Geodesics in Lorentzian Manifolds

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

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Master of Arts

in

Mathematics

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by

Amir Amin Botros

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Abstract

We present an extension of Geodesics in Lorentzian Manifolds (Semi-Riemannian Manifolds or pseudo-Riemannian Manifolds). A geodesic on a Riemannian manifold is, locally, a length minimizing curve. On the other hand, geodesics in Lorentzian manifolds can be viewed as a distance between "events". They are no longer distance minimizing (instead, some are distance *maximizing*) and our goal is to illustrate over what time parameter geodesics in Lorentzian manifolds are defined. If all geodesics in timelike or spacelike or lightlike are defined for infinite time, then the manifold is called "geodesically complete", or simply, "complete". It is easy to show that $g(\sigma', \sigma')$ is constant if σ is a geodesic and g is a smooth metric on a manifold M, so one can characterize geodesics in terms of their *causal character*: if $g(\sigma', \sigma') < 0$, σ is timelike. If $g(\sigma', \sigma') > 0$, σ is spacelike. If $g(\sigma', \sigma') = 0$, then σ is lightlike or null. Geodesic completeness can be considered by only considering one causal character to produce the notions of spacelike complete, timelike complete, and null or lightlike complete. We illustrate that some of the notions are inequivalent.

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Chapter 1

Introduction

A geodesic on a Riemannian manifold is, locally, a length minimizing curve. In other words, a geodesic is a path that a non-accelerating particle would follow. For example, a geodesic in the Euclidean plane is a straight line and on the sphere, all geodesics are great circles. If we consider the distance function $||X||^2 = \sum_{i=1}^4 X_i^2 = t^2 + x^2 + y^2 + z^2$ on $\mathbb{R}^4$, we notice that it is positive definite (Riemannian). Moreover, the Hopf-Rinow Theorem states that a connected Riemannian manifold is geodesically complete if and only if it is complete as a metric space [Lee97].

In 1915 Albert Einstein introduced the general theory of relativity, which led to the need to consider manifolds whose metric is not positive definite (pseudo-Riemannian). In other words, pseudo-Riemannian manifolds are not metric spaces, since the distance function, $||X||^2 = -X_1^2 + \sum_2^4 X_i^2 = -t^2 + x^2 + y^2 + z^2$ is no longer positive definite and geodesics here can be viewed as a distance between "events". They are no longer distance minimizing (instead, some are distance *maximizing*) or zero.

There is no analog of the Hopf-Rinow Theorem in the pseudo-Riemannian case. So, completeness is defined as geodesic completeness. A manifold is *geodesically complete* if every geodesic extends for infinite time and here are 3 types of geodesic completeness.

1. *Timelike geodesically complete* if every timelike geodesic extends for infinite time.

2. *Spacelike geodesically complete* if every spacelike geodesic extends for infinite time.

3. *Lightlike geodesically complete* if every lightlike geodesic extends for infinite time.

It is our goal to show that these notions are inequivalent.

Here is an outline of the thesis. Chapter 2 contains Topological and Differential Geometric Preliminaries. We introduce topological spaces, smooth manifolds, and related notions that we will need.

Chapter 3 is about Inner Products and pseudo-Riemannian manifolds. We introduce inner products, timelike, lightlike, spacelike vectors, and pseudo-Riemannian manifolds.

Chapter 4 is about Connections and Geodesics Completeness with examples that demonstrate certainty of completeness are inequivalent. We introduce connections, Levi-Civita connections, Koszul formula, Geodesics Completeness, special manifold, examples, and finally, give a short summary of our goal.

Chapter 2

Topological and Differential Geometric Preliminaries

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Most of this material that I introduce here in this chapter can be found in a basic topology book. The concept of topological space grew out of the study of the real line and Euclidean space and the study of continuous functions on these spaces ([Mun00]). In this chapter we define what a topological space is, and we provide some examples.

Definition 2.1. *A topological space is a set X together with a collection of subsets S, called open sets, that satisfies the four conditions:*

1. *The empty set $\emptyset$ is in S.*
2. *X is in S.*
3. *The intersection of a finite number of sets in S is also in S.*
4. *The union of an arbitrary number of sets in S is also in S.*

Here we gave a basic example of topological space over a set to give some of the elementary concepts to understand a topological space.

Example 2.2. *If $X = \{1,2,3\}$ then $S = \{\phi, \{1\}, X\}$ is a topology on X.*

A Euclidean space or, more precisely, a Euclidean $n-$space, is the generalization of the notions “plane” and “space” (from elementary geometry) to arbitrary dimensions n. Thus Euclidean $2-$space is the plane, and Euclidean $3-$space is space. This generalization is obtained by extending the axioms of Euclidean geometry to allow n directions which are mutually perpendicular to each other. Euclidean $n-$space, sometimes called Cartesian space or simply $n-$space, is the space of all $n-$tuples of real numbers, $(x_1, x_2, ..., x_n)$. Such $n-$tuples are sometimes called points.

Definition 2.3. *Euclidean $n-$space $\mathbb{R}^n$ is the set of all $n-$tuples $p = (p_1, p_2, \ldots, p_n)$ of a real numbers.*

We work here over the real numbers, $\mathbb{R}$.

Definition 2.4. *For any x in $\mathbb{R}^n$ and $r > 0$, the open ball of radius r around x is subset $B_r(x) = \{y \in \mathbb{R}, such\ that \mid x - y \mid < r\}$.*

In a moment we will give an example of the standard topology on $\mathbb{R}^n$.

Example 2.5. *If $X = \mathbb{R}^n$ then let S be the set of arbitrary unions of open balls. This is the standard topology on $\mathbb{R}^n$ we will be using.*

In mathematical analysis, the smoothness of a function is a property measured by the number of derivatives it has which are continuous. A smooth function is a function that has derivatives of all orders everywhere in its domain.

Definition 2.6. *A real valued function f defined on an open set U of $\mathbb{R}^n$ is smooth if all mixed partial derivatives of f and of all orders exist and are continuous at every point of U.*

If $U \subseteq \mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is smooth then $f = \{f_1, f_2, \ldots, f_m\}$ where $f_i :\to \mathbb{R}$. These $f_i, i = 1, 2, ..., m$, are called the coordinate functions of f, and f is smooth if all of the coordinate functions of f are smooth [Shi04].

In mathematics, a continuous function is a function for which small changes in the input result in small changes in the output. Otherwise, a function is said to be a discontinuous function. A continuous function with a continuous inverse function is called a homeomorphism. Continuity of functions is one of the core concepts of topology.

Definition 2.7. *If X, Y are topological spaces, and $f : X \to Y$ then f is continuous if $f^{-1}(U)$ is open in X whenever U is open in Y. Let X, Y are topological spaces, A function $f : X \to Y$ is a homeomorphism if f and f^{-1} are continuous.*

Next we will define smooth overlap of topological spaces.

Definition 2.8. *If S is a topological space, an n-dimensional coordinate system at a point $p \in S$ is a homeomorphism $\phi : U \to \mathbb{R}^n$, where U is an open set containing p. If U and V are n-dimensional coordinate systems, we say U and V overlap smoothly if whenever $p \in U \cap V$, the transition functions $\phi_V \circ \phi_U^{-1} : \phi_U(U \cap V) \to \phi_V(U \cap V)$ are smooth, whenever $\phi_U : U \to \mathbb{R}^n$ and $\phi_V : V \to \mathbb{R}^n$ are the associated homeomorphisms. The points in an $n-$dimensional coordinate system are locally like Euclidean space $\mathbb{R}^n$.*

Here we would like to use coordinate systems and homeomorphism to describe what an atlas on a manifold is.

Definition 2.9. *An atlas A of dimension n on a space S is a collection of n-dimensional coordinate systems such that*

1. *Every point in S is located in some coordinate system in A.*
2. *All n-dimensional coordinate systems in A overlap smoothly.*

Here we will illustrate that a complete atlas is important.

Definition 2.10. *A complete atlas B on S if B contains each coordinate system in S that overlap smoothly with every coordinate system in B .*

The existence and uniqueness of a complete atlas on topological space will avoid unintended technicalities. the proof of the following lemma may be found in ([O'N83]).

Lemma 2.11. *Each atlas A on S is contained in a unique complete atlas.*

We introduce the differential geometric preliminaries that we will need, including manifolds.

Definition 2.12. *A manifold M is a Hausdorff topological space that locally resembles Euclidean space. This means that for all $p \in M$ there exists an open set $U \subseteq M$, $p \in U$ and homeomorphism $\phi : U \to \mathbb{R}^n$.*

Here we will only deal with smooth manifolds which we define. Also, we assume all manifolds here are connected.

Definition 2.13. *A smooth manifold M is a Hausdorff topological space furnished with a complete atlas.*

If we let M be a smooth manifold, then denote $\mathcal{F}(\mathcal{M})$ as the set of all smooth real-valued functions on M. Some important objects on manifolds may now be discussed. These include tangent vectors, the tangent space, the tangent bundle and vector fields.

Definition 2.14. *Let p be a point of a manifold M. A tangent vector to M at p is a real valued function $v : \mathcal{F}(\mathcal{M}) \to \mathbb{R}$ that satisfies*

1. *$\mathbb{R}$-Liner: $v(af + bg) = av(f) + bv(g)$, and*
2. *Leibnizian: $v(fg) = v(f)g(p) + f(p)v(g)$,*
   for all $a, b \in \mathbb{R}$ and $f, g \in \mathcal{F}(M)$.

