\sigma_x$ and the distance $dx$ in the x direction, then the stress at B is increased by amount $(\partial\sigma/\partial x) \, dx$. Here we are assuming that the stress components and their first derivatives are continuous. Therefore the stress at B is

$$\sigma_{xB} = \sigma_x + \frac{\partial \sigma_x}{\partial x} \, dx.$$ 

Similarly, the stress at C and D are
\[ \sigma_{XC} = \sigma_X + \frac{\partial \sigma_X}{\partial y} \ dy \]

\[ \sigma_{XD} = \sigma_{XB} + \frac{\partial \sigma_{XB}}{\partial y} \ dy \]

\[ = \sigma_X + \frac{\partial \sigma_X}{\partial x} \ dx + \frac{\partial \sigma_X}{\partial y} \ dy \]

We omit the second order term \((dx \ dy)\) because it is small compared to the first order terms. By neglecting the small quantities of higher order, we have the following result for the resultant force.

Fig 2.8 Resultant Force on an Element

\[ P_1 = \left( \frac{\sigma_{XA} + \sigma_{XB}}{2} \right) \ dy \]

\[ = \left( \frac{\sigma_X + \sigma_X + \frac{\partial \sigma_X}{\partial x} \ dy}{2} \right) \ dy . \]

Assuming the depth of the prism in the \(z\) direction is unity. By simplifying we get

\[ P_1 = \sigma_X \ dy + \frac{1}{2} \frac{\partial \sigma_X}{\partial y} \ dy^2. \]
Similarly

\[ P_2 = \left( \frac{\sigma_{xx} + \sigma_{xd}}{2} \right) dy \]

\[ = \left( \frac{\sigma_x + \frac{\partial \sigma_x}{\partial x} dx + \sigma_x + \frac{\partial \sigma_x}{\partial x} dx + \frac{\partial \sigma_x}{\partial y} dy}{2} \right) dy. \]

By the same assumption for \( P_1 \) we have

\[ P_2 = \sigma_x dy + \frac{\partial \sigma_x}{\partial x} dx \ dy + \frac{1}{2} \frac{\partial \sigma_x}{\partial y} dy^2 \]

so the resultant force on the element is

\[ P_2 - P_1 = \frac{\partial \sigma_x}{\partial x} dx \ dy. \]

From the conclusion above we see that

Total force = \( \Sigma \) of all the forces on all the volume elements.

If \( dv \) is the volume element, \( F \) is the force per element, then

\[ \text{total force} = \int F \ dv. \]

Also, by Newton’s third law the total resultant force is zero if various parts of the portion of the body act on each other. Therefore, the resultant force can be represented as the sum of forces acting on all the surface elements, i.e., as an integral over the surface. For any portion of the body each of the three components \( \int F_i \ dv \) at the resultant of all the internal stresses can be transformed into an integral over the surface. \( F_i \) is the divergence of a tensor of rank two.

\[ F_i = \frac{\partial \sigma_{ik}}{\partial x_k} \]

\[ \int F_i \ dv = \int \frac{\partial \sigma_{ik}}{\partial x_k} \ dv = \int \sigma_{ik} \ df_k \]
where $\sigma_{ik}$ is the stress tensor and $\sigma_{ik} \, df_k$ is the $i$th component of the force on the surface element $df$.

The stress tensor is symmetric i.e., $\sigma_{ik} = \sigma_{ki}$.

Generally not only a normal stress but also shearing stresses act on each surface element.

In equilibrium the internal stresses in every volume element must balance.

i.e, $F_i = 0$.

From the discussion above we come to an important conclusion. We can use a simplified stress system that consists of an uniform stress distribution on each face to derive the equilibrium equations. In doing so we can represent the force by a single vector applied at the center of each face.

---

Here we assume $\sigma_x = \tau_{yz} = \tau_{zx} = \tau_{yx} = F_z = 0$.

We also assume $\sigma_x$, $\sigma_y$, $\tau_{xy}$, $\tau_{yx}$ and the body force $F_x$ and $F_y$ are independent of $z$. The situation satisfying all of the above conditions is called plane stress.
Let $\Sigma x$-Forces = 0 and assume a unit depth.

We have

\[
F_x \, dx \, dy + \left[ \sigma_x + \frac{\partial \sigma_x}{\partial x} \right] dy - \sigma_x \, dy + \left[ \tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} \right] dx - \tau_{xy} \, dx = 0 .
\]

After simplifying we get

\[
\left[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x \right] dx \, dy = 0 .
\]

This implies

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0 \quad \text{\(\because\) } dx \, dy \neq 0 \quad (12) .
\]

Similarly when $\Sigma y$-Forces = 0 we get

\[
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + F_y = 0 .
\]

Continuing the process we have the equilibrium equations in the three dimensional case.

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x &= 0 \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + F_y &= 0 . \\
\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + F_z &= 0
\end{align*}
\]

The equilibrium equations we just developed are for a non-uniformly stressed body.

How about the equations of equilibrium for a uniform gravitational field? The general equations of equilibrium for isotropic bodies is

\[
\frac{\partial \sigma_{ik}}{\partial x_k} + \rho g_i = 0
\]
\[ \rho - \text{density} \]
\[ g - \text{gravitational acceleration vector.} \]

The expression for the stress tensor is:

\[
- \frac{\partial \sigma_{ik}}{\partial x_k} = \frac{E \sigma}{(1 + \sigma)(1 - 2\sigma)} \frac{\partial u_i}{\partial x_i} + \frac{E}{1 + \sigma} \frac{\partial u_k}{\partial x_k}
\]

\( u_{ik} \) - strain tensor.

Substituting

\[
u_{ik} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right).
\]

We obtain the equations of equilibrium in the form

\[
\frac{E}{2(1 + \sigma)} \frac{\partial^2 u_i}{\partial x_i^2} + \frac{E}{2(1 + \sigma)(1 - 2\sigma)} \frac{\partial^2 u_i}{\partial x_i \partial x_j} + \rho g_t = 0.
\]

Rewrite the equation in vector notation. The term \( \frac{\partial^2 u_i}{\partial x_i^2} \) are components of the vector \( \Delta u \)

and \( \frac{\partial u_k}{\partial x_i} = \text{div } u \), thus the equations of equilibrium become

\[
\Delta U + \frac{1}{1 - 2\sigma} \text{grad div } u = - \rho g \frac{2(1 + \sigma)}{E}.
\]

Landau and Lifshitz [4] point out a very important case, that is where the deformation of the body is caused, not by body forces, but by forces applied to its surface.
The equation of equilibrium then becomes

\[(1 - 2\sigma) \Delta \mathbf{u} + \text{grad} \text{ div } \mathbf{u} = 0. \tag{13}\]

Taking the divergence of equation (13) and using the identity

\[\text{div } \text{grad} = \Delta\]

we have \(\Delta \text{ div } \mathbf{u} = 0\).

Thus, \(\text{div } \mathbf{u}\) is a harmonic function. Taking the Laplacian of equation (13), we have

\[\Delta \Delta \mathbf{u} = 0.\]

Thus in a uniform gravitational field under the condition of equilibrium the displacement vector satisfies the so-called biharmonic equation.

