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Symmetric representation of the elements of finite groups

Barbara Hope Gwinn-Edwards

Symmetric Representation of the Elements of Finite Groups

A Thesis

Presented to the

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In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

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Abstract

The main purpose of this thesis is to construct finite groups as homomorphic images of infinite semi-direct products, $2^{*n} : N$, where 2^{*n} is a free product of n copies of the cyclic group C_2 extended by N, a group of permutations on n letters. We constructed several finite homomorphic images of the semi-direct products $2^{*4} : S_4$, $3^{*5} : S_5$, and $2^{*6} : L_2(5)$. In particular, we constructed S_5, $2 \times S_5$, S_7, $S_7 \times 3$, $2 \times M_{12}$, and M_{12}. Moreover, every element of M_{12}, usually written on 12 letters, can be expressed as permutations of $L_2(5)$ on six letters followed by a word in terms of the 6 symmetric generators of length at most 6.

ACKNOWLEDGEMENTS

I would first like to thank Dr. Zahid Hasan who spent many, many hours teaching me about group theory, the construction of symmetric groups, the details of M_{12}, and math in general. His patience seemed to be infinite and I tested it often. Also I would like to thank my committee members, Dr. John Sarli and Dr. Rollie Trapp for their time spent reviewing my work. Additionally, I am deeply grateful to the faculty of CSUSB's math department for their excellent instruction and obvious dedication to student learning. I aspire to convey the same passion for math and caring attitude toward my students that they have shown to me. Finally I would like to thank my husband, Craig Edwards, for his support and technical assistance during the hundreds of hours I spent working on this thesis.

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Chapter 1

Introduction

The two general methods for working with groups, permutations and matrices, are inconvenient or unmanageable for large finite groups and in particular for the larger sporadic groups, for example for the Monster group, the least degree of a matrix representation is 196881 and it takes n^3 operations to multiply two matrices of dimension n and the least degree of any permutation representation is 10^{20}. Matrix multiplication for matrices of large size is very time-consuming and, although MAGMA ([C$^+$05]), and other group theory packages, handle permutations of reasonably large size quite efficiently, recording and transmitting elements is inconvenient. The main purpose of this paper is to give an alternative and more efficient method for working with groups. Double coset enumeration can be performed on groups that possess generating sets of involutions. Curtis has constructed several sporadic groups by manual double coset enumeration; for references see [CHB96], [CH96] and [Cur07]. Now any finite group generated by a conjugacy class of involutions, and hence all finite non-abelian simple groups have symmetric generating sets of involutions (see [Bra97]). It is this technique of double coset enumeration that allows us to write elements in a much more concise manner.

1.1 Symmetric Generation of a Group

Let G be a group and

$$T = \{t_1, t_2, \ldots, t_n\} \subseteq G,$$

then we define

$$\overline{T} = \{T_1, T_2, \ldots, T_n\},$$

where $T_i = \langle t_i \rangle$, the cyclic subgroup generated by t_i; we further define $N = N_G(\overline{T})$, the set normalizer in G of $\overline{T}$.
If the following two conditions hold:

(i) $G = \langle T \rangle$, and

(ii) N permutes $\overline{T}$ transitively, not necessarily faithfully,

then, following Curtis and Hasan [CH96], we say that T is a *symmetric generating set* for G . In these circumstances we call N the *control subgroup.* Note that (i) and (ii) imply that G is a homomorphic image of the (infinite) *progenitor*

$$m^{*n} : N,$$

where m^{*n} represents a free product of n copies of the cyclic group C_m, m being the order of t_i, and N is a group of automorphisms of m^{*n} which permutes the n cyclic subgroups by conjugation. Thus, for $\pi \in N$, we have

$$t_i^{\pi} = t_j^{r},$$

where r is an integer coprime to m. Of course, if $m = 2$ then N will simply act by conjugation as permutations of the n involutory symmetric generators. Now, since above elements of N can be gathered on the left, every element of the progenitor can be represented as πw, where $\pi \in N$ and w is a word in the symmetric generators. Indeed this representation is unique provided w is simplified so that adjacent symmetric generators are distinct. Thus any additional relator by which we must factor the progenitor to obtain G must have the form

$$\pi w(t_1, t_2, .., t_n),$$

where $\pi \in N$ and w is a word in T. In the next section we describe how a particular factor group

$$\frac{m^{*n} : N}{\pi_1 w_1, .., \pi_s w_s}$$

may be identified.

1.2 Manual Double Coset Enumeration

We seek homomorphic images of the progenitor

$$2^{*n} : N,$$

(where N is now a transitive permutation group on n letters), which act faithfully on N and on the generators of the free product. It is convenient to identify the n free generators and N with their respective images. Thus

$$\frac{2^{*n} : N}{\pi_1 w_1, .., \pi_s w_s} \cong \langle N, T \mid t_i^2 = 1, t_i^\pi = t_{\pi(i)}, \pi_1 w_1 = .. = \pi_s w_s = 1 \rangle,$$

where $\pi \in N, t_i \in T$. Following [Cur07] we are allowing i to stand for the symmetric generator t_i in expressions such as the above relation. By a slight abuse of notation we also allow i to denote the coset Nt_i, ij the coset Nt_it_j etc., and we write, for instance,

$$ij \sim k \text{ to mean } Nt_it_j = Nt_k.$$

Writing $ij = k$ would be the much stronger statement that $t_it_j = t_k$. Now since

$$t_i\pi = \pi t_{\pi(i)},$$

(or $i\pi = \pi i^\pi$ as we shall more commonly write), the permutations involved in any element of G can be gathered on the left. Thus any element of G can be written as a permutation belonging to N followed by a word in the symmetric generators. Indeed, as mentioned in the last section, in the case of the progenitor itself this representation is unique provided the obvious cancellations are performed. Thus, if NgN is a double coset of N in G, we have

$$NgN = N\pi wN = NwN,$$

where $g = \pi w \in G$, with $\pi \in N$, and w is a word in the t_i. We denote this double coset by $[w]$, e.g. $[01]$ denotes the double coset Nt_0t_1N. The double coset $NeN = N$, where e is the identity element, is denoted by $[\star]$. Furthermore we define

$$N^i = C_N(t_i);\ N^{ij} = C_N(\langle t_i, t_j \rangle) \text{ etc},$$

single point and two point stabilizers in N respectively. The coset stabilizing subgroup, $N^{(w)}$, of N is given by,

$$N^{(w)} = \{\pi \in N : Nw\pi = Nw\},$$

for w a word in the symmetric generators. Clearly $N^w \le N^{(w)}$, and the number of cosets in the double coset $[w] = NwN$ is given by $\mid N \mid / \mid N^{(w)} \mid$, since

$$\begin{aligned} Nw\pi_1 \neq Nw\pi_2 &\iff Nw\pi_1{\pi_2}^{-1} \neq Nw \\ &\iff \pi_1{\pi_2}^{-1} \notin N^{(w)} \\ &\iff N^{(w)}\pi_1{\pi_2}^{-1} \neq N^{(w)} \\ &\iff N^{(w)}\pi_1 \neq N^{(w)}\pi_2. \end{aligned}$$

In order to obtain the index of N in G we shall perform a manual double coset enumeration of G over N; thus we must find all double cosets $[w]$ and work out how many single cosets each of them contains. We shall know that we have completed the double coset enumeration when the set of right cosets obtained is closed under right multiplication. Moreover, the completion test above is best performed by obtaining the orbits of $N^{(w)}$ on the symmetric generators. We need only identify, for each $[w]$, the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit. We will decompose the image G into double cosets NgN, where $g \in 2^{*n} : N$ and find a set $\{g_1, g_2, \ldots\}$ of elements of G such that

$$G = Ng_1N \cup Ng_2N \cup \ldots.$$

But for each i, we have $g_i = \pi_i w_i$, where $\pi_i \in N$ and w_i is a word in the t_i, and so the double coset deomposition simplifies to

$$G = N \cup Nw_2N \cup Nw_3N \cup \ldots,$$

where w_1 is chosen to be the identity. When the set of relations by which we are factoring is empty this gives the double coset decomposition of the progenitor $2^{*n} : N$, and in this case there are infinitely many double cosets corresponding to the orbits of N on the ordered k-tuples of the letters of $\Omega = \{t_1, \ldots, t_n\}$ which have no adjacent repetitions, where $k \in \mathbb{N} = \{1, 2, \ldots\}$.

We used the technique described above to construct several finite groups. We constructed the group S_5, the symmetric group of degree 5, in Chapter 2, the group $2 \times S_5$ in Chapter 3, the group S_7, the symmetric group of degree 7, in Chapter 4, the group $S_7 \times 3$ in Chapter 5, the group $2 \times M_{12}$ in Chapter 6, and the group M_{12}, the second largest of the Mathieu sporadic groups, in Chapter 7.

Chapter 2

Construction of S_5

Consider the progenitor

$2^{*4} : S_4 \cong \langle x, y, t | x^4 = y^2 = (xy)^3 = 1 = [y, t] = [t^x, y] \rangle$, where
$x \sim (0, 1, 2, 3)$,
$y \sim (2, 3)$, and
$t \sim t_0$.

From [CHB96] we find the following relations:

1. $[(0, 1, 2, 3)t_0]^a = [(0, 1, 2, 3)t_0]^5 = 1$
2. $[(0, 1, 2)t_0]^b = [(0, 1, 2)t_0]^4 = 1$
3. $[(0, 1)(2, 3)t_0]^c = [(0, 1)(2, 3)t_0]^6 = 1$
4. $[(0, 1)t_0]^d = [(0, 1)t_0]^3 = 1$
5. $[(0, 1)t_0t_2]^e = [(0, 1)t_0t_2]^4 = 1$

Based on a computer, it is known, see [CHB96], that the progenitor $2^{*4} : S_4$ factored by relations (1) through (5), although relation (1) suffices, is isomorpic to S_5. We will construct by hand S_5 using the technique of manual double coset enumeration of $G \cong \frac{2^{*4}:S_4}{[(0,1,2,3)t_0]^5}$ over S_4.

Expanding relation (4) gives us:

(4) $[(0, 1)t_0]^3 = 1$

$(0,1)t_0(0,1)t_0(0,1)t_0 = 1$
$(0,1)^3 t_0{}^{(0,1)^2} t_0{}^{(0,1)} t_0 = 1$
$(0,1)t_0t_1t_0 = 1$
$(0,1)t_0 = t_0t_1 \qquad (4)$

We now perform the manual double coset enumeration of $G = \frac{2^{*4}:S_4}{[(0,1,2,3)t_0]^5}$ over S_4 over S_4.

We start with the double coset representative word of length zero, $NeN = N$, denoted [*].

Next we consider the double coset with word length one. Since N is transitive on $\Omega = \{0,1,2,3\}$, $Nt_0N = \{Nt_0{}^n | n \in N\} = \{Nt_0, Nt_1, Nt_2, Nt_3\}$ denoted [0].

Now we determine for [0] to which coset Nt_0t_i belongs for one t_i from each orbit of N^0 one Ω. Since $N^0 = \{n \in N | t_0{}^n = t_0\} = \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\} = \langle (2,3), (1,2,3) \rangle \cong S_3$, $N^0 = S_3$ has orbits $\{0\}, \{1,2,3\}$, on Ω.

First let's consider t_0, $Nt_0t_0 = N \in NeN$ since t_0 is of order 2, so the generator t_0 from the 1 orbit takes [0] back to a right coset in [*].

Next let's consider t_1, a representative from the orbit $\{1,2,3\}$, $Nt_0t_1 = N(0,1)t_0$ $= Nt_0 \in Nt_0N = [0]$ (from relation 4). So the generator t_1 a representative from the 3-orbit takes [01] back to a single coset in [0].

Our double coset enumeration must be complete since the set of right cosets is closed under right multiplication by the symmetric generators.

Thus we have the Cayley diagram that is shown in Figure 2.1.

The maximum possible index of N in $G \cong \frac{2^{*4}:S_4}{[(0,1,2,3)t_0]^5}$ is $\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} = 1 + 4 = 5$. Thus $|G| \leq 5 \times |N| = 5 \times 24 = 120$. In order to show $|G| = 120$, we consider G as a subgroup of S_5 acting on five symbols that we have found, and label as follows:

1. N
2. Nt_0
3. Nt_1
4. Nt_2
5. Nt_3

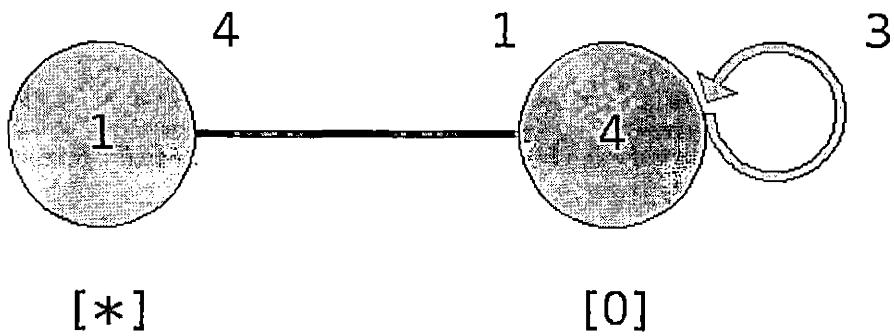

Figure 2.1: Cayley Diagram of S_5 over S_4

For this purpose we compute the action of the control group N as well as the action of t_0, t_1, t_2, and t_3 on the five cosets. These permutations are as follows:

$t_0 : (1,2)$ $\quad t_1 : (1,3)$ $\quad t_2 : (1,4)$ $\quad t_3 : (1,5)$

$x : (2,3,4,5)$ $\quad y : (4,5)$

It readily checks that the order of $\langle x, y, t\rangle$, a subgroup of the symmetric group S_5 acting on the five right cosets of N in G, is 120. Since the order of xy is three, $N = \langle x, y\rangle \cong S_4$ and t_0 has exactly four conjugates under conjugation by N, we conclude that $\langle x, y, t\rangle$ is a homomorphic image of the progenitor $2^{*4} : S_4$.

Thus if the original five relations hold in $\langle x, y, t\rangle$, then $\langle x, y, t\rangle$ is a homomorphic image of G and this will give $|G| \geq |\langle x, y, t\rangle| = 120$.

Verify relation (1) $t_0t_3t_2t_1t_0 = (0,3,2,1)$ by conjugating the four symmetric generators by $t_0t_3t_2t_1t_0$. By multiplying the permutations listed above, we find $t_0t_3t_2t_1t_0$ $= (2,5,4,3)$. Thus when we conjugate t_0, t_1, t_2, and t_3 by $t_0t_3t_2t_1t_0$ we obtain:

$t_0^{t_0t_3t_2t_1t_0} = t_3$ $\quad t_1^{t_0t_3t_2t_1t_0} = t_0$

$t_2^{t_0t_3t_2t_1t_0} = t_1$ $\quad t_3^{t_0t_3t_2t_1t_0} = t_2$

So $t_0t_3t_2t_1t_0$ acts as the permutation $(0,3,2,1)$.

Verify relation (2) $t_0t_2t_1t_0 = (0,2,1)$ by conjugating the four symmetric generators.

$t_0^{t_0t_2t_1t_0} = t_2$ $\quad t_1^{t_0t_2t_1t_0} = t_0$

$t_2^{t_0t_2t_1t_0} = t_1$ $\quad t_3^{t_0t_2t_1t_0} = t_3$

So $t_0t_2t_1t_0$ acts as the permutation $(0,2,1)$.

Verify relation (3) $t_1t_0t_1t_0t_1t_0 = 1$ by conjugating the generators.
$t_0^{t_1t_0t_1t_0t_1t_0} = t_0$ $\quad t_1^{t_1t_0t_1t_0t_1t_0} = t_1$
$t_2^{t_1t_0t_1t_0t_1t_0} = t_2$ $\quad t_3^{t_1t_0t_1t_0t_1t_0} = t_3$
So $t_1t_0t_1t_0t_1t_0$ acts as the identity.

Verify relation (4) $t_0t_1t_0 = (0,1)$ by conjugating the generators.
$t_0^{t_0t_1t_0} = t_1$ $\quad t_1^{t_0t_1t_0} = t_0$
$t_2^{t_0t_1t_0} = t_2$ $\quad t_3^{t_0t_1t_0} = t_3$
So $t_0t_1t_0$ acts as the permutation $(0,1)$.

Verify relation (5) $t_1t_2t_0t_2t_1t_2t_0t_2 = 1$ by conjugating the generators.
$t_0^{t_1t_2t_0t_2t_1t_2t_0t_2} = t_0$ $\quad t_1^{t_1t_2t_0t_2t_1t_2t_0t_2} = t_1$
$t_2^{t_1t_2t_0t_2t_1t_2t_0t_2} = t_2$ $\quad t_3^{t_1t_2t_0t_2t_1t_2t_0t_2} = t_3$
So $t_1t_2t_0t_2t_1t_2t_0t_2$ acts as the identity.

Thus $G/ker\phi \cong \langle x, y, t\rangle$ and $|G| \geq |\langle x, y, t\rangle| = 120$. As shown earlier, $|G| \leq 120$. Hence $|G| = 120$.

Moreover, $a = (1,3,5,4,2)$ and $b = (1,2)$ are in G with $S_5 \cong \langle a, b\rangle$ since $S_5 \cong \{a, b|a^5, b^2, (ab)^4, (a^{-1}ab)^2, (a^{-2}ba^2b)^2\}$. So $\langle a, b\rangle \leq G$, but $|\langle a, b\rangle| = |G|$, therefore $G = \langle a, b\rangle \cong S_5$.

Chapter 3

Construction of $2 \times S_5$

Consider the progenitor

$2^{*4} : S_4 \cong \langle x, y, t | x^4 = y^2 = (xy)^3 = 1 = [y, t] = [t^x, y] \rangle$, where
$x \sim (0, 1, 2, 3)$,
$y \sim (2, 3)$, and
$t \sim t_0$.

From [CHB96] we find the following relations:

1. $[(0, 1, 2, 3)t_0]^a = [(0, 1, 2, 3)t_0]^{10} = 1$

2. $[(0, 1, 2)t_0]^b = [(0, 1, 2)t_0]^4 = 1$

3. $[(0, 1)(2, 3)t_0]^c = [(0, 1)(2, 3)t_0]^6 = 1$

4. $[(0, 1)t_0]^d = [(0, 1)t_0]^6 = 1$

5. $[(0, 1)t_0t_2]^e = [(0, 1)t_0t_2]^4 = 1$

Based on a computer, it is known, see [CHB96], that the progenitor $2^{*4} : S_4$, factored by relations (1) through (5), although relation (2) suffices, is isomorphic to $2 \times S_5$. We will construct by hand $2 \times S_5$ using the technique of manual double coset enumeration of $G \cong \frac{2^{*4}:S_4}{[(0,1,2)t_0]^4}$ over S_4.

Expanding relations (2) and (3) gives us:

(2) $[(0, 1, 2)t_0]^4 = 1$

$(0,1,2)t_0(0,1,2)t_0(0,1,2)t_0(0,1,2)t_0 = 1$

$(0,1,2)^4 t_0^{(0,1,2)^3} t_0^{(0,1,2)^2} t_0^{(0,1,2)} t_0 = 1$

$(0,1,2)t_0t_2t_1t_0 = 1$

$(0,1,2)t_0t_2 = t_0t_1 \qquad (2)$

(3) $[(0,1)(2,3)t_0]^6 = 1$

$(0,1)(2,3)t_0(0,1)(2,3)t_0(0,1)(2,3)t_0(0,1)(2,3)t_0(0,1)(2,3)t_0(0,1)(2,3)t_0 = 1$

$[(0,1)(2,3)]^6 t_0^{[(0,1)(2,3)]^5} t_0^{[(0,1)(2,3)]^4} t_0^{[(0,1)(2,3)]^3} t_0^{[(0,1)(2,3)]^2} t_0^{[(0,1)(2,3)]} t_0 = 1$

$t_1t_0t_1t_0t_1t_0 = 1$

$t_1t_0t_1t_0 = t_0t_1$

$t_1t_0t_1 = t_0t_1t_0 \qquad (3)$

We now perform the manual double coset enumeration of $G = \frac{2^{*4}:S_4}{[(0,1,2)t_0]^4}$ over S_4

We start with the double coset representative word of length zero, $NeN = N$, denoted $[*]$.

Next we consider the double coset with word length one. Since N is transitive on $\Omega = \{0,1,2,3\}$, $Nt_0N = \{Nt_0{}^n | n \in N\} = \{Nt_0, Nt_1, Nt_2, Nt_3\}$ denoted [0].

Now we determine for [0] to which coset Nt_0t_i belongs for one t_i from each orbit of N^0 on Ω. Since$N^0 = \{n \in N | t_0{}^n = t_0\} = \{e, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\} = \langle (2,3), (1,2,3) \rangle \cong S_3$

$N^0 = S^3$ has orbits $\{0\}, \{1,2,3\}$, on Ω.

First let's consider t_0, $Nt_0t_0 = N \in NeN$ since t_0 is of order 2, so the generator t_0, a representative from the 1-orbit takes [0] back to $[*]$.

Next let's consider t_1, a representative from the orbit $\{1,2,3\}$.

$Nt_0t_1 = N(0,1,2)t_0t_2 = Nt_0t_2$ (from relation 2)

$Nt_0t_1 = N(0,1,3)t_0t_3 = Nt_0t_3$ (from relation 2 conjugated by (2,3)).

So, $Nt_0t_1 = Nt_0t_2 = Nt_0t_3$.

Similarly, $Nt_3t_1 = Nt_3t_2 = Nt_3t_0$ (conjugating the abpve equation by (0,3)).

$Nt_2t_1 = Nt_2t_0 = Nt_2t_3$ (conjugating the above equation by (0,2)).

Finally, $Nt_1t_0 = Nt_1t_2 = Nt_1t_3$ (conjugating the above equation by (0,1)).

Thus the generator t_1 a representative from the 3-orbit takes [0] to a single coset in [01]. There are four unique single cosets in [01] each with three names.

Now we determine for [01] to which coset $Nt_0t_1t_i$ belongs for one t_i from each orbit of $N^{(01)}$ on Ω.

$N^{01} = \{n \in N^0 | (t_0t_1)^n = t_0t_1\} = \langle(2,3)\rangle$

$N^{(01)} = \{\pi \in N | N(t_0t_1)^\pi = Nt_0t_1\} = \langle(2,3),(1,2,3)\rangle$ since $Nt_0t_1 = Nt_0t_2 = Nt_0t_3$.

So the orbits of $N^{(01)}$ are $\{0\}$ and $\{1,2,3\}$ on Ω.

First let's consider t_1 as a representative from the orbit $\{1,2,3\}$.

$Nt_0t_1t_1 = Nt_0 \in Nt_0N$ since the t_i's are of order two. So the generator t_1, a representative from the 3-orbit takes [01] back to [0].

Now let's consider t_0. $Nt_0t_1t_0 = Nt_1t_0t_1$ from relation 3. We can conjugate this relation by various permutations to obtain the following equations.

$Nt_2t_1t_2 = Nt_1t_2t_1$ (conjugating by (0,2))
$Nt_3t_1t_3 = Nt_1t_3t_1$ (conjugating by (0,3))
$Nt_0t_2t_0 = Nt_2t_0t_2$ (conjugating by (1,2))
$Nt_0t_3t_0 = Nt_3t_0t_3$ (conjugating by (1,3))
$Nt_3t_2t_3 = Nt_2t_3t_2$ (conjugating by (0,3)(1,2))

We note that:

$N\underline{t_0t_1}t_0 = N\underline{(0,1,2)}t_0t_2t_0 = Nt_0t_2t_0$ (by relation 2).
$N\underline{t_0t_1}t_0 = N\underline{(0,1,3)}t_0t_3t_0 = Nt_0t_3t_0$ (conjugating relation 2 by $(2,3)$).
$N\underline{t_1t_0}t_1 = N\underline{(1,0,2)}t_1t_2t_1 = Nt_1t_2t_1$ (conjugating relation 2 by $(0,1)$).
$N\underline{t_1t_0}t_1 = N\underline{(1,0,3)}t_1t_3t_1 = Nt_1t_3t_1$ (conjugating relation 2 by $(0,1)(2,3)$).
$N\underline{t_2t_1}t_2 = N\underline{(2,1,3)}t_2t_3t_2 = Nt_2t_3t_2$ (conjugating relation 2 by $(0,2,3)$).

Hence, $Nt_0t_1t_0 = Nt_1t_0t_1 = Nt_2t_1t_2 = Nt_1t_2t_1 = Nt_3t_1t_3 = Nt_1t_3t_1 = Nt_0t_2t_0 = Nt_2t_0t_2 = Nt_0t_3t_0 = Nt_3t_0t_3 = Nt_3t_2t_3 = Nt_2t_3t_2 \in Nt_0t_1t_0N$. There is one unique single coset in [010] with twelve names.

Now we determine for [010] to which coset $Nt_0t_1t_0t_i$ belongs for one t_i from each orbit of $N^{(010)}$ on Ω.

$N^{010} = \{n \in N^{01} | (t_0t_1t_0)^n = t_0t_1t_0\} = \langle(2,3)\rangle$.

$N^{(010)} = \{\pi \in N | N(t_0t_1t_0)^\pi = Nt_0t_1t_0\} = \langle(0,1),(0,2),(0,3)\rangle \cong S_4$ since all of the single cosets within $Nt_0t_1t_0N$ are equal.

So the orbit of $N^{(010)}$ is $\{0,1,2,3\}$ on Ω, i.e., $N^{(010)}$ is transitive on $\Omega = \{0,1,2,3\}$.

$Nt_0t_1t_0t_0 = Nt_0t_1 \in Nt_0t_1N$, so t_0, a representative from the 4-orbit, [010] back to a single coset in [01], and since [0100] = [01], we must have completed the double coset enumeration..

Thus we have the Cayley diagram that is shown in Figure 3.1.

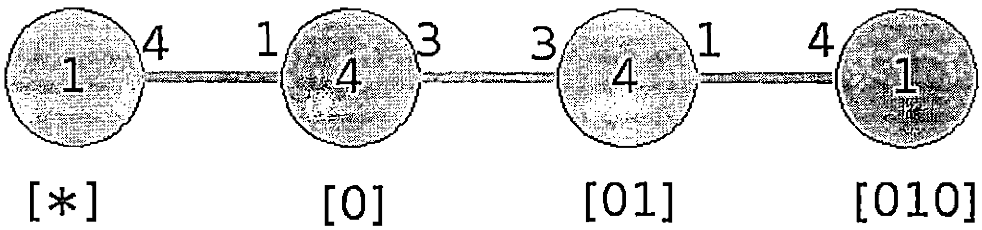

Figure 3.1: Cayley Diagram of $2 \times S_5$ over S_4

The maximum possible index of N in $G \cong \frac{2^{*4}:S_4}{[(0,1,2)t_0]^4}$ is $\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(01)}|} + \frac{|N|}{|N^{(010)}|} = 1+4+4+1 = 10$. Thus $|G| \leq 10 \times |N| = 10 \times 24 = 240$. In order to show $|G| = 240$, we consider G as a subgroup of S_{10} acting on ten cosets that we have found, and labeled as follows:

1. N
2. Nt_0
3. Nt_1
4. Nt_2
5. Nt_3
6. $Nt_0t_1 = Nt_0t_2 = Nt_0t_3$
7. $Nt_1t_0 = Nt_1t_2 = Nt_1t_3$
8. $Nt_2t_0 = Nt_2t_1 = Nt_2t_3$
9. $Nt_3t_0 = Nt_3t_1 = Nt_3t_2$
10. $Nt_0t_1t_0 = Nt_1t_0t_1 = Nt_2t_1t_2 = Nt_1t_2t_1 = Nt_3t_1t_3 = Nt_1t_3t_1 =$
    $Nt_0t_2t_0 = Nt_2t_0t_2 = Nt_0t_3t_0 = Nt_3t_0t_3 = Nt_3t_2t_3 = Nt_2t_3t_2$

For this purpose we compute the action of the control group N as well as the action of t_0, t_1, t_2, and t_3 on the ten cosets. These permutations follow.

t_0 : t_1 :

$(1,2)(3,7)(4,8)(5,9)(6,10)$ $(1,3)(4,8)(5,9)(2,6)(7,10)$

t_2 : t_3 :

$(1,4)(5,9)(2,6)(3,7)(8,10)$ $(1,5)(2,6)(3,7)(4,8)(9,10)$

$x:(2,3,4,5)(6,7,8,9)$ $y:(4,5)(8,9)$

It readily checks that the order of $\langle x,y,t\rangle$, a subgroup of the symmetric group S_{10} acting on the ten right cosets of N in G, is 240. Since the order of xy is three, $N=\langle x,y\rangle \cong S_4$ and t_0 has exactly four conjugates under conjugation by N, we conclude that $\langle x,y,t\rangle$ is a homomorphic image of the progenitor $2^{*4}:S_4$.

Thus if the original five relations hold in $\langle x,y,t\rangle$, then $\langle x,y,t\rangle$ is a homomorphic image of G and this will give $|G|\geq|\langle x,y,t\rangle|=240$.

Verify relation (1) $t_1t_0t_3t_2t_1t_0t_3t_2t_1t_0=(0,2)(1,3)$ by conjugating the four symmetric generators by $t_1t_0t_3t_2t_1t_0t_3t_2t_1t_0$. By multiplying the permutations listed above, we find $t_1t_0t_3t_2t_1t_0t_3t_2t_1t_0=(2,4)(3,5)(6,8)(7,9)$. Thus when we conjugate t_0,t_1,t_2, and t_3 by $t_1t_0t_3t_2t_1t_0t_3t_2t_1t_0$ we obtain:

$t_0^{t_1t_0t_3t_2t_1t_0t_3t_2t_1t_0}=t_2$ $t_1^{t_1t_0t_3t_2t_1t_0t_3t_2t_1t_0}=t_3$

$t_2^{t_1t_0t_3t_2t_1t_0t_3t_2t_1t_0}=t_0$ $t_3^{t_1t_0t_3t_2t_1t_0t_3t_2t_1t_0}=t_1$

So $t_1t_0t_3t_2t_1t_0t_3t_2t_1t_0$ acts as the permutation $(0,2)(1,3)$.

Verify relation (2) $t_0t_2t_1t_0=(0,2,1)$ by conjugating the four symmetric generators.

$t_0^{t_0t_2t_1t_0}=t_2$ $t_1^{t_0t_2t_1t_0}=t_0$

$t_2^{t_0t_2t_1t_0}=t_1$ $t_3^{t_0t_2t_1t_0}=t_3$

So $t_0t_2t_1t_0$ acts as the permutation $(0,2,1)$.

Verify the equivalent relations (3) and (4) $t_1t_0t_1t_0t_1t_0=1$ by conjugating the four symmetric generators.

$t_0^{t_1t_0t_1t_0t_1t_0}=t_0$ $t_1^{t_1t_0t_1t_0t_1t_0}=t_1$

$t_2^{t_1t_0t_1t_0t_1t_0}=t_2$ $t_3^{t_1t_0t_1t_0t_1t_0}=t_3$

So $t_1t_0t_1t_0t_1t_0$ acts as the identity.

Verify relation (5) $t_1t_2t_0t_2t_1t_2t_0t_2=1$ by conjugating the four symmetric generators.

$t_0^{t_1t_2t_0t_2t_1t_2t_0t_2}=t_0$ $t_1^{t_1t_2t_0t_2t_1t_2t_0t_2}=t_1$

$t_2^{t_1t_2t_0t_2t_1t_2t_0t_2}=t_2$ $t_3^{t_1t_2t_0t_2t_1t_2t_0t_2}=t_3$

So $t_1t_2t_0t_2t_1t_2t_0t_2$ acts as the identity.

Thus $G/ker\phi\cong\langle x,y,t\rangle$ and $|G|\geq|\langle x,y,t\rangle|=240$. As shown earlier, $|G|\leq 240$. Hence $|G|=240$.

Moreover, $a = (1,6,7,9,8)(2,5,10,3,4), b = (1,5)(2,9)(3,6)(4,7)(8,10)$ and $c = (1,10)(2,9)(3,6)(4,7)(5,8)$ are in G with $S_5 \times 2 \cong \langle a,b,c\rangle$ since $S_5 \times 2 \cong \{a,b,c|a^5,\ b^2,\ (ab)^4,\ (a^{-1}ab)^2,\ (a^{-2}ba^2b)^2,\ c^2, [c,a], [c,b]\}$. So $\langle a,b,c\rangle \leq G$, but $|\langle a,b,c\rangle| = |G|$, therefore $G = \langle a,b,c\rangle \cong S_5 \times 2$.

Chapter 4

Construction of S_7

Consider the progenitor

$3^{*5} : S_5 \cong \langle x, y, t | x^5 = y^2 = (xy)^4 = [x, y]^3 = 1 = t^3 = [t, y] = [t^x, y] = [t^{x^2}, y] \rangle$, where

$x \sim (0, 1, 2, 3, 4)$

$y \sim (3, 4)$, and

$t \sim t_0$.

The following relations from [Bra97] may be used for the purpose of manual double coset enumeration:

1. $[(0, 1, 2)(3, 4)t_0]^a = [(0, 1, 2)(3, 4)t_0]^{10} = 1$
2. $[(0, 1, 2, 3, 4)t_0]^b = [(0, 1, 2, 3, 4)t_0]^7 = 1$
3. $[(0, 1, 2, 3)t_0]^c = [(0, 1, 2, 3)t_0]^6 = 1$
4. $[(0, 1, 2)t_0]^d = [(0, 1, 2)t_0]^5 = 1$
5. $[(0, 1)t_0]^e = [(0, 1)t_0]^4 = 1$
6. $[t_0^{-1}t_1]^f = [t_0^{-1}t_1]^3 = 1$
7. $[(0, 1, 2)t_0^{-1}t_1]^p = [(0, 1, 2)t_0^{-1}t_1]^2 = 1$
8. $[(0, 1)t_0t_2]^q = [(0, 1)t_0t_2]^6 = 1$
9. $[(0, 1)t_0t_2^{-1}]^r = [(0, 1)t_0t_2^{-1}]^4 = 1$

Based on a computer, it is known, see [Bra97], that the progenitor $3^{*5}:S_5$, factored by relations (1) through (9), although relation (2) suffices, is isomorphic to S_7. We will construct by hand S_7 using the technique of manual double coset enumeration of $G \cong \frac{3^{*5}:S_5}{[(0,1,2,3,4)t_0]^7}$ over S_5.

Expanding relations (4), (5), (6), and (7) gives us:

(4) $[(0,1,2)t_0]^5 = 1$

$(0,1,2)t_0(0,1,2)t_0(0,1,2)t_0(0,1,2)t_0(0,1,2)t_0$

$(0,1,2)^5 t_0{}^{(0,1,2)^4} t_0{}^{(0,1,2)^3} t_0{}^{(0,1,2)^2} t_0{}^{(0,1,2)} t_0$

$(0,2,1)t_1t_0t_2t_1t_0 = 1$

$(0,2,1)t_1t_0t_2 = t_0{}^{-1}t_1{}^{-1}$ (4)

(5) $[(0,1)t_0]^4 = 1$

$(0,1)t_0(0,1)t_0(0,1)t_0(0,1)t_0 = 1$

$(0,1)^4 t_0{}^{(0,1)^3} t_0{}^{(0,1)^2} t_0{}^{(0,1)} t_0 = 1$

$t_1t_0t_1t_0 = 1$

$t_1t_0 = t_0^{-1}t_1^{-1}$ (5)

(6) $[t_0^{-1}t_1]^f = [t_0^{-1}t_1]^3 = 1$

$t_0^{-1}t_1t_0^{-1}t_1t_0^{-1}t_1 = 1$

$t_0^{-1}t_1t_0^{-1} = t_1{}^{-1}t_0t_1{}^{-1} = 1$ (6)

(7) $[(0,1,2)t_0^{-1}t_1]^2 = 1$

$(0,1,2)t_0^{-1}t_1(0,1,2)t_0^{-1}t_1 = 1$

$(0,1,2)^2(t_0^{-1}t_1)^{(0,1,2)}t_0^{-1}t_1 = 1$

$(0,2,1)t_1^{-1}t_2t_0^{-1}t_1 = 1$

$(0,2,1)t_1^{-1}t_2 = t_1^{-1}t_0$ (7)

We now perform the manual double coset enumeration of $G = \frac{3^{*5}:S_5}{[(0,1,2,3,4)t_0]^7}$ over S_5

We start with the double coset representative word of length zero, $NeN = N$, denoted [*].

Next we consider the double cosets with word length one. N is transitive on $T = \{t_0, t_1, t_2, t_3, t_4\} = \{0, 1, 2, 3, 4\}$ and therefore on their inverses $\bar{T} = \{t_0{}^{-1}, t_1{}^{-1}, t_2{}^{-1}, t_3{}^{-1}, t_4{}^{-1}\} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$, thus $Nt_0N = \{Nt_0{}^n | n \in N\} = \{Nt_0, Nt_1, Nt_2, Nt_3,$

$Nt_4\} = \{N0, N1, N2, N3\}$ denoted [0] and $Nt_0^{-1}N = \{N(t_0^{-1})^n | n \in N\} = \{Nt_0^{-1}, Nt_1^{-1}, Nt_2^{-1}, Nt_3^{-1}, Nt_4^{-1}\} = \{N\bar{0}, N\bar{1}, N\bar{2}, N\bar{3}, N\bar{4}\}$ denoted $[\bar{0}]$.

Now we determine for [0] and $[\bar{0}]$ to which coset Nt_0t_i and $Nt_0^{-1}t_i$ belongs for one t_i from each orbit of $N^0 = N^{\bar{0}}$ on T and $\bar{T}$. Since $N^0 = \{n \in N | t_0{}^n = t_0\} = \langle(1.2), (1,3).(1,4)\rangle \cong S_4$, $N^0 = S^4 = N^{\bar{0}}$ has orbits $\{0\}$ and $\{1, 2, 3\}$ on T, and $\{\bar{0}\}$ and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ on $\bar{T}$.

So we need to consider the double cosets $[00], [01], [0\bar{0}], [0\bar{1}], [\bar{0}0], [\bar{0}1], [\overline{00}], [\overline{01}]$.

$00 = \bar{0} \Longrightarrow [00] = [\bar{0}]$, so the generator t_0 takes [0] back to a single coset in $[\bar{0}]$.

$\overline{00} = 0 \Longrightarrow [\overline{00}] = [0]$, so the generator $t_0{}^{-1}$ takes $[\bar{0}]$ to a single coset in [0].

$0\bar{0} = e = \bar{0}0 \Longrightarrow [0\bar{0}] = [*] = [\bar{0}0]$, so the generator $t_0{}^{-1}$ takes [0] to a single coset in [*] and t_0 takes $[\bar{0}]$ to a single coset in [*].

By relation (5), $10 = \overline{01}$, and $10 \in [01]$, so $[\overline{01}] = [01]$.

Thus we need to consider the double cosets $[01], [0\bar{1}], [\bar{0}1]$.

There are four ways, t_1, t_2, t_3, t_4, to go from [0] to single cosets in [01]. Likewise, there are four ways, $t_1{}^{-1}, t_2{}^{-1}, t_3{}^{-1}, t_4{}^{-1}$, to go from $[\bar{0}]$ to single cosets in $[\overline{01}] = [01]$. There are twenty distince single cosets in [01] as listed below.

$$[01] = N01N = \{N01, N02, N03, N04, N10, N12, N13, N14, N20, N21, N23, N24, N30, N31, N32, N34, N40, N41, N42, N43\}$$

$N^{(01)} \geq N^{01} = \{n \in N^0 | (t_0t_1)^n = t_0t_1\} = \langle(2,3), (2,4)\rangle = S_3$

$N^{(01)} = \{\pi \in N | N(t_0t_1)^\pi = Nt_0t_1\} = \langle(2,3), (2,4)\rangle$ since all twenty of the single cosets in [01] are distinct. Thus $N^{(01)}$ has orbits $\{0\}, \{1\}, \{2, 3, 4\}$ and $\{\bar{0}\}, \{\bar{1}\}, \{\bar{2}, \bar{3}, \bar{4}\}$. So we need to analyze the double cosets $[010], [011], [012], [01\bar{0}], [01\bar{1}], [01\bar{2}]$.

Now, $N010 = N\bar{1}$ by relation (5), and $N\bar{1} \in [\bar{0}]$, so $[010] = [\bar{0}]$. So t_0 takes [01] to a single coset in $[\bar{0}]$.

Again, by relation (5), we know $N1 = N\overline{010}$. So $N01\bar{0} = N0\overline{0100} = N\bar{1}0 \in [\bar{0}1]$. So $[01\bar{0}] = [\bar{0}1]$ and $t_0{}^{-1}$ take [01] to single cosets in $[\bar{0}1]$.

$N011 = N0\bar{1}$, since the t_i's are of order three. Hence, $[011] = [0\bar{1}]$ and t_1 takes [01] to a single coset in $[0\bar{1}]$.

$N01\bar{1} = N0$, since $\bar{1} = t_1{}^{-1}$ is the inverse of $1 = t_1$. Hence, $[01\bar{1}] = [0]$ and $t_1{}^{-1}$ take$[01]$ to a single coset in $[0]$.

By relation (4), $(0,2,1)102 = \overline{01}$. If we conjugate this relation by $(0,1)$ we obtain $(1,2,0)012 = \overline{10}$. Thus $N(1,2,0)012 = N\overline{10} \Longrightarrow N012 = N\overline{10} \in [01]$. So $[012] = [01]$. Hence t_2, a representative from the 3-orbit, takes $[01]$ back to a single coset in $[01]$.

By relation (7), $(0,2,1)\bar{1}2 = \bar{1}0$, or equivalently $(0,2,1)2\bar{0} = 2\bar{1}$. If we conjugate this relation by $(1,2)$, we obtain $(0,1,2)1\bar{0} = 1\bar{2}$. It follows that $N0\underline{1\bar{2}} = N0\underline{(0,1,2)1\bar{0}} = N\underline{11}\bar{0} = N\overline{10} \in [01]$. Hence $[01\bar{2}] = [01]$ and $t_2{}^{-1}$ takes $[01]$ back to a single coset in $[01]$.

Now let's consider $[0\bar{1}]$. By relation (7) we know $(0,2,1)2\bar{0} = 2\bar{1}$. If we conjugate this relation by $(2,0)$ we obtain $(2,0,1)0\bar{2} = 0\bar{1}$. Hence $N0\bar{1} = N(2,0,1)0\bar{2} = N0\bar{2}$. Similarly if we conjugate by $(1,3)$ and $(1,4)$ we derive $N0\bar{1} = N0\bar{2} = N0\bar{3} = N0\bar{4}$. We can then conjugate this relation by $(0,1),(0,2),(0,3)$, and $(0,4)$ to obtain the following set of equationss:

$N0\bar{1} = N0\bar{2} = N0\bar{3} = N0\bar{4}$
$N1\bar{0} = N1\bar{2} = N1\bar{3} = N1\bar{4}$
$N2\bar{1} = N2\bar{0} = N2\bar{3} = N2\bar{4}$
$N3\bar{1} = N3\bar{2} = N3\bar{0} = N3\bar{4}$
$N4\bar{1} = N4\bar{2} = N4\bar{3} = N4\bar{0}$

Hence there are five distinct single cosets in $[0\bar{1}]$.

Similarly we can prove there are five distinct single cosets in $[\bar{0}1]$. By relation (7) we know $(0,2,1)\bar{1}2 = \bar{1}0$. If we conjugate this relation by $(0,1)$ we obtain $(1,2,0)\bar{0}2 = \bar{0}1$, and hence $N\bar{0}1 = N(1,2,0)\bar{0}2 = N\bar{0}2$. Similarly if we conjugate by $(1,3)$ and $(1,4)$ we then obtain $N\bar{0}1 = N\bar{0}2 = N\bar{0}3 = N\bar{0}4$. Again we can conjugate this relation by $(0,1),(0,2),(0,3)$, and $(0,4)$ to obtain the following set of equations.

$N\bar{0}1 = N\bar{0}2 = N\bar{0}3 = N\bar{0}4$
$N\bar{1}0 = N\bar{1}2 = N\bar{1}3 = N\bar{1}4$
$N\bar{2}1 = N\bar{2}0 = N\bar{2}3 = N\bar{2}4$
$N\bar{3}1 = N\bar{3}2 = N\bar{3}0 = N\bar{3}4$
$N\bar{4}1 = N\bar{4}2 = N\bar{4}3 = N\bar{4}0$

Hence there are five distinct single cosets in $[\bar{0}1]$

Now let's consider $[\bar{0}1]$ and $[0\bar{1}]$.

$N^{(\bar{0}1)} \geq N^{\bar{0}1} = \langle (2,3),(2,4) \rangle = S_3$
$N^{(\bar{0}1)} = \langle (1,2),(1,3),(1,4) \rangle = S_4$ since $N\bar{0}1 = N\bar{0}2 = N\bar{0}3 = N\bar{0}4$.
So the orbits of $N^{(\bar{0}1)}$ are $\{0\},\{1,2,3,4\},\{\bar{0}\}$, and $\{\bar{1},\bar{2},\bar{3},\bar{4}\}$.

Similarly, $N^{(0\bar{1})} \geq N^{0\bar{1}} = \langle (2,3),(2,4) \rangle = S_3$
$N^{(0\bar{1})} = \langle (1,2),(1,3),(1,4) \rangle = S_4$ since $N0\bar{1} = N0\bar{2} = N0\bar{3} = N0\bar{4}$.
So the orbits of $N^{(0\bar{1})}$ are $\{0\},\{1,2,3,4\},\{\bar{0}\}$, and $\{\bar{1},\bar{2},\bar{3},\bar{4}\}$.

Thus we need to consider $[\bar{0}10], [\bar{0}11], [\bar{0}1\bar{0}], [\bar{0}1\bar{1}], [0\bar{1}0], [0\bar{1}1], [0\bar{1}\bar{0}]$, and $[0\bar{1}\bar{1}]$.

$N\bar{0}1\bar{1} = N\bar{0} \Longrightarrow [\bar{0}1\bar{1}] = [\bar{0}]$. So ${t_1}^{-1}$, a representative from one of the 4-orbits, takes $[\bar{0}1]$ back to a single coset in $[\bar{0}]$.

Similarly, $N0\bar{1}1 = N0 \Longrightarrow [0\bar{1}1] = [0]$. So t_1, a representative from the other 4-orbit, takes $[0\bar{1}]$ to a single coset in $[0]$.

$N\bar{0}11 = N\overline{01} \Longrightarrow [\bar{0}11] = [\overline{01}]$. So t_1 takes $[\bar{0}1]$ to a single coset in $[\overline{01}]$.

Similarly, $N0\overline{11} = N01 \Longrightarrow [0\overline{11}] = [01]$. So ${t_1}^{-1}$ takes $[0\bar{1}]$ to a single coset in $[01]$.

Now by relation (5) we know $10 = \overline{01}$. Hence $\underline{\bar{0}}10 = \overline{\underline{\underline{0}}01} = \underline{0}\bar{1} \Longrightarrow [\bar{0}10] = [0\bar{1}]$. Thus t_0 takes $[\bar{0}1]$ to a single coset in $[0\bar{1}]$.

Similarly, if we conjugate relation (5) by $(0,1)$ we obtain the relation $01 = \overline{10}$. Hence $\underline{0}\overline{10} = \underline{\underline{0}0}1 = \underline{\bar{0}}1 \Longrightarrow [0\overline{10}] = [\bar{0}1]$. Thus ${t_0}^{-1}$ takes $[0\bar{1}]$ to a single coset in $[\bar{0}1]$.

This leaves $[\bar{0}1\bar{0}]$ and $[0\bar{1}0]$ to consider. By relation (5), $1 = \overline{010}$. So it follows $\bar{0}\underline{1}\bar{0} = \overline{0\underline{\underline{0}}100} = \underline{0}\bar{1}0$. So $[\bar{0}1\bar{0}] = [0\bar{1}0]$. Hence ${t_0}^{-1}$ takes $[\bar{0}1]$ to a single coset in $[0\bar{1}0]$, while t_0 takes $[0\bar{1}]$ to a single coset in $[0\bar{1}0]$.

Now we previously proved: $N0\bar{1} = N0\bar{2} = N0\bar{3} = N0\bar{4}$
$N1\bar{0} = N1\bar{2} = N1\bar{3} = N1\bar{4}$
$N2\bar{1} = N2\bar{0} = N2\bar{3} = N2\bar{4}$
$N3\bar{1} = N3\bar{2} = N3\bar{0} = N3\bar{4}$
$N4\bar{1} = N4\bar{2} = N4\bar{3} = N4\bar{0}$

Hence we know:
$N0\bar{1}\underline{0} = N0\bar{2}\underline{0} = N0\bar{3}\underline{0} = N0\bar{4}\underline{0}$
$N1\bar{0}\underline{1} = N1\bar{2}\underline{1} = N1\bar{3}\underline{1} = N1\bar{4}\underline{1}$

$N2\bar{1}\underline{2} = N2\bar{0}\underline{2} = N2\bar{3}\underline{2} = N2\bar{4}\underline{2}$
$N3\bar{1}\underline{3} = N3\bar{2}\underline{3} = N3\bar{0}\underline{3} = N3\bar{4}\underline{3}$
$N4\bar{1}\underline{4} = N4\bar{2}\underline{4} = N4\bar{3}\underline{4} = N4\bar{0}\underline{4}$

By relation (6) we know $\bar{0}1\bar{0} = \bar{1}0\bar{1}$, and hence $1\bar{0}1 = 0\bar{1}0$. If we conjugate this relation by $(1,2),(1,3)$, and $(1,4)$ respectively, we obtain $0\bar{2}0 = 2\bar{0}2, 0\bar{3}0 = 3\bar{0}3$, and $0\bar{4}0 = 4\bar{0}4$. Therefore, $N0\bar{1}0 = N0\bar{2}0 = N0\bar{3}0 = N0\bar{4}0 = N1\bar{0}1 = N1\bar{2}1 = N1\bar{3}1 = N1\bar{4}1 = N2\bar{1}2 = N2\bar{0}2 = N2\bar{3}2 = N2\bar{4}2 = N3\bar{1}3 = N3\bar{2}3 = N3\bar{0}3 = N3\bar{4}3 = N4\bar{1}4 = N4\bar{2}4 = N4\bar{3}4 = N4\bar{0}4$ and $[0\bar{1}0]$ contains one unique single coset.

$N^{(0\bar{1}0)} \geq N^{0\bar{1}0} = \langle (2,3),(2,4) \rangle = S_3$

$N^{(0\bar{1}0)} = \langle (0,2),(1,2),(2,3),(2,4) \rangle = S_5$ since all twenty single coset names are equivalent in $[0\bar{1}0]$. Hence the orbits of $[0\bar{1}0]$ are $\{0,1,2,3,4\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3},\bar{4}\}$. So we need to consider $[0\bar{1}00]$ and $[0\bar{1}0\bar{0}]$.

Now $0\bar{1}00 = 0\bar{1}\bar{0}$. But we previously proved $[0\bar{1}\bar{0}] = [\bar{0}1]$. So t_0, a representative from one of the 5-orbits, takes $[0\bar{1}0]$ back to a single coset in $[\bar{0}1]$.

Also $0\bar{1}\underline{0}\underline{\bar{0}} = 0\bar{1}$, so ${t_0}^{-1}$, a representative from the other 5-orbit, takes $[0\bar{1}0]$ back to a single coset in $[0\bar{1}]$.

Our double coset enumeration must be complete since the set of right cosets is closed under right multiplication by the symmetric generators.

Thus we have the Cayley diagram that is shown in Figure 4.1.

The maximum possible index of N in $G \cong \frac{3^{*5}:S_5}{[(0,1,2,3,4)t_0]^7}$ is $\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(\bar{0})}|} + \frac{|N|}{|N^{(01)}|} + \frac{|N|}{|N^{(0\bar{1})}|} + \frac{|N|}{|N^{(\bar{0}1)}|} + \frac{|N|}{|N^{(0\bar{1}0)}|} = 1+5+5+20+5+5+1 = 42$. Thus $|G| \leq 42 \times |N| = 42 \times 120 = 5040$. In order to show $|G| = 5040$, we consider G as a subgroup of S_{42} acting on 42 cosets that we have found, and labeled as follows:

1.	N	8.	$N\bar{1}$	15.	$N04$	22.	$N23$	29.	$N41$	36.	$N4\bar{0}$
2.	$N0$	9.	$N\bar{2}$	16.	$N10$	23.	$N24$	30.	$N42$	37.	$N\bar{0}1$
3.	$N1$	10.	$N\bar{3}$	17.	$N12$	24.	$N30$	31.	$N43$	38.	$N\bar{1}0$
4.	$N2$	11.	$N\bar{4}$	18.	$N13$	25.	$N31$	32.	$N0\bar{1}$	39.	$N\bar{2}0$
5.	$N3$	12.	$N01$	19.	$N14$	26.	$N32$	33.	$N1\bar{0}$	40.	$N\bar{3}0$
6.	$N4$	13.	$N02$	20.	$N20$	27.	$N34$	34.	$N2\bar{0}$	41.	$N\bar{4}0$
7.	$N\bar{0}$	14.	$N03$	21.	$N21$	28.	$N40$	35.	$N3\bar{0}$	42.	$N0\bar{1}0$

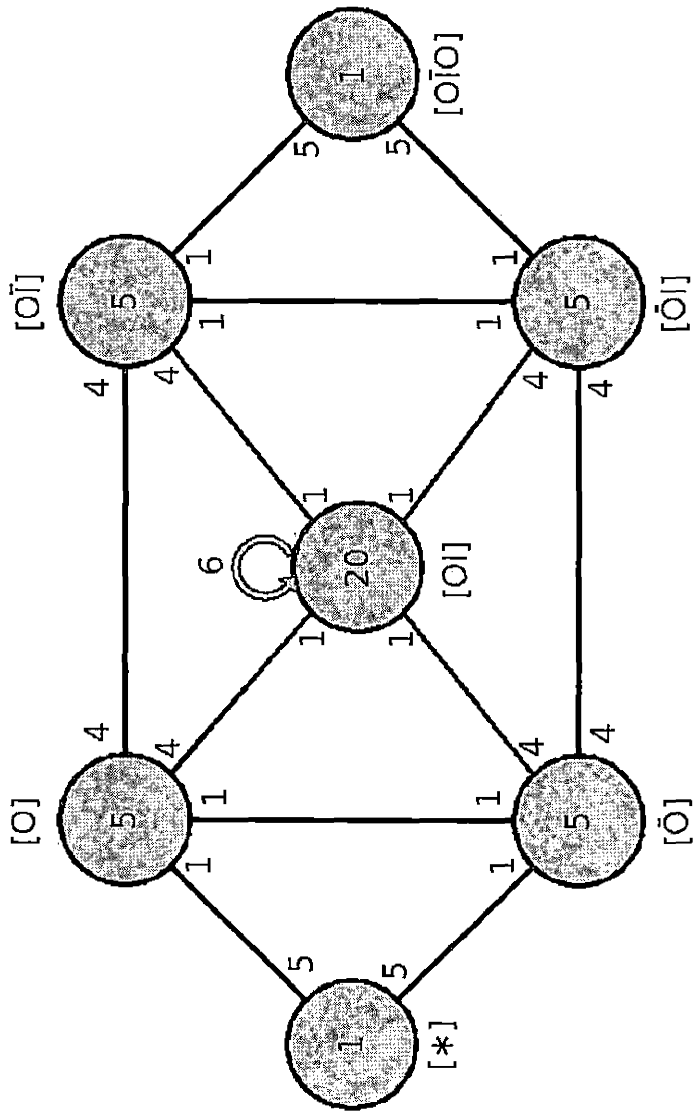

Figure 4.1: Cayley Diagram of S_7 over S_5

For this purpose we compute the action of the control group N as well as the action of t_0, t_1, t_2, t_3, and t_4 on the 42 cosets. These permutations are as follows:

$t_0 : (1, 2, 7)(3, 16, 33)(4, 20, 34)(5, 24, 35)(6, 28, 36)(8, 38, 12)(9, 39, 13)$
$(10, 40, 14)(11, 41, 15)(32, 42, 37)$

$t_1 : (1, 3, 8)(4, 21, 34)(5, 25, 35)(6, 29, 36)(2, 12, 32)(9, 39, 17)(10, 40, 18)$
$(11, 41, 22)(7, 37, 20)(8, 38, 21)(43, 42, 39)$

$t_2 : (1,4,9)(5,26,35)(6,30,36)(2,13,32)(3,17,33)(10,40,22)(11,41,23)$
$(7,37,20)(8,38,21)(34,42,39)$

$t_3 : (1,5,10)(6,31,36)(2,14,32)(3,18,33)(4,22,34)(11,41,27)(7,37,24)$
$(8,38,25)(9,39,26)(35,42,40)$

$t_4 : (1,6,11)(2,15,32)(3,19,33)(4,23,34)(5,27,35)(7,37,28)(8,38,29)$
$(9,39,30)(10,40,31)(36,42,41)$

$x : (2,3,4,5,6)(7,8,9,10,11)(12,17,22,27,28)(13,18,23,24,29)$
$(14,19,20,25,30)(15,16,21,26,31)(32,33,34,35,36)(37,38,39,40,41)$

$y : (5,6)(10,11)(14,15)(18,19)(22,23)(24,28)(25,29)(26,30)(27,31)(35,36)$
$(40,41)$

It readily checks that the order of $\langle x,y,t\rangle$, a subgroup of the symmetric group S_{42} acting on the 42 right cosets of N in G, is 5040. Visibly $|x| = 5$ and $|y| = 2$, additionally $|xy| = 4$ and $[x,y]^3 = 1$, hence $\langle x,y\rangle \cong S_5$. If we conjugate t by S_5 we see that t has exactly five conjugates. We conclude that $\langle x,y,t\rangle$ is a homomorphic image of the progenitor $3^{*5} : S_5$.

Thus if the original nine relations hold in $\langle x,y,t\rangle$, then $\langle x,y,t\rangle$ is a homomorphic image of G and this will give $|G| \geq |\langle x,y,t\rangle| = 5040$.

Verify relation (1) $t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0 = (0,2,1)$ by conjugating the five symmetric generators by $t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0$. By multiplying the permutations listed above, we find $t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0 = (2,4,3)\ (7,9,8)\ (12,20,17)\ (13,21,16)\ (14,22,18)\ (15,23,19)\ (24,26,25)\ (28,30,29)\ (32,34,33)\ (37,39,38)$. Thus when we conjugate t_0, t_1, t_2, t_3 and t_4 by $t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0$ we obtain:

$t_0^{t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_2$ $\quad$ $t_1^{t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_0$
$t_2^{t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_1$ $\quad$ $t_3^{t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_3$
$t_4^{t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_4$

So $t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0$ acts as the permutation $(0,2,1)$.

Verify relation (2) $t_1t_0t_4t_3t_2t_1t_0 = (3,1,4,2,0)$ by conjugating the five symmetric generators.

$t_0^{t_1t_0t_4t_3t_2t_1t_0} = t_3$ $\quad$ $t_1^{t_1t_0t_4t_3t_2t_1t_0} = t_4$
$t_2^{t_1t_0t_4t_3t_2t_1t_0} = t_0$ $\quad$ $t_3^{t_1t_0t_4t_3t_2t_1t_0} = t_1$
$t_4^{t_1t_0t_4t_3t_2t_1t_0} = t_2$

So $t_1t_0t_4t_3t_2t_1t_0$ acts as the permutation $(3,1,4,2,0)$.

Verify relation (3) $t_1t_0t_3t_2t_1t_0 = (0,2)(1,3)$ by conjugating the five symmetric generators.

$t_0^{t_1t_0t_3t_2t_1t_0} = t_2$ $\quad$ $t_1^{t_1t_0t_3t_2t_1t_0} = t_3$
$t_2^{t_1t_0t_3t_2t_1t_0} = t_0$ $\quad$ $t_3^{t_1t_0t_3t_2t_1t_0} = t_1$
$t_4^{t_1t_0t_3t_2t_1t_0} = t_4$

So $t_1t_0t_3t_2t_1t_0$ acts as the permutation $(0,2)(1,3)$.

Verify relation (4) $t_1t_0t_2t_1t_0 = (0,1,2)$.

$t_0^{t_1t_0t_2t_1t_2} = t_2$ $\quad$ $t_1^{t_1t_0t_2t_1t_0} = t_3$
$t_2^{t_1t_0t_2t_1t_0} = t_0$ $\quad$ $t_3^{t_1t_0t_2t_1t_0} = t_1$
$t_4^{t_1t_0t_2t_1t_0} = t_4$

So $t_1t_0t_2t_1t_0$ acts as the permutation $(0,2)(1,3)$.

Verify relation (5) $t_1t_0t_1t_0 = 1$ by conjugating the five symmetric generators.

$t_0^{t_1t_0t_1t_0} = t_0$ $\quad$ $t_1^{t_1t_0t_1t_0} = t_1$
$t_2^{t_1t_0t_1t_0} = t_2$ $\quad$ $t_3^{t_1t_0t_1t_0} = t_3$
$t_4^{t_1t_0t_1t_0} = t_4$

So $t_1t_0t_1t_0$ acts as the identity.

Verify relation (6) $t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1 = 1$ by conjugating the five symmetric generators.

$t_0^{t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1} = t_0$ $\quad$ $t_1^{t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1} = t_1$
$t_2^{t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1} = t_2$ $\quad$ $t_3^{t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1} = t_3$
$t_4^{t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1} = t_4$

So $t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1$ acts as the identity.