Definition 2.15. *At each point $p \in M$, let $T_p(M)$ be a set of all tangent vectors to M at p. Then $T_p(M)$ is called tangent space to M at p.*

Definition 2.16. *Addition and scalar multiplication in $T_p(M)$ is defined as*

1. *$(v + w)(f) = v(f) + w(f)$,*
2. *$(av)(f) = av(f)$,*
   for all v,$w \in T_p(M)$, $f \in \mathcal{F}(M)$, $a \in \mathbb{R}$ and make the tangent space $T_p(M)$ a vector space over the real numbers $\mathbb{R}$.

Definition 2.17. *The tangent bundle $T(M)$ for a manifold M is the disjoint union of all its tangent spaces.*

The tangent bundle can be made into a smooth manifold as well, for details, see ([O'N83]).

Definition 2.18. *A vector field V on a manifold M is a function that assigns to each point $p \in M$ a tangent vector V_p to M at p.*

V can be imagined as a collection of arrows, one at each point of M. If V is a vector field on M and $f \in \mathcal{F}(M)$, then Vf denotes the real-valued function on M given by $(Vf)(p) = V_p(f)$ for all $p \in M$.

Here we will give a proposition that we will need later in Theorem (4.5). Its proof can be found in [O'N83].

Proposition 2.19. *Let M be a semi-Riemannian manifold. If $V \in \mathfrak{X}(M)$, and let $V^* =< V, X >$ for all $X \in \mathfrak{X}(M)$. Then the funtion $V \longrightarrow V^*$ is an $\mathcal{F}(M)-$linear isomorphism form $\mathfrak{X}(M)$ to $\mathfrak{X}^*(M)$.*

Chapter 3

Inner Products and Pseudo Riemannian Manifolds

We introduce here the linear algebraic notion of a symmetric bilinear form, it is a generalization of the dot product. In a vector space, it is a way to multiply vectors together, with the result of this multiplication being a scalar. More precisely, for a real vector space, a symmetric bilinear form $< .,. >$ satisfies the following definition.

Definition 3.1. *Define $< .,. >$ a symmetric bilinear form on V to be a function $< .,. > : V \times V \to \mathbb{R}$, and satisfying*

1. $< x+y,z >=< x,z > + < y,z >$ *and if* $\alpha \in \mathbb{R}$,*then* $< \alpha x,y >= \alpha < x,y >$, *and*
2. $< x,y >=< y,x >$ *for all* $x,y,z \in V$.

It is easily seen that $< .,. >$ is linear in the second slot:

$$\begin{aligned} < x,y+z > &= < y+z,x > \\ &= < y,x > + < z,x > \\ &= < x,y > + < x,z > . \end{aligned}$$

We know in most of linear algebra, functional analysis, and related areas of mathematics, a norm is a function that is generally assumed to assign a strictly positive length or size to each vector in a vector space.

Definition 3.2. *Define the norm of* v *as* $||v||^2 =< v, v >$.

We will give here an example of symmetric bilinear form that challenges the notion that a norm must always be nonnegative.

Example 3.3. *Let* $V = \mathbb{R}^2 =$ *span* $\{e_1, e_2\}$. *Define a symmetric bilinear form as follows:* $< e_1, e_1 >= 1$, $< e_2, e_2 >= -1$ *and* $< e_1, e_2 >= 0$ *find*

1. $< xe_1 + ye_2, xe_1 - ye_2 >$, *and*

2. $||xe_1 + ye_2||^2$.

Solution:

1.

$$\begin{aligned} < xe_1 + ye_2, xe_1 - ye_2 > &= < xe_1, xe_1 > - < xe_1, ye_2 > \\ &\quad + < ye_2, xe_1 > - < ye_2, ye_2 > \\ &= x^2 + y^2. \end{aligned}$$

2.

$$\begin{aligned} ||xe_1 + ye_2||^2 &= < xe_1 + ye_2, xe_1 + ye_2 > \\ &= < xe_1, xe_1 > + < xe_1, ye_2 > \\ &\quad + < ye_2, xe_1 > + < ye_2, ye_2 > \\ &= x^2 - y^2. \end{aligned}$$

We can see from this norm, $||xe_1 + ye_2||^2 = x^2 - y^2$ *that if*
$x > y$, *then the norm is positive (+).*
$x < y$, *then the norm is negative (-).*
$x = y$, *then the norm is zero (0).*

We will give here a characterization of certain symmetric bilinear forms.

Definition 3.4. *Let* $< ., . >$ *be symmetric and bilinear. Then*

1. *If* $< v, v >$ *greater than 0 if* $v \neq 0$ *and* $< v, v >= 0$ *if* $v = 0$, *then* $< ., . >$ *is positive definite.*

2. *For all* $v \in V$, *there exists* $w \in V$ *such that* $< v, w > \neq 0$, *then* $< ., . >$ *is non-degenerate.*

3. *If* $< ., . >$ *is non-degenerate, then we call* $< ., . >$ *an inner product.*

We will give these specific definitions for inner products which characterize all non-zero vectors in an inner product space.

Definition 3.5. *If* $< .,. >$ *is an inner product, and* $x \neq 0, x \in V$*,*

1. *If* $< x, x >$ *less than 0 , then* x *is* timelike.
2. *If* $< x, x >$ *greater than 0 , then* x *is* spacelike.
3. *If* $< x, x >$ *equals 0 , then* x *is* lightlike *or* null.

The following theorem illustrates that orthonormal bases exist in inner product spaces and we can find the proof in [Gil01].

Theorem 3.6. *If* $< .,. >$ *is an inner product , there exists a basis* $\{e_1, e_2, \ldots, e_p, f_1, f_2, \ldots, f_q\}$ *when* $p + q = n = dimV$ *such that*
$< e_i, e_i >= -1$,
$< e_i, e_j >= 0$, *if* $i \neq j$,
$< f_i, f_i >= 1$,
$< f_i, f_j >= 0$, *if* $i \neq j$,
$< e_i, f_j >= 0 \; \forall i, j$.
We call this an orthonormal basis. For every orthonormal basis, the numbers p and q are the same. So, (p, q) *is invariantly defined, and* (p, q) *is called the* signature *of the inner product. If this is* $(0, n)$*, the inner product is* positive definite*. If it is* $(1, n - 1)$*, then it is* Lorentzian*. Otherwise, if it is* (p, q) *for* $p, q \geq 2$ *, this is referred to as the* higher signature setting*.*

We will give example to illustrate the previous theorem.

Example 3.7. *Let* $< e_1, e_1 >= -1$ *and* $< f_1, f_1 >=< f_2, f_2 >=< f_3, f_3 >= 1$ *on* $R^4 =$ *space* $\{e_1, f_1, f_2, f_3\}$ *. The signature of* $< .,. >$ *is* $(1, 3)$ *and is thus Lorentzian.*

1. *If* $v = xf_1 + xe_1$*, then*

$$\begin{aligned} ||v||^2 &= < v, v > \\ &= < xf_1 + xe_1, xf_1 + xe_1 > \\ &= x^2 - x^2 \\ &= 0. \end{aligned}$$

Thus, v is null.

2. *Let $v = yf_1 + xe_1$*

   (a) *If $y < x$ then*

   $$||v||^2 = < yf_1 + xe_1, yf_1 + xe_1 > = y^2 - x^2 < 0.$$

   So v is timelike.

   (b) *If $y > x$ then*

   $$||v||^2 = < yf_1 + xe_1, yf_1 + xe_1 > = y^2 - x^2 > 0.$$

   So v is spacelike.

   (c) *If $y = x$ then*

   $$||v||^2 = < yf_1 + xe_1, yf_1 + xe_1 > = y^2 - x^2 = 0.$$

   So v is lightlike or null.

But how does this relate to our topic? We will give here specific definitions about this issue.

Definition 3.8. *If M is a smooth manifold, a metric g on M is an inner product of the same signature on each tangent space of M. g is smooth if the components of g in one (and hence any) coordinate system are smooth.*

We mention here that M and g must be smooth.

Definition 3.9. *If (M, g) is a smooth manifold.*

1. *If the signature of g is $(0, n)$, then (M, g) is Riemannian.*
2. *If the signature of g is $(1, n-1)$ or $(n-1, 1)$ then (M, g) is Lorentzian.*
3. *If the signature of g is (p, q), with $p, q \geq 2$, then (M, g) is of higher signature. It is a pseudo-Riemannian manifold if it is not Riemannian.*

The following proposition illustrates how the signature (p, q) and (q, p) are related.

Proposition 3.10. *Let (M,g) be a pseudo-Riemannian manifold of signature (p, q), then the signature of $-g$ is (q, p).*

Proof. We know that if the signature of g is (p, q), then there exists a basis

$$\{e_1, e_2, \ldots, e_p, f_1, f_2, \ldots, f_q\}$$

for $T_p(M)$ at any $p \in M$ such that
$g(e_i, e_i) = -1$,
$g(e_i, e_j) = 0$, if $i \neq j$,
$g(f_i, f_i) = 1$,
$g(f_i, f_j) = 0$, if $i \neq j$,
$g(e_i, f_j) = 0$, for all i, j.
Then, an orthonormal basis relative to $-g$ is $\{f_1, f_2, \ldots, f_q, e_1, e_2, \ldots, e_p\}$: such that
$(-g)(e_i, e_i) = -g(e_i, e_i) = +1$,
$(-g)(e_i, e_j) = -g(e_i, e_j) = 0$,
$(-g)(f_i, f_i) = -g(f_i, f_i) = -1$,
$(-g)(f_i, f_j) = -g(f_i, f_j) = 0$,
$(-g)(e_i, f_j) = -g(e_i, f_j) = 0$.
Therefore the signature of -g is (q, p). □

Chapter 4

Connection and Geodesics Completeness with Examples in Timelike, Spacelike, and Lightlike

4.1 Connection

A connection is supposed to be differentiation in a given direction as measured by a tangent vector. The next example illustrates a connection.