Examples for the Chapter

(1) Calculate the values of the principal stresses and the angles between the principal axes and the x-axis for the element shown in the figure. Assume the following stresses are known.

\[\sigma_{x'} = 5000 \text{ psi } \quad \tau_{x'y'} = -2500 \text{ psi } \quad \sigma_{y'} = 0\]

\[\sigma_{y'} = \sigma_x \cos^2 \alpha + \sigma_y \sin^2 \alpha + 2 \tau_{xy} \sin \alpha \cos \alpha\]

\[\sigma_{y'} = \sigma_x \sin^2 \alpha + \sigma_y \cos^2 \alpha - 2\tau_{xy} \sin \alpha \cos \alpha\]
solution: \[
\sigma_x + \sigma_y = \sigma_x + \sigma_y = 5000 \text{ psi}
\]
\[
\tau_{xy} = (\tau_y - \tau_x) \sin 15^\circ \cos 15^\circ + \tau_{xy} (\cos^2 15^\circ - \sin^2 15^\circ)
\]
\[
\sigma_x - \sigma_y = \sigma_x \cos 2\alpha + \sigma_y (- \cos 2\alpha) + 4\tau_{xy} \sin \alpha \cos \alpha
\]
\[
-2500 = (5000 - 2\sigma_x) \frac{1}{2} \cdot \frac{1}{2} + \tau_{xy} \frac{\sqrt{3}}{2}
\]
\[
5000 = \frac{\sqrt{3}}{2} \sigma_x - \frac{\sqrt{3}}{2} (5000 - \sigma_x) + 2\tau_{xy} \cdot \frac{1}{2}
\]
\[
-10,000 = 5000 - 2\sigma_x + 2\sqrt{3} \tau_{xy}
\]
\[
10,000 = \sqrt{3} \sigma_x - 5000 \sqrt{3} + \sqrt{3} \sigma_x + 2\tau_{xy}
\]
\[
-15,000 = -2\sigma_x + 2\sqrt{3} \tau_{xy}
\]
\[
10,000 + 5000 \sqrt{3} = 2\sqrt{3} \sigma_x + 2\tau_{xy}
\]
\[
-15,000 = -2\sigma_x + 2\sqrt{3} \tau_{xy}
\]
\[
10,000 \sqrt{3} + 5000 = 6 \sigma_x + 2\sqrt{3} \tau_{xy}
\]
\[-2,000 - 10,000 \sqrt{3} = -8\sigma_x\]
\[-37320.5 = -8\sigma_x\]
\[\sigma_x = 4665 \text{ psi.}\]
\[\sigma_y = 5,000 - 4665 = 335 \text{ psi.}\]
\[
\tau_{xy} = \frac{-15000 + 2\sigma_x}{2\sqrt{3}} = \frac{-15000 + 2 \times 4665}{2\sqrt{3}} = -1638 \text{ psi.}
\]
\[ \sigma_{\text{MAX}} = \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \]
\[ = \frac{5000}{2} + \sqrt{\left(\frac{5000}{2}\right)^2 + (-1638)^2} \]
\[ = 5489 \text{ psi} \]

\[ \sigma_{\text{MIN}} = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \]
\[ = -489 \text{ psi} \]

\[ \tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(-1638)}{5000} = -0.6555 \]
\[ 2\alpha = -33^0 \]
\[ \alpha = -16.5^0 \]

(2) Show that the following state of stress is in equilibrium

\[ \sigma_x = 3x^2 + 3y^2 - z \quad \tau_{xy} = z - 6xy - \frac{3}{4} \]
\[ \sigma_y = 3y^2 \quad \tau_{xz} = x + y - \frac{3}{2} \]
\[ \sigma_z = 3x + y - z + \frac{5}{4} \quad \tau_{yz} = 0 \]

solution:

\[ \frac{\partial \sigma_x}{\partial x} = 6x \quad \frac{\partial \tau_{xy}}{\partial y} = -6x \quad \frac{\partial \tau_{xz}}{\partial z} = 0 \quad F_x = 0 \]

\[ \therefore \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0 \]

\[ \therefore \frac{\partial \sigma_y}{\partial y} = 6y \quad \frac{\partial \tau_{xy}}{\partial x} = -6y \quad \frac{\partial \tau_{yz}}{\partial z} = 0 \quad F_y = 0 \]
\[
\begin{align*}
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + F_y &= 0 \\
\frac{\partial \sigma_z}{\partial z} &= -1 \\
\frac{\partial \tau_{xz}}{\partial x} &= 1 \\
\frac{\partial \tau_{yz}}{\partial y} &= 0 \\
F_z &= 0
\end{align*}
\]

Therefore the state of stress is in equilibrium.

(3) For the state of stress in problem (2) at the specific point

\( x = \frac{1}{2}, \quad y = 1, \quad z = \frac{3}{4} \), determine the principal stresses.

solution:

\[
\begin{align*}
\sigma_x &= 3 \cdot \frac{1}{4} + 3 - \frac{3}{4} = 3 \\
\tau_{xy} &= \frac{3}{4} - 6 \cdot \frac{1}{2} - \frac{3}{4} = -3 \\
\sigma_y &= 3 \\
\tau_{xz} &= \frac{1}{2} + 1 - \frac{3}{2} = 0 \\
\sigma_z &= \frac{3}{2} + 1 - \frac{3}{4} + \frac{5}{4} = 3 \\
\tau_{yz} &= 0 \\
\sigma_p^3 - (\sigma_x + \sigma_y + \sigma_z) \sigma_p^2 + (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2) \sigma_p \\
&- (\sigma_x \sigma_y \sigma_z + 2 \tau_{xy} \tau_{xz} \tau_{yz} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{xz}^2 - \sigma_z \tau_{xy}^2) = 0 \\
\sigma_p^3 - (3 + 3 + 3) \sigma_p^2 + (9 + 9 + 9 - 9) \sigma_p - [27 - 3 (-3)]^2 &= 0 \\
\sigma_p^3 - 9 \sigma_p^2 + 18 \sigma_p &= 0 \\
\sigma_p (\sigma_p^2 - 9 \sigma_p + 18) &= 0 \\
\sigma_p &= \sigma_1 = 0
\end{align*}
\]
\[ \sigma_p^2 - 9\sigma_p + 18 = 0 \]

\[ (\sigma_p - 3)(\sigma_p - 6) = 0 \]

\[ \sigma_p = \sigma_2 = 3 \quad \sigma_p = \sigma_3 = 6 \]
A body is said to be strained or deformed when the relative positions of points in the body are changed.

The vector distance from the initial location to the final location of a point is defined as the displacement of the point. We use \( u, v, \) and \( w \) to denote its components; \( u, v, \) and \( w \) are functions of \( x, y, \) and \( z \).

![Diagram of normal strain in a bar](image)

Figure 3.1 Normal Strain in a Bar

Applying stress to the bar, points A, B move to points \( A'B' \). \( \varepsilon \) is the normal strain as the unit change in length.

\[
\varepsilon_x = \frac{\partial u}{\partial x} \frac{dx}{dx} = \frac{\partial u}{\partial x}.
\]

To consider a body in a state of plane strain, we have

\[
u = u(x,y), \quad v = v(x,y), \quad w = 0.
\]
Let $\gamma$ be the shear strain as the change in the original right angle between two axes.

\[
\gamma = \frac{\partial u}{\partial y}\frac{dy}{dy} + \frac{\partial v}{\partial y}\frac{dy}{dy} - \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial v}{\partial x}\frac{dx}{dx}
\]

Figure 3.2 Translation and Deformation of a Two-Dimensional Element.

From figure 3.2 we see the unit change of length in the x direction is

\[
\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{A'B' - dx}{dx}
\]

The unit change of the length in the y direction is

\[
\varepsilon_y = \frac{A'D' - AD}{AD} = \frac{A'D' - dy}{dy}
\]

The change of shear strain is

\[
\beta + \vartheta - \lambda = \frac{\pi}{2}
\]

\[\Rightarrow \gamma_{xy} = \frac{\pi}{2} - \beta = \vartheta - \lambda.\]

Note: we assign "-" to the clockwise direction and "+" to the counterclockwise direction.

