Effects of mathematics professional development on growth in teacher mathematical content knowledge

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EFFECTS OF MATHEMATICS PROFESSIONAL DEVELOPMENT ON GROWTH IN TEACHER MATHEMATICAL CONTENT KNOWLEDGE

A Project
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Teaching:
Mathematics

by
Carol Elizabeth Cronk
June 2012
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ON GROWTH IN TEACHER MATHEMATICAL
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ABSTRACT

Teachers in one southern California school participated in a three-year professional development project designed to increase student performance by assisting the teachers in providing more balanced math instruction. The goals of this study were (1) to evaluate the change in teacher mathematics knowledge for teaching in the area of fractions; (2) to compare the growth in teacher understanding of fractions to teacher percentage of participation in project activities; (3) to determine common fraction misconceptions among the teachers; and (4) to determine to what extent these misconceptions were ameliorated by project activities. Data was collected regarding the modified Learning Mathematics for Teaching (LMT) assessments and amount of participation in project activities, and interviews were conducted to gain insights into reasons for particular results in the assessments. Findings from the modified LMT assessments include an insignificant correlation between participation in the project for all teachers and modified LMT scores, but in separate analysis of grade levels, there was a significant correlation for the 4th- to 8th-grade level span. In addition, there was a significant increase in scores on modified LMT assessments over time. Findings from interviews indicate that teachers felt they had improved their teaching of fractions through involvement in project activities. Analysis of the participant interviews revealed more growth in content knowledge than the modified LMT scores would seem to indicate. One conjecture as to the reason for this discrepancy is that teachers may not have been as proficient in
writing about mathematics as they were in verbal explanation. It is suggested that additional studies on larger groups of teachers could lead to a better understanding of the length and intensity required of professional development in order to support significant improvement of teachers' mathematical understanding and their ability to teach mathematics conceptually.
ACKNOWLEDGEMENTS

I would like to thank the members of my project committee: Dr. Joseph Jesunathadas (project chair), Dr. Davida Fischman, and Dr. Laura Wallace; they have provided extensive support academically as well as emotionally. I also want to thank Ron Williams, for being a great listener, and to Marie, who calmly and patiently helped with my editing.

I would like to acknowledge the five teachers who were willing to spend time with me and answer questions about what they learned in the project. Without their willingness to give of their time, this project would not have been possible.

I want to give special thanks to my daughter, Caren, and son-in-law, Jacob, for their understanding and patience. I am especially grateful to my husband, Allen, for his never-ending support and encouragement.

Last, but not least, I want to thank the California Post-Secondary Commission (CPEC) who funded the project studied in my project.
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CHAPTER ONE
INTRODUCTION

Background

The quest to determine to what extent mathematics instruction can be improved has not only been of keen interest to researchers, but has also been a long and frustrating endeavor, fueled by a high stakes accountability system for those working in K-12 education. The goal of improved student performance has often proven elusive. School districts have frequently changed textbook adoptions and supplementary and intervention materials, and provided various mathematics professional development opportunities for teachers in an attempt to improve the quality of instruction, and as a result, to improve student performance. To further exacerbate matters, some states such as California have frameworks that emphasize conceptual understanding, but the high-stakes, multiple-choice assessments given to students primarily require memorization of procedures.

Statement of the Problem

In the United States, there is still an achievement gap between mathematics grade-level expectations and actual student mathematics attainment. In addition, even though more students are proficient in mathematics than a few years ago, there is still a significant performance gap between the
White and Asian American students and the Latino and African American students (California Department of Education, 2011a). There may be a variety of reasons that can be attributed to this problem. One problem is that many teachers do not have a deep knowledge of the mathematical content they are teaching. Researchers continue to study the relationship between teacher mathematical knowledge and student performance in an attempt to help close this achievement gap. With all the efforts expended by stakeholders to help improve student performance in mathematics, the national data on student improvement does reflect modest increases in student mathematics scores in the United States, but, unfortunately, the United States still lags way behind many other countries in mathematics performance. The Programme for International Student Assessment (PISA, n.d.) is an assessment given to 15-year-old students internationally to assess their literacy in multiple subject areas. The last PISA was administered in 2009, and its results ranked the United States 31st in mathematics achievement, just a few places higher than in 2006 when it ranked 35th. For the 2009 assessment, China ranked highest with a score of 600, the average score for countries who are members of the Organization of Economic Cooperation and Development (OECD) was 496, and the United States had a score of 487 (Fleischman, Hopstock, Pelczar, Shelley, & Xie, 2010). While the United States’ average score rose from 474 to 487 (PISA, 2006), these results show that the United States still has a long way to go in improving student mathematics achievement. Therefore, it becomes increasingly important to
obtain a better understanding of the teacher qualities and skills necessary in
order for student performance to improve.

Purpose of the Study

In response to this need to know more about what teachers understand
about mathematics and learning, this study was conducted to investigate whether
or not professional development has a positive effect on teachers’ mathematical
and pedagogical knowledge. The study was also conducted to determine if there
was a correlation between teachers’ scores on fraction items on project
assessments and the percentage of participation time in professional
development activities. The specific goals are:

1. To study the changes in teachers’ mathematics knowledge for teaching
   fractions,
2. To compare the growth in teachers’ mathematical knowledge for teaching
   fractions in relation to the percentage of their attendance in project
   activities,
3. To determine participants’ common misconceptions about fractions and to
   what extent these were ameliorated as a result of participating in the
   project.

An assessment instrument called Learning Mathematics for Teaching
(LMT) was used to examine teachers’ mathematical content knowledge and to
determine the impact of the professional development activities on teachers’
knowledge for teaching. The LMT is a multiple-choice instrument; however, for this project, modifications were made so that participants would explain and justify their responses (Appendix A). Subsequently, the teachers were interviewed to examine their perceptions regarding the professional development received.

Context: Program and Participant Description

In this study, the participants were teachers in a California Postsecondary Education Commission (CPEC) funded project. The participants were all teaching at a K-8 science, technology engineering, and mathematics (STEM) school within a culturally diverse school district. Due to budget cuts, there was a great deal of teacher turnover, and only about one-fourth of the teachers participated in the project for its duration.

Participants in the project participated in three-hour monthly “Math Explorations” where they discussed mathematics content aligned to the California mathematics standards and research-based instructional strategies. Participants also took part in a two-week intensive summer institute each year of the grant and were released one day a month during the school year to participate in grade-level or grade-span lesson studies. All professional-development and lesson-study sessions were facilitated by project personnel who included Institute of Higher Learning (IHE) departments of mathematics and education faculty and county office EL and mathematics K-12 personnel. The
levels of participants' mathematics content knowledge were determined by their responses on the Learning Mathematics for Teaching (LMT) multiple-choice questions and their written justifications for selecting those answers. Participants were given the assessment at the beginning of the project, once each year, and again at the end of the project.

Significance of Study

Although billions of dollars are spent nationally each year on teacher professional development (Federal Coordinating Council for Science, Engineering, and Technology, 1993), the mathematics performance of students in the United States continues to lag behind the rest of the developed world. Educators in California and the rest of the states in the nation have worked hard to show progress in their Annual Yearly Progress (AYP) reports as the NCLB deadline of 2014 for all students to be proficient in language arts and mathematics quickly approaches. California’s student achievement data over the past 12 years has shown that students’ math scores on the state tests have improved, but that these scores are still not as high as their language arts scores (California Department of Education, 2011a). There is increasing pressure from school administrators and the public at large to continue to increase student scores in mathematics, and many teachers feel pressured to “teach to the test.”

Beginning in spring 2015, students will no longer be assessed by the California Standards Test (CST). Instead, students across the country will be
assessed by a new assessment system that will be aligned to the California Common Core State Standards for mathematics (Smarter Balanced Assessment Consortium, 2012). This new assessment system will include not only multiple-choice items, but free-response items as well. The content of mathematics in these standards is much deeper conceptually than in the current state standards. Teachers will not only be responsible for teaching computations, they will have to teach problem solving and mathematical reasoning at a much deeper level.

The findings from this research provided some information on what aspects of the professional development in the project may have increased teacher performance on the LMT, as well as to what extent teachers’ perceived mathematical and pedagogical needs were met.

Limitations of the Study

The most significant limitation of this study was the small number of teachers who participated and completed all three years of the project. There were only seven teachers who were part of the project from the beginning to the end. This was due to several factors; most significantly, there was a high turnover of teachers each year due to district layoffs, transfers, and retirements. Qualitative analysis was used to support and clarify findings from quantitative analyses, in particular transcripts and recordings from participant interviews, because of the limitations of the small sample size.
CHAPTER TWO
LITERATURE REVIEW

What Kinds of Knowledge Should a Mathematics Teacher Have?

For more than 100 years, many documents have been created which list the content teachers should know in order to provide effective mathematics instruction for their students (Mewborn, 2001). Concern over the content teachers should know in order to be effective teachers is not new. As part of the state board examination in 1875, potential teachers took a 1000-point test that included written arithmetic and mental arithmetic. Each content area of the test was composed of questions that were 95% content and 5% pedagogy (Shulman, 1986). However, efforts to distinguish between teacher’s content knowledge and pedagogical knowledge and how each affects student achievement is a fairly recent change (Shulman, 1986).

Beginning in the 1960s and continuing through the 1970s, a number of quantitative studies were carried out to determine a connection between teacher content knowledge and student achievement (Begle, 1972, 1979; Eisenberg, 1977; Mewborn, 2001). These studies looked at measures of teacher knowledge such as courses taken, grade point average, and whether or not the teacher had majored in mathematics in college. These studies, however, failed to find a significant correlation between mathematics knowledge and student achievement. Mixed method studies were also conducted during the 1960s through the 1980s
to try to characterize the strengths and weaknesses of teachers' content knowledge in specific concepts such as fractions and geometry (Baturo & Nason, 1996; Graeber, Tirosh, & Glover, 1989; Mewborn, 2001). Most of these studies were conducted with pre-service teachers. The results from studies with elementary teachers suggested that elementary teachers knew facts and could use to algorithms to compute but did not have conceptual understanding of the mathematics. These teachers tended to have fragmented understanding of mathematical topics and were unable to transfer knowledge from one domain to another (Schoenfield & Kilpatrick, 2008; Hill & Ball, 2004).