Example 4.1. *Let ∇ be a directional derivative on $\mathbb{R}^3$, f is a function and x is a direction when $x \in \mathbb{R}^3$ and* $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, *then* $\nabla_x(f) = \nabla f \cdot x = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{\partial f}{\partial x}$.
Thus, $\nabla_x : \mathcal{F}(R^3) \longrightarrow \mathcal{F}(R^3)$.

A connection is intended to measures a rate of change of a lot of objects in direction x, and it is not just functions.

Definition 4.2. *A connection ∇ on a smooth manifold M is a function,* $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, *satisfying* ($\nabla 1$) *to* ($\nabla 3$) *below. We call the covariant derivative of Y in direction X is equal to Z and we donate that as $\nabla_X Y = Z$.*

($\nabla 1$) ∇ is $\mathcal{F}(M)$ - liner in X, that is, if $f, g : M \to \mathbb{R}$ and $X_1, X_2, Y \in \mathfrak{X}(M)$, then $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$.

(∇2) $\mathbb{R}$-liner in Y, that is, for all $\alpha, \beta \in \mathbb{R}$, and $X, Y_1, Y_2 \in \mathfrak{X}(M)$,
$\nabla_X(\alpha Y_1 + \beta Y_2) = \alpha \nabla_X Y_1 + \beta \nabla_X Y_2$.

(∇3) If $f : M \to \mathbb{R}, \nabla_X(fY) = X(f)Y + f\nabla_X Y$, for all $X, Y \in \mathfrak{X}(M)$.

There are two other properties that a connection may or may not satisfy.

(∇4) $\nabla_X Y - \nabla_Y X = [X, Y]$. (Torsion Free)

(∇5) If g is a metric on M, then $Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$. (Metric Compatible, Riemannian)

We will show here a lemma that we will use in the following Theorem (4.5) and we can find it in [O'N83].

Lemma 4.3. *The bracket operation on $\mathfrak{X}(M)$ has the following properties:*

1. $\mathbb{R}$*-bilinearity:*

$$[f_1X + f_2Y, Z] = f_1[X, Z] + f_2[Y, Z],$$

$$[Z, f_1X + f_2Y] = f_1[Z, X] + f_2[Z, Y].$$

2. *Skew-symmetry:*

$$[X, Y] = -[Y, X].$$

3. *Jacobi identity:*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

We will prove here a lemma that the bracket operation on $\mathfrak{X}(M)$ though $\mathbb{R}$-bilinear, is not $\mathcal{F}(M)-$bilinear. Furthermore, we will use it in the following Theorem (4.5).

Lemma 4.4. *The fact of the bracket operation on $\mathfrak{X}(M)$ which is the following equation*

$$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X.$$

If g is the constant function, $g = 1$, then the formula above takes a simpler form:

$$[fX, Y] = f[X, Y] - Y(f)X.$$

Proof.

$$\begin{aligned}[fX, gY](h) &= (fX)(gY)(h) - (gY)(fX)(h) \\ &= fX(gY(h)) - gY(fX(h)) \\ &= fX(g)Y(h) + fg(XY(h)) - gY(f)X(h) - fgYX(h) \\ &= ((f(X(g))Y - gY(f)X + fg[X, Y])(h).\end{aligned}$$

So,

$$[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X.$$

When g is the constant function, $g = 1$, then the formula above takes a simpler form, since $X(g) = 0$:

$$[fX, Y] = f[X, Y] - Y(f)X.$$

□

The following fundamental result, theorem has been called the miracle of Semi-Riemannian geometry [O'N83].

Theorem 4.5. *There exists a unique connection* ∇ *on* (M, g) *satisfying* $(\nabla 1)$ *to* $(\nabla 5)$*. This is characterized by the Koszul formula:*

$$\begin{aligned}2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad -g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]).\end{aligned}$$

This is called the Levi-Civita connection on M*.*

Proof. We will show the Levi-Civita connection is unique. We suppose ∇ satisfies connection $(\nabla 4)$ and $(\nabla 5)$ from the previous definition and we will show the Koszul formula holds. We will use $(\nabla 5)$ on the first three terms and $(\nabla 4)$ on the last three terms of the RHS of the Koszul formula.

$$\begin{aligned}RHS &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad -g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]). \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ &\quad -g(\nabla_Z X, Y) - g(X, \nabla_Z Y) - g(X, \nabla_Y Z) + g(X, \nabla_Z Y) \\ &\quad +g(Y, \nabla_Z X) - g(Y, \nabla_X Z) + g(Z, \nabla_X Y) - g(Z, \nabla_Y X). \\ &= 2g(\nabla_X Y, Z).\end{aligned}$$

Therefore, if ∇ satisfies ($\nabla 1$) to ($\nabla 5$), then the Koszul formula holds.

Now we show there exists such a connection. If $Z \longrightarrow F(X,Y,Z)$ is $\mathcal{F}(\mathcal{M})$-Linear, hence it is a one-form for fixed X and $Y \in \mathfrak{X}(M)$.
If $F(X,Y,fZ) = fF(X,Y,Z)$ for all $f \in \mathcal{F}(M)$, then $F(X,Y,fZ)$ is $\mathcal{F}(M)$-linear, and hence a 1-form. We will show this by using the Koszul formula, Lemma (4.3), and Lemma (4.4). I mean the following two facts from those lemmas,

1. $[fX,Y] = f[X,Y] - Y(f)X$, and
2. $[Y,fX] = -[fX,Y]$.

So,

$$\begin{aligned} F(X,Y,fZ) &= X(Y,fZ) + Y(fZ,X) - fZ(X,Y) \\ &\quad -(X,[Y,fZ]) + (Y,[fZ,X]) + (fZ,[X,Y]) \\ &= Xf(Y,Z) + fX(Y,Z) + Yf(Z,X) + fY(Z,X) \\ &\quad -fZ(X,Y) + (X,f[Z,Y]) - (X,YfZ) \\ &\quad +(Y,f[Z,X]) - (Y,XfZ) + (fZ,[X,Y]) \\ &= fF(X,Y,Z). \end{aligned}$$

Now we know that F is $\mathcal{F}(M)-$linear, hence a 1-form. By Proposition (2.19) there is a unique vector field,$\nabla_X Y$ such that $2g(\nabla_X Y, Z) = F(X,Y,Z)$ for all Z. Thus the Koszul formula holds and we can use the Koszul formula to prove the connection satisfies ($\nabla 1$) to ($\nabla 5$).

1. Our proof for the ($\nabla 1$):
   We have to show the following equation,

   $$2g(\nabla_{f_1X_1+f_2X_2}Y, Z) = 2g(f_1\nabla_{X_1}Y + f_2\nabla_{X_2}Y, Z).$$

   We will use here the Lemma (4.3) and the Lemma (4.4). I mean the following two facts from those lemmas,

   (a) $[fX,Y] = f[X,Y] - Y(f)X$, and
   (b) $[Y,fX] = -[fX,Y]$.

$$
\begin{aligned}
LHS &= (f_1X_1 + f_2X_2)(g(Y,Z)) + y(g(Z, f_1X_1 + f_2X_2)) - Z(g(f_1X_1 + f_2X_2, Y)) \\
&\quad -g(f_1X_1 + f_2X_2, [Y,Z]) + g(Y, [Z, f_1X_1 + f_2X_2]) + g(Z, [f_1X_1 + f_2X_2, Y]) \\
&= f_1X_1(g(Y,Z)) + f_2X_2(g(Y,Z)) + Y(f_1)(g(Z,X_1)) + Y(f_2)(g(Z,X_2)) \\
&\quad +f_1Y(g(Z,X_1)) + f_2Y(g(Z,X_2)) - Z(f_1)(g(X_1,Y)) - Z(f_2)(g(X_2,Y)) \\
&\quad -f_1Z(g(X_1,Y)) - f_2Z(g(X_2,Y)) - f_1g(X_1,[Y,Z]) - f_2g(X_2,[Y,Z]) \\
&\quad +g(Y, f_1[Z,X_1]) + g(Y, Z(f_1)X_1) + g(Y, f_2[Z,X_2]) + g(Y, (Z(f_2)X_2) \\
&\quad g + (Z, f_1[X_1,Y]) - g(Z, Y(f_1)X_1) + g(Z, f_2[X_2,Y]) - g(Z, Y(f_2)X_2) \\
&= 2g(f_1\nabla_{X_1}Y, Z) + 2g(f_2\nabla_{X_2}Y, Z) \\
&= 2g(f_1\nabla_{X_1}Y + f_2\nabla_{X_2}Y, Z) \\
&= RHS.
\end{aligned}
$$