The displacement components of point A are $u$ and $v$.

The displacement components of point B are $u + (\partial u/\partial x)dx$ and $v + (\partial v/\partial x)$. 

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The displacement components of point D are \( u + (\partial u/\partial y)dy \) and \( v + (\partial v/\partial y)dy \).

\[
\therefore \varepsilon_x = \frac{A'B' - dx}{dx}
\]

\[
\therefore (A'B')^2 = [dx(1 + \varepsilon_x)]^2
\]

\[
= \left( dx + \frac{\partial u}{\partial x} dx \right)^2 + \left( \frac{\partial v}{\partial x} dx \right)^2
\]

\[
\varepsilon_x^2 + 2\varepsilon_x + 1 = 1 + 2 \frac{\partial u}{\partial x} \left( \frac{\partial y}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2
\]

Since \( \left( \frac{\partial u}{\partial x} \right)^2 \) and \( \left( \frac{\partial v}{\partial x} \right)^2 \) are relatively small compared to \( \frac{\partial y}{\partial x} \), so we can neglect them.

\[
\Rightarrow \varepsilon_x = \frac{\partial y}{\partial x} .
\]

and similarly \( \varepsilon_y = \frac{\partial v}{\partial y} . \)

also

\[
\tan \theta = \frac{\left( \frac{\partial v}{\partial x} \right) dx}{dx + \left( \frac{\partial u}{\partial x} \right) dx} .
\]

Since \( \theta \) is a small angle, we let \( \tan \theta = \theta \)

and \( \frac{\partial u}{\partial x} \) is small compared to 1, so we omit it. Therefore we have

\[
\theta = \frac{\partial v}{\partial x} .
\]
and similarly

\[ \lambda = - \frac{\partial u}{\partial y}, \]

thus the shear strain \( \gamma_{xy} \) is

\[ \gamma_{xy} = \theta - \lambda = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \]

From above we can conclude the strain-displacement relations in three dimensional cases are:

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} \\
\varepsilon_z &= \frac{\partial w}{\partial z}
\end{align*}
\]

\[
\begin{align*}
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\
\gamma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}
\end{align*}
\]

We should notice here that the stress components are distributed over a deformed body, so the coordinates for the equilibrium equations \( x, y \) and \( z \) are based on the coordinates of the deformed body; on the other hand the coordinates \( x, y \) and \( z \) for the strain-displacement are for the undeformed body. However we can still let \( x, y \) and \( z \) represent the undeformed coordinates in both equations since the deformation is infinitesimal. But this would not be the case in finite elasticity because we are dealing with potentially large deformations.

From the strain-displacement relations we observe that the equation groups have six equations for the strain components as a function and three displacement components only.
To derive the strain components from the equation groups we use $u,v$ and $w$ as functions of $x,y$ and $z$. That will give us six equations for three unknown. These equation groups usually do not have any solution for $u,v$ and $w$ unless the six strain components are related. That is why we need compatibility equations. To develop the equations we simply differentiate the first equation of strain-displacement relations twice with respect to $y$ and the second equation twice with respect to $x$ and add the results, this will give us the following equation.

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 v}{\partial x^2 \partial y}.$$ 

Differentiating the fourth equation with respect to $x$ and $y$, we get

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 v}{\partial x^2 \partial y^2}.$$ 

Putting these two equations together we have the first compatibility equation

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}.$$ 

By similar approaches we develop the remaining five compatibility equations

$$\frac{\partial^2 \varepsilon_y}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial y \partial x}$$

$$\frac{\partial^2 \varepsilon_x}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{xz}}{\partial z \partial x}$$

$$2 \frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial z} \left( - \frac{\partial \gamma_{xy}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial y} + \frac{\partial \gamma_{yx}}{\partial z} \right)$$

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\[
2 \frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial \gamma_y}{\partial x} - \frac{\partial \gamma_y}{\partial y} + \frac{\partial \gamma_x y}{\partial z} \right)
\]

\[
2 \frac{\partial^2 \varepsilon_x}{\partial x \partial y} = \frac{\partial}{\partial z} \left( \frac{\partial \gamma_x}{\partial x} + \frac{\partial \gamma_y}{\partial y} - \frac{\partial \gamma_x y}{\partial z} \right).
\]

One important fact is that the strain components must satisfy the compatibility equations for the existence of solutions of displacement components.

We already know that the state of stress at a point is uniquely determined if the stress components on two planes are given. This fact also applies to state of strain at a point. This can be seen from the transformation equations for the strain on any two planes such as \(x'y'\) plane.

For normal strain:

On \(x'\) direction:

\[
\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\alpha + \frac{\gamma_{xy}}{2} \sin 2\alpha
\]

On \(y'\) direction:

\[
\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\alpha - \frac{\gamma_{xy}}{2} \sin 2\alpha
\]

For shear strain:

\[
\gamma_{x'y'} = (\varepsilon_y - \varepsilon_x) \sin 2\alpha + \gamma_{xy} \cos 2\alpha
\]
From the above equations we can tell that the state of strain is uniquely determined in any direction if the strain components $\varepsilon_x$, $\varepsilon_y$, and $\gamma_{xy}$ on two planes are given. Also, by replacing $\sigma$ by $\varepsilon$ and $\tau$ by $\gamma/2$, the equations for stress are converted to the strain relations. Therefore we develop the following equations.

For the directions of principal strain:

$$\tan 2\alpha = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

For the magnitudes of the principal strain:

$$\varepsilon_1 \text{ or } \varepsilon_2 = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \frac{1}{2} \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2}$$

If we integrate the strain-displacement relations to obtain the displacements, then we know there are certain constants of integration. This is the same as for rigid body translations and rotations.

Fig 3.3 Rotation of an Element
From Fig 3.3 we see that if the element is rotated as a rigid body through a small angular displacement $w_{zo}$, we have

$$w_{zo} = \frac{\partial y}{\partial x} = -\frac{\partial u}{\partial y}.$$  

During this rigid body movement no strain occurs. If both rigid body displacements and deformation (strain) occur, then

$$w_z = \frac{1}{2} \left( \frac{\partial y}{\partial x} - \frac{\partial u}{\partial y} \right)$$

where $w_z$ is called the rotation which represents the average of the angular displacement of $dx$ and the angular displacement of $dy$. 
Examples for the Chapter

1. Determine $\varepsilon_n$, $\varepsilon_t$, and $\gamma_{tn}$ if $\gamma_{xy} = 0.002828$ and $\varepsilon_x = \varepsilon_y = 0$ for the element shown.

$$\varepsilon_n = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\alpha + \frac{\gamma_{xy}}{2}\sin 2\alpha$$

$$= \frac{0.02828}{2}\sin 2 \times 22.5^\circ$$

$$= 0.001414 \times \frac{\sqrt{2}}{2} = 0.0009998$$

$$\varepsilon_t = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\alpha - \frac{\gamma_{xy}}{2}\sin 2\alpha$$

$$= -\frac{0.002828}{2}\sin 2 \times 22.5^\circ$$

$$= -0.009998.$$  

$$\gamma_{tn} = (\varepsilon_y - \varepsilon_x)\sin 2\alpha + \gamma_{xy}\cos 2\alpha$$

$$= \gamma_{xy}\cos 2\alpha = 0.002828 \cos 45^\circ$$

$$= 0.0019996.$$
2. Given the following system of strains:

\[ \varepsilon_x = 5 + x^2 + y^2 + x^4 + y^4 \]
\[ \varepsilon_y = 6 + 3x^2 + 3y^2 + x^4 + y^4 \]
\[ \gamma_{xy} = 10 + 4xy(x^2 + y^2 + 2) = 10 + 4x^3y + 4xy^3 + 8xy \]
\[ \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0 \]

determine if the system of strain is possible. If this strain distribution is possible, find the displacement components in terms of \( x \) and \( y \), assuming that the displacement and rotation at the origin are zero.