Verify relation (7) $t_1{}^{-1}t_2t_0{}^{-1}t_1 = (0,1,2)$ by conjugating the five symmetric generators.

$t_0^{t_1{}^{-1}t_2t_0{}^{-1}t_1} = t_1$ $\quad$ $t_1^{t_1{}^{-1}t_2t_0{}^{-1}t_1} = t_2$
$t_2^{t_1{}^{-1}t_2t_0{}^{-1}t_1} = t_0$ $\quad$ $t_3^{t_1{}^{-1}t_2t_0{}^{-1}t_1} = t_3$
$t_4^{t_1{}^{-1}t_2t_0{}^{-1}t_1} = t_4$

So $t_1{}^{-1}t_2t_0{}^{-1}t_1$ acts as the permutation $(0,1,2)$.

Verify relation (8) $t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2 = 1$ by conjugating the five symmetric generators.

$t_0^{t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2} = t_0$ $\quad$ $t_1^{t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2} = t_1$
$t_2^{t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2} = t_2$ $\quad$ $t_3^{t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2} = t_3$
$t_4^{t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2} = t_4$

So $t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2$ acts as the identity.

Finally, verify relation (9) $t_1t_2^{-1}t_0t_2^{-1}t_1t_2^{-1}t_0t_2^{-1} = 1$ by conjugating the five symmetric generators.

$t_0^{t_1t_2^{-1}t_0t_2^{-1}t_1t_2^{-1}t_0t_2^{-1}} = t_0$ $\quad$ $t_1^{t_1t_2^{-1}t_0t_2^{-1}t_1t_2^{-1}t_0t_2^{-1}} = t_1$

$t_2^{t_1t_2^{-1}t_0t_2^{-1}t_1t_2^{-1}t_0t_2^{-1}} = t_2$ $\quad$ $t_3^{t_1t_2^{-1}t_0t_2^{-1}t_1t_2^{-1}t_0t_2^{-1}} = t_3$

$t_4^{t_1t_2^{-1}t_0t_2^{-1}t_1t_2^{-1}t_0t_2^{-1}} = t_4$

So $t_1t_2^{-1}t_0t_2^{-1}t_1t_2^{-1}t_0t_2^{-1}$ acts as the identity.

Thus $G/ker\phi \cong \langle x, y, t\rangle$ and $|G| \geq |\langle x, y, t\rangle| = 5040$. As shown earlier, $|G| \leq 5040$. Hence $|G| = 5040$.

Moreover, $a = (1,4,22,36,33,19,3)(2,6,27,41,39,24,13)$ (5, 16, 14, 31, 35, 20, 25)

$(7,29,10,8,34,42,32)(9,21,11,38,28,17,18)$(12, 23, 37, 15, 30, 40, 26)

and $b = (5,11)(7,17)(12,24)(13,25)(18,31)(19,32)$ $(26,33)$ $(34,38)$ $(35,41)$ $(36,42)$ $(39,40)$

are in G with $S_7 \cong \langle a, b\rangle$ since $S_7 \cong \{a, b|a^7, b^2, (ab)^6, (a^{-2}(ab)^2)^3, (a^{-2}ba^2b)^2\}$. So $\langle a, b\rangle \leq G$, but $|\langle a, b\rangle| = |G|$, therefore $G = \langle a, b\rangle \cong S_7$.

Chapter 5

Construction of $S_7 \times 3$

Consider the progenitor

$3^{*5} : S_5 \cong \langle x, y, t | x^5 = y^2 = (xy)^4 = [x, y]^3 = 1 = t^3 = [t, y] = [t^x, y] = [t^{x^2}, y] \rangle$, where

$x \sim (0, 1, 2, 3, 4)$

$y \sim (3, 4)$, and

$t \sim t_0$.

The following relations from [Bra97] may be used for the purpose of manual double coset enumeration:

1. $[(0, 1, 2)(3, 4)t_0]^a = [(0, 1, 2)(3, 4)t_0]^{30} = 1$
2. $[(0, 1, 2, 3, 4)t_0]^b = [(0, 1, 2, 3, 4)t_0]^{21} = 1$
3. $[(0, 1, 2, 3)t_0]^c = [(0, 1, 2, 3)t_0]^6 = 1$
4. $[(0, 1, 2)t_0]^d = [(0, 1, 2)t_0]^{15} = 1$
5. $[(0, 1)t_0]^e = [(0, 1)t_0]^{12} = 1$
6. $[t_0^{-1}t_1]^f = [t_0^{-1}t_1]^3 = 1$
7. $[(0, 1, 2)t_0^{-1}t_1]^p = [(0, 1, 2)t_0^{-1}t_1]^2 = 1$
8. $[(0, 1)t_0t_2]^q = [(0, 1)t_0t_2]^6 = 1$
9. $[(0, 1)t_0t_2^{-1}]^r = [(0, 1)t_0t_2^{-1}]^4 = 1$

Based on a computer, it is known, see [Bra97], that the progenitor $3^{*5}:S_5$, factored by relations (1) through (9), although relation (3) suffices, is isomorphic to $S_7 \times 3$. We will construct by hand $S_7 \times 3$ using the technique of manual double coset enumeration of $G \cong \frac{3^{*5}:S_5}{[(0,1,2,3)t_0]^6}$ over S_5.

Expanding relations (3), (6), and (7) gives us:

(3) $[(0,1,2,3)t_0]^6 = 1$

$(0,1,2,3)t_0(0,1,2,3)t_0(0,1,2,3)t_0(0,1,2,3)t_0(0,1,2,3)t_0(0,1,2,3)t_0$

$(0,1,2,3)^6 t_0{}^{(0,1,2,3)^5} t_0{}^{(0,1,2,3)^4} t_0{}^{(0,1,2,3)^3} t_0{}^{(0,1,2,3)^2} t_0{}^{(0,1,2)} t_0$

$(0,2)(1,3)t_1t_0t_3t_2t_1t_0 = 1$

$(0,2)(1,3)t_1t_0t_3 = t_0{}^{-1}t_1{}^{-1}t_2{}^{-1}$ (3)

(6) $[t_0{}^{-1}t_1]^3 = 1$

$t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1 = 1$

$t_0{}^{-1}t_1t_0{}^{-1} = t_1{}^{-1}t_0t_1{}^{-1}$ (6)

(7) $[(0,1,2)t_0{}^{-1}t_1]^2 = 1$

$(0,1,2)^2(t_0{}^{-1}t_1)^{(0,1,2)}t_0{}^{-1}t_1 = 1$

$(0,2,1)t_1{}^{-1}t_2t_0{}^{-1}t_1 = 1$

$(0,2,1)t_1{}^{-1}t_2 = t_1{}^{-1}t_0$

or equivalently $(0,2,1)t_2t_0{}^{-1} = t_2t_1{}^{-1}$ (7)

We now perform the manual double coset enumeration of $G = \frac{3^{*5}:S_5}{[(0,1,2,3)t_0]^6}$ over S_5

We start with the double coset representative word of length zero, $NeN = N$, denoted $[*]$.

Next we consider the double cosets with word length one. N is transitive on $T = \{t_0, t_1, t_2, t_3, t_4\} = \{0,1,2,3,4\}$ and therefore on their inverses $\bar{T} = \{t_0{}^{-1}, t_1{}^{-1}, t_2{}^{-1}, t_3{}^{-1}, t_4{}^{-1}\} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$, thus $Nt_0N = \{Nt_0{}^n | n \in N\} = \{Nt_0, Nt_1, Nt_2, Nt_3, Nt_4\} = \{N0, N1, N2, N3\}$ denoted $[0]$ and $Nt_0^{-1}N = \{N(t_0^{-1})^n | n \in N\} = \{Nt_0^{-1}, Nt_1^{-1}, Nt_2^{-1}, Nt_3^{-1}, Nt_4^{-1}\} = \{N\bar{0}, N\bar{1}, N\bar{2}, N\bar{3}, N\bar{4}\}$ denoted $[\bar{0}]$.

Now we determine for $[0]$ and $[\bar{0}]$ to which coset Nt_0t_i and $Nt_0^{-1}t_i$ belongs for one t_i from each orbit of $N^0 = N^{\bar{0}}$ on T and $\bar{T}$. Since $N^0 = \{n \in N | t_0{}^n = t_0\} =$

$\langle (1.2),(1,3).(1,4)\rangle \cong S_4$, $N^0 = S^4 = N^{\overline{0}}$ has orbits $\{0\}$, and $\{1,2,3\}$ on T, and $\{\overline{0}\}$ and $\{\overline{1},\overline{2},\overline{3},\overline{4}\}$ on $\bar{T}$.

So we need to consider the double cosets $[00],[01],[0\overline{0}],[0\overline{1}],[\overline{0}0],[\overline{0}1],[\overline{0}\overline{0}],[\overline{0}\overline{1}]$.

$00 = \overline{0} \Longrightarrow [00] = [\overline{0}]$, so t_0 takes $[0]$ to a single coset in $[\overline{0}]$.

$\overline{0}\overline{0} = 0 \Longrightarrow [\overline{0}\overline{0}] = [0]$, so $t_0{}^{-1}$ takes $[\overline{0}]$ to a single coset in $[0]$.

$0\overline{0} = e = \overline{0}0 \Longrightarrow [0\overline{0}] = [*] = [\overline{0}0]$, so $t_0{}^{-1}$ takes $[0]$ to a single coset in $[*]$ and t_0 takes $[\overline{0}]$ to a single coset in $[*]$.

Thus we need to consider $[01],[0\overline{1}],[\overline{0}1]$, and $[\overline{0}\overline{1}]$.

t_1, a representative from one of the 4-orbits, takes $[0]$ to a single coset in $[01]$. There are twenty distinct single cosets in $[01]$.

$$[01] = N01N = \{N01,N02,N03,N04,N10,N12,N13,N14,N20,N21,N23,$$
$$N24,N30,N31,N32,N34,N40,N41,N42,N43\}$$

$t_1{}^{-1}$, a representative from the other 4-orbit, takes $[\overline{0}]$ to a single coset in $[\overline{0}\overline{1}]$. There are twenty distinct single cosets in $[\overline{0}\overline{1}]$.

$$[\overline{0}\overline{1}] = N\overline{0}\overline{1}N = \{N\overline{0}\overline{1},N\overline{0}\overline{2},N\overline{0}\overline{3},N\overline{0}\overline{4},N\overline{1}\overline{0},N\overline{1}\overline{2},N\overline{1}\overline{3},N\overline{1}\overline{4},N\overline{2}\overline{0},N\overline{2}\overline{1},N\overline{2}\overline{3},$$
$$N\overline{2}\overline{4},N\overline{3}\overline{0},N\overline{3}\overline{1},N\overline{3}\overline{2},N\overline{3}\overline{4},N\overline{4}\overline{0},N\overline{4}\overline{1},N\overline{4}\overline{2},N\overline{4}\overline{3}\}$$

$t_1{}^{-1}$ takes $[0]$ to a single coset in $[0\overline{1}]$.

By relation (7), $(0,2,1)\overline{1}2 = \overline{1}0$

$\Longrightarrow \underline{2}(0,2,1)\overline{1}2 = \underline{2}\overline{1}0$

$\Longrightarrow (0,2,1)\underline{1\overline{1}}2 = 2\overline{1}0$

$\Longrightarrow (0,2,1)2\underline{\overline{0}} = 2\overline{1}0\underline{\overline{0}}$

$\Longrightarrow (0,2,1)2\overline{0} = 2\overline{1}$

Conjugating by $(0,2)$, we find $(2,0,1)0\overline{2} = 0\overline{1}$

Hence, $N(2,0,1)0\overline{2} = N0\overline{2} = N0\overline{1}$

Conjugating this equation by $(1,3)$ and $(1,4)$ we find $N0\overline{1} = N0\overline{2} = N0\overline{3} = N0\overline{4}$.

Now if we conjugate this equation by $(0,1),(0,2),(0,3)$, and $(0,4)$ respectively we obtain the following set of equations:

$N0\overline{1} = N0\overline{2} = N0\overline{3} = N0\overline{4}$

$N1\overline{0} = N1\overline{2} = N1\overline{3} = N1\overline{4}$

$N2\overline{0} = N2\overline{1} = N2\overline{3} = N2\overline{4}$
$N3\overline{0} = N3\overline{1} = N3\overline{2} = N3\overline{4}$
$N4\overline{0} = N4\overline{1} = N4\overline{2} = N4\overline{3}$

Hence there are five distinct single cosets in $[0\overline{1}]$.

t_1 takes $[\overline{0}]$ to a single coset in $[\overline{0}1]$.

By relation (7), $(0,2,1)\overline{1}2 = \overline{1}0$

Conjugating by $(0,1)$ we obtain $(1,2,0)\overline{0}2 = \overline{0}1$.
$\Longrightarrow N(1,2,0)\overline{0}2 = N\overline{0}2 = N\overline{0}1$

Conjugating this equation by $(1,3)$ and $(1,4)$, we find $N\overline{0}1 = N\overline{0}2 = N\overline{0}3 = N\overline{0}4$.

Now if we conjugate this equation by $(0,1), (0,2), (0,3)$, and $(0,4)$ we obtain the following set of equations:

$N\overline{0}1 = N\overline{0}2 = N\overline{0}3 = N\overline{0}4$
$N\overline{1}0 = N\overline{1}2 = N\overline{1}3 = N\overline{1}4$
$N\overline{2}0 = N\overline{2}1 = N\overline{2}3 = N\overline{2}4$
$N\overline{3}0 = N\overline{3}1 = N\overline{3}2 = N\overline{3}4$
$N\overline{4}0 = N\overline{4}1 = N\overline{4}2 = N\overline{4}3$.

Hence there are five distinct single cosets in $[\overline{0}1]$.

Now let's consider the double cosets $[0\overline{1}]$ and $[\overline{0}1]$.

$N^{(0\overline{1})} \geq N^{0\overline{1}} = \{n \in N^0 | (t_0 t_1{}^{-1})^n = t_0 t_1{}^{-1}\} = \langle (2,3),(2,4)\rangle = S_3$

$N^{(0\overline{1})} = \{\pi \in N | N(t_0 t_1{}^{-1})^\pi = N t_0 t_1{}^{-1}\} = \langle (1,2),(2,3),(2,4)\rangle$ since $N0\overline{1} = N0\overline{2} = N0\overline{3} = N0\overline{4}$. Thus $N^{(0\overline{1})}$ has orbits $\{0\},\{1,2,3,4\}$ and $\{\overline{0}\},\{\overline{1},\overline{2},\overline{3},\overline{4}\}$. So we need to analyze the double cosets $[0\overline{1}0], [0\overline{1}1], [0\overline{1}\overline{0}]$, and $[0\overline{1}\overline{1}]$.

Similarly, $N^{(\overline{0}1)} = \{\pi \in N | N(t_0{}^{-1}t_1)^\pi = N t_0{}^{-1} t_1\} = \langle (1,2),(2,3),(2,4)\rangle$ since $N\overline{0}1 = N\overline{0}2 = N\overline{0}3 = N\overline{0}4$. Thus $N^{(\overline{0}1)}$ has orbits $\{0\},\{1,2,3,4\}$ and $\{\overline{0}\},\{\overline{1},\overline{2},\overline{3},\overline{4}\}$. So we need to analyze the double cosets $[\overline{0}10], [\overline{0}11], [\overline{0}1\overline{0}]$, and $[\overline{0}1\overline{1}]$.

$N0\overline{1}1 = N0$, since $\overline{1} = t_1{}^{-1}$ is the inverse of $1 = t_1$. Hence, $[0\overline{1}1] = [0]$ and t_1, a representative from one of the 4-orbits, takes $[0\overline{1}]$ back to a single coset in $[0]$. Similarly, $N\overline{0}1\overline{1} = N\overline{0}$. Hence, $t_1{}^{-1}$, a representative from the other 4-orbit, takes $[\overline{0}1]$ back to a single coset in $[\overline{0}]$.

$N0\bar{1}\bar{1} = N01$ since the t_i's are of order three. Similarly, $N\bar{0}11 = N\bar{0}\bar{1}$. Hence $[0\bar{1}\bar{1}] = [01]$ and $t_1{}^{-1}$ takes $[0\bar{1}]$ to a single coset in $[01]$. Similarly, $N\bar{0}11 = N\bar{0}\bar{1}$. Hence $[\bar{0}11] = [\bar{0}\bar{1}]$ and t_1 takes $[\bar{0}1]$ to a single coset in $[\bar{0}\bar{1}]$.

Thus we need to consider $[0\bar{1}0], [0\bar{1}\bar{0}], [\bar{0}1\bar{0}]$, and $[\bar{0}10]$.

Let's start with $[0\bar{1}0]$.

By relation (6), $\bar{0}1\bar{0} = \bar{1}0\bar{1}$.

$\Longrightarrow \underline{0}\bar{0}1\bar{0}\underline{1} = \underline{0}\bar{1}0\bar{1}\underline{1}$

$\Longrightarrow 1\bar{0}1 = 0\bar{1}0$

So $N1\bar{0}1 = N0\bar{1}0$. Now if we conjugate this equation by $(0,2),(0,3),(0,4)$, $(1,2),(1,3),(1,4),(0,3)(1,2),(0,4)(1,2)$, and $(0,4)(1,3)$ we obtain the following equations:

$N1\bar{0}1 = N0\bar{1}0$	$N2\bar{1}2 = N1\bar{2}1$
$N3\bar{1}3 = N1\bar{3}1$	$N4\bar{1}4 = N1\bar{4}1$
$N0\bar{2}0 = N2\bar{0}2$	$N0\bar{3}0 = N3\bar{0}3$
$N0\bar{4}0 = N4\bar{0}4$	$N3\bar{2}3 = N2\bar{3}2$
$N4\bar{2}4 = N2\bar{4}2$	$N4\bar{3}4 = N3\bar{4}3$

Additionally, by relation (7) $(0,2,1)\bar{1}2 = \bar{1}0$.

$\Longrightarrow N1\bar{0}1 = N0\underline{\bar{1}0} = N0\underline{(0,2,1)\bar{1}2} = N(0,2,1)2\bar{1}2 = N2\bar{1}2 = N1\bar{2}1$

Hence, $N1\bar{0}1 = N1\bar{0}1 = N2\bar{1}2 = N1\bar{2}1$. If we conjugate this equation by $(0,3),(0,4)$, and $(1,2)$, we derive the following equation:

$N1\bar{0}1 = N1\bar{0}1 = N2\bar{1}2 = N1\bar{2}1 = N1\bar{3}1 = N3\bar{1}3 = N1\bar{4}1 = N4\bar{1}4 = N2\bar{0}2 = N0\bar{2}0$.

If we now conjugate this equation by $(1,3)$ and $(1,4)$ we then obtain the following:

$N1\bar{0}1 = N1\bar{0}1 = N2\bar{1}2 = N1\bar{2}1 = N1\bar{3}1 = N3\bar{1}3 = N1\bar{4}1 = N4\bar{1}4 = N2\bar{0}2 = N0\bar{2}0 = N3\bar{0}3 = N0\bar{3}0 = N4\bar{0}4 = N0\bar{4}0 = N2\bar{3}2 = N3\bar{2}3 = N3\bar{4}3 = N4\bar{3}4 = N2\bar{4}2 = N4\bar{2}4$

Hence $[0\bar{1}0]$ contains one distinct single coset and t_0 takes us from $[0\bar{1}]$ to $[0\bar{1}0]$.

Similarly, we can prove $[\bar{0}1\bar{0}]$ contains one distinct single coset. By relation (6), $\bar{0}1\bar{0} = \bar{1}0\bar{1}$. So $N\bar{0}1\bar{0} = N\bar{1}0\bar{1}$. If we then conjugate by $(0,2),(0,3),(0,4)$,

$(1,2),(1,3),(1,4),(0,3)(1,2),(0,4)(1,2)$, and $(0,4)(1,3)$ we obtain the following equations:

$$N\bar{1}0\bar{1} = N\bar{0}1\bar{0} \qquad N\bar{2}1\bar{2} = N\bar{1}2\bar{1}$$
$$N\bar{3}1\bar{3} = N\bar{1}3\bar{1} \qquad N\bar{4}1\bar{4} = N\bar{1}4\bar{1}$$
$$N\bar{0}2\bar{0} = N\bar{2}0\bar{2} \qquad N\bar{0}3\bar{0} = N\bar{3}0\bar{3}$$
$$N\bar{0}4\bar{0} = N\bar{4}0\bar{4} \qquad N\bar{3}2\bar{3} = N\bar{2}3\bar{2}$$
$$N\bar{4}2\bar{4} = N\bar{2}4\bar{2} \qquad N\bar{4}3\bar{4} = N\bar{3}4\bar{3}$$

Again, by relation (7) $(0,2,1)\bar{1}2 = \bar{1}0$. So $N\bar{0}1\bar{0} = N\underline{\bar{1}0}\bar{1} = N\underline{(0,2,1)\bar{1}2}\bar{1} = N\bar{1}2\bar{1} = N\bar{2}1\bar{2}$. If we conjugate this equation by $(0,3)$, $(0,4)$, and $(1,2)$ we obtain the following equation:

$N\bar{0}1\bar{0} = N\bar{1}0\bar{1} = N\bar{1}2\bar{1} = N\bar{2}1\bar{2} = N\bar{3}1\bar{3} = N\bar{1}3\bar{1} = N\bar{4}1\bar{4} = N\bar{1}4\bar{1} = N\bar{0}2\bar{0} = N\bar{2}0\bar{2}$

If we now conjugate this equation by $(1,3)$ and $(1,4)$, we find $N\bar{0}1\bar{0} = N\bar{1}0\bar{1} = N\bar{1}2\bar{1} = N\bar{2}1\bar{2} = N\bar{3}1\bar{3} = N\bar{1}3\bar{1} = N\bar{4}1\bar{4} = N\bar{1}4\bar{1} = N\bar{0}2\bar{0} = N\bar{2}0\bar{2} = N\bar{0}3\bar{0} = N\bar{3}0\bar{3} = N\bar{3}2\bar{3} = N\bar{2}3\bar{2} = N\bar{4}3\bar{4} = N\bar{3}4\bar{3} = N\bar{0}4\bar{0} = N\bar{4}0\bar{4} = N\bar{4}2\bar{4} = N\bar{2}4\bar{2}$.

Hence $[\bar{0}1\bar{0}]$ contains one distinct single coset and t_0^{-1} takes $[\bar{0}1]$ to a single coset in $[\bar{0}1\bar{0}]$.

Now consider $[0\overline{10}]$ and $[\bar{0}10]$. We already proved that $N0\bar{1} = N0\bar{2} = N0\bar{3} = N0\bar{4}$. It follows that $N0\overline{10} = N0\overline{20} = N0\overline{30} = N0\overline{40}$. If we conjugate by $(0,1)$, $(0,2)$, $(0,3)$, and $(0,4)$, we obtain the following:

$$N0\overline{10} = N0\overline{20} = N0\overline{30} = N0\overline{40}$$
$$N1\overline{21} = N1\overline{31} = N1\overline{41} = N1\overline{01}$$
$$N2\overline{02} = N2\overline{12} = N2\overline{32} = N2\overline{42}$$
$$N3\overline{03} = N3\overline{13} = N3\overline{23} = N3\overline{43}$$
$$N4\overline{04} = N4\overline{14} = N4\overline{24} = N4\overline{34}$$

Hence $[0\overline{10}]$ contains five distinct single cosets and t_0^{-1} takes $[0\bar{1}]$ to a single coset in $[0\overline{10}]$.

Similarly. we can prove $[\bar{0}10]$ contains five distinct cosets. We already proved that $N\bar{0}1 = N\bar{0}2 = N\bar{0}3 = N\bar{0}4$. It follows that $N\bar{0}10 = N\bar{0}20 = N\bar{0}30 = N\bar{0}40$. If we conjugate by $(0,1),(0,2),(0,3)$, and $(0,4)$, we obtain the following:

$$N\bar{0}10 = N\bar{0}20 = N\bar{0}30 = N\bar{0}40$$
$$N\bar{1}21 = N\bar{1}31 = N\bar{1}41 = N\bar{1}01$$

$N\overline{2}02 = N\overline{2}12 = N\overline{2}32 = N\overline{2}42$

$N\overline{3}03 = N\overline{3}13 = N\overline{3}23 = N\overline{3}43$

$N\overline{4}04 = N\overline{4}14 = N\overline{4}24 = N\overline{4}34$

Hence $[\overline{0}10]$ contains five distinct single cosets and t_0 takes $[\overline{0}1]$ to a single coset in $[\overline{0}10]$.

Now consider $[0\overline{1}0]$ and $[\overline{0}1\overline{0}]$. $N^{(0\overline{1}0)} \geq N^{0\overline{1}0} = \{n \in N^{0\overline{1}} | (t_0 t_1{}^{-1} t_0)^n = t_0 t_1{}^{-1} t_0\} = \langle(2,3),(2,4)\rangle = S_3$

$N^{(0\overline{1}0)} = \{\pi \in N | N(t_0 t_1{}^{-1} t_0)^\pi = N t_0 t_1{}^{-1} t_0\} = \langle(0,2),\ (1,2),\ (2,3),\ (2,4)\rangle$ since all the single cosets in $[0\overline{1}0]$ are equal. Thus $N^{(0\overline{1}0)}$ has orbits $\{0,\ 1,\ 2,\ 3,\ 4\}$ and $\{\overline{0},\ \overline{1},\ \overline{2},\ \overline{3},\ \overline{4}\}$. So we need to analyze the double cosets $[0\overline{1}00]$, and $[0\overline{1}0\overline{0}]$.

Similarly $N^{(\overline{0}1\overline{0})} \geq N^{\overline{0}1\overline{0}} = \{n \in N^{\overline{0}1} |\ (t_0{}^{-1} t_1 t_0{}^{-1})^n = t_0{}^{-1} t_1 t_0{}^{-1}\} = \langle(2,3), (2,4)\rangle = S_3$

$N^{(\overline{0}1\overline{0})} = \{\pi \in N | N(t_0{}^{-1} t_1 t_0{}^{-1})^\pi = N t_0{}^{-1} t_1 t_0{}^{-1}\} = \langle(0,2),(1,2),(2,3),(2,4)\rangle$ since all the single cosets in $[\overline{0}1\overline{0}]$ are equal. Thus $N^{(\overline{0}1\overline{0})}$ has orbits $\{0,1,2,3,4\}$ and $\{\overline{0},\overline{1},\overline{2},\overline{3},\overline{4}\}$. So we need to analyze the double cosets $[\overline{0}1\overline{0}0]$, and $[\overline{0}1\overline{0}\overline{0}]$.

$N0\overline{1}\underline{00} = N0\overline{1}$. So $t_0{}^{-1}$, a representative from one the 4-orbits takes $[0\overline{1}0]$ back to a single coset in $[0\overline{1}]$. Similarly, $N\overline{0}1\underline{00} = N\overline{0}1$. So t_0, a representative from the other 4-orbit takes $[\overline{0}1\overline{0}]$ back to a single coset in $[\overline{0}1]$.

$N0\overline{1}00 = N0\overline{1}\overline{0}$. So t_0 takes $[0\overline{1}0]$ to a single coset in $[0\overline{1}\overline{0}]$. Similarly, $N\overline{0}1\overline{0}\overline{0} = \overline{0}10$. So $t_0{}^{-1}$ takes $[\overline{0}1\overline{0}]$ to a single coset in $[\overline{0}10]$.

Now let's consider $[01]$ and $[\overline{01}]$.

$N^{(01)} \geq N^{01} = \{n \in N^0 | (t_0 t_1)^n = t_0 t_1\} = \langle(2,3),(2,4)\rangle = S_3$

$N^{(01)} = \{\pi \in N | N(t_0 t_1)^\pi = N t_0 t_1\} = \langle((2,3),(2,4)\rangle$ since all the single cosets in $[01]$ are unique. Thus $N^{(01)}$ has orbits $\{0\},\{1\},\{2,3,4\},\{\overline{0}\},\{\overline{1}\}$ and $\{\overline{1},\overline{2},\overline{3},\overline{4}\}$.

Similarly,$N^{(\overline{01})} \geq N^{\overline{01}} = \{n \in N^{\overline{0}} | (t_0{}^{-1-} t_1{}^{-1})^n = t_0{}^{-1} t_1{}^{-1}\} = \langle(2,3),(2,4)\rangle = S_3$

$N^{(\overline{01})} = \{\pi \in N | N(t_0{}^{-1} t_1{}^{-1})^\pi = N t_0{}^{-1} t_1{}^{-1}\} = \langle((2,3),(2,4)\rangle$ since all the single cosets in $[\overline{01}]$ are unique. Thus $N^{(\overline{01})}$ has orbits $\{0\},\{1\},\{2,3,4\},\{\overline{0}\},\{\overline{1}\}$ and $\{\overline{1},\overline{2},\overline{3},\overline{4}\}$.

So we need to analyze the double cosets $[010],[011],[012],[01\overline{0}],[01\overline{1}],[01\overline{2}],[\overline{01}0],[\overline{01}1],[\overline{01}2],[\overline{010}],[\overline{011}]$, and $[\overline{012}]$.

$01\bar{1} = 0$, so $[01\bar{1}] = [0]$. Hence $t_1{}^{-1}$ takes $[01]$ back to a single coset in $[0]$.
$\overline{01}1 = \bar{0}$, so $[\overline{01}1] = [\bar{0}]$. Hence t_1 takes $[\overline{01}]$ back to a single coset in $[\bar{0}]$.
$011 = 0\bar{1}$, so $[011] = [0\bar{1}]$. Hence t_1 takes $[01]$ to a single coset in $[0\bar{1}]$.
$\overline{01}\bar{1} = \bar{0}1$, so $[\overline{01}\bar{1}] = [\bar{0}1]$. Hence $t_1{}^{-1}$ takes $[\overline{01}]$ to a single coset in $[\bar{0}1]$.

We previously proved using relation (7) $(2,0,1)0\bar{2} = 0\bar{1}$. If we conjugate this relation by $(0,1,2)$, we derive $N(0,1,2)1\bar{0} = N1\bar{2}$. So it follows $N0\underline{1\bar{2}} = N0\underline{(0,1,2)1\bar{0}} = N(0,1,2)11\bar{0} = N\underline{11}\bar{0} = N\overline{10} \in [\overline{01}]$. Hence $[01\bar{2}] = [\overline{01}]$. So $t_2{}^{-1}$, a representative from one of the 3-orbits, takes $[01]$ to a single coset in $[\overline{01}]$.

Similarly, by relation (7) $(0,2,1)\bar{1}2 = \bar{1}0$, equivalently $\bar{1}2 = (0,1,2)\bar{1}0$. So if follows $N\bar{0}\underline{\bar{1}2} = N\bar{0}\underline{(0,1,2)\bar{1}0} = N(0,1,2)\bar{1}\bar{1}0 = N(0,1,2)10 = N10 \in [01]$. Hence $[\overline{01}2] = [01]$ and t_2, a representative from the other 3-orbit, takes $[\overline{01}]$ to a singel coset in $[01]$.

This leaves $[010], [01\bar{0}], [012], [\overline{01}0], [\overline{010}]$, and $[\overline{012}]$ to consider.

t_2 takes $[01]$ to a single coset in $[012]$.

By relation (3), $(0,2)(1,3)103 = \overline{012}$. So $N\overline{012} = N(0,2)(1,3)103 = N103 \in [012]$. Hence $[\overline{012}] = [012]$. So $t_2{}^{-1}$ takes $[\overline{01}]$ to a single coset in $[012]$.

t_0 takes $[01]$ to a single coset in $[010]$. Likewise, $t_0{}^{-1}$ takes $[\overline{01}]$ to a single coset in $[\overline{010}]$. $t_0{}^{-1}$ takes $[01]$ to a single coset in $[01\bar{0}]$. Also, t_0 takes $[\overline{01}]$ to a single coset in $[\overline{01}0]$

Now let's consider the double coset $[012]$.

By relation (3), $(0,2)(1,2)103 = \overline{012}$.
$\Longrightarrow N103 = N\underline{(0,2)(1,2)103} = N\underline{\overline{012}}$
$\Longrightarrow N103 = N\overline{01}\underline{\bar{2}} = N\overline{01}\underline{22}$
$\Longrightarrow N103 = N\bar{0}\underline{\bar{1}2}2 = N\bar{0}\underline{(0,1,2)\bar{1}}02$ (by relation 7)
$\Longrightarrow N103 = N(0,1,2)\bar{1}\bar{1}02 = N(0,1,2)102 = N102$

So $N103 = N102$. If we conjugate this equation by $(2,3)$ we derive $N103 = N102 = N104$. If we now conjugate this equation by $(1,0)$, $(0,2)$, $(0,3)$, $(0,4)$, $(1,2)$, $(1,2)$, $(1,4)$, $(1,0,2)$, $(1,0,3)$, $(1,0,4)$, $(1,2,0)$, $(1,2,0,3)$, $(1,2,0,4)$, $(1,3,0)$, $(1,3,0,2)$, $(1,3,0,4)$, $(1,4,0)$, $(1,4,0,2)$, and $(1,4,0,3)$, we find the following equations:

$N103 = N102 = N104$ $\qquad$ $N012 = N013 = N014$

$N120 = N123 = N124$	$N132 = N130 = N134$
$N142 = N143 = N140$	$N201 = N203 = N204$
$N302 = N301 = N304$	$N402 = N403 = N401$
$N021 = N023 = N024$	$N032 = N031 = N034$
$N042 = N043 = N041$	$N210 = N213 = N214$
$N231 = N230 = N234$	$N241 = N243 = N240$
$N312 = N310 = N314$	$N321 = N320 = N324$
$N342 = N340 = N341$	$N412 = N413 = N410$
$N421 = N423 = N420$	$N432 = N431 = N430$

Hence [012] has twenty distinct single cosets.

$N^{(012)} \geq N^{012} = \{n \in N^{01} | (t_0t_1t_2)^n = t_0t_1t_2\} = \langle(3,4)\rangle = S_2$

$N^{(012)} = \{\pi \in N | N(t_0t_1t_2)^\pi = Nt_0t_1t_2\} = \langle(2,3),(2,4)\rangle$ since $N012 = N013 = N014$. Thus $N^{(012)}$ has orbits $\{0\},\{1\},\{2,3,4\},\{\overline{0}\},\{\overline{1}\}$, and $\{\overline{2},\overline{3},\overline{4}\}$. So we need to analyze the double cosets $[0120],[0121],[0122],[012\overline{0}],[012\overline{1}]$, and $[012\overline{2}]$.

$N012\overline{2} = N01 \implies [012\overline{2}] = [01]$. Hence $t_2{}^{-1}$, a representative from one of the 3-orbits, takes [012] to a single coset in [01].

$[0122] = [01\overline{2}] = [\overline{01}]$ as proved previously. Hence t_2, a representative from the other 3-orbit, takes [012] to a single coset in $[\overline{01}]$

Now by relation (7) $(0,2,1)\overline{1}2 = \overline{1}0$

$\implies \underline{2}(0,2,1)\overline{1}2 = \underline{2}\overline{1}0$

$\implies (0,2,1)\underline{1\overline{1}}2 = 2\overline{1}0$

$\implies (0,2,1)2 = 2\overline{1}0$

$\implies (0,2,1)2\underline{\overline{0}} = 2\overline{1}\underline{\underline{0\overline{0}}}$

$\implies (0,2,1)2\overline{0} = 2\overline{1}$

$\implies 2\overline{0} = (0,1,2)2\overline{1}$

So it follows $N01\underline{2\overline{0}} = N01\underline{(0,1,2)2\overline{1}} = N1\underline{22}\overline{1} = N1\overline{2}\overline{1} \in [0\overline{1}\overline{0}]$. So $[012\overline{0}] = [0\overline{1}\overline{0}]$ and $t_0{}^{-1}$ takes [012] to a single coset in $[0\overline{1}\overline{0}]$.

We previously proved by relation (7) that $(0,1,2)2\overline{1} = 2\overline{0}$ and equivalently $2\overline{1} = (0,2,1)2\overline{0}$. So it follows that $N012\overline{1} = N01(1,0,2)2\overline{0} = N(1,0,2)202\overline{0} = N202\overline{0} \in [010\overline{1}]$. So it follows that $[012\overline{1}] = [010\overline{1}]$. Thus, $t_1{}^{-1}$, takes [012] to a single coset in $[010\overline{1}]$.

Now let's consider [0120].

By relation (6) $\bar{0}1\bar{0} = \bar{1}0\bar{1}$.

$\Longrightarrow N\bar{0}1\underline{\bar{0}0} = N\bar{1}0\bar{1}\underline{0}$

$\Longrightarrow N\bar{0}1\underline{1} = N\bar{1}0\bar{1}0\underline{1}$

$\Longrightarrow N\bar{0}1\underline{\bar{0}1} = N\bar{1}0\bar{1}01\underline{\bar{0}1}$

$\Longrightarrow N\overline{010}1 = N\bar{1}0\bar{1}0\underline{1\bar{0}}1$

$\Longrightarrow N\overline{010}1 = N\bar{1}0\bar{1}0\underline{(0,2,1)1\bar{2}}1$

Since relation (7) conjugated by $(1,2)$ is $1\bar{0} = (0,2,1)1\bar{2}$.

$\Longrightarrow N\overline{010}1 = N(0,2,1)\bar{0}2\underline{\bar{0}2}1\bar{2}1$

$\Longrightarrow N\overline{010}1 = N(0,2,1)\bar{0}2\underline{(1,0,2)\bar{0}1}1\bar{2}1$

Since relation (7) conjugated by $(0,1)$ is $\bar{0}2 = (1,0,2)\bar{0}1$.

$\Longrightarrow N\overline{010}1 = N(0,2,1)(1,0,2)\bar{2}1\bar{0}\underline{11}\bar{2}1 = N(0,1,2)\bar{2}1\underline{\bar{0}\bar{1}\bar{2}}1$

$\Longrightarrow N\overline{010}1 = N(0,1,2)\bar{2}1\underline{(0,2)(1,3)103}1$ by relation (3)

$\Longrightarrow N\overline{010}1 = N(0,1,2)(0,2)(1,3)\underline{\bar{0}3}1031$

$\Longrightarrow N\overline{010}1 = N(0,3,1)\underline{(3,2,0)\bar{0}2}1031$

Since relation (7) conjugated by $(1,0,3)$ is $(3,2,0)\bar{0}2 = \bar{0}3$.

$\Longrightarrow N\overline{010}1 = N(0,2)(3,1)\bar{0}21\underline{031}$

$\Longrightarrow N\overline{010}1 = N(0,2)(3,1)\bar{0}21\underline{(3,2)(0,1)\overline{302}}$

Since relation (3) conjugated by $(1,0,3)$ is $031 = (3,2)(0,1)\overline{302}$.

$\Longrightarrow N\overline{010}1 = N(0,2)(3,1)(3,2)(0,1)\bar{1}3\underline{0\overline{3}}\overline{02}$

$\Longrightarrow N\overline{010}1 = N(0,3)(1,2)\bar{1}3\underline{(2,0,3)0\overline{2}}\overline{02}$

Since relation (7) conjugated by $(2,0,3)0\bar{2} = 0\bar{3}$.

$\Longrightarrow N\overline{010}1 = N(0,3)(1,2)(2,0,3)\underline{\bar{1}2}0\overline{202}$

$\Longrightarrow N\overline{010}1 = N(0,2,1)\underline{(0,1,2)\bar{1}0}0\overline{202}$ by relation (7).

$\Longrightarrow N\overline{010}1 = N(0,2,1)(0,1,2)\bar{1}\underline{00}\overline{202}$

$\Longrightarrow N\overline{010}1 = N\overline{1\underline{0}202}$

$\Longrightarrow N\overline{010}1 = N\overline{\underline{102}02}$

$\Longrightarrow N\overline{010}1 = N\underline{(1,2)(0,3)013}\overline{02}$

Since relation (3) conjugated by $(0,1)$ is $(1,2)(0,3)013 = \overline{102}$.

$\Longrightarrow N\overline{010}1 = N(1,2)(0,3)01\underline{3\overline{02}}$

$\Longrightarrow N\overline{010}1 = N(1,2)(0,3)01\underline{(0,1,3)3\overline{1}}\overline{2}$

Since relation (7) conjugated by $(3,2)$ is $(0,1,3)3\bar{1} = 3\bar{0}$.

$\Longrightarrow N\bar{0}1\bar{0}1 = N(1,2)(0,3)(0,1,3)1\underline{3\bar{3}}\bar{1}\bar{2} = N(1,2,3)1\underline{\bar{3}\bar{1}\bar{2}}$

$\Longrightarrow N\bar{0}1\bar{0}1 = N(1,2,3)1\underline{(3,2)(1,0)130}$

Since relation (3) conjugated by $(0,3)$ is $(3,2)(1,0)130 = \bar{3}\bar{1}\bar{2}$.

$\Longrightarrow N\bar{0}1\bar{0}1 = N(1,2,3)(3,2)(1,0)0130 = N(1,3,0)0130 = N0130 \in [0120]$

$\Longrightarrow [0120] = [\bar{0}1\bar{0}1]$

Hence, t_0 takes $[012]$ to a single coset in $[\bar{0}1\bar{0}1]$.

As proved above, $\bar{0}1\bar{0}1 = (1,3,0)0130$.

If we conjugate this relation by $(1,2)$ we obtain $\bar{0}1\bar{0}1 = (1,2,0)0120$

$\Longrightarrow N\bar{0}1\bar{0}1 = N(1,2,0)0120$

$\Longrightarrow N\bar{0}1\bar{0}1\underline{\bar{0}} = N(1,2,0)012\underline{0\bar{0}} = N(1,2,0)012$

$\Longrightarrow N\bar{0}1\bar{0}1\bar{0}\underline{1} = N(1,2,0)012\underline{1}$

$\Longrightarrow N\bar{0}1\bar{0}\underline{1\bar{0}1} = N(1,2,0)0121$

$\Longrightarrow N\bar{0}1\bar{0}\underline{0\bar{1}\bar{0}} = N(1,2,0)0121$ by relation (6)

$\Longrightarrow N\bar{0}1\underline{\bar{0}0}\bar{1}\bar{0} = N(1,2,0)0121$

$\Longrightarrow N\bar{0}\underline{1\bar{1}}0 = N(1,2,0)0121$

$\Longrightarrow N\bar{0}10 = N(1,2,0)0121 = N0121$

$\Longrightarrow [\bar{0}10] = [0121]$

Hence t_1 takes $[012]$ to a single coset in $[\bar{0}10]$.

Now let's consider $[0\bar{1}\bar{0}]$ and $[\bar{0}10]$.

$N^{(0\bar{1}\bar{0})} \geq N^{0\bar{1}\bar{0}} = \{n \in N^{0\bar{1}} | (t_0{t_1}^{-1}{t_0}^{-1})^n = t_0{t_1}^{-1}{t_0}^{-1}\} = \langle(2,3),(2,4)\rangle = S_3$.

$N^{(0\bar{1}\bar{0})} = \{\pi \in N | N(t_0{t_1}^{-1}{t_0}^{-1})^\pi = Nt_0{t_1}^{-1}{t_0}^{-1}\} = \langle(1,2),(2,3),(2,4)\rangle = S_4$ since $N0\bar{1}\bar{0} = N0\bar{2}\bar{0} = N0\bar{3}\bar{0} = N0\bar{4}\bar{0}$. Thus $N^{(0\bar{1}\bar{0})}$ has orbits $\{0\}, \{1,2,3,4\}, \{\bar{0},\}$, and $\{\bar{1},\bar{2},\bar{3},\bar{4}\}$.

Similarly $N^{(\bar{0}10)} \geq N^{\bar{0}10} = \{n \in N^{\bar{0}1} | ({t_0}^{-1}t_1t_0)^n = {t_0}^{-1}t_1t_0\} = \langle(2,3),(2,4)\rangle = S_3$.

$N^{(\bar{0}10)} = \{\pi \in N | N({t_0}^{-1}t_1t_0)^\pi = N{t_0}^{-1}t_1t_0\} = \langle(1,2),(2,3),(2,4)\rangle = S_4$ since $N\bar{0}10 = N\bar{0}20 = N\bar{0}30 = N\bar{0}40$. Thus $N^{(\bar{0}10)}$ has orbits $\{0\}, \{1,2,3,4\}, \{\bar{0},\}$, and $\{\bar{1},\bar{2},\bar{3},\bar{4}\}$.

So we need to analyze the double cosets $[0\bar{1}\bar{0}0]$, $[0\bar{1}\bar{0}1]$, $[0\bar{1}\bar{0}\bar{0}]$, $[0\bar{1}\bar{0}\bar{1}]$, $[\bar{0}100]$, $[\bar{0}101]$, $[\bar{0}10\bar{0}]$, and $[\bar{0}10\bar{1}]$.

$[0\bar{1}\bar{0}0] = [0\bar{1}] \Longrightarrow t_0$ takes $[0\bar{1}\bar{0}]$ back to a single coset in $[0\bar{1}]$.

$[\bar{0}10\bar{0}] = [\bar{0}1] \implies t_0{}^{-1}$ takes $[\bar{0}10]$ back to a single coset in $[\bar{0}1]$.
$[0\bar{1}\bar{0}\bar{0}] = [0\bar{1}0] \implies t_0{}^{-1}$ takes $[0\bar{1}\bar{0}]$ to a single coset in $[0\bar{1}0]$.
$[\bar{0}100] = [\bar{0}1\bar{0}] \implies t_0$ takes us from $[\bar{0}10]$ to $[\bar{0}1\bar{0}]$.

Now let's consider $[0\bar{1}\bar{0}1]$.

By relation (6), $\bar{0}1\bar{0} = \bar{1}0\bar{1}$
$\implies N\underline{\underline{0\bar{0}}}1\bar{0} = N\underline{0}\bar{1}0\bar{1}$
$\implies N1\bar{0} = N0\bar{1}0\bar{1}$
$\implies N1\bar{0}\underline{1} = N0\bar{1}0\underline{\underline{\bar{1}1}}$
$\implies N1\bar{0}1 = N0\bar{1}0$
$\implies N1\bar{0}1\underline{0} = N0\bar{1}\underline{\underline{00}}$
$\implies N1\bar{0}10 = N0\bar{1}\bar{0}$
$\implies N1\bar{0}101 = N0\bar{1}\bar{0}1$
$\implies N0\bar{1}\bar{0}1 = N1\underline{\bar{0}1}01 = N1\underline{(1,2,0)\bar{0}2}01$

Since relation (7) conjugated by $(0,1)$ is $(1,2,0)\bar{0}2 = \bar{0}1$.
$\implies N0\bar{1}\bar{0}1 = N(1,2,0)2\bar{0}201$
$\implies N0\bar{1}\bar{0}1 = N(1,2,0)\underline{2\bar{0}2}01$
$\implies N0\bar{1}\bar{0}1 = N(1,2,0)\underline{0\bar{2}0}01$

Since relation (6) conjugated by $(0,2)$ is $2\bar{0}2 = 0\bar{2}0$.
$\implies N0\bar{1}\bar{0}1 = N(1.2.0)0\bar{2}\underline{00}1$
$\implies N0\bar{1}\bar{0}1 = N(1,2,0)0\bar{2}\underline{\underline{01}}$
$\implies N0\bar{1}\bar{0}1 = N(1,2,0)0\bar{2}\underline{\bar{0}1}$
$\implies N0\bar{1}\bar{0}1 = N(1,2,0)0\bar{2}\underline{(1,2,0)\bar{0}2}$

Since relation (7) conjugated by $(0,1)$ is $(1,2,0)\bar{0}2 = \bar{0}1$.
$\implies N0\bar{1}\bar{0}1 = N(1,0,2)1\underline{\bar{0}0}2$
$\implies N0\bar{1}\bar{0}1 = N(1,0,2)102 = N102 \in [012]$

Hence $[0\bar{1}\bar{0}1] = [012]$ and t_1, t_2, t_3, and t_4 take us from $[0\bar{1}\bar{0}]$ to $[012]$.

Similarly, we can prove$[\bar{0}10\bar{1}] = [012]$, since $\bar{0}1\bar{0} = \bar{1}0\bar{1}$ by relation (6).
$\implies N\bar{0}1\underline{\underline{\bar{0}0}} = N\bar{1}0\bar{1}\underline{0}$
$\implies N\bar{0}1\underline{0\bar{1}} = N\bar{1}0\bar{1}0\underline{0\bar{1}}$
$\implies N\bar{0}10\bar{1} = N\bar{1}0\bar{1}\underline{00}\bar{1} = N\bar{1}0\overline{101}$
$\implies N\bar{0}10\bar{1} = N\bar{1}0\overline{10}\underline{\bar{0}0}\bar{1}$

$\Longrightarrow N\bar{0}10\bar{1} = N\bar{1}0\bar{1}0\underline{0\bar{1}} = N\bar{1}0\bar{1}0\underline{(2,0,1)0\bar{2}}$

Since $0\bar{1} = (2,0,1)0\bar{2}$ relation (7) conjugated by $(0,2)$.

$\Longrightarrow N\bar{0}10\bar{1} = N(2,0,1)\bar{2}1\bar{2}10\bar{2}$

$\Longrightarrow N\bar{0}10\bar{1} = N(2,0,1)\underline{\bar{1}2\bar{1}}10\bar{2}$

Since $\bar{2}1\bar{2} = \bar{1}2\bar{1}$, relation (6) conjugated by $(0,2)$.

$\Longrightarrow N\bar{0}10\bar{1} = N(2,0,1)\bar{1}2\underline{\bar{1}1}0\bar{2} = N(2,0,1)\underline{\bar{1}2}0\bar{2}$

$\Longrightarrow N\bar{0}10\bar{1} = N(2,0,1)\underline{(0,1,2)\bar{1}0}0\bar{2}$

Since $\bar{1}2 = (0,1,2)\bar{1}0$, relation (7).

$\Longrightarrow N\bar{0}10\bar{1} = N(2,0,1)(0,1,2)\overline{102} = N(2,1,0)\underline{\overline{102}}$

$\Longrightarrow N\bar{0}10\bar{1} = N(2,1,0)\underline{(1,2)(0,3)013} = N(1,3,0)013 = N013$

Since $(1,2)(0,3)013 = \overline{102}$, relation(3) conujugated by $(0,1)$.

$\Longrightarrow N\bar{0}10\bar{1} = N013 \in [012]$

Hence $[\bar{0}10\bar{1}] = [012]$, and $t_1{}^{-1}$, a reprentative from one of the 4-orbits takes $[\bar{0}10]$ to a single coset in $[012]$.

Now let's consider $[0\overline{101}]$.

As proven below, $\overline{01}\bar{0}10 = 1\overline{01}$.

$\Longrightarrow \overline{010}1\underline{0\bar{0}} = 1\overline{01}\underline{\bar{0}}$

$\Longrightarrow \overline{01}01 = 1\overline{010}$

$\Longrightarrow N\overline{01}01 = N1\overline{010} \in [0\overline{101}]$

$\Longrightarrow [\overline{01}01] = [0\overline{101}]$

Hence $[0\overline{101}] = [\overline{01}01]$, and $t_1{}^{-1}$ takes $[0\overline{10}]$ to a single coset in $[\overline{01}01]$.

Let's consider $[\bar{0}101]$.

As proven previously, we know $\overline{01}01 = 1\overline{010}$.

$\Longrightarrow \overline{010}\underline{1\bar{1}} = 1\overline{0101}\underline{\bar{1}}$.

$\Longrightarrow \overline{01}\underline{0\bar{0}} = 1\overline{0101}\underline{\bar{0}}$.

$\Longrightarrow \bar{0}\underline{\bar{1}1} = 1\overline{01010}\underline{1}$.

$\Longrightarrow \bar{0}\underline{101} = 1\overline{0101}01\underline{101}$.

$\Longrightarrow \bar{0}101 = 1\overline{0101}0\underline{11}01$.

$\Longrightarrow \bar{0}101 = 1\overline{010}\underline{\overline{10}}\bar{1}01$.

$\Longrightarrow \bar{0}101 = 1\overline{01}0\underline{01}\bar{0}01$ by relation (6).

$\Longrightarrow \bar{0}101 = 1\overline{01}\underline{00}1\underline{\bar{0}0}1 = 1\overline{01}0\underline{11} = 1\bar{0}\underline{10\bar{1}}$

$\Longrightarrow \overline{0}101 = 1\overline{0}\underline{\overline{0}1\overline{0}}$ 0.5in by relation (6)

$\Longrightarrow \overline{0}101 = 1\underline{\overline{00}}1\overline{0} = 1\underline{0}1\overline{0}$

$\Longrightarrow N\overline{0}101 = N101\overline{0} \in [010\overline{1}]$

Hence $[\overline{0}101] = [010\overline{1}]$, and t_1, a representative from the other 4-orbit, takes $[\overline{0}10]$ to a single coset in $[010\overline{1}]$.

Now let's consider the double cosets $[010]$ and $[\overline{010}]$.

$010 = 012\underline{\overline{2}}0$

$\Longrightarrow 010 = \underline{012}\overline{2}0 = \underline{(1,3)(0,2)\overline{103}}\overline{2}0$

Since relation (3) conjugated by $(1,0)(3,2)$ is $(1,3)(0,2)\overline{103} = 012$.

$\Longrightarrow 010 = (1,3)(0,2)\overline{103}\underline{\overline{2}0}$

$\Longrightarrow 010 = (1,3)(0,2)\overline{103}\underline{(0,1,2)\overline{2}1}$

Since relation (7) conjugated by $(1,2)$ is $(0,1,2)\overline{2}1 = \overline{2}0$.

$\Longrightarrow 010 = (1,3)(0,2)(0,1,2)\underline{\overline{213}}\overline{2}1$

$\Longrightarrow 010 = (1,3,2)\underline{(2,3)(1,0)120}\overline{2}1$

Since relation (3) conjugated by $(0,2,3)$ is $(2,3)(1,0)120 = \overline{213}$.

$\Longrightarrow 010 = (1,2,0)120\overline{2}1 = (1,2,0)1\overline{2}\underline{\overline{2}0\overline{2}}1$

$\Longrightarrow 010 = (1,2,0)120\overline{2}1 = (1,2,0)1\overline{2}\underline{\overline{0}2\overline{0}}1$

Since relation (6) conjugated by $(1,2)$ is $\overline{2}0\overline{2} = \overline{0}2\overline{0}$.

$\Longrightarrow 010 = (1,2,0)1\overline{20}\underline{2\overline{0}}1$

$\Longrightarrow 010 = (1,2,0)1\overline{20}\underline{(0,1,2)2\overline{1}}1$ by relation (7).

$\Longrightarrow 010 = (1,2,0)(0,1,2)2\overline{01}2 = (1,0,2)\underline{2\overline{0}}\overline{1}2$

$\Longrightarrow 010 = (1,0,2)\underline{(0,1,2)2\overline{1}}\overline{1}2$ by relation (7).

$\Longrightarrow 010 = 2\underline{\overline{11}}2 = 212$

If we conjugate this equation by $(2,3)$ and $(2,4)$ we find $010 = 212 = 313 = 414$. Hence $N010 = N212 = N313 = N414$.

If we conjugate the above equation by $(0,1), (1,2), (1,3)$, and $(1,4)$, we obtain the following equivalent cosets:

$N010 = N212 = N313 = N414$ $N101 = N202 = N303 = N404$

$N020 = N121 = N323 = N424$ $N030 = N131 = N232 = N434$

$N040 = N141 = N242 = N343$

Hence there are five distinct single cosets in $[010]$.

Now as we proved above, $010 = 212$.

$\Longrightarrow 01\underline{0\overline{0}} = 212\overline{0}$

$\Longrightarrow 0\underline{1\overline{1}} = 21\overline{2}\overline{0}\overline{1}$

$\Longrightarrow 0\overline{0} = 212\overline{010}$

$\Longrightarrow \overline{2}e = \overline{2}212\overline{010}$

$\Longrightarrow \overline{212} = \overline{2}2\overline{010}$

$\Longrightarrow \overline{212} = \overline{010}$

If we conjugate this equation by $(2,3)$ and $(2,4)$ we find $\overline{212} = \overline{010} = \overline{313} = \overline{414}$

If we conjugate this equation by $(1,0), (1,2), (1,3)$, and $(1,4)$ we obtain the following equivalent cosets:

$N\overline{212} = N\overline{010} = N\overline{313} = N\overline{414}$

$N\overline{202} = N\overline{101} = N\overline{303} = N\overline{404}$

$N\overline{121} = N\overline{020} = N\overline{323} = N\overline{424}$

$N\overline{232} = N\overline{030} = N\overline{131} = N\overline{434}$

$N\overline{242} = N\overline{040} = N\overline{343} = N\overline{141}$

Hence there are five distinct single cosets in $[\overline{010}]$.

$N^{(010)} \geq N^{010} = \{n \in N^{01} | (t_0t_1t_0)^n = t_0t_1t_0\} = \langle (2,3),(2,4)\rangle = S_3$

$N^{(010)} = \{\pi \in N | N(t_0t_1t_0)^\pi = Nt_0t_1t_0\} = \langle (0,2),(2,3),(2,4)\rangle$ since $N010 = N212 = N313 = N414$. Thus $N^{(010)}$ has orbits $\{1\}, \{0,2,3,4\}, \{\overline{1}\}$, and $\{\overline{0},\overline{2},\overline{3},\overline{4}\}$.

Similarly, $N^{(\overline{010})} \geq N^{\overline{010}} = \{n \in N^{\overline{01}} | (t_0{}^{-1}t_1{}^{-1}t_0{}^{-1})^n = t_0{}^{-1}t_1{}^{-1}t_0{}^{-1}\} = \langle (2,3),(2,4)\rangle = S_3$

$N^{(\overline{010})} = \{\pi \in N | N(t_0{}^{-1}t_1{}^{-1}t_0{}^{-1})^\pi = Nt_0{}^{-1}t_1{}^{-1}t_0{}^{-1}\} = \langle (0,2),(2,3),(2,4)\rangle$. since $N\overline{010} = N\overline{212} = N\overline{313} = N\overline{414}$. Thus $N^{(\overline{010})}$ has orbits $\{1\}, \{0,2,3,4\}, \{\overline{1}\}$, and $\{\overline{0},\overline{2},\overline{3},\overline{4}\}$.

So we need to analyze the double cosets $[0101]$, $[0100]$, $[010\overline{1}]$, $[010\overline{0}]$, $[\overline{010}1]$, $[\overline{010}0]$, $[\overline{0101}]$, and $[\overline{0100}]$.

$[0100] = [01\overline{0}]$, so t_0, a representative from one of the 4-orbits, takes $[010]$ to single coset in $[01\overline{0}]$.

$[\overline{0100}] = [\overline{01}0]$ so $t_0{}^{-1}$, a representative from the other 4-orbit, takes $[\overline{010}]$ to a single coset in $[\overline{01}0]$.

$[010\overline{0}] = [01]$. so ${t_0}^{-1}$ takes $[010]$ to a single coset in $[01]$.

$[\overline{010}\,\overline{0}] = [\overline{01}]$ so t_0 takes $[\overline{010}]$ to a single coset in $[\overline{01}]$.

This leaves $[010\overline{1}]$, $[\overline{010}1], [0101]$ and $[\overline{010}\,\overline{1}]$ to consider. ${t_1}^{-1}$ takes $[010]$ to a single coset in $010\overline{1}]$, while t_1 takes $[\overline{010}]$ to a single coset in $[\overline{010}1]$. Similarly, t_1 takes $[010]$ to a single coset in $[0101]$, while ${t_1}^{-1}$ takes $[\overline{010}]$ to a single coset in $[\overline{010}\,\overline{1}]$.

Now let's consider $[01\overline{0}]$ and $[\overline{01}0]$.

By relation (7) conjugated by $(1,2)$, $1\overline{0} = (0,2,1)1\overline{2}$.

$\Longrightarrow N\underline{0}1\overline{0} = N\underline{0}(0,2,1)1\overline{2}$

$\Longrightarrow N01\overline{0} = N(0,2,1)21\overline{2} = N21\overline{2}$

If we conjugate the resulting equation by $(2,3)$ and $(2,4)$, we obtain the following equation.

$N01\overline{0} = N21\overline{2} = N31\overline{3} = N41\overline{4}$

If we now conjugate this equation by $(0,1),(1,2),(1,3)$, and $(1,4)$ we find the following equations:

$N10\overline{1} = N20\overline{2} = N30\overline{3} = N40\overline{4}$

$N02\overline{0} = N12\overline{1} = N32\overline{3} = N42\overline{4}$

$N03\overline{0} = N23\overline{2} = N13\overline{1} = N43\overline{4}$

$N04\overline{0} = N24\overline{2} = N34\overline{3} = N14\overline{1}$

Hence there are five distinct single cosets in $[01\overline{0}]$.

Similarlly by relation (7), $\overline{1}0 = (0,2,1)\overline{1}2$.

$\Longrightarrow N\overline{01}0 = N\overline{0}(0,2,1)\overline{1}2$

$\Longrightarrow N\overline{01}0 = N(0,2,1)\overline{21}2 = N\overline{21}2$

If we conjugate this equation by $(2,3)$ and $(2,4)$, we obtain the following equation.

$N\overline{01}0 = N\overline{21}2 = N\overline{31}3 = \overline{41}4$

If we now conjugate this equation by $(0,1),(1,2),(1,3)$, and $(1,4)$ we find the following equations:

$N\overline{10}1 = N\overline{20}2 = N\overline{30}3 = \overline{40}4$

$N\overline{02}0 = N\overline{12}1 = N\overline{32}3 = \overline{42}4$

$N\overline{03}0 = N\overline{23}2 = N\overline{13}1 = \overline{43}4$

$N\overline{0}4\overline{0} = N\overline{2}4\overline{2} = N\overline{3}4\overline{3} = \overline{1}4\overline{1}$

Hence there are five distinct single cosets in $[\overline{0}1\overline{0}]$.