2. Our proof for the $(\nabla 2)$:

   We have to show the following equation for all $\alpha, \beta \in R$,

$$
2g(\nabla_X(\alpha Y_1 + \beta Y_2), Z) = 2g(\nabla_X \alpha Y_1, Z) + 2g(\nabla_X \alpha Y_2, Z).
$$

$$
\begin{aligned}
LHS &= X(g(\alpha Y_1 + \beta Y_2), Z) + (\alpha Y_1 + \beta Y_2)(g(Z,X)) - Z(g(X, \alpha Y_1 + \beta Y_2)) \\
&- g(X, [\alpha Y_1 + \beta Y_2], Z) + g(\alpha Y_1 + \beta Y_2, [Z,X]) + g(Z, [\alpha Y_1 + \beta Y_2]) \\
&= X(g(\alpha Y_1, Z)) + (g(\beta Y_2.Z)) + \alpha Y_1(g(Z,X)) + \beta Y_2(g(Z,X)) \\
&- Z(g(X,\alpha Y_1)) - Z(g(X,\beta Y_2)) - g(X,[\alpha Y_1, Z]) - g(X,[\beta Y_2, Z]) \\
&+ g(\alpha Y_1, [Z,X]) + g(\beta Y_2, [Z,X]) + g(Z,[X,\alpha Y_1]) + g(Z,[X,\beta Y_2]) \\
&= 2g(\nabla_X \alpha Y_1, Z) + 2g(\nabla_X \alpha Y_2, Z) \\
&= RHS.
\end{aligned}
$$

3. Our proof for the $(\nabla 3)$:

   If $f : M \to \mathbb{R}, \nabla_X(fY) = X(f)Y + f\nabla_X Y$, for all X.

   We know $2(\nabla_X fY, Z) = F(X, fY, Z)$, then using Lemma (4.3) and the Lemma

(4.4).

$$\begin{aligned}
2(\nabla_X fY, Z) &= X(fY, Z) + fY(Z, X) - Z(X, fY) \\
&\quad -(X, [fY, Z]) + (fY, [Z, X]) + (Z, [X, fY]) \\
&= Xf \cdot (Y, Z) + fX(Y, Z) + fY(Z, X) \\
&\quad -Zf \cdot (X, Y) - fZ(X, Y) - (X, f[Y, Z]) \\
&\quad -(X, ZfY) + f(Y, [Z, X]) - (Z, f[Y, X]) + (Z, XfY) \\
&= 2Xf \cdot (Y, Z) + f \cdot F(X, Y, Z) \\
&= 2Xf \cdot (Y, Z) + f \cdot 2(\nabla_X Y, Z).
\end{aligned}$$

Therefore,

$$\nabla_X fY = XfY + f\nabla_X Y.$$

4. Our proof for the $(\nabla 4)$:

We will show the following, $\nabla_X Y - \nabla_Y X = [X, Y]$. (Torsion Free).

We interchange X and Y in the Koszul formula and we will subtract it from the original Koszul formula which is

$$\begin{aligned}
2g(\nabla_X Y, Z) - 2g(\nabla_Y X, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\
&\quad -g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) \\
&\quad -Y(g(X, Z)) - X(g(Z, Y)) + Z(g(Y, X)) \\
&\quad +g(Y, [X, Z]) - g(X, [Z, Y]) - g(Z, [Y, X]). \\
2g(\nabla_X Y, Z) - 2g(\nabla_Y X, Z) &= 2g([X, Y], Z).
\end{aligned}$$

Therefore,

$$(\nabla_X Y) - (\nabla_Y X) = [X, Y].$$

5. Our proof for the $(\nabla 5)$:

If g is a metric on M, then $Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$.

It calls (Metric Compatible, Riemannian).

We interchange Y and Z in the Koszul formula and we will add it to the original Koszul formula which is

$$\begin{aligned}
2g(\nabla_X Y, Z) + 2g(\nabla_X Z, Y) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\
&\quad -g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) \\
&\quad +X(g(Z, Y)) + Z(g(Y, X)) - Y(g(X, Z)) \\
&\quad -g(X, [Z, Y]) + g(Z, [Y, X]) + g(Y, [X, Z]). \\
2g(\nabla_X Y, Z) + 2g(\nabla_X Z, Y) &= 2gX(Y, Z)
\end{aligned}$$

Therefore,

$$(\nabla_X Y, Z) + (\nabla_X Z, Y) = X(Y, Z).$$

□

The following lemma illustrates the computing of the Levi-Civita connection.

Lemma 4.6. *If* ∇ *is the Levi-Civita connection, and*
$X = \partial_x, Y = \partial_y, Z = \partial_z$*, so* $([\partial_x, \partial_y] = [\partial_x, \partial_z] = [\partial_y, \partial_z] = 0)$*, then*
$2g(\nabla_{\partial_x}\partial_y, \partial_z) = \frac{\partial}{\partial_x} g(\partial_y, \partial_z) + \frac{\partial}{\partial_y} g(\partial_x, \partial_z) - \frac{\partial}{\partial_z} g(\partial_x, \partial_y)$.

Proof. By using the Koszul formula (4.5) we will find the following,

$$\begin{aligned} 2g(\nabla_{\partial_x}\partial_y, \partial_z) &= \frac{\partial}{\partial_x} g(\partial_y, \partial_z) + \frac{\partial}{\partial_y} g(\partial_x, \partial_z) - \frac{\partial}{\partial_z} g(\partial_x, \partial_y) + 0 + 0 + 0. \\ 2g(\nabla_{\partial_x}\partial_y, \partial_z) &= \frac{\partial}{\partial_x} g(\partial_y, \partial_z) + \frac{\partial}{\partial_y} g(\partial_x, \partial_z) - \frac{\partial}{\partial_z} g(\partial_x, \partial_y). \end{aligned}$$

□

We denote $g_{ij} = g(\partial_{x_i}, \partial_{x_j})$ and $g_{ij/k} = \frac{\partial}{\partial_{x_k}} g(\partial_{x_i}, \partial_{x_j})$, then
$2g(\nabla_{\partial x_i}\partial x_j, \partial x_k) = g_{jk/i} + g_{ik/j} - g_{ij/k}$.
We call $\Gamma_{ijk} = g(\nabla_{\partial_{x_i}}\partial_{x_j}, \partial_{x_k}) = \frac{1}{2}(g_{jk/i} + g_{ik/j} - g_{ij/k})$ the Christoffel symbols of the 2^{nd}-kind. If we write $\nabla_{\partial_{x_i}}\partial_{x_j} = \sum_{l=1}^{n} \Gamma_{ij}^{l} \partial_{x_l}$ then we also call Γ_{ij}^{l} is Christoffel symbols of the 1^{st}-kind.

Here is the lemma to show what happens if a connection ∇ is torsion free.

Lemma 4.7. *If* ∇ *is torsion free, then* $\Gamma_{ijk} = \Gamma_{jik}$*.*
When $X = \partial_{x_i}, Y = \partial_{x_j}, Z = \partial_{x_k}$

Proof. $[\partial_{x_i}, \partial_{x_j}] = 0$, so by $(\nabla 4)$,
$\nabla_{\partial_{x_i}}\partial_{x_j} = \nabla_{\partial_{x_j}}\partial_{x_i}$, then
$\Gamma_{ijk} = g(\nabla_{\partial_{x_i}}\partial_{x_j}, \partial_{x_k}) = g(\nabla_{\partial_{x_i}}\partial_{x_j}, \partial_{x_k}) = \Gamma_{jik}$. □

The following example is simply to illustrate how we compute ∇.

Example 4.8. *Let* $M = \mathbb{R}^3$ *with coordinates* (x_1, x_2, x_3)*, and suppose that*
$g(\partial_{x_1}, \partial_{x_1}) = x_1^2 + 2x_3^2 + 1$,
$g(\partial_{x_2}, \partial_{x_2}) = x_2^2 + 1$,
$g(\partial_{x_3}, \partial_{x_3}) = 1$ *, and*
$g(\partial_{x_i}, \partial_{x_j}) = 0$ *when* $i \neq j$ *. We compute:*

1. $\nabla_{\partial_{x_2}}\partial_{x_1} = \Gamma^1_{21}\partial_{x_1} + \Gamma^2_{21}\partial_{x_2} + \Gamma^3_{11}\partial_{x_3}$.

2. $\nabla_{\partial_{x_1}}\partial_{x_1} = \Gamma^1_{11}\partial_{x_1} + \Gamma^2_{11}\partial_{x_2} + \Gamma^3_{11}\partial_{x_3}$.

3. $\nabla_{\partial_{x_1}+x_2^2\partial_{x_2}}(x_2^2\partial_{x_1})$.