More recent studies in the 1990s and in following years looked at comparisons between different groups of teachers: elementary versus secondary, pre-service versus in-service teachers, and U.S. teachers versus teachers in other countries (Ball, 1991; Ball & Wilson, 1990; Mewborn, 2001). These studies found one common thread amongst the different groups: conceptual knowledge is low in all of these different U.S. populations. And through these studies, it became more apparent that teacher knowledge and its implementation in the classroom are much more complex than once thought. However, these recent findings did not reveal which characteristics of mathematics teachers successfully enhanced student achievement. Determining whether mathematics teachers will be successful is not as straightforward as previously thought. Even so, as late as 2002, the United States Department of Education (USDE) still argued that a teacher's capability to do general mathematics was the most
important qualification being able to teach mathematics: “We have found that rigorous research indicates that verbal ability and content knowledge are the most important attributes of highly qualified teachers” (U.S. Department of Education, 2002, p. 19).

Researchers continue to seek to determine what is important for mathematics teachers to know; this includes not only the mathematical content, but also the pedagogy a teacher must possess in order to teach mathematics in a way that increases student performance. Researchers have categorized these different types of teacher knowledge in several different ways. It comes as no surprise that what an educated adult, who is not a math major, knows about mathematics is going to be different from the variety of knowledge a mathematics teacher needs to know in order to teach effectively. However, honing in on exactly how it should be different for it to enhance students' understanding of mathematics is the challenge.

One might think that secondary teachers who have had more mathematics classes would provide better mathematics instruction than elementary teachers who have not had as many classes. It may be that the solution is to provide more conceptual instruction in college courses. However, studies have shown that having more mathematical knowledge or having been taught conceptually does not necessarily transfer to classroom practice (Mewborn, 2001). This study did not address change in student performance.
So how can the types of teacher content knowledge be categorized? Early on, one way of categorizing subject matter knowledge teachers must know had been to separate the knowledge into two types: lesson structure knowledge and subject matter knowledge (Leinhardt & Smith, 1985). However, just one year later, Shulman (1986) suggested separating mathematics knowledge for teaching into three domains: content knowledge, pedagogical content knowledge, and curricular knowledge.

In Shulman's (1986) work, the first type of knowledge, content knowledge, refers to the amount and organization of knowledge a person has, and requires going beyond knowledge of the facts or concepts in a domain. This type of knowledge is often represented in the higher levels of Bloom's cognitive taxonomy and Costa's Levels of Inquiry (Costa, 2001). A mathematics teacher needs to not only know how to do the mathematics, but be able to explain how one concept connects to another and why the mathematics is valuable. An example of this is elementary division. Ball (1990) found that many elementary teachers were unable to distinguish between partitive and quotitive division. What they were able to describe was partitive division only. For secondary mathematics teachers, the level of content knowledge needed should be at least the same as those who have a mathematics degree, but who are not in the teaching field.
The second type of knowledge described by Shulman (1986) is pedagogical content knowledge (PCK). Magnusson, Krajcik, and Borko (1999) described PCK as follows:

...a teacher's understanding of how to help students understand specific subject matter. It includes knowledge of how particular subject matter topics, problems, and issues can be organized, represented and adapted to the diverse interests and abilities of learners, and then presented instruction. . . the defining feature of pedagogical content knowledge is its conceptualization as the results of a transformation of knowledge from other domains. (Magnusson et al., 1999, p. 96)

This type of knowledge allows teachers to choose and use the most useful illustrations, examples, and explanations to represent a mathematical idea (Shulman, 1986). In addition, pedagogical content knowledge assists teachers in knowing what makes a concept easy or difficult for students and the most useful ways to help students of different ages and knowledge backgrounds to understand new ideas.

Curricular knowledge is the third type of knowledge, and this is the knowledge of the full range of programs available to teach different topics at different levels. Curricular knowledge involves both lateral and vertical knowledge of what students are learning. So, a mathematics teacher should know what the students learn before they are in their class and what they will be learning later. In addition, a mathematics teacher should also be familiar with the
curriculum their students are learning in other subject areas, and be able to relate the mathematics to what the students are learning in those other subject areas.

In response to two hypotheses on what teachers need to know to be effective teachers Ball, Thames, and Phelps (2008) adapted Shulman's (1986) categories for their own research. The two hypotheses were: 1) teachers need to know and understand the mathematics they will teach and some additional college mathematics, and 2) teachers need to know and understand what they will teach at a much deeper level than they will teach to their students, along with some amount of pedagogical content knowledge. However, their study is unclear in both instances regarding the specifics of what teachers need to know.

In the work of Ball et al. (2008), Shulman's (1986) third category of curricular knowledge was placed within pedagogical content knowledge. In this new model, pedagogical content knowledge was divided into three domains: knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum. KCS involves both knowing about how students learn and knowing about mathematics. Teachers use KCS when they anticipate in advance what students are likely to think and what concepts in mathematics instructions they will find confusing. KCT is knowing about teaching and knowing about mathematics. KCT includes how instruction is designed and the choices that are made as to what examples to start with and what examples will help deepen students' understanding of the content.
The third major category, content matter knowledge, was also divided into three more specific domains: specialized content knowledge (SCK), common content knowledge (CCK), and horizon content knowledge. CCK is the knowledge and skills that are used in settings outside of teaching and SCK is the mathematical knowledge and skill unique to teaching. It is not used for any other purpose and can involve tasks that require knowledge beyond what is taught to students.

It is important to note that this more detailed categorization has several limitations when it comes to determining where certain situations fit into the domains (Ball et al. 2008). The first problem has to do with how a teacher handles a situation. For instance, when a teacher analyzes a student misconception, he/she might use specialized content knowledge or knowledge of content and students. Another problem has to do with common understanding of the boundaries of each category. For example, CCK is the mathematical knowledge that teachers share in common with other professions. So, a problem such as $\frac{4}{7}$ of 3 could be thought of as being in either of two categories: specialized content knowledge or common content knowledge. However, despite such limitations, the domains are useful for studying the relationships between teachers' knowledge and student achievement.
Effective Professional Development and Teacher Mathematics Content Knowledge

Fennema and Franke (1992) argued that teachers need to have deep content knowledge to influence learning in a positive way. In addition, they added that since mathematics is made up of many abstractions, an effective teacher must know how to connect those abstractions to real life for the student or learning with understanding will not take place. However, as mentioned before, it is not always clear what specific types of mathematics content teachers need to know. Most measures to determine a teacher's knowledge have only tested skills, and measuring math skills alone is not enough to know whether or not a teacher can deliver content in a way that supports students in making sense of mathematics (Hill & Ball, 2004).

What is viewed as effective professional development has changed over time, and so it is important to provide a definition for professional development. Little (1987) described professional development as "any activity that is intended partly or primarily to prepare paid staff members for improved performance in present or future roles in the school districts" (p. 491). Until 10 years ago, most professional development was in the form of lectures, college classes, conferences, and special institutes. However, in the last decade, more professional development has been based in discourse and community practice (Desimone, 2009). A lot of professional development is in the form of formal and informal professional learning communities as well as lesson study groups.
Many projects that are focused on helping teachers improve student performance incorporate several types of professional development with the same group of teachers. Desimone (2009) suggested that there are five critical components necessary to increase teacher knowledge, skills, and practice through professional development. The five components are content, active learning, coherence, duration, and collective participation. The first component could be the most influential one. Desimone stated that:

A compilation of evidence in the past decade points to the link between activities that focus on subject matter content and how students learn that content with increases in teacher knowledge and skills, improvements in practice, and, to a more limited extent, increases in student achievement. (Desimone, 2009, p. 184)

Active learning is the alternative to passively listening to a lecture. Examples of active learning would include teachers watching expert teachers teach and having a discussion regarding the lesson afterwards, or having the expert teachers observe them, then having a discussion about the lesson, as well as the expert teachers providing feedback. It could also include reviewing and discussing student work (Banilower & Shimkus, 2004; Borko, 2004; Carey & Frechtling, 1997; Darling-Hammond, 1997; Lieberman, 1996).

Teachers' buy-in to professional development is contingent upon whether they are convinced that the professional development they receive is relevant to their current instructional practice and is responsive to administrative and
legislative pressures for student achievement. Teachers need to see that the professional development is consistent, at least in some ways, with what they already know and believe about student learning. They also need to feel that the information received is tied to school, district, and/or state reforms (Desimone, 2009). In a time of high stakes accountability, any information that does not appear aligned to directives by their sites or districts can cause teachers to be resistant to professional development.

“Duration” refers to the length of time it takes for professional development to make a long-lasting intellectual and pedagogical change. This change requires activities spread over at least a semester or intensive time over a summer with follow-ups, and it also requires a substantial number of hours (Cohen & Hill, 2001; Fullan, 1993; Guskey, 1994; Supovitz & Turner, 2000). A review of nine studies found that sustained professional development was related to increases in student achievement. Professional development of less than 14 hours showed no effects on student learning, professional development lasting more than 14 hours showed significant positive effects, but the largest effects on student learning were found for programs between 30 and 100 hours spread out over 6–12 months (Yoon, Duncan, Lee, Scarloss, & Shapley, 2007).

The last of the five components is collaborative participation. To encourage discourse and active participation by teachers, it is important to have teachers that share a commonality. This can be accomplished by including
Desimone (2009) proposed a core theory of action for professional development that incorporates these components:

1. Teachers experience effective professional development.
2. The professional development increases teachers' knowledge and skills and/or changes their attitudes and beliefs.
3. Teachers use their new knowledge and skills, attitudes, and beliefs to improve the content of their instruction or their approach to pedagogy, or both.
4. The instructional changes foster increased student learning.