$N^{(01\overline{0})} \geq N^{01\overline{0}} = \{n \in N^{01} | (t_0 t_1 t_0^{-1})^n = t_0 t_1 t_0^{-1}\} = \langle (2,3), (2,4) \rangle = S_3$

$N^{(01\overline{0})} = \{\pi \in N | \ N(t_0 t_1 t_0^{-1})^\pi = N t_0 t_1 t_0^{-1}\} = \langle (0,2), (2,3), (2,4) \rangle$ since $N01\overline{0}$ $= N21\overline{2} = N31\overline{3} = N41\overline{4}$. Thus $N^{(01\overline{0})}$ has orbits $\{1\}$, $\{0, 2, 3, 4\}$, $\{\overline{1}\}$, and $\{\overline{0}, \overline{2}, \overline{3}, \overline{4}\}$.

$N^{(\overline{0}\overline{1}0)} \geq N^{\overline{0}\overline{1}0} = \{n \in N^{\overline{0}\overline{1}} | \ (t_0^{-1} t_1^{-1} t_0)^n = t_0^{-1} t_1^{-1} t_0\} = \langle (2,3), (2,4) \rangle = S_3$

$N^{(\overline{0}\overline{1}0)} = \{\pi \in N | N(t_0^{-1} t_1^{-1} t_0)^\pi = N t_0^{-1} t_1^{-1} t_0\} = \langle (0,2), (2,3), (2,4) \rangle$ since $N\overline{0}\overline{1}0 = N\overline{2}\overline{1}2 = N\overline{3}\overline{1}3 = N\overline{4}\overline{1}4$. Thus $N^{(\overline{0}\overline{1}0)}$ has orbits $\{1\}, \{0,2,3,4\}, \{\overline{1}\}$, and $\{\overline{0}, \overline{2}, \overline{3}, \overline{4}\}$.

So we need to consider $[01\overline{0}1]$, $[01\overline{0}0]$, $[01\overline{0}\overline{1}]$, $[01\overline{0}\overline{0}]$, $[\overline{0}\overline{1}01]$, $[\overline{0}\overline{1}00]$, $[\overline{0}\overline{1}0\overline{1}]$, and $[\overline{0}\overline{1}0\overline{0}]$.

$[01\overline{0}0] = [01]$ so t_0, a representative from oen of the 4-orbits, takes $[[01\overline{0}]$ to a single coset in $[01]$.

$[\overline{0}\overline{1}0\overline{0}] = [\overline{0}\overline{1}]$ so t_0^{-1}, a representative from the other 4-orbit, takes $[\overline{0}\overline{1}0]$ back to a single coset in $[\overline{0}\overline{1}]$.

$[01\overline{0}\overline{0}] = [010]$ so t_0^{-1} takes $[01\overline{0}]$ to a single coset in $[010]$.

$[\overline{0}\overline{1}00] = [\overline{0}\overline{1}\overline{0}]$ so t_0 takes $[\overline{0}\overline{1}0]$ to a single coset in $[\overline{0}\overline{1}\overline{0}]$.

Now let's consider $[\overline{0}\overline{1}0\overline{1}]$. By relation (6) $\overline{0}1\overline{0} = \overline{1}0\overline{1}$

$\Longrightarrow N\overline{0}1\underline{\overline{0}0} = N\overline{1}0\overline{1}\underline{0}$

$\Longrightarrow N\underline{01}\overline{0}1 = N\underline{01}\overline{1}0\overline{1}0$

$\Longrightarrow N01\overline{0}1 = N0\underline{1\overline{1}}0\overline{1}0 = N\underline{00}\overline{1}0 = N\overline{0}\overline{1}0$

Hence $[01\overline{0}1] = [\overline{0}\overline{1}0]$, and t_1 takes $[01\overline{0}]$ to a single coset in $[\overline{0}\overline{1}0]$.

Similarly, let's consider $[\overline{0}\overline{1}0\overline{1}]$. By relation (6) $\overline{0}1\overline{0} = \overline{1}0\overline{1}$

$\Longrightarrow N\underline{\overline{0}}\overline{1}0\overline{1} = N\underline{\overline{0}}\overline{0}1\overline{0}$

$\Longrightarrow N\overline{0}\overline{1}0\overline{1} = N01\overline{0}$

Hence $[\overline{0}\overline{1}0\overline{1}] = [01\overline{0}]$, and t_1^{-1} takes $[\overline{0}\overline{1}0]$ to a single coset in $[01\overline{0}]$.

This leaves $[01\overline{0}\overline{1}]$ and $[\overline{0}\overline{1}01]$ to consider.

t_1^{-1} takes $[01\overline{0}]$ to a single coset in $[01\overline{0}\overline{1}]$.

By relation (6) $1\overline{0} = 0\overline{1}0\overline{1}$

$\Longrightarrow N\underline{0}1\overline{0\underline{1}} = N\underline{0}0\overline{1}01\overline{\underline{1}}$

$\Longrightarrow N01\overline{01} = N\underline{00}\overline{1}0\overline{\underline{11}} = N\overline{01}01$

Hence $[01\overline{01}] = [\overline{01}01]$ and t_1 takes $[\overline{010}]$ to a single coset in $[01\overline{01}]$.

Now let's consider $[0101]$ and $[\overline{0101}]$.

We previously proved $010 = 212 = 313 = 414$. Hence $010\underline{1} = 212\underline{1} = 313\underline{1} = 414\underline{1}$. If we conjugate this equation by $(1,0), (1,2), (1,3)$, and $(1,4)$, we obtain the following equations:

$0101 = 2121 = 3131 = 4141$

$1010 = 2020 = 3030 = 4040$

$0202 = 1212 = 3232 = 4242$

$0303 = 2323 = 1313 = 4343$

$0404 = 2424 = 3434 = 1414$

Now $0\underline{101} = 0\underline{202}$ since we previously proved $101 = 202$.

$\Longrightarrow 0101 = 2121 = 3131 = 4141 = 0202 = 1212 = 3232 = 4242$

If we conjugate this equation by $(1,3)$ and $(1,4)$ we find $0101 = 2121 = 3131 = 4141 = 0202 = 1212 = 3232 = 4242 = 0303 = 2323 = 1313 = 4343 = 0404 = 2424 = 3434 = 1414$.

Hence all single cosets in $[0101]$ are equal.

Also we previously proved $\overline{010} = \overline{212} = \overline{313} = \overline{414}$. Therefore it follows $\overline{010\underline{1}} = \overline{212\underline{1}} = \overline{313\underline{1}} = \overline{414\underline{1}}$. If we conjugate this equation by $(1,0), (1,2), (1,3)$, and $(1,4)$ we obtain the following equations:

$\overline{0101} = \overline{2121} = \overline{3131} = \overline{4141}$

$\overline{1010} = \overline{2020} = \overline{3030} = \overline{4040}$

$\overline{0202} = \overline{1212} = \overline{3232} = \overline{4242}$

$\overline{0303} = \overline{2323} = \overline{1313} = \overline{4343}$

$\overline{0404} = \overline{2424} = \overline{3434} = \overline{1414}$

Now $\overline{0\underline{101}} = \overline{0\underline{202}}$ since we previously proved $\overline{101} = \overline{202}$.

$\Longrightarrow \overline{0101} = \overline{2121} = \overline{3131} = \overline{4141} = \overline{0202} = \overline{1212} = \overline{3232} = \overline{4242}$

If we conjugate the equation by $(1,3)$ and $(1,4)$ we find $\Longrightarrow \overline{0101} = \overline{2121} = \overline{3131} = \overline{4141} = \overline{0202} = \overline{1212} = \overline{3232} = \overline{4242} = \overline{0303} = \overline{2323} = \overline{1313} = \overline{4343} = \overline{0404} = \overline{2424} = \overline{3434} = \overline{1414}$

Hence all single cosets in $[\overline{0101}]$ are equal.

$N^{(0101)} \geq N^{0101} = \{n \in N^{010} | (t_0 t_1 t_0 t_1)^n = t_0 t_1 t_0 t_1\} = \langle (2,3), (2,4) \rangle = S_3$

$N^{(0101)} = \{\pi \in N | N(t_0 t_1 t_0 t_1)^\pi = N t_0 t_1 t_0 t_1\} = \langle (0,2), (1,2), (2,3), (2,4) \rangle$ since all cosets in $[0101]$ are equal. Thus $N^{(0101)}$ has orbits $\{0, 1, 2, 3, 4\}$, and $\{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$.

Similarly, $N^{(\overline{0101})} \geq N^{\overline{0101}} = \{n \in N^{\overline{010}} |\ (t_0{}^{-1} t_1{}^{-1} t_0{}^{-1} t_1{}^{-1})^n = t_0{}^{-1}\ t_1{}^{-1}\ t_0{}^{-1}\ t_1{}^{-1}\} = \langle (2,3),\ (2,4) \rangle = S_3$

$N^{(\overline{0101})} = \{\pi \in N |\ N(t_0{}^{-1} t_1{}^{-1} t_0{}^{-1} t_1{}^{-1})^\pi = N t_0{}^{-1} t_1{}^{-1} t_0{}^{-1} t_1{}^{-1}\} = \langle (0,2),$ $(1,2),\ (2,3),\ (2,4) \rangle$ since all cosets in $[\overline{0101}]$ are equal. Thus $N^{(\overline{0101})}$ has orbits $\{0,\ 1,\ 2,\ 3,\ 4\}$, and $\{\overline{0},\ \overline{1},\ \overline{2},\ \overline{3},\ \overline{4}\}$.

Hence we need to consider $[01011], [0101\overline{1}], [\overline{0101}1]$, and $[\overline{01011}]$.

$[01011] = [010\overline{1}]$, so t_0, t_1, t_2, t_3, and t_4 all take us from $[0101]$ to $[010\overline{1}]$.

$[\overline{01011}] = [\overline{010}1]$, so $t_0{}^{-1}, t_1{}^{-1}, t_2{}^{-1}, t_3{}^{-1}$, and $t_4{}^{-1}$ all take us from $[\overline{0101}]$ to $[\overline{010}1]$.

$[0101\overline{1}] = [010]$, so $t_1{}^{-1}$, a representative from one of the 5-orbits, takes $[0101]$ back to a single coset in $[010]$.

$[\overline{0101}1] = [\overline{010}]$, so t_1, a representative from the other 5-orbit, takes $[\overline{0101}]$ back to a single coset in $[\overline{010}]$.

Now let's consider $[010\overline{1}]$ and $[\overline{010}1]$.

We previously proved $N010 = N212 = N313 = N414$. Therefore it follows $N010\underline{\overline{1}} = N212\underline{\overline{1}} = N313\underline{\overline{1}} = N414\underline{\overline{1}}$ If we conjugate this equation by $(0,1), (1,2), (1,3)$, and $(1,4)$, we obtain the five distinct single cosets in $[010\overline{1}]$.

$N010\overline{1} = N212\overline{1} = N313\overline{1} = N414\overline{1}$

$N101\overline{0} = N202\overline{0} = N303\overline{0} = N404\overline{0}$

$N020\overline{2} = N121\overline{2} = N323\overline{2} = N424\overline{2}$

$N030\overline{3} = N232\overline{3} = N131\overline{3} = N434\overline{3}$

$N040\overline{4} = N242\overline{4} = N343\overline{4} = N141\overline{4}$

Similarly we previously proved $N\overline{010} = N\overline{212} = N\overline{313} = N\overline{414}$. Therefore it follows that $N\overline{010}\underline{1} = N\overline{212}\underline{1} = N\overline{313}\underline{1} = N\overline{414}\underline{1}$. If we conjugate this equation by $(0,1), (1,2), (1,3)$, and $(1,4)$ we obtain the five distinct single cosets in $[\overline{010}1]$.

$N\overline{010}1 = N\overline{212}1 = N\overline{313}1 = N\overline{414}1$.

$N\bar{1}0\bar{1}0 = N\bar{2}0\bar{2}0 = N\bar{3}0\bar{3}0 = N\bar{4}0\bar{4}0$.

$N\bar{0}2\bar{0}2 = N\bar{1}2\bar{1}2 = N\bar{3}2\bar{3}2 = N\bar{4}2\bar{4}2$.

$N\bar{0}3\bar{0}3 = N\bar{2}3\bar{2}3 = N\bar{1}3\bar{1}3 = N\bar{4}3\bar{4}3$.

$N\bar{0}4\bar{0}4 = N\bar{2}4\bar{2}4 = N\bar{3}4\bar{3}4 = N\bar{1}4\bar{1}4$.

$N^{(010\bar{1})} \geq N^{010\bar{1}} = \{n \in N^{010} | (t_0t_1t_0t_1{}^{-1})^n = t_0t_1t_0t_1{}^{-1}\} = \langle(2,3),(2,4)\rangle = S_3$

$N^{(010\bar{1})} = \{\pi \in N | N(t_0t_1t_0t_1{}^{-1})^\pi = Nt_0t_1t_0t_1t_0{}^{-1}\} = \langle(0,2),(2,3),(2,4)\rangle$ since $N010\bar{1} = N212\bar{1} = N313\bar{1} = N414\bar{1}$. Thus $N^{(010\bar{1})}$ has orbits $\{1\}, \{0,2,3,4\}, \{\bar{1}\}$, and $\{\bar{0},\bar{2},\bar{3},\bar{4}\}$.

Similarly, $N^{(\bar{0}\bar{1}\bar{0}1)} \geq N^{\bar{0}\bar{1}\bar{0}1} = \{n \in N^{\bar{0}\bar{1}\bar{0}} | (t_0{}^{-1}t_1{}^{-1}t_0{}^{-1}t_1)^n = t_0{}^{-1}t_1{}^{-1}t_0{}^{-1}t_1\} = \langle(2,3),(2,4)\rangle = S_3$

$N^{(\bar{0}\bar{1}\bar{0}1)} = \{\pi \in N| \ N(t_0{}^{-1}t_1{}^{-1}t_0{}^{-1}t_1)^\pi = Nt_0{}^{-1}t_1{}^{-1}t_0{}^{-1}t_1\} = \langle(0,2),\ (2,3),\ (2,4)\rangle$ since $N\bar{0}1\bar{0}1 = N\bar{2}1\bar{2}1 = N\bar{3}1\bar{3}1 = N\bar{4}1\bar{4}1$. Thus $N^{(\bar{0}\bar{1}\bar{0}1)}$ has orbits $\{1\}$, $\{0,\ 2,\ 3,\ 4\}, \{\bar{1}\}$, and $\{\bar{0},\ \bar{2},\ \bar{3},\ \bar{4}\}$.

Hence we need to consider $[010\bar{1}1]$, $[010\bar{1}0]$, $[010\bar{1}\bar{1}]$, $[010\bar{1}\bar{0}]$, $[\bar{0}\bar{1}\bar{0}11]$, $[\bar{0}\bar{1}\bar{0}10]$, $[\bar{0}\bar{1}\bar{0}1\bar{1}]$, and $[\bar{0}\bar{1}\bar{0}1\bar{0}]$.

$[010\bar{1}1] = [010]$, so t_1 takes $[010\bar{1}]$ back to a single coset in $[010]$.

$[\bar{0}\bar{1}\bar{0}1\bar{1}] = [\bar{0}\bar{1}\bar{0}]$, so $t_1{}^{-1}$ takes $[\bar{0}\bar{1}\bar{0}1]$ back to a single coset in $[\bar{0}\bar{1}\bar{0}]$.

$[010\bar{1}\bar{1}] = [0101]$, so $t_1{}^{-1}$ takes $[010\bar{1}$ to a single coset in $[0101]$.

$[\bar{0}\bar{1}\bar{0}11] = [\bar{0}\bar{1}\bar{0}\bar{1}]$, so t_1 takes $[\bar{0}\bar{1}\bar{0}1]$ to a single coset in $[\bar{0}\bar{1}\bar{0}\bar{1}]$

Let's consider $[010\bar{1}0]$. By relation (6) $0\bar{1}0 = 1\bar{0}1$.

$\Longrightarrow N0\bar{1}0 = N1\bar{0}1$

$\Longrightarrow N\underline{01}0\bar{1}0 = N\underline{01}1\bar{0}1$

$\Longrightarrow N010\bar{1}0 = N0\underline{11}\bar{0}1$

$\Longrightarrow N010\bar{1}0 = N0\bar{1}\underline{\bar{0}1}$

$\Longrightarrow N010\bar{1}0 = N0\bar{1}\underline{(1,2,0)\bar{0}2}$

Since relation (7) conjugated by $(0,1)$ is $(1,2,0)\bar{0}2 = \bar{0}1$.

$\Longrightarrow N010\bar{1}0 = N(1,2,0)1\bar{2}\bar{0}2$

$\Longrightarrow N010\bar{1}0 = N(1,2,0)\underline{1\bar{2}}\bar{0}2$

$\Longrightarrow N010\bar{1}0 = N(1,2,0)\underline{(0,1,2)1\bar{0}}\bar{0}2$

Since relation (7) conjugated by $(1,2)$ is $(0,1,2)1\bar{0} = 1\bar{2}$.

$\Longrightarrow N010\bar{1}\bar{0} = N(1,0,2)1\bar{0}\bar{0}2$

$\Longrightarrow N010\bar{1}\bar{0} = N(1,0,2)102 = N102 \in [012]$

$\Longrightarrow [010\bar{1}\bar{0}] = [012]$

Hence t_0, a representative from one of the 4-orbits, takes $[010\bar{1}]$ to a single coset in $[012]$.

Similarly, let's consider $[\bar{0}\bar{1}\bar{0}1\bar{0}]$. By relation (6) $\bar{0}1\bar{0} = \bar{1}0\bar{1}$.

$\Longrightarrow N\bar{0}1\bar{0} = N\bar{1}0\bar{1}$

$\Longrightarrow N\underline{\bar{0}\bar{1}}\bar{0}1\bar{0} = N\underline{\bar{0}\bar{1}}\bar{1}0\bar{1}$

$\Longrightarrow N\bar{0}\bar{1}\bar{0}1\bar{0} = N\bar{0}\underline{\bar{1}\bar{1}}0\bar{1}$

$\Longrightarrow N\bar{0}\bar{1}\bar{0}1\bar{0} = N\underline{\bar{0}1}0\bar{1}$

$\Longrightarrow N\bar{0}\bar{1}\bar{0}1\bar{0} = N\underline{(1,2,0)\bar{0}}20\bar{1}$

$\Longrightarrow N\bar{0}\bar{1}\bar{0}1\bar{0} = N(1,2,0)\bar{0}2\underline{0\bar{1}}$

$\Longrightarrow N\bar{0}\bar{1}\bar{0}1\bar{0} = N(1,2,0)\bar{0}2\underline{(2,0,1)0\bar{2}}$

Since relation (7) conjugated by $(0,2)$ is $(2,0,1)0\bar{2} = (0\bar{1}$

$\Longrightarrow N\bar{0}\bar{1}\bar{0}1\bar{0} = N(1,2,0)(2,0,1)\bar{1}\underline{00}\bar{2}$

$\Longrightarrow N\bar{0}\bar{1}\bar{0}1\bar{0} = N(1,0,2)\underline{\overline{102}}$

$\Longrightarrow N\bar{0}\bar{1}\bar{0}1\bar{0} = N(1,0,2)\underline{(1,2)(0,3)013}$

Since relation (3) conjugated by $(0,1)$ is $(1,2)(0,3)013 = \overline{102}$

$\Longrightarrow N\bar{0}\bar{1}\bar{0}1\bar{0} = N(1,3,0)013 = N013 \in [012]$

$\Longrightarrow [\bar{0}\bar{1}\bar{0}1\bar{0}] = [012]$

Hence ${t_0}^{-1}$ takes $[\bar{0}\bar{1}\bar{0}1]$ to a single coset in $[012]$.

Now let's consider $[010\bar{1}\bar{0}]$. As previously proved $\bar{0}\bar{1}\bar{0}1 = (1,3,0)0130$.

$\Longrightarrow \underline{0}\bar{0}\bar{1}\bar{0}1\underline{\bar{1}} = \underline{0}(1,3,0)0130\underline{\bar{1}}$

$\Longrightarrow \bar{1}\bar{0} = (1,3,0)\underline{1}0130\bar{1}$

$\Longrightarrow \underline{010}\bar{1}\bar{0} = \underline{010}(1,3,0)10130\bar{1}$

$\Longrightarrow 010\bar{1}\bar{0} = (1,3,0)13\underline{11}0130\bar{1}$

$\Longrightarrow 010\bar{1}\bar{0} = (1,3,0)13\bar{1}0130\bar{1}$

$\Longrightarrow 010\bar{1}\bar{0} = (1,3,0)1\underline{3\bar{1}}0130\bar{1}$

$\Longrightarrow 010\bar{1}\bar{0} = (1,3,0)1\underline{(0,3,1)3\bar{0}}0130\bar{1}$

Since relation (7) conjugated by $(3,2)$ is $(0,3,1)3\bar{0} = 3\bar{1}$.

$\Longrightarrow 010\bar{1}\bar{0} = (1,3,0)(0,3,1)03\underline{\bar{0}0}130\bar{1} = 0313\underline{0\bar{1}}$

$\Longrightarrow 010\bar{1}\bar{0} = 0313\underline{(2,0,1)0\bar{2}}$

Since relation (7) conjugated by $(0,2)$ is $(2,0,1)0\overline{2} = 0\overline{1}$.

$\Longrightarrow 010\overline{10} = (2,0,1)13\underline{230}\overline{2}$

$\Longrightarrow 010\overline{10} = (2,0,1)13\underline{(3,1)(2,0)\overline{321}}\overline{2}$

since relation (3) conjugated by $(1,2)(0,3)$ is $(3,1)(2,0)\overline{321} = 230$.

$\Longrightarrow 010\overline{10} = (2,0,1)(3,1)(2,0)31\overline{3212}$

$\Longrightarrow 010\overline{10} = (0,3,1)3\underline{1\overline{3}}\overline{212}$

$\Longrightarrow 010\overline{10} = (0,3,1)3\underline{(0,1,3)1\overline{0}}\overline{212}$

Since relation (7) conjugated by $(2,1,3)$ is $(0,1,3)1\overline{0} = 1\overline{3}$.

$\Longrightarrow 010\overline{10} = (0,3,1)(0,1,3)01\overline{0212} = 01\underline{\overline{021}}\overline{2}$

$\Longrightarrow 010\overline{10} = 01\underline{(0,1)(2,3)203}\overline{2}$

Since relation (3) conjugated by $(1,2)$ is $(0,1)(2,3)203 = \overline{021}$.

$\Longrightarrow 010\overline{10} = (0,1)(2,3)1020\underline{3\overline{2}}$

$\Longrightarrow 010\overline{10} = (0,1)(2,3)1020\underline{(0,3,2)3\overline{0}}$

Since relation (7) conjugated by $(2,3,1)$ is $(0,3,2)3\overline{0} = 3\overline{2}$.

$\Longrightarrow 010\overline{10} = (0,1)(2,3)(0,3,2)130\underline{33}\overline{0} = (0,1,3)1\underline{3}0\overline{30}$

$\Longrightarrow 010\overline{10} = (0,1,3)1\overline{3}\underline{30\overline{3}}\overline{0}$

$\Longrightarrow 010\overline{10} = (0,1,3)1\overline{3}\underline{030}\overline{0}$

Since relation (6) conjugated by $(1,3)$ is $\overline{3}0\overline{3} = \overline{0}3\overline{0}$.

$\Longrightarrow 010\overline{10} = (0,1,3)1\overline{3}0\overline{3}\underline{\overline{00}}$

$\Longrightarrow 010\overline{10} = (0,1,3)\underline{1}\overline{3}03\underline{0}$

$\Longrightarrow 010\overline{10} = (0,1,3)\overline{\underline{11}3}030$

$\Longrightarrow 010\overline{10} = (0,1,3)\overline{1\underline{13}}\underline{0}30$

$\Longrightarrow 010\overline{10} = (0,1,3)\overline{1}\underline{(1,0)(3,2)312}30$

Since relation (3) conjugated by $(0,1,3,2)$ is $(1,0)(3,2)312 = \overline{130}$.

$\Longrightarrow 010\overline{10} = (0,1,3)(1,0)(3,2)\overline{0}31230 = (1,2,3)\overline{0}31230$

$\Longrightarrow 010\overline{10} = (1,2,3)\overline{0}3\underline{123}0$

$\Longrightarrow 010\overline{10} = (1,2,3)\overline{0}3\underline{(2,0)(1,3)\overline{210}}0$

Since relation (3) conjugated by $(0,2)$ is $(2,0)(1,3)\overline{210} = 123$.

$\Longrightarrow 010\overline{10} = (1,2,3)(2,0)(1,3)\overline{2}1\overline{21}\underline{00} = (1,0,2)\underline{\overline{2}1\overline{2}}\overline{1}$

$\Longrightarrow 010\overline{10} = (1,0,2)\underline{\overline{1}2\overline{1}}\overline{1}$

Since relation (6) conjugated by $(0,2)$ is $\overline{2}1\overline{2} = \overline{1}2\overline{1}$.

$\Longrightarrow 010\overline{10} = (1,0,2)\overline{1}2\underline{\overline{11}} = (1,0,2)\overline{1}2\underline{1}$

$\Longrightarrow N010\overline{1}\overline{0} = N(1,0,2)\overline{1}21 = N\overline{1}21 \in [\overline{0}10]$

$\Longrightarrow [010\overline{1}\overline{0}] = [\overline{0}10]$

Hence ${t_0}^{-1}$, a representative from the other 4-orbit, takes $[010\overline{1}]$ to a single coset in $[\overline{0}10]$.

Now let's consider $[\overline{0}\overline{1}\overline{0}10]$. As proven above, we know $010\overline{1}\overline{0} = (1,0,2)\overline{1}21$.

$\Longrightarrow \underline{\overline{1}\overline{0}}010\overline{1}\overline{0} = \underline{\overline{1}\overline{0}}(1,0,2)\overline{1}21$

$\Longrightarrow \overline{1}\underline{\overline{0}0}10\overline{1}\overline{0} = (1,0,2)\overline{0}\overline{2}\overline{1}21$

$\Longrightarrow \underline{\overline{1}1}0\overline{1}\overline{0} = (1,0,2)\overline{0}\overline{2}\overline{1}21$

$\Longrightarrow \underline{0}0\overline{1}0\underline{10} = \underline{0}(1,0,2)\overline{0}\overline{2}\overline{1}21\underline{10}$

$\Longrightarrow \overline{0}\overline{1}\overline{0}10 = (1,0,2)2\overline{0}\overline{2}\underline{\overline{1}2\overline{1}}0$

$\Longrightarrow \overline{0}\overline{1}\overline{0}10 = (1,0,2)2\overline{0}\overline{2}\underline{\overline{2}1\overline{2}}0$

Since relation (6) conjugated by $(0,2)$ is $\overline{2}1\overline{2} = \overline{1}2\overline{1}$.

$\Longrightarrow \overline{0}\overline{1}\overline{0}10 = (1,0,2)2\overline{0}\underline{\overline{2}\overline{2}}1\overline{2}0$

$\Longrightarrow \overline{0}\overline{1}\overline{0}10 = (1,0,2)\underline{2\overline{0}2}1\overline{2}0$

$\Longrightarrow \overline{0}\overline{1}\overline{0}10 = (1,0,2)\underline{0\overline{2}0}1\overline{2}0$

Since relation (6) conjugated by $(1,2)$ is $0\overline{2}0 = 2\overline{0}2$.

$\Longrightarrow \overline{0}\overline{1}\overline{0}10 = (1,0,2)0\overline{2}0\underline{1\overline{2}}0$

$\Longrightarrow \overline{0}\overline{1}\overline{0}10 = (1,0,2)0\overline{2}0\underline{(0,1,2)1\overline{0}}0$

Since relation (7) conjugated by $(1,2)$ is $(0,1,2)1\overline{0} = 1\overline{2}$.

$\Longrightarrow \overline{0}\overline{1}\overline{0}10 = (1,0,2)(0,1,2)1\overline{0}\underline{1\overline{1}\overline{0}0} = 1\overline{0}\overline{1}$

$\Longrightarrow N\overline{0}\overline{1}\overline{0}10 = N1\overline{0}\overline{1} \in [0\overline{1}\overline{0}]$

$\Longrightarrow [\overline{0}\overline{1}\overline{0}10] = [0\overline{1}\overline{0}]$

Hence t_0 takes $[\overline{0}\overline{1}01]$ to a single coset in $[0\overline{1}\overline{0}]$.

Now let's consider $[01\overline{0}\overline{1}]$

We previously proved that $N01\overline{0} = N21\overline{2} = N31\overline{3} = N41\overline{4}$

$\Longrightarrow N01\overline{0}\underline{\overline{1}} = N21\overline{2}\underline{\overline{1}} = N31\overline{3}\underline{\overline{1}} = N41\overline{4}\underline{\overline{1}}$

$N0\underline{1\overline{0}}\overline{1} = N0\underline{(0,2,1)1\overline{2}}\overline{1}$

Since relation (7) conjugated by $(1,2)$ is $(0,1,2)1\overline{0} = 1\overline{2}$.

$N0\underline{1\overline{0}}\overline{1} = N0\underline{(0,2,1)1\overline{2}}\overline{1} = N(0,2,1)21\overline{2}\overline{1} = N21\overline{2}\overline{1}$

If we conjugate this relation by $(0,3)$ and $(0,4)$ we obtain the following equations:

$N01\overline{01} = N21\overline{21} = N31\overline{31} = N41\overline{41}$
$N10\overline{10} = N20\overline{20} = N30\overline{30} = N40\overline{40}$
$N02\overline{02} = N12\overline{12} = N32\overline{32} = N42\overline{42}$
$N03\overline{03} = N13\overline{13} = N23\overline{23} = N43\overline{43}$
$N04\overline{04} = N14\overline{14} = N24\overline{24} = N34\overline{34}$

$N01\overline{01} = N012\underline{2\overline{2}}\overline{01}$

$\Longrightarrow N01\overline{01} = N012\underline{\overline{201}}$

$\Longrightarrow N01\overline{01} = N012\underline{(2,1)(0,3)023}$

Since relation (3) conjugated by $(0,2,1)$ is $(2,1)(0,3)023 = \overline{201}$.

$\Longrightarrow N01\overline{01} = N(2,1)(0,3)\underline{321}023$

$\Longrightarrow N01\overline{01} = N(2,1)(0,3)\underline{(2,0)(3,1)\overline{230}}023$

Since relation (3) conjugated by $(1,3)(0,2)$ is $(2,0)(3,1)\overline{230} = 321$.

$\Longrightarrow N01\overline{01} = N(2,3)(1,0)\overline{23}\underline{\overline{0}0}23 = N(2,3)(1,0)\overline{23}23$

$\Longrightarrow N01\overline{01} = N(2,3)(1,0)\overline{23}\underline{\overline{1}1}23$

$\Longrightarrow N01\overline{01} = N(2,3)(1,0)\overline{231}\underline{123}$

$\Longrightarrow N01\overline{01} = N(2.3)(1,0)\overline{231}\underline{(0,2)(1,3)\overline{210}}$

Since relation (3) conjugated by $(0,2)$ is $(0,2)(1,3)\overline{210} = 123$.

$\Longrightarrow N01\overline{01} = N(2,3)(1,0)(0,2)(1,3)\overline{013210} = N(2,1)(3,0)\overline{013210}$

$\Longrightarrow N01\overline{01} = N(2,1)(3,0)\underline{\overline{013}}\overline{210}$

$\Longrightarrow N01\overline{01} = N(2,1)(3,0)\underline{(0,3)(1,2)102}\overline{210}$

Since relation (3) conjugated by $(2,3)$ is $(0,3)(1,2)102 = \overline{013}$

$\Longrightarrow N01\overline{01} = N(2,1)(3,0)(0,3)(1,2)10\underline{2\overline{2}}\overline{10} = N10\overline{10}$

So $N01\overline{01} = N21\overline{21} = N31\overline{31} = N41\overline{41} = N10\overline{10} = N20\overline{20} = N30\overline{30} = N40\overline{40}$

If we conjugate this equation by $(1,2), (1,3)$, and $(1,4)$ we obtain the following: $N01\overline{01} = N21\overline{21} = N31\overline{31} = N41\overline{41} = N10\overline{10} = N20\overline{20} = N30\overline{30} = N40\overline{40} = N02\overline{02} = N12\overline{12} = N32\overline{32} = N42\overline{42} = N03\overline{03} = N13\overline{13} = N23\overline{23} = N43\overline{43} = N04\overline{04} = N14\overline{14} = N24\overline{24} = N34\overline{34}$

Hence, all single cosets in $[01\overline{01}]$ are equal.

$N^{(01\overline{01})} \geq N^{01\overline{01}} = \{n \in N^{01\overline{0}} \mid (t_0t_1{t_0}^{-1}{t_1}^{-1})^n = t_0t_1{t_0}^{-1}{t_1}^{-1}\} = \langle(2,3), (2,4)\rangle = S_3$

$N^{(01\overline{01})} = \{\pi \in N \mid N(t_0t_1t_0{}^{-1}t_1{}^{-1})^{\pi} = Nt_0t_1t_0{}^{-1}t_1{}^{-1}\} = \langle(0,2),\ (1,2),\ (2,3),\ (2,4)\rangle$ since all single cosets in $[01\overline{01}]$ are equal. Thus $N^{(01\overline{01})}$ has orbits $\{0,\ 1,\ 2,\ 3,\ 4\}$ and $\{\bar{0},\ \bar{1},\ \bar{2},\ \bar{3},\ \bar{4}\}$.

Hence we need to consider $[01\overline{01}1]$ and $[01\overline{011}]$.

$[01\overline{01}1] = [01\bar{0}]$, so t_1, a representative from one of the 5-orbits, takes $[01\overline{01}]$ to a single coset in $[01\bar{0}]$.

$[01\overline{011}] = [01\bar{0}1]$, and by relation (6) $\bar{0}1 = \bar{1}0\bar{1}0$

$\Longrightarrow N\bar{0}1 = N\bar{1}0\bar{1}0$

$\Longrightarrow N\underline{01}\bar{0}1 = N\underline{01}\bar{1}0\bar{1}0$

$\Longrightarrow N01\bar{0}1 = N0\underline{1\bar{1}}0\bar{1}0$

$\Longrightarrow N01\bar{0}1 = N\underline{00}\bar{1}0$

$\Longrightarrow N01\bar{0}1 = N\overline{0}\bar{1}0$

$\Longrightarrow [01\overline{011}] = [01\bar{0}1] = [\overline{0}\bar{1}0]$

Hence $t_1{}^{-1}$, a representative from the other 5-orbit, takes $[01\overline{01}]$ to a single coset in $[\overline{0}\bar{1}0]$.

Our double coset enumeration must be complete since the set of right cosets is closed under right multiplication by the symmetric generators.

Thus we have the Cayley diagram that is shown in Figure 5.1.

The maximum possible index of N in $G \cong \frac{3^{*5}:S_5}{[(0,1,2,3)t_0]^6}$ is $\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(\bar{0})}|} + \frac{|N|}{|N^{(01)}|} + \frac{|N|}{|N^{(\overline{01})}|} + \frac{|N|}{|N^{(0\bar{1})}|} + \frac{|N|}{|N^{(\bar{0}1)}|} + \frac{|N|}{|N^{(0\bar{1}0)}|} + \frac{|N|}{|N^{(\bar{0}1\bar{0})}|} + \frac{|N|}{|N^{(0\bar{1}\bar{0})}|} + \frac{|N|}{|N^{(\bar{0}10)}|} + \frac{|N|}{|N^{(012)}|} + \frac{|N|}{|N^{(010)}|} + \frac{|N|}{|N^{(\overline{010})}|} + \frac{|N|}{|N^{(01\bar{0})}|} + \frac{|N|}{|N^{(\overline{01}0)}|} + \frac{|N|}{|N^{(0101)}|} + \frac{|N|}{|N^{(0\bar{1}0\bar{1})}|} + \frac{|N|}{|N^{(010\bar{1})}|} + \frac{|N|}{|N^{(0\bar{1}01)}|} + \frac{|N|}{|N^{(01\overline{01})}|} = 1+5+5+20+20+5+5+1+1+5+5+20+5+5+5+5+1+1+5+5+1 = 126.$ Thus $|G| \leq 126 \times |N| = 126 \times 120 = 15120$. In order to show $|G| = 15120$, we consider G as a subgroup of S_{126} acting on 126 cosets that we have found, and labeled as follows:

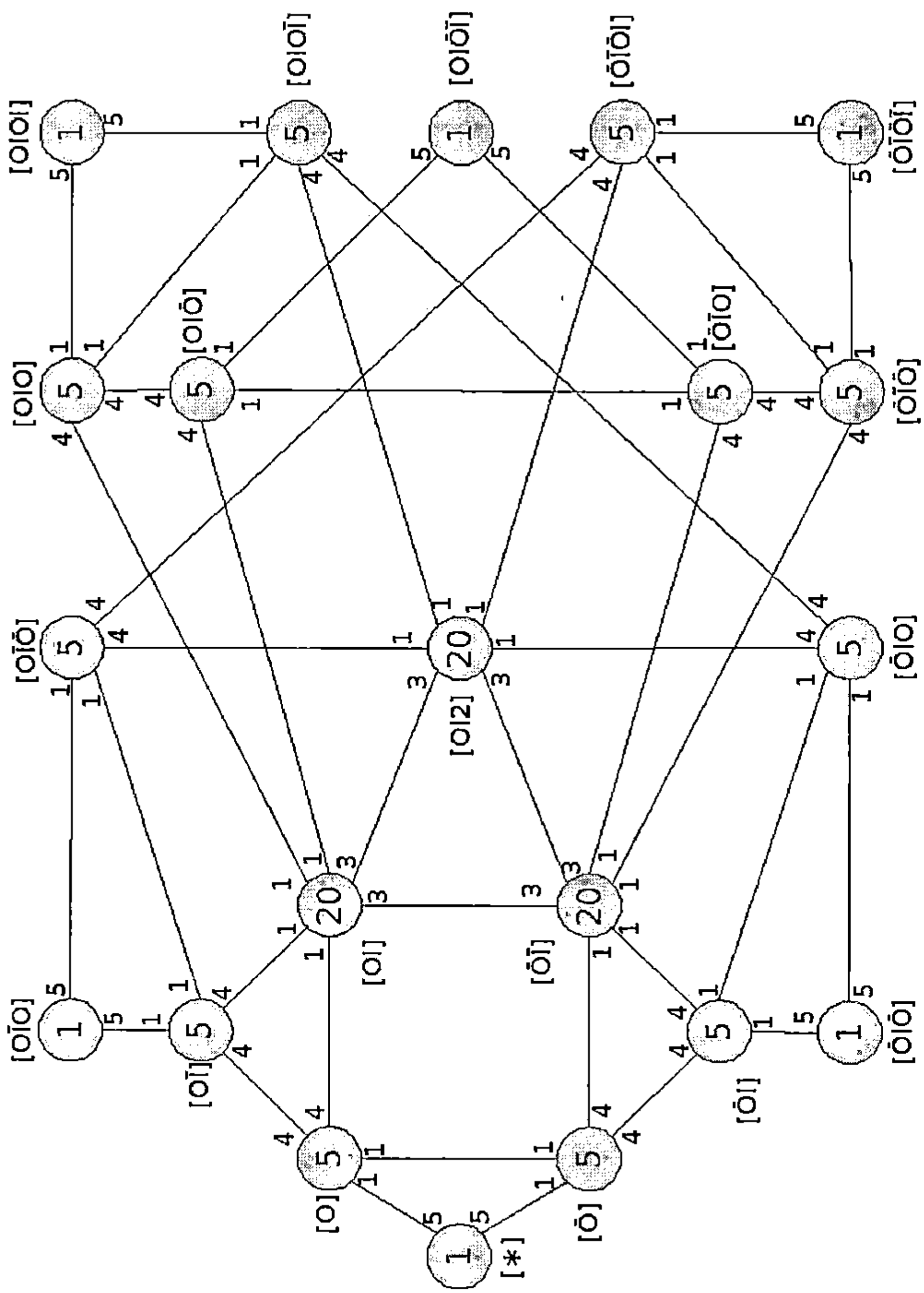

Figure 5.1: Cayley Diagram of $S_7 \times 3$ over S_5

1. N	22. $N21$	43. $N03$	64. $N\overline{14}1$	85. $N02$	106. $N201$
2. $N0$	23. $N04$	44. $N141$	65. $N\overline{141}$	86. $N412$	107. $N031$
3. $N\bar{0}$	24. $N0\bar{1}$	45. $N43$	66. $N\overline{41}$	87. $N243$	108. $N010\bar{1}$
4. $N1$	25. $N30$	46. $N14\bar{1}$	67. $N\overline{24}$	88. $N\overline{42}$	109. $N0101$
5. $N4$	26. $N3\bar{2}$	47. $N0\bar{1}0$	68. $N\overline{01}0$	89. $N121$	110. $N404\bar{0}$
6. $N\bar{1}$	27. $N01$	48. $N0\overline{10}$	69. $N\overline{010}$	90. $N12\bar{1}$	111. $N01\overline{01}$
7. $N\bar{4}$	28. $N34$	49. $N41$	70. $N\overline{23}$	91. $N401$	112. $N2\overline{32}$
8. $N2$	29. $N\bar{2}3$	50. $N24$	71. $N430$	92. $N234$	113. $N\overline{01}01$
9. $N10$	30. $N\overline{20}$	51. $N12$	72. $N42$	93. $N132$	114. $N3\overline{13}$
10. $N1\bar{2}$	31. $N\bar{0}1$	52. $N010$	73. $N312$	94. $N\bar{1}21$	115. $N\overline{141}4$
11. $N3$	32. $N\overline{21}$	53. $N01\bar{0}$	74. $N\overline{13}$	95. $N\bar{4}04$	116. $N021$
12. $N40$	33. $N\overline{04}$	54. $N340$	75. $N13$	96. $N012$	117. $N301$
13. $N4\bar{0}$	34. $N\bar{3}2$	55. $N23$	76. $N142$	97. $N\overline{23}2$	118. $N141\bar{4}$
14. $N\bar{2}$	35. $N\overline{30}$	56. $N\overline{43}$	77. $N324$	98. $N\overline{232}$	119. $N\bar{2}32$
15. $N\bar{1}2$	36. $N\overline{01}$	57. $N\overline{31}$	78. $N102$	99. $N\overline{40}4$	120. $N\bar{3}13$
16. $N\overline{10}$	37. $N\overline{34}$	58. $N\overline{14}$	79. $N232$	100. $N\overline{404}$	121. $N\overline{404}0$
17. $N\bar{3}$	38. $N31$	59. $N\bar{0}10$	80. $N23\bar{2}$	101. $N\overline{02}$	122. $N\overline{0101}$
18. $N\bar{4}0$	39. $N14$	60. $N\bar{0}1\bar{0}$	81. $N404$	102. $N423$	123. $N121\bar{2}$
19. $N\overline{40}$	40. $N32$	61. $N\overline{32}$	82. $N40\bar{4}$	103. $N\overline{12}1$	124. $N\overline{232}3$
20. $N20$	41. $N213$	62. $N123$	83. $N1\overline{21}$	104. $N\overline{121}$	125. $N\overline{121}2$
21. $N2\bar{3}$	42. $N\overline{12}$	63. $N\overline{03}$	84. $N4\overline{04}$	105. $N041$	126. $N232\bar{3}$

For this purpose we compute the action of the control group N as well as the action of t_0, t_1, t_2, t_3, and t_4 on the 126 cosets. These permutations are as follows:

t_0 : (1, 2, 3)(4, 9, 10)(5, 12, 13)(6, 15, 16) (7, 18, 19) (8, 20, 21) (11, 25, 26) (14, 29, 30) (17, 34, 35) (22, 41, 42) (23, 44, 46) (24, 47, 48) (27, 52, 53) (28, 54, 56) (31, 59, 60) (32, 51, 62) (33, 64, 65) (36, 68, 69) (37, 45, 71) (38, 73, 74) (39, 76, 66) (40, 77, 70) (43, 79, 80) (49, 86, 58) (50, 87, 88) (55, 92, 61) (57, 75, 93) (63, 97, 98) (67, 72, 102) (78, 94, 108) (81, 109, 110) (82, 99, 111) (83, 96, 113) (84, 105, 115) (85, 89, 90) (91, 95, 118) (100, 121, 122) (101, 103, 104) (106, 119, 123) (107, 124, 114) (112, 116, 125) (117, 120, 126)

t_1 : (1, 4, 6) (2, 27, 24) (3, 31, 36) (5, 49, 13) (7, 18, 66) (8, 22, 21) (9, 81, 82) (10, 47, 83) (11, 38, 26) (12, 91, 33) (14, 29, 32) (15, 94, 60) (16, 99, 100) (17, 34, 57)

(19, 23, 105) (20, 106, 101) (25, 117, 63) (28, 54, 56) (30, 85, 116) (35, 43, 107) (37, 45, 71) (39, 44, 46) (40, 77, 70) (41, 119, 123) (42, 103, 104) (48, 78, 121) (50, 87, 88) (51, 89, 90) (52, 109, 108) (53, 68, 111) (55, 92, 61) (58, 64, 65) (59, 110, 96) (62, 125, 112) (67, 72, 102) (69, 113, 122) (73, 120, 126) (74, 97, 98) (75, 79, 80) (76, 115, 84) (86, 95, 118) (93, 124, 114)

t_2 : (1, 8, 14) (2, 85, 24) (3, 31, 101) (4, 51, 10) (5, 72, 13) (6, 15, 42) (7, 18, 88) (9, 78, 36) (11, 40, 26) (12, 91, 33) (16, 27, 96) (17, 34, 61) (19, 23, 105) (20, 81, 82) (21, 47, 112) (22, 52, 53) (25, 117, 63) (28, 54, 56) (29, 119, 60) (30, 99, 100) (32, 68, 69) (35, 43, 107) (37, 45, 71) (38, 73, 74) (39, 76, 66) (41, 113, 83) (44, 46, 50) (48, 106, 121) (49, 86, 58) (55, 79, 80) (57, 75, 93) (59, 110, 116) (62, 94, 108) (64, 65, 67) (70, 97, 98) (77, 120, 126) (84, 87, 115) (89, 109, 123) (90, 103, 111) (92, 124, 114) (95, 118, 102) (104, 125, 122)

t_3 : (1, 11, 17) (2, 43, 24) (3, 31, 63) (4, 75, 10) (5, 45, 13) (6, 15, 74) (7, 18, 56) (8, 55, 21) (9, 78, 36) (12, 91, 33) (14, 29, 70) (16, 27, 96) (19, 23, 105) (20, 106, 101) (22, 41, 42) (25, 81, 82) (26, 47, 114) (28, 44, 46) (30, 85, 116) (32, 51, 62) (34, 120, 60) (35, 99, 100) (37, 64, 65) (38, 52, 53) (39, 76, 66) (40, 89, 90) (48, 117, 121) (49, 86, 58) (50, 87, 88) (54, 115, 84) (57, 68, 69) (59, 110, 107) (61, 103, 104) (67, 72, 102) (71, 95, 118) (73, 113, 83) (77, 125, 112) (79, 109, 126) (80, 97, 111) (92, 119, 123) (93, 94, 108) (98, 124, 122)

t_4 : (1, 5, 7) (2, 23, 24) (3, 31, 33) (4, 39, 10) (6, 15, 58) (8, 50, 21) (9, 78, 36) (11, 28, 26) (12, 81, 82) (13, 47, 84) (14, 29, 67) (16, 27, 96) (17, 34, 37) (18, 95, 60) (19, 99, 100) (20, 106, 101) (22, 41, 42) (25, 117, 63) (30, 85, 116) (32, 51, 62) (35, 43, 107) (38, 73, 74) (40, 77, 70) (44, 109, 118) (45, 79, 80) (46, 64, 111) (48, 91, 121) (49, 52, 53) (54, 120, 126) (55, 92, 61) (56, 97, 98) (57, 75, 93) (59, 110, 105) (65, 115, 122) (66, 68, 69) (71, 124, 114) (72, 89, 90) (76, 94, 108) (83, 86, 113) (87, 119, 123) (88, 103, 104) (102, 1425, 112)

x : (2, 4, 8, 11, 5)(3, 6, 14, 17, 7) (9, 22, 40, 45, 23) (10, 21, 26, 13, 24) (12, 27, 51, 55, 28) (15, 29, 34, 18, 31) (16, 32, 61, 56, 33) (19, 36, 42, 70, 37) (20, 38, 72, 43, 39) (25, 49, 85, 75, 50) (30, 57, 88, 63, 58) (35, 66, 101, 74, 67) (41, 77, 71, 105, 78) (44, 81, 52, 89, 79) (46, 82, 53, 90, 80) (48, 83, 112, 114, 84) (54, 91, 96, 62, 92) (59, 94, 119, 120, 95) (64, 99, 68, 103, 97) (65, 100, 69, 104, 98) (73, 102, 107, 76, 106) (86, 116, 93, 87, 117) (108, 123, 126, 118, 110) (113, 125, 124, 115, 121)

y : (5, 11)(7, 17)(12, 25)(13, 26)(18, 34)(19, 35)(23, 43)(28, 45) (33, 63) (37, 56) (38, 49) (39, 75) (40, 72) (44, 79) (46, 80) (50, 55) (54, 71) (57, 66) (58, 74) (61, 88) (64, 97) (65, 98) (67, 70) (73, 86) (76, 93) (77, 102) (84, 114) (87, 92) (91, 117) (95, 120) (105, 107) (115, 124) (118, 126)

It readily checks that the order of $\langle x, y, t\rangle$, a subgroup of the symmetric group S_{126} acting on the 126 right cosets of N in G, is 15120. Visibly $|x| = 5$ and $|y| = 2$, additionally $|xy| = 4$ and $[x, y]^3 = 1$, hence $\langle x, y\rangle \cong S_5$. If we conjugate t by S_5 we see that t has exactly five conjugates. We conclude that $\langle x, y, t\rangle$ is a homomorphic image of the progenitor $3^{*5} : S_5$.

Thus if the original nine relations hold in $\langle x, y, t\rangle$, then $\langle x, y, t\rangle$ is a homomorphic image of G and this will give $|G| \geq |\langle x, y, t\rangle| = 15120$.

Verify relation (1) $t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0 = 1$ by conjugating the five symmetric generators by $t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0$. By multiplying the permutations listed above, we find $t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0 =$ Identity. Thus when we conjugate t_0, t_1, t_2, t_3 and t_4 by $t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0\ t_2\ t_1\ t_0$ we obtain:

$$t_0^{t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_0$$
$$t_1^{t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_1$$
$$t_2^{t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_2$$
$$t_3^{t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_3$$
$$t_4^{t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_4$$

So $t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0$ acts as the identity.

Verify relation (2) $t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0 = (4, 3, 2, 1, 0)$ by conjugating the five symmetric generators. By multiplying the permutations listed above, we find $(2, 5, 11, 8, 4)(3, 7, 17, 14, 6)(9, 23, 45, 40, 22)(10, 24, 13, 26, 21)$

$(12, 28, 55, 51, 27)(15, 31, 18, 34, 29)$ (16, 33, 56, 61, 32) (19, 37, 70, 42, 36) (20, 39, 43, 72, 38) (25, 50, 75, 85, 49) (30, 58, 63, 88, 57) (35, 67, 74, 101, 66) (41, 78, 105, 71, 77) (44, 79, 89, 52, 81) (46, 80, 90, 53, 82) (48, 84, 114, 112, 83) (54, 92, 62, 96, 91) (59, 95, 120, 119, 94) (64, 97, 103, 68, 99) (65, 98, 104, 69, 100) (73, 106, 76, 107, 102)

(86, 117, 87, 93, 116) (108, 110, 118, 126, 123) (113, 121, 115, 124, 125). Thus when we conjugate t_0, t_1, t_2, t_3 and t_4 by $t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0$ we obtain:

$t_0^{t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0} = t_4$

$t_1^{t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0} = t_0$

$t_2^{t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0} = t_1$

$t_3^{t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0} = t_2$

$t_4^{t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0} = t_3$

So $t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0t_4t_3t_2t_1t_0$ acts as the permutation $(4,3,2,1,0)$.

Verify relation (3) $t_1t_0t_3t_2t_1t_0 = (0,2)(1,3)$ by conjugating the five symmetric generators.

$t_0^{t_1t_0t_3t_2t_1t_0} = t_2$ $\quad$ $t_1^{t_1t_0t_3t_2t_1t_0} = t_3$

$t_2^{t_1t_0t_3t_2t_1t_0} = t_0$ $\quad$ $t_3^{t_1t_0t_3t_2t_1t_0} = t_1$

$t_4^{t_1t_0t_3t_2t_1t_0} = t_4$

So $t_1t_0t_3t_2t_1t_0$ acts as the permutation $(0,2)(1,3)$.

Verify relation (4) $t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0 = 1$ by conjugating the five symmetric generators.

$t_0^{t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_0$

$t_1^{t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_1$

$t_2^{t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_2$

$t_3^{t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_3$

$t_4^{t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0} = t_4$

So $t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0$ acts as the identity.

Verify relation (5) $t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0 = 1$ by conjugating the five symmetric generators.

$t_0^{t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0} = t_0$ $\quad$ $t_1^{t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0} = t_1$

$t_2^{t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0} = t_2$ $\quad$ $t_3^{t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0} = t_3$

$t_4^{t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0} = t_4$

So $t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0t_1t_0$ acts as the identity.

Verify relation (6) $t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1 = 1$ by conjugating the five symmetric generators.

$t_0^{t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1} = t_0$ $\quad$ $t_1^{t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1} = t_1$

$t_2^{t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1} = t_2$ $\quad$ $t_3^{t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1} = t_3$

$t_4^{t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1} = t_4$

So $t_0{}^{-1}t_1t_0{}^{-1}t_1t_0{}^{-1}t_1$ acts as the identity.

Verify relation (7) $t_1{}^{-1}t_2t_0{}^{-1}t_1 = (0,1,2)$ by conjugating the five symmetric generators.

$t_0^{t_1{}^{-1}t_2t_0{}^{-1}t_1} = t_1$ $\quad t_1^{t_1{}^{-1}t_2t_0{}^{-1}t_1} = t_2$
$t_2^{t_1{}^{-1}t_2t_0{}^{-1}t_1} = t_0$ $\quad t_3^{t_1{}^{-1}t_2t_0{}^{-1}t_1} = t_3$
$t_4^{t_1{}^{-1}t_2t_0{}^{-1}t_1} = t_4$

So $t_1{}^{-1}t_2t_0{}^{-1}t_1$ acts as the permutation $(0,1,2)$.

Verify relation (8) $t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2 = 1$ by conjugating the five symmetric generators.

$t_0^{t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2} = t_0$ $\quad t_1^{t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2} = t_1$
$t_2^{t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2} = t_2$ $\quad t_3^{t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2} = t_3$
$t_4^{t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2} = t_4$

So $t_1t_2t_0t_2t_1t_2t_0t_2t_1t_2t_0t_2$ acts as the identity.

Finally, verify relation (9) $t_1t_2{}^{-1}t_0t_2{}^{-1}t_1t_2{}^{-1}t_0t_2{}^{-1} = 1$ by conjugating the five symmetric generators as follows:

$t_0^{t_1t_2{}^{-1}t_0t_2{}^{-1}t_1t_2{}^{-1}t_0t_2{}^{-1}} = t_0$ $\quad t_1^{t_1t_2{}^{-1}t_0t_2{}^{-1}t_1t_2{}^{-1}t_0t_2{}^{-1}} = t_1$
$t_2^{t_1t_2{}^{-1}t_0t_2{}^{-1}t_1t_2{}^{-1}t_0t_2{}^{-1}} = t_2$ $\quad t_3^{t_1t_2{}^{-1}t_0t_2{}^{-1}t_1t_2{}^{-1}t_0t_2{}^{-1}} = t_3$
$t_4^{t_1t_2{}^{-1}t_0t_2{}^{-1}t_1t_2{}^{-1}t_0t_2{}^{-1}} = t_4$

So $t_1t_2{}^{-1}t_0t_2{}^{-1}t_1t_2{}^{-1}t_0t_2{}^{-1}$ acts as the identity.

Thus $G/ker\phi \cong \langle x,y,t\rangle$ and $|G| \geq |\langle x,y,t\rangle| = 15120$. As shown earlier, $|G| \leq 15120$. Hence $|G| = 15120$.

Moreover, $a = (1,78,87,98,52,116,71)(2,58,90,57,$ 119, 63, 115) (3, 23, 118, 83, 51, 55, 97) (4, 42, 70, 124, 59, 33, 46)(5, 53, 32, 61, 120, 121, 19) (6, 85, 45, 122, 9, 50, 126) (7, 110, 49, 112, 75, 103, 25) (8, 74, 125, 35, 95, 82, 66) (10, 106, 54, 111, 96, 102, 34) (11, 113, 30, 37, 109, 36, 88) (12, 84, 108, 22, 40, 17, 99) (13, 31, 76, 104, 73, 89, 107) (14, 43, 64, 48, 39, 123, 38) (15, 62, 92, 26, 81, 105, 65) (16, 67, 80, 94, 101, 56, 47) (18, 69, 41, 77, 79, 24, 91) (20, 28, 60, 27, 72, 114, 68) (21, 117, 44, 100, 86, 29, 93),

$b = (1,13)(2,19)(4,66)(5,46)(7,60)(8,88)(11,56)$(12, 48) (15, 86) (18, 65) (23, 110) (28, 126) (29, 102) (31, 91) (33, 82) (34, 71) (37, 80) (39, 108) (44, 111) (45, 114) (47, 115) (49, 83) (50, 123) (53, 58) (54, 98) (64, 122) (67, 90) (69, 76) (72, 112) (84, 118) (87, 104) (95, 109) (100, 105),

and $c = (1,109,122)(2,110,100)(3,81,121)$ (4, 108, 69) (5, 118, 65) (6, 52, 113) (7, 44, 115) (8, 123, 104) (9, 78, 36) (10, 94, 68) (11, 126, 98) (12, 91, 33) (13, 95, 64)

(14, 89, 125) (15, 53, 83) (16, 27, 96) (17, 79, 124) (18, 46, 84) (19, 23, 105) (20, 106, 101) (21, 119, 103) (22, 41, 42) (24, 59, 99) (25, 117, 63) (26, 120, 97) (28, 54, 56) (29, 90, 112) (30, 85, 116) (31, 82, 48) (32, 51, 62) (34, 80, 114) (35, 43, 107) (37, 45, 71) (38, 73, 74) (39, 76, 66) (40, 77, 70) (47, 60, 111) (49, 86, 58) (50, 87, 88) (55, 92, 61) (57, 75, 93) (67, 72, 102)

are in G with $S_7 \times 3 \cong \langle a, b, c\rangle$ since $S_7 \times 3 \cong \{a, b, c | a^7, b^2, (ab)^6, (a^{-2}(ab)^2)^3,$ $(a^{-2}ba^2b)^2, c^3, [c, a], [c, b] \}$. So $\langle a, b, c\rangle \leq G$, but $|\langle a, b, c\rangle| = |G|$, therefore $G = \langle a, b, c\rangle \cong$ $S_7 \times 3$.

Chapter 6

Construction of $2 \times M_{12}$

Consider the progenitor

$$2^{*6} : L_2(5) \cong \langle x, y, t | x^5 = y^3 = (xy)^2 = 1 = t^2 = [t, x] = [t^{yx^2}, xy] \rangle,$$

where

$$x \sim (0, 1, 2, 3, 4)$$
$$y \sim (\infty, 0, 1)(2, 4, 3), \text{ and}$$
$$t \sim t_\infty.$$

The following relations from [Cur07] may be used for the purpose of manual double coset enumeration:

1. $[(0, 1, 2, 3, 4)t_0]^8 = 1$
2. $[(0, 2, 4, 1, 3)t_0]^8 = 1$
3. $[(\infty, 0, 1)(2, 4, 3)t_\infty]^8 = 1$
4. $[(\infty, 0, 1)(2, 4, 3)t_2]^8 = 1$
5. $[(\infty, 0)(1, 4)t_\infty]^6 = 1$
6. $[(\infty, 0)(1, 4)t_1]^6 = 1$

Based on a computer, it is known, see [Cur07], that the progenitor $2^{*6} : L_2(5)$ factored by relations (1) through (6), although relations (1),(3), and (6) suffice, is isomorphic to $2 \times M_{12}$. We will construct by hand $2 \times M_{12}$ using the technique of manual double coset enumeration of $G \cong \frac{2^{*6}:L_2(5)}{[(0,1,2,3,4)t_0]^8,[(\infty,0,1)(2,4,3)t_\infty]^8,[(\infty,0)(1,4)t_1]^6}$ over $L_2(5)$.