We first need to exam if they satisfy compatibility equations.

\[ \frac{\partial^2 \varepsilon_x}{\partial y^2} = 2y + 4y^3 \quad \frac{\partial^2 \varepsilon_x}{\partial y^2} = 2 + 12y^2 \]
\[ \frac{\partial \varepsilon_x}{\partial x} = 6x + 4x^3 \quad \frac{\partial^2 \varepsilon_x}{\partial x^2} = 6 + 12x^2 \]
\[ \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 8 + 12x^2 + 12y^2 \]
\[ \frac{\partial \gamma_{xy}}{\partial y} = 4x^3 + 12xy^2 + 8x \quad \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 12x^2 + 12y^2 + 8 \]

Therefore

\[ \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_x}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 12x^2 + 12y^2 + 8, \text{ the equation is satisfied.} \]

\[ \frac{\partial \varepsilon_y}{\partial z} = 0 \quad \frac{\partial^2 \varepsilon_y}{\partial z^2} = 0 \quad \frac{\partial \varepsilon_z}{\partial y} = 0 \quad \frac{\partial^2 \varepsilon_z}{\partial y^2} = 0 \quad \frac{\partial \gamma_{zy}}{\partial y} = 0 \quad \frac{\partial \gamma_{xz}}{\partial y} = 0 \quad \frac{\partial \gamma_{yz}}{\partial y} = 0 \]

Therefore
\[
\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = 0, \text{ the equation is satisfied}
\]

Similarly
\[
\frac{\partial^2 \varepsilon_x}{\partial x^2} + \frac{\partial^2 \varepsilon_z}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}
\]

Also
\[
\frac{\partial \varepsilon_y}{\partial z} = 0, \quad \frac{\partial^2 \varepsilon_x}{\partial y \partial z} = 0
\]

\[
- \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} = 0 + 0 + 0 = 0
\]

Therefore
\[
2 \frac{\partial^2 \varepsilon_y}{\partial y \partial z} = \frac{\partial}{\partial z} \left( - \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) = 0, \text{ so the equation is satisfied.}
\]

Similarly
\[
2 \frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right)
\]

\[
2 \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{zx}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial z} \right)
\]

So the strain distribution is possible. Hence
\[
\varepsilon_x = \frac{\partial y}{\partial x} = 5 + x^2 + y^2 + x^4 + y^4
\]

\[
u = \int \left( 5 + x^2 + y^2 + x^4 + y^4 \right) dx = 5x + \frac{1}{3}x^3 + xy^2 + \frac{1}{5}x^5 + xy^4 + c
\]

\[
\varepsilon_y = \frac{\partial y}{\partial y} = 6 + 3x^2 + 3y^2 + x^4 + y^4
\]
\[ v = \int \left( 6 + 3x^2 + 3y^2 + x^4 + y^4 \right) \, dy = 6y + 3x^2 y + y^3 + x^4 y + \frac{1}{5} y^5 + c \]

\[ \varepsilon_z = \frac{\partial w}{\partial z} = 0 \]

w = c.
CHAPTER 4
STRESS - STRAIN RELATION

The two types of field equations that we learn from chapter 2 and 3 are equilibrium equations and the strain - displacement relations. The equilibrium equations are based on statics and continuity of stress which involve stress components only. The strain-displacement relations are based on continuity of displacements and infinitesimal deformations which involve strain and displacement components. In this chapter we want to see the relationship between the strain components and the stress components. To see this we first need to define what elastic is. An elastic body is a body that will go back to its original dimensions after the forces acting on it are removed. The reason we need to define this is because the relationship between the stress components and the strain components depends on the properties of the particular solid. In this chapter we only consider the behavior of isotropic, homogeneous materials.

Hooke’s Law generalized the stress-strain relations. Here is what Hooke’s Law states:

\[ \sigma_x = E \varepsilon_x \]

E is constant and is called the modulus of elasticity or Young’s modulus.

There are two approaches to determine the stress-strain relations, one is the “mathematical” approach and another is the “semiempirical” approach. The mathematical approach gives the following linear equations for the relation between the stress and strain components.

\[ \sigma_x = c_{11} \varepsilon_x + c_{12} \varepsilon_y + c_{13} \varepsilon_z + c_{14} \gamma_{xy} + c_{15} \gamma_{yz} + c_{16} \gamma_{zx} \]
\[ \sigma_y = c_{21} \varepsilon_x + c_{22} \varepsilon_y + c_{23} \varepsilon_z + c_{24} \gamma_{xy} + c_{25} \gamma_{yz} + c_{26} \gamma_{zx} \]
\[ \sigma_z = c_{31} \varepsilon_x + c_{32} \varepsilon_y + c_{33} \varepsilon_z + c_{34} \gamma_{xy} + c_{35} \gamma_{yz} + c_{36} \gamma_{zx} \]
The coefficients $c_{ij} \ldots c_{66}$ represent material properties if we assume an isotropic material. These constants must be the same for an orthogonal coordinate system of any orientation at the point. For instance, isotropy requires that $c_{11}$, the constant that measures the stress component $\sigma_x$ which was caused by $\varepsilon_x$ (the normal stress in the $x$ direction), should also relate to $\sigma_x$ and $\varepsilon_x$.

In this chapter we only consider the “semiempirical” approach which is guided by experimental evidence to develop stress-strain relations (Hooke’s Law). In this approach we assume most engineering materials are under small strains.
\[ \varepsilon_y = \varepsilon_z = -\nu(\sigma_y/E) \]

\[ \nu = \frac{3K - 2\mu}{2(3K + \mu)} \]

where \( \nu \) is Poisson’s ratio

K is the modulus of compression and \( \mu \) is the modulus of rigidity. \( K \) and \( \mu \) are always positive, so Poisson’s ratio can vary between -1 (for \( K = 0 \)) and \( \frac{1}{2} \) (for \( \mu = 0 \)). Also, there is no known substance for which \( \nu < 0 \) (material which would expand transversely when stretched longitudinally). So, in practice, Poisson’s ratio varies only between 0 and \( \frac{1}{2} \).

To determine the strain component \( \varepsilon_x \), we first apply \( \sigma_x \), the changing on length AB is \( (1/E)\sigma_x \), then apply \( \sigma_y \). Hence the additional change in length AB is

\[ -(\nu/E)\sigma_y (1+\sigma_z/E) \]

Here the elastic strain \( (1/E)\sigma_x \) is dropped because it is small compare to unity. When applying \( \sigma_z \), we ignore the small quantities of higher order, the changing of length AB will become

\[ -(\nu/E)\sigma_z \]

Hence the total strain in the x direction is

\[ \varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \]

Similarly

\[ \varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \]
and

\[ \varepsilon_z = \frac{1}{E} [\sigma_z - \nu (\sigma_x - \sigma_y)]. \]

Figure 4.3 Element under Pure Shear

From figure 4.3 we can see the relation between elastic stress-strain under a two dimensional state of pure shear is

\[ \gamma_{xy} = \frac{1}{G} \tau_{xy} \]

similarly

\[ \gamma_{yx} = \frac{1}{G} \tau_{yx} \]

\[ \gamma_{zx} = \frac{1}{G} \tau_{zx} \]

\( G \) is an elastic constant and called the modulus of elasticity in shear. So far we have introduced three constants, \( E, \nu, \) and \( G \). The relations between them are