Solution:

$g(\nabla_{\partial_{x_2}}\partial_{x_1}, \partial_{x_1}) = \Gamma_{211} = \frac{1}{2}(g_{11/2} + g_{12/1} - g_{21/1}) = \frac{1}{2}(0+0+0) = 0.$

$g(\nabla_{\partial_{x_2}}\partial_{x_1}, \partial_{x_2}) = \Gamma_{212} = \frac{1}{2}(g_{12/2} + g_{22/1} - g_{21/1}) = \frac{1}{2}(0+0+0) = 0.$

$g(\nabla_{\partial_{x_2}}\partial_{x_1}, \partial_{x_3}) = \Gamma_{231} = \frac{1}{2}(g_{31/2} + g_{12/3} - g_{23/1}) = \frac{1}{2}(0+0+0) = 0.$

$g(\nabla_{\partial_{x_2}}\partial_{x_1}, \partial_{x_1}) = g(\Gamma^1_{21}\partial_{x_1} + \Gamma^2_{21}\partial_{x_2} + \Gamma^3_{11}\partial_{x_3}, \partial_{x_1}) = 0.$

$g(\Gamma^1_{21}\partial_{x_1}, \partial_{x_1}) + g(\Gamma^2_{21}\partial_{x_2}, \partial_{x_1}) + g(\Gamma^3_{21}\partial_{x_3}, \partial_{x_1}) = 0.$

$\Gamma^1_{21}g(\partial_{x_1}, \partial_{x_1}) + \Gamma^2_{21}g(\partial_{x_2}, \partial_{x_1}) + \Gamma^3_{21}g(\partial_{x_3}, \partial_{x_1}) = 0.$

$\Gamma^1_{21}\cdot(x_1^2 + x_3^2 + 1) + \Gamma^2_{21}\cdot 0 + \Gamma^3_{21}\cdot 0 = 0.$

$\Gamma^1_{21} = 0.$

$g(\nabla_{\partial_{x_2}}\partial_{x_1}, \partial_{x_2}) = g(\Gamma^1_{21}\partial_{x_1} + \Gamma^2_{21}\partial_{x_2} + \Gamma^3_{11}\partial_{x_3}, \partial_{x_2}) = 0.$

$g(\Gamma^1_{21}\partial_{x_1}, \partial_{x_2}) + g(\Gamma^2_{21}\partial_{x_2}, \partial_{x_2}) + g(\Gamma^3_{21}\partial_{x_3}, \partial_{x_2}) = 0.$

$\Gamma^1_{21}g(\partial_{x_1}, \partial_{x_2}) + \Gamma^2_{21}g(\partial_{x_2}, \partial_{x_2}) + \Gamma^3_{21}g(\partial_{x_3}, \partial_{x_2}) = 0.$

$\Gamma^1_{21}\cdot 0 + \Gamma^2_{21}\cdot(x_2^2 + 1) + \Gamma^3_{21}\cdot 0 = 0.$

$\Gamma^2_{21} = 0.$

$g(\nabla_{\partial_{x_2}}\partial_{x_1}, \partial_{x_3}) = g(\Gamma^1_{21}\partial_{x_1} + \Gamma^2_{21}\partial_{x_2} + \Gamma^3_{11}\partial_{x_3}, \partial_{x_3}) = 0.$

$g(\Gamma^1_{21}\partial_{x_1}, \partial_{x_3}) + g(\Gamma^2_{21}\partial_{x_2}, \partial_{x_3}) + g(\Gamma^3_{21}\partial_{x_3}, \partial_{x_3}) = 0.$

$\Gamma^1_{21}g(\partial_{x_1}, \partial_{x_3}) + \Gamma^2_{21}g(\partial_{x_2}, \partial_{x_3}) + \Gamma^3_{21}g(\partial_{x_3}, \partial_{x_3}) = 0.$

$\Gamma^1_{21}\cdot 0 + \Gamma^2_{21}\cdot 0 + \Gamma^3_{21}\cdot 1 = 0.$

$\Gamma^3_{21} = 0.$

$\nabla_{\partial_{x_2}}\partial_{x_1} = \Gamma^1_{21}\partial_{x_1} + \Gamma^2_{21}\partial_{x_2} + \Gamma^3_{11}\partial_{x_3} = 0 + 0 + 0 = 0.$

$\nabla_{\partial_{x_2}}\partial_{x_1} = 0.$

If $[X, Y] = 0$ *, and* ∇ *is Torsion Free , then* $\nabla_X Y - \nabla_Y X = [X, Y] = 0,$

$\nabla_X Y = \nabla_Y X \Longrightarrow \nabla_{\partial_{x_i}}\partial_{x_j} = \nabla_{\partial_{x_j}}\partial_{x_i}.$

Back to our example then $\nabla_{\partial_{x_2}}\partial_{x_1} = \nabla_{\partial_{x_1}}\partial_{x_2} = 0.$

We compute $\nabla_{\partial_{x_1}}\partial_{x_1} = \Gamma^1_{11}\partial_{x_1} + \Gamma^2_{11}\partial_{x_2} + \Gamma^3_{11}\partial_{x_3}$.
$g(\nabla_{\partial x_1}\partial x_1, \partial x_1) = \Gamma_{111} = \frac{1}{2}(g_{11/1} + g_{11/1} - g_{11/1}) = \frac{1}{2}g_{11/1} = \frac{1}{2}(2x_1) = x_1$.
$g(\nabla_{\partial_{x_1}}\partial_{x_1}, \partial_{x_2}) = \Gamma_{112} = \frac{1}{2}(g_{12/1} + g_{12/1} - g_{12/1}) = 0$.
$g(\nabla_{\partial_{x_1}}\partial_{x_1}, \partial_{x_3}) = \Gamma_{113} = \frac{1}{2}(g_{13/1} + g_{13/1} - g_{11/3}) = \frac{-1}{2}g_{11/3} = \frac{-1}{2}(4x_3) = -2x_3$.

$$\begin{aligned} g(\nabla_{\partial_{x_1}}\partial_{x_1}, \partial_{x_1}) &= g(\Gamma^1_{11}\partial_{x_1} + \Gamma^2_{11}\partial_{x_2} + \Gamma^3_{11}\partial_{x_3}, \partial_{x_1}) \\ &= g(\Gamma^1_{11}\partial_{x_1}, \partial_{x_1}) + g(\Gamma^2_{11}\partial_{x_2}, \partial_{x_1}) + g(\Gamma^3_{11}\partial_{x_3}, \partial_{x_1}) \\ &= \Gamma^1_{11}g(\partial_{x_1}, \partial_{x_1}) + 0 + 0 = \Gamma^1_{11}(x_1^2 + 2x_3^2 + 1) + 0 + 0 = x_1. \end{aligned}$$

Then $\Gamma^1_{11} = \frac{x_1}{x_1^2+2x_3^2+1}$.

$$\begin{aligned} g(\nabla_{\partial_{x_1}}\partial_{x_1}, \partial_{x_2}) &= g(\Gamma^1_{11}\partial_{x_1} + \Gamma^2_{11}\partial_{x_2} + \Gamma^3_{11}\partial_{x_3}, \partial_{x_2}) \\ &= g(\Gamma^1_{11}\partial_{x_1}, \partial_{x_2}) + g(\Gamma^2_{11}\partial_{x_2}, \partial_{x_2}) + g(\Gamma^3_{11}\partial_{x_3}, \partial_{x_2}) \\ &= 0 + \Gamma^2_{11}g(\partial_{x_2}, \partial_{x_2}) + 0 = \Gamma^2_{11}(x_2^2 + 1) = 0. \end{aligned}$$

Then $\Gamma^2_{11} = 0$.

$$\begin{aligned} g(\nabla_{\partial_{x_1}}\partial_{x_1}, \partial_{x_3}) &= g(\Gamma^1_{11}\partial x_1 + \Gamma^2_{11}\partial x_2 + \Gamma^3_{11}\partial_{x_3}, \partial_{x_3}) \\ &= g(\Gamma^1_{11}\partial_{x_1}, \partial_{x_3}) + g(\Gamma^2_{11}\partial_{x_2}, \partial_{x_3}) + g(\Gamma^3_{11}\partial_{x_3}, \partial_{x_3}) \\ &= 0 + 0 + \Gamma^3_{11}g(\partial_{x_3}, \partial_{x_3}) \\ &= \Gamma^3_{11} \cdot 1 = \Gamma^3_{11}. \end{aligned}$$

So $\Gamma^3_{11} = -2x_3$.
Therefore, $\nabla_{\partial_{x_1}}\partial_{x_1} = \frac{x_1}{x_1^2+2x_3^2+1}\partial_{x_1} - 2x_3\partial_{x_3}$.

We compute $\nabla_{\partial_{x_1}+x_2^2\partial_{x_2}}(x_2^2\partial_{x_1})$.

$$\begin{aligned} \nabla_{\partial_{x_1}+x_2^2\partial_{x_2}}(x_2^2\partial_{x_1}) &= \nabla_{\partial_{x_1}}(x_2^2\partial_{x_1}) + \nabla_{x_2^2\partial_{x_2}}(x_2^2\partial_{x_1}) \\ &= \nabla_{\partial_{x_1}}(x_2^2\partial_{x_1}) + x_2^2\nabla_{\partial_{x_2}}(x_2^2\partial_{x_1}) \\ &= \frac{\partial}{\partial_{x_1}}(x_2^2)\partial_{x_1} + x_2^2\nabla_{\partial_{x_1}}\partial_{x_1} + x_2^2[\frac{\partial}{\partial_{x_2}}(x_2^2)\partial_{x_1} + x_2^2\nabla_{\partial_{x_2}}\partial_{x_1}] \\ &= 0 + x_2^2[\frac{x_1}{x_1^2+2x_3^2+1}\partial_{x_1} - 2x_3\partial_{x_3}] + x_2^2[2x_2\partial_{x_1} + x_2^2 \cdot 0] \\ &= (\frac{x_1x_2^2}{x_1+2x_3^2+1} + 2x_2^3)\partial_{x_1} - (2x_2^2x_3)\partial_{x_3}. \end{aligned}$$

4.2 Geodesics and Completeness

In this section we will give a special definition of a manifold that we will use for the remainder of this thesis. These manifolds are defined in Definition (4.17). Prior to that, we introduce the notions of geodesics, and completeness on pseudo-Riemannian manifolds. The following theorem is to illustrate when a Riemannian manifold is geodesically complete.