Expanding relations (1), (2), (3), (4), (5), and (6) gives us:

(1) $[(0,1,2,3,4)t_0]^8 = 1$

$(0,1,2,3,4)t_0(0,1,2,3,4)t_0(0,1,2,3,4)t_0(0,1,2,3,4)t_0(0,1,2,3,4)\,t_0\,(0,1,2,3,4)\,t_0\,(0,1,2,3,4)\,t_0\,(0,1,2,3,4)t_0 = 1$

$(0,1,2,3,4)^8 t_0^{(0,1,2,3,4)^7} t_0^{(0,1,2,3,4)^6} t_0^{(0,1,2,3,4)^5}\ t_0^{(0,1,2,3,4)^4}\ t_0^{(0,1,2,3,4)^3}\ t_0^{(0,1,2,3,4)^2}\ t_0^{(0,1,2,3,4)}\ t_0 = 1$

$(0,3,1,4,2)t_2t_1t_0t_4t_3t_2t_1t_0 = 1$

$(0,3,1,4,2)t_2t_1t_0t_4 = t_0t_1t_2t_3$ (1)

(2) $[(0,2,4,1,3)t_0]^8 = 1$

$(0,2,4,1,3)t_0(0,2,4,1,3)t_0(0,2,4,1,3)t_0(0,2,4,1,3)t_0(0,2,4,1,3)t_0\,(0,2,4,1,3)\,t_0\,(0,2,4,1,3)\,t_0\,(0,2,4,1,3)\,t_0 = 1$

$(0,1,2,3,4)^8 t_0^{(0,1,2,3,4)^7} t_0^{(0,1,2,3,4)^6}\ t_0^{(0,1,2,3,4)^5}\ t_0^{(0,1,2,3,4)^4}\ t_0^{(0,1,2,3,4)^3}\ t_0^{(0,1,2,3,4)^2}\ t_0^{(0,1,2,3,4)}\ t_0 = 1$

$(0,1,2,3,4)t_4t_2t_0t_3t_1t_4t_2t_0 = 1$

$(0,1,2,3,4)t_4t_2t_0t_3 = t_0t_2t_4t_1$ (2)

(3) $[(\infty,0,1)(2,4,3)t_\infty]^8 = 1$

$(\infty,0,1)(2,4,3)t_\infty(\infty,0,1)(2,4,3)t_\infty\,(\infty,0,1)\,(2,4,3)\,t_\infty\,(\infty,0,1)\,(2,4,3)\,t_\infty\,(\infty,0,1)\,(2,4,3)\,t_\infty\,(\infty,0,1)\,(2,4,3)\,t_\infty\,(\infty,0,1)\,(2,4,3)\,t_\infty\,(\infty,0,1)\,(2,4,3)\,t_\infty = 1$

$(\infty,0,1)(2,4,3)^8 t_\infty^{(\infty,0,1)(2,4,3)^7}\ t_\infty^{(\infty,0,1)(2,4,3)^6}\ t_\infty^{(\infty,0,1)(2,4,3)^5}\ t_\infty^{(\infty,0,1)(2,4,3)^4}\ t_\infty^{(\infty,0,1)(2,4,3)^3}\ t_\infty^{(\infty,0,1)(2,4,3)^2}\ t_\infty^{(\infty,0,1)(2,4,3)}\ t_\infty = 1$

$(\infty,1,0)(2,3,4)t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty = 1$

$(\infty,1,0)(2,3,4)t_0t_\infty t_1t_0 = t_\infty t_0t_1t_\infty$ (3)

(4) $[(\infty,0,1)(2,4,3)t_2]^8 = 1$

$(\infty,0,1)(2,4,3)t_2(\infty,0,1)(2,4,3)t_2(\infty,0,1)(2,4,3)t_2(\infty,0,1)(2,4,3)t_2\,(\infty,0,1)\,(2,4,3)\,t_2\,(\infty,0,1)\,(2,4,3)\,t_2\,(\infty,0,1)\,(2,4,3)t_2\,(\infty,0,1)\,(2,4,3)\,t_2 = 1$

$(\infty,0,1)(2,4,3)^8 t_2^{(\infty,0,1)(2,4,3)^7} t_2^{(\infty,0,1)(2,4,3)^6}$

$t_2^{(\infty,0,1)(2,4,3)^5} t_2^{(\infty,0,1)(2,4,3)^4} t_2^{(\infty,0,1)(2,4,3)^3} t_2^{(\infty,0,1)(2,4,3)^2}\ t_2^{(\infty,0,1)(2,4,3)}\ t_2 = 1$

$(\infty,1,0)(2,3,4)t_4t_2t_3t_4t_2t_3t_4t_2 = 1$

$(\infty,1,0)(2,3,4)t_4t_2t_3t_4 = t_2t_4t_3t_2$ (4)

(5) $[(\infty,0)(1,4)t_\infty]^6 = 1$

$(\infty,0)(1,4)t_\infty(\infty,0)(1,4)t_\infty(\infty,0)(1,4)t_\infty(\infty,0)(1,4)t_\infty\ (\infty,0)\ (1,4)\ t_\infty\ (\infty,0)\ (1,4)t_\infty = 1$

$(\infty,0)(1,4)^6 t_\infty{}^{(\infty,0)(1,4)^5} t_\infty{}^{(\infty,0)(1,4)^4}$
$t_\infty{}^{(\infty,0)(1,4)^3} t_\infty{}^{(\infty,0)(1,4)^2} t_\infty{}^{(\infty,0)(1,4)} t_\infty = 1$

$t_0 t_\infty t_0 t_\infty t_0 t_\infty = 1$

$t_0 t_\infty t_0 = t_\infty t_0 t_\infty \qquad (5)$

(6) $[(\infty,0)(1,4)t_1]^6 = 1$

$(\infty,0)(1,4)t_1(\infty,0)(1,4)t_1(\infty,0)(1,4)\ t_1\ (\infty,0)\ (1,4)\ t_1\ (\infty,0)\ (1,4)\ t_1\ (\infty,0)\ (1,4)\ t_1 = 1$

$(\infty,0)(1,4)^6 t_1{}^{(\infty,0)(1,4)^5} t_1{}^{(\infty,0)(1,4)^4}$
$t_1{}^{(\infty,0)(1,4)^3} t_1{}^{(\infty,0)(1,4)^2} t_1{}^{(\infty,0)(1,4)} t_1 = 1$

$t_4 t_1 t_4 t_1 t_4 t_1 = 1$

$t_4 t_1 t_4 = t_1 t_4 t_1 \qquad (6)$

We now perform the manual double coset enumeration of

$G = \frac{2^{*6}:L_2(5)}{[(0,1,2,3,4)t_0]^8, [(\infty,0,1)(2,4,3)t_\infty]^8, [(\infty,0)(1,4)t_1]^6}$ over $L_2(5)$

We start with the double coset representative word of length zero, $NeN = N$, denoted $[*]$.

Next we consider the double cosets with word length one. N is transitive on $\Omega = \{t_\infty, t_0,\ t_1,\ t_2,\ t_3,\ t_4\} = \{\infty, 0,\ 1,\ 2,\ 3,\ 4\}$, thus $Nt_\infty N = \{Nt_\infty{}^n \mid n \in N\} = \{Nt_\infty, Nt_0,$ $Nt_1,\ Nt_2,\ Nt_3,, Nt_4\} = \{N\infty,\ N0,\ N1,\ N2,\ N3\}$ denoted $[\infty]$.

Now we determine for $[\infty]$ to which coset $Nt_\infty t_i$ belongs for one t_i from each orbit of N^∞ on Ω. Since $N^\infty = \{n \in N | t_\infty{}^n = t_\infty\} = \langle (0,1.2,3,4), (2,5)(3,4)\rangle$, N^∞ has orbits $\{\infty\}, \{0,1,2,3,4\}$ on Ω.

So we need to consider the double cosets $[\infty\infty]$, and $[\infty 0]$.

$\infty\infty = e \Longrightarrow [\infty\infty] = [*]$, since t_∞ is of order 2. So t_∞ takes $[\infty]$ to the single coset in $[*]$.

Thus we need to consider $[\infty 0]$.

The generator t_0, a representative from the 5-orbit, takes $[\infty]$ to a single coset in $[\infty 0]$. There are 30 distinct single cosets in $[\infty 0]$.

Now let's consider the double coset of word length two, $Nt_\infty t_0 N$, denoted $[\infty 0]$. $N^{\infty 0} = N^{(\infty 0)} = \langle (1,4)(2,3)\rangle$. $[\infty 0]$ has orbits $\{0\}, \{\infty\}, \{1,4\}$, and $\{2,3\}$ on Ω. Hence

we need to consider $[\infty 00], [\infty 0\infty], [\infty 01]$, and $[\infty 02]$.

First $[\infty 00] = [\infty]$, since t_0 is of order 2. So the generator t_0 takes $[\infty 0]$ back to a single coset in $[\infty]$.

Next consider $Nt_\infty t_0 t_\infty$, denoted as $[\infty 0\infty]$. By relation (5), we know $Nt_\infty t_0 t_\infty = Nt_0 t_\infty t_0$. $N^{(\infty 0\infty)} = \langle (1,4)(2,3), (2,3)(0,\infty)\rangle$ so there are 15 distinct cosets in $[\infty 0\infty]$.

Now consider $Nt_\infty t_0 t_1 N$ denoted $[\infty 01]$. The generator t_1, a representative from one of the 2-orbits, takes $[\infty 0]$ to a single coset in $[\infty 01]$. $[\infty 01]$ has 60 distinct cosets.

Finally, consider $Nt_\infty t_0 t_2 N$ denoted $[\infty 02]$. The generator t_2, a representative from the other 2-orbit, takes $[\infty 0]$ to a single coset in $[\infty 02]$. $[\infty 02]$ has 60 distinct cosets.

Next let's study the double cosets of word length three. Start with $[\infty 0\infty]$. $N^{(\infty 0\infty)}$ has orbits $\{1,4\}$, $\{2,3\}$, and $\{0,\infty\}$ on Ω. So we need to consider $[\infty 0\infty\infty]$, $[\infty 0\infty 1]$, and $[\infty 0\infty 2]$.

$[\infty 0\infty\infty] = [\infty 0]$, so t_∞, a representative from one of the 2-orbits, takes $[\infty 0\infty]$ back to a single coset in $[\infty 0]$.

t_1, a representative from another 2-orbit, takes $[\infty 0\infty]$ to a single coset in $[\infty 0\infty 1]$. Now by relation (5), $\infty 0\infty = 0\infty 0$, so $N\infty 0\infty 1 = N0\infty 01$. $N^{(\infty 0\infty 1)} = \langle (2,3)(0,\infty)\rangle$ so there are 30 distinct single cosets in $[\infty 0\infty 1]$.

Similarly, t_2, a representative from the remaining 2-orbit, takes $[\infty 0\infty]$ to a single coset in $[\infty 0\infty 2]$, and $N\infty 0\infty 2 = N0\infty 02$. $N^{(\infty 0\infty 2)} = \langle (1,4)(0,\infty)\rangle$ so there are 30 distinct single cosets in $[\infty 0\infty 2]$.

Now consider $[\infty 01]$. $N^{(\infty 01)} = N^{\infty 01} = e$ and has orbits of $\{\infty\}, \{0\}, \{1\}$, $\{2\}$, $\{3\}$, and $\{4\}$ on Ω. So we need to look at $[\infty 01\infty], [\infty 010], [\infty 011].[\infty 012], [\infty 013]$, and $[\infty 014]$.

First note that $[\infty 011] = [\infty 0]$, since the t_i's are of order 2. Hence t_1 takes $[\infty 01]$ back to a single coset in $[\infty 0]$.

Next look at $[\infty 01\infty]$. t_∞ takes $[\infty 01]$ to a single coset in $[\infty 01\infty]$. By relation (3) we know that $N(\infty,1,0),(2,3,4)0\infty 10 = N\infty 01\infty$. Hence $N0\infty 10 = N\infty 01\infty$. $N^{(\infty 01\infty)} = \langle (2,3)(0,\infty)\rangle$ so there are 30 distinct single cosets in $[\infty 01\infty]$.

t_0 takes $[\infty 01]$ to a single coset in $[\infty 010]$. By relation (5), we know $0\infty 0 = \infty 0\infty$. If we conjugate this relation by $(1,\infty)(3,4) \in L_2(5)$, we derive $010 = 101$. Hence $\underline{\infty}010 = \underline{\infty}101$, and $N\infty 010 = N\infty 101$. $N^{(\infty 010)} = \langle (1,0)(2,4)\rangle$ so there are 30 distinct single cosets in $[\infty 010]$.

t_2 takes $[\infty 01]$ to a single coset in $[\infty 012]$. There are 60 distinct single cosets in $[\infty 012]$.

t_3 takes $[\infty 01]$ to a single coset in $[\infty 013]$. By relation relation (2), $(0,1,2,3,4)$ $4203 = 0241$, if we conjugate by $(4,\infty,2,0,1) \in L_2(5)$, we obtain $(1,4,0,3,\infty)$ $\infty 013$ $= 10\infty 4$. Hence $N(1,4,0,3,\infty)$ $\infty 013 = N10\infty 4$. Thus it follows $N\infty 013 = N10\infty 4$. $N^{(\infty 013)} = \langle(1,\infty)(3,4)\rangle$ so there are 30 distinct single cosets in $[\infty 013]$.

t_4 takes $[\infty 01]$ to a single coset in $[\infty 014]$. There are 60 distinct single cosets in $[\infty 014]$.

Now let's turn our attention to $[\infty 02]$. $N^{(\infty 02)} = e$, and has orbits $\{\infty\}$, $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, and $\{4\}$ on Ω. So we need to look at $[\infty 02\infty], [\infty 020], [\infty 021].[\infty 022], [\infty 023]$, and $[\infty 024]$.

Start with $[\infty 022] = [\infty 0]$, hence t_2 takes $[\infty 02]$ back to a single coset in $[\infty 0]$.

Next, t_∞ takes $[\infty 02]$ to a single coset in $[\infty 02\infty]$. By relation (4) we know $(\infty,1,0)(2,3,4)4234 = 2432$, if we conjugate this relation by $(2,0,4,\infty,3) \in L_2(5)$, we obtain $(3,1,4)(0,2,\infty)\infty 02\infty = 0\infty 20$. Hence $N(3,1,4)(0,2,\infty)\infty 02\infty = N0\infty 20$ and it follows that $N\infty 02\infty = N0\infty 20$. $N^{(\infty 02\infty)} = \langle(1,4)(0,\infty)\rangle$ so there are 30 distinct single cosets in $[\infty 02\infty]$.

t_0 takes $[\infty 02]$ to a single coset in $[\infty 020]$. By relation (5) we know $0\infty 0 = \infty 0\infty$. If we conjugate this relation by $(1,3)(2,\infty) \in L_2(5)$, we derive $020 = 202$. Hence we know $N\underline{\infty}020 = N\underline{\infty}202$. $N^{(\infty 020)} = \langle(2,0)(3,4)\rangle$ so there are 30 distinct single cosets in $[\infty 020]$.

t_1 takes $[\infty 02]$ to a single coset in $[\infty 021]$. By relation (1) we know $(0,3,1,4,2)$ $2104 = 0123$. If we conjugate by $(1,0,2,\infty,4) \in L_2(5)$ we find that $(2,3,0,1,\infty)$ $\infty 021$ $= 20\infty 3$. Hence $N(2,3,0,1,\infty)$ $\infty 021 = N20\infty 3$ and it follows that $N\infty 021 = N20\infty 3$. $N^{(\infty 021)} = \langle(1,3)(2,\infty)\rangle$ so there are 30 distinct single cosets in $[\infty 021]$.

t_3 takes $[\infty 02]$ to a single coset in $[\infty 023]$. There are 60 distinct single cosets in $[\infty 023]$.

Similarly, t_4 takes $[\infty 02]$ to a single coset in $[\infty 024]$ and there are 60 distinct single cosets in $[\infty 024]$.

Now let's consider $[\infty 0\infty 1]$. $N^{(\infty 0\infty 1)}$ has orbits $\{1\}, \{4\}, \{2,3\}$, and $\{0,\infty\}$ on Ω. So we need to look at $[\infty 0\infty 11], [\infty 0\infty 14], [\infty 0\infty 12]$, and $[\infty 0\infty 10]$

First $[\infty 0\infty 11] = [\infty 0\infty]$ so t_1 takes $[\infty 0\infty 1]$ back to a single coset in $[\infty 0\infty]$.

t_4 takes $[\infty 0 \infty 1]$ to a single coset in $[\infty 0 \infty 14]$. By relation (5) we know $\infty 0 \infty = 0 \infty 0$, so it follows that $\infty 0 \infty 14 = 0 \infty 014$. Hence $N \infty 0 \infty 14 = N 0 \infty 014$. $N^{(\infty 0 \infty 14)} = \langle (2,3)(0,\infty) \rangle$ so there are 30 distinct single cosets in $[\infty 0 \infty 14]$.

t_2, a representative from one of the 2-orbits, takes $[\infty 0 \infty 1]$ to a single coset in $[\infty 0 \infty 12]$. There are 60 distinct single cosets in $[\infty 0 \infty 12]$.

t_0, a representative from the other 2-orbit, takes $[\infty 0 \infty 1]$ to a single coset in $[\infty 0 \infty 10]$. By relation (3) we know $(\infty,1,0)(2,3,4)0\infty 10 = \underline{\infty 0} 1\infty$. But by relation (5) we know $0\infty 0\infty = \infty 0$. So $(\infty,1,0)(2,3,4)0\infty 10 = \underline{0\infty 0\infty} 1\infty$

$\implies \underline{0}(\infty,1,0)(2,3,4)0\infty 10 = \underline{\underline{0}}0\infty 0\infty 1\infty$

$\implies (\infty,1,0)(2,3,4)\infty 0\infty 10 = \underline{\infty 0 \infty} 1\infty$. But again recall that by relation (5) we can write $(\infty,1,0)(2,3,4)\infty 0\infty 10 = 0\infty 01\infty$. Hence it follows that $N(\infty,1,0)(2,3,4)\infty 0\infty 10 = N0\infty 01\infty$

$\implies N\infty 0\infty 10 = N0\infty 01\infty$.

$N^{(\infty 0\infty 10)} = \langle (2,3)(0,\infty) \rangle$ so $[\infty 0\infty 10]$ contains 30 distinct single cosets.

Now let's consider $[\infty 0\infty 2]$. $N^{(\infty 0\infty 2)}$ has orbits $\{2\},\{3\},\{1,4\}$, and $\{0,\infty\}$ on Ω. So we need to look at $[\infty 0\infty 22], [\infty 0\infty 23], [\infty 0\infty 21]$, and $[[\infty 0\infty 20]$.

First, $[\infty 0\infty 22] = [\infty 0\infty]$ So t_2 takes $[\infty 0\infty 2]$ back to a single coset in $[\infty 0\infty]$.

Next t_3 takes $[\infty 0\infty 2]$ to a single coset in $[\infty 0\infty 23]$. By relation (5) $\infty 0\infty = 0\infty 0$ so $\infty 0\infty \underline{23} = 0\infty 0\underline{23}$. Hence if follows that $N\infty 0\infty 23 = N0\infty 023$. $N^{(\infty 0\infty 23)} = \langle (1,4)(0,\infty) \rangle$ so $[\infty 0\infty 23]$ contains 30 distinct single cosets.

t_1, a representative from one of the 2-orbits, takes $[\infty 0\infty 2]$ to a single coset in $[\infty 0\infty 21]$ which has 60 distinct single cosets.

t_0, a representative from the other 2-orbit, takes $[\infty 0\infty 2]$ to a single coset in $[\infty 0\infty 20]$. Now relation (4) is $(\infty,1,0)(2,3,4)4234 = 2432$, if we conjugate this relation by $(1,4,0)(2,\infty,3) \in L_2(5)$, we obtain $(3,4,1)(\infty,2,0)0\infty 20 = \infty 02\infty$. Now it follows that $\underline{0}(3,4,1)(\infty,2,0)0\infty 20 = \underline{0}\infty 02\infty$.

$\implies (3,4,1)(\infty,2,0)\infty 0\infty 20 = 0\infty 02\infty$

$\implies N(3,4,1)(\infty,2,0)\infty 0\infty 20 = N0\infty 02\infty$

$\implies N\infty 0\infty 20 = N0\infty 02\infty$.

$N^{(\infty 0\infty 20)} = \langle (1,4)(0,\infty) \rangle$ so there are 30 distinct single cosets in $[\infty 0\infty 20]$.

Now consider $[\infty 01\infty]$, $N^{(\infty 01\infty)}$ has orbits $\{1\},\{4\},\{2,3\}$, and $\{0,\infty\}$ on Ω. So we need to look at $[\infty 01\infty 1], [\infty 01\infty 4], [\infty 01\infty 2]$ and $[\infty 01\infty\infty]$.

First $[\infty 01\infty\infty\infty] = [\infty 01]$, so t_∞, a representative from one of the 2-orbits takes $[\infty 01\infty]$ back to a single coset in $[\infty 01]$.

Next consider $[\infty 01\infty 1]$. By relation (3) $\infty 01\infty = (\infty,1,0)(2,3,4)0\infty 10$, so it follows $\infty 01\infty 1 = (\infty,1,0)(2,3,4)0\infty\underline{101}$.

$\Longrightarrow \infty 01\infty 1 = (\infty,1,0)(2,3,4)0\infty\underline{010}$ by relation (5) conjugated by $(1,3,2,4,\infty) \in L_2(5)$

$\Longrightarrow \infty 01\infty 1 = (\infty,1,0)(2,3,4)\underline{0\infty 0}10$

$\Longrightarrow \infty 01\infty 1 = (\infty,1,0)(2,3,4)\underline{\infty 0\infty}10$ by relation (5).

$\Longrightarrow N\infty 01\infty 1 = N(\infty,1,0)(2,3,4)\infty 0\infty 10$

$\Longrightarrow N\infty 01\infty 1 = N\infty 0\infty 10$. So it follows that $[\infty 01\infty 1] = [\infty 0\infty 10]$.

Now t_4 takes $[\infty 01\infty]$ to a single coset in $[\infty 01\infty 4]$. By relation (3) we know $(\infty,1,0)(2,3,4)0\infty 10 = \infty 01\infty$

$\Longrightarrow (\infty,1,0)(2,3,4)0\infty 10\underline{4} = \infty 01\infty\underline{4}$. So it follows that $N(\infty,1,0)(2,3,4)0\infty 104 = \infty 01\infty 4$

$\Longrightarrow N0\infty 104 = N\infty 01\infty 4$.

$N^{(\infty 01\infty 4)} = \langle (2,3)(0,\infty)\rangle$ so there are 30 distinct single cosets in $[\infty 01\infty 4]$.

t_2, a representative from the other 2-orbit, takes $[\infty 01\infty]$ to a single coset in $[\infty 01\infty 2]$. By relation (2) conjugated by $(1,2)(4,\infty) \in L_2(5)$ we obtain $(0,2,1,3,\infty)\infty 103 = 01\infty 2$.

$\Longrightarrow \underline{\infty}(0,2,1,3,\infty)\infty 103 = \underline{\infty}01\infty 2$

$\Longrightarrow (0,2,1,3,\infty)0\infty 103 = \infty 01\infty 2 \Longrightarrow N(0,2,1,3,\infty)0\infty 103 = N\infty 01\infty 2 \Longrightarrow N0\infty 103 = N\infty 01\infty 2$.

$N^{(\infty 01\infty 2)} = \langle (2,3)(0,\infty)\rangle$ so it follows that there are 30 distinct single cosets in $[\infty 01\infty 2]$.

Now let's consider the double coset $[\infty 010]$. $N^{(\infty 010)}$ has orbits $\{3\},\{\infty\},\{1,0\}$, and $\{2,4\}$ on Ω. So we need to look at $[\infty 0103], [\infty 010\infty], [\infty 0100]$, and $[\infty 0102]$.

Start with $[\infty 0100] = [\infty 01]$, so t_0, a representative from one of the 2-orbits, takes $[\infty 010]$ back to a single coset in $[\infty 01]$.

Next consider $[\infty 0102]$. t_2, a representative from the other 2-orbit, takes $[\infty 010]$ to a single coset in $[\infty 0102]$. There are 60 distinct single cosets in $\infty 0102]$.

t_3 takes $[\infty 010]$ to a single coset in $[\infty 0103]$. If we conjugate relation (5) by $(1,\infty)(3,4) \in L_2(5)$, we find $010 = 101$. Hence it follows $N\underline{\infty}010\underline{3} = N\underline{\infty}101\underline{3}$.

$N^{(\infty 0103)} = \langle (1,0)(2,4) \rangle$ so there are 30 distinct single cosets in $[\infty 0103]$.

t_∞ takes $[\infty 010]$ to a single coset in $[\infty 0103]$. By a previous proof, we know $\infty 010 = \infty 101$, so if follows that $N\infty 010\underline{\infty} = N\infty 101\underline{\infty}$. Further we can see that $N\infty 010\infty = N\infty 0\underline{\infty\infty}10\infty$.

$\implies N\infty 0\infty\underline{\infty 10\infty} = N\infty 0\infty\underline{(\infty,0,1)(4,3,2)1\infty 01}$ by relation (3) conjugated by $(1,0)(2,4) \in L_2(5)$.

$\implies N\infty 010\infty = N(\infty,0,1)(4,3,2)0\underline{101}\infty 1\infty 01$
$= N(\infty,0,1)(4,3,2)0\underline{010}\infty 1\infty 01$ by relation (5) conjugated by $(1,\infty)(2,4) \in L_2(5)$.

$\implies N\infty 010\infty = N(\infty,0,1)(4,3,2)\underline{00}10\infty 1\infty 01 = N(\infty,0,1)(4,3,2)10\infty 1\infty 01$
$= N10\infty 1\infty 01$.

$\implies N\infty 010\infty = N\infty 101\infty = N1\underline{0\infty 0}1 = N1\underline{\infty 0\infty}1$ by relation (5). Furthermore it follows $N1\infty 0\infty 1 = N1\infty 0\infty 1\underline{00} = N1\infty\underline{0\infty 10}0 = N1\infty\underline{(\infty,0,1)(2,4,3)\infty 01\infty 0}$ by relation (3).

$\implies N1\infty 0\infty 1 = N(\infty,0,1)(2,4,3)\underline{\infty 0\infty}01\infty 0 = N(\infty,0,1)(2,4,3)\underline{0\infty 0}01\infty 0$ by relation (5).

$\implies N1\infty 0\infty 1 = N(\infty,0,1)(2,4,3)0\infty\underline{00}1\infty 0 = N(\infty,0,1)(2,4,3)0\infty 1\infty 0$.
Hence it follows $N1\infty 0\infty 1 = N0\underline{\infty 1\infty}0 = N0\underline{1\infty 1}0$ by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$. So finally we have that $N\infty 010\infty = N\infty 101\infty = N10\infty 01 = N1\infty 0\infty 1 = N01\infty 10 = N0\infty 10\infty$. $N^{(\infty 010\infty)} = \langle (1,\infty,0)(2,4,3), (1,0)(2,4) \rangle$ so there are 10 distinct single cosets in $[\infty 010\infty]$.

Now we will consider $[\infty 012]$. $N^{(\infty 012)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$ on Ω. So we need to consider $[\infty 0121]$, $[\infty 0122]$, $[\infty 0123]$, $[\infty 0124]$, $[\infty 0120]$, and $[\infty 012\infty]$.

First note that $[\infty 0122] = [\infty 01]$ so t_2 takes $[\infty 012]$ back to a single coset in $[\infty 01]$.

Next, t_1 takes $[\infty 012]$ to a single coset in $[\infty 0121]$. There are 60 distinct single cosets in $[\infty 0121]$.

t_3 takes $[\infty 012]$ to a single coset in $[\infty 0123]$. By relation (1) we know $(0,3,1,4,2)$ $2104 = 0123$, so $N\infty 0123 = N\infty(0,3,1,4,2)2104 = N(0,3,1,4,2)\ \infty 2104 = N\infty 2104$. $N^{(\infty 0123)} = \langle (2,0)(3,4) \rangle$ so there are 30 distinct single cosets in $[\infty 0123]$.

t_4 takes $[\infty 012]$ to a single coset in $[\infty 01\infty 4]$, since $\infty 0124$
$= (1,3,4,\infty,0)\infty 23\infty 1$ thus $N\infty 0124 = N\infty 23\infty 1 \in [\infty 01\infty 4]$ since $N\infty 01\infty 4$ conju-

gated by $(1,3,0,2,4) \in L_2(5)$ is $N\infty 23\infty 1$. To prove $\infty 0124 = (1,3,4,\infty,0)\infty 23\infty 1$. move the relation to the one side of the equal sign and prove it equals identity.

$(1,0,\infty,4,3)\infty 012\underline{41\infty 3}2\infty = (1,0,\infty,4,3)\infty 012\underline{(1,0,\infty,4,3)\infty 140}2\infty$ (by relation (1) conjugated by $(2,4,3,0,\infty) \in L_2(5)$)

$= (1,\infty,3,0,4)4\underline{\infty 02\infty}1402\infty = (1,\infty,3,0,4)4\underline{(1,3,4)(2,0,\infty)0\infty 20}1402\infty$ (by relation (4) conjugated by $(2,0,4,\infty,3) \in L_2(5)$)

$= (1,2,0)(3,\infty,4)10\underline{\infty 201}402\infty = (1,2,0)(3,\infty,4)10\underline{(1,2,4,0,\infty)02\infty 4}402\infty$ (by relation (1) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)2\infty 02\infty \underline{44}02\infty = (1,4,3)(2,\infty,0)2\infty 02\infty 02\infty = e$, by relation (4) conjugated by $(1,3,0)(2,\infty,4) \in L_2(5)$.

t_0 takes $[\infty 012]$ to a single coset in $[\infty 0120]$. Relation (4) conjugated by $(1,\infty,4)(2,0,3) \in L_2(5)$ is $(4,\infty,3)(0,2,1)\ 1021 = 0120$, so $N\infty\ \underline{0120}$ $= N\infty\ \underline{(4,\infty,3)(0,2,1)1021} = N(4,\infty,3)(0,2,1)31021 = N31021$. So $N\infty 0120$ $= N31021$. $N^{(\infty 0120)} = \langle(1,0)(3,\infty)\rangle$ so $[\infty 0120]$ has 30 distinct single cosets.

Finally t_∞ takes a single coset in $[\infty 012]$ to $[\infty 012\infty]$. There are 60 distinct single cosets in $[\infty 012\infty]$.

Now consider $[\infty 013]$. $N^{(\infty 013)}$ has orbits $\{2\}$, $\{0\}$, $\{1,\infty\}$, and $\{3,4\}$ on Ω. Hence we need to examine $[\infty 0132]$, $[\infty 0130]$, $[\infty 0131]$, and $[\infty 0133]$.

$[\infty 0133] = [\infty 01]$, so t_3, a representative from one of the 2-orbits, takes $[\infty 013]$ back to a single coset in $[\infty 01]$.

t_2 takes $[\infty 013]$ to a single coset in $[\infty 0123]$, since $\infty 0132 = (1,4,0,3,\infty)23041$ thus $N\infty 0132 = N23041 \in [\infty 0123]$ since $N\infty 0123$ conjugated by $(1,0,3)(2,5,\infty) \in L_2(5)$ is $N23041$. To prove $\infty 0132 = (1,4,0,3,\infty)23041$. move the relation to the left side and prove it equals identity.

$(1,4,0,3,\infty)2304\underline{1231}0\infty = (1,4,0,3,\infty)2304\underline{(1,3,2)(4,0,\infty)2132}0\infty$ (by relation (4) conjugated by $(1,4)(0,\infty) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)12\infty\underline{0213}20\infty = (1,0,2)(3,4,\infty)12\infty\underline{(1,0,3,2,\infty)120\infty 2}0\infty$ (by relation (1) conjugated by $(1,2,0)(3,\infty,4) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)0\infty\underline{11}20\infty 20\infty = (1,3,4)(2,0,\infty)0\infty 20\infty 20\infty = e$ (by relation (4) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$).

t_0 takes $[\infty 013]$ to a single coset in $[\infty 0130]$. If we conjugate relation (2) by $(1,4,\infty,2,0) \in L_2(5)$, we find $N\underline{\infty 0130} = N\underline{(\infty,3,0,4,1)10\infty 40} = N10\infty 40$. $N^{(\infty 0130)} =$

$\langle(1,\infty)(3,4)\rangle$ so there are 30 distinct single cosets in $[\infty 0130]$.

t_1, a representative from the other 2-orbit, takes $[\infty 013]$ to a single coset in $[\infty 0131]$. There are 60 distinct single cosets in $[\infty 0131]$.

Now consider $[\infty 014]$. $N^{(\infty 014)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$ on Ω. So we need to consider $[\infty 0141]$, $[\infty 0142]$, $[\infty 0143]$, $[\infty 0144]$, $[\infty 0140]$, and $[\infty 014\infty]$.

$[\infty 0144] = [\infty 01]$, so t_4 takes $[\infty 014]$ back to a single coset in $[\infty 01]$.

t_1 takes $[\infty 014]$ to a single coset in $[\infty 0141]$. By relation (6) $N\infty 0\underline{141} = N\infty 0\underline{414}$. $N^{(\infty 0141)} = \langle(1,4)(2,3)\rangle$ so there are 30 distinct single cosets in $[\infty 0141]$.

t_2 takes $[\infty 014]$ to a single coset in $[\infty 0142]$. There are 60 distinct single cosets in $[\infty 0142]$.

t_3 takes $[\infty 014]$ to a single coset in $[\infty 0143]$. $N\infty 0143 = N10\infty 34$ since $\infty 0143 = (1,\infty,3,0,4)10\infty 34$. To prove this relation move the relation to one side of the equal sign, and prove it equals identity.

$(1,\infty,3,0,4)10\infty\underline{343}410\infty = (1,\infty,3,0,4)10\infty\underline{434}410\infty$ (by relation (5) conjugated by $(1,\infty,3,0,4) \in L_2(5)$)

$= (1\infty,3,0,4)10\infty 43\underline{44}10\infty = (1,\infty,3,0,4)10\infty 4310\infty = e$ (by relation (4) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$).

$N^{(\infty 0143)} = \langle(1,\infty)(3,4)\rangle$ so there are 30 distinct single cosets in $[\infty 0143]$.

t_0 takes $[\infty 014]$ to a single coset in $[\infty 0140]$. If we conjugate relation (4) by $(1,\infty,2)(3,4,0) \in L_2(5)$ we derive $0140 = (2,3,\infty)(1,0,4)1041$.

$\Longrightarrow N\infty 0140 = N\infty(2,3,\infty)(1,0,4)1041 = N(2,3,\infty)(1,0,4)21041$

$\Longrightarrow N\infty 0140 = N(2,3,\infty)(1,0,4)\underline{21041}$

$= N(2,3,\infty)(1,0,4)\underline{(2,4,1,3,0)01231} = N(2,0,1)(3,\infty,4)01231$

$\Longrightarrow N\infty 0140 = N(2,0,1)(3,\infty,4)0\underline{1231}$

$= N(2,0,1)(3,\infty,4)0\underline{(0,\infty,4)(2,1,3)2132}$ by relation (4) conjugated by $(1,4)(0,\infty) \in L_2(5)$.

$\Longrightarrow N\infty 0140 = N(2,\infty,0,3,4)\infty 2132 = N\infty 2132$.

$N^{(\infty 0140)} = \langle(2,0)(3,4)\rangle$ so there are 30 distinct single cosets in $[\infty 0140]$.

t_∞ takes $[\infty 014]$ to a single coset in $[\infty 0\infty 21]$, since $\infty 014\infty$ $= (1,2,4.0,\infty)41402$ thus $N\infty 014\infty = N41402 \in [\infty 0\infty 21]$ since $N\infty 0\infty 21$ conjugated by $(1,2,0)(3,\infty,4) \in L_2(5)$ is $N41402$. To prove $\infty 014\infty = (1,2,4.0,\infty)41402$ move the relation to one side of the equal sign and prove it equals identity

$\implies (1,2,4,0,\infty)414\underline{02\infty4}10\infty = (1,2,4,0,\infty)414\underline{(1,\infty,0,4,2)\infty201}10\infty$ (by relation (1) conjugated by $(1,2,0,\infty,3) \in L_2(5)) = 2\underline{\infty2\infty}20\underline{11}0\infty = 2\underline{2\infty\infty2}2\underline{00}\infty$ (by relation (5) conjugated by $(1,3,0,2,4) \in L_2(5)) = \underline{22}\infty\underline{22}\infty = \infty\infty = e$.

Now consider $[\infty024]$. $N^{(\infty024)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$ on Ω. So we need to look at $[\infty0241]$, $[\infty0242]$, $[\infty0243]$, $[\infty0244]$, $[\infty0240]$, and $[\infty024\infty]$.

First $[\infty0244] = [\infty02]$, so t_4 takes $[\infty024]$ back to a single coset in $[\infty02]$.

t_1 takes $[\infty024]$ to a single coset in $[\infty0241]$. By relation (2), $N\infty0\underline{241} = N\infty\underline{(0,1,2,3,4)4203}$. Hence $N\infty0241 = N\infty4203$. $N^{(\infty0241)} = \langle(1,3)(4,0)\rangle$ so $[\infty0241]$ has 30 distinct single cosets.

Now $\infty0\underline{242} = \infty0\underline{424} \in [\infty0131]$ since $N\infty0131$ conjugated by $(1,4)(2,3) \in L_2(5)$ is $N\infty0424$. Hence t_2 takes $[\infty024]$ to a single coset in $[\infty0131]$.

t_3 takes $[\infty024]$ to a single coset in $[\infty0243]$. $N\infty0243 = N(\infty,3,2,0,4)\ 432\infty0$ $= N432\infty0$ since $\infty0243 = (\infty,3,2,0,4)\ 432\infty0$. To prove this relation move it to one side of the equal sign and prove it equals identity.

$(\infty,3,2,0,4)432\underline{\infty0342}0\infty = (\infty,3,2,0,4)432\underline{(2,3,\infty,4,0)30\infty22}0\infty$ (by relation (1) conjugated by $(1,0,3,2,\infty) \in L_2(5) = 0\infty\underline{33}0\infty\underline{22}0\infty = 0\underline{\infty0\infty}0\infty = 0\underline{0\infty0}0\infty$ by relation (5) $= \underline{00}\infty\underline{00}\infty = \underline{\infty\infty} = e$.

$N^{(\infty0243)} = \langle(3,0)(4,\infty)\rangle$ so $[\infty0243]$ has 30 distinct single cosets.

t_0 takes $[\infty024]$ to a single coset in $[\infty0143]$, since $\infty0240$ $= (1,2,3)(4,\infty,0)2401\infty$ thus $N\infty0240 = N2401\infty \in [\infty0143]$ since $N2401\infty$ conjugated by $(1,0,4)(2,3,\infty) \in L_2(5)$ is $N2401\infty$. To prove $\infty0240 = (1,2,3)(4,\infty,0)2401\infty$ move the relation to one side of the equal sign and prove it equals identity.

$(1,2,3)(4,\infty,0)240\underline{1\infty04}20\infty = (1,2,3)(4,\infty,0)240\underline{(1,4,\infty,2,0)0\infty122}0\infty$ $= (1,0,\infty)(2,3,4)0\infty10\infty1\underline{22}0\infty = (1,0,\infty)(2,3,4)0\infty10\infty10\infty = e$ by relation (3).

t_∞ takes $[\infty024]$ to a single coset in $[\infty024\infty]$ which contains 60 distinct single cosets.

Let's consider $[\infty023]$. $N^{(\infty023)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$ on Ω. So we need to look at $[\infty0231]$, $[\infty0232]$, $[\infty0233]$, $[\infty0234]$, $[\infty0230]$, and $[\infty023\infty]$.

$[\infty0233] = [\infty02]$, so t_3 takes us from $[\infty023]$ back to a single coset in $[\infty02]$.

t_1 takes $[\infty023]$ to a single coset in $[\infty0120]$, since $\infty0231 = (1,\infty,4,2,3)3\infty20\infty$ thus $N\infty0231 = N3\infty20\infty \in [\infty0120]$ since $\infty0120$ conjugated by $(1,2,0,\infty,3) \in L_2(5)$ is $N3\infty20\infty$. To prove $\infty0231 = (1,\infty,4,2,3)3\infty20\infty$ move the relation to one side of

the equal sign and prove it equals identity.

$(1,\infty,4,2,3)3\infty20\underline{\infty132}0\infty = (1,\infty,4,2,3)3\infty20\underline{1,4,3,\infty,2)31\infty4}0\infty$ (by relation (1) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= (1,2,\infty,3,4)\infty2\underline{1031}\infty40\infty = (1,2,\infty,3,4)\infty2\underline{(1,3,0)(2,\infty,4)0130}\infty40\infty$ (by relation (3) conujugated by $(1,3,4,\infty,0) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)4\infty013\underline{0\infty40}\infty = (1,\infty,0)(2,4,3)4\infty013$ $\underline{(1,2,3)(4,\infty,0)\infty04\infty}\infty$ (by relation (3) conjugated by $(1,4)(2,3) \in L_2(5)$)

$= (1,0,2,\infty,4)\infty0421\infty04\underline{\infty\infty} = (1,0,2,\infty,4)\infty042\underline{1\infty04} = (1,0,2,\infty,4)$ $\infty042$ $\underline{(1,4,\infty,2,0)0\infty12}$ (by relation (2) conjugated by $(1,2,\infty,3,4) \in L_2(5)$

$= 21\infty\underline{00}\infty12 = 12\underline{\infty\infty}12 = 2112 = 22 = e.$

t_2 takes $[\infty023]$ to a single coset in $[\infty0232]$. $N\infty0\underline{232} = N\infty0\underline{323}$ by relation (5) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$.

t_4 takes $[\infty023]$ to a single coset in $[\infty0142]$, since $\infty0234 = (1,3,0)(2,\infty,4)\ 320\infty1$. Thus $N\infty0234 = N(1,3,0)(2,\infty,4)320\infty1 = N320\infty1 \in [\infty0142]$ since $N\infty0142$ conjugated by $(1,0,2)(3,4,\infty) \in L_2(5)$ is $N320\infty1$. To prove $\infty0234 = (1,3,0)(2,\infty,4)320\infty1$ move the relation to one side of the equal sign and prove it equals identity.

$(1,3,0)(2,\infty,4)320\infty143\underline{11}120\infty = (1,3,0)(2,\infty,4)320\infty\underline{1431}120\infty$ (by relation (3) conjugated by $(1,3,\infty)(2,0,4) \in L_2(5)$

$= (1,4,0,3,\infty)40\infty241\underline{3412}0\infty = (1,4,0,3,\infty)40\infty241\underline{(1,3,2,4,\infty)143}\infty0\infty$ (by relation (2) conjugated by $(1,2,4)(3,\infty,0) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)\infty014\infty3143\underline{\infty0\infty} = (1,\infty,3)(2,4,0)\infty014\infty3143\underline{0\infty0}$ (by relation (5))

$= (1,\infty,3)(2,4,0)\infty014\infty31\underline{430\infty}0 = (1,\infty,3)(2,4,0)\infty014\infty31$ $\underline{(2,0,4,\infty,3)0}3420$ (by relation (1) conjugated by $(1,3,\infty)(2,0,4) \in L_2(5)$)

$= (1,3)(2,\infty)341\infty\underline{3210}3420 = (1,3)(2,\infty)341\infty\underline{(1,3,0,2,4)1234}3420$ (by relation (1) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,0,2,\infty,4)013\infty12\underline{343}420 = (1,0,2,\infty,4)013\infty12\underline{434}420$ (by relation (6) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,0,2,\infty,4)0\underline{13\infty1}243\underline{4}420 = (1,0,2,\infty,4)0\underline{(1,\infty,3)(2,4,0)31\infty3}24320$ (by relation (4) conjugated by $(1,4,3,\infty,2) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)231\infty\underline{324}320 = (1,2,3)(4,\infty,0)231\infty\underline{(1,\infty,0)(2,3,4)234}220$

(by relation (4) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$)

$= (1,3,0,2,4)3401234\underline{22}0 = (1,3,0,2,4)34012340 = e$ by relation (1) conjugated by $(1,4)(2,3) \in L_2(5)$.

By relation (3) conjugated by $(1,3)(2,\infty) \in L_2(5)$ we know $N\infty\underline{0230}$ $= N\infty\underline{(2,0,3)(\infty,4,1)2032}$. So $N\infty0230 = N42032 \in [\infty0130]$ since $N\infty0130$ conjugated by $(1,0,2,\infty,4) \in L_2(5)$ is $N42032$. Hence t_0 takes $[\infty023]$ to a single coset in $[\infty0130]$.

By relation (2) conjugated by $(1,4,0,3,\infty) \in L_2(5)$, we know $N\infty\underline{023\infty} =$ $N\infty\underline{(0,\infty,2,4,3)3204} = N(0,\infty,2,4,3)23204 = N23204 \in [\infty0\infty12]$ since $N\infty0\infty12$ conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$ is $N23204$. Hence t_∞ takes $[\infty023]$ to a single coset in $[\infty0\infty12]$.

Now consider $[\infty021]$. $N^{(\infty021)}$ has orbits $\{4\}$, $\{0\}$, $\{1,3\}$, and $\{2,\infty\}$ on Ω. So we need to examine $[\infty0214]$, $[\infty0210]$, $[\infty0211]$, and $[\infty0212]$.

Start with $[\infty0211] = [\infty02]$, so t_1 a representative from one of the 2-orbits takes $[\infty021]$ back to $[\infty02]$.

t_4 takes $[\infty021]$ to a single coset in $[\infty0241]$, since $\infty0214 = (1,3,\infty,0,2)4301\infty$ thus $N\infty0214 = N4301\infty \in [\infty0241]$ since $\infty0241$ conjugated by $(1,\infty,4)(2,0,3) \in L_2(5)$ is $N4301\infty$. To prove $\infty0214 = (1,3,\infty,0,2)4301\infty$ move the relation to one side of the equal sign and prove it equals identity.

$(1,3,\infty,0,2)430\underline{1\infty312}0\infty = (1,3,\infty,0,2)430\ \underline{(1,4,\infty)(2,3,0)\infty14\infty}\ 20\infty$ (by relation (4) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= (1,0,3)(2,4,\infty)\infty02\infty14\ \underline{\infty20\infty} = (1,0,3)(2,4,\infty)\infty02\infty14$ $\underline{(1,4,3)(2,\infty,0)2\infty02}$ (by relation (4) conjugated by $(3,0)(4,\infty) \in L_2(5)$)

$= (1,2,3,4,0)02\infty\ \underline{0432}\ \infty02 = (1,2,3,4,0)02\infty\ \underline{(1,3,0,2,4)3401}\ \infty02$ (by relation (1) conjugated by $(1,4,2,0,3) \in L_2(5)$).

$= (1,4,2,0,3)2\ \underline{4\infty34}\ 01\infty02 = (1,4,2,0,3)2\ \underline{(1,2,0)(3,\infty,4)\infty43\infty}\ 01\infty02$ (by relation (3) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)0\infty43\ \underline{\infty01\infty}\ 02 = (1,3,2)(4,0,\infty)0\infty43$ $\underline{(1,0,\infty)(2,3,4)0\infty10}$ 02 (by relation (3))

$= (1,4,\infty,2,0)\infty1240\infty1\ \underline{00}2 = (1,4,\infty,2,0)\infty\ \underline{1240}\ \infty12 = (1,4,\infty,2,0)\infty$ $\underline{(1,0,2,\infty,4)421\infty}\ \infty12$ (by relation (2) conjugated by $(1,\infty,3,0,4) \in L_2(5)$)

$= \underline{44}21\ \underline{\infty\infty}12 = 2\underline{11}2 = \underline{22} = e$.

Now by relation (4) conjugated by $(1,3)(4,0) \in L_2(5)$ we know $N\infty\underline{0210} =$

$N\infty\underline{(\infty,4,3)(2,0,1)2012} = N(\infty,4,3)(2,0,1)42012 = N42012 \in [\infty0140]$ since $N\infty0140$ conjugated by $(0,2,\infty,4,1) \in L_2(5)$ is $N42012$. So t_0 takes $[\infty021]$ to a single coset in $[\infty0140]$.

By relation (5) conjugated by $(1,\infty)(2,0) \in L_2(5)$ we know $212 = 121$, so $[\infty0212] = [\infty0121]$ So t_2 a representative from the other 2-orbit, takes $[\infty021]$ to a single coset in $[\infty0121]$.

Now consider $[\infty020]$. $N^{(\infty020)}$ has orbits $\{1\}$, $\{\infty\}$, $\{2,0\}$, and $\{3,4\}$ on Ω. So we need to examine $[\infty0201]$, $[\infty020\infty]$, $[\infty0200]$, and $[\infty0203]$.

First $[\infty0200] = [\infty02]$. So t_0, a representative from one of the 2-orbits, takes $[\infty020]$ back to $[\infty02]$.

Now t_1 takes $[\infty020]$ to a single coset in $[\infty0201]$. By relation (6) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$ we know $020 = 202$, so it follows that $N\infty0201 = N\infty2021$. $N^{(\infty0201)} = \langle(2,0)(3,4)\rangle$ so there are 30 distinct single cosets in $[\infty0201]$.

t_∞ takes $[\infty020]$ to a single coset in $[\infty020\infty]$. By relation (6) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$ we know $N\infty\ \underline{020}\ \infty = N\infty\ \underline{202}\ \infty$. Additionally, $N\infty020\infty = N\infty0\ \underline{\infty\infty}\ 20\infty = N\infty0\infty\ \underline{\infty20\infty} = N\infty0\infty\ \underline{(4,3,1)(2,\infty,0)2\infty02}$ (by relation (4) conjugated by $(3,0)(4,\infty) \in L_2(5)$).

$\Longrightarrow N\infty020\infty = N(4,3,1)(2,\infty,0)0\ \underline{202}\ \infty02 = N(4,3,1)(2,\infty,0)0\ \underline{020}\ \infty02$ by relation (6) conjugated by $(1,0)(4,2) \in L_2(5)$.

$\Longrightarrow N\infty020\infty = N(4,3,1)(2,\infty,0)\ \underline{00}20\infty02 = N(4,3,1)(2,\infty,0)20\infty02 = N2\underline{0\infty0}2 = N2\underline{\infty0\infty}2$ by relation (5). So we know $N\infty020\infty = N\infty202\infty = N20\infty02 = N2\infty0\infty2$.

$N2\infty0\infty2 = N2\infty\underline{22}0\infty2 = N2\infty2\underline{20\infty2} = N2\infty2\underline{(1,4,3)(2,\infty,0)02\infty0}$ by relation (4) conjugated by $(1,4,0,3,\infty) \in L_2(5)$.

$\Longrightarrow N2\infty0\infty2 = N(1,4,3)(2,\infty,0)\underline{\infty0\infty}02\infty0 = N(1,4,3)(2,\infty,0)\underline{0\infty0}02\infty0 = N(1,4,3)(2,\infty,0)0\infty\underline{00}2\infty0 = N(1,4,3)(2,\infty,0)0\infty2\infty0 = N0\underline{\infty2\infty}0 = N0\underline{2\infty2}0$ by relation 6 conjugated by $(1,\infty,4,2,3) \in L_2(5)$. Hence $N\infty020\infty = N\infty202\infty = N20\infty02 = N2\infty0\infty2 = N0\infty2\infty0 = N02\infty20$.

$N^{(\infty020\infty)} = \langle(1,4)(0,\infty),(1,4,3)(2,\infty,0)\rangle$ so there are 10 distinct single cosets in $[\infty020\infty]$.

Finally t_3, a representative from the other 2-orbit, takes $[\infty020]$ to a single coset in $[\infty0203]$. There are 60 distinct single cosets in $[\infty0203]$.

Now consider $[\infty02\infty]$. $N^{(\infty02\infty)}$ has orbits $\{2\}$, $\{3\}$, $\{1,4\}$, and $\{0,\infty\}$ on Ω. So we need to examine $[\infty02\infty2]$, $[\infty02\infty3]$, $[\infty02\infty1]$, and $[\infty02\infty\infty]$.

$[\infty02\infty\infty] = [\infty02]$, so t_∞, a representative from one of the 2-orbits, takes $[\infty02\infty]$ back to a single coset in $[\infty02]$.

Now $N\infty02\infty2 = N\underline{00}\infty02\infty2 = N0\underline{0\infty0}2\infty2 = N0\underline{\infty0\infty}2\infty2$ by relation (5).

$\Longrightarrow N\infty02\infty2 = N0\infty0\underline{\infty2\infty}2 = N0\infty0\underline{2\infty2}2$ by relation (5) conjugated by $(2,4,3,0,\infty) \in L_2(5)$

$\Longrightarrow N\infty02\infty2 = N0\infty02\infty\underline{22} = N0\infty02\infty \in [\infty0\infty20]$ since $N\infty0\infty20$ conjugated by $(1,4)(0,\infty) \in L_2(5)$ is $N0\infty02\infty$. So t_2 takes $[\infty02\infty]$ to a single coset in $[\infty0\infty20]$.

t_3 takes $[\infty02\infty]$ to a single coset in $[\infty0243]$, since $\infty02\infty3$ $= (1,\infty,4,2,3)\infty1304$ thus $N\infty02\infty3 = N\infty1304 \in [\infty0243]$ since $N\infty0243$ conjugated by $(1,2,3,4,0) \in L_2(5)$ is $N\infty1304$. To prove $\infty02\infty3 = (1,\infty,4,2,3)\infty1304$ move the relation to one side of the equal sign and prove it equals identity

$\Longrightarrow (1,\infty,4,2,3)\infty1\underline{3043}\infty20\infty$

$= (1,\infty,4,2,3)\infty1\underline{(1,\infty,2)(3,4,0)0340}\infty20\infty$ (by relation (4) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,2,4)(3,\infty,0)2\infty034\underline{0\infty20}\infty = (1,2,4)(3,\infty,0)2\infty034$ $\underline{(1,4,3)(2,\infty,0)\infty02\infty}\ \infty$ (by relation (4) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$)

$= (1,\infty,2,3,0)\infty0213\infty02\underline{\infty\infty} = (1,\infty,2,3,0)\infty\ \underline{0213}\ \infty02 = (1,\infty,2,3,0)\infty$ $\underline{(1,0,3,2,\infty)120\infty}\ \infty02$ (by relation (1) conjugated by $(1,2,0)(3,\infty,4) \in L_2(5)$)

$= \underline{11}20\underline{\infty\infty}02 = 2\underline{00}2 = 22 = e.$

Finally, t_1, a representative from the other 2-orbit, takes $[\infty02\infty]$ to a single coset in $[\infty02\infty1]$. By relation (4) conjugated by $(2,0,4,\infty,3) \in L_2(5)$ we know $(3,1,4)(0,2,\infty)$ $\infty02\infty = 0\infty20$. So $N\underline{\infty02\infty}1 = N\underline{(3,4,1)(0,\infty,2)0\infty20}1 = (3,4,1)(0,\infty,2)0\ \underline{\infty201} =$ $(3,4,1)(0,\infty,2)0\ \underline{(\infty,1,2,4,0)02\infty4}$ by relation (1) conjugated by $(1,2,0,\infty,3) \in L_2(5)$.

$\Longrightarrow N\infty02\infty1 = N(3,0,1)(2,\infty,4)\ \underline{\infty02\infty4} = N(3,0,1)(2,\infty,4)$ $\underline{(3,4,1)(0,\infty,2)0\infty204}$ by relation (4) conjugated by $(2,0,4,\infty,3) \in L_2(5)$

$\Longrightarrow N\infty02\infty1 = N(3,\infty,1,4,0)0\infty204 = N0\infty204$. $N^{(\infty02\infty1)} = \langle(1,4)(0,\infty)\rangle$ so there are 30 distinct single cosets in $[\infty02\infty1]$.

Now consider $[\infty01\infty2]$. $N^{(\infty01\infty2)}$ has orbits $\{1\}$, $\{4\}$, $\{2,3\}$, and $\{0,\infty\}$.

$[\infty01\infty22] = [\infty01\infty]$ since t_2 is of order 2. So t_2 a representative from on of

the 2-orbits takes $[\infty01\infty2]$ back to a single coset in $[\infty01\infty]$.

t_1 takes $[\infty01\infty2]$ to a single coset in $[\infty01\infty21]$.

We previously proved that $N\infty01\infty2 = N0\infty103$, so $N\infty01\infty2\underline{1} = N0\infty103\underline{1}$.

Now $N\infty\underline{01\infty2}1 = N\infty\ \underline{(0,2,1,3,\infty)\infty1031}$ by relation (2) conjugated by $(1,3,2)(4,0,\infty) \in L_2(5)$

$\Longrightarrow N\infty01\infty21 = N(0,2,1,3,\infty)0\infty\ \underline{1031} = N(0,2,1,3,\infty)0\infty$ $\underline{(0,1,3)(2,\infty,4)0130}$ by relation (3) conjugated by $(1,3,4,\infty,0) \in L_2(5)$.

$\Longrightarrow N\infty01\infty21 = N(0,\infty,1)(2,3,4)\ \underline{1401}30 = N(0,\infty,1)(2,3,4)$ $\underline{(\infty,2,3)(4,1,0)410430}$ by relation (2) conjugated by $(1,3,0,2,4) \in L_2(5)$.

$\Longrightarrow N\infty01\infty21 = N(0,2,\infty)(1,4,3)41\ \underline{0430} = N(0,2,\infty)(1,4,3)41$ $\underline{(1,\infty,2)(4,0,3)4034}$ by relation (4) conjugated by $(1,2,4,0,\infty) \in L_2(5)$.

$\Longrightarrow N\infty01\infty21 = N(1,0)(3,\infty)\ \underline{0\infty40}34 = N(1,0)(3,\infty)$ $\underline{(\infty,0,4)(3,1,2)\infty04\infty}34 = N(1,4,\infty)(2,3,0)\infty04\infty34 = N\infty04\infty34$.

So we now know $N\infty01\infty21 = N\infty04\infty34$.

Now $N\infty\underline{04\infty34} = N\infty\underline{(\infty,4,3)(2,0,1)\infty43\infty}$ by relation (3) conjugated by $(1,3)(4,0) \in L_2(5)$.

$\Longrightarrow N\infty04\infty34 = N(\infty,4,3)(1,2,0)\ \underline{41\infty4}3\infty = N(\infty,4,3)(1,2,0)$ $\underline{(0,2,3)(1,4,\infty)14\infty13}\infty$ by relation (4) conjugated by $(1,3,\infty,0,2) \in L_2(5)$.

$\Longrightarrow N\infty04\infty34 = N(1,3)(4,0)1\underline{4\infty13}\infty = N(1,3)(4,0)1\underline{(4,3,\infty,2,1)1\infty42}\infty$ by relation (1) conjugated by $(1,\infty,0)(2,4,3) \in L_2(5)$.

$\Longrightarrow N\infty04\infty34 = N(1,\infty,2)(3,4,0)41\ \underline{\infty42\infty} = N(1,\infty,2)(3,4,0)41$ $\underline{(1,0,3)(4,\infty,2)4\infty24}$ by relation (4) conjugated by $(1,3,2,4,\infty) \in L_2(5)$.

$\Longrightarrow N\infty04\infty34 = N(1,2,0)(3,\infty,4)\ \underline{\infty04\infty}24 = N(1,2,0)(3,\infty,4)$ $\underline{(\infty,4,0)(3,2,1)0\infty4}024$ by relation (3) conjugated by $(1,4)(2,3) \in L_2(5)$.

$\Longrightarrow N\infty04\infty34 = N(2,\infty,0,3,4)0\infty4024 = N0\infty4024$

Hence $N\infty04\infty34 = N0\infty4024 = N\infty04\infty34 = N0\infty1031$.

$N^{(\infty01\infty21)} = \langle(1,4)(0,\infty),(2,3)(0,\infty)\rangle$ so there are 15 distinct single cosets in $[\infty01\infty21]$.

t_4 takes $[\infty01\infty2]$ to a single coset in $[\infty01\infty24]$.

$N\infty\underline{01\infty2}4 = N\infty\underline{(0,2,1,3,\infty)\infty1034}$ by relation (2) conjugated by $(1,3,2)(4,0,\infty) \in L_2(5)$

$\Longrightarrow N\infty01\infty24 = N(0,2,1,3,\infty)0\infty1034 = N0\infty1034$.

$N^{(\infty 01\infty 24)} = \langle (2,3)(0,\infty)\rangle$ so there are 30 distinct single cosets in $[\infty 01\infty 24]$.

t_0, a representative from the 2-orbit, takes $[\infty 01\infty 2]$ to a single coset in $[\infty 01\infty 20]$. There are 60 distinct single cosets in $[\infty 01\infty 20]$.

Now consider $[\infty 01\infty 4]$. $N^{(\infty 01\infty 4)}$ has orbits $\{1\}$, $\{4\}$, $\{2,3\}$, and $\{0,\infty\}$ on Ω. So we need to consider $[\infty 01\infty 41]$, $[\infty 01\infty 44]$, $[\infty 01\infty 42]$, and $[\infty 01\infty 40]$.

First $[\infty 01\infty 44] = [\infty 01\infty]$, since t_4 has order 2. So t_4 takes $[\infty 01\infty 4]$ back to a single coset in $[\infty 01\infty]$.

Next t_1 takes $[\infty 01\infty 4]$ to a single coset in $[\infty 01\infty 41]$. $N\underline{\infty 01\infty}41$ $= N\underline{(\infty,1,0)(2,3,4)0\infty 10}41$ by relation (3).

$\Longrightarrow N\infty 01\infty 41 = N0\infty 1041$

Additionally, $N\infty 0\underline{1\infty 41} = N\infty 0\underline{(3,0,2)(1,4,\infty)\infty 14\infty}$ by relation (4) conjugated by $(1,0,2)(3,4,\infty) \in L_2(5)$.

$\Longrightarrow N\infty 01\infty 41 = N(3,0,2)(1,4,\infty)12\infty 14\infty = N12\infty 14\infty$

If we conjugate the new relation we found above by $(1,0,3,2,\infty) \in L_2(5)$ we now find $N0\infty 1041 = N130140$.