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Therefore the stress-strain relations, also known as Hooke’s law, have the following equations

\[ E \frac{E}{2(1 + \nu)} \]

The result is

\[ \sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \]
\[ \sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \]
\[ \sigma_z = 2G\varepsilon_z + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \]
\[ \tau_{xy} = G\gamma_{xy} \]
\[ \tau_{yz} = G\gamma_{yz} \]

These equations can be solved for the stress components in terms of the strain components.
\[ \tau_{zx} = G\gamma_{zx} \]

Where the constants \( G \) and \( \lambda \) are called Lame’s constant and

\[ \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}. \]

Besides the elastic constant we have introduced so far, there is another very important elastic constant called bulk modulus of elasticity. This constant is useful in a state of hydrostatic pressure in the physical field. To see how the constant is developed, we have

\[ \sigma_x = \sigma_y = \sigma_z = -p \quad (p > 0) \]

\[ \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \]

\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x - \nu (\sigma_y + \sigma_z) \right] \]

\[ = \frac{1}{E} p \left[ -1 + 2\nu \right] \]
Let $\varepsilon$ be the volumetric strain defined as the unit change in volume (change in volume divided by the original volume) and

$$\varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z.$$  

If the initial volume of a prism is $dx\,dy\,dz$ with its dimension $dx, dy, dz$, then the volume after straining will become

$$(dx + \varepsilon_x \, dx) \, (dy + \varepsilon_y \, dy) \, (dz + \varepsilon_z \, dz)$$  

$$= dx \, (1 + \varepsilon_x) \, dy \, (1 + \varepsilon_y) \, dz \, (1 + \varepsilon_z)$$  

$$= (1 + \varepsilon_x + \varepsilon_y + \varepsilon_z) \, dx\,dy\,dz$$

Again we neglect the higher order terms here.

Since $\varepsilon_x = \varepsilon_y = \varepsilon_z = -\left(\frac{1 - 2\nu}{E}\right)\rho$

so $\varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z$

$$= -3\left(\frac{1 - 2\nu}{E}\right)\rho$$

$$= -\frac{3}{E}(1 - 2\nu)p$$

$$= -\frac{1}{K}p$$

where $K = E/\left[3(1 - 2\nu)\right]$ is the bulk modulus of elasticity.
Examples for the Chapter

1. A square Duralumin plate is loaded as shown, where

\[ \sigma_x = \sigma_y = \tau_{xy} = 15,000 \text{ psi}. \]

If \( E = 10^3 \text{ psi} \) and \( v = 0.3 \), determine the change in length of the diagonal \( ab \).

\[
\begin{align*}
\sigma_y & \\
\tau_{xy} & \\
\sigma_x &
\end{align*}
\]

\[
\begin{align*}
\text{Solution:} & \\
\left( \sigma - \frac{\sigma_x + \sigma_y}{2} \right)^2 + \tau^2 & = \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \tau_{xy}^2 \\
\left( \sigma - \frac{15,000 + 15,000}{2} \right)^2 + \tau^2 & = \left( \frac{15,000 - 15,000}{2} \right)^2 + 15,000^2 \\
\left( \sigma - 15,000 \right)^2 + \tau^2 & = 15,000^2
\end{align*}
\]

so by using Mohr’s circle of stress, we can find the stresses on planes normal and parallel to \( ab \). Mohr’s equation tell us

- the center of the circle is \( (15,000, 0) \).
- the radius of the circle is \( 15,000 \).
Therefore $\sigma_1 = 30,000$psi $\quad \sigma_2 = 0$

\[
\sigma_b = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\alpha = 15000\text{psi}
\]

\[
\sigma_a = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2}\sin 2\alpha = 15000\text{psi}
\]

\[
\tau_{ab} = \frac{\sigma_2 - \sigma_1}{2}\sin 2\alpha
\]

\[
\varepsilon_a = \frac{1}{E}\left[\sigma_a - \nu \sigma_b\right]
\]

\[
= \frac{1}{10^7}\left[15000 - 0.3 \times 15000\right]
\]

\[
= 1.05 \times 10^{-3}\text{in}
\]

\[
\varepsilon_b = \frac{1}{E}\left[\sigma_b - \nu \sigma_a\right]
\]

\[
= 1.05 \times 10^{-3}\text{in}
\]

The original length of AB is $\sqrt{2} = 1.414\text{ in}$. After the pressure is applied, it became

\[
\sqrt{\left(1 + 1.05 \times 10^{-3}\right)^2 + \left(1 + 1.05 \times 10^{-3}\right)^2} = 1.415\text{(in)}
\]

Therefore length change of AB is

\[
\varepsilon_{ab} = 1.415 - 1.414 = 0.001\text{ (in)}.
\]

2. If a medium is initially unstrained and is then subjected to a constant positive temperature change, the normal strains are expressed by

\[
\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu (\sigma_y + \sigma_z)\right] + \alpha T
\]

\[
\ldots
\]

\[
\ldots
\]
where $\alpha$ is the coefficient of linear expansion and $T$ is the temperature rise. The temperature change does not affect the shear strain components.

\[ \epsilon_y = \epsilon_z = \alpha T (1 + \nu) \]

A bar restrained in the $x$ direction only, and free to expand in the $y$ and $z$ directions as shown, is subjected to a uniform temperature rise $T$. Show that the only nonvanishing stress (the bar is in a state of uniform stress) and strain components are

\[ \sigma_x = -E\alpha T \]
\[ \sigma_y = \sigma_z = \sigma_x = 0 \]
\[ \epsilon_y = \epsilon_z = \alpha T (1 + \nu) \]

Solution: since the bar is restrained in the $x$ direction only and free on the $y$ and $z$ direction, i.e, there is no stress acting on the $y$ and $z$ direction, so $\sigma_y = \sigma_z = 0$. Also $\epsilon_x = 0$ because there is no length changed in the $x$ direction. Therefore, by the given formula, we have

\[ \epsilon_x = \frac{1}{E} \left[ \sigma_x - \nu (\sigma_y + \sigma_z) \right] + \alpha T \]
\[ 0 = \frac{1}{E} \sigma_x + \alpha T \]
\[ 0 = \sigma_x + E\alpha T \]
\[ \therefore \sigma_x = -E\alpha T \]

\[ \epsilon_y = \frac{1}{E} \left[ \sigma_y - \nu (\sigma_z + \sigma_x) \right] + \alpha T \]
\[ = \frac{1}{E} \left( -\nu \sigma_x \right) + \alpha T \]
\[ = \nu \alpha T + \alpha T \]
\[ = \alpha T (1 + \nu) \]
\[ \varepsilon_z = \frac{1}{E} \left[ \sigma_z - \nu \left( \sigma_x + \sigma_y \right) \right] + \alpha T \]
\[ = \frac{1}{E} \left( -\nu \sigma_z \right) + \alpha T \]
\[ = \nu \alpha T + \alpha T \]
\[ = \alpha T (1 + \nu) \]

3. Determine the stress and strain components if the bar in the preceding problem is restrained in the x and y directions but is free to expand in the z direction.