(Desimone, 2009, p. 184)

Desimone's (2009) model assumed that teachers' attitudes and beliefs will change as they participate in professional development. However, before beliefs can change it is important to consider when planning professional development what makes professional development effective as well as what can hinder its effectiveness. The common purpose of professional development programs is to alter professional practices, beliefs, and improve student learning. The ineffectiveness of some professional development programs can be attributed to two crucial factors not being taken into account: 1) what motivates teachers to engage in professional development and 2) the process by which instructional change in teachers generally occurs (Guskey, 1986).
According to Guskey (1986), a key to facilitating change in instruction was the order in which the outcomes occur most often. In programs where the desire is to first change beliefs and attitudes among the teachers, a concerted effort is made to build relationships with the staff members, and the staff members are included in the decision-making process. However, this process does not significantly change teachers' attitudes or secure their commitment (John & Hayes, 1980). Instead, teachers do not tend to change their attitudes until they have proof that what they have implemented in their classrooms has improved student achievement. For example, teachers who have worked with students from disadvantaged backgrounds, often feel that their students are incapable of learning. Guskey (1986) suggested that teachers first need to be provided professional development that gives them strategies to take back and use in their classrooms immediately. Then, after teachers feel successful with the new strategies and see an improvement in student achievement, then they are more likely to change their beliefs and attitudes about their teaching.

Since increased teacher content knowledge is assumed to support student achievement, many programs have provided teachers with professional development that is aimed at deepening their mathematics content knowledge. This is no surprise as many providers of professional development are content experts. However, supporting teachers in improving student achievement is much more complicated than solely conducting monthly mathematics content seminars or holding two-week intensive summer institutes. In fact, research is
somewhat limited on which aspects of professional development have the most
effect on teacher understanding of mathematics as well as which aspects play a
role in increasing pedagogical knowledge teaching, and equally as important,
how to change teacher beliefs and attitudes.

Connections to Teacher Professional Development

It is important to note that many studies have been done in which
researchers have attempted to look for correlations between student
achievement and teacher PCK. For example, Carpenter, Fennema, Peterson,
and Carey (1988) investigated the PCK for children’s problem solving using 40
first grade teachers and their students. The measures of this study focused on
the teachers’ knowledge of distinctions between problem types and the ability to
predict performance of specific students on different problems.

Teachers were assessed in two ways. The first type of problem was the
‘Writing Word Problems.’ Teachers had to write problems that were represented
by different number sentences. The second type of assessment was the
Relative Problem Difficulty test where teachers had to determine which of a pair
of problems would be more difficult for their students. Teachers also had to
demonstrate how different students in their class would solve particular addition
and subtraction word problems.

It was found that teachers’ general knowledge of problems, the strategies
needed to solve them, and the difficulty of the problems were significantly
correlated with student achievement. In addition, there was no correlation between teacher ability to predict what strategy a student would use and student achievement. On the other hand, there was a significant correlation between teachers' ability to predict how successful their students would be on specific problems. While this study provided some important information on teacher mathematical knowledge and understanding, the instruments were specific to this particular study.

Another study, one that lasted three years, was conducted with teachers who were teaching in high-risk schools. The professional development included providing teachers effective strategies for facilitating problem-solving, deepening teachers' content knowledge in relation to authentic assessment for the grade levels being taught, and understanding the NCTM standards and principles as well as their state standards that are relevant to second and third grade (Bailey, 2010). Some of the key professional development strategies employed were those commonly believed to improve instruction and used frequently in mathematics professional development. For example, emphasized in the professional development was the teaching of concepts through lessons that address multiple standards. In addition, participants were encouraged to find more than one strategy to solve problems. Lastly, after unpacking the standards, teachers evaluated the lessons to find materials that aligned to the concepts for their grade level.
Teachers in this study by Bailey (2010) were given the ITQ (Improving Teacher Quality Survey), which included both math content knowledge and self-efficacy items. The survey was given both pre- and post-treatment. The results showed that teachers improved in both their self-efficacy in regards to teaching and their mathematics content knowledge. Implications from the ITQ study by Bailey included that professional development for mathematics teachers should include the components described above as well as a recommendation that researchers, teachers, educators, and school administrators all need to work collaboratively to design the professional development.

Development of Learning Mathematics for Teaching Instrument

What had been needed was an instrument that could be used more generally in studies to determine how much of each type of mathematical knowledge (PCK, MKT, etc.) a teacher possesses, and if that knowledge is sufficient to increase the academic performance of the majority of that teacher's students. A study by Hill and Ball (2004) documented the first use of an instrument for evaluating teachers designed to measure teacher content knowledge. The instrument was built on the assumption that there is a construct or multiple constructs that can be called Mathematics Knowledge for Teaching (MKT) and that scales could be developed that measure this knowledge. In addition, there is a second construct that can be called Pedagogical Content Knowledge (PCK). PCK would include familiarity with topics that children find
interesting, representations that are most useful for teaching a concept, and
typical student errors and misconceptions.

The first MKT items were developed around specific math content:
number concepts, operations, patterns, functions, and algebra. The rationale
behind this content was that number concepts and operations made up a large
portion of the K–6 curricula. Patterns, functions, and algebra were selected as
topics as they were relatively new concepts for elementary students. For the
purposes of developing items, two sub-categories were used. These sub­
categories were Common Knowledge of Content (CKC) and Specialized
Knowledge of Content (SKC) (Hill & Ball, 2004). The difference between the two
types is that common knowledge is the ability of mathematics teachers to
compute and solve problems and specialized knowledge is the ability to
demonstrate alternative ways of solving mathematics problems as well as the
ability to recognize and evaluate unconventional strategies of solving problems.

The item writers of the first MKT items sought to serve three purposes with
the items developed in the categories listed above. The first was to have
measures in which growth in content knowledge could be measured. The
second was to sharpen the researchers’ ideas about what knowledge and skills
in mathematics teachers need to have. The last was to be able to find items that
could be used as pilot items that could be used to help clarify the organization
and characteristics of MKT.
Correlation of Learning Mathematics for Teaching Results, Teacher Instruction, and Student Achievement

The resulting instrument was given to elementary school teachers attending summer mathematics professional development institutes to see if a large-scale project could be evaluated using this instrument. Although the effectiveness of the LMT would still need more study, as some of the measures were still under development at the time of this current study, the use of the instrument, showed that teachers can and do learn mathematical content for teaching in summer professional development programs. The teachers showed an increase in their scores between the pre- and post-assessments based on the LMT. This was true independent of what the mathematical content focus was for the professional development. However, it is important to note that all of the instructors were carefully selected for the summer professional development; all were knowledgeable in mathematics at the college level.

In a subsequent study, (Hill, Ball, Goffney, Blunk, & Rowan, 2008) researchers focused on whether an instrument that measures teacher content knowledge in a multiple-choice format such as the LMT, gave accurate information about the teachers' mathematical knowledge. This 30-item instrument consisted of items that were “balanced across content domains (13 number items, 13 operations items, 4 pre-algebra items), and specialized (16 items) and common (14 items) content knowledge” (Hill et al., 2008, p. 109) was given to teachers who were later videotaped in the classroom.
This study found that there was a correlation between high scores on the assessment and the teachers’ abilities to avoid mathematical errors and provide classroom instruction that is “rich in representations, explanations, reasoning, and meaning” (Hill et al., 2008, p. 117).

In addition, a case study involving 10 teachers was conducted as researchers looked for an association between teachers’ MKT as measured by the LMT and the Mathematical Quality of Instruction (MQI) as evaluated through classroom observations (Hill, et al., 2008). The researchers were interested in finding where teachers get their content knowledge and instructional strategies. The 10 teachers chosen for this study all agreed to be involved in mathematics professional development. In addition, the teachers agreed to take pencil-and-paper LMT assessments as pre- and post-tests and nine classroom assessments were videotaped. Although there were many limitations to this study, including the fact that these particular teachers volunteered for the study and were not randomly chosen, a positive correlation between the teachers' LMT scores and the quality of their mathematics lessons were evidenced through the videotaped lessons. In addition, in some instances, it appeared that extensive professional development improved both teacher mathematical knowledge and the quality of instruction.

The LMT was used in a large-scale study where teachers received professional development in mathematics through intensive workshops, such as Mathematics Professional Development Institutes (MPDI). The MPDIs were
three-week intensive summer institutes offered in California from 2000–2003. Twenty-three thousand teachers received subject-matter specific professional development in English language development and in mathematics. Overall, teachers showed growth in their math content knowledge as evidenced by the LMT assessment. However, it is not known which aspects of the professional development were most effective in strengthening teacher content knowledge and improving student performance (Hill, Schilling, & Ball, 2004).

The formula for being an effective math teacher is not a simple one. For over a century, stakeholders and researchers have been trying to improve student mathematical performance by modifying the ingredients in the formula. What has been learned in the quest for improved mathematics instruction is that there are categories of teacher knowledge, and these categories include conceptual knowledge of the mathematics as well as pedagogical competence. The LMT is an instrument that can assist professional developers in finding out whether provided professional development has increased teacher understanding in several different categories. However, there is still much to learn about whether or not increased teacher knowledge in any particular area is associated with improved student performance.
CHAPTER THREE
METHODOLOGY

As stated in the introduction, the goals of this study were (1) to evaluate the change in teacher mathematics knowledge for teaching fractions; (2) to compare the growth in teachers' mathematical knowledge for teaching fractions to the percentage of their attendance in project activities; (3) to determine and analyze common mathematical misconceptions in the area of fractions among the teachers, and (4) to what extent they were ameliorated by the provided professional development.