So now we have the following equations:

$N\infty 01\infty 41 = N0\infty 1041 = N130140 = N12\infty 14\infty$

If we conjugate this equation by $(1,\infty,2,3,0) \in L_2(5)$, then conjugate the resulting equation by $(1,3)(2,\infty) \in L_2(5)$ we then find the following:

$N\infty 01\infty 41 = N0\infty 1041 = N130140 = N12\infty 14\infty = N21\infty 24\infty = N\infty 32\infty 42$ $= N203243 = N023043 = N3\infty 2342 = N310340$

$N^{(\infty 01\infty 41)} = \langle (1,0)(3,\infty), (1,3,\infty,0,2)\rangle$ so there are 6 distinct single cosets in $[\infty 01\infty 41]$.

Now let's look at $[\infty 01\infty 42]$. As previously shown $N\infty 0124 = N(\infty,3,2)(1,4,0)$ $\infty 23\infty 1$.

$\Longrightarrow N\infty 0124\underline{4} = N(\infty,3,2)(1,4,0)\infty 23\infty 1\underline{4}$.

$\Longrightarrow N\infty 012 = N(\infty,3,2)(1,4,0)\infty 23\infty 14 = N\infty 23\infty 14 \in [\infty 01\infty 42]$ since $N\infty 01\infty 42$ conjugated by $(1,3,0,2,4) \in L_2(5)$ is $N\infty 23\infty 14$.

So t_2, a representative from one of the 2-orbits, takes $\infty 01\infty 4]$ to a single coset in $[\infty 012]$.

t_0, a representative from the other 2-orbit, takes $[\infty 01\infty 4]$ to a single coset in $[\infty 01\infty 40]$. There are 60 distinct single cosets in $[\infty 01\infty 40]$.

Now consider $[\infty 0102]$. $N^{(\infty 0102)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to consider $[\infty 01021]$, $[\infty 01022]$, $[\infty 01023]$, $[\infty 01024]$, $[\infty 01020]$ and $[\infty 0102\infty]$.

$[\infty 01022] = [\infty 010]$ so t_2 takes $[\infty 0102]$ back to a single coset in $[\infty 010]$.

t_1 takes $[\infty 0102]$ to a single coset in $[\infty 01403]$, since $\infty 01021 =$ $(1,3,2,4,\infty)\infty 40342$ thus $N\infty 01021 = N\infty 40342 \in [\infty 01403]$ since $N\infty 01403$ conjugated by $(1,0,4,3,2) \in L_2(5)$ is $N\infty 40342$. To prove $\infty 01021 = (1,3,2,4,\infty))\infty 40342$ move the relation to one side of the equal sign and prove it equals identity.

$(1,3,2,4,\infty)\infty 4034\underline{212010}\infty = (1,3,2,4,\infty)\infty 4034\underline{121101}\infty$ (by relation (6) conjugated by $(2,4)(3,\infty)$ and $(2,\infty)(4,0) \in L_2(5)$)

$= (1,3,2,4,\infty)\infty 403412\underline{11}01\infty = (1,3,2,4,\infty)\infty 4034\ \underline{1201}\infty = (1,3,2,4,\infty)$ $\infty 4034\ \underline{(1,0,2)(3,4,\infty)2102}\infty$ (by relation (4) conjugated by $(1,\infty,3,0,4) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)3\underline{\infty 24\infty}2102\infty = (1,4,3)(2,\infty,0)3\ \underline{(1,3,0)(2,\infty,4)2\infty 42}$ 2102∞ (by relation (4) conjugated by $(0,3,1)(2,4,\infty) \in L_2(5)$)

$= (1,2,4,0,\infty)02\underline{\infty 4102}\infty = (1,2,4,0,\infty)02\underline{(1,\infty,0,4,2)14\infty 22}\infty$ (by relation (1) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$)

$= 4\underline{11}4\infty\underline{22}\infty = \underline{44\infty\infty} = e$.

t_3 takes $[\infty 0102]$ to a single coset in $[\infty 01023]$ since $N\infty 010231 = N\infty 4342$ $\in [\infty 0102]$. To prove $\infty 01023 = (2\infty)(4,0)\infty 43421$ we will move the relation to one side of the equal sign and prove it equals identity.

$(2,\infty)(4,0)\infty 434\underline{2132}010\infty = (2,\infty)(4,0)\infty 434\underline{(1,2,3)(4,\infty,0)1231}010\infty$ (by relation (4) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (1,2,0,\infty,3)0\underline{\infty 1\infty}1231\underline{010}\infty = (1,2,0,\infty,3)01\underline{\infty 1}1231\underline{101}\infty$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$ and $(1,\infty,0)(2,4,3) \in L_2(5)$)

$= (1,2,0,\infty,3)01\infty\underline{11}23\underline{11}01\infty = (1,2,0,\infty,3)01\infty 2301\infty = e$ (by relation (2) conjugated by $(1,3,2)(4,0,\infty) \in L_2(5)$).

$N^{(\infty 01023)} = \langle(1,3)(4,0)\rangle$ so there are 30 distinct single cosets in $[\infty 01023]$.

t_4 takes $[\infty 0102]$ to a single coset in $[\infty 01024]$. There are 60 distinct single cosets in $[\infty 01024]$.

t_0 takes $[\infty 0102]$ to a single coset in $[\infty 01020]$. Now $N\infty\underline{01}020 = N\infty\underline{10}120$ by relation (6) conjugated by $(2,\infty)(4,0) \in L_2(5)$.

$\implies N\infty 01020 = N\infty 1\underline{0120} = N\infty 1\underline{(4,\infty,3)(1,0,2)1021}$ by relation (4) conjugated by $(1,3,2)(4,0,\infty) \in L_2(5)$.

$\Longrightarrow N\infty01020 = N(1,0,2)(3,4,\infty)3\underline{01}021 = N(1,0,2)(3,4,\infty)3\underline{10}121$ by relation (6) conjugated by $(2,\infty)(4.0) \in L_2(5)$.

$\Longrightarrow N\infty01020 = N310121$.

$N^{(\infty01020)} = \langle(1,0)(3,\infty)\rangle$ so there are 30 distinct single cosets in $[\infty01020]$.

Now let's look at $[\infty0102\infty]$. $N\infty0\underline{102\infty} = N\infty0\underline{(1,\infty,0,4,2)2014}$ by relation (1) conjugated by $(1,0,2)(4,\infty,3) \in L_2(5)$

$\Longrightarrow N\infty0102\infty = N(1,\infty,0,4,2)\underline{0420}14 = N(1,\infty,0,4,2)$ $\underline{(0,2,4)(\infty,3,1)402414}$ by relation (3) conjugated by $(1,2)(4,\infty) \in L_2(5)$.

$\Longrightarrow N\infty0102\infty = N(1,3)(2,\infty)4\underline{0241}4 = N(1,3)(2,\infty)4\underline{(0,1,2,3,4)42034}$ by relation (2).

$\Longrightarrow N\infty0102\infty = N(1,4,0)(2,\infty,3)042034 = N042034 \in [\infty01\infty20]$ since $N\infty01\infty20$ conjugated by $(1,2,3)(4,\infty,0)$ is $N042034$.

So t_∞ takes $[\infty0102]$ to a single coset in $[\infty01\infty20]$.

Consider $[\infty0103]$. $N^{(\infty0103)}$ has orbits $\{3\}$, $\{\infty\}$, $\{1,0\}$, and $\{2,4\}$. So we need to look at $[\infty01033]$, $[\infty0103\infty]$, $[\infty01031]$, and $[\infty01032]$.

$[\infty01033] = [\infty010]$, so t_3 takes $[\infty0103]$ back to a single coset in $[\infty010]$.

t_∞ takes $[\infty0103]$ to a single coset in $[\infty0103\infty]$. $N\infty\underline{010}3\infty = N\infty\underline{101}3\infty$ by relation (6) conjugated by $(2,\infty)(4,0) \in L_2(5)$. $N^{(\infty0103\infty)} = \langle(1,0)(3,4)\rangle$, so there are 30 distinct single cosets in $[\infty0103\infty]$.

t_1, a representative from one of the 2-orbits, takes $[\infty0103]$ to a single coset in $[\infty01031]$. There are 60 distinct single cosets in $[\infty01031]$.

t_2, a representative from the other 2-orbit, takes $[\infty0103]$ to a single coset in $[\infty01032]$. There are 60 distinct single cosets in $[\infty01032]$.

Now consider $[\infty010\infty]$. $N^{(\infty010\infty)}$ has orbits $\{1,0,\infty\}$ and $\{2,3,4\}$. So we need to look at $[\infty010\infty\infty]$ and $[\infty010\infty2]$.

$[\infty010\infty\infty] = [\infty010]$, so t_∞, a representative from one of the 3-orbits, takes $[\infty010\infty]$ back to a single coset in $[\infty010]$.

t_2, a representative from the other 3-orbit, take $[\infty010\infty]$ to a single coset in $[\infty010\infty2]$.

$N\infty010\infty2 = N\infty010\infty\underline{11}2 = N\infty0\underline{10\infty1}12 = N\infty0\underline{(1,\infty,0)(2,4,3)01\infty0}12$ by relation (3) conjugated by $(1,\infty,0)(2,4,3) \in L_2(5)$.

$\Longrightarrow N\infty010\infty2 = N(1,\infty,0)(2,4,3)0\underline{101}\infty012 = N(1,\infty,0)(2,4,3)0\underline{010}\infty012$ by relation (6) conjugated by $(2,\infty)(4,0) \in L_2(5)$.

$\Longrightarrow N\infty010\infty2 = N(1,\infty,0)(2,4,3)\underline{00}10\infty012 = N(1,\infty,0)(2,4,3)10\infty012 = N10\infty012$

$N^{(\infty010\infty2)} = \langle(1,\infty)(3,4)\rangle$ so there are 30 distinct single cosets in $[\infty010\infty2]$.

Now consider $[\infty0121]$. $N^{(\infty0121)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty01211]$, $[\infty01212]$, $[\infty01213]$, $[\infty01214]$, $[\infty01210]$, and $[\infty0121\infty]$.

$[\infty01211] = [\infty012]$, so t_1 takes $[\infty0121]$ back to a single coset in $[\infty012]$.

$N\infty01\underline{212} = N\infty01\underline{121}$ by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$.

$\Longrightarrow N\infty01212 = N\infty0\underline{11}21 = N\infty021 \in [\infty021]$.

So t_2 takes $[\infty0121]$ to a single coset in $[\infty021]$.

t_3 takes $[\infty0121]$ to a single coset in $[\infty01213]$. There are 60 distinct single cosets in $[\infty01213]$.

t_4 takes $[\infty0121]$ to a single coset in $[\infty01\infty24]$, since $\infty01214$ $= (1,3,4,\infty,0)\infty23\infty41$ thus $N\infty01214 = N\infty23\infty41 \in [\infty01\infty24]$ since $N\infty01\infty24$ conjugated by $(1,3,0,2,4) \in L_2(5)$ is $N\infty23\infty41$. To prove $\infty01214$ $= (1,3,4,\infty,0)\infty23\infty41$ move the relation to one side of the equal sign and prove it equals identity.

$(1,3,4,\infty,0)\infty23\infty\underline{414}1210\infty = (1,3,4,\infty,0)\infty23\infty\underline{141}1210\infty$ (by relation (4))

$= (1,3,4,\infty,0)\infty23\infty14\underline{11}210\infty = (1,3,4,\infty,0)\infty23\infty\ \underline{1421}\ 0\infty$

$= (1,3,4,\infty,0)\infty23\infty\ \underline{(1,2,4)(3,\infty,0)4124}\ 0\infty$ (by relation (3) conjugated by $(1,2,0)(3,\infty,4) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)0\underline{4\infty04}1240\infty = (1,\infty,3)(2,4,0)0\underline{(1,3,2)(4,0,\infty)\infty40\infty}$ 1240∞ (by relation (3) conjugated by $(1,0,4,3,2) \in L_2(5)$)

$= (1,4,\infty,2,0)\underline{\infty\infty}40\infty1240\infty = (1,4,\infty,2,0)40\infty1240\infty = e$ (by relation (2) conjugated by $(1,2,0,\infty,3) \in L_2(5)$).

$N\infty0\underline{121}0 = N\infty0\underline{212}0$ by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$.

$\Longrightarrow N\infty01210 = N\underline{\infty02}120 = N\underline{(\infty,1,0,3,2)20\infty}320$ by relation (1) conjugated by $(1,0,2,\infty,4) \in L_2(5)$.

$\Longrightarrow N\infty01210 = N(\infty,1,0,3,2)20\underline{\infty320} = N(\infty,1,0,3,2)20\underline{(\infty,0,3,4,2)23\infty4}$ by relation (2) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$.

$\Longrightarrow N\infty01210 = N(\infty,1,3)(2,0,4)\underline{\infty323\infty}4 = N(\infty,1,3)(2,0,4)$

$\underline{(\infty, 3, 2)(4, 0, 1)230324}$ by a previously proven relation, $\infty010\infty$ $= (\infty, 0, 1)(4, 3, 2)10\infty01$, conjugated by $(1, 2)(3, 0) \in L_2(5)$.

$\Longrightarrow N\infty01210 = N(1, 2)(4.\infty)230324 = N230324 \in [\infty010\infty2]$ since $N\infty010\infty2$ conjugated by $(1, 0, 3)(2, 4, \infty) \in L_2(5)$ is $N230324$.

So t_0 takes $[\infty0121]$ to a single coset in $[\infty010\infty2]$.

t_∞ takes $[\infty0121]$ to a single coset in $[\infty01032]$, since $\infty0121\infty$ $= (1, \infty)(2, 0)431302$ thus $N\infty0121\infty = N(1, \infty)(2, 0)431302 = N431302 \in [\infty01032]$ since $N\infty01032$ conjugated by $(3, 0)(4, \infty) \in L_2(5)$ is $N431302$.. To prove $\infty0121\infty$ $= (1, \infty)(2, 0)431302$ move the relation to one side of the equal sign and prove it equals identity.

$(1, \infty)(2, 0)43\underline{1302}\infty1210\infty = (1, \infty)(2, 0)43\underline{(1, 2, 3, 4, 0)0314}\infty1210\infty$ (by relation (2) conjugated by $(1, 2, 3, 4, 0) \in L_2(5)$)

$= (1, \infty, 2)(3, 4, 0)0403\underline{14\infty01}210\infty = (1, \infty, 2)(3, 4, 0)\ 0403$ $\underline{(1, \infty, 4)(2, 0, 3)41\infty4}\ 210\infty$ (by relation (4) conjugated by $(1, 3, \infty, 0, 2) \in L_2(5)$)

$= (1, 4, 3)(2, \infty, 0)\underline{313}241\infty43210\infty = (1, 4, 3)(2, \infty, 0)\underline{131}241\infty43210\infty$ (by relation (6) conjugated by $(2, 0)(3, 4) \in L_2(5)$)

$= (1, 4, 3)(2, \infty, 0)13\underline{1241}\infty43210\infty = (1, 4, 3)(2, \infty, 0)13\ \underline{(1, 4, 2)(3, 0, \infty)2142}$ $\infty43210\infty$ (by relation (3) conjugated by $(1, 4, \infty)(2, 3, 0) \in L_2(5)$)

$= (1, 2, 3, 4, 0)4021\underline{42\infty04}210\infty = (1, 2, 3, 4, 0)4021\ \underline{(1, 0, 3)(4, \infty, 2)24\infty2}\ 210\infty$ (by relation (4) conjugated by $(2, 4)(3, \infty) \in L_2(5)$)

$= (1, 4, 3, \infty, 2)\infty3\underline{4024}\infty\underline{22}10\infty = (1, 4, 3, \infty, 2)\infty3\underline{(1, 3, \infty)(2, 0, 4)0420}\infty10\infty$ (by relation (3) conjugated by $(1, 2)(4, \infty) \in L_2(5)$)

$= ((1, 2, 3)(4, \infty, 0)1\infty0420\underline{\infty10}\infty = ((1, 2, 3)(4, \infty, 0)1\infty042$ $\underline{(1, \infty, 0)(2, 4, 3)\infty01\infty}\infty$ (by relation (3))

$= (1, 4, 0, 3, \infty)\infty1034\infty01\underline{\infty\infty} = (1, 4, 0, 3, \infty)\infty1034\infty01 = e$ by relation (2) conjugated by $(1, 4, \infty, 2, 0) \in L_2(5)$.

Now consider $[\infty0123]$. $N^{(\infty0123)}$ has orbits $\{1\}$, $\{\infty\}$, $\{2, 0\}$, and $\{3, 4\}$. So we need to look at $[\infty01231]$, $[\infty0123\infty]$, $[\infty01232]$, $[\infty01233]$,

First $[\infty01233] = [\infty012]$. So t_3, a representative from one of the 2-orbits, takes $[\infty0123]$ back to a single coset in $[\infty012]$.

t_1 takes $[\infty0123]$ to a single coset in $[\infty01231]$.

$N\infty\underline{0123}1 = N\infty\underline{(0, 3, 1, 4, 2)2104}1$ by relation (1).

$\Longrightarrow N\infty01231 = N(0,3,1,4,2)\infty21041 = N\infty21041 \in [\infty21041]$.

$N^{(\infty01231)} = \langle(2,0)(3,4)\rangle$ so there are 30 distinct single cosets in $[\infty01231]$.

As previously shown, $\infty0132 = (1,\infty,0)(2,4,3)23041$. So it follows that $N\infty0132\underline{2} = N(1,\infty,0)(2,4,3)\ 23041\underline{2}$

$\Longrightarrow N\infty013 = N(1,\infty,0)(2,4,3)230412 = N230412 \in [\infty0123\infty]$ since $N\infty0123\infty$ conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$ is $N230412$.

So t_∞ takes $[\infty0123]$ to a single coset in $[\infty013]$.

t_2, a representative from the other 2-orbit, takes $[\infty0123]$ to a single coset in $[\infty01232]$.

$N\infty01232 = N314\infty2\infty = N2403\infty3$ since $\infty01232 = (0,2,4,1,3)\ 314\infty2\infty$ and $\infty01232 = (1,3,4,\infty,0)\ 2403\infty3$.

To prove $\infty01232 = (0,2,4,1,3)314\infty2\infty$ we will first move the relation to one side of the equal sign and then prove it equals identity.

$(0,2,4,1,3)314\underline{\infty2\infty}23210\infty = (0,2,4,1,3)314\ \underline{2\infty2}\ 23210\infty$ (by relation (5) conjugated by $(1,3,4,2,0,\infty) \in L_2(5)$)

$= (0,2,4,1,3)3142\infty\underline{22}3210\infty = (0,2,4,1,3)3142\infty\ \underline{3210}\infty = (0,2,4,1,3)\ 3142\infty\ \underline{(1,3,0,2,4)1234}\infty$ (by relation (1) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,0,4,3,2)03\underline{14\infty1}234\infty = (1,0,4,3,2)03\ \underline{(1,\infty,4)(2,0,3)41\infty4}\ 234\infty$ (by relation (4) conjugated by $(1,2,4)(3,\infty,0) \in L_2(5)$)

$= (1,3,0)(2,\infty,4)324\underline{1\infty42}34\infty = (1,3,0)(2,\infty,4)\ 324\ \underline{(1,2,\infty,3,4)4\infty13}\ 34\infty$ (by relation (1) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (1,4,\infty)(2,3,0)4\infty14\infty1\underline{33}4\infty = (1,4,\infty)(2,3,0)4\infty14\infty14\infty = e$ (by relation (4) conjugated by $(1,3)(2,\infty) \in L_2(5)$).

To prove $\infty01232 = (1,3,4,\infty,0)2403\infty3$ we will first move the relation to one side of the equal sign and then prove it equals identity.

$(1,3,4,\infty,0)2403\infty\underline{323}210\infty = (1,3,4,\infty,0)\ 2403\infty\ \underline{232}\ 210\infty$ (by relation (5) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,3,4,\infty,0)\ 2403\infty23\ \underline{22}\ 10\infty = (1,3,4,\infty,0)2\ \underline{403\infty}\ 2310\infty = (1,3,4,\infty,0)\ \underline{2(1,3,4,\infty,0)3041}\ 2310\infty$ (by relation (1) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (1,4,0,3,\infty)230412\underline{310\infty} = (1,4,0,3,\infty)230412\underline{(1,4,0,3,\infty)0134}$ (by relation (2) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,0,\infty,4,3)2\underline{\infty304}20134 = (1,0,\infty,4,3)2\underline{(1,0,\infty,4,3)03\infty1}20134$ (by rela-

tion (1) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5))$

$= (1,\infty,3,0,4)203\infty12\underline{0134} = (1,\infty,3,0,4)203\infty12\underline{(1,\infty,3,0,4)310\infty}$ (by relation (2) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5))$

$= (1,3,4,\infty,0)240\underline{3\infty23}10\infty = (1,3,4,\infty,0)240\underline{(1,4,0)(2,\infty,3)\infty32\infty}10\infty$ (by relation (3) conjugated by $(1,2)(3,0) \in L_2(5))$

$= (1,2,\infty)(3,0,4)\underline{\infty01\infty3}2\infty10\infty = (1,2,\infty)(3,0,4)\ \underline{(1,0,\infty)(2,3,4)0\infty10}$ $32\infty10\infty$ (by relation (3))

$= (1,3,\infty,0,2)0\underline{\infty1032}\infty10\infty = (1,3,\infty,0,2)0\underline{(1,2,0,\infty,3)01\infty2}2\infty10\infty$ (by relation (2) conjugated by $(1,2)(4,\infty) \in L_2(5))$

$= \infty01\infty\underline{22}\infty10\infty = \infty01\underline{\infty\infty}10\infty = \infty0\underline{11}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

$N^{(\infty01232)} = \langle(1,0,4)(2,3,\infty)\rangle$ so there are 20 distinct single cosets in $[\infty01232]$.

Now consider $[\infty0120]$. $N^{(\infty0120)}$ has orbits $\{2\}$, $\{4\}$, $\{1,0\}$, and $\{3,\infty\}$. So we need to look at $[\infty01202]$, $[\infty01204]$, $[\infty01200]$, and $[\infty01203]$.

First $[\infty01200] = [\infty012]$. So t_0, a representative from one of the 2-orbits, takes $[\infty0120]$ back to a single coset in $[\infty012]$.

$[\infty01\underline{202}] = [\infty01\underline{020}]$ (by relation (6) conjugated by $(1,0)(3,4) \in L_2(5))$. So t_2 takes $[\infty0120]$ to a single coset in $[\infty01020]$.

t_4 takes $[\infty0120]$ to a single coset in $[\infty01204]$. $N\infty\ \underline{0120}4$ $= N\infty\ \underline{(1,0,2)(3,4,\infty)1021}4 = N(1,0,2)(3,4,\infty)\ 310214 = N310214$. If we conjugate this relation by $(1,4)(2,3)$ and again by $(2.\infty)(4,0) \in L_2(5)$, we obtain $N\infty01204 = N310214$, $N\infty04301 = N240341$, and $N241\infty40 = N314\infty10$.

We can prove $N\infty01204 = N\infty04301$ by proving $\infty01204 = (1,2,\infty,3,4)$ $\infty04301$. We will do this by moving the relation to one side of the equal sign and proving it equals identity.

$(1,2,\infty,3,4)\infty0\underline{4301}40210 = (1,2,\infty,3,4)\infty0\ \underline{(1,\infty,2)(3,4,0)4034}\ 40210$ (by relation (4) conjugated by $(1,2,4,0,\infty) \in L_2(5))$

$= (3,0)(4,\infty)2403\underline{414}0210\infty = (3,0)(4,\infty)2403\underline{141}0210\infty$ (by relation (6))

$= (3,0)(4,\infty)240314\underline{1021}0\infty = (3,0)(4,\infty)240314\underline{(1,2,0)(3,\infty,4)0120}0\infty$ (by relation (4) conjugated by $(1,\infty,4)(2,0,3) \in L_2(5))$

$= (1,2,0,\infty,3)031\infty23012\underline{00}\infty = (1,2,0,\infty,3)03\ \underline{1\infty23}\ 012\infty = (1,2,0,\infty,3)$ $03\underline{(1,3,\infty,0,2)2\infty10}\ 012\infty$ (by relation (2) conjugated by $(1,0,2,\infty,4) \in L_2(5))$

$= 2\infty2\infty\underline{11}2\infty = \underline{2\infty2}\infty2\infty = \underline{\infty2\infty}\infty2\infty$ (by relation (5) conjugated by

$(2,0)(3,4) \in L_2(5))$

$= \infty 2\underline{\infty\infty}2\infty = \infty\underline{22}\infty = \underline{\infty\infty} = e.$

Now $N\infty 01204 = N241\infty 40$ since $\infty 01204 = (1,0,2,\infty,4)\ 241\infty 40$. We will prove this by moving the relation to one side of the equal sign and proving it equals identity.

$(1,0,2,\infty,4)241\infty 4\underline{040}210\infty = (1,0,2,\infty,4)\ 241\infty 4\ \underline{404}\ 210\infty$ (by relation (5) conjugated by $(1,2,3)(4,\infty,0) \in L_2(5))$

$= (1,0,2,\infty,4)241\infty\underline{44}04210\infty = (1,0,2,\infty,4)24\ \underline{1\infty 04}\ 210\infty = (1,0,2,\infty,4)$ $24\underline{(1,4,\infty,2,0)0\infty 12}\ 210\infty$ (by relation (2) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5))$

$= 0\infty 0\infty\underline{11}0\infty = 0\infty 0\underline{\infty 0\infty} = 0\infty 0\underline{0\infty 0}$ (by relation (5))

$= 0\infty\underline{00}\infty 0 = 0\underline{\infty\infty}0 = \underline{00} = e$

Hence $N\infty 01204 = N241\infty 40 = N\infty 04301 = N310214 = N314\infty 10$ $= N241\infty 40 = N240341$. $N^{(\infty 01204)} = \langle (1,0)(3,\infty), (1,0,4)(2,3,\infty)\rangle$ so there are 10 distinct single cosets in $[\infty 01204]$.

As previously shown $\infty 0231 = (\infty,4,0)(1,3,2)3\infty 20\infty$. If we conjugate this relation by $(1,3,\infty,0,2) \in L_2(5)$ we obtain $021\infty = (0,4,2)(1,3,\infty)\infty 01203$ so $N\infty 01203$ $= N021\infty \in [\infty 023]$, since $N\infty 023$ conjugated by $(1,3,\infty,0,2) \in L_2(5)$ is $N021\infty$. So t_3, a reprentative from the other 2-orbit, takes $[\infty 0120]$ to a single coset in $[\infty 023]$.

Now consider $[\infty 012\infty]$. $N^{(\infty 012\infty)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty 012\infty 1]$, $[\infty 012\infty 2]$, $[\infty 012\infty 3]$, $[\infty 012\infty 4]$, $[\infty 012\infty 0]$, and $[\infty 012\infty\infty]$,

First $[\infty 012\infty\infty] = [\infty 012]$. So t_∞ take $[\infty 012\infty]$ back to a single coset in $[\infty 012]$.

t_1 takes $[\infty 012\infty]$ to a single coset in $[\infty 01\infty 40]$, since $\infty 012\infty 1 = (2,\infty,0,3,4)$ 143124 thus $N\infty 012\infty 1 = N143124 \in [\infty 01\infty 40]$ since $N\infty 01\infty 40$ conjugated by $(1,3,\infty)(2,0,4) \in L_2(5)$ is $N143124$. To prove $\infty 012\infty 1 = (2,\infty,0,3,4)143124$ move the relation to one side of the equal sign and prove it equals identity.

$(2,\infty,0,3,4)\underline{1431}241\infty 210\infty = (2,\infty,0,3,4)\underline{(1,3,4)(2,0,\infty)4134}$ $241\infty 210\infty$ (by relation (3) conjugated by $(1,3,0)(2,\infty,4) \in L_2(5))$

$= (1,3)(4,0)413\underline{424}1\infty 210\infty = (1,3)(4,0)413\underline{242}1\infty 210\infty$ (by relation (6) conjugated by $(1,2)(3,0) \in L_2(5))$

$= (1,3)(4,0)41324\underline{21\infty 2}10\infty = (1,3)(4,0)41324\underline{(1,2,\infty)(3,0,4)12\infty 1}10\infty$ (by

relation (3) conjugated by $(1,\infty)(2,0) \in L_2(5)$)

$= (1,0,3,2,\infty)320\infty312\infty0\infty = e$ (by a previously proved relation $(1,\infty,0,4,2)$ $\infty014\infty20414 = e$ conjugated by $(1,0,2)(3,4,\infty) \in L_2(5)$).

t_2 takes $[\infty012\infty]$ to a single coset in $[\infty01\infty20]$, since $\infty012\infty2$ $= (1,3,\infty,0,2)0\infty103\infty$ thus $N\infty012\infty2 = N0\infty103\infty \in [\infty01\infty20]$ since $N\infty01\infty20$ conjugated by $(2,3)(0,\infty) \in L_2(5)$ is $N0\infty103\infty$. To prove $\infty012\infty2$ $= (1,3,\infty,0,2)0\infty103\infty$ move the relation to one side of the equal sign and prove it equals identity.

$(1,3,\infty,0,2)0\infty103\infty\underline{2\infty2}10\infty = (1,3,\infty,0,2)0\infty103\infty\underline{\infty2\infty}10\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,3,\infty,0,2)0\infty103\underline{\infty\infty}2\infty10\infty = (1,3,\infty,0,2)\ \underline{0\infty10}\ 32\infty10\infty$

$= (1,3,\infty,0,2)\ \underline{(1,\infty,0)(2,4,3)\infty01\infty}\ 32\infty10\infty$ (by relation (3))

$= (1,2,\infty)(3,0,4)\infty01\infty\underline{32\infty1}0\infty = (1,2,\infty)(3,0,4)\infty01\infty\ \underline{(1,2,0,\infty,3)\infty230}$ 0∞ (by relation (2) conjugated by $(1,0,\infty,4,3) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)3\infty23\infty23\infty = e$ (by relation (3) conjugated by $(1,2)(3,0) \in L_2(5)$).

t_3 takes $[\infty012\infty]$ to a single coset in $[\infty012\infty3]$.

$N\infty012\infty3 = N10\infty214$ since $\infty012\infty3 = (1,3,2,4,\infty)10\infty214$.

To prove $\infty012\infty3 = (1,3,2,4,\infty)10\infty214$ we will first move the relation to one side of the equal sign and then prove it equals identity.

$(1,3,2,4,\infty)10\infty214\underline{3\infty21}0\infty = (1,3,2,4,\infty)10\infty214\ \underline{(1,\infty,4,2,3)2\infty34}\ 0\infty$ (by relation (2) conjugated by $(1,4,3)(2,\infty,0) \in L_2(5)$)

$= \infty043\infty\underline{22}\infty340\infty = \infty043\underline{\infty\infty}340\infty = \infty04\underline{33}40\infty = \infty0\underline{44}0\infty = \infty\underline{00}\infty =$ $\underline{\infty\infty} = e$.

t_4 takes $[\infty012\infty]$ to a single coset in $[\infty012\infty4]$.

$N\infty012\infty4 = N0\infty4201$ since $\infty012\infty4 = (1,\infty,0,4,2)0\infty4201$.

To prove $\infty012\infty4 = (1,\infty,0,4,2)0\infty4201$ we will first move the relation to one side of the equal sign and then prove it equals identity.

$(1,\infty,0,4,2)0\infty42\underline{014\infty}210\infty = (1,\infty,0,4,2)0\infty42\underline{(1,2,4,0,\infty)4102}210\infty$ (by relation (1) conjugated by $(2,0,4,\infty,3) \in L_2(5)$)

$= \infty10\underline{44}10\underline{22}10\infty = \infty10\underline{1010}\infty = \infty10\underline{0100}\infty$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= \infty 1\underline{00}1\underline{00}\infty = \infty\underline{11}\infty = \underline{\infty\infty} = e.$

t_0 takes $[\infty 012\infty]$ to $[\infty 01024]$, since $\infty 012\infty 0 = (1,\infty,0)(2,4,3)1303\infty 4$ thus $N\infty 012\infty 0 = N1303\infty 4 \in [\infty 01024]$ since $N\infty 01024$ conjugated by $(1,0,3,2,\infty) \in L_2(5)$ is $N1303\infty 4$. To prove $\infty 012\infty 0 = (1,\infty,0)(2,4,3)1303\infty 4$ move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,0)(2,4,3)1\underline{303}\infty 40\infty 210\infty = (1,\infty,0)(2,4,3)1\underline{030}\infty 40\infty 210\infty$ (by relation (5) conjugated by $(2,3)(3,\infty) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)103\underline{0\infty 40}\infty 210\infty = (1,\infty,0)(2,4,3)103$
$\underline{(1,2,3)(4,\infty,0)\infty 04\infty}$ $\infty 210\infty$ (by relation (3) conjugated by $(1,4)(2,3) \in L_2(5)$)

$= (1,0,2,\infty,4)24\underline{1\infty 04}210\infty = (1,0,2,\infty,4)24\underline{(1,4,\infty,2,0)0\infty 12}210\infty$ (by relation (2) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= 0\infty 0\infty 1\underline{22}10\infty = 0\infty 0\infty\underline{11}0\infty = 0\infty 0\underline{\infty 0\infty} = 0\infty 0\underline{0\infty 0}$ (by relation (5))

$= 0\infty\underline{00}\infty 0 = 0\underline{\infty\infty}0 = \underline{00} = e.$

Now consider $[\infty 0130]$. $N^{(\infty 0130)}$ has orbits $\{2\}$, $\{0\}$, $\{1,\infty\}$, and $\{3,4\}$. So we need to look at $[\infty 01302]$, $[\infty 01300]$, $[\infty 01301]$, and $[\infty 01303]$.

First $[\infty 01300] = [\infty 013]$. So t_0 takes $[\infty 0130]$ back to a single coset in $[\infty 013]$.

t_2 takes $[\infty 0130]$ to a single coset in $[\infty 01302]$.

$N\infty 01302 = N0\infty 12\infty 3 = N\infty 10314 = N10\infty 402 = N01\infty 214 = N1\infty 04\infty 3$ since by relation (2) conjugated by $(1,4,\infty,2,0) \in L_2(5)$ we know $(1,4,0,3,\infty)\infty 013 = 10\infty 4$.

$\Longrightarrow N(1,4,0,3,\infty)\infty 013\underline{02} = N10\infty 4\underline{02}.$

$\Longrightarrow N\infty 01302 = N10\infty 402$

If we conjugate this equation by $(2,3)(0,\infty)$ and $(1,0)(2,4) \in L_2(5)$ we obtain $N0\infty 12\infty 3 = N1\infty 04\infty 3$ and $N\infty 10314 = N01\infty 214$.

Now $N\infty 01302 = (1,\infty,3,0,4)10\infty 402$ (as previously shown)

$= (1,\infty,3,0,4)10\underline{\infty 402} = (1,\infty,3,0,4)10\underline{(2,4,3,0,\infty)04\infty 3}$ (by relation (2) conjugated by $(1,3,2,4,\infty) \in L_2(5)$)

$= (1,2,4)(3,\infty,0)1\infty 04\infty 3$

So $N\infty 01302 = N1\infty 04\infty 3$

Now $N\infty 0\underline{1302} = N\infty 0$ $\underline{(1,2,3,4,0)0314}$ (by relation (2) conjugated by $(1,4)(2,3) \in L_2(5)$)

$= N(1,2,3,4,0)\infty 10314 = N\infty 10314.$

So now we have $N\infty01302 = N10\infty402 = N0\infty12\infty3 = N1\infty04\infty3$ $= N\infty10314 = N01\infty214$.

$N^{(\infty01302)} = \langle(1,\infty)(3,4)\rangle$ so there are 10 distinct single cosets in $[\infty01302]$.

t_1, a representative from one of the 2-orbits, takes $[\infty0130]$ to a single coset in $[\infty023]$ because we previously proved $\infty0230 = (2,0,3)(1,\infty,4)42032$

$\Longrightarrow N\infty023 = N(2,0,3)(1,\infty,4)420320 = N420320 \in [\infty01301]$ since $N\infty01301$ conjugated by $(1,0,2,\infty,4) \in L_2(5)$ is $N420320$.

t_3, a represetative from the other 2-orbit, takes $[\infty0130]$ to a single coset in $[\infty01031$ since $N\infty01\underline{303} = N\infty01\underline{030}$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= N\infty\underline{01}030 = N\infty\underline{01}030$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$N\infty01030 \in [\infty01031]$ since $N\infty01031$ conjugated by $(1,0)(2,4) \in L_2(5)$ is $N\infty01030$.

Now consider $[\infty0131]$. $N^{(\infty0131)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$ and $\{\infty\}$. So we need to look at $[\infty01311]$, $[\infty01312]$, $[\infty01313]$, $[\infty01314]$, $[\infty01310]$, and $[\infty0131\infty]$.

First $[\infty01311] = [\infty013]$. So t_1 takes $[\infty0131]$ back to a single coset in $[\infty013]$.

t_2 takes $[\infty0131]$ to a single coset in $[\infty012\infty3]$, since $N\infty01312$ $= N(1,4,0,3,\infty)230421 = N230421 \in [\infty012\infty3]$ since $N\infty012\infty3$ conjugated by $(1,0,3)(2,3,\infty) \in L_2(5)$ is $N230421$. To prove $\infty01312 = (1,4,0,3,\infty)230421$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,0,3,\infty)2304\underline{212}1310\infty = (1,4,0,3,\infty)2304\underline{121}1310\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,4,0,3,\infty)230412\underline{11}310\infty = (1,4,0,3,\infty)2304\ \underline{1231}0\infty = (1,4,0,3,\infty)2304$ $\underline{(1,3,2)(4,0,\infty)2132}\ 0\infty$ (by relation (4) conjugated by $(1,4)(0,\infty) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)12\underline{\infty021}320\infty = (1,0,2)(3,4,\infty)12\underline{(1,0,3,2,\infty)20\infty3}320\infty$ (by relation (1) conjugated by $(1,4)(0,\infty) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)0\infty20\infty\underline{33}20\infty = (1,3,4)(2,0,\infty)0\infty\ \underline{20\infty2}0\infty$ $= (1,3,4)(2,0,\infty)0\infty\ \underline{(1,4,3)(2,\infty,0)02\infty0}\ 0\infty$ (by relation (4) conjugated by $(1,3,\infty)(2,0,4) \in L_2(5)$)

$= 2\underline{00}2\infty\underline{00}\infty = \underline{22\infty\infty} = e$.

t_3 takes $[\infty0131]$ to a single coset in $[\infty024]$ since by relation $N\infty01\underline{313} = N\infty01\underline{131}$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= N\infty01\underline{11}31 = N\infty031 \in [\infty024]$ (since $N\infty024$ conjugated by $(1,4)(2,3) \in$

$L_2(5)$ is $N\infty031$.

t_4 takes $[\infty0131]$ to a single coset in $[\infty01\infty20]$ since $N\infty01314$ $= N(1,0,3)(2,4,\infty)01\infty041 = N01\infty041 \in [\infty01\infty20]$ since $N\infty01\infty20$ conjugated by $(1,\infty,0)(2,4,3) \in L_2(5)$ is $N01\infty041$.

To prove $\infty01314 = (1,0,3)(2,4,\infty)01\infty041$ we move the relation to one side of the equal sign and prove it equals identity.

$(1,0,3)(2,4,\infty)01\infty0\underline{414}1310\infty = (1,0,3)(2,4,\infty)01\infty0\underline{141}1310\infty$ (by relation (6))

$= (1,0,3)(2,4,\infty)01\infty014\underline{11}310\infty = (1,0,3)(2,4,\infty)\underline{01\infty0}\ 14310\infty$ $= (1,0,3)(2,4,\infty)\underline{(1,0,\infty)(2,3,4)10\infty1}\ 14310\infty$ (by relation (3) conjugated by $(1,\infty,0)(2,4,3) \in L_2(5)$)

$= (1,\infty,3,0,4)10\infty\underline{11}4310\infty = (1,\infty,3,0,4)10\infty4310\infty = e$ (by relation (2) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$).

t_0 takes $[\infty0131]$ to a single coset in $[\infty01310]$. $N\infty01310 = N431013$ since $\infty01310 = (1,3,0)(2,\infty,4)431013$ by moving the relation to one side of the equal sign and proving it equals identity.

$(1,3,0)(2,\infty,4)4310\underline{1301}310\infty = (1,3,0)(2,\infty,4)4310\ \underline{(1,0,3)(2,4,\infty)3103}$ 310∞ (by relation (1) conjugated by $(1,0)(3,\infty) \in L_2(5)$)

$= \infty10\underline{33}1010\infty = \infty10\underline{1010}\infty = \infty10\underline{0101}0\infty$ (by relation (6) conjugated by $(1,0)(3,\infty) \in L_2(5)$)

$= \infty1\underline{00}1\underline{00}\infty = \infty\underline{11}\infty = \underline{\infty\infty} = e$.

$N^{(\infty01310)} = \langle(3,0)(4,\infty)\rangle$ so there are 30 distinct single cosets in $[\infty01310]$.

t_∞ takes $[\infty0131]$ to a single coset in $[\infty0131\infty]$. There are 60 distinct single cosets in $[\infty0131\infty]$.

Now consider $[\infty0141]$. $N^{(\infty0141)}$ has orbits $\{0\}$, $\{\infty\}$, $\{1,4\}$, and $\{2,3\}$. So we need to look at $[\infty01410]$, $[\infty0141\infty]$, $[\infty01411]$, and $[\infty01412]$.

First $[\infty01411] = [\infty014]$. So t_1, a representative from one of the 2-orbits, takes $[\infty0141]$ back to a single coset in $[\infty014]$.

t_0 takes $[\infty0141]$ to a single coset in $[\infty010\infty2]$, since $N\infty01410$ $= N(1,\infty)(2,0)30203\infty = N30203\infty \in [\infty010\infty2]$ since $N\infty010\infty2$ conjugated by $(1,2,\infty.3.4) \in L_2(5)$ is $N30203\infty$. To prove $\infty01410 = (1,\infty)(2,0)30203\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty)(2,0)3\underline{020}3\infty01410\infty = (1,\infty)(2,0)3\underline{202}3\infty01410\infty$ (by relation (5) conjugated by $(1,3)(2,0) \in L_2(5)$)

$= (1,\infty)(2,0)32\underline{023\infty}01410\infty = (1,\infty)(2,0)32\underline{(2,4,3,0,\infty)3204}01410\infty$ (by relation (2) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)0432\underline{040141}0\infty = (1,2,\infty)(3,0,4)0432\underline{404414}0\infty$ (by relation (6) and relation (6) conjugated by $(1,0)(3,\infty) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)043240\underline{44}140\infty = (1,2,\infty)(3,0,4)0432\ \underline{4014}\ 0\infty$ $= (1,2,\infty)(3,0,4)0432\ \underline{(1,0,4)(2,3,\infty)0410}\ 0\infty$ (by relation (4) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (1,3,4,\infty,0)41\infty3041\underline{00}\infty = (1,3,4,\infty,0)41\infty3041\infty = e$ by relation (1) conjugated by $(2,4,3,0,\infty) \in L_2(5)$.

t_∞ takes $[\infty0141]$ to a single coset in $[\infty0103\infty]$, since $N\infty0141\infty$ $= N(1,4)(2,3)0232\infty0 = N0232\infty0 \in [\infty0103\infty]$ since $N\infty0103\infty$ conjugated by $(1,3,\infty.0,2) \in L_2(5)$ is $N0232\infty0$. To prove $\infty0141\infty = (1,4)(2,3)0232\infty0$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4)(2,3)0232\infty0\infty1410\infty = (1,4)(2,3)0232\infty\underline{11}\ 0\infty1410\infty$ $= (1,4)(2,3)0232\infty1\ \underline{10\infty1}\ 410\infty = (1,4)(2,3)0232\infty1\ \underline{(1,\infty,0)(2,4,3)01\infty0}\ 410\infty$ (by relation (3) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (1,3,4,\infty,0)1424\underline{0\infty0}1\infty0410\infty = (1,3,4,\infty,0)1424\underline{\infty0\infty}1\infty0410\infty$ (by relation (5))

$= (1,3,4,\infty,0)1424\infty0\underline{\infty1\infty}0410\infty = (1,3,4,\infty,0)1424\infty0\underline{1\infty1}0410\infty$ (by relation (6) conjugated by $(3,0)(4,\infty) \in L_2(5)$)

$= (1,3,4,\infty,0)1424\infty01\infty\underline{1041}0\infty = (1,3,4,\infty,0)1424\infty01\infty$ $\underline{(1,4,0)(2,\infty,3)0140}0\infty$ (by relation (4) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,2,\infty,)(3,0,4)40\infty0314301 4\underline{00}\infty = (1,2,\infty,)(3,0,4)40\infty03143\ \underline{014\infty} =$ $(1,2,\infty,)(3,0,4)40\infty03143\ \underline{(1,2,4,0,\infty)4102}$ (by relation (1) conjugated by $(2,4)(3,\infty)$ $\in L_2(5)$)

$= (1,4,3,\infty,2)0\infty1\infty\underline{3203}4102 = (1,4,3,\infty,2)0\infty1\infty\underline{(1,4,\infty)(2,3,0)2302}4102$ (by relation (3) conjugated by $(1,0,2)(3,4,\infty) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)21412\underline{3024}102 = (1,\infty,3)(2,4,0)21412\ \underline{(1,2,3,4,0)2031}102$ (by relation (2) conjugated by $(1,4,2,0,3) \in L_2(5)$)

$= (1,\infty,4)(2,0,3)32023230 3\underline{11}02 = (1,\infty,4)(2,0,3)320232\ \underline{0302}$

$= (1,\infty,4)(2,0,3)320232\,\underline{303}2$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,\infty,4)(2,0,3)3202\underline{323}032 = (1,\infty,4)(2,0,3)3202\underline{232}032$ (by relation (5) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (1,\infty,4)(2,0,3)3202\underline{22}32032 = (1,\infty,4)(2,0,3)32032032 = e$ by relation (3) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$.

t_2, a representative from the other 2-orbit, takes $[\infty 0141]$ to a single coset in $[\infty 01024]$, since $N\infty 01412 = N(1,2,\infty,3,4)43\infty 301 = N43\infty 301 \in [\infty 01024]$ since $N\infty 01024$ conjugated by $(1,\infty.4)(2,0,3) \in L_2(5)$ is $N43\infty 301$. To prove $\infty 01412 = (1,2,\infty,3,4)43\infty 301$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,\infty,3,4)4\underline{3\infty 30}\underline{121}410\infty = (1,2,\infty,3,4)4\underline{\infty 3\infty}0\underline{212}410\infty$ (by relation (5) conjugated by $(1,2)(3,0)$ and $(1,4,\infty,2,0) \in L_2(5)$)

$= (1,2,\infty,3,4)4\infty\underline{3\infty 02}12410\infty = (1,2,\infty,3,4)4\infty\underline{(3,2,\infty,1,0)0\infty 31}12410\infty$ (by relation (1) conjugated by $(1\infty,4)(2,0,3) \in L_2(5)$)

$= (1,\infty,2)(3,4,0)410\infty 3\underline{11}2410\infty = (1,\infty,2)(3,4,0)410\infty 32410\infty = e$ (by a previously proved relation).

Now consider $[\infty 0142]$. $N^{(\infty 0142)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$ and $\{\infty\}$. So we need to look at $[\infty 01421]$, $[\infty 01422]$, $[\infty 01423]$, $[\infty 01424]$, $[\infty 01420]$, and $[\infty 0142\infty]$.

First $[\infty 01422] = [\infty 014]$. So t_2 takes $[\infty 0142]$ back to a single coset in $[\infty 014]$.

t_1 takes $[\infty 0142]$ to a single coset in $[\infty 01023]$, since $N\infty 01421$ $= N(1,2,\infty,3,4)43\infty 301 = N43\infty 301 \in [\infty 01023]$ since $N\infty 01023$ conjugated by $(1,\infty.4)(2,0,3) \in L_2(5)$ is $N43\infty 301$. To prove $\infty 01421 = (1,2,\infty,3,4)43\infty 301$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,\infty,3,4)43\infty 302\underline{12410}\infty = (1,2,\infty,3,4)43\infty 302$ $\underline{(1,4,2)(3,0,\infty)21420}\infty$ (by relation (3) conjugated by $(1,4,\infty)(2,3,0) \in L_2(5)$)

$= (2,3)(0,\infty)203\underline{0\infty 12}1420\infty = (2,3)(0,\infty)203\underline{(0,2,\infty,4,1)1\infty 04}1420\infty$ (by relation (2) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)\infty 231\infty 04\underline{1420}\infty = (1,0,4)(2,3,\infty)\infty 231\infty 04$ $\underline{(1,0,4,3,2)2413}\infty$ (by relation (2) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,4,0,3,\infty)\infty 120\infty\underline{432}413\infty = (1,4,0,3,\infty)\infty 120\infty$ $\underline{(1,0,\infty)(2,3,4)3423}13\infty$ (by relation (2) conjugated by $(1,\infty,0)(2,4,3) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)103\infty 13\underline{4231}3\infty = (1,2,3)(4,\infty,0)103\infty 13$

$\underline{(4,1,2,\infty,3)324\infty}3\infty$ (by relation (1) conjugated by $(1,2,3)(4,\infty,0) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)204\underline{3243}24\infty3\infty = (1,\infty,0)(2,4,3)204\ \underline{(1,0,\infty)(3,4,2)2342}$ $24\infty3\infty$ (by relation (4) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$)

$= 3\infty\underline{22}34\underline{22}4\infty3\infty = 3\infty3\underline{44}\infty3\infty = 3\infty3\underline{\infty3\infty} = 3\infty3\underline{3\infty3}$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= 3\infty\underline{33}\infty3 = 3\underline{\infty\infty}3 = 33 = e.$

t_3 takes $[\infty0142]$ to a single coset in $[\infty023]$, since we previously proved $\infty0234 = (1,3,0)(2,\infty,4)320\infty1$. So it follows that $N\infty023 = N(1,3,0)(2,\infty,4)320\infty14 = N320\infty14 \in [\infty01423]$ since $N\infty01423$ conjugated by $(1,0,2)(3,4,\infty) \in L_2(5)$ is $N320\infty14$.

t_4 takes $[\infty0142]$ to a single coset in $[\infty01\infty40]$, since $N\infty01424 = N(1,3,4,\infty,0)\infty23\infty12 = N\infty23\infty12 \in [\infty01\infty40]$ since $N\infty01023$ conjugated by $(1,3,0,2,4) \in L_2(5)$ is $N\infty01023$. To prove $\infty01424 = (1,3,4,\infty,0)\infty23\infty12$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,4,\infty,0)\infty23\infty12\underline{424}10\infty = (1,3,4,\infty,0)\infty23\infty12\underline{242}10\infty$ (by relation (6) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,3,4,\infty,0)\infty23\infty1\underline{22}4210\infty = (1,3,4,\infty,0)\infty23\infty\ \underline{1421}0\infty$ $= (1,3,4,\infty,0)\infty23\infty\ \underline{(1,2,4)(\infty,0,3)41240}\infty$ (by relation (3) conjugated by $(1,2,\infty)(3,0,4) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)\underline{04\infty0}41240\infty = (1,\infty,3)(2,4,0)\underline{(1,3,2)(4,0,\infty)40\infty4}$ 41240∞ (by relation (3) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (1,4,\infty,2,0)40\infty\underline{44}1240\infty = (1,4,\infty,2,0)40\infty1240\infty = e$ (by relation (2) conjugated by $(1,2,0,\infty,3) \in L_2(5)$).

t_0 takes $[\infty0142]$ to a single coset in $[\infty01231]$, since $N\infty01420 = N(1,2,3)(4,\infty,0)10\infty24\infty = N10\infty24\infty \in [\infty01231]$ since $N\infty01231$ conjugated by $(1,\infty)(3,4) \in L_2(5)$ is $N10\infty24\infty$. To prove $\infty01420 = (1,2,3)(4,\infty,0)10\infty24\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,3)(4,\infty,0)10\underline{\infty24\infty}02410\infty = (1,2,3)(4,\infty,0)10$ $\underline{(1,3,0)(2,\infty,4)2\infty42}\ 02410\infty$ (by relation (4) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= (1,\infty)(2,0)312\infty4\underline{202}410\infty = (1,\infty)(2,0)312\infty4\underline{020}410\infty$ (by relation (5) conjugated by $(1,4,3)(2,\infty,0) \in L_2(5)$)

$= (1,\infty)(2,0)312\infty402\underline{0410}\infty = (1,\infty)(2,0)312\infty402\ \underline{(3,2,\infty)(0,1,4)4014}\infty$

(by relation (4) conjugated by $(1,2,0,\infty,3) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)24\infty\underline{301\infty}4014\infty = (1,3,2)(4,0,\infty)24\infty\ \underline{(3,\infty,0,2,1)1032}$ 4014∞ (by relation (2) conjugated by $(1,\infty,4)(2,0,3) \in L_2(5)$)

$= (1,\infty,4,2,3)14\underline{010}324014\infty = (1,\infty,4,2,3)14\underline{101}324014\infty$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (1,\infty,4,2,3)\underline{141}01324014\infty = (1,\infty,4,2,3)\underline{414}01324014\infty$ (by relation (6))

$= (1,\infty,4,2,3)4\underline{1401}324014\infty = (1,\infty,4,2,3)4\underline{(\infty,2,3)(1,0,4)4104}324014\infty$ (by relation (4) conjugated by $(1,2,)(3,0) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)\underline{141}04324014\infty = (1,2,\infty)(3,0,4)\underline{414}04324014\infty$ (by relation (6))

$= (1,2,\infty)(3,0,4)41\underline{404}324014\infty = (1,2,\infty)(3,0,4)41\underline{040}324014\infty$ (by relation (5) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)41040324\underline{014\infty} = (1,2,\infty)(3,0,4)41040324\underline{(1,2,4,0,\infty)4102}$ (by relation (1) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,4,3,\infty,2)02\infty0\infty34\underline{04102} = (1,4,3,\infty,2)02\infty0\infty34\underline{(1,4,0)(2,\infty,3)4014}2$ (by relation (4) conjugated by $(1,2,0,\infty,3) \in L_2(5)$)

$= (1,0)(2,4)1\infty3132\underline{040}142 = (1,0)(2,4)1\infty3132\underline{404}142$ (by relation (5) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (1,0)(2,4)1\infty313240\underline{414}2 = (1,0)(2,4)1\infty313240\underline{141}2$ (by relation (6))

$= (1,0)(2,4)1\infty3132\underline{4014}12 = (1,0)(2,4)1\infty3132\underline{(1,0,4)(2,3,\infty)0410}12$ (by relation (4) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (1,4,3,\infty,2)02\infty0\underline{\infty304}1012 = (1,4,3,\infty,2)02\infty0\underline{(1,0,\infty,4,3)03\infty1}1012$ (by relation (1) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)\infty24\infty03\infty\underline{11}012 = (1,3,4)(2,0,\infty)\infty24\underline{\infty03\infty}012$

$= (1,3,4)(2,0,\infty)\infty24\underline{(1,4,2)(3,0,\infty)0\infty300}12$ (by relation (4) conjugated by $(2,\infty)(4,0) \in L_2(5)$)

$= (1,0,3,2,\infty)3120\infty3\underline{00}12 = (1,0,3,2,\infty)3120\infty312 = e$ (by relation (1) conjugated by $(2,3,\infty,4,0) \in L_2(5)$).

t_∞ takes [∞0142] to a single coset in [∞0142∞] which has 60 distinct single cosets.

Now consider [∞0143]. $N^{(\infty0143)}$ has orbits $\{2\}$, $\{0\}$, $\{1,\infty\}$, and $\{3,4\}$. So we need to look at [∞01432], [∞01430], [∞01431], and [∞01433].

First $[\infty 01433] = [\infty 014]$. So t_3, a representative from one of the 2-orbits, takes $[\infty 0143]$ back to a single coset in $[\infty 014]$.

t_2 takes $[\infty 0143]$ to a single coset in $[\infty 01204]$, since $N\infty 01432$ $= N(1,4,\infty,2,0)203\infty 04 = N203\infty 04 \in [\infty 01204]$. To prove $\infty 01432$ $= (1,4,\infty,2,0)203\infty 04$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,\infty,2,0)2\underline{03\infty 0}423410\infty = (1,4,\infty,2,0)2\underline{(1,4,2)(3,0,\infty)30\infty 3}$ 423410∞ (by relation (4) conjugated by $(1,2,0)(3,\infty,4) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)130\infty\underline{3423}410\infty = (1,2,\infty)(3,0,4)130\infty$ $\underline{(1,\infty,0)(2,4,3)4324}410\infty$ (by relation (4) conjugated by $(2,3)(0,\infty) \in L_2(5)$)

$= (1,4,2,0,3)\infty 210432\underline{44}10\infty = (1,4,2,0,3)\infty\underline{2104}3210\infty = (1,4,2,0,3)\infty$ $\underline{(1,3,0,2,4)0123}3210\infty$ (by relation (1))

$= \infty 0012\underline{33}210\infty = \infty 0012\underline{22}10\infty = \infty 00\underline{11}0\infty = \infty 0\underline{00}\infty = \underline{\infty\infty} = e$.

t_0 takes $[\infty 0143]$ to a single coset in $[\infty 01430]$. As we previously proved $\infty 0143 =$ $(1,\infty,3,0,4)10\infty 34$. So it follows that $N\infty 01430 = N(1,\infty,3,0,4)10\infty 340 = N10\infty 340$. $N^{(\infty 01430)} = \langle(1,\infty)(3,4)\rangle$ so there are 30 distinct single cosets in $[\infty 01430]$.

t_∞, a representative from the other 2-orbit, takes $[\infty 0143]$ back to a single coset in $[\infty 024]$. Since as we previously proved $\infty 0240 = (1,2,3)(4,\infty,0)2401\infty$. Therefore it follows $N\infty 024 = N(1,2,3)(4,\infty,0)2401\infty 0 = N2401\infty 0$. If we conjugate $N2401\infty$ by $(1,4,0)(2,\infty,3) \in L_2(5)$, we get $N\infty 0143$. Hence $N2401\infty \in [\infty 0143]$.

Now consider $[\infty 0140]$. $N^{(\infty 0140)}$ has orbits $\{1\}$, $\{\infty\}$, $\{2,0\}$, and $\{3,4\}$. So we need to look at $[\infty 01401]$, $[\infty 0140\infty]$, $[\infty 01400]$, and $[\infty 01403]$.

First $[\infty 01400] = [\infty 014]$. So t_0, a representative from one of the 2-orbits, takes $[\infty 0140]$ back to a single coset in $[\infty 014]$.

t_1 takes $[\infty 0140]$ back to a single coset in $[\infty 021]$. Since as we previously proved $\infty 0210 = (1,2,0)(3,\infty,4)42012$. Therefore it follows $N\infty 021 = N(1,2,0)(3,\infty,4)420120$ $= N420120$. If we conjugate $N420120$ by $(1,4,\infty,2,0) \in L_2(5)$, we get $N\infty 01401$. Hence $N420120 \in [\infty 01401] = [\infty 021]$.

t_∞ takes $[\infty 0140]$ to a single coset in $[\infty 0140\infty]$. As we previously proved $\infty 0140 = (2,\infty,0,3,4)\infty 2132$. So it follows that $N\infty 0140\infty = N(2,\infty,0,3,4)\infty 2132\infty$ $= N\infty 2132\infty$. If we conjugate this relation by $(1,2,0)(3,\infty,4)$, $(1,0,2)(3,4,\infty)$, $(2,0)(3,\infty)$, and $(2,0)(3,4) \in L_2(5)$, we get $N\infty 0140\infty = N\infty 2132\infty = N412314 =$

$N402\infty04 = N320\infty23 = N310413$. $N^{(\infty0140\infty)} = \langle(1,0)(3,\infty),(1,2,0)(3,\infty,4)\rangle$ so there are 10 distinct single cosets in $[\infty0140\infty]$.

t_3, a representative from the other 2-orbit, takes $[\infty0140]$ to a single coset in $[\infty01403]$. There are 60 distinct single cosets in $[\infty01403]$.

Now consider $[\infty0\infty14]$. $N^{(\infty0\infty14)}$ has orbits $\{1\}$, $\{4\}$, $\{2,3\}$, and $\{0,\infty\}$. So we need to look at $[\infty0\infty141]$, $[\infty0\infty144]$, $[\infty0\infty142]$, and $[\infty0\infty140]$.

First $[\infty0\infty144] = [\infty0\infty1]$. So t_4 takes $[\infty0\infty14]$ back to a single coset in $[\infty0\infty1]$.

t_1 takes $[\infty0\infty14]$ to a single coset in $[\infty0\infty141]$. As we previously proved $N\infty0\infty14 = N0\infty014$. Therefore it follows $N\infty0\infty\underline{141} = N0\infty0\underline{141} = N\infty0\infty\underline{414} = N0\infty0\underline{414}$ (by relation (6)). If we conjugate this equation by $(1,2,0)(3,\infty,4)$ and $(1,0,3)(2,4,\infty) \in L_2(5)$ we obtain $N414232 = N141232 = N141323 = N414323$ and $N2320\infty0 = N3230\infty0 = N323\infty0\infty = N232\infty0\infty$. To show these single cosets are all equal we will prove $N\infty0\infty141 = N(2,3)(0,\infty)414232 = N414232$ and $N\infty0\infty141 = N(1,4)(2,3)2320\infty0 = N2320\infty0$. Hence $N\infty0\infty141 = N0\infty0141 = N\infty0\infty414 = N0\infty0414 = N414232 = N141232 = N141323 = N414323 = N2320\infty0 = N3230\infty0 = N323\infty0\infty = N232\infty0\infty$.