Theorem 4.9. *(Hopf-Rinow) A connected Riemannian manifold is geodesically complete if and only if it is complete as a metric space [Lee97].*

However, there is no analogue to this in the pseudo-Riemannian setting. In fact, only certain geodesics can measure a suitable "distance" on manifolds, and some of these must be distance *maximizing.*

The following definition is to define geodesics on a special family of manifolds defined in Definition (4.17).

Definition 4.10. *Suppose that* $\sigma : (-\epsilon, \epsilon) \to M$ *is a smooth path, then* $\sigma' = \frac{d}{dt}(\sigma)$ *is a vector field in* $T_{\sigma(t)}M$*,* σ *is a geodesic if* $\nabla_{\sigma'}\sigma' = 0$.

The following theorem illustrates that geodesics exist.

Theorem 4.11. *[O'N83]*
Given $p \in M$*,* $v \in T_pM$*, there exists a geodesic* $\sigma : (-\epsilon, \epsilon) \longrightarrow M$ *with* $\sigma(0) = p$*,* $\sigma'(0) = v$.

We will give here an example to illustrate the existence of a geodesic between two different points and the length is zero. The point here is to illustrate that distance minimizing curves need not behave as one might expect in pseudo-Riemannian manifolds.

Example 4.12. *If* (M, g) *is pseudo-Riemannian, with* $L(\gamma) = \int_0^1 \sqrt{|g(\gamma', \gamma')|}\partial t$ *as the length* $g(\gamma)$ *on* $M = \mathbb{R}^2$*, with metric as follows:*
$g(\partial_x, \partial_x) = -1, g(\partial_t, \partial_t) = 1, \gamma(0,0) = 0, \gamma(1,1) = 1,$
$\gamma(t) = (t,t) \longrightarrow \gamma' = (1,1) = \partial_x + \partial_t,$
$g(\gamma', \gamma') = g(\partial_x + \partial_t, \partial_x + \partial_t) = 0,$ *then* $L(\gamma) = 0$.
Here we see $L((0,0),(1,1)) = 0$ *but* $(0,0) \neq (1,1)$

The following definition illustrates that a geodesic can sometimes be extendible.

Definition 4.13. *If* $\sigma : (-\epsilon, \epsilon) \longrightarrow M$ *is a geodesic on* (M, g)*, then it is extendible if there exists an interval* $(-\epsilon, \epsilon) \subsetneqq I$ *and* σ *is a geodesic on* I *as well. Also* σ *extends for infinite time if* $I = \mathbb{R}$*.*

The following definition is to define geodesic completeness on pseudo-Riemannian manifolds.

Definition 4.14. *(M,g) complete if all geodesics extend for infinite time.*

Lemma 4.15. *If* ∇ *is the Levi-Civita connection on* M*, and* σ *is a geodesic, then* $||\sigma'||^2$ *is constant.*

Proof. $\nabla_{\sigma'}(g(\sigma', \sigma')) = g(\nabla_{\sigma'}\sigma', \sigma') + g(\sigma', \nabla_{\sigma'}\sigma') = 0 + 0 = 0.$ □

Therefore, if σ is a geodesic we call:
σ spacelike if $||\sigma'(0)||^2 > 0$,
σ timelike if $||\sigma'(0)||^2 < 0$,
σ lightlike or null if $||\sigma'(0)||^2 = 0$.

Now we know that a manifold is *geodesically complete* if every geodesic extends for infinite time. On the other hand, the following definition is to illustrates the geodesic completeness in spacelike, timelike, and lightlike.

Definition 4.16. (M, g) *is*

1. timelike geodesically complete *if every timelike geodesic extends for infinite time.*

2. spacelike geodesically complete *if every spacelike geodesic extends for infinite time.*

3. lightlike geodesically complete *if every lightlike geodesic extends for infinite time.*

Our goal in this thesis is to illustrate that completeness in spacelike, timelike, and lightlike are inequivalent. To do this, we introduce the following family of manifolds.

Definition 4.17. *Let* $f(x) : \mathbb{R} \longrightarrow \mathbb{R}$ *be a smooth and non vanishing define the following metric* g *on* $M = \{(t, x) \in \mathbb{R}^2 | x > 0\}$:
$g(\partial_x, \partial_x) = 1,\ g(\partial_t, \partial_t) = -f^{-2}(x),\ g(\partial_x, \partial_t) = g(\partial_t, \partial_x) = 0$ *[Hub09].*

These manifolds were introduced in [Hub09] specifically for the purpose of demonstrating that the various notions of completeness on pseudo-Riemannian manifolds are inequivalent. The following theorem is to show the computation of the Levi-Civita connection and geodesics on manifold that defined in Definition (4.17). We refer to these manifolds for the remainder of the thesis.

Theorem 4.18. *Let $f(x) : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth and non vanishing. Define on (M,g) which is defined on (4.17). Then*

$$\nabla_{\partial_x}\partial_x = 0, \nabla_{\partial_x}\partial_t = \nabla_{\partial_t}\partial_x = \frac{-f'(x)}{f(x)} \cdot \partial_t, \nabla_{\partial_t}\partial_t = \frac{-f'(x)}{f^3(x)} \cdot \partial_x.$$

Proof. $g(\nabla_{\partial_x}\partial_x, \partial_x) = \Gamma_{xxx} = \frac{1}{2}(g_{xx/x} + g_{xx/x} - g_{xx/x}) = 0.$
$g(\nabla_{\partial_x}\partial_x, \partial_t) = \Gamma_{xxt} = \frac{1}{2}(g_{xt/x} + g_{tx/x} - g_{xx/t}) = 0.$
$g(\nabla_{\partial_x}\partial_t, \partial_x) = \Gamma_{xtx} = \frac{1}{2}(g_{tx/x} + g_{xx/t} - g_{xt/x}) = 0.$
$g(\nabla_{\partial_x}\partial_t, \partial_t) = \Gamma_{xtt} = \frac{1}{2}(g_{tt/x} + g_{tx/t} - g_{xt/t}) = \frac{f'(x)}{f^3(x)}.$
$g(\nabla_{\partial_t}\partial_t, \partial_x) = \Gamma_{ttx} = \frac{1}{2}(g_{tx/t} + g_{xt/t} - g_{tt/x}) = \frac{-f'(x)}{f^3(x)}.$
$g(\nabla_{\partial_t}\partial_t, \partial_t) = \Gamma_{xtt} = \frac{1}{2}(g_{tt/t} + g_{tt/t} - g_{tt/t}) = 0.$
We know that:
$\nabla_{\partial_x}\partial_x = \Gamma^x_{xx}\partial x + \Gamma^t_{xx}\partial_t,$
$\nabla_{\partial_x}\partial_t = \nabla_{\partial_t}\partial_x = \Gamma^x_{xt}\partial_x + \Gamma^t_{xt}\partial_t,$
$\nabla_{\partial_t}\partial_t = \Gamma^x_{tt}\partial_x + \Gamma^t_{tt}\partial_t$, then

$g(\nabla_{\partial_x}\partial_x, \partial x) = g(\Gamma^x_{xx}\partial_x + \Gamma^t_{xx}\partial_t, \partial_x) = g(\Gamma^x_{xx}\partial_x, \partial_x) + g(\Gamma^t_{xx}\partial_t, \partial_x) = 0.$
$\Gamma^x_{xx}g(\partial_x, \partial_x) + \Gamma^t_{xx}g(\partial_t, \partial_x) = \Gamma^x_{xx} \cdot 1 + \Gamma^t_{xx} \cdot 0 = 0.$
$\Gamma^x_{xx} = 0.$

$g(\nabla_{\partial_x}\partial_x, \partial_t) = g(\Gamma^x_{xx}\partial_x + \Gamma^t_{xx}\partial_t, \partial_t) = g(\Gamma^x_{xx}\partial_x, \partial_t) + g(\Gamma^t_{xx}\partial t, \partial t) = 0.$
$\Gamma^x_{xx}g(\partial_x, \partial_t) + \Gamma^t_{xx}g(\partial_t, \partial_t) = \Gamma^x_{xx} \cdot 0 + \Gamma^t_{xx} \cdot \frac{-1}{f^2(x)} = 0.$
$\Gamma^t_{xx} = 0.$

$\nabla_{\partial_x}\partial_x = \Gamma^x_{xx}\partial_x + \Gamma^t_{xx}\partial_t = 0 \cdot \partial_x + 0 \cdot \partial_t = 0.$
$\nabla_{\partial_x}\partial_x = 0.$

$g(\nabla_{\partial_x}\partial_t, \partial_x) = g(\Gamma^x_{xt}\partial_x + \Gamma^t_{xt}\partial_t, \partial_x) = g(\Gamma^x_{xt}\partial_x, \partial_x) + g(\Gamma^t_{xt}\partial_t, \partial_x) = 0.$

$\Gamma^x_{xt} g(\partial_x, \partial_x) + \Gamma^t_{xx} g(\partial_t, \partial_x) = \Gamma^x_{xt} \cdot 1 + \Gamma^t_{xt} \cdot 0 = 0.$
$\Gamma^x_{xt} = 0.$