Project Professional Development

The professional development project was a three-year project at one school site in a K–8 district. The school is in an urban area with a high population of English Language learners (ELLs). At the onset of the project, all teachers agreed to be part of the project as a requirement of teaching at that particular school site. However, after the first year, with layoffs and mandatory reassignments, the new teachers to the school did not make the same commitment although they were required to participate in project activities.

One of the goals of the professional development project was to reduce the two following district-wide achievement gaps by 30%: a) the gap between the White student population and the rest of the school’s population, and b) the gap
between the English Learner (EL) population and the non-EL population. Another goal was to increase math conceptual content knowledge and math knowledge for teaching among the teachers as well as to increase use of effective pedagogical skills, especially for English Language Learners and Standard English Learners (SELS). A third goal was to promote cultural and systemic change that would result in ongoing improvement that promotes long-term sustainability. The final project goal was to create a replicable professional development model with evidence-based successes.

Throughout the project, teachers were provided sustained professional development through several different components. Each year of the project participants attended a summer institute. Throughout the school year, participants attended a monthly afternoon professional development session referred to as Math Explorations. In addition, small groups of teachers participated, once a month, in school-day lesson study sessions where they studied content, planned and conducted research lessons, and debriefed and reflected on those research lessons.

The summer institute content was different each year (Table 1). Multiple members of the project leadership team facilitated each summer institute. As with lesson study and Math Explorations, the content for the professional development throughout the project was chosen largely based on the leadership team's perception of what mathematical and pedagogical content teachers needed. Students' CST scores in the year prior to the beginning of the program,

27
teacher requests, and classroom observations by project staff played a role in the
design of the professional development framework, content, and learning
activities.

Table 1

Mathematics Content of Summer Institutes

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>-Fractions, decimals, and place values</td>
<td>-Underlying structure of mathematics: relational thinking, equivalence, justification, mathematical reasoning</td>
<td>-Grades K–2: addition and subtraction</td>
</tr>
<tr>
<td>-Proportional reasoning</td>
<td>-Grades 3–5: fractions</td>
<td></td>
</tr>
<tr>
<td>-Patterns, graphs, and equations</td>
<td>-Fractions: estimation, part to whole, comparing and ordering, operations, equivalence</td>
<td>-Grades 6–8: fractions, decimal, and percent representations</td>
</tr>
<tr>
<td>-Problem solving and mathematical reasoning</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As mentioned above, the participants met once a month during the workweek for Math Explorations. The mathematics content for these sessions varied as well. Table 2 shows the content for each year of Math Explorations. In addition to content, pedagogy and English Learner instructional strategies were integrated into the professional development.
Table 2

*Mathematics Content of Math Explorations*

<table>
<thead>
<tr>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>-Fractions</td>
<td>-Place value</td>
<td>-Patterns and functions</td>
</tr>
<tr>
<td></td>
<td>-Expressions and equations – use of equal sign and equivalent equations</td>
<td>-What is a function?: recursive vs. explicit, linear vs. non-linear</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-Five ways to represent functions</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-Fractions</td>
</tr>
</tbody>
</table>

Lesson study topics varied for each lesson study group. The lesson study groups were organized by grade level. Some lesson study groups were composed of teachers from several grade levels and other groups were composed of teachers from just one grade level. In Math Explorations and summer institutes, teachers from different grade levels were combined more often than not. During the last summer institute, K–2 was separated from 3–5 and 6–8 for all content sessions.

A combination of the modified LMT results and an analysis of the participant interviews were used to determine the effectiveness of the provided professional development in strengthening teachers’ knowledge of fraction content knowledge. The information documented in this study helped in determining how to better meet the mathematical needs of elementary teachers.
As the project had multiple components, the researcher also wanted to determine which aspects of the professional development the participants found most influential in improving their content knowledge in the area of fractions.

There were some difficulties in conducting an uninterrupted sequence of progressively more conceptual and substantive content in the professional development because of changes in the project school's staffing. Beginning with the second year of the project, unexpected school staffing changes were made. Unfortunately, both the second and third years had a high turnover of teachers for a variety of reasons including teacher retirement, layoffs, and re-assignment. Therefore, there were only seven teachers involved in the project for the entire three years. With a very small sample size, results from the modified LMT were not enough for any statistical significance. In addition to the modified LMT scores, transcripts of participant interviews with the evaluator as well as additional, semi-structured interviews conducted by the researcher were used as data for this study.

Population

The target population for this study was K–8 teachers in an urban elementary school. A total of 59 teachers participated for one or more years in the program, with about 23 teachers participating in any given year. These teachers were involved in an intensive three-year mathematics professional development program.
Participant data from those who had taken multiple modified LMT assessments were used for the purpose of comparing amount of participation to gains in assessment scores. There were 25 teachers who took more than one assessment. Of those who took the assessment, 22 were female and three were male. Table 3 shows the distribution of grade levels of these 25 teachers based on the last grade they taught while participating in the project. The teachers listed under other included one teacher who was teaching science during the last year, a reading coach, a Resource Specialist (RSP teacher), and an Outreach Counselor. All of these teachers hold a California multiple subject credentials.

Table 3

*Teachers Who Took More than One Modified Learning Mathematics for Teaching Assessment*

<table>
<thead>
<tr>
<th>Grade</th>
<th>K</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Teachers</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

However, analysis of the participation data revealed that only seven teachers were involved in the study for the full three years. These seven participants were chosen for further analysis of their responses on the modified
LMT assessments and of their responses to the exit interviews conducted with the project's internal evaluator. Additionally, semi-structured interviews were conducted with five of the teachers who participated in all three years of the project and who were willing to be interviewed by the researcher. This sample of the seven teachers is described below.

Sample

Although other participants had been in the study for a majority of the project's duration, data from these participants were not used for this study. The sample of seven participants included six female teachers and one male teacher. All seven teachers had California Clear Multiple Subject Credentials. None of the seven had a single subject mathematics credential or any other single subject credential. Three of the teachers described themselves as Caucasian, two as Hispanic, one as Hispanic/Caucasian, and one as being of multiple ethnicities. Some of the sample teachers were assigned to different grade levels over the years; Table 4 represents the number of teachers assigned to each grade level in each year of the project.
Table 4

**Grade Level Assignments of Teachers Part of Project All Three Years**

<table>
<thead>
<tr>
<th>Grade Level</th>
<th>2008-2009</th>
<th>2009-2010</th>
<th>2010-2011</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Not teaching fractions</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

**Instrument**

Three instruments were used in the collection of data. All of the participants in the project were given a modified version of the LMT at the beginning of the project and then once a year until the end of the three-year project for a total of four assessments. Table 5 shows the number of fraction items on each Modified Learning Mathematics for Teaching assessment as well as how many items were aligned to each cognitive level. Some items addressed
more than one cognitive level. The assessments given later during the project had a higher percentage of items that required participant conceptual understanding. The modified version of the LMT consisted of both multiple choice response questions as well as written justifications. For purposes of this study, only the items with fractions were analyzed.

Table 5

Cognitive Level of Items on Modified Learning Mathematics for Teaching

<table>
<thead>
<tr>
<th>Assessments</th>
<th>Total Number of Items on Fractions</th>
<th>Conceptual Understanding</th>
<th>Problem/Solving Reasoning</th>
<th>Pedagogical Content Knowledge</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. January, 2009</td>
<td>10</td>
<td>8</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2. December, 2009</td>
<td>10</td>
<td>7</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>4. July, 2011</td>
<td>30</td>
<td>30</td>
<td>5</td>
<td>12</td>
</tr>
</tbody>
</table>
The modified LMT data was analyzed using the Rasch model for the dichotomous and polytomous data. This model was used to measure both the difficulties of the items and the achievement of teacher performance on a linear scale (Pallant & Tennant, 2007).

The second instrument was an interview protocol used with all available project participants at the end of the three years for purposes of evaluating the full project. The teachers' responses to the interview questions and transcripts were made available for this research study. The interview protocol is provided in Appendix B.

The third instrument was a semi-structured interview. During these semi-structured interviews, the researcher asked participants to solve problems and to clarify and elaborate on their responses from the prior interviews with the project's internal evaluation.

Five of the seven participants were selected for the semi-structured interviews. They were chosen because they had discussed fractions more than briefly in their exit interviews (instrument two). Two questions from the LMT item bank were included as part of the interview. The participants had never seen the first question prior to being interviewed. The participants had seen the second question previously on one of the four modified LMT assessments. Both the modified LMT questions, as well as the rest of the interview protocol used during the additional interviews by the researcher, are in Appendix C.
Data Analysis

The scores on the modified LMT assessments for each teacher were compared to determine gains using the table of specifications (Appendix D) to select the items whose mathematical content included fractions. These gains were further analyzed to determine if there was a correlation between time in the project and gains in knowledge of fractions. In addition, the teachers were separated into primary and upper elementary/middle school to see if the correlation was higher for one grade span than for the other. These analyses were done to determine whether or not there was a relationship in growth of teachers' mathematical knowledge for teaching fraction and percentage of participant attendance in project activities.

Since some items on the modified LMT assessments required explanations, the researcher selected the items that not only involved fractions, but also those that were used on more than one assessment, to analyze them more closely. Notes were taken on the teacher responses to fraction items that were answered on multiple assessments as to whether the responses for each participant stayed the same or changed, and if the responses changed, how the responses changed.

To add to the modified LMT item data, the researcher utilized the scripts of the seven teachers that had been in the project for the full three years. The conversations involving fractions were separated from the rest of the scripts. Information on how teacher thinking about fractions had changed, including the
ameliorating of misconceptions, changes in classroom instruction, and teacher personal growth in their understanding of fractions, was collected from these scripts.

Finally, five participants provided additional data in the semi-structured interviews. The researcher recorded the interviews and used the recordings to gather additional information about changes in understanding of fractions and the transfer of this understanding to the classroom. Furthermore, the interview data were used to find out which components of the project they felt were most helpful in enhancing their understanding and their students' understanding of fractions.