$N^{(\infty0\infty141)} = \langle(1,4)(0,\infty),(1,4)(2,3),(1,\infty,2)(3,4,0)\rangle$, so there are 5 distinct single cosets in $[\infty0\infty141]$.

$\infty0141\infty = (1,4)(2,3)0232\infty0$ (by a previously proved relation)

$\Longrightarrow \underline{0}\infty0141\infty\underline{\infty} = \underline{0}(1,4)(2,3)0232\infty0\underline{\infty}$

$\Longrightarrow 0\infty0141\underline{\infty\infty} = (1,4)(2,3)\underline{00}232\infty0\infty$

$\Longrightarrow \underline{0\infty0}141 = (1,4)(2,3)232\underline{\infty0\infty}$

$\Longrightarrow \underline{\infty0\infty}141 = (1,4)(2,3)232\underline{0\infty0}$ (by relation (5)).

Hence $N\infty0\infty141 = N2320\infty0$. To prove $\infty0\infty141 = (2,3)(0,\infty)414232$ we will move the relation to one side of the equal sign and prove it equals identity.

$(2,3)(0,\infty)41423214\underline{22}1\infty0\infty = (2,3)(0,\infty)41423\underline{2142}21\infty0\infty$

$= (2,3)(0,\infty)41423\underline{(1,2,4)(3,\infty,0)1241}21\infty0\infty$ (by relation (3) conjugated by $(1,4,\infty,2,0) \in L_2(5)$)

$= (1,2,\infty,3,4)1214\infty124\underline{121}\infty0\infty = (1,2,\infty,3,4)1214\infty124\underline{212}\infty0\infty$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,2,\infty,3,4)1214\infty1\underline{242}12\infty0\infty = (1,2,\infty,3,4)1214\infty1\underline{424}12\infty0\infty$ (by rela-

tion (6) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,2,\infty,3,4)12\underline{14\infty1}42412\infty0\infty = (1,2,\infty,3,4)12\underline{(1,\infty,4)(2,0,3)41\infty4}$ $42412\infty0\infty$ (by relation (4) conjugated by $(1,3,\infty,0,2) \in L_2(5)$)

$= (1,0,3)(2,4,\infty)\infty\underline{041\infty44}2412\infty0\infty = (1,0,3)(2,4,\infty)\infty\underline{(1,0,\infty,4,3)1403}$ $2412\infty0\infty$ (by relation (1) conjugated by $(1,4,3,\infty,2) \in L_2(5)$)

$= (1,\infty,2,3,0)\underline{414}032412\infty0\infty = (1,\infty,2,3,0)\underline{141}032412\infty0\infty$ (by relation (6))

$= (1,\infty,2,3,0)14\underline{1032}412\infty0\infty = (1,\infty,2,3,0)14\underline{(1,2,0,\infty,3)301\infty}412\infty0\infty$ (by relation (2) conjugated by $(1,2,0)(3,\infty,4) \in L_2(5)$)

$= (1,3,\infty,0,2)24301\infty\underline{412\infty0}\infty = (1,3,\infty,0,2)24301\infty\underline{(1,3,2,4,\infty)2143}0\infty$ (by relation (2) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)4\infty203\underline{121}430\infty = (1,2,3)(4,\infty,0)4\infty203\underline{212}430\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)4\infty20321\underline{2430}\infty = (1,2,3)(4,\infty,0)4\infty20321$ $\underline{(2,0,4,\infty,3)342\infty}\infty$ (by relation (1) conjugated by $(1,4,\infty)(2,3,0) \in L_2(5)$)

$= (1,0,\infty,4,3)\infty3\underline{0420}1342\underline{\infty\infty} = (1,0,\infty,4,3)\infty3\underline{(1,\infty,3)(2,4,0)4024}1342$ (by relation (3) conjugated by $(1,2,3)(4,\infty,0) \in L_2(5)$)

$= (1,2,4)(3,\infty,0)314\underline{0241}342 = (1,2,4)(3,\infty,0)314\underline{(1,2,3,4,0)4203}342$ (by relation (2))

$= (1,3,\infty)(2,0,4)4204203\underline{3}42 = (1,3,\infty)(2,0,4)42042042 = e$ by relation (3) conjugated by $(1,0,4)(2,3,\infty) \in L_2(5)$.

t_3, a representative from one of the 2-orbits, takes $[\infty0\infty14]$ to a single coset in $[\infty01\infty40]$, since $N\infty0\infty14 = N(1,2,0)(3,\infty,4)032013 = N032013 \in [\infty01\infty40]$. To prove $\infty0\infty14 = (1,2,0)(3,\infty,4)032013$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,0)(3,\infty,4)\underline{0320}1341\infty0\infty = (1,2,0)(3,\infty,4)\ \underline{(1,4\infty)(2,3,0)3023}$ $1341\infty0\infty$ (by relation (3) conjuugated by $(1,2,4)(3,\infty,0) \in L_2(5)$)

$= (1,3)(4,0)302\underline{313}241\infty0\infty = (1,3)(4,0)302\underline{131}241\infty0\infty$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,3)(4,0)30213\underline{1241}\infty0\infty = (1,3)(4,0)30213\underline{(1,4,2)(3,0,\infty)2142}\infty0\infty$ (by relation (3) conjugated by $(1,4,\infty)(2,3,0) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)0\infty1402142\underline{\infty0\infty} = (1,0,2)(3,4,\infty)0\infty1402142\underline{0\infty0}$ (by relation (5))

$= (1,0,2)(3,4,\infty)0\infty1402\underline{1420}\infty0 = (1,0,2)(3,4,\infty)0\infty1402$ $\underline{(1,0,4,3,2)2413}\infty0$ (by relation (2) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,4,\infty,2,0)4\infty03\underline{4124}13\infty0 = (1,4,\infty,2,0)4\infty03\underline{(1,4,2)(3,0,\infty)1421}13\infty0$ (by relation (3) conjugated by $(1,2,0)(3,\infty,4) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)23\infty0142\underline{11}3\infty0 = (1,2,\infty)(3,0,4)23\infty01423\infty0 = e$ (by the previously proved relation $(1,3,0)(2,\infty,4)320\infty14320\infty = e$ conjugated by $(2,3)(0,\infty) \in L_2(5)$).

t_0, a representative from the other 2-orbit, takes $[\infty0\infty14]$ to a single coset in $[\infty0131\infty]$, since $N\infty0\infty140 = N(1,\infty,0)(2,4,3)2\infty3432 = N2\infty3432 \in [\infty0131\infty]$. To prove $\infty0\infty140 = (1,\infty,0)(2,4,3)2\infty3432$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,0)(2,4,3)2\infty34\underline{3204}1\infty0\infty = (1,\infty,0)(2,4,3)2\infty34$ $\underline{(3,\infty,0,3,4)023\infty}\ 1\infty0\infty$ (by relation (2) conjugated by $(1,4,0,3,\infty) \in L_2(5)$)

$= (1,0)(3,\infty)\infty042\underline{023\infty}1\infty0\infty = (1,0)(3,\infty)\infty042\ \underline{(2,3,4,0,\infty)3204}\ 1\infty0\infty$ (by relation (2) conjuated by $(1,\infty)(3,4) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)\underline{2\infty34}32041\infty0\infty = (1,\infty,0)(2,4,3)\underline{(1,3,2,4,\infty)3\infty21}$ $32041\infty0\infty$ (by relation (2) conjugated by $(1,4,3)(2,\infty,0) \in L_2(5)$)

$= (2,\infty,0,3,4)3\infty\underline{2132}041\infty0\infty = (2,\infty,0,3,4)3\infty\ \underline{(1,2,3)(4,\infty,0)1231}$ $041\infty0\infty$ (by relation (4) conjugated by $(1,4)(0,\infty) \in L_2(5)$)

$= (1,2,0)(3,\infty,4)101231\underline{041\infty0\infty} = (1,2,0)(3,\infty,4)10123\ \underline{(1,4,0)(2,\infty,3)0140}$ $\underline{0\infty0}$ (by relation (4) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$ and by relation (5))

$= (1,\infty,0,4,2)\underline{414}\infty2014\underline{00}\infty0 = (1,\infty,0,4,2)\underline{141}\infty2014\infty0 = e$ (by relation (6) and by a previously proved relation $(1,2,4,0,\infty)41402\infty410\infty = e$ conjugated by $(1,4)(0,\infty) \in L_2(5)$).

Now consider $[\infty0\infty12]$. $N^{(\infty0\infty12)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$ and $\{\infty\}$. So we need to look at $[\infty0\infty121]$, $[\infty0\infty122]$, $[\infty0\infty123]$, $[\infty0\infty124]$, $[\infty0\infty120]$, and $[\infty0\infty12\infty]$.

First $[\infty0\infty122] = [\infty0\infty1]$. So t_2 takes $[\infty0\infty12]$ back to a single coset in $[\infty0\infty1]$.

t_1 takes $[\infty0\infty12]$ to a single coset in $[\infty0\infty121]$. There are 60 distinct single cosets in $[\infty0\infty121]$.

t_3 takes $[\infty0\infty12]$ to a single coset in $[\infty01232]$, since $N\infty0\infty123$

$= N(1,\infty)(2,0)01\infty424 = N01\infty424 \in [\infty01232]$. To prove $\infty0\infty123$ $= (1,\infty)(2,0)01\infty424$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty)(2,0)01\infty42\underline{4321\infty0\infty} = (1,\infty)(2,0)01\infty42\underline{(1,3,0,2,4)2340}\underline{0\infty0}$ (by relation (1) conjugated by $(1,3)(4,0) \in L_2(5)$ and by relation (5))

$= (1,\infty,3,0,4)23\infty1423\underline{400}\infty0 = (1,\infty,3,0,4)23\infty14234\infty0 = e$ (by a previously proved relation $e = (1,4,0,3,\infty)230412310\infty$ conjugated by $(1,4)(0,\infty) \in L_2(5)$).

t_4 takes $[\infty0\infty12]$ to a single coset in $[\infty023]$ since as we previously proved $\infty023\infty = (2,4.3,0.\infty)23204$. Therefore it follows $N\infty023 = N(2,4.3,0.\infty)23204\infty = N23204\infty$. If we conjugate $N23204\infty$ by $(1,3,0)(2,\infty,4) \in L_2(5)$, we get $N\infty0\infty124$. Hence $N23204\infty \in [\infty0\infty124] = [\infty023]$.

t_0 takes $[\infty0\infty12]$ to a single coset in $[\infty0142\infty]$ since $N\infty0\infty120$ $= N(1,4,0,3,\infty)3021\infty3 = N3021\infty3 \in [\infty0142\infty]$. To prove $\infty0\infty120$ $= (1,4,0,3,\infty)3021\infty3$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,0,3,\infty)302\underline{1\infty3}021\infty0\infty = (1,4,0,3,\infty)302\ \underline{(1,0,\infty,4,3)3\infty14}$ $21\infty0\infty$ (by relation (1) conjugated by $(1,\infty,2,3,0) \in L_2(5)$)

$= (1,3,4,\infty,0)1\infty23\infty1\underline{421\infty}0\infty = (1,3,4,\infty,0)1\infty23\infty1\underline{(1,4,\infty,2,0)1240}0\infty$ (by relation (2) conjugated by $(1,\infty,3,0,4) \in L_2(5)$)

$= (1,3,\infty)(2,0,4)4203\underline{2412}\underline{400}\infty = (1,3,\infty)(2,0,4)4203$ $\underline{(1,4,2)(3,0,\infty)4214}4\infty$ (by relation (3) conjugated by $(2,\infty)(4,0) \in L_2(5)$)

$= (1,0,2,\infty,4)21\infty0421\underline{44}\infty = (1,0,2,\infty,4)21\infty0421\infty = e$ (by relation (2) conjugated by $(1,4,2)(3,0,\infty) \in L_2(5)$).

t_∞ takes $[\infty0\infty12]$ to a single coset in $[\infty01032]$ since $N\infty0\infty12\infty$ $= N(1,2,3,4,0)0\infty4\infty32 = N0\infty4\infty32 \in [\infty01032]$. To prove $\infty0\infty12\infty$ $= (1,2,3,4,0)0\infty4\infty32$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,3,4,0)0\infty\underline{4\infty32}\infty21\infty0\infty = (1,2,3,4,0)0\infty\ \underline{(2,\infty,0,3,4)3\infty40}$ $\infty21\infty0\infty$ (by relation (2) conjugated by $(1,2,\infty)(3,0,4) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)\underline{303}\infty40\infty21\infty0\infty = (1,\infty,0)(2,4,3)\underline{030}\infty40\infty21\infty0\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)03\underline{0\infty40}\infty21\infty0\infty = (1,\infty,0)(2,4,3)03$

$\underline{(1,2,3)(4,\infty,0)\infty 04\infty}\ \infty 21\infty 0\infty$ (by relation (3) conjugated by $(1,4)(0,\infty) \in L_2(5)$)

$= (1,0,2,\infty,4)41\infty 04\underline{\infty\infty}21\infty 0\infty = (1,0,2,\infty,4)41\infty 0\ \underline{421\infty}0\infty$

$= (1,0,2,\infty,4)41\infty 0\ \underline{(1,4,\infty,2,0)12400}\infty$ (by relation (2) conjugated by $(1,\infty,3,0,4) \in L_2(5)$)

$= \infty 42\underline{11}24\underline{00}\infty = \infty 4\underline{22}4\infty = \infty\underline{44}\infty = \underline{\infty\infty} = e$.

Now consider $[\infty 0\infty 10]$. $N^{(\infty 0\infty 10)}$ has orbits $\{1\}$, $\{4\}$, $\{2,3\}$, and $\{0,\infty\}$. So we need to look at $[\infty 0\infty 101]$, $[\infty 0\infty 104]$, $[\infty 0\infty 102]$, and $[\infty 0\infty 100]$.

First $[\infty 0\infty 100] = [\infty 0\infty 1]$. So t_0, a representative from one of the 2-orbits, takes $[\infty 0\infty 10]$ back to a single coset in $[\infty 0\infty 1]$.

t_1 takes $[\infty 0\infty 10]$ to a single coset in $[\infty 01\infty]$ since as we previously proved $\infty 01\infty 1 = (1,0.\infty)(2,3,4)\infty 0\infty 10$. Therefore it follows $N\infty 01\infty$ $= N(1,0.\infty)(2,3,4)\infty 0\infty 101 = N\infty 0\infty 101 \in [\infty 0\infty 101] = [\infty 01\infty]$.

t_4 takes $[\infty 0\infty 10]$ to a single coset in $[\infty 0\infty 104]$. As we previously proved $(1,0,\infty)(2,3,4)\infty 0\infty 10 = 0\infty 01\infty$. So it follows that $N(1,0,\infty)(2,3,4)\infty 0\infty 104 = N\infty 0\infty 104 = N0\infty 01\infty 4$. There are 30 distinct single cosets in $[\infty 0\infty 104]$.

t_2, a representative from the other 2-orbit, takes $[\infty 0\infty 10]$ to a single coset in $[\infty 01031]$ since $N\infty 0\infty 102 = N(1,\infty)(2,0)04\infty 43\infty = N04\infty 43\infty \in [\infty 01031]$. To prove $\infty 0\infty 102 = (1,\infty)(2,0)04\infty 43\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty)(2,0)04\infty\underline{43\infty 2}01\infty 0\infty = (1,\infty)(2,0)04\infty\ \underline{(1,\infty,4,2,3)\infty 341}$ $01\infty 0\infty$ (by relation (2) conjugated by $(1,2,3)(4,\infty,0) \in L_2(5)$)

$= (1,4,2,0,3)024\infty 34\underline{101\infty 0\infty} = (1,4,2,0,3)024\infty 34\underline{0100\infty 0}$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (1,4,2,0,3)024\infty 3401\underline{00}\infty 0 = (1,4,2,0,3)02\ \underline{4\infty 34}\ 01\infty 0 = (1,4,2,0,3)02$ $\underline{(1,2,0)(3,\infty,4)\infty 43\infty}\ 01\infty 0$ (by relation (3) conjugated by $(1,3,2)(4,0,\infty) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)10\infty 43\underline{\infty 01\infty}0 = (1,3,2)(4,0,\infty)10\infty 43$ $\underline{(1,0,\infty)(2,3,4)0\infty 100}$ (by relation (3))

$= (1,4,\infty,2,0)0\infty 1240\infty 1\underline{00} = (1,4,\infty,2,0)0\infty 1240\infty 1 = e$ (by relation (2) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$).

Now consider $[\infty 0\infty 20]$. $N^{(\infty 0\infty 20)}$ has orbits $\{2\}$, $\{3\}$, $\{1,4\}$, and $\{0,\infty\}$. So we need to look at $[\infty 0\infty 202]$, $[\infty 0\infty 203]$, $[\infty 0\infty 201]$, and $[\infty 0\infty 200]$.

First $[\infty 0 \infty 200] = [\infty 0 \infty 2]$. So t_0, a representative from one of the 2-orbits, takes $[\infty 0 \infty 20]$ back to a single coset in $[\infty 0 \infty 2]$.

t_2 takes $[\infty 0 \infty 20]$ to a single coset in $[\infty 02 \infty]$ since as we previously proved $\infty 02 \infty 2 = 0 \infty 02 \infty$. Therefore it follows $N \infty 02 \infty = N 0 \infty 02 \infty 2$. If we conjugate $N 0 \infty 02 \infty 2$ by $(1,4)(0,\infty) \in L_2(5)$ we obtain $N \infty 0 \infty 202$. Hence $N \infty 0 \infty 202 \in [0 \infty 02 \infty 2]$ $= [\infty 02 \infty]$.

t_3 takes $[\infty 0 \infty 20]$ to a single coset in $[\infty 012 \infty 4]$ since $N \infty 0 \infty 203$ $N(1,2,3,4,0) \infty 043 \infty 1 = N \infty 043 \infty 1 \in [\infty 012 \infty 4]$. To prove $\infty 0 \infty 203$ $= (1,2,3,4,0) \infty 043 \infty 1$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,3,4,0)\infty 043 \infty \underline{1302} \infty 0 \infty = (1,2,3,4,0) \infty 043 \infty \underline{(1,2,3,4,0)0314} \infty 0 \infty$ (by relation (2) conjugated by $(1,2,3,4,0) \in L_2(5)$)

$= (1,3,0,2,4)\infty 1 \underline{04 \infty 0} 314 \infty 0 \infty = (1,3,0,2,4) \infty 1\ \underline{(1,3,2)(4,0,\infty)40 \infty 4}$ $314 \infty 0 \infty$ (by relation (3) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (1,2,0)(3,\infty,4)434\underline{0 \infty 43}14 \infty 0 \infty = (1,2,0)(3,\infty,4)434\ \underline{(1,4,0,3,\infty)4 \infty 01}$ $14 \infty 0 \infty$ (by relation (2) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)0 \infty 04 \infty 0\underline{11}4 \infty 0 \infty = (1,2,3)(4,\infty,0)0 \infty 0\ \underline{4 \infty 04}\ \infty 0 \infty$ $= (1,2,3)(4,\infty,0)0 \infty 0\ \underline{(1,3,2)(4,0,\infty)\infty 40 \infty}\ \infty 0 \infty$ (by relation (3) conjugated by $(1,0,\infty,4,3) \in L_2(5)$)

$= \infty 4 \underline{\infty \infty} 40 \underline{\infty \infty} 0 \infty = \infty \underline{4400} \infty = \underline{\infty \infty} = e.$

t_1, a representative from the other 2-orbit, takes $[\infty 0 \infty 20]$ to a single coset in $[\infty 01403]$ since $N \infty 0 \infty 201 = N(1,2,4,0,\infty)1 \infty 02 \infty 4 \in [\infty 01403]$. To prove $\infty 0 \infty 201$ $= (1,2,4,0,\infty)1 \infty 02 \infty 4$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,4,0,\infty)1 \infty \underline{02 \infty 4}102 \infty 0 \infty = (1,2,4,0,\infty)1 \infty\ \underline{(1,\infty,0,4,2)\infty 201}$ $102 \infty 0 \infty$ (by relation (1) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= \infty 0 \infty 20\underline{11}02 \infty 0 \infty = \infty 0 \infty 2\underline{00}2 \infty 0 \infty = \infty 0 \infty \underline{22} \infty 0 \infty = \infty 0 \underline{\infty \infty} 0 \infty =$ $\infty \underline{00} \infty = \underline{\infty \infty} = e.$

Now consider $[\infty 0 \infty 21]$. $N^{(\infty 0 \infty 21)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty 0 \infty 211]$, $[\infty 0 \infty 212]$, $[\infty 0 \infty 213]$, $[\infty 0 \infty 214]$, $[\infty 0 \infty 210]$, and $[\infty 0 \infty 21 \infty]$.

First $[\infty 0 \infty 211] = [\infty 0 \infty 2]$. So t_1 takes $[\infty 0 \infty 21]$ back to a single coset in

$[\infty 0 \infty 2]$.

t_2 takes $[\infty 0 \infty 21]$ to a single coset in $[\infty 0 \infty 121]$ since $N\infty 0\infty \underline{212} = N\infty 0 \infty \underline{121} \in [\infty 0 \infty 121]$.

t_3 takes $[\infty 0 \infty 21]$ to a single coset in $[\infty 014]$ since as we previously proved $\infty 014\infty = (1.\infty, 3, 0, 4)41402$. Therefore it follows $N\infty 014 = N(1.\infty, 3, 0, 4)41402\infty = N41402\infty \in [\infty 0 \infty 213] = [\infty 014]$.

t_4 takes $[\infty 0 \infty 21]$ to a single coset in $[\infty 0 \infty 214]$. $N\infty 0 \infty 214 = N3234\infty 0 = N141032$ since $N\infty 0 \infty 214 = N(1, 4, 3, \infty, 2)3234\infty 0 = N3234\infty 0$ and $N\infty 0\infty 214 = N(1, \infty, 2, 3, 0)141032 = N141032$. $N^{(\infty 0 \infty 214)} = \{(1, \infty, 3)(2, 4, 0)\}$, so there are 20 distinct single cosets in $[\infty 0 \infty 214]$.

To prove $\infty 0 \infty 214 = (1, 4, 3, \infty, 2)3234\infty 0$ we will move the relation to one side of the equal sign and prove it equals identity.

$(1, 4, 3, \infty, 2)\underline{323}4\infty 0412\infty 0\infty = (1, 4, 3, \infty, 2)\underline{232}4\infty 0412\infty 0\infty$ (by relation (5) conjugated by $(1, \infty, 3)(2, 4, 0) \in L_2(5)$)

$= (1, 4, 3, \infty, 2)2\underline{324\infty}0412\infty 0\infty = (1, 4, 3, \infty, 2)2\ \underline{(1, 4, 3, \infty, 2)4231}\ 0412\infty 0\infty$ (by relation (1) conjugated by $(1, 2, 4)(3, \infty, 0) \in L_2(5)$)

$= (1, 3, 2, 4, \infty)142310\underline{412\infty}0\infty = (1, 3, 2, 4, \infty)142310\underline{(1, 3, 2, 4, \infty)2143}0\infty$ (by relation (2) conjugated by $(1, \infty, 0, 4, 2) \in L_2(5)$)

$= (1, 2, \infty, 3, 4)3\infty 4\underline{2302}1430\infty = (1, 2, \infty, 3, 4)3\infty 4\underline{(1, \infty, 4)(2, 0, 3)3203}1430\infty$ (by relation (3) conjugated by $(1, 0, 3)(2, 4, \infty) \in L_2(5)$)

$= (1, 0, 3)(2, 4, \infty)2413\underline{2031}430\infty = (1, 0, 3)(2, 4, \infty)2413\underline{(1, 0, 4, 3, 2)3024}430\infty$ (by relation (2) conjugated by $(2, 0)(3, 4) \in L_2(5)$)

$= (1, 4, \infty)(2, 3, 0)13\underline{0230}2\underline{44}30\infty = (1, 4, \infty)(2, 3, 0)13\ \underline{(1, \infty, 4)(2, 0, 3)2032}$ $2\underline{44}30\infty$ (by relation (3) conjugated by $(1, 3, \infty, 0, 2) \in L_2(5)$)

$= \infty\underline{22}03\underline{22}30\infty = \infty 0\underline{33}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

To prove $\infty 0 \infty 214 = (1, \infty, 2, 3, 0)141032$ we will move the relation to one side of the equal sign and prove it equals identity.

$(1, \infty, 2, 3, 0)\underline{141}032412\infty 0\infty = (1, \infty, 2, 3, 0)\underline{414}032412\infty 0\infty$ (by relation (6))

$= (1, \infty, 2, 3, 0)4\underline{1403}2412\infty 0\infty = (1, \infty, 2, 3, 0)4\ \underline{(1, 3, 4, \infty, 0)041\infty}\ 2412\infty 0\infty$ (by relation (1) conjugated by $(1, 4, \infty, 2, 0) \in L_2(5)$)

$= (1, 0, 3)(2, 4, \infty)\infty 041\infty\underline{2412}\infty 0\infty = (1, 0, 3)(2, 4, \infty)\infty 041\infty$ $\underline{(1, 4, 2)(3, 0, \infty)4214}\ \infty 0\infty$ (by relation (3) conjugated by $(2, \infty)(4, 0) \in L_2(5)$)

$= (1,\infty)(3,4)3\infty2\underline{434}214\infty0\infty = (1,\infty)(3,4)3\infty2\underline{343}214\infty0\infty$ (by relation (6) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,\infty)(3,4)\underline{3\infty23}43214\infty0\infty = (1,\infty)(3,4)\underline{(1,4,0)(2,\infty,3)}\infty32\infty43214\infty0\infty$ (by relation (3) conjugated by $(1,2,0,\infty,3) \in L_2(5)$)

$= (1,3,0)(2,\infty,4)\infty32\infty\underline{4321}4\infty0\infty = (1,3,0)(2,\infty,4)\infty32\infty\ \underline{(1,3,0,2,4)2340}$ $4\infty0\infty$ (by relation (1) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (1,0,3,2,\infty)\infty04\infty23\underline{404\infty0\infty} = (1,0,3,2,\infty)\infty04\infty23\underline{0400\infty0}$ (by relation (6) conjugated by $(1,0)(3,\infty) \in L_2(5)$ and relation (5))

$= (1,0,3,2,\infty)\underline{\infty04\infty}2304\underline{00}\infty0 = (1,0,3,2,\infty)\underline{(1,3,2)(4,0,\infty)}0\infty402304\infty0$ (by relation (3) conjugated by $(1,4)(2,3) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)0\underline{\infty402}304\infty0 = (1,\infty,3)(2,4,0)0\underline{(2,4,3,0,\infty)}04\infty3304\infty0$ (by relation (2) conjugated by $(1,2,4,0,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)\infty04\infty\underline{33}04\infty0 = (1,2,3)(4,\infty,0)\infty04\infty04\infty0 = e$ by relation (3) conjugated by $(1,4)(0,\infty) \in L_2(5)$.

t_0 takes $[\infty0\infty21]$ to a single coset in $[\infty0131\infty]$ since $N\infty0\infty210$ $= N(2,4)(3,\infty)241312 = N241312 \in [\infty0131\infty]$. To prove $\infty0\infty210$ $= (2,4)(3,\infty)241312$, we will move the relation to one side of the equal sign and prove it equals identity.

$(2,4)(3,\infty)24131\underline{2012\infty0\infty} = (2,4)(3,\infty)24131\ \underline{(1,0,2)(3,4,\infty)0210}\ \underline{0\infty0}$ (by relation (4) conjugated by $(1,3)(4,0) \in L_2(5)$ and relation (5))

$= (1,0,2,\infty,4)1\infty04\underline{00}21\underline{00}\infty0 = (1,0,2,\infty,4)1\infty0421\infty0 = e$ (by relation (2) conjugated by $(1,2,\infty,3,4) \in L_2(5)$).

t_∞ takes $[\infty0\infty21]$ to a single coset in $[\infty01\infty40]$ since $N\infty0\infty21\infty$ $= N(1,2)(4,\infty)2302\infty3 = N2302\infty3 \in [\infty01\infty40]$. To prove $\infty0\infty21\infty$ $= (1,2)(4,\infty)2302\infty3$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2)(4,\infty)\underline{2302}\infty3\infty12\infty0\infty = (1,2)(4,\infty)\ \underline{(1,\infty,4)(2,0,3)3203}$ $\infty3\infty12\infty0\infty$ (by relation (3) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (1,0,3,2,\infty)320\underline{3\infty3}\infty12\infty0\infty = (1,0,3,2,\infty)320\underline{\infty3\infty}\infty12\infty0\infty$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,0,3,2,\infty)320\infty3\underline{\infty\infty}12\infty0\infty = (1,0,3,2,\infty)320\infty312\infty0\infty = e$ (by a prevously proved relation $(1,\infty,0,4,2)\infty014\infty20414 = e$ conjugated by $(1,0,2)(3,4,\infty) \in$

$L_2(5)$).

Now consider $[\infty 0 \infty 23]$. $N^{(\infty 0 \infty 23)}$ has orbits $\{2\}$, $\{3\}$, $\{1,4\}$, and $\{0,\infty\}$. So we need to look at $[\infty 0 \infty 232]$, $[\infty 0 \infty 233]$, $[\infty 0 \infty 231]$, and $[\infty 0 \infty 230]$.

First $[\infty 0 \infty 233] = [\infty 0 \infty 2]$. So t_3 takes $[\infty 0 \infty 23]$ back to a single coset in $[\infty 0 \infty 2]$.

t_2 takes $[\infty 0 \infty 23]$ to a single coset in $[\infty 0 \infty 232]$. $N\infty 0 \infty 232 = N\infty 0 \infty 323 = N0\infty 0232 = N0\infty 0323 = N414\infty 0\infty = N141\infty 0\infty = N4140\infty 0 = N1410\infty 0 = N323141 = N323414 = N232141 = N232414$ since we previously proved $N\infty 0\infty 23 = N0\infty 023$. Therefore it follows that $N\infty 0\infty \underline{232} = N0\infty 0\underline{232} = N\infty 0\infty \underline{323} = N0\infty 0\underline{323}$ (by relation (5) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$). If we conjugate this equation by $(1,3,0)(2,\infty,4)$ and $(1,0,2)(3,4,\infty) \in L_2(5)$ we find that $N414\infty 0\infty = N141\infty 0\infty = N4140\infty 0 = N1410\infty 0$ and $N323141 = N232141 = N323414 = N232414$. To show that all of these single cosets are equal we can prove $\infty 0\infty 232 = (1,4)(2,3)414\infty 0\infty$ and $\infty 0\infty 232 = (1,4)(0,\infty)323141$. $N^{(\infty 0\infty 232)} = \langle (1,4)(0,\infty),(1,4)(2,3),(1,\infty,2)(3,4,0)\rangle$, so there are 5 distinct single cosets in $[\infty 0\infty 232]$.

To prove $\infty 0\infty 232 = (1,4)(2,3)414\infty 0\infty$ we will use a previously proved relation, $\infty 0\infty 141 = (1,4)(2,3)2320\infty 0$.

$\implies (\infty 0\infty 141)^{-1} = ((1,4)(2,3)2320\infty 0)^{-1}$

$\implies 1^{-1}4^{-1}1^{-1}\infty^{-1}0^{-1}\infty^{-1} = 0^{-1}\infty^{-1}0^{-1}2^{-1}3^{-1}2^{-1}((1,4)(2,3))^{-1}$

$\implies 141\infty 0\infty = 0\infty 0232(1,4)(2,3)$

$\implies 141\infty 0\infty\underline{(1,4)(2,3)} = 0\infty 0232\underline{(1,4)(2,3)(1,4)(2,3)}$

$\implies 141\infty 0\infty(1,4)(2,3) = \underline{0\infty 0}232$

$\implies (1,4)(2,3)414\infty 0\infty = \underline{\infty 0\infty}232$ (by relation (5))

To prove $\infty 0\infty 232 = (1,4)(0,\infty)323141$ we will use two previously proved relations, $\infty 0\infty 141 = (1,4)(2,3)2320\infty 0$ and $\infty 0\infty 141 = (2,3)(0,\infty)414232$.

$\implies (1,4)(2,3)2320\infty 0 = (2,3)(0,\infty)414232$

$\implies ((1,4)(2,3)2320\infty 0)^{-1} = ((2,3)(0,\infty)414232)^{-1}$

$\implies 0^{-1}\infty^{-1}0^{-1}2^{-1}3^{-1}2^{-1}((1,4)(2,3))^{-1}((1,4)(2,3))^{-1}$

$= 2^{-1}3^{-1}2^{-1}4^{-1}1^{-1}4^{-1}((2,3)(0,\infty)^{-1}$

$\implies 0\infty 0232(1,4)(2,3)\underline{(1,4)(2,3)} = 232414(2,3)(0,\infty)\underline{(1,4)(2,3)}$

$\implies \underline{0\infty 0}232 = 232414(1,4)(0,\infty)$

$\implies \underline{\infty 0\infty}232 = (1,4)(0,\infty)\underline{232}141$ (by relation (5))

$\Longrightarrow \infty 0\infty 232 = (1,4)(0,\infty)\underline{323}141$ (by relation (5) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$).

t_1, a representative from one of the 2-orbits, takes $[\infty 0\infty 23]$ to a single coset in $[\infty 0142\infty]$ since $N\infty 0\infty 231 = N(1,0,3,2,\infty)320\infty 13 = N320\infty 13 \in [\infty 0142\infty]$. To prove $\infty 0\infty 231 = (1,0,3,2,\infty)320\infty 13$, we will move the relation to one side of the equal sign and prove it equals identity.

$\Longrightarrow (1,0,3,2,\infty)320\infty 1\underline{313}2\infty 0\infty = (1,0,3,2,\infty)320\infty 1\underline{131}2\infty 0\infty$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,0,3,2,\infty)320\infty\underline{11}312\infty 0\infty = (1,0,3,2,\infty)3\ \underline{20\infty 3}\ 12\infty 0\infty$

$= (1,0,3,2,\infty)3\ \underline{(1,\infty,2,3,0)\infty 021}\ 12\infty 0\infty$ (by relation (1) conjugated by $(1,0,2,\infty,4) \in L_2(5)$)

$= \underline{0\infty 0}2\underline{11}2\infty 0\infty = \infty 0\infty\underline{22}\infty 0\infty$ (by relation (5))

$= \infty 0\underline{\infty\infty}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e$.

t_0, a representative from the other 2-orbit, takes $[\infty 0\infty 23]$ to a single coset in $[\infty 01032]$ since $N\infty 0\infty 230 = N(1,3,\infty,0,2)0232\infty 1 = N0232\infty 1 \in [\infty 01032]$. To prove $\infty 0\infty 230 = (1,3,\infty,0,2)0232\infty 1$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,\infty,0,2)0232\infty 1\underline{0320}\infty 0 = (1,3,\infty,0,2)0232\infty 1\ \underline{(1,4,\infty)(2,3,0)3023}\infty 0$ (by relation (3) conjugated by $(1,2,4)(3,\infty,0) \in L_2(5)$)

$= (1,0,3)(2,4,\infty)230\underline{3143}023\infty 0 = (1,0,3)(2,4,\infty)230\ \underline{(1,3,4)(2,0,\infty)1341}\ 023\infty 0$ (by relation (3) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$)

$= (1,\infty,0,4,2)04\infty 13\underline{4102}3\infty 0 = (1,\infty,0,4,2)04\infty 13\ \underline{(1,\infty,0,4,2)014\infty}3\infty 0$ (by relation (1) conjugated by $(2,0,4,\infty,3) \in L_2(5)$)

$= (1,0,2,\infty,4)42\underline{0\infty 30}14\infty 3\infty 0 = (1,0,2,\infty,4)42\ \underline{(1,2,4)(3,\infty,0)\infty 03\infty}\ 14\infty 3\infty 0$ (by relation (4) conjugated by $(1,4,\infty,2,0) \in L_2(5)$)

$= (1,3,\infty)(2,0,4)14\infty\underline{03\infty 1}4\infty 3\infty 0 = (1,3,\infty)(2,0,4)14\infty\ \underline{(1,3,4,\infty,0)\infty 304}\ 4\infty 3\infty 0$ (by relation (1) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,4,2)(3,0,\infty)3\underline{\infty 0\infty}30\underline{\infty 3\infty}0 = (1,4,2)(3,0,\infty)3\underline{0\infty}030\underline{3\infty 3}0$ (by relation (5) and relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,4,2)(3,0,\infty)30\infty\underline{030}3\infty 30 = (1,4,2)(3,0,\infty)30\infty\underline{303}3\infty 30$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,4,2)(3,0,\infty)30\infty 30\underline{33}\infty 30 = (1,4,2)(3,0,\infty)30\infty 30\infty 30 = e$ (by relation

(4) conjugated by $(1,4,\infty,2,0) \in L_2(5)$).

Now consider $[\infty 0241]$. $N^{(\infty 0241)}$ has orbits $\{2\}$, $\{\infty\}$, $\{1,3\}$, and $\{4,0\}$. So we need to look at $[\infty 02411]$, $[\infty 02412]$, $[\infty 02414]$, and $[\infty 0241\infty]$.

First $[\infty 02411] = [\infty 024]$. So t_1, a representative from one of the 2-orbits, takes $[\infty 0241]$ back to a single coset in $[\infty 024]$.

t_2 takes $[\infty 0241]$ to a single coset in $[\infty 01023]$ since $N\infty 02412$ $= N(2,3)(0,\infty)2030\infty 1 = N2030\infty 1 \in [\infty 01023]$. To prove $\infty 02412$ $= (2,3)(0,\infty)2030\infty 1$, we will move the relation to one side of the equal sign and prove it equals identity.

$(2,3)(0.\infty)2\underline{030}\infty 121420\infty = (2,3)(0.\infty)2\underline{303}\infty 121420\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (2,3)(0.\infty)23\underline{03\infty 1}21420\infty = (2,3)(0.\infty)23\underline{(1,3,4,\infty,0)\infty 304}21420\infty$ (by relation (1) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,3,2,4,\infty)24\infty 30\underline{4214}20\infty = (1,3,2,4,\infty)24\infty 30\underline{(1,2,4)(3,\infty,0)2412}20\infty$ (by relation (3) conjugated by $(2,3,\infty,4,0) \in L_2(5)$)

$= (1,\infty,2)(3,4,0)410\infty 3241\underline{22}0\infty = (1,\infty,2)(3,4,0)410\infty 32410\infty = e$ (by a previously proved relation $(1,0,3)(2,4,\infty)\infty 02341\infty 023 = e$ conjugated by $(1,2,0)(3,\infty,4) \in L_2(5)$).

t_∞ takes $[\infty 0241]$ to a single coset in $[\infty 021]$ since as we previously proved $\infty 0214 = (1,3.\infty,0,2)4301\infty$. Therefore it follows $N\infty 021 = N(1,3.\infty,0,2)4301\infty 4 = N4301\infty 4 \in [\infty 0241\infty] = [\infty 021]$.

t_4, a representative from the other 2-orbit, takes $[\infty 0241]$ to a single coset in $[\infty 0\infty 214]$ since $N\infty 02414 = N(1,0,2)(3,4,\infty)4043\infty 2 = N4043\infty 2 \in [\infty 0\infty 214]$. To prove $\infty 02414 = (1,0,2)(3,4,\infty)4043\infty 2$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,2)(3,4,\infty)4043\infty 2\underline{414}20\infty = (1,0,2)(3,4,\infty)4043\infty 2\underline{141}20\infty$ (by relation (6))

$= (1,0,2)(3,4,\infty)404\underline{3\infty 21}4120\infty = (1,0,2)(3,4,\infty)404\ \underline{(1,\infty,4,2,3)2\infty 34}$ 4120∞ (by relation (2) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= 20\underline{22}\infty 3\underline{44}120\infty = 20\infty 3120\infty = e$ (by relation (1) conjugated by $(1,0,\infty,4,3) \in L_2(5)$).

Now consider $[\infty 0243]$. $N^{(\infty 0243)}$ has orbits $\{1\}$, $\{2\}$, $\{3,0\}$, and $\{4,\infty\}$. So we

need to look at $[\infty 02431]$, $[\infty 02432]$, $[\infty 02433]$, and $[\infty 02434]$.

First $[\infty 02433] = [\infty 024]$. So t_3, a representative from one of the 2-orbits, takes $[\infty 0243]$ back to a single coset in $[\infty 024]$.

t_1 takes $[\infty 0243]$ to a single coset in $[\infty 02431]$. $N\infty 02431 = N10234\infty = N432\infty 01 = N4\infty 2310 = N1320\infty 4 = N3124\infty 0 = N34210\infty = N\infty 42013 = N012\infty 43 = N0\infty 2134$ since we previously proved $N\infty 0243 = N432\infty 0$. Therefore it follows that $N\infty 02431 = N432\infty 01$. If we conjugate this equation by $(1,0,\infty,4,3)$, $(1,\infty)(3,4)$, $(1,4,0,3,\infty)$, and $(1,3)(4,0) \in L_2(5)$ we find that $N4\infty 2310 = N3124\infty 0$, $N10234\infty = N34210\infty$, $N1320\infty 4 = N0\infty 2134$ and $N\infty 42013 = N012\infty 43$. To show that all of these single cosets are equal we will prove $\infty 02431 = (1,2,\infty)(3,0,4)4\infty 2310$, $\infty 02431 = (0,2)(4,3)10234\infty$, $\infty 02431 = (2,\infty)(4,0)1320\infty 4$, and $\infty 02431 = (1,2,3,4,0)\infty 42013$.

To prove $\infty 02431 = (1,2,\infty)(3,0,4)4\infty 2310$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,\infty)(3,0,4)4\infty 231\underline{0134}20\infty = (1,2,\infty)(3,0,4)4\infty 231$ $\underline{(1,\infty,3,0,4)310\infty}\ 20\infty$ (by relation (2) conjugated by $(1,4,3,\infty,2) \in L_2(5)$)

$= (1,2,3,4,0)13\underline{20\infty 3}10\infty 20\infty = (1,2,3,4,0)13\ \underline{(1,\infty,2,3,0)\infty 021}\ 10\infty 20\infty$ (by relation (1) conjugated by $(1,0,2,\infty 4) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)\underline{\infty 0\infty}02\underline{11}0\infty 20\infty = (1,3,4)(2,0,\infty)\underline{0\infty 0}020\infty 20\infty$ (by relation (5))

$= (1,3,4)(2,0,\infty)0\infty\underline{00}20\infty 20\infty = (1,3,4)(2,0,\infty)0\infty 20\infty 20\infty = e$ (by relation (4) conjugated by $(2,0,4,\infty,3) \in L_2(5)$).

To prove $\infty 02431 = (0,2)(4,3)10234\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(0,2)(4,3)1023\underline{4\infty 13}420\infty = (0,2)(4,3)1023\ \underline{(1,4,3,\infty,2)1\infty 42}\ 420\infty$ (by relation (1) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (1,4,\infty,2,0)401\underline{\infty 1\infty}42420\infty = (1,4,\infty,2,0)401\underline{1\infty 1}42420\infty$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,4,\infty,2,0)40\underline{11}\infty 142420\infty = (1,4,\infty,2,0)40\infty 1\ \underline{424}20\infty$
$= (1,4,\infty,2,0)40\infty 1\ \underline{242}20\infty$ (by relation (5) conjugated by $(2,\infty)(4.0) \in L_2(5)$)

$= (1,4,\infty,2,0)40\infty 124\underline{22}0\infty = (1,4,\infty,2,0)40\infty 1240\infty = e$ (by relation (2) conjugated by $(1,2,0,\infty 3) \in L_2(5)$).

To prove $\infty 02431 = (2,\infty)(4,0)1320\infty 4$, we will move the relation to one side

of the equal sign and prove it equals identity.

$(2,\infty)(4,0)1320\infty\underline{4134}20\infty = (2,\infty)(4,0)1320\infty\underline{(1,4,3)(2,\infty,0)1431}20\infty$ (by relation (3) conjugated by $(1,3,0)(2,\infty,4) \in L_2(5)$)

$= (1,4,2,0,3)41\underline{\infty201}43120\infty = (1,4,2,0,3)41\underline{(1,2,4,0,\infty)02\infty4}43120\infty$ (by relation (1) conjugated by $(1,2,0,\infty,3) \in L_2(5)$)

$= (1,0,3,2,\infty)0\underline{202}\infty3120\infty = (1,0,3,2,\infty)0\underline{020}\infty3120\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,0,3,2,\infty)\underline{00}20\infty3120\infty = (1,0,3,2,\infty)20\infty3120\infty = e$ (by relation (1) conjugated by $(1,0,\infty,4,3) \in L_2(5)$).

To prove $\infty02431 = (1,0,3,2,\infty)320\infty13$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,3,2,\infty)320\infty1\underline{313}420\infty = (1,0,3,2,\infty)320\infty1\underline{131}420\infty$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,0,3,2,\infty)320\infty\underline{11}31420\infty = (1,0,3,2,\infty)3\underline{20\infty3}1420\infty = (1,0,3,2,\infty)3$ $\underline{(1,\infty,2,3,0)\infty021}1420\infty$ (by relation (1) conjugated by $(1,0,2,\infty,4) \in L_2(5)$)

$= \underline{0\infty0}2\underline{11}2\infty0\infty = \underline{\infty0\infty}22\infty0\infty$ (by relation (5))

$= \infty0\infty\underline{22}\infty0\infty = \infty0\underline{\infty\infty}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

$N^{(\infty0\infty232)} = \langle(1,4)(0,\infty),(1,4)(2,3),(1,\infty,2)(3,4,0)\rangle$, so there are 6 distinct single cosets in $[\infty0\infty232]$.

t_2 takes $[\infty0243]$ to $[\infty02\infty]$ since as we previously proved $\infty02\infty3$ $= (1.\infty,4,2,3)\infty1304$. Therefore it follows $N\infty02\infty = N(1.\infty,4,2,3)\infty13043$ $= N\infty13043 \in [\infty02432] = [\infty02\infty]$.

t_4, a representative from the other 2-orbit, takes $[\infty0243]$ to $[\infty0142\infty]$ since $N\infty02434 = N(1,3,0)(2,\infty,4)320\infty13 = N320\infty13 \in [\infty0142\infty]$.

To prove $\infty02434 = (1,3,0)(2,\infty,4)320\infty13$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,0)(2,\infty,4)320\infty13\underline{434}20\infty = (1,3,0)(2,\infty,4)320\infty13\underline{343}20\infty$ (by relation (6) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,3,0)(2,\infty,4)320\infty1\underline{33}4320\infty = (1,3,0)(2,\infty,4)320\infty14320\infty = e$ by a previously proved relation.

Now consider $[\infty024\infty]$. $N^{(\infty024\infty)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty024\infty1]$, $[\infty024\infty2]$, $[\infty024\infty3]$, $[\infty024\infty4]$, $[\infty024\infty0]$, and

$[\infty024\infty\infty\infty]$.

First $[\infty024\infty\infty\infty] = [\infty024]$. So t_∞ takes $[\infty024\infty]$ back to a single coset in $[\infty024]$.

t_2 takes $[\infty024\infty]$ to a single coset in $[\infty0142\infty]$ since $N\infty024\infty2$ $= N(1,4,3)(2,\infty,0)04\infty210 = N04\infty210 \in [\infty0142\infty]$. To prove $\infty024\infty2$ $= (1,4,3)(2,\infty,0)04\infty210$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,3)(2,\infty,0)04\infty2\underline{102\infty}420\infty = (1,4,3)(2,\infty,0)04\infty2\ \underline{(1,\infty,0,4,2)2014}$ 420∞ (by relation (1) conjugated by $(1,0)(3,\infty) \in L_2(5)$)

$= (1,2,0)(3,\infty,4)4201201\underline{44}20\infty = (1,2,0)(3,\infty,4)4201\ \underline{2012}0\infty$ $= (1,2,0)(3,\infty,4)4201\ \underline{(1,0,2)(3,4,\infty)0210}0\infty$ (by relation (4) conjugated by $(1,3)(4,0)$ $\in L_2(5)$)

$= \infty12\underline{00}21\underline{00}\infty = \infty1\underline{22}1\infty = \infty\underline{11}\infty = \underline{\infty\infty} = e.$

t_3 takes $[\infty024\infty]$ to a single coset in $[\infty0\infty104]$ since $N\infty024\infty3$ $= N(1,3,0,2,4)\infty0\infty104 = N\infty0\infty104 \in [\infty0\infty104]$. To prove $\infty024\infty3$ $= (1,3,0,2,4)\infty0\infty104$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,0,2,4)\infty\underline{0\infty10}43\infty420\infty = (1,3,0,2,4)\infty\ \underline{(1,\infty,0)(2,4,3)\infty01\infty}$ $43\infty420\infty$ (by relation (3))

$= (1,2,3)(4,\infty,0)0\infty01\underline{\infty43\infty}420\infty = (1,2,3)(4,\infty,0)0\infty01$ $\underline{(1,0,2)(3,4,\infty)4\infty34}\ 420\infty$ (by relation (3) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (2,4,3,0,\infty)2\underline{3204}\infty3\underline{44}20\infty = (2,4,3,0,\infty)2\underline{(2,\infty,0,3,4)023\infty}\infty320\infty$ (by relation (2) conjugated by $(1,4,0,3,\infty) \in L_2(5)$)

$= \infty023\underline{\infty\infty}320\infty = \infty02\underline{33}20\infty = \infty0\underline{22}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

t_4 takes $[\infty024\infty]$ to a single coset in $[\infty01213]$ since $N\infty024\infty4$ $= N(1,3,0,2,4)203\infty31 = N203\infty31 \in [\infty01213]$. To prove $\infty024\infty4$ $= (1,3,0,2,4)203\infty31$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,0,2,4)20\underline{3\infty3}14\infty420\infty = (1,3,0,2,4)20\underline{\infty3\infty}14\infty420\infty$ (by relation (5) conjugated by $(1,2,)(3,0) \in L_2(5)$)

$= (1,3,0,2,4)\underline{20\infty3}\infty14\infty420\infty = (1,3,0,2,4)\underline{(1,\infty,2,3,0)\infty021}\infty14\infty420\infty$ (by relation (1) conjugated by $(1,0,2,\infty,4) \in L_2(5)$)

$= (1,0,3)(2,4,\infty)\infty 02\underline{1\infty 1}4\infty 420\infty = (1,0,3)(2,4,\infty)\infty 02\underline{\infty 1\infty}4\infty 420\infty$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,0,3)(2,4,\infty)\infty 02\infty 1\underline{\infty 4\infty}420\infty = (1,0,3)(2,4,\infty)\infty 02\infty 1\underline{4\infty 4}420\infty$ (by relation (5) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (1,0,3)(2,4,\infty)\underline{\infty 02\infty 14\infty 4}420\infty = (1,0,3)(2,4,\infty)\ \underline{(1,4,0,3,\infty)0\infty 204}$ $4\infty 20\infty$ (by a previously proved relation)

$= (1,3,4)(2,0,\infty)0\infty 20\infty 20\infty = e$ (by relation (4) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$).

t_0 takes $[\infty 024\infty]$ to a single coset in $[\infty 01024]$ since $N\infty 024\infty 0$ $= N(1,3,\infty)(2,0,4)032341 = N032341 \in [\infty 01024]$. To prove $\infty 024\infty 0$ $= (1,3,\infty)(2,0,4)032341$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,\infty)(2,0,4)03234\underline{10\infty 4}20\infty = (1,3,\infty)(2,0,4)03234$ $\underline{(1,4,0,3,\infty)\infty 0}1320\infty$ (by relation (2) conjugated by $(1,4,\infty,2,0) \in L_2(5)$)

$= (1,\infty,4,2,3)3\infty 2\infty\underline{0\infty 0}1320\infty = (1,\infty,4,2,3)3\infty 2\infty\underline{\infty 0\infty}1320\infty$ (by relation (5))

$= (1,\infty,4,2,3)3\infty 2\underline{\infty\infty}0\infty 1320\infty = (1,\infty,4,2,3)3\infty 20\infty 1320\infty = e$ (by a previous proof).

t_1 takes $[\infty 024\infty]$ to a single coset in $[\infty 01\infty 24]$ since $N\infty 024\infty 1$ $= N(1,3,2,4,\infty)134120 = N134120 \in [\infty 01\infty 24]$. To prove $\infty 024\infty 1$ $= (1,3,2,4,\infty)134120$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,2,4,\infty)1\underline{3412}01\infty 420\infty = (1,3,2,4,\infty)1\underline{(1,3,2,4,\infty)143\infty 0}1\infty 420\infty$ (by relation (2) conjugated by $(1,2,4)(3,\infty,0) \in L_2(5)$)

$= (1,2,\infty,3,4)\underline{3143}\infty 01\infty 420\infty = (1,2,\infty,3,4)\underline{(1,3,4)(2,0,\infty)1341}$ $\infty 01\infty 420\infty$ (by relation (3) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$)

$= (1,0,\infty 4,3)134\underline{1\infty 01}\infty 420\infty = (1,0,\infty 4,3)134\ \underline{(1,0,\infty)(3,4,2)\infty 10\infty}$ $\infty 420\infty$ (by relation (3) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$)

$= (1,\infty,2,3,0)04\underline{2\infty 10\infty\infty}420\infty = (1,\infty,2,3,0)04\ \underline{(1,2,0,\infty,3)1\infty 23}\ 420\infty$ (by relation (2) conjugated by $(1,0,2,\infty,4) \in L_2(5)$)

$= (1,3,\infty,0,2)\infty 4\underline{1\infty 23}420\infty = (1,3,\infty,0,2)\infty 4\underline{(1,3,\infty,0,2)2\infty 10}420\infty$ (by relation (2) conjugated by $(1,3,0)(2,\infty,4) \in L_2(5)$)

$= (1,\infty,2,3,0)042\infty 1\underline{0420}\infty = (1,\infty,2,3,0)042\infty 1\underline{(1,\infty,3)(2,4,0)4024}\infty$ (by relation (3) conjugated by $(1,2,3)(4,\infty,0) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)2043\underline{\infty 4024}\infty = (1,3,2)(4,0,\infty)2043\underline{(2,4,3,0,\infty)04\infty 34}\infty$ (by relation (2) conjugated by $(1,2,4,0,\infty) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)4\infty 3\underline{00}4\infty 34\infty = (1,0,2)(3,4,\infty)4\infty 34\infty 34\infty = e$ (by relation (3) conjugated by $(1,3,2)(4,0,\infty) \in L_2(5)$).

Now consider $[\infty 0232]$. $N^{(\infty 0232)}$ has orbits $\{0\}$, $\{\infty\}$, $\{1,4\}$, and $\{2,3\}$. So we need to look at $[\infty 02320]$, $[\infty 0232\infty]$, $[\infty 02321]$, and $[\infty 02322]$.

First $[\infty 02322] = [\infty 023]$. So t_2, a representative from one of the 2-orbits, takes $[\infty 0232]$ back to a single coset in $[\infty 023]$.

t_0 takes $[\infty 0232]$ to a single coset in $[\infty 01310]$ since $N\infty 02320 = N(1,4,\infty)(2,3,0)130203 = N130203 \in [\infty 01310]$. To prove $\infty 02320 = (1,4,\infty)(2,3,0)130203$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,\infty)(2,3,0)13\underline{020}302320\infty = (1.4,\infty)(2,3,0)13\underline{202}302320\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,4,\infty)(2,3,0)1320\underline{2302}320\infty = (1.4,\infty)(2,3,0)1320\ \underline{(1,\infty,4)(2,0,3)3203}\ 320\infty$ (by relation (3) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= \infty 20\underline{33}20\underline{33}20\infty = \infty 20\underline{202}0\infty = \infty 20\underline{0200}\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= \infty 2\underline{00}2\underline{00}\infty = \infty\underline{22}\infty = \underline{\infty\infty} = e$.

t_∞ takes $[\infty 0232]$ to a single coset in $[\infty 0232\infty]$. $N\infty 0\underline{232}\infty = N\infty 0\underline{323}\infty$ by relation (5) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$. There are 30 distinct single cosets in $[\infty 0232\infty]$.

t_1, a representative from the other 2-orbit, takes $[\infty 0232]$ to a single coset in $[\infty 01024]$ since $N\infty 02321 = N(1,4)(0,\infty)\infty 32314 = N\infty 32314 \in [\infty 01024]$. To prove $\infty 02321 = (1,4)(0,\infty)\infty 32314$ we will use a previously proved relation, $\infty 0\infty 232 = (1,4)(0,\infty)323141$.

$\Longrightarrow \underline{\infty 0\infty}232 = (1,4)(0,\infty)323141$.

$\Longrightarrow \underline{0\infty 0}232 = (1,4)(0,\infty)323141$ (by relation (5))

$\Longrightarrow \underline{0}0\infty 0232 = \underline{0}(1,4)(0,\infty)323141$

$\Longrightarrow \infty 0232\underline{1} = (1,4)(0,\infty)\infty 323141\underline{1}$

$\Longrightarrow \infty02321 = (1,4)(0,\infty)\infty32314$.

Now consider $[\infty0201]$. $N^{(\infty0201)}$ has orbits $\{1\}$, $\{\infty\}$, $\{2,0\}$, and $\{3,4\}$. So we need to look at $[\infty02011]$, $[\infty0201\infty]$, $[\infty02012]$, and $[\infty02013]$.

First $[\infty02011] = [\infty020]$. So t_1 takes $[\infty0201]$ back to a single coset in $[\infty020]$.

t_∞ takes $[\infty0201]$ to a single coset in $[\infty0232\infty]$ since $N\infty0201\infty = N(2,0)(3,4)1\infty3431 = N1\infty3431 \in [\infty0232\infty]$. To prove $\infty0201\infty = (2,0)(3,4)1\infty3431$, we will move the relation to one side of the equal sign and prove it equals identity.

$(2,0)(3,4)1\infty3431\infty1020\infty = (2,0)(3,4)1\infty3431\infty\ \underline{33}1020\infty = (2,0)(3,4)1\infty34\ \underline{31\infty3}\ 31020\infty = (2,0)(3,4)1\infty34\ \underline{(1,3,\infty)(2,0,4)13\infty1}\ 31020\infty$ (by relation (4) conjugated by $(1,0,4)(2,3,\infty) \in L_2(5)$)

$= (1,3,2,4,\infty)3\underline{1\infty21}3\infty131020\infty = (1,3,2,4,\infty)3\ \underline{(1,2,\infty)(3,0,4)\infty12\infty}\ 3\infty131020\infty$ (by relation (3) conjugated by $(1,2,4,0,\infty) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)0\infty12\underline{\infty3\infty131}020\infty = (1,0,4)(2,3,\infty)0\infty12\ \underline{3\infty3}\ \underline{313}\ 020\infty$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$ and relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)0\infty123\infty\underline{33}13\underline{020}\infty = (1,0,4)(2,3,\infty)0\infty123\infty13\underline{202}\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)0\infty123\underline{\infty132}02\infty = (1,0,4)(2,3,\infty)0\infty123\ \underline{(1,4,3,\infty,2)\infty402}\ 02\infty$ (by relation (1) conjugated by $(2,3)(0,\infty) \in L_2(5)$)

$= (1,0,3,2,\infty)024\underline{1\infty31}\infty402\infty = (1,0,3,2,\infty)024\ \underline{(1,3,\infty)(2,0,4)\infty13\infty}\ \infty402\infty$ (by relation (4) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (1,4,2)(3,0,\infty)402\infty13402\infty = e$ by a previously proved relation, $(1,3,0)(2,\infty,4)320\infty\ 14320\infty = e$ conjugated by $(2,0)(3,4) \in L_2(5)$.

t_2, a representative from one of the 2-orbits, takes $[\infty0201]$ to a single coset in $[\infty01403]$ since $N\infty02012 = N(1.4,3)(2,\infty,0)3\infty24\infty1 = N3\infty24\infty1 \in [\infty01403]$. To prove $\infty02012 = (1.4,3)(2,\infty,0)3\infty24\infty1$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,3)(2,\infty,0)3\underline{\infty24\infty}121020\infty = (1,4,3)(2,\infty,0)3\ \underline{(1,3,0)(2,\infty,4)2\infty42}\ 121020\infty$ (by relation (4) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= (1,2,4,0,\infty)02\infty42\underline{121}020\infty = (1,2,4,0,\infty)02\infty42\underline{212}020\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,2,4,0,\infty)02\infty4\underline{22}12\underline{020}\infty = (1,2,4,0,\infty)02\infty412\underline{202}\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,2,4,0,\infty)02\infty41\underline{22}202\infty = (1,2,4,0,\infty)02\infty4102\infty = e$ (by relation (1) conjugated by $(1,2,0,\infty,3) \in L_2(5)$).

t_3, a representative from the 2-orbit, takes $[\infty0201]$ to a single coset in $[\infty0131\infty]$ since $N\infty02013 = N(1.3)(4,0)413034 = N413034 \in [\infty0131\infty]$. To prove $\infty02013 = (1.3)(4,0)413034$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3)(4,0)41303\underline{4310}20\infty = (1,3)(4,0)41303\underline{(1,4,0,3,\infty)134\infty}20\infty$ (by relation (2) conjugated by $(1,0,4)(2,3,\infty) \in L_2(5)$)

$= (1,\infty)(3,4)04\infty\underline{3\infty13}4\infty20\infty = (1,\infty)(3,4)04\infty\ \underline{(1,\infty,3)(2,4,0)\infty31\infty}$ $4\infty20\infty$ (by relation (4) conjugated by $(1,2,3)(4,\infty,0) \in L_2(5)$)

$= (1,3,0,2,4)20\underline{3\infty3}1\infty4\infty20\infty = (1,3,0,2,4)20\underline{\infty3\infty}1\infty4\infty20\infty$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,3,0,2,4)20\infty3\underline{\infty1\infty}4\infty20\infty = (1,3,0,2,4)20\infty3\underline{1\infty1}4\infty20\infty$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,3,0,2,4)20\infty31\infty\underline{14\infty2}0\infty = (1,3,0,2,4)20\infty31\infty\underline{(1,2,4,0,\infty)\infty410}0\infty$ (by relation (1) conjugated by $(1,4,0))(2,\infty,3) \in L_2(5)$)

$= (1,3,\infty)(2,0,4)\underline{4\infty13}21\infty41\underline{00}\infty = (1,3,\infty)(2,0,4)\underline{(1,4,3,\infty,2)1\infty42}$ $21\infty41\infty$ (by relation (1) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (1,\infty,4)(2,0,3)1\infty4\underline{22}1\infty41\infty = (1,\infty,4)(2,0,3)\underline{1\infty41}\ \infty41\infty$

$= (1,\infty,4)(2,0,3)\ \underline{(1,4,\infty)(2,3,0)\infty14\infty}\ \infty41\infty$ (by relation (4) conjugated by $(1,0,2)(3,4,\infty) \in L_2(5)$)

$= \infty14\underline{\infty\infty}41\infty = \infty1\underline{44}1\infty = \infty\underline{11}\infty = \underline{\infty\infty} = e$.