Solution: since the bar is restrained in the x and y direction, so there is no changing length in the x and y direction and no acting stress on the z direction.

so \( \sigma_z = \varepsilon_x = \varepsilon_y = 0 \)

\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x - \nu \left( \sigma_y + \sigma_x \right) \right] + \alpha T \]
\[ 0 = \frac{1}{E} \left( \sigma_x - \nu \sigma_x \right) + \alpha T \]
\[ \sigma_x = \nu \sigma_x - E \alpha T \quad (1) \]

\[ \varepsilon_y = \frac{1}{E} \left[ \sigma_y - \nu \left( \sigma_z + \sigma_x \right) \right] + \alpha T \]
\[ 0 = \frac{1}{E} \left( \sigma_y - \nu \sigma_x \right) + \alpha T \]
\[ \sigma_y - \nu \sigma_x = -E \alpha T \quad (2) \]

Substituting equation (1) into equation (2), we have

\[ \sigma_y - \nu (\nu \sigma_x - E \alpha T) = -E \alpha T \]
\[ \sigma_y - \nu^2 \sigma_x + \nu E \alpha T = -E \alpha T \]
\[ \sigma_y \left( 1 - \nu^2 \right) = -E \alpha T (1 + \nu) \]
\[ \sigma_y \left( 1 + \nu \right) \left( 1 - \nu \right) = -E \alpha T (1 + \nu) \]
Putting $\sigma_y$ into equation (1), we have

\[ \sigma_x = \nu \frac{E\alpha T}{\nu - 1} - E\alpha T \]
\[ = E\alpha T \frac{\nu - \nu + 1}{\nu - 1} \]
\[ = \frac{E\alpha T}{\nu - 1} \]

\[ \varepsilon_z = \frac{1}{E} \left[ \sigma_z - \nu \left( \sigma_x + \sigma_y \right) \right] + \alpha \]
\[ = \frac{1}{E} \left[ -\nu \left( \frac{E\alpha T}{\nu - 1} + \frac{E\alpha T}{\nu - 1} \right) \right] + \alpha T \]
\[ = \frac{\alpha T (1 + \nu)}{(1 - \nu)} \]
CHAPTER 5
FORMULATION OF PROBLEMS IN ELASTICITY

In the previous three chapters we introduced the 15 governing equations which are equilibrium equations, strain-displacement relations and the stress strain relations. These equations are sometimes called field equations. The compatibility equations are not governing equations since they are derived from the strain-displacement equations. The governing equations enable us to solve various types of boundary value problems in elasticity. To solve an elasticity problem we need to determine the stress, strain, and displacement functions satisfying the field equations and the boundary conditions, and the solution is unique which we will prove in the end of the chapter. In this chapter we only focus on the mathematical formulation of elasticity problems instead of their solutions.

Figure 5.1 Surface Forces.

The following are the stress boundary condition equations.

\[
\tau_x^{\mu} = \sigma_{x0} \mu_x + \tau_{xy0} \mu_y + \tau_{x0} \mu_z
\]

\[
\tau_y^{\mu} = \tau_{y0} \mu_x + \sigma_{y0} \mu_y + \tau_{y0} \mu_z
\]
\[ \tau_z^\mu = \tau_{x_0} \mu_x + \tau_{y_0} \mu_y + \sigma_{z_0} \mu_z \]

\( \mu \) — a unit vector normal to the surface and going outward,

\( x_0, y_0, z_0 \) — coordinates of points on the boundary surface.

\( \sigma_{x_0}, \tau_{x_0}, \text{ etc.} \) — stress components evaluated at the boundary \((x_0, y_0, z_0)\).

\( \mu_x, \mu_y, \mu_z \) — direction cosines of the unit outward normal vector \( \mu \) with respect to \( x, y, \) and \( z \), respectively (or components of \( \mu \) along the three coordinate axes).

\( T_x^\mu, T_y^\mu, T_z^\mu \) — prescribed surface force.

**Definition**

A first boundary-value problem in elasticity is a problem in which the stress is prescribed over the entire boundary.

A second boundary-value problem in elasticity is a problem in which the displacement components are prescribed over the entire boundary surface.

A mixed boundary-value problem is a problem in which the stress components are prescribed over part of the boundary and the displacement components over the rest of the boundary.

Now we are going to develop the governing equations for elastic bodies under plain strain. First let us review the governing equations we have developed in the previous chapters.

For the strain components we have

\[ \varepsilon_x = \frac{\partial u}{\partial x} \]

\[ \varepsilon_y = \frac{\partial v}{\partial y} \]  \hspace{1cm} (1)
\[
\gamma_{xy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}
\]

\[
\varepsilon_x = \gamma_{yz} = \gamma_{xz} = 0
\]

From generalized Hooke’s Law we find

\[
\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y)
\]

\[
\sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y)
\]

\[
\sigma_z = \lambda(\varepsilon_x + \varepsilon_y)
\]

\[
= \nu(\sigma_x + \sigma_y)
\]

\[
\tau_{xy} = G\gamma_{xy}
\]

\[
\tau_{xz} = \tau_{yz} = 0.
\]

Note: We don’t need to worry about \(\sigma_z\) even thought it is not zero because \(\sigma_z\) does not appear in any of the governing equations. Also for plane strain problems \(\varepsilon_z\) vanishes, but \(\sigma_z\) does not.

For plain strain the equations of equilibrium is

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0
\]

\[
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0
\]

\[
F_z = 0 \quad (\text{since } \sigma_z \text{ is only a function of } x \text{ and } y, \text{ and } \\
\tau_{xz} = \tau_{yz} = 0).
\]

The compatibility equations in term of strain is

\[
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}.
\]
The compatibility equations in terms of stress is

\[ \nabla^2 (\sigma_x + \sigma_y) = -\frac{1}{1-v} \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \right) \]  
(5)

where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \).

The following governing equations are the most convenient way to solve second boundary-value problems in elasticity for plane strain.

\[ GV^2 u + \left( \lambda + G \right) \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + F_x = 0 \]  
(6)

\[ GV^2 v + \left( \lambda + G \right) \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + F_y = 0. \]

Equations (3) and (5) give a complete system of three equations in terms of \( \sigma_x, \sigma_y, \) and \( \tau_{xy} \). This system of equations are most useful for solving first boundary value problem.

We already know the governing equations for the three-dimensional problems. We have 15 equations and 15 unknown quantities. The method we use to find solutions is to seek expressions for the stress, strain, and displacement components which satisfy these equations and the prescribed boundary conditions. These equation groups, however, are not convenient to apply, so we will reduce them to systems of equations which are easier to use in the various types of boundary value problems. Here is how to do it.

Substituting the strain displacement relations into the stress-strain relations, we obtain six stress-displacement relations
\[ \sigma_x = \lambda \varepsilon_x + 2G \frac{\partial u}{\partial y} \]
\[ \sigma_y = \lambda \varepsilon_y + 2G \frac{\partial v}{\partial y} \]
\[ \sigma_z = \lambda \varepsilon_z + 2G \frac{\partial w}{\partial z} \]
\[ \tau_{xy} = G \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]
\[ \tau_{xz} = G \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \]
\[ \tau_{zx} = G \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \]  

(7)

Where \( \varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \).

Together with the three equations of equilibrium, we have a system of nine equations and nine unknowns. We can further reduce the nine equations into three equations in terms of three displacement components. This can be done by substituting equations (7) into the equilibrium equations to eliminate the stress variables. We now have

\[ \left( \lambda + G \right) \frac{\partial \varepsilon_x}{\partial x} + G \nabla^2 u + F_x = 0 \]
\[ \left( \lambda + G \right) \frac{\partial \varepsilon_y}{\partial y} + G \nabla^2 v + F_y = 0 \]
\[ \left( \lambda + G \right) \frac{\partial \varepsilon_z}{\partial z} + G \nabla^2 w + F_z = 0 \]  

(8)

where \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \)

The equations (8) are called equations of equilibrium in terms of displacement or Navier’s equations. These equations are much easier to apply. Once we obtain the solutions for \( u, v, w \), we can use Eqs. (7) or stress-strain relations to determine the stress components.
The compatibility equations in terms of stress in three dimensions are

\[ \nabla^2 \sigma_x + \frac{1}{1 + \nu} \frac{\partial^2 \Theta}{\partial x^2} = -\nu \left[ \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right] - 2 \frac{\partial F_x}{\partial x} \]

\[ \nabla^2 \tau_{yx} + \frac{1}{1 + \nu} \frac{\partial^2 \Theta}{\partial y \partial z} = -\left( \frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial z} \right) \]

The solutions for any linear elasticity problem are unique. This means for a given surface force and body force distribution, there is only one solution for the stress components consistent with equilibrium and compatibility. Now let’s prove it.