The teachers in this project were immersed in professional development for three years. The researcher used the three instruments described to determine any changes in teacher fraction knowledge, teachers' pedagogical knowledge of fractions, and whether or not any misconceptions were resolved for the participants. In the next chapter are the findings from these three instruments.
CHAPTER FOUR
FINDINGS

Data were obtained from different sources. Analysis of the data provided information about the effectiveness of the project. Descriptive statistics of the LMT measures were obtained for teachers and fraction items. The modified LMT results and teacher interviews provided data on determining whether there was growth in teacher understanding of fractions. The measures of participants who were administered multiple tests were correlated with the amount of participation in project activities. Lastly, both interviews of the seven teachers who were involved in the project for the entire period of the project with the internal evaluation, as well as additional interviews with five of the seven teachers, provided information on teacher misconceptions and teacher perceptions of project effectiveness in increasing their understanding of fractions.

Modified LMT Data

The items of the modified LMT that involved fractions were separated from the rest of the items. The multiple choice items of the modified LMT were scored dichotomously and the explanations for the choices were scored according to an Assessment Rubric (Appendix E). Participants were given separate scores for two parts of the modified LMT assessment.

Each of the four modified LMT assessments were linked for analyzes purposes with common items across the modified LMT assessments. In total,
there were 108 LMT items. Of those 108 items, 46 included understanding of fractions. Those 46 items were calibrated and analyzed separately using Winsteps® (Linacre, 2011). The analysis provided a variable map (Figure 1) to plot the participants’ ability location and the level of item difficulty in logits. The letter before each number on the right refers to the individual assessments with A being the first assessment and D being the last one. Participant numbers remained consistent for the four LMT assessments.

Figure 1. Wright Map of Participants and Fraction Items
Table 6 contains the descriptive statistics for all teacher measures and item calibrations. The mean teacher measure was about 0.5 logits below the mean fraction item calibration. The standard deviation was within the expected range. The range in teacher measures (i.e., difference between the maximum and minimum) was almost 7.5 logits. This included two scores, at either extreme of the distribution, that were outliers. The highest score was approximately 1.5 logits greater than the next highest score, and the lowest score was approximately 2.5 logits less than the score greater than it. The range and standard deviation would have been less without the two outliers. The Person reliability measure of .76 was moderately high.
Table 6

Descriptive Statistics of Teacher Scores on Fraction Items

Summary of 103 Measured Person

<table>
<thead>
<tr>
<th>TOTAL SCORE</th>
<th>COUNT</th>
<th>MEASURE</th>
<th>MODEL ERROR</th>
<th>INFIT MNSQ</th>
<th>ZSTD</th>
<th>OUTFIT MNSQ</th>
<th>ZSTD</th>
</tr>
</thead>
<tbody>
<tr>
<td>MEAN</td>
<td>11.9</td>
<td>-.45</td>
<td>.50</td>
<td>1.09</td>
<td>.1</td>
<td>1.02</td>
<td>.1</td>
</tr>
<tr>
<td>S.D.</td>
<td>8.2</td>
<td>1.17</td>
<td>.14</td>
<td>.48</td>
<td>1.0</td>
<td>.41</td>
<td>.8</td>
</tr>
<tr>
<td>MAX</td>
<td>39.0</td>
<td>3.00</td>
<td>1.08</td>
<td>2.88</td>
<td>2.4</td>
<td>2.25</td>
<td>1.9</td>
</tr>
<tr>
<td>MIN</td>
<td>1.0</td>
<td>-4.61</td>
<td>.30</td>
<td>.31</td>
<td>-2.5</td>
<td>.24</td>
<td>-1.5</td>
</tr>
</tbody>
</table>

REAL RMSE .57  TRUE SD 1.02  SEPARATION 1.79  PERSON RELIABILITY .76
MODEL RMSE .52  TRUE SD 1.05  SEPARATION 2.02  PERSON RELIABILITY .80
S.E. OF PERSON MEAN = .12

Table 7 shows the descriptive statistics for just the fraction items on the four modified LMT assessments. The standard deviation was slightly high probably due to the two outliers described above.
Table 7

Descriptive Statistics of All Fraction Items

<table>
<thead>
<tr>
<th>TOTAL</th>
<th>COUNT</th>
<th>MEASURE</th>
<th>ERROR</th>
<th>INFIT</th>
<th>MNSQ</th>
<th>ZSTD</th>
<th>OUTFIT</th>
<th>MNSQ</th>
<th>ZSTD</th>
</tr>
</thead>
<tbody>
<tr>
<td>MEAN</td>
<td>26.6</td>
<td>39.9</td>
<td>.00</td>
<td>.41</td>
<td>.98</td>
<td>-.2</td>
<td>.94</td>
<td>.0</td>
<td></td>
</tr>
<tr>
<td>S.D.</td>
<td>17.5</td>
<td>18.4</td>
<td>1.72</td>
<td>.21</td>
<td>.43</td>
<td>1.5</td>
<td>.41</td>
<td>1.2</td>
<td></td>
</tr>
<tr>
<td>MAX</td>
<td>83.0</td>
<td>76.0</td>
<td>3.24</td>
<td>1.34</td>
<td>3.15</td>
<td>5.6</td>
<td>2.15</td>
<td>4.0</td>
<td></td>
</tr>
<tr>
<td>MIN.</td>
<td>6.0</td>
<td>20.0</td>
<td>-3.97</td>
<td>.15</td>
<td>.23</td>
<td>-2.9</td>
<td>.04</td>
<td>-2.3</td>
<td></td>
</tr>
</tbody>
</table>

REAL RMSE .48  TRUE SD 1.65 SEPARATION 3.47 RELIABILITY .92
MODEL RMSE .46 TRUE SD 1.66 SEPARATION 3.59 RELIABILITY .93
S.E. OF ITEM MEAN = .26

Of the 59 participants in the project, only 26 took the modified LMT more than one time. The mean teacher measure increased between all assessments, except between the second and the third. However, the overall growth, in logits, had a substantial growth of .98 (Table 8). The standard deviation is fairly consistent across all four assessments. The minimum and maximum do not show any particular pattern. The highest maximum was on the third assessment, and the lowest minimum was on the last assessment.
Table 8

*Descriptive Statistics for Participants’ Ability Measures with Two or More Assessments*

<table>
<thead>
<tr>
<th>Year</th>
<th>1/09</th>
<th>12/09</th>
<th>7/10</th>
<th>7/11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>-0.91</td>
<td>-0.36</td>
<td>-0.41</td>
<td>0.07</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>1.01</td>
<td>1.02</td>
<td>1.21</td>
<td>1.36</td>
</tr>
<tr>
<td>Minimum</td>
<td>-2.68</td>
<td>-2.34</td>
<td>-2.52</td>
<td>-4.61</td>
</tr>
<tr>
<td>Maximum</td>
<td>1.43</td>
<td>1.56</td>
<td>3.00</td>
<td>1.49</td>
</tr>
</tbody>
</table>

Individual data for the seven participants who were in the project for the entire three years are given in Table 9. Five of the participants showed growth as measured in logits from the first to the last assessment. The largest growth was approximately 3.5 logits. The other two participants showed a slight decrease, with the larger decrease almost 0.5 of a logit.
Table 9

*Learning Mathematics for Teaching Fraction Scores in Logits for Participants in Project All Three Years*

<table>
<thead>
<tr>
<th>Participant</th>
<th>1/09</th>
<th>12/09</th>
<th>7/10</th>
<th>7/11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Participant 24</td>
<td>-0.35</td>
<td>-0.63</td>
<td>-0.5</td>
<td>0.23</td>
</tr>
<tr>
<td>Participant 11</td>
<td>1.43</td>
<td>0.81</td>
<td>1.08</td>
<td>0.92</td>
</tr>
<tr>
<td>Participant 10</td>
<td>-2.68</td>
<td>-0.98</td>
<td>-1.28</td>
<td></td>
</tr>
<tr>
<td>Participant 30</td>
<td>-2.09</td>
<td>1.1</td>
<td>1.32</td>
<td>1.4</td>
</tr>
<tr>
<td>Participant 12</td>
<td>0.75</td>
<td>1.56</td>
<td>1.08</td>
<td>1.49</td>
</tr>
<tr>
<td>Participant 18</td>
<td>-0.5</td>
<td></td>
<td>-0.68</td>
<td>-1.4</td>
</tr>
<tr>
<td>Participant 21</td>
<td>-0.18</td>
<td>0.64</td>
<td>0.31</td>
<td>0.47</td>
</tr>
</tbody>
</table>

Note: Missing scores are when participants were not available for the administration of the assessment, but were still part of the project.

Descriptive statistics were also determined for the seven participants who were involved in project activities all three years (Table 10). The change in the mean, in logits, was even greater for this small sample than it was for the previous sample. The increase in logits between the first assessment and the last assessment was over 1.5. As would be expected, the standard deviation for this small sample was very slight. The maximum scores changed very little, but the scores at the low end of the scale increased by about 2.25 logits between the
first and last assessments. This increase in the minimum accounts for the increase in the mean.

Table 10

<table>
<thead>
<tr>
<th>YEAR</th>
<th>1/09</th>
<th>12/09</th>
<th>7/10</th>
<th>7/11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>-0.42</td>
<td>0.42</td>
<td>0.19</td>
<td>0.99</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>1.49</td>
<td>1.00</td>
<td>1.02</td>
<td>0.54</td>
</tr>
<tr>
<td>Minimum</td>
<td>-2.68</td>
<td>-0.98</td>
<td>-1.28</td>
<td>0.23</td>
</tr>
<tr>
<td>Maximum</td>
<td>1.43</td>
<td>1.56</td>
<td>1.32</td>
<td>1.49</td>
</tr>
</tbody>
</table>

Correlation of Participation and Scores on Modified Learning Mathematics for Teaching Fraction Items

The Pearson product-moment correlation was computed to determine the magnitude of the correlation between the gains made in the area of fractions by each of the 26 participants who had taken multiple, modified LMT assessments and each of those participant's participation ratios. The gain was determined by taking the difference between each participant's first and last LMT assessments.
The participation ratio was determined by dividing the total number of days a participant was involved in project activities during the three years of the project by the total number of days of participation possible. The Pearson correlation between the participation ratio and the gains in LMT measures was $r = .22$ (see plot in Figure 2). The proportion of common variance between the two variables was .048 implying that less than 5% of the variance in the LMT was associated with the proportion of time spent in the project’s professional development.