Now consider $[\infty020\infty]$. $N^{(\infty020\infty)}$ has orbits $\{1,3,4\}$, and $\{2,0,\infty\}$. So we need to look at $[\infty020\infty1]$ and $[\infty020\infty\infty]$.

First $[\infty020\infty\infty] = [\infty020]$. So t_∞, a representative from one of the 3-orbits, takes $[\infty020\infty]$ back to a single coset in $[\infty020]$.

t_1, a representative from the other 3-orbit, takes $[\infty020\infty]$ to a single coset in $[\infty01310]$ since $N\infty020\infty1 = N(1.0)(3,\infty)04\infty3\infty4 = N04\infty3\infty4 \in [\infty01310]$. To prove $\infty020\infty1 = (1.0)(3,\infty)04\infty3\infty4$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0)(3,\infty)04\infty3\underline{\infty41\infty}020\infty = (1,0)(3,\infty)04\infty3\ \underline{(1,4,\infty)(2,3,0)4\infty14}$ 020∞ (by relation (4) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)2\infty104\underline{\infty14}020\infty = (1,2,3)(4,\infty,0)2\infty104$ $\underline{(1,3,4,\infty,0)41\infty3}\ 20\infty$ (by relation (1) conjugated by $(2,4,3,0,\infty) \in L_2(5)$)

$= (1,2,4,0,\infty)\underline{2031}\infty41\infty320\infty = (1,2,4,0,\infty)\underline{(1,0,4,3,2)3024}\ \infty41\infty320\infty$ (by relation (2) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (2,3)(0,\infty)302\underline{4\infty4}1\infty320\infty = (2,3)(0,\infty)302\underline{\infty4\infty}1\infty320\infty$ (by relation (5) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (2,3)(0,\infty)3\underline{02\infty4}\infty1\infty320\infty = (2,3)(0,\infty)3\underline{(1,\infty,0,4,2)\infty201}\infty1\infty320\infty$ (by relation (1) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= (1,\infty,4,2,3)3\infty20\underline{1\infty1}\infty320\infty = (1,\infty,4,2,3)3\infty20\underline{\infty1\infty}\infty320\infty$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,\infty,4,2,3)3\infty20\infty1\underline{\infty\infty}320\infty = (1,\infty,4,2,3)3\infty20\infty320\infty = e$ (by a previously proved relation).

Now consider $[\infty0203]$. $N^{(\infty0203)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty02031]$, $[\infty02032]$, $[\infty02033]$, $[\infty02034]$, $[\infty02030]$ and $[\infty0203\infty]$.

First $[\infty02033] = [\infty020]$. So t_3 takes $[\infty0203]$ back to a single coset in $[\infty020]$.

t_1 takes $[\infty0203]$ to a single coset in $[\infty01231]$ since $N\infty02031$ $= N(1,\infty,4,2,3)34\infty20\infty = N34\infty20\infty \in [\infty01231]$. To prove $\infty02031$ $= (1,\infty,4,2,3)34\infty20\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,4,2,3)34\underline{\infty20\infty}13020\infty = (1,\infty,4,2,3)34\ \underline{(1,4,3)(2,\infty,0)2\infty02}$ 13020∞ (by relation (4) conjugated by $(1,3,0)(2,\infty,4) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)132\underline{\infty021}3020\infty = (1,0,2)(3,4,\infty)132\ \underline{(1,0,3,2,\infty)20\infty3}$ 3020∞ (by relation (1) conjugated by $(1,0,\infty,4,3) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)02\infty20\infty\underline{33020}\infty = (1,3,4)(2,0,\infty)02\infty20\infty\underline{202}\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)02\infty20\infty202\infty = e$ by a previously proved relation, $(1,3,4)(2,0,\infty)2\infty0\infty\ 20\infty2\infty0 = e$ conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$.

t_2 takes $[\infty0203]$ to a single coset in $[\infty01430]$ since $N\infty02032$ $= N(1,0,4)(2,3,\infty)0321\infty3 = N0321\infty3 \in [\infty01430]$. To prove $\infty02032$

$= (1,0,4)(2,3,\infty)0321\infty3$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,4)(2,3,\infty)0321\infty\underline{323}020\infty = (1,0,4)(2,3,\infty)0321\infty\underline{232}020\infty$ (by relation (5) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)0321\infty23\underline{2020}\infty = (1,0,4)(2,3,\infty)0321\infty23\underline{0200}\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)0321\infty2302\underline{00}\infty = (1,0,4)(2,3,\infty)0321\infty2302\infty = e$ (by a previously proved relation, $(1,2,3)(4,\infty,0)2401\infty0420\infty = e$, conjugated by $(2,0)(3,4) \in L_2(5)$).

t_4 takes $[\infty0203]$ to a single coset in $[\infty01024]$ since $N\infty02034$ $= N(1,4,\infty)(2,3,0)04\infty412 = N04\infty412 \in [\infty01024]$. To prove $\infty02034$ $= (1,4,\infty)(2,3,0)04\infty412$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,\infty)(2,3,0)04\infty41\underline{243}020\infty = (1,4,\infty)(2,3,0)04\infty41$ $\underline{(2,0,4,\infty,3)342\infty}\ 20\infty$ (by relation (1) conjugated by $(1,4,\infty)(2,3,0) \in L_2(5)$)

$= (1,\infty)(3,4,)4\infty3\infty134\underline{2\infty2}0\infty = (1,\infty)(3,4,)4\infty3\infty134\underline{\infty2\infty}0\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,\infty)(3,4,)4\infty3\infty\underline{134\infty}2\infty0\infty = (1,\infty)(3,4,)4\infty3\infty\ \underline{(1,\infty,3,0,4)4310}$ $2\infty0\infty$ (by relation (2) conjugated by $(1,\infty,2,3,0) \in L_2(5)$)

$= (1,3)(4,0)130\underline{343}102\infty0\infty = (1,3)(4,0)130\underline{434}102\infty0\infty$ (by relation (5) conjugated by $(3,0)(4,\infty) \in L_2(5)$)

$= (1,3)(4,0)1\underline{30434}102\infty0\infty = (1,3)(4,0)1\underline{(1,\infty,2)(3,4,0)03404}102\infty0\infty$ (by relation (4) conjugated by $(1,2,3,4,0) \in L_2(5)$)

$= (1,4,3,\infty,2)\infty0340\underline{4102}\infty0\infty = (1,4,3,\infty,2)\infty0340\underline{(1,\infty,0,4,2)014\infty}\infty0\infty$ (by relation (1) conjugated by $(2,0,4,\infty,3) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)04324014\underline{\infty\infty}0\infty = (1,2,\infty)(3,0,4)0432\ \underline{4014}0\infty$ $= (1,2,\infty)(3,0,4)0432\ \underline{(1,0,4)(2,3,\infty)0410}\ 0\infty$ (by relation (4) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (1,3,4,\infty,0)41\infty3041\underline{00}\infty = (1,3,4,\infty,0)41\infty3041\infty = e$ (by relation (1) conjugated by $(2,4,3,0,\infty) \in L_2(5)$).

t_0 takes $[\infty0203]$ to a single coset in $[\infty01031]$ since $N\infty02030$ $= N(1,\infty,4)(2,0,3)402032 = N402032 \in [\infty01031]$. To prove $\infty02030$

$= (1,\infty,4)(2,0,3)402032$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,4)(2,0,3)40\underline{2032}03020\infty = (1,\infty,4)(2,0,3)40\ \underline{(1,4,\infty)(2,3,0)0230}$ 03020∞ (by relation (3) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= \infty2023\underline{00}3\underline{020}\infty = \infty20233\underline{202}\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= \infty202\underline{33}202\infty = \infty202\underline{22}02\infty = \infty2\underline{00}2\infty = \infty\underline{22}\infty = \underline{\infty\infty} = e$.

t_∞ takes $[\infty0203]$ to a single coset in $[\infty01213]$ since $N\infty0203\infty$ $= N(2,3)(0,\infty)32\infty4\infty1 = N32\infty4\infty1 \in [\infty01213]$. To prove $\infty0203\infty$ $= (2,3)(0,\infty)32\infty4\infty1$, we will move the relation to one side of the equal sign and prove it equals identity.

$(2,3)(0,\infty)32\infty4\infty\underline{1\infty30}20\infty = (2,3)(0,\infty)32\infty4\infty\ \underline{(1,0,\infty,4,3)3\infty14}$ 20∞ (by relation (1) conjugated by $(1,\infty,2,3,0) \in L_2(5)$)

$= (1,0,4,3,2)12\underline{434}3\infty1420\infty = (1,0,4,3,2)12\underline{343}3\infty1420\infty$ (by relation (6) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,0,4,3,2)1234\underline{33}\infty1420\infty = (1,0,4,3,2)123\ \underline{4\infty14}20\infty = (1,0,4,3,2)123$ $\underline{(1,\infty,4)(2,0,3)\infty41\infty}\ 20\infty$ (by relation (4) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (1,3,0)(2,\infty,4)\infty\underline{02\infty4}1\infty20\infty = (1,3,0)(2,\infty,4)\infty\underline{(1,\infty,0,4,2)\infty201}$ $1\infty20\infty$ (by relation (1) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)0\infty20\underline{11}\infty20\infty = (1,3,4)(2,0,\infty)0\infty20\infty20\infty = e$ by relation (4) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$).

Now consider $[\infty02\infty1]$. $N^{(\infty02\infty1)}$ has orbits $\{2\}$, $\{3\}$, $\{1,4\}$, and $\{0,\infty\}$. So we need to look at $[\infty02\infty12]$, $[\infty02\infty13]$, $[\infty02\infty11]$, and $[\infty02\infty10]$.

First $[\infty02\infty11] = [\infty02\infty]$. So t_1, a representative from one of the 2-orbits, takes $[\infty02\infty1]$ back to a single coset in $[\infty02\infty]$.

t_2 takes $[\infty02\infty1]$ to a single coset in $[\infty01\infty21]$ since $N\infty02\infty12$ $= N(1,\infty,2)(3,4,0)23\infty21\infty = N23\infty21\infty \in [\infty01\infty21]$. To prove $\infty02\infty12$ $= (1,\infty,2)(3,4,0)23\infty21\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,2)(3,4,0)23\infty21\underline{\infty21\infty}20\infty = (1,\infty,2)(3,4,0)23\infty21$ $\underline{(1,2,\infty)(3,0,4)2\infty12}\ 20\infty$ (by relation (3) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= \infty01\infty\underline{22}\infty1\underline{22}0\infty = \infty01\underline{\infty\infty}10\infty = \infty0\underline{11}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e$.

t_3 takes $[\infty 02\infty 1]$ to a single coset in $[\infty 012\infty 3]$ since $N\infty 02\infty 13 = N(1,3,2,4,\infty)431240 = N431240 \in [\infty 012\infty 3]$. To prove $\infty 02\infty 13 = (1,3,2,4,\infty)431240$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,2,4,\infty)43\underline{1240}31\infty 20\infty = (1,3,2,4,\infty)43\underline{(1,0,2,\infty,4)421}\infty 31\infty 20\infty$ (by relation 2) conjugated by $(1,0)(3,\infty) \in L_2(5))$

$= (1,3,\infty,0,2)1342\underline{1\infty 31}\infty 20\infty = (1,3,\infty,0,2)1342\ \underline{(1,3,\infty)(2,0,4)\infty 13\infty}$ $\infty 20\infty$ (by relation (4) conjugated by $(1,2)(4,\infty) \in L_2(5))$

$= (1,\infty 4,2,3)3\infty 20\infty 13\underline{\infty\infty}20\infty = (1,\infty 4,2,3)3\infty 20\infty 1320\infty = e$ (by a previously proved relation).

t_0, a representative from the other 2-orbit, takes $[\infty 02\infty 1]$ to a single coset in $[\infty 01213]$ since $N\infty 02\infty 10 = N(1,\infty)(2,0)2\infty 3034 = N2\infty 3034 \in [\infty 01213]$. To prove $\infty 02\infty 10 = (1,\infty)(2,0)2\infty 3034$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty)(2,0)2\infty 30\underline{3401}\infty 20\infty = (1,\infty)(2,0)2\infty 30\ \underline{(1,4,2,0,3)0432}\ \infty 20\infty$ (by relation (1) conjugated by $(1,4,2,0,3) \in L_2(5))$

$= (1,\infty,4,2,3)0\infty 1\underline{30432}\infty 20\infty = (1,\infty,4,2,3)0\infty 1\ \underline{(1,\infty,2)(3,4,0)0340}$ $2\infty 20\infty$ (by relation (4) conjugated by $(1,2,3,4,0) \in L_2(5))$

$= (1,2,4)(3,\infty,0)3\underline{2\infty 03}402\infty 20\infty = (1,2,4)(3,\infty,0)3\ \underline{(2,3,\infty,4,0)0\infty 24}$ $402\infty 20\infty$ (by relation (1) conjugataed by $(1,\infty)(2,0) \in L_2(5))$

$= (1,3,4,)(2,0,\infty)\infty 0\infty 2\underline{44}02\infty 20\infty = (1,3,4,)(2,0,\infty)\underline{\infty 0\infty}\ \underline{202}\infty 20\infty = (1,3,4,)(2,0,\infty)\underline{0\infty 0}\ \underline{020}\infty 20\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5))$

$= (1,3,4,)(2,0,\infty)0\infty\underline{00}20\infty 20\infty = (1,3,4,)(2,0,\infty)0\infty 20\infty 20\infty = e$ (by relation (4) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5))$.

Now consider $[\infty 01\infty 21]$. $N^{(\infty 01\infty 21)}$ has orbits $\{1,4\}$, $\{2,3\}$, and $\{0,\infty\}$. So we need to look at $[\infty 01\infty 211]$, $[\infty 01\infty 212]$, and $[\infty 01\infty 21\infty]$.

First $[\infty 01\infty 211] = [\infty 01\infty 2]$. So t_1, a representative from one of the 2-orbits, takes $[\infty 01\infty 21]$ back to a single coset in $[\infty 01\infty 2]$.

t_2, a representative from another 2-orbit, takes $[\infty 01\infty 21]$ to a single coset in $[\infty 01\infty 212]$. By a previous proof we know $N\infty 01\infty 21 = N0\infty 4024$, so it follows that $N\infty 01\infty 212 = N0\infty 40242$. There are 30 distinct single cosets in $[\infty 01\infty 212]$.

t_∞, a representative from the remaining 2-orbit, takes $[\infty 01\infty 21]$ to a single coset

in $[\infty 02\infty]$ since as we previously proved $\infty 02\infty 12 = (1,\infty,2)(3,4,0)23\infty 21\infty$. Therefore it follows $N\infty 02\infty 1 = N(1,\infty,2)(3,4,0)23\infty 21\infty 2 = N23\infty 21\infty 2 \in [\infty 01\infty 21\infty] = [\infty 02\infty 1]$.

Now consider $[\infty 01\infty 24]$. $N^{(\infty 01\infty 24)}$ has orbits $\{1\}$, $\{4\}$, $\{2,3\}$ and $\{0,\infty\}$. So we need to look at $[\infty 01\infty 241]$, $[\infty 01\infty 244]$, $[\infty 01\infty 242]$, and $[\infty 01\infty 24\infty]$.

First $[\infty 01\infty 244] = [\infty 01\infty 2]$. So t_4 takes $[\infty 01\infty 24]$ back to a single coset in $[\infty 01\infty 2]$.

t_1 takes $[\infty 01\infty 24]$ to a single coset in $[\infty 01\infty 241]$. $N\infty 01\infty 241 = N0\infty 10341$ $= N\infty 04\infty 314 = N23024\infty 0 = N23\infty 210\infty = N1421\infty 32 = N1431023 = N4124032$ $= N0\infty 40214 = N32\infty 340\infty = N4134\infty 23 = N32031\infty 0$ since we previously proved $N\infty 01\infty 24 = N0\infty 1034$. Therefore it follows that $N\infty 01\infty 241 = N0\infty 10341$. If we conjugate this equation by $(1,4)(2,3)$, $(1,0,3)(2,4,\infty)$, $(1,\infty,2)(3,4,0)$, $(1,2,\infty)(3,0,4)$ and $(1,3,\infty)(2,0,4) \in L_2(5)$ we find that $N\infty 04\infty 314 = N0\infty 40214$, $N23024\infty 0 = N32031\infty 0$, $N23\infty 210\infty = N32\infty 340\infty$, $N1421\infty 32 = N4124032$, and $N1431023 = N4134\infty 23$. To show that all of these single cosets are equal we will prove $\infty 01\infty 241$ $= (1,4,\infty)(2,3,0)\infty 04\infty 314$, $\infty 01\infty 241 = (1,4,\infty,2,0)1421\infty 32$, $\infty 01\infty 241$ $= (1,\infty,3)(2,4,0)1431023$, $\infty 01\infty 241 = (1,\infty)(3,4)23024\infty 0$, and $\infty 01\infty 241$ $= (1,2,0,\infty,3)23\infty 210\infty$.

To prove $\infty 01\infty 241 = (1,4,\infty)(2,3,0)\infty 04\infty 314$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,\infty)(2,3,0)\infty 04\infty 31\underline{4142}\infty 10\infty = (1,4,\infty)(2,3,0)\infty 04\infty 31\ \underline{141}$ $2\infty 10\infty$ (by relation (6))

$= (1,4,\infty)(2,3,0)\infty 04\infty 3\underline{11}412\infty 10\infty = (1,4,\infty)(2,3,0)\infty 0\ \underline{4\infty 34}\ 12\infty 10\infty$ $= (1,4,\infty)(2,3,0)\infty 0\ \underline{(1,2,0)(3,\infty,4)\infty 43\infty}\ 12\infty 10\infty$ (by relation (3) conjugated by $(1,3,2)(4,0,\infty) \in L_2(5)$)

$= (1,3)(2,\infty)41\infty 43\underline{\infty 12\infty}10\infty = (1,3)(2,\infty)41\infty 43\underline{(1,\infty,2)(3,4,0)1\infty 21}10\infty$ (by relation (3) conjugated by $(1,2,3,4,0) \in L_2(5)$)

$= (1,4,0,3,\infty)0\infty 2041\infty 2\underline{11}0\infty = (1,4,0,3,\infty)0\infty 2041\infty 20\infty = e$ (by a previously proved relation).

To prove $\infty 01\infty 241 = (1,4,\infty,2,0)1421\infty 32$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,\infty,2,0)1421\infty 32142\underline{\infty 10\infty} = (1,4,\infty,2,0)1421\infty 32142$

$\underline{(1,\infty,0)(2,4,3)1\infty01}$ (by relation (3) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)\infty34\infty024\underline{\infty341}\infty01 = (1,3,2)(4,0,\infty)\infty34\infty024$ $\underline{(1,3,2,4,\infty)43\infty2}\ \infty01$ (by relation (2) conjugated by $(2,3)(0,\infty) \in L_2(5)$)

$= (1,2,3,4,0)1\underline{2\infty10}4\infty43\infty2\infty01 = (1,2,3,4,0)1\ \underline{(1,2,0,\infty,3)1\infty23}$ $4\infty43\infty2\infty01$ (by relation (2) conjugated by $(1,0,2,\infty,4) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)\underline{21\infty2}34\infty43\infty2\infty01 = (1,0,2)(3,4,\infty)\ \underline{(1,2,\infty)(3,0,4)12\infty1}$ $34\infty43\infty2\infty01$ (by relation (3) conjugated by $(1,\infty,2,3,0) \in L_2(5)$)

$= (1,4)(0,\infty)12\infty13\underline{4\infty4}3\infty2\infty01 = (1,4)(0,\infty)12\infty13\underline{\infty4\infty}3\infty2\infty01$ (by relation (6) conjugated by $(1,\infty)(2,0) \in L_2(5)$)

$= (1,4)(0,\infty)12\infty13\infty4\underline{\infty3\infty}2\infty01 = (1,4)(0,\infty)12\infty13\infty4\underline{3\infty3}2\infty01$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,4)(0,\infty)12\infty13\infty43\underline{\infty32\infty}01 = (1,4)(0,\infty)12\infty13\infty43$ $\underline{(1,0,4)(2,3,\infty)3\infty23}01$ (by relation (3) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (2,3,\infty,4,0)0320\infty21\infty\underline{3\infty23}01 = (2,3,\infty,4,0)0320\infty21\infty$ $\underline{(1,3,\infty,0,2)32\infty1}01$ (by relation (2) conjugated by $(1,0,\infty,4,3) \in L_2(5)$)

$= (1,3,0)(2,\infty,4)\underline{2\infty12}0130\infty32\infty = (1,3,0)(2,\infty,4)\ \underline{(1,\infty,2)(4,0,3)\infty21\infty}$ $0130\infty32\infty$ (by relation (3) conjugated by $(2,4,3,0,\infty) \in L_2(5)$)

$= (1,4)(0,\infty)\infty\underline{21\infty0}130\infty32\infty = (1,4)(0,\infty)\infty\underline{(1,4,\infty,2,0)\infty124}130\infty32\infty$ (by relation (2) conjugated by $(1,0,2)(3,4,\infty) \in L_2(5)$)

$= (1,\infty)(2,0)2\infty\underline{124}130\infty32\infty = (1,\infty)(2,0)2\infty\underline{(1,2,3,4,0)1420}30\infty32\infty$ (by relation (2) conjugated by $(1,3,0,2,4) \in L_2(5)$)

$= (1,\infty,2)(3,4,0)3\infty\underline{214200}\infty32\infty = (1,\infty,2)(3,4,0)3\infty\ \underline{(1,2,4)(3,\infty,0)1241}$ $\infty32\infty$ (by relation (3) conjugated by $(1,4,\infty,2,0) \in L_2(5)$)

$= (1,0,\infty,4,3)\infty01241\infty32\infty = e$ (by a previously proved relation).

To prove $\infty01\infty241 = (1,\infty,3)(2,4,0)1431023$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,3)(2,4,0)143102\underline{3142}\infty01\infty = (1,\infty,3)(2,4,0)143102$ $\underline{(1,0,4,3,2)4130}\ \infty01\infty$ (by relation (2) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,\infty,2,3,0)0320\underline{414}130\infty10\infty = (1,\infty,2,3,0)0320\ \underline{141}\ 130\infty10\infty$ (by relation (6))

$= (1,\infty,2,3,0)032014\underline{11}3\underline{0\infty10}\infty = (1,\infty,2,3,0)0320143$ $\underline{(1,\infty,0)(2,4,3)\infty01\infty}\infty$ (by relation (3))

$= (1,0,\infty,4,3)1241\infty3\underline{\infty\infty}2\infty01\underline{\infty\infty} = (1,0,\infty,4,3)1241\infty32\infty01 = e$ (by a previously proved relation $(1,3,4,\infty,0)\infty23\infty14210\infty = e$ conjugated by $(1,\infty)(3,4) \in L_2(5)$).

To prove $\infty01\infty241 = (1,\infty)(3,4)23024\infty0$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty)(3,4)\underline{2302}4\infty0142\infty10\infty = (1,\infty)(3,4)\underline{(1,\infty,4)(2,0,3)3203}$
$4\infty0142\infty10\infty$ (by relation (3) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (1,4,2,0,3)3203\underline{4\infty0}142\infty10\infty = (1,4,2,0,3)320\ \underline{(1,\infty,3,0,4)\infty431}$
$142\infty10\infty$ (by relation (2) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (2,4)(3,\infty)024\infty43\underline{11}142\infty10\infty = (2,4)(3,\infty)024\infty\ \underline{434}\ 2\infty10\infty$
$= (2,4)(3,\infty)024\infty\ \underline{343}2\infty10\infty$ (by relation (6) conjugated by $(1,3)(4,\infty) \in L_2(5)$)

$= (2,4)(3,\infty)024\infty34\underline{32\infty1}0\infty = (2,4)(3,\infty)024\infty34\underline{(1,2,0,\infty,3)\infty2300}\infty$ (by relation (2) conjugated by $(3,0)(4,\infty) \in L_2(5)$)

$= (1,2,4,0,\infty)\infty04314\infty23\underline{00}\infty = (1,2,4,0,\infty)\infty04314\infty23\infty = e$ (by a previously proved relation $(1,0,\infty,4,3)\infty01241\infty32\infty = e$ conjugated by $(1,4)(2,3) \in L_2(5)$).

To prove $\infty01\infty241 = (1,2,0,\infty,3)23\infty210\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,0,\infty,3)23\infty2\underline{10\infty1}42\infty10\infty = (1,2,0,\infty,3)23\infty2$
$\underline{(1,\infty,0)(2,4,3)01\infty0}\ 42\infty10\infty$ (by relation (3) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (1,4,3,\infty,2)42040\underline{1\infty04}2\infty10\infty = (1,4,3,\infty,2)42040\ \underline{(1,4,\infty,2,0)0\infty12}$
$2\infty10\infty$ (by relation (2) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)\infty01\infty10\infty1\underline{22}\infty10\infty = (1,\infty,0)(2,4,3)\infty01\infty10\ \underline{\infty1\infty}10\infty$
$= (1,\infty,0)(2,4,3)\infty01\infty10\ \underline{1\infty1}10\infty$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)\infty01\infty101\infty\underline{110}\infty = (1,\infty,0)(2,4,3)\infty01\infty\ \underline{101}\infty0\infty$
$= (1,\infty,0)(2,4,3)\infty01\infty\ \underline{010}\infty0\infty$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)\infty01\infty010\underline{\infty0\infty} = (1,\infty,0)(2,4,3)\infty01\infty010\underline{0\infty0}$ (by relation (5))

$= (1,\infty,0)(2,4,3)\infty01\infty01\underline{00}\infty0 = (1,\infty,0)(2,4,3)\infty0\ \underline{1\infty01}\infty0$
$= (1,\infty,0)(2,4,3)\infty0\ \underline{(1,0,\infty)(2,3,4)\infty10}\infty\infty0$ (by relation (3) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$)

$= 1\underline{\infty\infty}10\underline{\infty\infty}0 = \underline{1100} = e$.

$N^{(\infty01\infty241)} = \langle(1,4)(0,\infty),(1,4)(2,3),(1,\infty,2)(3,4,0)\rangle$, so there are 5 distinct

single cosets in $[\infty01\infty241]$.

t_2, a representative from one of the 2-orbits, takes $[\infty01\infty24]$ to a single coset in $[\infty0121]$ since as we previously proved $\infty01214 = (1,3,4,\infty,0)\infty23\infty41$. Therefore it follows $N\infty0121 = N(1,3,4,\infty,0)\infty23\infty414 = N\infty23\infty414 \in [\infty01\infty242] = [\infty0121]$.

t_∞, a representative from the other 2-orbit, takes $[\infty01\infty24]$ to a single coset in $[\infty024\infty]$ since as we previously proved $\infty024\infty1 = (1,3,2,4,\infty)134120$. Therefore it follows $N\infty024\infty = N(1,3,2,4,\infty)1341201 = N1341201 \in [\infty01\infty24\infty] = [\infty024\infty]$.

Now consider $[\infty01\infty20]$. $N^{(\infty01\infty20)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty01\infty201]$, $[\infty01\infty202]$, $[\infty01\infty203]$, $[\infty01\infty204]$, $[\infty01\infty200]$, and $[\infty01\infty20\infty]$.

First $[\infty01\infty200] = [\infty01\infty2]$. So t_0 takes $[\infty01\infty20]$ back to a single coset in $[\infty01\infty2]$.

t_1 takes $[\infty01\infty20]$ to a single coset in $[\infty01\infty201]$. There are 60 distinct single cosets in $[\infty01\infty201]$.

t_2 takes $[\infty01\infty20]$ to a single coset in $[\infty0131]$ since as we previously proved $\infty01314 = (1,0,3)(2,4,\infty)01\infty041$. Therefore it follows $N\infty0131$ $= N(1,0,3)(2,4,\infty)01\infty0414 = N01\infty0414 \in [\infty01\infty202] = [\infty0131]$.

t_3 takes $[\infty01\infty20]$ to a single coset in $[\infty012\infty]$ since as we previously proved $\infty012\infty2 = (1,3,\infty,0,2)0\infty103\infty$. Therefore it follows $N\infty012\infty$ $= N(1,3,\infty,0,2)0\infty103\infty2 = N0\infty103\infty2 \in [\infty01\infty203] = [\infty012\infty]$.

t_4 takes $[\infty01\infty20]$ to a single coset in $[\infty0102]$ since as we previously proved $\infty0102\infty = (1,4,0)(2,\infty,3)042034$. Therefore it follows $N\infty0102$ $= N(1,4,0)(2,\infty,3)042034\infty = N042034\infty \in [\infty01\infty204] = [\infty0102]$.

t_∞ takes $[\infty01\infty20]$ to a single coset in $[\infty01\infty20\infty]$. $N\infty01\infty20\infty$ $= N1421\infty41 = N23\infty2132$ since $\infty01\infty20\infty = (1,0,2)(3,4,\infty)1421\infty41$ and $\infty01\infty20\infty = (1,3,\infty)(2,0,4)23\infty2132$.

To prove $\infty01\infty20\infty = (1,0,2)(3,4,\infty)1421\infty41$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,2)(3,4,\infty)142\underline{1\infty41}\infty02\infty10\infty = (1,0,2)(3,4,\infty)142$ $\underline{(1,4,\infty)(2,3,0)\infty14\infty}$ $\infty02\infty10\infty$ (by relation (4) conjugated by $(1,0,2)(3,4,\infty)$ $\in L_2(5))$

$= (1,2,4)(3,\infty,0)4\infty\underline{3\infty14\infty\infty}02\infty = (1,2,4)(3,\infty,0)4\infty$ $\underline{(1,3,4,\infty,0)1\infty30}$

$02\infty10\infty$ (by relation (1) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)\underline{\infty01\infty3}\underline{00}2\infty10\infty = (1,2,\infty)(3,0,4)\underline{(1,0,\infty)(2,3,4)0\infty10}$ $32\infty10\infty$ (by relation (3))

$= (1,3,\infty,0,2)0\infty1032\infty10\infty = e$ (by a previously proved relation).

To prove $\infty01\infty20\infty = (1,3,\infty)(2,0,4)23\infty2132$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,\infty)(2,0,4)23\infty213\underline{2\infty02}\infty10\infty = (1,3,\infty)(2,0,4)23\infty213$ $\underline{(1,3,4)(2,0,\infty)\infty20\infty}\ \infty10\infty$ (by relation (4) conjugated by $(3,0)(4,\infty) \in L_2(5)$)

$= (1,4,0)(2,\infty,3)042034\infty20\underline{\infty\infty}10\infty = (1,4,0)(2,\infty,3)042034\ \underline{\infty2010}\infty =$ $(1,4,0)(2,\infty,3)042034\ \underline{(1,2,4,0,\infty)2\infty41}0\infty$ (by relation (1) conjugated by $(1,\infty,3,0,4) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)\infty\underline{04\infty32}\infty40\infty = (1,0,2)(3,4,\infty)\infty\ \underline{(2,\infty,0,3,4)\infty402}$ $2\infty40\infty$ (by relation (2) conjugated by $(1,3,2,4,\infty) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)0\infty40\underline{22}\infty40\infty = (1,3,2)(4,0,\infty)0\infty40\infty40\infty = e$ (by relation (3) conjugated by $(1,4)(2,3) \in L_2(5)$).

$N^{(\infty01\infty20\infty)} = \langle(1,\infty,2)(3,4,0)\rangle$ so there are 20 distinct single cosets in $[\infty01\infty20\infty]$.

Now consider $[\infty01\infty41]$. $N^{(\infty01\infty41)}$ has orbits $\{4\}$ and $\{1,2,3,0,\infty\}$. So we need to look at $[\infty01\infty411]$ and $[\infty01\infty414]$.

First $[\infty01\infty411] = [\infty01\infty4]$. So t_1, a representative from the 5-orbit, takes $[\infty01\infty41]$ back to a single coset in $[\infty01\infty4]$.

t_4 takes $[\infty01\infty41]$ to a single coset in $[\infty01\infty414]$. By a previous proof we know $N\infty01\infty41 = N21\infty24\infty = N3\infty2342 = N203243 = N12\infty14\infty = N310340 = N\infty32\infty42 = N0\infty2041 = N023043 = N130140$. So it follows that $N\infty01\infty414 = N21\infty24\infty4 = N3\infty23424 = N2032434 = N12\infty14\infty4 = N3103404 = N\infty32\infty424 = N0\infty20414 = N0230434 = N1301404$.

$N^{(\infty01\infty414)} = \langle(1,0)(3,\infty),(1,3,\infty,0,2)\rangle$ so there are 6 distinct single cosets in $[\infty01\infty414]$.

Now consider $[\infty01\infty40]$. $N^{(\infty01\infty40)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty01\infty401]$, $[\infty01\infty402]$, $[\infty01\infty403]$, $[\infty01\infty404]$, $[\infty01\infty400]$, and $[\infty01\infty40\infty]$.

First $[\infty 01\infty 400] = [\infty 01\infty 4]$, so t_0 takes $[\infty 01\infty 40]$ back to a single coset in $[\infty 01\infty 4]$.

t_1 takes $[\infty 01\infty 40]$ to a single coset in $[\infty 0\infty 14]$ since as we previously proved $\infty 0\infty 142 = (1,2,0)(3,\infty,4)032013$. Therefore it follows $N\infty 0\infty 14$ $= N(1,2,0)(3,\infty,4)0320132 = N0320132 \in [\infty 01\infty 401] = [\infty 0\infty 14]$.

t_2 takes $[\infty 01\infty 40]$ to a single coset in $[\infty 0142]$ since as we previously proved $\infty 01424 = (1,3,4,\infty,0)\infty 23\infty 12$. Therefore it follows $N\infty 0142$ $= N(1,3,4,\infty,0)\infty 23\infty 124 = N\infty 23\infty 124 \in [\infty 01\infty 402] = [\infty 0142]$.

t_3 takes $[\infty 01\infty 40]$ to a single coset in $[\infty 01\infty 403]$. There are 60 distinct single cosets in $[\infty 01\infty 403]$.

t_4 takes $[\infty 01\infty 40]$ to a single coset in $[\infty 0\infty 21]$ since as we previously proved $\infty 0\infty 21\infty = (1,2)(4,\infty)2302\infty 3$. Therefore it follows $N\infty 0\infty 21$ $= N(1,2)(4,\infty)2302\infty 3\infty = N2302\infty 3\infty \in [\infty 01\infty 404] = [\infty 0\infty 21]$.

t_∞ takes $[\infty 01\infty 40]$ to a single coset in $[\infty 012\infty]$ since as we previously proved $\infty 012\infty 1 = (2,\infty,0,3,4)143124$. Therefore it follows $N\infty 012\infty$ $= N(2,\infty,0,3,4)1431241 = N1431241 \in [\infty 01\infty 40\infty] = [\infty 012\infty]$.

Now consider $[\infty 01023]$. $N^{(\infty 01023)}$ has orbits $\{2\}$, $\{\infty\}$, $\{1,3\}$, and $\{4,0\}$. So we need to look at $[\infty 010231]$, $[\infty 01023\infty]$, $[\infty 010233]$,and $[\infty 010234]$.

First $[\infty 010233] = [\infty 0102]$. So t_3, a representative from one of the 2-orbits, takes $[\infty 01023]$ back to a single coset in $[\infty 0102]$.

t_∞ takes $[\infty 01023]$ to a single coset in $[\infty 0241]$ since as we previously proved $\infty 02412 = (2,3)(0,\infty)2030\infty 1$. Therefore it follows $N\infty 0241 = N(2,3)(0,\infty)2030\infty 12 = N2030\infty 12 \in [\infty 01023\infty] = [\infty 0241]$.

t_4, a representative from the other 2-orbit, takes $[\infty 01023]$ to a single coset in $[\infty 0142]$ since as we previously proved $\infty 01421 = (1,2,\infty,3,4)43\infty 302$. Therefore it follows $N\infty 0142 = N(1,2,\infty,3,4)43\infty 3021 = N43\infty 3021 \in [\infty 010234] = [\infty 0142]$.

t_2 takes $[\infty 01023]$ to a single coset in $[\infty 010232]$. By a previous proof we know $N\infty 01023 = N\infty 43421$. So it follows that $N\infty 010232 = N\infty 434212$. $N^{(\infty 010232)} = \langle (1,3)(4,0) \rangle$, so there are 30 distinct single cosets in $[\infty 010232]$.

Now consider $[\infty 01024]$. $N^{(\infty 01024)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty 010241]$, $[\infty 010242]$, $[\infty 010243]$, $[\infty 010244]$, $[\infty 010240]$, and $[\infty 01024\infty]$.

First $[\infty010244] = [\infty0102]$, so t_4 takes $[\infty01024]$ back to a single coset in $[\infty0102]$.

t_1 takes $[\infty01024]$ to a single coset in $[\infty012\infty]$ since as we previously proved $\infty012\infty0 = (1,\infty,0)(2,4,3)\ 1303\infty4$. Therefore it follows $N\infty012\infty$ $= N(1,\infty,0)(2,4,3)1303\infty40 = N1303\infty40 \in [\infty010241] = [\infty012\infty]$.

t_2 takes $[\infty01024]$ to a single coset in $[\infty0232]$ since as we previously proved $\infty02321 = (1,4)(0,\infty)\infty32314$. Therefore it follows $N\infty0232 = N(1,4)(0,\infty)\infty323141 = N\infty323141 \in [\infty010242] = [\infty0232]$.

t_3 takes $[\infty01024]$ to a single coset in $[\infty0141]$ since as we previously proved $\infty01412 = (1,2,\infty,3,4)43\infty301$. Therefore it follows $N\infty0141$ $= N(1,2,\infty,3,4)43\infty3012 = N43\infty3012 \in [\infty010243] = [\infty0141]$.

t_0 takes $[\infty01024]$ to a single coset in $[\infty0203]$ since as we previously proved $\infty02034 = (1,4,\infty)(2,3,0)04\infty412$. Therefore it follows $N\infty0203$ $= N(1,4,\infty)(2,3,0)04\infty4124 = N04\infty4124 \in [\infty010240] = [\infty0203]$.

t_∞ takes $[\infty01024]$ to a single coset in $[\infty024\infty]$ since as we previously proved $\infty024\infty0 = (1,3,\infty)(2,0,4)032341$. Therefore it follows $N\infty024\infty$ $= N(1,3,\infty)(2,0,4)0323410 = N0323410 \in [\infty01024\infty] = [\infty024\infty]$.

Now consider $[\infty01020]$. $N^{(\infty01020)}$ has orbits $\{2\}$, $\{4\}$, $\{1,0\}$, and $\{3,\infty\}$. So we need to look at $[\infty010202]$, $[\infty010204]$, $[\infty010200]$,and $[\infty010203]$.

First $[\infty010200] = [\infty0102]$. So t_0, a representative from one of the 2-orbits, take $[\infty01020]$ back to a single coset in $[\infty0102]$.

t_2 takes $[\infty01020]$ to a single coset in $[\infty0120]$ since as we previously proved $\infty01202 = \infty01020$. Therefore it follows $N\infty0120 = N\infty010202 \in [\infty010202] = [\infty0120]$.

t_3, a representative from the other 2-orbit, takes $[\infty01020]$ to a single coset in $[\infty010203]$. There are 60 distinct single cosets in $[\infty010203]$.

t_4 takes $[\infty01020]$ to a single coset in $[\infty010204]$. $N\infty010204 = N3101214 = N4121013 = N402010\infty = N320212\infty = N\infty212023$ since previously we proved that $N\infty01020 = N310121$. Therefore it follows that $N\infty010204 = N3101214$. If we conjugate this equation by $(1,2,0)(3,\infty,4)$ and $(1,2)(4,\infty) \in L_2(5)$ we find $N4121013 = N\infty212023$ and $N402010\infty = N320212\infty$. To show that all of these single cosets are equal we will prove $\infty010204 = 4121013$ and $\infty010204 = (1,0,2)(3,4,\infty)402010\infty$.

To prove $\infty010204 = 4121013$, we will move the relation to one side of the equal

sign and prove it equals identity.

$4\underline{121}013402010\infty = 4\underline{212}\ 01340\ 2010\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5))$

$= 42\underline{1201}3402010\infty = 42\underline{(1,0,2)(3,4,\infty)2102}3402010\infty$ (by relation (4) conjugated by $(1,4,2)(3,0,\infty) \in L_2(5))$

$= (1,0,2)(3,4,\infty)\infty1210\underline{2340}2010\infty = (1,0,2)(3,4,\infty)\infty1210\ \underline{(1,4,2,0,3)4321}$ 2010∞ (by relation (1) conjugated by $(1,3,0,2,4) \in L_2(5))$

$= (1,3,2,4,\infty)\infty404\underline{343}212\underline{010}\infty = (1,3,2,4,\infty)\infty404\underline{434}212\underline{101}\infty$ (by relation (6) conjugated by $(1,3)(2,\infty)$ and $(1,\infty)(3,4) \in L_2(5))$

$= (1,3,2,4,\infty)\infty40\underline{44}342\underline{121}01\infty = (1,3,2,4,\infty)\infty40342\underline{212}01\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5))$

$= (1,3,2,4,\infty)\infty4034\underline{22}1201\infty = (1,3,2,4,\infty)\infty40341201\infty = e$ (by a previously proved relation $(2,\infty,0,3,4)\infty21320410\infty = e$ conjugated by $(1,0)(2,4) \in L_2(5))$.

To prove $\infty010204 = (1,0,2)(3,4,\infty)402010\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,2)(3,4,\infty)4\underline{020}10\infty402010\infty = (1,0,2)(3,4,\infty)4\underline{202}10\infty402010\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5))$

$= (1,0,2)(3,4,\infty)42\underline{0210}\infty402010\infty = (1,0,2)(3,4,\infty)42\ \underline{(1,2,0)(3,\infty,4)2012}$ $\infty402010\infty$ (by relation (4) conjugated by $(1,4,2,0,3) \in L_2(5))$

$= 302012\infty4\underline{020}10\infty = 302012\infty4\ \underline{202}\ 10\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5))$

$= 30201\underline{2\infty42}0210\infty = 30201\underline{(1,0,3)(2,4,\infty)\infty24\infty}0210\infty$ (by relation (4) conjugated by $(1,3,4,\infty,0) \in L_2(5))$

$= (1,0,3)(2,4,\infty)13\underline{430\infty}24\infty0210\infty = (1,0,3)(2,4,\infty)13\ \underline{(2,0,4,\infty,3)0342}$ $24\infty0210\infty$ (by relation (1) conjugated by $(1,3,\infty)(2,0,4) \in L_2(5))$

$= (1,4,3)(2,\infty,0)12034\underline{22}4\infty0210\infty = (1,4,3)(2,\infty,0)1203\ \underline{44}\ \infty0210\infty$
$= (1,4,3)(2,\infty,0)120\ \underline{3\infty02}10\infty = (1,4,3)(2,\infty,0)120\ \underline{(1,0,3,2,\infty)0\infty31}\ 10\infty$ (by relation (1) conjugated by $(1,\infty,4)(2,0,3) \in L_2(5))$

$= (1,4,2)(3,0,\infty)0\infty30\infty30\infty = e$ (by relation (4) conjugated by $(2,\infty)(4,0) \in L_2(5))$.

$N^{(\infty010204)} = \langle(1,0)(3,\infty),(1,2,0)(3,\infty,4)\rangle$ so there are 10 distinct single cosets in $[\infty010204]$.

Now consider $[\infty0103\infty]$. $N^{(\infty0103\infty)}$ has orbits $\{3\}$, $\{\infty\}$, $\{1,0\}$, and $\{2,4\}$. So we need to look at $[\infty0103\infty3]$, $[\infty0103\infty\infty]$, $[\infty0103\infty1]$,and $[\infty0103\infty2]$.

First $[\infty0103\infty\infty] = [\infty0103]$. So t_∞ takes $[\infty0103\infty]$ back to a single coset in $[\infty0103]$.

t_3 takes $[\infty0103\infty]$ to a single coset in $[\infty0141]$ since as we previously proved $\infty0141\infty = (1,4)(2,3)0232\infty0$. Therefore it follows $N\infty0141 = N(1,4)(2,3)0232\infty0\infty = N0232\infty0\infty \in [\infty0103\infty3] = [\infty0141]$.

t_1, a representative from one of the 2-orbits, takes $[\infty0103\infty]$ to a single coset in

$[\infty01\infty20\infty]$ since $N\infty0103\infty1 = N(1,2,4,0,\infty)2402342 = N2402342 \in [\infty01\infty20\infty]$. To prove $\infty0103\infty1 = (1,2,4,0,\infty)2402342$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,4,0,\infty)240\underline{23421}\infty3010\infty = (1,2,4,0,\infty)240\ \underline{(1,\infty,0)(2,4,3)3243}\ 1\infty3010\infty$ (by relation (4) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)431324\underline{31\infty3}010\infty = (1,3,2)(4,0,\infty)431324\ \underline{(1,3,\infty)(2,0,4)13\infty1}\ 010\infty$ (by relation (4) conjugated by $(1,0,4)(2,3,\infty) \in L_2(5)$)

$= (1,\infty,2,3,0)2\infty3\infty0213\infty\underline{1010}\infty = (1,\infty,2,3,0)2\infty3\infty0213\infty\underline{010}0\infty$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (1,\infty,2,3,0)2\infty\underline{3\infty02}13\infty01\underline{00}\infty = (1,\infty,2,3,0)2\infty\ \underline{(1,0,3,2,\infty)0\infty31}\ 13\infty01\infty$ (by relation (1) conjugated by $(1,\infty,4)(2,0,3) \in L_2(5)$)

$= \infty10\infty3\underline{11}3\infty01\infty = \infty10\infty\ \underline{33}\ \infty01\infty = \infty10\ \underline{\infty\infty}\ 01\infty = \infty1\ \underline{00}\ 1\infty = \infty\underline{11}\infty = \underline{\infty\infty} = e.$

t_2, a representative from the other 2-orbit, takes $[\infty0103\infty]$ to a single coset in $[\infty010203]$ since $N\infty0103\infty2 = N(1,2,4,0,\infty)3\infty2\infty0\infty1 = N3\infty2\infty0\infty1 \in [\infty010203]$. To prove $\infty0103\infty2 = (1,2,4,0,\infty)3\infty2\infty0\infty1$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,4,0,\infty)3\infty2\underline{\infty0\infty}12\infty3010\infty = (1,2,4,0,\infty)3\infty2\underline{0\infty0}12\infty3010\infty$ (by relation (5))

$= (1,2,4,0,\infty)3\underline{\infty20\infty}012\infty3010\infty = (1,2,4,0,\infty)3\ \underline{(1,4,3)(2,\infty,0)2\infty02}\ 012\infty3010\infty$ (by relation (4) conjugated by $(1,3,0)(2,\infty,4) \in L_2(5)$)

$= (1,\infty,4,2,3)12\infty020\underline{12\infty3}010\infty = (1,\infty,4,2,3)12\infty020\ \underline{(1,3,2,4,\infty)\infty214}\ 010\infty$ (by relation (2) conjugated by $(1,3,4,\infty,0) \in L_2(5)$)

$= 341040\infty214\underline{010}\infty = 341040\infty214\ \underline{101}\infty$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= 341040\infty2\underline{141}01\infty = 341040\infty2\underline{414}01\infty$ (by relation (6))

$= 34104\underline{0\infty24}1401\infty = 34104\underline{(2,0,4,\infty,3)2\infty03}1401\infty$ (by relation (1) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (2,0,4,\infty,3)2\underline{\infty14\infty}2\infty031401\infty = (2,0,4,\infty,3)2\ \underline{(1,\infty,4)(2,0,3)1\infty41}$ $2\infty031401\infty$ (by relation (4) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= (1,\infty,2,3,0)01\infty\underline{412\infty}031401\infty = (1,\infty,2,3,0)01\infty\ \underline{(1,3,2,4,\infty)2143}$ 031401∞ (by relation (2) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (3,0)(4,\infty)031214\underline{303}1401\infty = (3,0)(4,\infty)031214\underline{030}1401\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (3,0)(4,\infty)03121403\underline{0140}1\infty = (3,0)(4,\infty)03121403\underline{(1,0,4)(2,3,\infty)1041}1\infty$ (by relation (4) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$)

$= (1,0,\infty)(2,3,4)4\infty\underline{030}14\infty104\underline{11}\infty = (1,0,\infty)(2,3,4)4\infty\underline{303}14\infty104\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,0,\infty)(2,3,4)4\infty3\underline{0314}\infty104\infty = (1,0,\infty)(2,3,4)4\infty3\ \underline{(1,0,4,3,2)1302}$ $\infty104\infty$ (by relation (2) conjugated by $(1,4)(2,3) \in L_2(5)$)

$= (1,4)(0,\infty)3\infty2130\underline{2\infty10}4\infty = (1,4)(0,\infty)3\infty2130\underline{(1,2,0,\infty,3)1\infty23}4\infty$ (by relation (2) conjugated by $(1,0,2,\infty,4) \in L_2(5)$)

$= (1,4,2,0,3)1302\underline{1\infty1}\infty234\infty = (1,4,2,0,3)1302\underline{\infty1\infty}\infty234\infty$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,4,2,0,3)1302\infty1\underline{\infty\infty}234\infty = (1,4,2,0,3)1302\infty\ \underline{1234}\infty$
$= (1,4,2,0,3)1302\infty\ \underline{(1,4,2,0,3)3210}\infty$ (by relation (1) conjugated by $(1,2,3,4,0) \in L_2(5)$)

$= (1,2,3,4,0)4130\infty3210\infty = e$ (by a previously proved relation, $(1,\infty,3,0,4)\infty013214032 = e$ conjugated by $(1,3,0)(2,\infty,4) \in L_2(5)$).

Now consider $[\infty01031]$. $N^{(\infty01031)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$ and $\{\infty\}$. So we need to look at $[\infty010311]$, $[\infty010312]$, $[\infty010313]$, $[\infty010314]$, $[\infty010310]$, and $[\infty01031\infty]$.

First $[\infty010311] = [\infty0103]$. So t_1 takes $[\infty01031]$ back to a single coset in $[\infty0103]$.

t_2 takes $[\infty01031]$ to a single coset in $[\infty01\infty212]$ since $N\infty010312$

$= N(2,\infty,0,3,4)1\infty21424 = N1\infty21424 \in [\infty01\infty212]$. To prove $\infty010312$ $= (2,\infty,0,3,4)1\infty21424$, we will move the relation to one side of the equal sign and prove it equals identity.

$(2,\infty,0,3,4)1\infty214\underline{24}213010\infty = (2,\infty,0,3,4)1\infty214\underline{42}413010\infty$ (by relation (6) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (2,\infty,0,3,4)1\infty21\underline{442413}010\infty = (2,\infty,0,3,4)1\infty21\ \underline{(1,2,3,4,0)1420}\ 010\infty$ (by relation (2) conjugated by $(1,3,0,2,4) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)2\infty3\underline{214200}10\infty = (1,2,\infty)(3,0,4)2\infty3\ \underline{(1,2,4)(3,\infty,0)1241}\ 10\infty$ (by relation (3) conjugated by $(1,4,\infty,2,0) \in L_2(5)$)

$= (1,4,\infty2,0)40\infty1240\infty = e$ (by relation (2) conjugated by $(1,2,0,\infty,3) \in L_2(5)$).

t_3 takes $[\infty01031]$ to a single coset in $[\infty0130]$ since we previously proved $\infty01303 = \infty10130$. Therefore it follows $N\infty0130 = N\infty101303 \in [\infty010313] = [\infty0130]$.

t_4 takes $[\infty01031]$ to a single coset in $[\infty0\infty10]$ since we previously proved $\infty0\infty102 = (1,\infty)(2,0)04\infty43\infty$. Therefore it follows $N\infty0\infty10$ $= N(1,\infty)(2,0)04\infty43\infty2 = N04\infty43\infty2 \in [\infty010314] = [\infty0\infty10]$.

t_0 takes $[\infty01031]$ to a single coset in $[\infty0203]$ since we previously proved $\infty02030$ $= (1,\infty,4)(2,0,3)402032$. Therefore it follows $N\infty0203 = N(1,\infty,4)(2,0,3)4020320$ $= N4020320 \in [\infty010310] = [\infty0203]$.

t_∞ takes $[\infty01031]$ to a single coset in $[\infty010203]$ since $N\infty01031\infty$ $= N(1,\infty)(2,0)2414\infty43 = N2414\infty43 \in [\infty010203]$. To prove $\infty01031\infty$ $= (1,\infty)(2,0)2414\infty43$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty)(2,0)2414\infty43\infty\underline{1301}0\infty = (1,\infty)(2,0)2414\infty43\infty$ $\underline{(1,0,3)(2,4,\infty)3103}0\infty$ (by relation (3) conjugated by $(1,0,3,2,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)4\infty0\infty2\infty1231\underline{030}\infty = (1,2,3)(4,\infty,0)4\infty0\infty2\infty1231\underline{303}\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)4\infty0\infty2\infty12\underline{313}03\infty = (1,2,3)(4,\infty,0)4\infty0\infty2\infty12\underline{131}03\infty$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)4\infty0\infty2\infty\underline{121}3103\infty = (1,2,3)(4,\infty,0)4\infty0\infty2\infty\underline{212}3103\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)4\infty0\infty\underline{2\infty2}123103\infty = (1,2,3)(4,\infty,0)4\infty0\infty\underline{\infty2\infty}123103\infty$

(by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)4\infty0\underline{\infty\infty}2\infty123103\infty = (1,2,3)(4,\infty,0)4\ \underline{\infty02\infty}\ 123103\infty$ $= (1,2,3)(4,\infty,0)4\ \underline{(1,3,4)(2,0,\infty)0\infty20}\ 123103\infty$ (by relation (4) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$)

$= (1,0)(2,4)10\infty\underline{2012}3103\infty = (1,0)(2,4)10\infty\underline{(1,0,2)(3,4,\infty)0210}3103\infty$ (by relation (4) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (1,2,\infty,3,4)023021\underline{0310}3\infty = (1,2,\infty,3,4)023021\ \underline{(1,3,0)(2,\infty,4)3013}\ 3\infty$ (by relation (3) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)1\infty01\infty01\infty = e$ (by relation (3) conjugated by $(1,0)(2,4) \in L_2(5)$).

Now consider [∞01032]. $N^{(\infty01032)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$ and $\{\infty\}$. So we need to look at [∞010321], [∞010322], [∞010323], [∞010324], [∞010320], and [∞01032∞].

First [∞010322] = [∞0103]. So t_2 takes [∞01032] back to a single coset in [∞0103].

t_1 takes [∞01032] to a single coset in [∞01∞403] since $N\infty010321$ $= N(1,3,4)(2,0,\infty)1\infty012\infty4 = N1\infty012\infty4 \in [\infty01\infty403]$. To prove $\infty010321 = (1,3,4)(2,0,\infty)1\infty012\infty4$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,4)(2,0,\infty)1\infty01\underline{2\infty41}23010\infty = (1,3,4)(2,0,\infty)1\infty01$ $\underline{(1,\infty,0,4,2)4\infty20}\ 23010\infty$ (by relation (1) conjugated by $(1,\infty,3)(2,4.0) \in L_2(5)$)

$= (1,3,2,4,\infty)\infty04\underline{\infty4\infty}2023010\infty = (1,3,2,4,\infty)\infty04\underline{4\infty4}2023010\infty$ (by relation (5) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (1,3,2,4,\infty)\infty0\underline{44}\infty42023010\infty = (1,3,2,4,\infty)\underline{\infty0\infty}\ 42023010\infty$ $= (1,3,2,4,\infty)\underline{0\infty0}\ 42023010\infty$ (by relation (5))

$= (1,3,2,4,\infty)0\underline{\infty042}023010\infty = (1,3,2,4,\infty)0\ \underline{(1,4,\infty,2,0)40\infty1}\ 023010\infty$ (by relation (2) conjugated by $(1,2,0,\infty,3) \in L_2(5)$)

$= (1,3,0)(2,\infty,4)14\underline{0\infty10}23010\infty = (1,3,0)(2,\infty,4)14\ \underline{(1,\infty,0)(2,4,3)\infty01\infty}$ 23010∞ (by relation (3) conjugated by $(2,3)(0,\infty) \in L_2(5)$)

$= (1,2,0,\infty,3)\infty3\infty\underline{01\infty2}3010\infty = (1,2,0,\infty,3)\infty3\infty\ \underline{(1,3,\infty,0,2)\infty103}$ 3010∞ (by relation (2) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= \underline{0\infty0}\infty10\underline{33}010\infty = \underline{\infty0\infty}\infty1\underline{00}10\infty$ (by relation (5))

$= \infty 0\underline{\infty\infty}\underline{11}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

t_3 takes $[\infty 01032]$ to a single coset in $[\infty 010232]$ since by relation (5) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5) N\infty 010\underline{323} = N\infty 010\underline{232} \in [\infty 010232]$.

t_4 takes $[\infty 01032]$ to a single coset in $[\infty 0121]$ since we previously proved $\infty 0121\infty = (1,\infty)(2,0)431302$. Therefore it follows $N\infty 0121 = N(1,\infty)(2,0)431302\infty = N431302\infty \in [\infty 010324] = [\infty 0121]$.

t_0 takes $[\infty 01032]$ to a single coset in $[\infty 0\infty 12]$ since we previously proved $\infty 0\infty 12\infty = (1,2,3,4,0)0\infty 4\infty 32$. Therefore it follows $N\infty 0\infty 12$
$= N(1,2,3,4,0)0\infty 4\infty 32\infty = N0\infty 4\infty 32\infty \in [\infty 010320] = [\infty 0\infty 12]$.

t_∞ takes $[\infty 01032]$ to a single coset in $[\infty 0\infty 23]$ since we previously proved $\infty 0\infty 230 = (1,3,\infty,0,2)0232\infty 1$. Therefore it follows $N\infty 0\infty 23$
$= N(1,3,\infty,0,2)0232\infty 10 = N0232\infty 10 \in [\infty 01032\infty] = [\infty 0\infty 23]$.

Now consider $[\infty 010\infty 2]$. $N^{(\infty 010\infty 2)}$ has orbits $\{2\}$, $\{0\}$, $\{1,\infty\}$, and $\{3,4\}$. So we need to look at $[\infty 010\infty 22]$, $[\infty 010\infty 20]$, $[\infty 010\infty 21]$, and $[\infty 010\infty 23]$.

First $[\infty 010\infty 22] = [\infty 010\infty]$. So t_2 takes $[\infty 010\infty 2]$ back to a single coset in $[\infty 010\infty]$.

t_0 takes $[\infty 010\infty 2]$ to a single coset in $[\infty 0141]$ since we previously proved $\infty 01410 = (1,\infty)(2,0)30203\infty$. Therefore it follows $N\infty 0141 = N(1,\infty)(2,0)30203\infty 0 = N30203\infty 0 \in [\infty 010\infty 20] = [\infty 0141]$.

t_1, a representative from one of the 2-orbits, takes $[\infty 010\infty 2]$ to a single coset in
$[\infty 01\infty 201]$ since $N\infty 010\infty 21 = N(1,3)(4,0)0320432 = N0320432 \in [\infty 01\infty 201]$. To prove $\infty 010\infty 21 = (1,3)(4,0)0320432$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3)(4,0)0\underline{3204}3212\infty 010\infty = (1,3)(4,0)0\underline{(2,\infty,0,3,4)023\infty}3212\infty 010\infty$ (by relation (2) conjugated by $(1,4,0,3,\infty) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)\underline{3023}\infty 3212\infty 010\infty = (1,4,3)(2,\infty,0)\ \underline{(1,\infty,4)(2,0,3)0320}$ $\infty 3212\infty 010\infty$ (by relation (3) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= (2,4)(3,\infty)03\underline{20\infty 3}212\infty 010\infty = (2,4)(3,\infty)03\ \underline{(1,\infty,2,3,0)\infty 021}$ $212\infty 010\infty$ (by relation (1) conjugated by $(1,0,2,\infty,4) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)10\infty 021\underline{212}\infty 010\infty = (1,\infty,0)(2,4,3)10\infty 021\ \underline{121}\ \infty 010\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)10\infty02\underline{11}21\infty010\infty = (1,\infty,0)(2,4,3)10\infty0\ \underline{22}\ 1\infty010\infty =$ $(1,\infty,0)(2,4,3)10\infty01\infty010\infty = e$ by a previously proved relation.

t_3, a representative from the other 2-orbit, takes $[\infty010\infty2]$ to a single coset in $[\infty0121]$ since we previously proved $\infty01210 = (1,3,4)(2,0,\infty)32\infty234$. Therefore it follows $N\infty0121 = N(1,3,4)(2,0,\infty)32\infty2340 = N32\infty2340 \in [\infty010\infty23] = [\infty0121]$.