Let \( T_x^\mu, T_y^\mu, T_z^\mu \) be a given surface force.

Let \( F_x, F_y, F_z \) be a given body force.

Assume there are two sets of stress components which both satisfy the governing equations and boundary conditions. Let the two sets of solutions be

\[ \sigma'_x, \cdots, \tau'_{yx} \quad \text{and} \quad \sigma''_x, \cdots, \tau''_{yx} \]

Now the first solution must satisfy the equilibrium equations

\[ \frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau'_{xy}}{\partial y} + \frac{\partial \tau'_{xz}}{\partial z} + F_x = 0 \quad (x,y,z) \]

The compatibility equations

\[ \nabla^2 \sigma'_x + \cdots = -2 \frac{\partial F_x}{\partial x} \quad (x,y,z) \]

\[ \nabla^2 \tau'_{yx} + \cdots = -\left( \frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial z} \right) \quad (x,y,z) \]

and the boundary conditions

\[ T_x^\mu = \sigma'_x \mu_x + \tau'_y \mu_y + \tau'_z \mu_z \quad (x,y,z) \]
Similarly

\[
\frac{\partial \tau^{''}_{xx} + \partial \tau^{''}_{yy} + \partial \tau^{''}_{zz}}{\partial x} + \frac{\partial \tau^{''}_{zy} + \partial \tau^{''}_{xz} + \partial \tau^{''}_{yz}}{\partial y} + \frac{\partial \tau^{''}_{yx} + \partial \tau^{''}_{zx} + \partial \tau^{''}_{yz}}{\partial z} + F_x = 0
\]

\[(x,y,z)\]

\[
\nabla^2 \sigma^{''}_x + \cdots = -2\frac{\partial \tau^{''}_{xx}}{\partial x}
\]

\[(x,y,z)\]

\[
\nabla^2 \tau^{''}_{xx} + \cdots = -\left( \frac{\partial F_x}{\partial y} + \frac{\partial F_x}{\partial z} \right)
\]

\[
T^{''}_x = \sigma^{''}_{xx} u_x + \tau^{''}_{xy} u_y + \tau^{''}_{xz} u_z .
\]

Let \( \sigma_x = \sigma^{'}_x - \sigma^{''}_x \), \( \sigma_{x_0} = \sigma^{'}_{x_0} - \sigma^{''}_{x_0} \), etc,

and subtracting the set of equations with double primes from those with single primes, we have

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau^{''}_{xy}}{\partial y} + \frac{\partial \tau^{''}_{xz}}{\partial z} = 0
\]

\[
\nabla^2 \sigma_x + \frac{1}{1 + \nu} \frac{\partial^2 \Theta}{\partial x^2} = 0
\]

\[
\nabla^2 \tau^{''} + \frac{1}{1 + \nu} \frac{\partial^2 \Theta}{\partial y \partial z} = 0 \quad \text{where} \quad \Theta = \sigma_x + \sigma_y + \sigma_z
\]

\[
\sigma_{x_0} u_x + \tau^{''}_{xy} u_y + \tau^{''}_{xz} u_z = 0.
\]

From the above equations we see there are no body force and surface force terms because of the linear character of equations and the same body forces and surface forces appear in both sets of equations. It tells us that a body with zero body forces and zero surface forces is in an unstressed state and the stresses at all points throughout the volume of the body are zero,
i.e. \[ \sigma_x = \sigma_x' - \sigma_x'' = 0 \quad (x,y,z) \]
\[ \tau_{yz} = \tau_{yz}' - \tau_{yz}'' = 0 \quad (x,y,z) \]

Therefore the two states of stress are identical so that the solution is unique. This proof is only for the first boundary-value problem (the stresses are prescribed on the entire boundary), but it is similar for the second boundary-value problem.

In the above proof we use the condition of linearity. This only happens in linear elasticity since the deformation is infinitesimal, so all the governing equations and boundary condition equations are linear equations. (An equation is linear if it contains only terms of up to the first degree in the dependent variables and their derivatives). We cannot use the same condition for finite elasticity which in general may deal with large deformations.

Examples for the chapter

1. Derive equations (6)

Substituting Eqs (1) into Eqs (2), we have

\[ \sigma_x = 2G \frac{\partial u}{\partial x} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \]
\[ \sigma_y = 2G \frac{\partial v}{\partial y} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \]
\[ \tau_{xy} = G \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \]

Combining these equations with equations (3) and eliminating the stress variables, we have

\[ \frac{\partial}{\partial x} \left[ 2G \frac{\partial u}{\partial x} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[ G \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] + F_x = 0 \]
By grouping the similar terms, we have
\[ G \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial^2 u}{\partial x^2} (\lambda + G) + \frac{\partial^2 v}{\partial x \partial y} (\lambda + G) + F_x = 0. \]

By similar approaching, we have
\[ GV^2 v + (\lambda + G) \left( \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) + F_y = 0. \]

2. Show that the stresses
\[
\begin{align*}
\sigma_x &= kxy \\
\sigma_y &= kx^2 \\
\sigma_z &= \nu kx(x+y) \\
\tau_{xy} &= -\frac{1}{2} \nu k^2 \\
\tau_{xz} &= \tau_{yz} = 0
\end{align*}
\]

where k is a constant, represent the solution to a plane strain problem with no body forces if the displacement components at the origin are zero. Determine the displacement components and the restraining force in the z direction. What surface force distribution must be applied to the prism shown in the figure so that the given stress components give the solution for this body?
solution: determine the displacement components

\[ \sigma_x = 2G\varepsilon_x + \lambda (\varepsilon_x + \varepsilon_y) \]
\[ kxy = 2G\varepsilon_x + \lambda \varepsilon_x + \lambda \varepsilon_y \]
\[ = (2G + \lambda)\varepsilon_x + \lambda \varepsilon_y \quad (1) \]
\[ \sigma_y = 2G\varepsilon_y + \lambda (\varepsilon_x + \varepsilon_y) \]
\[ kx^2 = 2G\varepsilon_y + \lambda \varepsilon_x + \lambda \varepsilon_y \]
\[ = (2G + \lambda)\varepsilon_y + \lambda \varepsilon_x \quad (2) \]
\[ \sigma_z = \lambda (\varepsilon_x + \varepsilon_y) \]
\[ = \nu(\sigma_x + \sigma_y) \]
\[ \nu kx(x+y) = \lambda \varepsilon_x + \lambda \varepsilon_y \quad (3) \]

From equation (3) we have

\[ \lambda \varepsilon_y = \nu kx(x+y) - \lambda \varepsilon_x \quad (4) \]