![Figure 2. Correlation of Participation Ratio and Teacher Gains](image)
The teachers were divided into three categories based on teaching assignments when they look their last modified LMT assessment. The three categories included those who were teaching primary grades (K–3), those who were teaching upper elementary (4–6), and those who had non-grade level assignments, such as resource teachers.

The Pearson correlation between the participation ratio and the gains in the modified LMT measures for the primary teachers was $r = 0.03$ and this correlation is shown in Figure 3. The proportion of variance = .009 implying that less than 1% of the variance in the modified LMT was associated with the proportion of time spent in the project's professional development. The Pearson correlation for the upper elementary was $r = 0.42$ (Figure 4). The proportion of common variance is 0.17 for this category; this was higher than that for the whole sample and for the primary teachers and implies that 17% of the variance in the modified LMT was associated with the proportion of time spent in the project's professional development.
Figure 3. Correlation of Participation Ratio and Teacher Gains (K-3)
Changes in Participant Explanations on Modified Learning Mathematics for Teaching Assessments

The written explanation of the fraction items on the modified LMT assessments for the seven participants who were in the project for the full three years were analyzed. In particular, the fraction items that occurred on more than one modified LMT assessment were examined to determine differences in how the items were answered.

Four out of seven participants showed evidence of increased sense-making of the mathematics in a complex problem on later assessments than on
earlier assessments. For example, one participant answered Item 61 (Figure 5) incorrectly the first time, but the second time she answered it correctly. Her first attempt was to multiply $\frac{1}{2}$ and $\frac{3}{4}$. This gave her the distance she needed to move along the number line, but she did not know what to do with her answer. The second time she answered this item, she realized she needed to move $\frac{1}{4}$ of the way from $\frac{1}{2}$ to $\frac{3}{4}$. She used the number line to get the correct answer, and then added an apology that she did not know how to use the algorithm to find the answer.

6. A group of Ms. Lee's students was following a set of directions to move a paper frog along a number line.

Their last direction took them to $\frac{1}{2}$. The next direction says:

Go $\frac{1}{3}$ of the way to $\frac{3}{4}$. What number will the frog land on?

The students disagreed about where the frog would land. Which answer should Ms. Lee accept as correct? (Mark ONE answer.)

a) $\frac{1}{12}$
b) $\frac{2}{3}$
c) $\frac{7}{12}$
d) $\frac{5}{6}$
e) $\frac{1}{4}$
f) I'm not sure.

Figure 5. Item 61
Two participants' answers improved over time although they did not achieve completely correct answers. When one participant responded to item 65 (Figure 6) on the first assessment, she was unable to solve the problem. She recognized that the smaller pieces could be used to make the larger ones, but was unable to get any further. On the second assessment, she was almost successful in solving the problem; she used variables to represent the pieces and used equations to find the value of each piece. The error she made was computational; when adding 6/9 and 2/9, her sum was 7/9 instead of 8/9. There was no evidence to determine whether or not her method for adding the two fractions was correct.

13) If \( \text{and} \) \( \text{together have area} \ 2 \), what is the area of \( \text{and?} \) \( \text{and?} \) \( \text{Explain your answer carefully.} \)

*Figure 6. Item 65*
Another change of response over time identified four of the participants answered items that were correct each time, but had less explanation in subsequent responses. For example, on Item 105 (Figure 7), one participant correctly answered the question, and explained her work to determine how many gallons could be purchased and what fractional part of the gas tank would be filled. The second time she answered the item, she gave an estimate of how much of the tank would be filled, and provided no explanation or computations.

Figure 7. Item 105

Interviews with Teachers

For the seven participants who were in the project all three years, the researcher used just the parts of the internal evaluator-participant scripts that
mentioned fractions. Additional interviews were conducted with five of these seven teachers to further analyze participant growth in understanding fraction concepts and pedagogy. Information gathered from participants interviewed by the internal evaluator and by the researcher using a semi-structured protocol, provided information about the participants in two different areas. The first one was the changes in participants' understanding of fractions. The second was how the teachers felt the professional development affected their understanding of fractions and how the changes in understanding had impacted their classroom instruction.

In the five interviews done by the researcher, participants were asked to solve the two questions in Appendix C. All interviewees answered the two questions correctly. All five remarked that they would have answered the first question, where they had to determine which student response provided a correct representation of the fraction \( \frac{3}{4} \), differently before being involved in the project. All said they would not have considered there to be more than a single way to represent a fraction pictorially. One participant said she would have insisted that the representation with a circle was the only correct representation.

For question two, participants had to decide which scenario could be used as a context for a division of fractions problem, \( 1 \frac{1}{4} \text{ divided by } \frac{1}{2} \). Three of the participants grappled with the second choice. They knew that doubling \$1.25\ would give the same answer as the division problem, but they also knew the scenario was a different multiplication problem. These three participants had a
difficult time deciding whether it was an appropriate example to use in their classroom. However, once they started talking about it, they decided there were more appropriate contexts that could be used. This means that the participants were concerned with more than just getting the right answer; the connection between the context and the algorithm was important.

In addition to the two math problems, two of the participants were asked to solve problems from the modified LMT that they had missed on their last assessment. Both teachers solved the problems correctly, and then wondered why they had been asked to solve the particular problems. In addition, neither teacher had any difficulty in justifying the answers to those problems.

The participants shared how their understanding of fractions was different as a result of the professional development they received through the project. Two of the participants mentioned how important understanding units was for them now during their first interviews with the internal evaluator and the same two participants as well as a third participant discussed their instruction of units in the classroom during the second interview with the researcher. This third participant said that a unit of a fraction has a very different meaning to him now than it did before participation in the project. In the past, he said he always looked for 'one of something' and labeled that one item as the unit. One of the other participants said she needed to get help from other teachers during the professional development and during lesson study to understand fractions, especially when it came to units. She said that she had difficulty understanding
the unit. For example, if given pattern blocks, and the unit is not the hexagon, but instead the trapezoid, that the rest of the pattern blocks now represent different fractional parts than they did when the unit was the hexagon. She also said it was difficult for her when the whole was not represented as a circle or as a bar. A third participant said that when collaborating with teachers at her grade level, the first thing they do when they encounter a difficult fraction problem is to consider the unit. They learned to examine the unit during the professional development in the project as a tool to help understand what the problem is asking.

There was evidence that the participants interviewed felt that the project changed the way they teach students fractions. One participant discussed teaching his students about dividing fractions. He said that he had always taught students to multiply by the reciprocal, and he added that students have no idea why multiplying by the reciprocal will produce the correct answer. He said he showed the students how to divide straight across (i.e., numerator by numerator and denominator by denominator) with a problem where the numbers were compatible and the answer worked out nicely. Then, he said he chose a problem that was taken from a worksheet, and showed the students that this new problem that did not have compatible numbers in the numerator and denominator, produced an answer that did not work out as nicely as the problem before. He said the students discovered it was more work to find the correct answer by having to change the numerator and denominator of the resulting complex
fraction into a simple fraction with decimals, and then simplifying the resulting fraction. It turned out, according to the participant, that there are about ten steps in using the divide across strategy with numbers that are not compatible, and there are only three steps to solving a fraction division problem using the reciprocal of the divisor and changing the problem to a multiplication problem. He said if he had not been in the project, he never would have thought of showing students both strategies so that they would understand that there is more than one way to divide fractions and why multiplying by a reciprocal is most commonly taught.

Another participant talked about how the project had helped her change her understanding of fractions, but that she was still having difficulty transferring her understanding to her students. This participant said she had increased her own understanding of how pictures help her understand the concepts, but she was still having trouble with the algorithms. She said that she was having the same issue with her students when teaching equivalence of fractions. She shared that she now tried to teach fractions without presenting the algorithm, but that she still had some problems. She said that when students draw pictures, they did not draw them to scale, and therefore, they were not able to tell when the fractional parts were equivalent. She added that she had found more success with manipulatives, but students were not allowed to use them on the state test, so she had resorted to telling students to halve or double numerators and denominators. She added that she had become very comfortable with
drawing pictures for her students, but the algorithms were still difficult for her, and through her own struggles, she learned how her students felt when they had trouble learning algorithms.

As to which aspects of the project the participants felt helped them the most, all five of the participants in the semi-structured interviews felt that all components were helpful. Three of the participants talked about how valuable the lesson study sessions were and how much they had learned about student understanding during the lesson study sessions. One of the three discussed the value of interviewing students to find out what they really know. Two of the participants apologized for not being able to discern the difference between what they learned in Math Explorations and what was learned in the summer institutes, because the content blended together for them. Both brought up learning about the same fraction activity, but neither could remember at which project activity they learned it. Finally, they added that both components were very valuable to them, along with the lesson study.
CHAPTER FIVE
DISCUSSION AND CONCLUSIONS

This study had clear limitations. The first was the small sample size. Having a large turnover each year resulted in there being only a small group of teachers who were involved in the project for a long enough period of time to produce change in their understanding of fractions. An even smaller subset of teachers than those that participated long enough to take more than one assessment was the tiny group of seven teachers who participated in the project all three years. However, this small group of teachers’ responses to the interviews provided some of the most informative data about the effectiveness of the project for those who participated long-term.