$g(\nabla_{\partial_x} \partial_t, \partial_t) = g(\Gamma^x_{xt}\partial_x + \Gamma^t_{xt}\partial_t, \partial_t) = g(\Gamma^x_{xt}\partial_x, \partial_t) + g(\Gamma^t_{xt}\partial_t, \partial_t) = \frac{f'(x)}{f(x)}.$
$\Gamma^x_{xt} g(\partial_x, \partial_t) + \Gamma^t_{xx} g(\partial_t, \partial_t) = \Gamma^x_{xt} \cdot 0 + \Gamma^t_{xt} \cdot \frac{-1}{f^2} = \frac{f'(x)}{f(x)}.$
$\Gamma^t_{xt} = \frac{-f'(x)}{f(x)}.$

$\nabla_{\partial_x} \partial_t = \nabla_{\partial_t} \partial_x = \Gamma^x_{xt}\partial - x + \Gamma^t_{xt}\partial_t = 0 \cdot \partial_x \frac{-f'(x)}{f(x)} \cdot \partial_t . \nabla_{\partial_x} \partial_t = \frac{-f'(x)}{f(x)} \cdot \partial_t.$

$g(\nabla_{\partial_t} \partial_t, \partial_x) = g(\Gamma^x_{tt}\partial_x + \Gamma^t_{tt}\partial_t, \partial_x) = g(\Gamma^x_{tt}\partial_x, \partial_x) + g(\Gamma^t_{tt}\partial_t, \partial_x) = \frac{-f'(x)}{f^3(x)}.$
$\Gamma^x_{tt} g(\partial_x, \partial_x) + \Gamma^t_{tt} g(\partial_t, \partial_x) = \Gamma^x_{tt} \cdot 1 + \Gamma^t_{tt} \cdot 0 = \frac{-f'(x)}{f^3(x)}.$
$\Gamma^x_{tt} = \frac{-f'(x)}{f^3(x)}.$

$g(\nabla_{\partial_t} \partial_t, \partial_t) = g(\Gamma^x_{tt}\partial x + \Gamma^t_{tt}\partial_t, \partial_t) = g(\Gamma^x_{tt}\partial_x, \partial_t) + g(\Gamma^t_{tt}\partial_t, \partial_t) = 0.$
$\Gamma^x_{tt} g(\partial_x, \partial_t) + \Gamma^t_{tt} g(\partial_t, \partial_t) = \Gamma^x_{tt} \cdot 1 + \Gamma^t_{tt} \cdot 0 = 0.$
$\Gamma^t_{tt} = 0.$

$$\begin{aligned}
\nabla_{\partial_t} \partial_t &= \Gamma^x_{tt}\partial_x + \Gamma^t_{tt}\partial_t \\
&= \frac{-f'(x)}{f^3(x)} \cdot \partial_x + 0 \cdot \partial_t \\
&= \frac{-f'(x)}{f^3(x)} \cdot \partial_x.
\end{aligned}$$

□

The following lemma is to illustrate the computation of $||\sigma'||^2$ for a geodesic σ on M.

Lemma 4.19. *If $\sigma(s)$ be a curve in M, $\sigma(s) = (t(s), x(s))$, and also $\sigma'(s) = t'(s) \cdot \partial_t + x'(s) \cdot \partial_x$. Then $||\sigma'||^2 = g(\sigma', \sigma') = (x')^2 - (\frac{t'}{f(x)})^2$.*

Proof.

$$\begin{aligned}
g(\sigma',\sigma') &= g(t'\partial_t + x'\partial_x, t'\partial_t + x'\partial_x) \\
&= g(t'\partial_t + x'\partial_x, t'\partial_t) + g(t'\partial_t + x'\partial_x, x'\partial_x) \\
&= g(t'\partial_t, t'\partial_t) + g(x'\partial_x, t'\partial_t) + g(t'\partial_t, x'\partial_x) + g(x'\partial_x, x'\partial_x) \\
&= (t')^2 g(\partial_t,\partial_t) + (x't')g(\partial_x,\partial_t) + (t'x')^2 g(\partial_t,\partial_x) + (x')^2 g(\partial_x,\partial_x) \\
&= (t')^2 \cdot \tfrac{-1}{f^2(x)} + x't'\cdot 0 + t'x'\cdot 0 + (x')^2\cdot 1 \\
&= -(\tfrac{t'}{f(x)})^2 + (x')^2 g(\sigma',\sigma') \\
&= -(\tfrac{t'}{f(x)})^2 + (x')^2.
\end{aligned}$$

Therefore,

$$g(\sigma',\sigma') = (x')^2 - (\frac{t'}{f(x)})^2. \tag{4.1}$$

□

The following theorem is to illustrate the computation of $\nabla_{\sigma'}\sigma'$.

Theorem 4.20. *If $\sigma(s)$ be a curve in M, $\sigma(s) = (t(s), x(s))$, and also $\sigma'(s) = t'(s)\cdot\partial_t + x'(s)\cdot\partial_x$. Then $\nabla_{\sigma'}\sigma' = (t'' - \frac{2x't'f'(x)}{f(x)})\partial_t + (x'' - \frac{(t')^2 f'(x)}{f^3(x)})\partial_x$.*

Proof.

$$\begin{aligned}
\nabla_{\sigma'}\sigma' &= \nabla_{\sigma'}(t'\partial_t + x'\partial_x) \\
&= \nabla_{\sigma'}(t'\partial_t) + \nabla_{\sigma'}(x'\partial_x) \\
&= t''\partial t + t'\nabla_{\sigma'}\partial_t + x''\partial_x + x'\nabla_{\sigma'}\partial_x \\
&= t''\partial_t + t'\nabla_{(t'\partial_t + x'\partial_x)}\partial_t + x''\partial_x + x'\nabla_{(t'\partial_t + x'\partial_x)}\partial_x \\
&= t''\partial_t + t'(t'\nabla_{\partial_t}\partial_t + x'\nabla_{\partial_x}\partial t) + x''\partial_x + x'(t'\nabla_{\partial_t}\partial_x + x'\nabla_{\partial_x}\partial_x) \\
&= t''\partial_t + (t')^2\nabla_{\partial_t}\partial_t + t'x'\nabla_{\partial_x}\partial_t + x''\partial_x + x't'\nabla_{\partial_t}\partial_x + (x')^2\nabla_{\partial_x}\partial_x \\
&= t''\partial_t + (t')^2\cdot\tfrac{-f'(x)}{f^3(x)}\cdot\partial_x + 2x't'\cdot\tfrac{-f'(x)}{f(x)} + x''\partial_x + (x)^2\cdot 0 \\
&= (t'' - \tfrac{2x't'f'(x)}{f(x)})\partial_t + (x'' - \tfrac{(t')^2 f'(x)}{f^3(x)})\partial_x \\
&= (t'' - \tfrac{2x't'f'(x)}{f(x)})\partial_t + (x'' - \tfrac{(t')^2 f'(x)}{f^3(x)})\partial_x.
\end{aligned}$$

□

The following theorem is to show the solution of $\nabla'_\sigma\sigma' = 0$ when the function, $f(x) = x^\alpha$. We will be using this f for the remainder of this thesis.

Theorem 4.21. *Let $f(x) = x^\alpha$ for some $\alpha \in \mathbb{R}$, then solution of $\nabla_{\sigma'}\sigma' = 0$ for $\sigma(s) = (t(s), x(s))$ are $t' = cx^{2\alpha}$ and $(x')^2 = c^2x^{2\alpha} + \epsilon$. for some $c > 0$ and $\epsilon \in R$. Furthermore, $||\sigma'||^2 = \epsilon$.*

Proof. When $f(x) = x^\alpha$, then $f'(x) = \alpha x^{\alpha-1}$.
And also from the previous theorem we find the following equations,

$$t'' - \frac{2x't'\alpha x^{\alpha-1}}{x^\alpha} = 0. \tag{4.2}$$

$$x'' - \frac{(t')^2\alpha s^{\alpha-1}}{x^{3\alpha}} = 0. \tag{4.3}$$

From Equation (4.2):
$t'' - \frac{2\alpha t'x'}{x} = 0$.
$\int \frac{t''}{t'} = 2\alpha \int \frac{x'}{x}$.
$\ln t' = 2\alpha \ln x + \ln c$.

$$t' = cx^{2\alpha}. \tag{4.4}$$

From Equation (4.4) into (4.3).
$x'' - \alpha(cx^{2\alpha})2x^{2\alpha-1} = 0$.
$x'' = \alpha c^2 x^{2\alpha-1}$.
$x''x' = \alpha c^2 x' x^{2\alpha-1}$.
$\frac{1}{2}(x')^2 = \frac{\alpha c^2 x^{2\alpha}}{2\alpha} + \frac{1}{2}\epsilon$.

$$(x')^2 = c^2x^{2\alpha} + \epsilon. \tag{4.5}$$

$$\epsilon = (x')^2 - (\frac{t'}{x^\alpha})^2. \tag{4.6}$$

□

Now we are close to showing our goal. According to the Equation (4.5) we can determine geodesic completeness for different values of α:

$$x' = \sqrt{c^2x^{2\alpha} + \epsilon}, so \tag{4.7}$$

$\frac{x'}{\sqrt{c^2x^{2\alpha}+\epsilon}} = 1$, and so $\int \frac{x'}{\sqrt{c^2x^{2\alpha}+\epsilon}} = \int 1 = s - s_0$.

The following theorem is to show that the manifolds in Definition (4.17) are not timelike geodesically complete.