Substituting equation (3) into equation (1), we have

\[ kxy = (2G + \lambda)\varepsilon_x + \nu kx(x+y) - \lambda \varepsilon_x \]
\[ = 2G\varepsilon_x + \nu kx(x+y) \]
\[ 2G\varepsilon_x = kxy - \nu kx(x+y) \]
\[ = 2ka^2 - 3\nu ka^2 \]
\[ = ka^2(2 - 3\nu) \]
Putting the solution into equation (4), we have

\[ \varepsilon_x = \frac{ka^2(2 - 3\nu)}{2G} \]

\[ \varepsilon_y = \frac{ka^2(6\nu G - 2\lambda + 3\lambda\nu)}{2\lambda G} \]

\[ \varepsilon_z = \frac{1}{E} \left[ \sigma_z - \nu(\sigma_x + \sigma_y) \right] 
\quad = \frac{1}{E} \left[ \nu kx(x + y) - \nu(kxy + kx^2) \right] 
\quad = 0 \]

\[ \gamma_{xy} = \frac{1}{G} \tau_{xy} 
\quad = -\frac{1}{2} k y^2 
\quad = -\frac{1}{2} k(2a)^2 \]

\[ \gamma_{xz} = \frac{1}{G} \tau_{xz} 
\quad = 0 \]

\[ \gamma_{yx} = \frac{1}{G} \tau_{yx} 
\quad = 0 \]

Determine the surface force.

\[ \mu_x = \mu_y = \mu_z = \cos 90^\circ = -1 \]

\[ T_x = \sigma_{x0} \mu_x + \tau_{xy0} \mu_y + \tau_{xz0} \mu_z 
\quad = -kxy + 1/2 ky^2 
\quad = -2ka^2 + 2ka^2 
\quad = 0 \]
\[ T_{y}^{\mu} = \tau_{x0} u_x + \sigma_{y0} u_y + \tau_{y0} u_z \]
\[ = -\frac{1}{2} ky^2 (-1) + kx^2 (-1) \]
\[ = \frac{1}{2} k (2a)^2 - ka^2 \]
\[ = ka^2 \]
\[ T_{z}^{\mu} = \tau_{x0} u_x + \tau_{y0} u_y + \sigma_{z0} u_z \]
\[ = \nu k x(x+y)(-1) \]
\[ = \nu k (a+2a)(-1) \]
\[ = -3\nu k a^2. \]
CHAPTER 6  
UNIFORM LOADING IN FINITE ELASTICITY

In the last chapter we talked about the uniqueness of solution for problems in linear elasticity. Can we reach the same conclusion in finite elasticity? That is the main goal we will focus on in this chapter. We will reach this goal by studying one of the most interesting and striking problems in finite elasticity: a cube loaded uniformly over its faces. Let’s consider a homogeneous, isotropic, incompressible body with constitutive equations, which means the equations define the particular idealized material[5].

\[
T = -\pi I + \beta B
\]  \hspace{1cm} (1)

where

\[
B = FF^T \quad \text{and} \quad \beta > 0 \quad \text{and} \quad \pi \quad \text{are constants.}
\]

\[
S = -\pi F^T + \beta F
\]  \hspace{1cm} (2)

We call the material field S defined by \( S=(\det F)TF^T \) the Piola-Kirchhoff stress.

Let B be a cube with faces parallel to the coordinate planes and loaded by the three pairs of equal and opposite forces with some magnitude which is applied normally and uniformly over the faces. Thus, the boundary condition on the faces with normal \( e_i = (1,0,0) \) is

\[
Se_i = \alpha e_i \quad \text{ (} \alpha \text{ = constant)} \hspace{1cm} (3)
\]

where \( \alpha > 0 \) if we are dealing with tension and \( \alpha < 0 \) if we are dealing with compression.

We are looking for the solution of the form

\[
S = \begin{bmatrix}
\alpha & 0 & 0 \\
0 & \alpha & 0 \\
0 & 0 & \alpha \\
\end{bmatrix} \quad \text{F=} \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3 \\
\end{bmatrix}
\]  \hspace{1cm} (4)

with \( \lambda_i > 0 \) is constant. Incompressibility requires
\[ \lambda_1 \lambda_2 \lambda_3 = 1 \quad (5) \]

Substituting (3) into (2) we get

\[ \alpha = \beta \lambda_k - \frac{\pi}{\lambda_k} \]

or

\[ \pi = \beta \lambda_k^2 - \alpha \lambda_k \]

\[ \alpha (\lambda_i - \lambda_j) = \beta (\lambda_i^2 - \lambda_j^2) \quad I = 1..k, \quad j = 1..k \]

so \( \lambda_i = \lambda_j \) or \( \frac{\alpha}{\beta} = \lambda_i + \lambda_j \)

Therefore

\[ \lambda_1 = \lambda_2 \quad \text{or} \quad \eta = \lambda_1 + \lambda_2 \quad (6) \]

\[ \lambda_2 = \lambda_3 \quad \text{or} \quad \eta = \lambda_2 + \lambda_3 \]

\[ \lambda_1 = \lambda_3 \quad \text{or} \quad \eta = \lambda_1 + \lambda_3 \]

where \( \eta = \frac{\alpha}{\beta} \)

Thus a solution of the form (4) is valid provided \( \alpha \) and \( \lambda_i \) are consistent with (5) and (6).

First, assume \( \alpha > 0 \) (tension)

Then \( \lambda_1 = \lambda_2 = \lambda_3 = 1 \) is always a solution which satisfies (5) and (6).

Next assume \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda \)

Then from (5)

\[ \varphi(\lambda) = \lambda_1 \lambda_2 \lambda_3 - 1 \]

\[ = (\eta - \lambda_3) \lambda_2 (\eta - \lambda_1) - 1 \]

\[ = (\eta - \lambda)^2 \lambda - 1 = 0 \]

Also

\[ \varphi'(\lambda) = -2(\eta - \lambda)\lambda + (\eta - \lambda)^2 \]

\[ = (\eta - \lambda)(\eta - 3\lambda) \]

\[ = 0 \]
hence the critical points of \( \eta \) are \( \frac{\eta}{3} \) and \( \eta \).

and

\[
\eta = \frac{4}{27} \eta^3 - 1
\]

Moreover,

\[
q(-\infty) = -\infty \quad q(\infty) = \infty \quad q(0) = -1
\]

\( \eta \) also has zero, one, or two roots in \( (0, \eta) \) according as \( q\left(\frac{\eta}{3}\right) \) is < 0, or = 0, or > 0. i.e;

\[
\frac{\alpha}{3} = \eta < \left(\frac{27}{4}\right)^{\frac{1}{3}} \quad \text{there are no other solutions}
\]

\[
\frac{\alpha}{3} = \eta = \left(\frac{27}{4}\right)^{\frac{1}{3}} \quad \text{there are three other solutions}
\]

\[
\frac{\alpha}{3} = \eta > \left(\frac{27}{4}\right)^{\frac{1}{3}} \quad \text{there are six other solutions.}
\]

Finally for \( \alpha < 0 \) (compression)

\( \lambda_1 = \lambda_2 = \lambda_3 = 1 \) is the only solution.

The number of solutions depends on the value of \( \eta \) at \( \eta/3 \). Let's look at the Figure.

when \( \eta(\eta/3) < 0 \), there is no intersection on \( \lambda \) axis. So there are no solutions. When \( \eta(\eta/3) = 0 \), there are one solution. But there are three other solutions since we can assume
$\lambda_1 = \lambda_2 = \lambda$ or $\lambda_1 = \lambda_2 = \lambda$ or $\lambda_2 = \lambda_3 = \lambda$. When $q(\eta/3) > 0$, there are two solutions. But there are six other solutions with the same reasons as the previous one. These results show that the solutions are not unique in finite elasticity which is not true for linear elasticity that we have showed in chapter 5. This concludes that a cube loaded with large tension uniformly over its faces will produce seven different solutions within the class of homogeneous deformation of the form.

This example is from “Topics in Finite Elasticity” by Morton E. Gurtin.