Beside the high turnover of teachers, the participants in this three-year project faced some additional challenges. One of these challenges was the involvement of teachers with additional professional development commitments. Some of the teachers were involved in a science grant in the district. That project also required a significant time commitment on the teachers’ part. Another challenge was that the school was involved in a variety of professional development activities and programs, and this project competed with many others for teachers’ and administrators’ time and effort. Teachers remarked that they were tired, and that it was difficult for them to consistently be engaged as they had school commitments almost every afternoon after teaching all day.
Teachers' prior knowledge of the subject matter may have also influenced which teachers made substantial growth in their understanding of fractions. According to Desimone (2009), cohesion is one of the five critical components for effective professional development. Many teachers came to the program without sufficient mathematics background to connect what was being taught during the professional development to what they needed to teach in the classroom. Teachers felt very stretched who came to the project without a strong mathematical background.

Lack of prior mathematical understanding may be partially responsible for the Pearson correlation being much lower for the primary teachers than it was for the upper elementary teachers. The upper elementary teachers use mathematics at a higher level in their daily instruction than teachers who teach in the primary grades.

Another possible explanation for the higher Pearson correlation at upper elementary might be the focus of the professional development during the first two years of the project. For the first two years of the project, most of the professional development activities, in both Math Explorations and in the summer institute, were conducted with the whole group; teachers were not divided into grade-level groups. During the third year, however, most of the professional development was separated and more focused on grade-level spans.

In addition to coherence, Desimone (2009) listed four other critical components that are necessary for effective professional development: content,
active learning, duration, and collective participation. These critical components were evident in the different aspects of this project. The mathematics content provided for the teachers was standards-based and grounded in research. The teachers found that it was helpful that all of the professional development, including lesson study sessions, allowed for collaborative work. Teachers worked together to solve problems and were given immediate feedback about their performance. Teachers found that the environment was non-threatening, and they were willing to share their solution approaches and misconceptions in a whole group setting without much concern of being judged by their peers or professional development instructors. This willingness to collaborate and take risks in lesson study and other project professional study carried over into their grade-level planning where collaboration took place for the first time in years.

Analysis of participants' explanations over time to the same items on the modified LMT produced mixed results and created additional questions for future study. It is clear that there were participants whose ability to explain their thinking about a problem did improve over time.

In addition to being able to explain their own thinking, on both the modified LMT assessments and on the two LMT items used for the semi-structured interviews, the responses indicated that teachers increased in what Shulman (1986) referred to as pedagogical content knowledge (PCK). Participants were able to show and identify more ways to teach and/or explain a problem than they reported being able to describe before the project. Specifically, when they were
asked during the semi-structured interviews to explain how they would teach a concept to students, these participants shared that they had increased the number of strategies they had to share.

An example of teachers' increased ability to utilize a variety of strategies is provided on an item in which participants were asked to explain what representations they would use to demonstrate why \( \frac{4}{5} \) is greater than \( \frac{6}{10} \). On the first assessment, one participant responded that she would use fraction bars to show why \( \frac{4}{5} \) was greater than \( \frac{6}{10} \). By the time the teachers took the last assessment, this same participant had also included in her explanations a traditional algorithm, number lines, fraction circles, and area models as well as the fraction bars. This ability to produce multiple instructional strategies for teaching content was also evident in the semi-structured interviews done by the researcher. Teachers stated that before being involved in the project they never would have realized that there was more than one way to solve any particular problem.

However, the participants' own growth in the understanding of fraction problems, where pedagogy was not the primary target of the question, was not always simple to discern by reading their responses and justifications to the modified LMT items. Research has shown that elementary teachers tend to have compartmentalized knowledge and cannot easily transfer their knowledge from one domain to another (Baturo & Nelson, 1996; Mewborn, 2001). In addition, these same studies tended to show that elementary teachers have a tendency to
know facts and be able to compute algorithms but not really understand the concept behind the mathematics they are teaching.

From the analysis of the test items, it was hard to tell why the explanations became better. It could have been because of an increased conceptual understanding of fractions, and the participants were better able to explain more about the mathematics because they knew more about the mathematics than simply how to compute the algorithm. Or, it may not be that the teachers increased their conceptual understanding, the more in depth responses could have also been because, at the start of the project, they were unfamiliar to writing about their mathematics, and communication of their thinking, in general, improved. It is also unclear whether the participation in the project helped them write more clearly about their thinking, because of opportunities provided in the project, or that they wrote better because they understood the expectations of the assessment better.

On the other hand, there were participants who showed no change at all in their explanations or, surprisingly, became briefer with each subsequent assessment. There could be several reasons for those explanations that actually became less effective over time. One explanation could be that the participants remembered the item, and it was no longer of interest to them. The first time the item was seen, it may have been a challenge, and the participants took their time thinking and writing about it. When the item was encountered again, the item
had lost its novelty, and the participant did not need to put much thought into answering the question.

Another explanation for shortened explanations could be that teachers increased their understanding of some concepts and did not feel the need for elaborate explanations. There is some evidence to support this explanation. One is that on some items from the first two modified LMT assessments the participants had to use drawings and diagrams to make sense of the problem. For the same item on later assessments, the participants used an algorithm and quickly solved the item. Since the participants did not use the algorithm the first time the item was attempted, it is unknown as to whether or not they learned the algorithm during the course of the project, or that they knew the algorithm before the start of the project and discovered the algorithm was applicable to that particular problem.

According to the research regarding the relationship between duration of professional development and likelihood of that professional development creating long-lasting intellectual and pedagogical change, more sustained professional development was preferable (Cohen & Hill, 2001; Fullan, 1993; Guskey, 1994; Supovitz & Turner, 2000). Therefore, the participant interviews were conducted with the participants who were involved in the project activities for all three years, and their responses supported the research that there are positive changes in teacher understanding with sustained professional development. The participant interviews seemed to provide more information,
not only about what the participants had learned about fractions and teaching fractions, but also about their perceptions of what they had learned.

These five participants had a strong perception that their instruction of fractions had changed for the better. Prior to participation in the professional development activities, most of the teachers had been using their textbooks as the sole source of instructional materials. For the most part, the textbook examples and lessons demonstrated procedures and had very weak conceptual development of mathematical topics. During an interview, one participant mentioned that she had moved away from using the textbook as her primary source of instruction and was now providing activities that supported student conceptual understanding through the use of multiple representations of fractions. Other participants mentioned that they were no longer providing mathematical algorithms without building connections to the concepts, and that students were generating algorithms on their own through experimentation with drawings and manipulatives. Overall, they said that they felt students were much more successful with fractions than they were before the project began. However, there is no direct evidence to support these teacher perceptions, as the CST data are not partitioned to report separate outcomes for fractions.

There is evidence, however, that the modified LMT data were not able to fully capture the actual growth in teacher understanding of fractions. First of all, teachers seemed to perform better on fraction items during professional development and during the semi-structured interviews than they did on the
assessments. The latter may be partly explained by the fraction data including all participants that had taken multiple assessments, and the interview data including only from those participants involved in the project for the entire three years. One question that arises from this is: “What is the minimum amount of participation needed for teacher change in knowledge and pedagogy to occur?”

Secondly, the modified LMT data includes both the objective scores from the multiple-choice answers as well as the more subjective scores from the Exemplars rubric. If the objective scores were analyzed separately, would the correlation between the fraction items and teacher participation be higher? This question, as well as other questions mentioned before, such as: Is professional development more successful for participants teaching upper grades as opposed to those teaching primary grades, and what is the percentage or participation required for professional development to facilitate teacher change, are questions that are worth studying in the future.
APPENDIX A

SAMPLE ASSESSMENT ITEMS:

MODIFIED LEARNING

MATHEMATICS FOR

TEACHING
1) Which of the following could be a unit for a fractions question?

a) 

b) 

c) 

d) 

18) Mrs. Cronk gave her class the following problem:

*Which would you rather have, \( \frac{6}{10} \) of a dollar or \( \frac{4}{5} \) of a dollar?*

When she read the student responses, she was surprised to discover that most of his students thought that \( \frac{6}{10} \) was worth more than \( \frac{4}{5} \). One student wrote: "If I had \( \frac{6}{10} \), I would have 2 more than \( \frac{4}{5} \). I would choose \( \frac{6}{10} \) so I would have more money." How would you recommend that she address this error in her next class?
8. Mr. Lewis asked his students to divide $\frac{6}{8}$ by $\frac{1}{2}$. Charlie said, "I have an easy method, Mr. Lewis. I just divide numerators and denominators. I get $\frac{6}{4}$, which is correct." Mr. Lewis was not surprised by this as he had seen students do this before. What did he know?

Circle ONE answer.

a) He knew that Charlie's method was wrong, even though he happened to get the right answer for this problem.

b) He knew that Charlie's answer was actually wrong.

c) He knew that Charlie's method was right, but that for many numbers this would produce a messy answer.

d) He knew that Charlie's method only works for some fractions

13) If $\square$ and $\square$ together have area 2, what is the area of $\square$ and $\triangle$? Explain your answer carefully.
1) Your gas tank is reading empty, but you are low on cash. You used your last $10 to buy gas at a station where you paid $2/gallon. If a full tank holds 14 gallons, put an arrow on the gas gauge to show how much gas you had in the tank after the purchase. EXPLAIN why you chose this placement.
6. A group of Ms. Lee's students was following a set of directions to move a paper frog along a number line.

Their last direction took them to $\frac{1}{2}$. The next direction says:

$$\text{Go } \frac{1}{3} \text{ of the way to } \frac{3}{4}. \text{ What number will the frog land on?}$$

The students disagreed about where the frog would land. Which answer should Ms. Lee accept as correct? (Mark ONE answer.)

a) $\frac{1}{12}$

b) $\frac{2}{3}$

c) $\frac{7}{12}$

d) $\frac{5}{6}$

e) $\frac{1}{4}$

f) I'm not sure.
20. Mrs. Bond's students were writing equivalent forms of the same number. However, students did not always find this easy. For each list below, indicate whether the expressions are equivalent forms of the same number. Circle EQUIVALENT, NOT EQUIVALENT, or I'M NOT SURE for each. EXPLAIN each of your choices.