Theorem 4.22. *The manifolds in Definition (4.17) for $f(x) = x^\alpha$ are not timelike complete for $\alpha = 1, \alpha = \frac{1}{2}$, and $\alpha = \frac{1}{4}$.*

Proof. Let $\epsilon = -1$, $c^2 = \frac{1}{\beta^{2\alpha}}$,
$x^\alpha = \beta^\alpha \sec\theta$, $\partial x = \frac{\beta}{\alpha}(\sec\theta)^{\frac{1}{\alpha}-1} \sec\theta \tan\theta \partial\theta$.
Then $\frac{\beta}{\alpha} \int (\sec\theta)^{\frac{1}{\alpha}} \partial\theta = s - s_0$.

1. Let $\alpha = \beta = c = 1$, then
   $\int \sec\theta\partial\theta = \ln|\sec\theta + \tan\theta| = s - s_0$.
   $\ln|x + \sqrt{x^2 - 1}| = s - s_0$.

   We claim the Left hand side(LHS) is always positive but the Right hand side(RHS) is not always positive because if we let RHS

   $$= -\ln 2 = \ln\frac{1}{2}. \tag{4.8}$$

   Then LHS$= \ln|x + \sqrt{x^2 - 1}| = \ln\frac{1}{2}$.
   $\sqrt{x^2 - 1} = \frac{1}{2} - x$.
   $x^2 - 1 = \frac{1}{4} + x^2 - x$.
   $x = \frac{5}{4}$.
   When we substitute $x = \frac{5}{4}$ on LHS, then

   $$\ln|\frac{5}{4} + \sqrt{\frac{25}{16} - 1}| = \ln|\frac{5}{4} + \frac{3}{4}| = \ln 2. \tag{4.9}$$

From Equations (4.8) and (4.9) we found contradiction.
Therefore, it is not geodesically complete.

2. Let $\alpha = \frac{1}{2}$ and $\beta = 1$, then
   $2\int \sec^2\theta \partial\theta = 2\tan\theta = s - s_0$.
   $2\sqrt{x-1} = s - s_0$.
   If $s - s_0 < 0$ then there is not exist x so that $2\sqrt{x-1} = s - s_0$.
   So this does not extend for infinite time.

3. Let $\alpha = \frac{1}{4}$ and $\beta = 1$
   $4\int \sec^4\theta \partial\theta = s - s_0$.
   $4(\tan\theta + \frac{1}{3}\tan^3\theta) = s - s_0$.
   $4(\sqrt{\sqrt{x}-1} + \frac{1}{3}(\sqrt{\sqrt{x}-1})^3 = s - s_0$.
   $\frac{4}{3}\sqrt{\sqrt{x}-1}(2+\sqrt{x}) = s - s_0$.
   We know $\sqrt{\sqrt{x}-1} \geq 0$ everywhere it is defined,
   and $(2+\sqrt{x}) \geq 0$ everywhere it is defined.
   So, $\frac{4}{3}\sqrt{\sqrt{x}-1}(2+\sqrt{x}) \geq 0$ everywhere it is defined. Therefore, the LHS ≥ 0 but the LHS is not always because if $s - s_0 < 0$ then there is not exist x so that
   $\frac{4}{3}\sqrt{\sqrt{x}-1}(2+\sqrt{x}) = s - s_0$.
   So this does not extend for infinite time.

□

The following theorem in the geodesically lightlike case is to show that the goedesics do extend for infinite time for one $\alpha \in \mathbb{R}$, but not for some other α.

Theorem 4.23. *The manifold in Definition (4.17) for $f(x) = x^\alpha$ are lightlike complete for $\alpha = 1$, but not lightlike complete for $\alpha = \frac{1}{2}$, or $\alpha = \frac{1}{4}$.*

Proof. $\int \frac{x'}{\sqrt{c^2x^{2\alpha}+\epsilon}} = \int 1 = s - s_0$.
Let $\epsilon = 0$, then
$\int \frac{x'}{cx^\alpha} = \frac{1}{c} \cdot \frac{x^{1-\alpha}}{1-\alpha} = s - s_0$ when $\alpha \neq 1$.
Or
$\int \frac{x'}{cx^\alpha} = \frac{1}{c}\ln x = s - s_0$ when $\alpha = 1$

1. When $\alpha = 1$

   $\frac{1}{c} \ln x = s - s_0$

   $\ln x = c(s - s_0)$

   $x = e^{c(s-s_0)}$

   It is geodesically complete.

2. when $\alpha = \frac{1}{2}$ and $c = 1$.

   $\frac{x^{\frac{1}{2}}}{\frac{1}{2}} = s - s_0$.

   $\sqrt{x} = \frac{1}{c}(s - s_0)$.

   The LHS is always positive but the RHS is not because if $s - s_0$ less than 0, then there is not exist x so that $\sqrt{x} = \frac{1}{c}(s - s_0)$.

   So this does not extend for infinite time.

3. when $\alpha = \frac{1}{4}$ and $c = 1$.

   $\frac{x^{\frac{3}{4}}}{\frac{3}{4}} = s - s_0$.

   $(\sqrt[4]{x})^3 = \frac{3}{4}(s - s_0)$.

   The LHS is always positive but the RHS is not because if $s - s_0$ less than 0, then there is not exist x so that $(\sqrt[4]{x})^3 = \frac{3}{4}(s - s_0)$.

   So this does not extend for infinite time.

□

The following theorem in the geodesically spacelike case is to show that the geodesics extend for infinite time for one $\alpha \in \mathbb{R}$, but not for some other α.

Theorem 4.24. *The manifold in Definition (4.17) for $f(x) = x^\alpha$ are spacelike complete for $\alpha = 1$, but not spacelike complete for $\alpha = \frac{1}{2}$, or $\alpha = \frac{1}{4}$.*

Proof. Let $\epsilon = 1$ and $c^2 = \frac{1}{\beta^{2\alpha}}$.

$\beta^\alpha \int \frac{x'}{\sqrt{x^{2\alpha} + \beta^{2\alpha}}} = s - s_0$.

Let $x^\alpha = \beta^\alpha \tan\theta$, then $\partial x = \frac{\beta}{\alpha} \tan^{\frac{1}{\alpha}-1} \sec\theta\partial\theta$

1. Let $\alpha = 1$.

   $\beta \int \sec \partial\theta = \beta \ln|\sec\theta + \tan\theta| = s - s_0$.

   $\beta \ln|\frac{\sqrt{x^2+\beta^2}}{\beta} + \frac{x}{\beta}| = s - s_0$.

   $(\sqrt{x^2 + \beta^2})^2 = (\beta e^{\frac{s-s_0}{\beta}} - x)^2$.

$x^2 + \beta^2 = \beta^2 e^{2\frac{(s-s_0)}{\beta}} + x^2 - 2\beta x e^{\frac{(s-s_0)}{\beta}}$.
$x = \beta \frac{(e^{\frac{(s-s_0)}{\beta}} - e^{\frac{-(s-s_0)}{\beta}})}{2} = \beta \sinh(\frac{s-s_0}{\beta})$.
It is geodesically complete.

2. Let $\alpha = \frac{1}{2}$ and $\beta = 1$, then
$2\int \tan\theta \sec\theta \partial\theta = 2\sec\theta = (s - s_0)$.
$\sqrt{x+1} = \frac{1}{2}(s - s_0)$.
The LHS is always positive but the RHS is not because if $s - s_0$ less than 0, then there is not exist x so that $\sqrt{x+1} = \frac{1}{2}(s - s_0)$.
So this does not extend for infinite time.

3. Let $\alpha = \frac{1}{4}$ and $\beta = 1$.
$4\int \tan^3\theta \sec\theta \partial\theta = s - s_0$.
$\frac{4}{3}\sec^3\theta - \sec\theta = s - s_0$.
$\frac{4}{3}\sqrt{\sqrt{x}+1} - \sqrt[4]{x} = s - s_0$.
If we multiply both sides by $(\frac{4}{3}\sqrt{\sqrt{x}+1} + \sqrt[4]{x})$, then
$\frac{16}{9}(\sqrt{x}+1) - \sqrt{x} = (\frac{4}{3}\sqrt{\sqrt{x}+1} + \sqrt[4]{x})(s - s_0)$.
$\frac{\frac{7\sqrt{x}}{9} + \frac{16}{9}}{\frac{4}{3}(\sqrt{\sqrt{x}+1} + \sqrt[4]{x})} = s - s_0$.

The LHS is always positive, but the RHS is not always positive(+) because if $s - s_0 < 0$, then there is not exist x so that $\frac{\frac{7\sqrt{x}}{9} + \frac{16}{9}}{\frac{4}{3}(\sqrt{\sqrt{x}+1} + \sqrt[4]{x})} = s - s_0$.
So this does not extend for infinite time.

□

4.3 Summary

The following summarizes our results concerning the completeness of the manifolds in Definition (4.17).

Summary 4.25. *For the manifolds, (M, g) in Definition (4.17),*

1. *Timelike: When $\epsilon = -1$ and $\alpha = 1$, $\frac{1}{2}$, $\frac{1}{4}$, then (M, g) is not geodesically complete.*

2. *Lightlike:* *When* $\epsilon = 0$ *and* $\alpha = 1$*, then* (M, g) *is geodesically complete, but when* $\alpha = \frac{1}{2}, \frac{1}{4}$*, then* (M, g) *is not geodesically complete.*

3. *Spacelike:* *When* $\epsilon = 1$ *and* $\alpha = 1$*, then* (M, g) *is geodesically complete, but when* $\alpha = \frac{1}{2}, \frac{1}{4}$*, then* (M, g) *is not geodesically complete.*

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