<table>
<thead>
<tr>
<th></th>
<th>Equivalent</th>
<th>Not equivalent</th>
<th>I'm not sure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) (\frac{30}{10}, 3.0, 300%)</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>b) (\frac{20}{5}, 0.4, 40%)</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>c) (\frac{7}{1000}, 0.007, 0.7%)</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>d) (\frac{6}{10}, 0.6, 6%)</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

APPENDIX B

INTERNAL EVALUATOR INTERVIEW PROTOCOL
Collect: Teacher Name, Grade level, average class size, number of years teaching at project school, number of years teaching, and quick survey regarding classroom practice:

Request permission to conduct and record the interview. Tell teacher the interview will take about an hour.

Begin interview:

1. If an observer enters your classroom when you are teaching math, what would you like for him/her to see?

2. Given what you said, how would you describe your teaching style? (Can you briefly describe your general approach to teaching math with this class?)

3. How comfortable do you feel about teaching math at this grade level? Why?
   a. How has participating in this project impacted you and your teaching of math?
   b. How has your comfort level changed since you began participating in this project?

Expectations:

a. What were your expectations when you joined this project as a participant?

b. How have your expectations changed as a consequence of participation in this project?

c. How satisfactorily has this project met your (evolving) expectations. What could have been done to have better met your expectations?
Project Effectiveness and Usefulness

Monthly Seminars

a. What words would you use to describe this project's professional development efforts?

b. In what ways / How has the math content introduced during monthly seminars impact your knowledge of mathematics? Give examples.

c. What math content presented to you during seminars was most relevant

i. Give an example of some math content that was relevant and impacted your math instruction.

ii. Give another example.

d. Give an example of a topic that impacted your understanding of math in a significant way.

e. How would you evaluate the way the project staff modeled instruction for teaching and engaging students in mathematics?

i. Give an example of an instructional strategy that impacted the way you teach mathematics.

ii. Give another example.

f. If not already brought up ask: One of the strategies used in the seminars was to have you do each math problem in multiple ways (at least in two different ways). How did this strategy help, excite, hinder, or frustrate you?

i. What was the benefit to you of seeing different groups do the same problem in different ways?
g. What do you do to address the needs of English language learners in your classroom during math instruction?

h. What were the benefits, if any, of the ELD content and instructional strategies addressed in monthly seminars and summer institutes?

i. *Give an example of an ELD strategy introduced through this project that impacted the way you teach mathematics.*

i. Teachers need a number of resources to help them in their work? How well were you supported with resources or guided to locate resources that are of value to you?

**Summer Institutes**

a. What words would you use to describe the usefulness and effectiveness of the summer institute?

b. How useful to you were the topics or math content of the summer institute?

i. *Give an example of a topic that impacted your understanding of math in a significant way.*

c. How adequately were the issues of ELD addressed in the summer institute? Give examples.

d. Describe the value of concept mapping as introduced to you in this project and how you have used it in lesson planning and teaching, if at all?
Lesson Study

a. According to you, what were the primary goals of lesson study and how satisfactorily were those goals met for you?

b. How would you describe the quality of the lesson study sessions?

c. What aspects of lesson study were of value to you as a teacher? How have these aspects impacted your students?

d. What aspects or components of lesson study were taken back to your classroom and implemented? How regularly are they (i.e., what you learned in lesson study) used in the class? *In what areas have you made changes in your teaching as a direct result of lesson study?*

i. Instructional strategies/practice

ii. Questioning?

iii. Formative assessments?

iv. Lesson planning?

v. Collaboration with your grade level team?

vi. Other

Leadership

a. Were you part of the teacher leadership team?

   i. If YES, what role did you play? How would you describe the value and impact of the leadership meeting and follow up activities to the success of this project?
ii. If NO, did your team leader inform you of issues addressed and/or decisions made at leadership meetings?

iii. How would you describe the value and impact of the leadership committee and its activities to the success of this project?

b. How important do you think it is to continue to have a leadership committee to maintain the momentum developed with this project? Explain.

**Implementation**

a. In what area(s) of teaching and learning math has this project been most influential to you personally?

b. With what aspects of the study weren't you satisfied? Why?

c. What plans have you made to implement what you've learned in this project to teaching mathematics? Give me at least one example.

d. What support do you need in order to implement what you have learned in this project?

e. How can you build on what you have learned through this project?

f. What other professional development activities have you been involved in while participating in this project?

APPENDIX C

LEARNING MATHEMATICS FOR TEACHING QUESTIONS

FOR ADDITIONAL INTERVIEWS BY RESEARCHER
First LMT question used for interview:

7. Which of the following story problems could be used to illustrate \( \frac{1}{4} \) divided by \( \frac{1}{2} \)? (Mark YES, NO, or I'M NOT SURE for each possibility.)

<table>
<thead>
<tr>
<th></th>
<th>Yes</th>
<th>No</th>
<th>I'm not sure</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>b)</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>c)</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

a) You want to split \( 1 \frac{1}{4} \) pies evenly between two families. How much should each family get?

b) You have $1.25 and may soon double your money. How much money would you end up with?

c) You are making some homemade taffy and the recipe calls for \( 1 \frac{1}{4} \) cups of butter. How many sticks of butter (each stick = \( \frac{1}{2} \) cup) will you need?
Second LMT question used for interviews:

3. Ms. Kelly was going to draw a picture of \( \frac{3}{4} \) on the blackboard. She asked how many circles she should draw to start, and to her surprise her students made different proposals.

Asa: I would draw four circles because the denominator tells you what the whole is.

José: I was thinking that fractions mean divide, and three circles is the whole thing. I would start with three circles, then divide them up.

Mina: I would draw one circle. One is the whole, and you break the whole up into four parts.

Ms. Kelly had planned to draw one circle, but now she was unsure. Which of these students is using a correct interpretation of fractions?

Circle ONE answer.

a) Only Asa.

b) Only José.

c) Only Mina.

d) Both Asa and Mina, but not José.

e) Asa, José, and Mina.
After teachers had read and responded to items, they were asked to respond to the following questions:

1. Explain why you chose that answer as the correct one. Why is it correct? For one of the other responses, what makes it incorrect?

2. What aspects of the item are easy; what aspects of the item are hard? (for you and for your students)

3. What implications would these thoughts have for your teaching?

4. How do you believe you would have responded to these questions four years ago, and in what ways has your thinking changed?

5. How has the study affected the way you think about fractions as a doer of mathematics?

6. How has the study influenced your teaching of fractions?

7. Explain in what ways different components of the study (summer institute, math explorations, and lesson study) have contributed to your understanding and teaching of fractions.

APPENDIX D

TABLE OF SPECIFICATIONS
<table>
<thead>
<tr>
<th>12 Specific goal</th>
<th>Item no.s on assessment 1 (109)</th>
<th>Running item no.s</th>
<th>No. items on assessment 1</th>
<th>Items on assessment 2 (12/09)</th>
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<td>PR: Fractions</td>
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<td>Understanding &quot;unit&quot;, or &quot;whole&quot;</td>
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<td>Solve problems involving fractions conceptually (without resorting to standard algorithms)</td>
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<td>Provide conceptual explanations for standard solution procedures</td>
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APPENDIX E

EXEMPLARS RUBRIC
Exemplars® Standards-Based Math Rubric (Cont.)*

<table>
<thead>
<tr>
<th>Practitioner</th>
<th>Reasoning and Proof</th>
<th>Communication</th>
<th>Connections</th>
<th>Representation</th>
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<td><strong>Problem Solving</strong></td>
<td>A correct strategy is chosen based on mathematical situation in the task.</td>
<td>Arguments are constructed with adequate mathematical basis.</td>
<td>A sense of audience or purpose is communicated, and/or Communication of an approach is evident through a methodical, organized, coherent sequenced and labeled response.</td>
<td>Appropriate and accurate mathematical representations are constructed and refined to solve problems or portray solutions.</td>
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<tr>
<td>Planning or monitoring of strategy is evident.</td>
<td>A systematic approach and/or justification of correct reasoning is present. This may lead to...</td>
<td>Formal math language is used throughout the solution to share and clarify ideas.</td>
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<tr>
<td>Evidence of solidifying prior knowledge and applying it to the problem-solving situation is present.</td>
<td>• classification of the task.</td>
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<td>Note: The practitioner must achieve a correct answer.</td>
<td>• exploration of mathematical phenomenon.</td>
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<td>* Deductive arguments are used to justify decisions and may result in formal proofs.</td>
<td>• noting patterns, structures and regularities.</td>
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<tr>
<td><strong>Expert</strong></td>
<td>Evidence is used to justify and support decisions made and conclusions reached. This may lead to...</td>
<td>A sense of audience and purpose is communicated, and/or Communication at the Practitioner level is achieved, and communication or argument is supported by mathematical properties.</td>
<td>Mathematical connections or observations are used to extend the solution.</td>
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<tr>
<td>An efficient strategy is chosen and progress towards a solution is evaluated.</td>
<td>• testing and accepting or rejecting of a hypothesis or conjecture.</td>
<td>Précise math language and symbolic notation are used to consolidate mathematical thinking and to communicate ideas.</td>
<td>Abstract or symbolic mathematical representations are constructed to analyze relationships, extend thinking, and clarify or interpret phenomena.</td>
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<tr>
<td>Adjustments in strategy, if necessary, are made along the way and/or alternative strategies are considered.</td>
<td>• explanation or phenomenon.</td>
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<td>Evidence of analyzing the situation in mathematical terms, and extending prior knowledge is present.</td>
<td>• generalizing and extending the solution to other cases.</td>
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<td>Note: The expert must achieve a correct answer.</td>
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</table>

*Based on revised NCTM standards.

REFERENCES