Now consider $[\infty01213]$. $N^{(\infty01213)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$ and $\{\infty\}$. So we need to look at $[\infty012131]$, $[\infty012132]$, $[\infty012133]$, $[\infty012134]$, $[\infty012130]$, and $[\infty01213\infty]$.

First $[\infty012133] = [\infty0121]$, so t_3 takes $[\infty01213]$ back to $[\infty0121]$.

t_1 takes $[\infty01213]$ to a single coset in $[\infty0203]$ since we previously proved $\infty0203\infty = (2,3)(0,\infty)32\infty4\infty1$. Therefore it follows $N\infty0203$ $= N(2,3)(0,\infty)32\infty4\infty1\infty = N32\infty4\infty1\infty \in [\infty012131] = [\infty0203]$.

t_2 takes $[\infty01213]$ to a single coset in $[\infty02\infty1]$ since we previously proved $\infty02\infty10 = (1,\infty)(2,0)2\infty3034$. Therefore it follows $N\infty02\infty1 = N(1,\infty)(2,0)2\infty30340$ $= N2\infty30340 \in [\infty012132] = [\infty02\infty1]$.

t_4 takes $[\infty01213]$ to a single coset in $[\infty024\infty]$ since we previously proved $\infty024\infty4 = (1,3,0,2,4)203\infty31$. Therefore it follows $N\infty024\infty$ $= N(1,3,0,2,4)203\infty314 = N203\infty314 \in [\infty012134] = [\infty024\infty]$.

t_0 takes $[\infty01213]$ to a single coset in $[\infty012130]$. $N\infty012130 = N03242\infty3$ $= N3\infty4140\infty$ since $\infty012130 = (1,0,4,3,2)03242\infty3$ and $\infty012130$ $= (1,0,2,\infty,4)3\infty4140\infty$.

To prove $\infty012130 = (1,0,4,3,2)03242\infty3$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,4,3,2)03242\infty\underline{303}1210\infty = (1,0,4,3,2)03242\infty\underline{030}1210\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,0,4,3,2)032\underline{42\infty0}301210\infty = (1,0,4,3,2)032\underline{(2,3,\infty,4,0)\infty24}3301210\infty$ (by relation (1) conjugated by $(1,2,\infty)(3,0,4) \in L_2(5)$)

$= (1,2)(4,\infty)2\infty3\infty24\underline{33}0\underline{121}0\infty = (1,2)(4,\infty)2\infty3\infty240\underline{212}0\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,2)(4,\infty)2\infty3\infty2402\underline{120\infty} = (1,2)(4,\infty)2\infty3\infty2402\underline{(1,\infty,2,3,0)0213}$ (by relation (1) conjugated by $(1,2,0)(3,\infty,4) \in L_2(5)$)

$= (1,3,0)(2,\infty,4)32023\underline{4130}213 = (1,3,0)(2,\infty,4)32023\ \underline{(1,2,3,4,0)3142}213$

(by relation (2) conjugated by $(1,0,4,3,2) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)4\underline{313}4314\underline{22}13 = (1,4,3)(2,\infty,0)4\underline{131}431413$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)4131431413 = e$ by a previously proved relation, $(1,\infty,0)(2,4,3)10\infty01\ \infty010\infty = e$ conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$.

To prove $\infty012130 = (1,0,2,\infty,4)3\infty4140\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,2,\infty,4)3\infty414\underline{0\infty0}31210\infty = (1,0,2,\infty,4)3\infty414\underline{\infty0\infty}31210\infty$ (by relation (5))

$= (1,0,2,\infty,4)3\infty414\infty\underline{0\infty31}210\infty = (1,0,2,\infty,4)3\infty\ 414\infty$ $\underline{(1,\infty,2,3,0)3\infty02}\ 210\infty$ (by relation (1) conjugated by $(1,\infty,4,2,3) \in L_2(5)$)

$= (3,0)(4,\infty)02\underline{4\infty4}23\infty0\underline{22}10\infty = (3,0)(4,\infty)02\underline{\infty4\infty}23\infty010\infty$ (by relation (5) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (3,0)(4,\infty)02\infty4\underline{\infty23\infty}010\infty = (3,0)(4,\infty)02\infty4\underline{(1,4,0)(2,\infty,3)2\infty32}010\infty$ (by relation (3) conjugated by $(1,3,0,2,4) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)\underline{1\infty30}2\infty32010\infty = (1,4,3)(2,\infty,0)\ \underline{(1,0,\infty,4,3)3\infty14}$ $2\infty32010\infty$ (by relation (1) conjugated by $(1,\infty,2,3,0) \in L_2(5)$)

$= (1,3,0,2,4)3\infty142\underline{\infty320}10\infty = (1,3,0,2,4)3\infty142\ \underline{(2,\infty,0,3,4)23\infty4}\ 10\infty$ (by relation (2) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$)

$= (1,4)(0,\infty)401\underline{2\infty2}3\infty410\infty = (1,4)(0,\infty)401\underline{\infty2\infty}3\infty410\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,4)(0,\infty)4\underline{01\infty2}\infty3\infty410\infty = (1,4)(0,\infty)4\underline{(1,3,\infty,0,2)\infty103}\infty3\infty410\infty$ (by relation (2) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (1,4,3,\infty,2)4\infty103\underline{\infty3\infty}410\infty = (1,4,3,\infty,2)4\infty103\underline{3\infty3}410\infty$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,4,3,\infty,2)4\infty10\underline{33}\infty3410\infty = (1,4,3,\infty,2)4\ \underline{\infty10\infty}\ 3410\infty$ $= (1,4,3,\infty,2)4\ \underline{(1,\infty,0)(2,4,3)1\infty01}\ 3410\infty$ (by relation (3) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,3,0)(2,\infty,4)31\underline{\infty013}410\infty = (1,3,0)(2,\infty,4)31\underline{(1,\infty,3,0,4)10\infty4}410\infty$ (by relation (2) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$)

$= (1,0,\infty)(2,3,4)0\infty10\infty\underline{44}10\infty = (1,0,\infty)(2,3,4)0\infty10\infty10\infty = e$ by relation (3).

$N^{(\infty 012130)} = \langle(1,4,2)(3,0,\infty)\rangle$, so there are 20 distinct single cosets in $[\infty 012130]$.

t_∞ takes $[\infty 01213]$ to a single coset in $[\infty 01\infty 201]$ since $N\infty 01213\infty = N(1,3)(4,0)1\infty 214\infty 2 = N1\infty 214\infty 2 \in [\infty 01\infty 201]$. To prove $\infty 01213\infty = (1,3)(4,0)1\infty 214\infty 2$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3)(4,0)1\underline{\infty 214}\infty 2\infty 31210\infty = (1,3)(4,0)1\ \underline{(1,\infty,4,2,3)12\infty 3}$ $\infty 2\infty 31210\infty$ (by relation (2) conjugated by $(1,4)(0,\infty) \in L_2(5)$)

$= (2,3,\infty,4,0)\infty 12\underline{\infty 3\infty}2\infty 31210\infty = (2,3,\infty,4,0)\infty 12\underline{3\infty 3}2\infty 31210\infty$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (2,3,\infty,4,0)\infty 123\underline{\infty 32\infty}31210\infty = (2,3,\infty,4,0)\infty 123\ \underline{(1,0,4)(2,3,\infty)3\infty 23}$ 31210∞ (by relation (3) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,0,3,2,\infty)20\underline{3\infty 3}\infty 2\underline{33}1210\infty = (1,0,3,2,\infty)20\underline{\infty 3\infty}\infty 21210\infty$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,0,3,2,\infty)20\infty 3\underline{\infty\infty 212}10\infty = (1,0,3,2,\infty)20\infty 3\underline{121}10\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,0,3,2,\infty)20\infty 312\underline{11}0\infty = (1,0,3,2,\infty)20\infty 3120\infty = e$ by relation (1) conjugated by $(1,0,\infty,4,3) \in L_2(5)$.

Now consider $[\infty 01231]$. $N^{(\infty 01231)}$ has orbits $\{1\}$, $\{\infty\}$, $\{2,0\}$, and $\{3,4\}$. So we need to look at $[\infty 012311]$, $[\infty 01231\infty]$, $[\infty 012310]$, and $[\infty 012314]$.

First $[\infty 012311] = [\infty 0123]$. So t_1 takes $[\infty 01231]$ back to a single coset in $[\infty 0123]$.

t_∞ takes $[\infty 01231]$ to a single coset in $[\infty 01231\infty]$. We previously proved that $N\infty 01231 = N\infty 21041$. Therefore it follows that $N\infty 01231\infty = N\infty 21041\infty$. $N^{(\infty 01231\infty)} = \langle(2,0)(3,4)\rangle$, so there are 30 distinct single cosets in $[\infty 01231\infty]$.

t_0, a representative from one of the 2-orbits, takes $[\infty 01231]$ to a single coset in $[\infty 0142]$ since we previously proved $\infty 01420 = (1,2,3)(4,\infty,0)10\infty 24\infty$. Therefore it follows $N\infty 0142 = N(1,2,3)(4,\infty,0)10\infty 24\infty 0 = N10\infty 24\infty 0 \in [\infty 012310] = [\infty 0142]$.

t_4, a representative from the other 2-orbit, takes $[\infty 01231]$ to a single coset in $[\infty 0203]$ since we previously proved $\infty 02031 = (1,\infty,4,2,3)34\infty 20\infty$. Therefore it follows $N\infty 0203 = N(1,\infty,4,2,3)34\infty 20\infty 1 = N34\infty 20\infty 1 \in [\infty 012314] = [\infty 0203]$.

Now consider $[\infty 01232]$. $N^{(\infty 01232)}$ has orbits $\{1,4,0\}$ and $\{2,3,\infty\}$. So we need

to look at $[\infty 012324]$ and $[\infty 012322]$.

First $[\infty 012322] = [\infty 0123]$. So t_2, a representative from one of the 3-orbits, takes $[\infty 01232]$ back to a single coset in $[\infty 0123]$.

t_4, a representative from the other 3-orbit, takes $[\infty 01232]$ to a single coset in $[\infty 0\infty 12]$ since we previously proved $\infty 0\infty 123 = (1,\infty)(2,0)01\infty 424$. Therefore it follows $N\infty 0\infty 12 = N(1,\infty)(2,0)01\infty 4243 = N01\infty 4243 \in [\infty 012324] = [\infty 0\infty 12]$.

Now consider $[\infty 01204]$. $N^{(\infty 01204)}$ has orbits $\{1,4,0\}$ and $\{2,3,\infty\}$. So we need to look at $[\infty 012044]$ and $[\infty 01204\infty]$.

First $[\infty 012044] = [\infty 0120]$. So t_4, a representative from one of the 3-orbits, takes $[\infty 01204]$ back to a single coset in $[\infty 0120]$.

t_∞, a representative from the other 3-orbit, takes $[\infty 01204]$ to a single coset in $[\infty 0143]$ since we previously proved $\infty 01432 = (1,4,\infty,2,0)203\infty 04$. Therefore it follows $N\infty 0143 = N(1,4,\infty,2,0)203\infty 042 = N203\infty 042 \in [\infty 01204\infty] = [\infty 0143]$.

Now consider $[\infty 012\infty 3]$. $N^{(\infty 012\infty 3)}$ has orbits $\{2\}$, $\{0\}$, $\{1,\infty\}$, and $\{3,4\}$. So we need to look at $[\infty 012\infty 33]$, $[\infty 012\infty 32]$, $[\infty 012\infty 30]$, and $[\infty 012\infty 3\infty]$.

First $[\infty 012\infty 33] = [\infty 012\infty]$. So t_3, a representative from one of the 2-orbits, take $[\infty 012\infty 3]$ back to a single coset in $[\infty 012\infty]$.

t_2 takes $[\infty 012\infty 3]$ to a single coset in $[\infty 012\infty 32]$. $N\infty 012\infty 32 = N3\infty 41301 = N4\infty 31421 = N03240\infty 4 = N2304214 = N10\infty 2142 = N12\infty 0130 = N314\infty 32\infty = N0423013 = N24032\infty 3 = N413\infty 40\infty = N\infty 210\infty 40$ since we previously proved $N\infty 012\infty 3 = N10\infty 214$. Therefore it follows that $N\infty 012\infty 32 = N10\infty 2142$. If we conjugate this equation by $(1,4,2)(3,0,\infty)$, $(1,2,4)(3,\infty,0)$, $(1,\infty)(2,0)$, $(1,4,0)(2,\infty,3)$ and $(1,2,3)(4,\infty,0) \in L_2(5)$ we find $N3\infty 41301 = N4\infty 31421$, $N03240\infty 4 = N2304214$, $N12\infty 0130 = N\infty 210\infty 40$, $N314\infty 32\infty = N413\infty 40\infty$, and $N0423013 = N24032\infty 3$. To show that all of these single cosets are equal we will prove $\infty 012\infty 32 = (1,\infty,4,2,3)314\infty 32\infty$, $\infty 012\infty 32 = (1,2,\infty,3,4)4\infty 31421$, $\infty 012\infty 32 = (1,\infty,2,3,0)12\infty 0130$, $\infty 012\infty 32 = (1,3)(4,0)03240\infty 4$, and $\infty 012\infty 32 = (1,0,3,2,\infty)0423013$.

To prove $\infty 012\infty 32 = (1,\infty,4,2,3)314\infty 32\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,4,2,3)314\infty 32\underline{\infty 23\infty}210\infty = (1,\infty,4,2,3)314\infty 32$
$\underline{(1,4,0)(2,\infty,3)2\infty 32}\ 210\infty$ (by relation (3) conjugated by $(1,3,0,2,4) \in L_2(5)$)

$= (1,3,4,\infty,0)24032\underline{\infty 2\infty}3\underline{22}10\infty = (1,3,4,\infty,0)24032\underline{2\infty 2}310\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,3,4,\infty,0)2403\underline{22}\infty 2310\infty = (1,3,4,\infty,0)2\ \underline{403\infty}\ 2310\infty = (1,3,4,\infty,0)2\ \underline{(1,3,4,\infty,0)3041}\ 2310\infty$ (by relation (1) conjugated by $(1,0,4)(2,3,\infty) \in L_2(5)$)

$= ((1,4,0,3,\infty)2304\underline{1231}0\infty = ((1,4,0,3,\infty)2304\underline{(1,3,2)(4,0,\infty)2132}0\infty$ (by relation (4) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)12\infty\underline{0213}20\infty = (1,0,2)(3,4,\infty)12\infty\underline{(1,0,3,2,\infty)120\infty}20\infty$ (by relation (1) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)0\infty\underline{11}20\infty 20\infty = (1,3,4)(2,0,\infty)0\infty 20\infty 20\infty = e$ by relation (4) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$.

To prove $\infty 012\infty 32 = (1,2,\infty,3,4)4\infty 31421$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,\infty,3,4)4\infty 3\underline{1421}23\infty 210\infty = (1,2,\infty,3,4)4\infty 3\underline{(1,2,4)(3,\infty,0)4124}\ 23\infty 210\infty$ (by relation (3) conjugated by $(1,2,\infty)(3,0,4) \in L_2(5)$)

$= (1,4,2,0,3)\underline{10\infty 4}12423\infty 210\infty = (1,4,2,0,3)\ \underline{(1,4,0,3,\infty)\infty 013}\ 12423\infty 210\infty$ (by relation (2) conjugated by $(1,4,\infty,2,0) \in L_2(5)$)

$= (1,0,\infty)(2,3,4)\infty 0\underline{131}2423\infty 210\infty == (1,0,\infty)(2,3,4)\infty 0\underline{313}2423\infty 210\infty$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,0,\infty)(2,3,4)\infty 03\underline{1324}23\infty 210\infty = (1,0,\infty)(2,3,4)\infty 03\ \underline{(1,4,3,\infty,2)231\infty}\ 23\infty 210\infty$ (by relation (1) conjugated by $(1,3,4,\infty,0) \in L_2(5)$)

$= (1,0,2,\infty,4)20\infty 231\underline{\infty 23\infty}210\infty = (1,0,2,\infty,4)20\infty 231\ \underline{(1,4,0)(2,\infty,3)2\infty 32}\ 210\infty$ (by relation (3) conjugated by $(1,3,0,2,4) \in L_2(5)$)

$= (2,3)(0,\infty)\underline{\infty 13\infty}242\infty 3\underline{22}10\infty = (2,3)(0,\infty)\ \underline{(1,\infty,3)(2,4,0)1\infty 31}\ 242\infty 310\infty$ (by relation (4) conjugated by $(1,0,2,\infty,4) \in L_2(5)$)

$= (1,\infty,2)(3,4,0)1\underline{\infty 312}42\infty 310\infty = (1,\infty,2)(3,4,0)1\ \underline{(1,\infty,2,3,0)13\infty 0}\ 42\infty 310\infty$ (by relation (1) conjugated by $(1,3,2)(4,0,\infty) \in L_2(5)$)

$= (1,2,\infty,3,4)\infty 13\infty 042\infty\underline{310\infty} = (1,2,\infty,3,4)\infty 13\infty 042\infty\underline{(1,4,0,3,\infty)0134}$ (by relation (2) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,2)(3,0)14\infty 13\underline{0210}134 = (1,2)(3,0)14\infty 13\underline{(1,2,0)(3,\infty,4)2012}134$ (by relation (4) conjugated by $(1,4,2,0,3) \in L_2(5)$)

$= (1,0,\infty,4,3)2342\infty 20\underline{12}134 = (1,0,\infty,4,3)2342\infty 20\underline{21}234$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,0,\infty,4,3)2342\infty\underline{202}1234 = (1,0,\infty,4,3)2342\infty\underline{020}1234$ (relation (5) conjugated by $(1,4,3)(2,\infty,0) \in L_2(5)$)

$= (1,0,\infty,4,3)2\underline{342\infty}0201234 = (1,0,\infty,4,3)2\underline{(2,3,\infty,4,0)243}00201234$ (by relation (1) conjugated by $(1,4,0,3,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)\underline{3243}201234 = (1,2,3)(4,\infty,0)\underline{(1,0,\infty)(2,3,4)2342}201234$ (by relation (4) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$)

$= (1,3,0,2,4)23401234 = e$ by relation (1) conjugated by $(1,3)(4,0) \in L_2(5)$.

To prove $\infty012\infty32 = (1,\infty,2,3,0)12\infty0130$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,2,3,0)12\infty01302\underline{3\infty21}0\infty = (1,\infty,2,3,0)12\infty01302$
$\underline{(1,\infty,4,2,3)2\infty34}\ 0\infty$ (by relation (2) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= (1,4,2)(3,0,\infty)\infty340\infty\underline{1032}\infty340\infty = (1,4,2)(3,0,\infty)\infty340\infty$
$\underline{(1,2,0,\infty,3)301\infty}\ \infty340\infty$ (by relation (2) conjugated by $(1,2,0)(3,\infty,4) \in L_2(5)$)

$= (1,4,0,3,\infty)314\infty\underline{33}01\underline{\infty\infty}340\infty = (1,4,0,3,\infty)31\ \underline{4\infty01}\ 340\infty$
$= (1,4,0,3,\infty)31\ \underline{(1,\infty,3,0,4)0\infty43}\ 340\infty$ (by relation (2) conjugated by $(2,\infty)(4,0) \in L_2(5)$)

$= \underline{0\infty0}\infty4\underline{33}40\infty = \underline{\infty0\infty}\infty440\infty$ (by relation (5))

$= \infty0\underline{\infty\infty44}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

To prove $\infty012\infty32 = (1,3)(4,0)03240\infty4$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3)(4,0)03240\infty42\underline{3\infty21}0\infty = (1,3)(4,0)03240\infty42\ \underline{(1,\infty,4,2,3)2\infty34}\ 0\infty$ (by relation (2) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= (2,3,\infty,4,0)01\underline{3204}232\infty340\infty = (2,3,\infty,4,0)01\ \underline{(2,\infty,0,3,4)023\infty}$
$232\infty340\infty$ (by relation (2) conjugated by $(1,4,0,3,\infty) \in L_2(5)$)

$= (2,4,3,0,\infty)310\underline{23\infty}232\infty340\infty = (2,4,3,0,\infty)310\ \underline{(1,4,0)(2,\infty,3)32\infty3}$
$32\infty340\infty$ (by relation (3) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,4,2,0,3)24132\infty\underline{33}2\infty340\infty = (1,4,2,0,3)24132\ \underline{\infty2\infty}\ 340\infty$
$= (1,4,2,0,3)24132\ \underline{2\infty2}\ 340\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,4,2,0,3)2413\underline{22}\infty2340\infty = (1,4,2,0,3)241\ \underline{3\infty23}\ 40\infty = (1,4,2,0,3)241$
$\underline{(1,4,0)(3,2,\infty)\infty32\infty}\ 40\infty$ (by relation (3) conjugated by $(1,2,0,\infty,3) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)\infty\underline{04\infty3}2\infty40\infty = (1,0,2)(3,4,\infty)\infty\ \underline{(2,\infty,0,3,4)\infty402}$
$2\infty40\infty$ (by relation (2) conjugated by $(1,3,2,4,\infty) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)0\infty40\underline{22}\infty40\infty = (1,3,2)(4,0,\infty)0\infty40\infty40\infty = e$ by relation (3) conjugated by $(1,4)(2,3) \in L_2(5)$.

To prove $\infty012\infty32 = (1,0,3,2,\infty)0423013$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,3,2,\infty)04230132\underline{3\infty21}0\infty = (1,0,3,2,\infty)04230132\ \underline{(1,\infty,4,2,3)2\infty34}\ 0\infty$ (by relation (2) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= (1,0)(2,4)02310\infty1\underline{32\infty3}40\infty = (1,0)(2,4)02310\infty1\ \underline{(1,0,4)(2,3,\infty)23\infty2}\ 40\infty$ (by relation (3) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (1,4,3,\infty,2)43\infty04\underline{202}3\infty240\infty = (1,4,3,\infty,2)43\infty04\underline{020}3\infty240\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,4,3,\infty,2)43\infty\underline{040}203\infty240\infty = (1,4,3,\infty,2)43\infty\underline{404}203\infty240\infty$ (by relation (5) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (1,4,3,\infty,2)\underline{43\infty4}04203\infty240\infty = (1,4,3,\infty,2)\ \underline{(1,0,2)(3,4,\infty)34\infty3}\ 04203\infty240\infty$ (by relation (3) conjugated by $(1,\infty,4)(2,0,3) \in L_2(5)$)

$= (1,\infty)(2,0)34\underline{\infty304}203\infty240\infty = (1,\infty)(2,0)34\ \underline{(1,0,\infty,4,3)03\infty1}\ 203\infty240\infty$ (by relation (1) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)1\underline{303}\infty1203\infty240\infty = (1,4,3)(2,\infty,0)1\ \underline{030}\ \infty1203\infty240\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)1030\underline{\infty12}03\infty240\infty = (1,4,3)(2,\infty,0)103\ \underline{(1,0,2,\infty,4)1\infty04}\ 03\infty240\infty$ (by relation (2) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= (2,4,3,0,\infty)0231\infty0\underline{403\infty}240\infty = (2,4,3,0,\infty)0231\infty0\ \underline{(1,3,4,\infty,0)3041}\ 240\infty$ (by relation (1) conjugated by $(1,0,4)(2,3,\infty) \in L_2(5)$)

$= (1,3)(2,\infty)124301304\underline{1240}\infty = (1,3)(2,\infty)124301304\ \underline{(1,0,2,\infty,4)421\infty}\infty$ (by relation (2) conjugated by $(1,0)(3,\infty) \in L_2(5)$)

$= (1,3,0,2,4)0\infty1\underline{3203}21421 = (1,3,0,2,4)0\infty1\underline{(1,4,\infty)(2,3,0)2302}21421$ (by relation (3) conjugated by $(1,0,2)(3,4,\infty) \in L_2(5)$)

$= (1,0,3,2,\infty)\underline{2142}30\underline{22}1421 = (1,0,3,2,\infty)\underline{(1,2,4)(0,3,\infty)1241}301421$ (by relation (3) conjugated by $(1,4,\infty,2,0) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)1\underline{2413}01421 = (1,3,4)(2,0,\infty)1\underline{(1,2,3,4,0)1420}01421$ (by relation (2) conjugated by $(1,3,0,2,4) \in L_2(5)$)

$= (1,4,2)(3,0,\infty)2142\underline{00}1421 = (1,4,2)(3,0,\infty)21421421 = e$ by relation (3) conjugated by $(1,4,\infty)(2,3,0) \in L_2(5)$.

$N^{(\infty 012\infty 32)} = \langle (1,\infty)(2,0), (2,0)(3,4), (1,3,2)(4,0,\infty)\rangle$, so there are 5 distincts single cosets in $[\infty 012\infty 32]$.

t_0 takes $[\infty 012\infty 3]$ to a single coset in $[\infty 02\infty 1]$ since we previously proved $\infty 02\infty 13 = (1,3,2,4,\infty)431240$. Therefore it follows $N\infty 02\infty 1$
$= N(1,3,2,4,\infty)4312403 = N4312403 \in [\infty 012\infty 30] = [\infty 02\infty 1]$.

t_∞, a representative from the other 2-orbit, takes $[\infty 012\infty 3]$ to a single coset in $[\infty 0131]$ since we previously proved $\infty 01312 = (1,4,0,3,\infty)230421$. Therefore it follows $N\infty 0131 = N(1,4,0,3,\infty)2304212 = N2304212 \in [\infty 012\infty 3\infty] = [\infty 0131]$.

Now consider $[\infty 012\infty 4]$. $N^{(\infty 012\infty 4)}$ has orbits $\{2\}$, $\{3\}$, $\{1,4\}$, and $\{0,\infty\}$. So we need to look at $[\infty 012\infty 42]$, $[\infty 012\infty 43]$, $[\infty 012\infty 44]$, and $[\infty 012\infty 40]$.

First $[\infty 012\infty 44] = [\infty 012\infty]$, so t_4, a representative from one of the 2-orbits, takes $[\infty 012\infty 4]$ back to a single coset in $[\infty 012\infty]$.

t_2 takes $[\infty 012\infty 4]$ to a single coset in $[\infty 0\infty 20]$ since we previously proved $\infty 0\infty 203 = (1,2,3,4,0)\infty 043\infty 1$. Therefore it follows $N\infty 0\infty 20$
$= N(1,2,3,4,0)\infty 043\infty 13 = N\infty 043\infty 13 \in [\infty 012\infty 42] = [\infty 0\infty 20]$.

t_3 takes $[\infty 012\infty 4]$ to a single coset in $[\infty 012\infty 43]$. $N\infty 012\infty 43 = N310234\infty$ $= N0\infty 42013 = N4\infty 02431 = N\infty 432\infty 01 = N134210\infty = N01320\infty 4 = N34\infty 2310 =$ $N43124\infty 0 = N10\infty 2134$ since we previously proved $N\infty 012\infty 4 = N0\infty 4201$. Therefore it follows that $N\infty 012\infty 43 = N0\infty 42013$. If we conjugate this equation by $(1,0)(3,\infty)$, $(1,0,\infty,4,3)$, $(1,3,4,\infty,0)$ and $(1,\infty,3,0,4) \in L_2(5)$ we find $N310234\infty = N134210\infty$, $N4\infty 02431 = N\infty 432\infty 01$, $N01320\infty 4 = N10\infty 2134$, and $N34\infty 2310 = N43124\infty 0$. To show that all of these single cosets are equal we will prove $\infty 012\infty 43$
$= (1,3,2,4,\infty)10\infty 2134$, $\infty 012\infty 43 = (1,3,0,2,4)\infty 432\infty 01$, $\infty 012\infty 43$
$= (1,2)(4,\infty)43124\infty 0$, and $\infty 012\infty 43 = (1,\infty,3,0,4)310234\infty$.

To prove $\infty 012\infty 43 = (1,3,2,4,\infty)10\infty 2134$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,2,4,\infty)10\infty 213\underline{434}\infty = (1,3,2,4,\infty)10\infty 213\underline{343}\infty$ (by relation (6) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,3,2,4,\infty)10\infty 21\underline{33}43\infty = (1,3,2,4,\infty)10\ \underline{\infty 214}3\infty = (1,3,2,4,\infty)10$ $\underline{(1,\infty,4,2,3)12\infty 33\infty}$

$= \infty 012\infty\underline{33}\infty 210\infty = \infty 012\underline{\infty\infty}210\infty = \infty 01\underline{22}10\infty = \infty 0\underline{11}0\infty = \infty\underline{00}\infty =$ $\underline{\infty\infty} = e$.

To prove $\infty012\infty43 = (1,3,0,2,4)\infty432\infty01$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,0,2,4)\infty432\underline{\infty0134}\infty210\infty = (1,3,0,2,4)\infty432\ \underline{(1,\infty,3,0,4)10\infty4}$ $4\infty210\infty$ (by relation (2) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)310210\infty\underline{44}\infty210\infty = (1,0,2)(3,4,\infty)310210\ \underline{\infty\infty}\ 210\infty =$ $(1,0,2)(3,4,\infty)310\ \underline{210}210\infty = (1,0,2)(3,4,\infty)310\ \underline{(1,2,0)(3,\infty,4)1201}\ 10\infty$ (by relation (4) conjugated by $(1,\infty,3,0,4) \in L_2(5)$)

$= \infty21\underline{11}20\underline{11}0\infty = \infty\underline{2200}\infty = \underline{\infty\infty} = e$.

To prove $\infty012\infty43 = (1,2)(4,\infty)43124\infty0$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2)(4,\infty)43124\underline{\infty034}\infty210\infty = (1,2)(4,\infty)43124\ \underline{(2,3,\infty,4,0)}\ \infty210\infty$ (by relation (1) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$)

$= (1,3,\infty,0,2)0\infty1\underline{3030}\underline{\infty2\infty}210\infty = (1,3,\infty,0,2)0\infty1\underline{03}00\underline{2\infty}2210\infty$ (by relation (5) conjugated by $(2,4)(3,\infty)$ and $(2,0)(3,4) \in L_2(5)$)

$= (1,3,\infty,0,2)0\infty103\underline{00}2\infty\underline{22}10\infty = (1,3,\infty,0,2)0\underline{\infty1032}\infty10\infty$ $= (1,3,\infty,0,2)0\underline{(1,2,0,\infty,3)01\infty2}2\infty10\infty$ (by relation (2) conjugated by $(1,3,2)(4,0,\infty) \in L_2(5)$)

$= \infty01\infty\underline{22}\infty10\infty = \infty01\underline{\infty\infty}10\infty = \infty0\underline{11}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e$.

To prove $\infty012\infty43 = (1,\infty,3,0,4)310234\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,3,0,4)3102\underline{34\infty3}4\infty210\infty = (1,\infty,3,0,4)3102\underline{(1,2,0)(3,\infty,4)43\infty4}$ $4\infty210\infty$ (by relation (3) conjugated by $(1,\infty,3,0,4) \in L_2(5)$)

$= (1,4,2,0,3)\infty21043\infty\underline{44}\infty210\infty = (1,4,2,0,3)\infty21043\underline{\infty\infty}210\infty$ $= (1,4,2,0,3)\infty21\underline{0432}10\infty = (1,4,2,0,3)\infty21\underline{(1,3,0,2,4)3401}10\infty$ (by relation (1) conjugated by $(1,4)(2,3) \in L_2(5)$)

$= \infty4\underline{33}40\underline{11}0\infty = \infty\underline{4400}\infty = \underline{\infty\infty} = e$.

$N^{(\infty012\infty43)} = \langle(1,0)(3,\infty),(1,4,0,3,\infty)\rangle$, so there are 6 distinct single cosets in $[\infty012\infty43]$.

t_0, a representative from the other 2-orbit, takes $[\infty012\infty4]$ to a single coset in $[\infty01\infty403]$ since $N\infty012\infty40 = N(2,3,\infty,4,0)2032401 = N2032401 \in [\infty01\infty403]$. To prove $\infty012\infty40 = (2,3,\infty,4,0)2032401$, we will move the relation to one side of the equal sign and prove it equals identity.

$(2,3,\infty,4,0)20324\underline{010}4\infty210\infty = (2,3,\infty,4,0)203241\underline{01}4\infty210\infty$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (2,3,\infty,4,0)203241\underline{014\infty}210\infty = (2,3,\infty,4,0)203241\underline{(1,2,4,0,\infty)4102}210\infty$ (by relation (1) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)4\infty340241\underline{22}10\infty = (1,2,3)(4,\infty,0)4\infty34024\ \underline{11}0\infty$
$= (1,2,3)(4,\infty,0)4\infty3\ \underline{4024}\ 0\infty = (1,2,3)(4,\infty,0)4\infty3\ \underline{(1,3,\infty)(2,0,4)0420}\ 0\infty$ (by relation (3) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (1,0,2,\infty,4)21\infty042\underline{00}1\infty = (1,0,2,\infty,4)21\infty0421\infty = e$ by relation (2) conjugated by $(1,4,2)(3,0,\infty) \in L_2(5)$.

Now consider $[\infty01302]$. $N^{(\infty01302)}$ has orbits $\{1,0,\infty\}$ and $\{2,3,4\}$. So we need to look at $[\infty013021]$ and $[\infty013022]$.

First $[\infty013022] = [\infty0130]$. So t_2, a representative from one of the 3-orbits, takes $[\infty01302]$ back to $[\infty0130]$.

t_1, a representative from the other 3-orbit, takes $[\infty01302]$ to a single coset in $[\infty01231\infty]$ since $N\infty013021 = N(2,\infty,0,3,4)4312014 = N4312014 \in [\infty01231\infty]$. To prove $\infty013021 = (2,\infty,0,3,4)4312014$, we will move the relation to one side of the equal sign and prove it equals identity.

$(2,\infty,0,3,4)43120\underline{141}20310\infty = (2,\infty,0,3,4)43120\underline{414}20310\infty$ (by relation (6))

$= (2,\infty,0,3,4)431204\underline{1420}310\infty = (2,\infty,0,3,4)431204\ \underline{(1,0,4,3,2)2413}\ 310\infty$ (by relation (2) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,0,2,\infty,4)320143241\underline{33}10\infty = (1,0,2,\infty,4)32014324\underline{11}0\infty$
$= (1,0,2,\infty,4)320143\underline{240\infty} = (1,0,2,\infty,4)320143\ \underline{(1,0,2,\infty,4)0421}$ (by relation (2) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (1,2,4,0,\infty)3\infty2\underline{0130}421 = (1,2,4,0,\infty)3\infty2\underline{(1,0,3)(2,4,\infty)1031}421$ (by relation (3) conjugated by $(1,3,4,\infty,0) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)1241\underline{0314}21 = (1,4,3)(2,\infty,0)1241\underline{(1,0,4,3,2)1302}21$ (by relation (2) conjugated by $(1,4)(2,3) \in L_2(5)$)

$= (1,3,0)(2,\infty,4)0130130\underline{22}1 = (1,3,0)(2,\infty,4)01301301 = e$ by relation (3) conjugated by $(1,3,2,4,\infty) \in L_2(5)$.

Now consider $[\infty01310]$. $N^{(\infty01310)}$ has orbits $\{1\}$, $\{2\}$, $\{3,0\}$, and $\{4,\infty\}$. So we need to look at $[\infty013101]$, $[\infty013102]$, $[\infty013100]$, and $[\infty013104]$.

First $[\infty013100] = [\infty0131]$. So t_0, a representative from one of the 2-orbits, takes $[\infty01310]$ back to a single coset in $[\infty0131]$.

t_1 takes $[\infty01310]$ to a single coset in $[\infty0232]$ since we previously proved $\infty02320 = (1,4,\infty)(2,3,0)130203$. Therefore it follows $N\infty0232 = N(1,4,\infty)(2,3,0)1302030 = N1302030 \in [\infty013101] = [\infty0232]$.

t_2 takes $[\infty01310]$ to a single coset in $[\infty020\infty]$ since we previously proved $\infty020\infty1 = (1,0)(3,\infty)04\infty3\infty4$. Therefore it follows $N\infty020\infty$ $= N(1,0)(3,\infty)04\infty3\infty41 = N04\infty3\infty41 \in [\infty013102] = [\infty020\infty]$.

t_4, a representative from the other 2-orbit, takes $[\infty01310]$ to a single coset in $[\infty01\infty201]$ since $N\infty013104 = N(1,\infty,0,4,2)4204\infty20 = N4204\infty20 \in [\infty01\infty201]$. To prove $\infty013104 = (1,\infty,0,4,2)4204\infty20$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,0,4,2)4\underline{204\infty}20401310\infty = (1,\infty,0,4,2)4\ \underline{(2,\infty,0,3,4)4023}\ 20401310\infty$ (by relation (2) conjugated by $(1,\infty)(2,0) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)2402\underline{3204}01310\infty = (1,0,2)(3,4,\infty)2402\ \underline{(2,\infty,0,3,4)023\infty}\ 01310\infty$ (by relation (2) conjugated by $(1,4,0,3,\infty) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)\infty23\infty023\infty01\underline{310\infty} = (1,3,2)(4,0,\infty)\infty23\infty\ 023\infty01\ \underline{(1,4,0,3,\infty)0134}$ (by relation (2) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)\underline{12\infty1}32\infty1340134 = (1,\infty,0)(2,4,3)\underline{(1,\infty,2)(3,4,0)21\infty2}\ 32\infty1340134$ (by relation (3) conjugated by $(1,\infty)(2,0) \in L_2(5)$)

$= (1,2,0,\infty,3)2\underline{1\infty23}2\infty1340134 = (1,2,0,\infty,3)2\ \underline{(1,3,\infty,0,2)2\infty10}\ 2\infty1340134$ (by relation (2) conjugated by $(1,3,0)(2,\infty,4) \in L_2(5)$)

$= 12\infty\underline{102\infty}1340134 = 12\infty\underline{(1,\infty,0,4,2)2014}1340134$ (by relation (1) conjugated by $(1,0)(3,\infty) \in L_2(5)$)

$= (1,\infty,0,4,2)\infty1\underline{020141}340134 = (1,\infty,0,4,2)\infty1\underline{202414}340134$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$ and relation (6))

$= (1,\infty,0,4,2)\infty12\underline{0241}4340134 = (1,\infty,0,4,2)\infty12\ \underline{(1,2,3,4,0)4203}\ 4340134$ (by relation (2))

$= (1,\infty)(3,4)\infty23420\underline{343}40134 = (1,\infty)(3,4)\infty23420\underline{434}40134$ (by relation (6) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,\infty)(3,4)\infty2342043\underline{44}0134 = (1,\infty)(3,4)\infty2342\ \underline{0430}134$
$= (1,\infty)(3,4)\infty2342\ \underline{(1,\infty,2)(3,4,0)4034}134$ (by relation (4) conjugated by $(1,\infty)(2,0)$

$\in L_2(5))$

$= (1,2)(3,0)21\underline{4014}034134 = (1,2)(3,0)21\underline{(1,0,4)(2,3,\infty)0410}034134$ (by relation (4) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5))$

$= (1,3,4)(2,0,\infty)3\underline{00}41\underline{00}34134 = (1,3,4)(2,0,\infty)34134134 = e$ by relation (3) conjugated by $(2,0)(4,\infty) \in L_2(5)$.

Now consider $[\infty0131\infty]$. $N^{(\infty0131\infty)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty0131\infty1]$, $[\infty0131\infty2]$, $[\infty0131\infty3]$, $[\infty0131\infty4]$, $[\infty0131\infty0]$, and $[\infty0131\infty\infty]$.

First $[\infty0131\infty\infty] = [\infty0131]$, so t_∞ takes $[\infty0131\infty]$ back to a single coset in $[\infty0131]$.

t_1 takes $[\infty0131\infty]$ to a single coset in $[\infty0201]$ since we previously proved $\infty02013 = (1,3)(4,0)413034$. Therefore it follows $N\infty0201 = N(1,3)(4,0)4130343 = N4130343 \in [\infty0131\infty1] = [\infty0201]$.

t_2 takes $[\infty0131\infty]$ to a single coset in $[\infty0\infty14]$ since we previously proved $\infty0\infty140 = (1,\infty,0)(2,4,3)2\infty3432$. Therefore it follows $N\infty0\infty14$ $= N(1,\infty,0)(2,4,3)2\infty34320 = N2\infty34320 \in [\infty0131\infty2] = [\infty0\infty14]$.

t_3 takes $[\infty0131\infty]$ to a single coset in $[\infty010203]$ since $N\infty0131\infty3$ $= N(2,0)(3,\infty)3\infty4\infty\ 1\infty0 = N3\infty4\infty1\infty0 \in [\infty010203]$. To prove $\infty0131\infty3 = (2,0)(3,\infty)3\infty4\infty1\infty0$, we will move the relation to one side of the equal sign and prove it equals identity.

$(2,0)(3,\infty)3\infty4\underline{\infty1\infty}03\infty1310\infty = (2,0)(3,\infty)3\infty4\underline{1\infty1}03\infty1310\infty$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5))$

$= (2,0)(3,\infty)3\underline{\infty41\infty}103\infty1310\infty = (2,0)(3,\infty)3\ \underline{(1,4,\infty)(2,3,0)4\infty14}$ $103\infty1310\infty$ (by relation (4) conjugated by $(1,3)(2,\infty) \in L_2(5))$

$= (1,4,0,3,\infty)04\infty\underline{14}103\infty1310\infty = (1,4,0,3,\infty)04\infty\underline{414}03\infty1310\infty$ (by relation (6))

$= (1,4,0,3,\infty)04\infty4\underline{1403}\infty1310\infty = (1,4,0,3,\infty)04\infty4\ \underline{(1,3,4,\infty,0)041\infty}$ $\infty1310\infty$ (by relation (1) conjugated by $(1,4,\infty,2,0) \in L_2(5))$

$= (1,\infty,3,0,4)1\underline{\infty0\infty}041\underline{\infty\infty}1310\infty = (1,\infty,3,0,4)1\underline{0\infty0}0411310\infty$ (by relation (5))

$= (1,\infty,3,0,4)10\infty\underline{00}4\underline{11}310\infty = (1,\infty,3,0,4)10\infty4310\infty = e$ by relation (2) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$.

t_4 takes $[\infty0131\infty]$ to a single coset in $[\infty0\infty21]$ since we previously proved $\infty0\infty210 = (2,4)(3,\infty)241312$. Therefore it follows $N\infty0\infty21 = N(2,4)(3,\infty)2413120 = N2413120 \in [\infty0131\infty4] = [\infty0\infty21]$.

t_0 takes $[\infty0131\infty]$ to a single coset in $[\infty01231\infty]$ since $N\infty0131\infty0$ $= N(1,2,\infty,3,4)1\infty24321 = N1\infty24321 \in [\infty01231\infty]$. To prove $\infty0131\infty0$ $= (1,2,\infty,3,4)1\infty24321$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,\infty,3,4)1\infty2\underline{4321}0\infty1310\infty = (1,2,\infty,3,4)1\infty2\ \underline{(1,3,0,2,4)2340}$ $0\infty1310\infty$ (by relation (1) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)3\underline{\infty423}4\underline{00}\infty1310\infty = (1,4,3)(2,\infty,0)3\ \underline{(1,2,\infty,3,4)24\infty1}$ $4\infty1310\infty$ (by relation (1) conjugated by $(1,4)(0,\infty) \in L_2(5)$)

$= (2,3)(0,\infty)424\underline{\infty14\infty}1310\infty = (2,3)(0,\infty)424\underline{(1,\infty,4)(2,0,3)1\infty41}1310\infty$ (by relation (4) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= (1,\infty,3,0,4)10\underline{11}\infty4\underline{11}310\infty = (1,\infty,3,0,4)10\infty4310\infty = e$ by relation (2) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$.

Now consider $[\infty0142\infty]$. $N^{(\infty0142\infty)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty0142\infty1]$, $[\infty0142\infty2]$, $[\infty0142\infty3]$, $[\infty0142\infty4]$, $[\infty0142\infty0]$, and $[\infty0142\infty\infty]$.

First $[\infty0142\infty\infty] = [\infty0142]$. So t_∞ takes $[\infty0142\infty]$ back to a single coset in $[\infty0142]$.

t_1 takes $[\infty0142\infty]$ to a single coset in $[\infty010203]$ since $N\infty0142\infty1$ $= N(1,2,\infty)(3,0,4)412101\infty = N412101\infty \in [\infty010203]$. To prove $\infty0142\infty1$ $= (1,2,\infty)(3,0,4)412101\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,\infty)(3,0,4)412101\ \underline{\infty1\infty}\ 2410\infty = (1,2,\infty)(3,0,4)412101\ \underline{1\infty1}\ 2410\infty$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)412101\underline{1}\underline{1}\infty12410\infty = (1,2,\infty)(3,0,4)4121\ \underline{0\infty12}410\infty$ $= (1,2,\infty)(3,0,4)4121\ \underline{(1,0,2,\infty,4)1\infty04}\ 410\infty$ (by relation (2) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)10\underline{\infty01\infty}0\underline{44}10\infty = (1,\infty,0)(2,4,3)10\ \underline{(1,0,\infty)(2,3,4)0\infty10}$ 010∞ (by relation (3))

$= \underline{0\infty0}\infty1\underline{00}10\infty = \underline{\infty0\infty}\infty110\infty$ (by relation (5))

$= \infty 0 \infty\infty \underline{11} 0\infty = \infty 0 \underline{\infty\infty} 0\infty = \infty \underline{00} \infty = \underline{\infty\infty} = e.$

t_2 takes $[\infty 0142 \infty]$ to a single coset in $[\infty 0 \infty 23]$ since we previously proved $\infty 0 \infty 231 = (1,0,3,2,\infty)320\infty 13$. Therefore it follows $N\infty 0\infty 23$ $= N(1,0,3,2,\infty)320\infty 131 = N320\infty 131 \in [\infty 0142\infty 2] = [\infty 0\infty 23]$.

t_3 takes $[\infty 0142\infty]$ to a single coset in $[\infty 0243]$ since we previously proved $\infty 02434 = (1,3,0)(2,\infty,4)320\infty 13$. Therefore it follows $N\infty 0243$ $= N(1,3,0)(2,\infty,4)320\infty 134 = N320\infty 134 \in [\infty 0142\infty 3] = [\infty 0243]$.

t_4 takes $[\infty 0142\infty]$ to a single coset in $[\infty 024\infty]$ since we previously proved $\infty 024\infty 2 = (1,4,3)(2,\infty,0)04\infty 210$. Therefore it follows $N\infty 024\infty$ $= N(1,4,3)(2,\infty,0)04\infty 2102 = N04\infty 2102 \in [\infty 0142\infty 4] = [\infty 024\infty]$.

t_0 takes $[\infty 0142\infty]$ to a single coset in $[\infty 0\infty 12]$ since we previously proved $\infty 0\infty 120 = (1,4,0,3,\infty)3021\infty 3$. Therefore it follows $N\infty 0\infty 12$ $= N(1,4,0,3,\infty)3021\infty 30 = N3021\infty 30 \in [\infty 0142\infty 0] = [\infty 0\infty 12]$.

Now consider $[\infty 01430]$. $N^{(\infty 01430)}$ has orbits $\{2\}$, $\{0\}$, $\{1,\infty\}$, and $\{3,4\}$. So we need to look at $[\infty 014302]$, $[\infty 014300]$, $[\infty 014301]$, and $[\infty 014303]$.

First $[\infty 014300] = [\infty 0143]$. So t_0 takes $[\infty 01430]$ back to a single coset in $[\infty 0143]$.

t_2 takes $[\infty 01430]$ to a single coset in $[\infty 010204]$ since $N\infty 014302$ $= N(1,3,0)(2,\infty,4)\infty 434240 = N\infty 434240 \in [\infty 010204]$. To prove $\infty 014302$ $= (1,3,0)(2,\infty,4)\infty 434240$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,0)(2,\infty,4)\infty 43\underline{4240}\underline{20}3410\infty = (1,3,0)(2,\infty,4)\infty 43\ \underline{242}\ \underline{202}\ 3410\infty$ (by relation (6) conjugated by $(1,2)(3,0) \in L_2(5)$ and relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,3,0)(2,\infty,4)\infty\underline{432422}023410\infty = (1,3,0)(2,\infty,4)\infty\ \underline{(1,0,\infty)(2,3,4)3423}$ 023410∞ (by relation (4) conjugated by $(1,\infty,0)(2,4,3) \in L_2(5)$)

$= (1,4,3,\infty,2)134\underline{2302}3410\infty = (1,4,3,\infty,2)134\ \underline{(1,\infty,4)(2,0,3)3203}\ 3410\infty$ (by relation (3) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (2,\infty,0,3,4)\infty 2132\underline{0334}10\infty = (2,\infty,0,3,4)\infty 21\ \underline{3204}\ 10\infty$ $= (2,\infty,0,3,4)\infty 21\ \underline{(2,\infty,0,3,4)023\infty}\ 10\infty$ (by relation (2) conjugated by $(1,4,0,3,\infty)$ $\in L_2(5)$)

$= (2,0,4,\infty,3)\underline{0\infty 10}23\infty 10\infty = (2,0,4,\infty,3)\underline{(1,\infty,0)(2,4,3)\infty 01\infty}23\infty 10\infty$

(by relation (3))

$= (1,\infty,2)(3,4,0)\infty\underline{01\infty2}3\infty10\infty = (1,\infty,2)(3,4,0)\infty\ \underline{(1,3,\infty,0,2)\infty103}$ $3\infty10\infty$ (by relation (2) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (1,0,\infty)(2,3,4)0\infty10\underline{33}\infty10\infty = (1,0,\infty)(2,3,4)0\infty10\infty10\infty = e$ by relation (3).

t_1, a representative of one of the 2-orbits, takes $[\infty01430]$ to a single coset in $[\infty0203]$ since we previously proved $\infty02032 = (1,0,4)(2,3,\infty)0321\infty3$. Therefore it follows $N\infty0203 = N(1,0,4)(2,3,\infty)0321\infty32 = N0321\infty32 \in [\infty014301] = [\infty0203]$.

t_3, a representative of the other 2-orbit, takes $[\infty01430]$ to a single coset in $[\infty01\infty403]$ since $N\infty014303 = N(1,2,4,0,\infty)421402\infty = N421402\infty \in [\infty01\infty403]$. To prove $\infty014303 = (1,2,4,0,\infty)421402\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,4,0,\infty)421402\infty\underline{303}410\infty = (1,2,4,0,\infty)421402\infty\underline{030}410\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,2,4,0,\infty)42140\underline{2\infty03}0410\infty = (1,2,4,0,\infty)42140\ \underline{(2,3,\infty,4,0)0\infty24}$ 0410∞ (by relation (1) conjugated by $(1,\infty)(2,0) \in L_2(5)$)

$= (1,3,\infty)(2,0,4)0310\underline{20\infty2}40410\infty = (1,3,\infty)(2,0,4)0310$ $\underline{(1,4,3)(2,\infty,0)02\infty0}\ 40410\infty$ (by relation (4) conjugated by $(1,4,0,3,\infty) \in L_2(5)$)

$= (3,0)(4,\infty)214202\infty\underline{040}410\infty = (3,0)(4,\infty)214202\infty\underline{404}410\infty$ (by relation (5) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (3,0)(4,\infty)\underline{2142}02\infty40\underline{44}10\infty = (3,0)(4,\infty)\underline{(1,2,4)(3,\infty,0)1241}02\infty4010\infty$ (by relation (3) conjugated by $(1,4,\infty,2,0) \in L_2(5)$)

$= (1,2,4,0,\infty)12\underline{4102}\infty4010\infty = (1,2,4,0,\infty)12\ \underline{(1,\infty,0,4,2)014\infty}\ \infty4010\infty$ (by relation (1) conjugated by $(2,0,4,\infty,3) \in L_2(5)$)

$= \infty\underline{1014\infty\infty}4010\infty = \infty\underline{010}\ 44010\infty$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= \infty010\underline{44}010\infty = \infty01\underline{00}10\infty = \infty0\underline{11}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e$.

Now consider $[\infty0140\infty]$. $N^{(\infty0140\infty)}$ has orbits $\{1,2,0\}$ and $\{3,4,\infty\}$. So we need to look at $[\infty0140\infty1]$ and $[\infty0140\infty\infty]$.

First $[\infty0140\infty\infty] = [\infty0140]$. So t_∞, a representative from one of the 3-orbits, takes $[\infty0140\infty]$ back to a single coset in $[\infty0140]$.

t_1, a representative from the other 3-orbit, takes $[\infty0140\infty]$ to a single coset

in $[\infty010232]$ since $N\infty0140\infty1 = N(1,4)(0,\infty)1303\infty2\infty = N1303\infty2\infty \in [\infty010232]$. To prove $\infty0140\infty1 = (1,4)(0,\infty)1303\infty2\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4)(0,\infty)1303\infty2\underline{\infty1\infty}0410\infty = (1,4)(0,\infty)1303\infty2\underline{1\infty1}0410\infty$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,4)(0,\infty)1303\infty21\infty\underline{1041}0\infty = (1,4)(0,\infty)1303\infty21\infty$ $\underline{(1,4,0)(2,\infty,3)01400}\infty$ (by relation (4) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,0,3,2,\infty)4212\underline{3\infty43}0140\underline{0}\infty = (1,0,3,2,\infty)4212$ $\underline{(1,0,2)(3,4,\infty)\infty34\infty}$ 014∞ (by relation (3) conjugated by $(1,4,2)(3,0,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)\infty101\infty\underline{34\infty0}14\infty = (1,2,3)(4,\infty,0)\infty101\infty$ $\underline{(1,\infty,3,0,4)\infty431}$ 14∞ (by relation (2) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (1,2,0)(4,\infty,0)3\underline{\infty4\infty}3\infty43\underline{11}4\infty = (1,2,0)(4,\infty,0)3\underline{4\infty4}3\infty434\infty$ (by relation (5) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (1,2,0)(4,\infty,0)34\underline{\infty43\infty}434\infty = (1,2,0)(4,\infty,0)34$ $\underline{(1,0,2)(3,4,\infty)4\infty34}$ 434∞ (by relation (3) conjugated by $(1,3)(4;0) \in L_2(5)$)

$= 4\underline{\infty4\infty}3\underline{44}34\infty = 4\underline{4\infty4}334\infty$ (by relation (5) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= \underline{44}\infty4\underline{33}4\infty = \infty\underline{44}\infty = \underline{\infty\infty} = e$.

Now consider $[\infty01403]$. $N^{(\infty01403)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty014031]$, $[\infty014032]$, $[\infty014033]$, $[\infty014034]$, $[\infty014030]$, and $[\infty01403\infty]$.

First $[\infty014033] = [\infty0140]$. So t_3 takes $[\infty01403]$ back to a single coset in $[\infty0140]$.

t_1 takes $[\infty01403]$ to a single coset in $[\infty0201]$ since we previously proved $\infty02012 = (1,4,3)(2,\infty,0)3\infty24\infty1$. Therefore it follows $N\infty0201 = N(1,4,3)(2,\infty,0)3\infty24\infty12 = N3\infty24\infty12 \in [\infty014031] = [\infty0201]$.

t_2 takes $[\infty01403]$ to a single coset in $[\infty0102]$ since we previously proved $\infty01021 = (1,3,2,4,\infty)\infty40342$. Therefore it follows $N\infty0102 = N(1,3,2,4,\infty)\infty403421 = N\infty403421 \in [\infty014032] = [\infty0102]$.

t_4 takes $[\infty01403]$ to a single coset in $[\infty01\infty212]$ since $N\infty01403 = N(1,3,4)(2,0,\infty)4204\infty0\infty = N4204\infty0\infty \in [\infty01\infty212]$. To prove $\infty01403 = (1,3,4)(2,0,\infty)4204\infty0\infty$, we will move the relation to one side of the equal sign and

prove it equals identity.

$(1,3,4)(2,0,\infty)42\underline{04\infty0}\infty30410\infty = (1,3,4)(2,0,\infty)42$ $\underline{(1,3,2)(4,0,\infty)40\infty4}\ \infty30410\infty$ (by relation (3) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)0140\infty\underline{4\infty4}30410\infty = (1,2,\infty)(3,0,4)0140\infty\ \underline{\infty4\infty}\ 30410\infty$ (by relation (5) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)0140\underline{\infty\infty}4\underline{\infty304}10\infty = (1,2,\infty)(3,0,4)01404$ $\underline{(1,0,\infty,4,3)03\infty1}\ 10\infty$ (by relation (1) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$)

$= (1,2,4)(3,\infty,0)\infty03\infty\underline{303}\infty\underline{11}0\infty = (1,2,4)(3,\infty,0)\infty03\infty\underline{030}\infty0\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,2,4)(3,\infty,0)\infty03\infty030\underline{\infty0\infty} = (1,2,4)(3,\infty,0)\infty03\infty030\underline{0\infty0}$ (by relation (5))

$= (1,2,4)(3,\infty,0)\infty03\infty03\underline{00}\infty0 = (1,2,4)(3,\infty,0)\infty03\infty03\infty0 = e$ by relation (4) conjugated by $(1,4,\infty,2,0) \in L_2(5)$.

t_0 takes $[\infty01403]$ to a single coset in $[\infty01\infty403]$ since $N\infty014030 = N(1,2,4,0,\infty)421402\infty = N421402\infty \in [\infty01\infty403]$. To prove $\infty014030 = (1,2,4,0,\infty)421402\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,4,0,\infty)\underline{4214}02\infty030410\infty = (1,2,4,0,\infty)\underline{(1,2,4)(3,\infty,0)2412}$ $02\infty030410\infty$ (by relation (3) conjugated by $(2,3,\infty,4,0) \in L_2(5)$)

$= (1,4,3,\infty,2)241\underline{202}\infty030410\infty = (1,4,3,\infty,2)241\underline{020}\infty030410\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,4,3,\infty,2)24102\underline{0\infty0}30410\infty = (1,4,3,\infty,2)24102\underline{\infty0\infty}30410\infty$ (by relation (5))

$= (1,4,3,\infty,2)24102\infty0\underline{\infty304}10\infty = (1,4,3,\infty,2)24102\infty0\ \underline{(1,0,\infty,4,3)03\infty1}$ 10∞ (by relation (1) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)230\infty24\underline{\infty03\infty11}0\infty = (1,3,4)(2,0,\infty)230\infty24$ $\underline{(1,4,2)(0,\infty,3)0\infty300}\infty$ (by relation (4) conjugated by $(2,\infty)(4,0) \in L_2(5)$)

$= (1,0,3,2,\infty)1\underline{0\infty31}20\infty3\underline{00}\infty = (1,0,3,2,\infty)1\ \underline{(1,\infty,2,3,0)3\infty02}\ 20\infty3\infty$ (by relation (1) conjugated by $(1,\infty,4,2,3) \in L_2(5)$)

$= \infty3\infty0\underline{22}0\infty3\infty = \infty3\infty\underline{00}\infty3\infty = \infty3\underline{\infty\infty}3\infty = \infty\underline{33}\infty = \underline{\infty\infty} = e.$

t_∞ takes $[\infty01403]$ to $[\infty0\infty20]$ since we previously proved $\infty0\infty201 = (1,2,4,0,\infty)1\infty02\infty4$. Therefore it follows $N\infty0\infty20 = N(1,2,4,0,\infty)1\infty02\infty41$

$= N1\infty02\infty41 \in [\infty01403\infty] = [\infty0\infty20]$.

Now consider $[\infty0\infty141]$. $N^{(\infty0\infty141)}$ has one orbit $\{1,2,3,4,0,\infty\}$. So we need to look at $[\infty0\infty1411]$.

First $[\infty0\infty1411] = [\infty0\infty14]$. So t_1, a representative of the 6-orbit, takes $[\infty0\infty141]$ back to a single coset in $[\infty0\infty14]$.

Now consider $[\infty0\infty121]$. $N^{(\infty0\infty121)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty0\infty1211]$, $[\infty0\infty1212]$, $[\infty0\infty1213]$, $[\infty0\infty1214]$, $[\infty0\infty1210]$, and $[\infty0\infty121\infty]$.

First $[\infty0\infty1211] = [\infty0\infty12]$, so t_1 takes $[\infty0\infty121]$ back to a single coset in $[\infty0\infty12]$.

t_2 takes $[\infty0\infty121]$ to a single coset in $[\infty0\infty21]$ since we previously proved $\infty0\infty212 = \infty0\infty121$. Therefore it follows $N\infty0\infty21 = N\infty0\infty1212 \in [\infty0\infty1212] = [\infty0\infty21]$.

t_3 takes $[\infty0\infty121]$ to a single coset in $[\infty010232]$ since $N\infty0\infty1213 = N2303414 \in [\infty010232]$. To prove $\infty0\infty1213 = 2303414$, we will move the relation to one side of the equal sign and prove it equals identity.

$2\underline{303}4143121\infty0\infty = 2\underline{030}4143121\infty0\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= 20\underline{3041}43121\infty0\infty = 20\underline{(1,0,\infty,4,3)403\infty}43121\infty0\infty$ (by relation (1) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (1,0,\infty,4,3)2\infty403\underline{\infty431}21\infty0\infty = (1,0,\infty,4,3)2\infty403\ \underline{(1,4,0,3,\infty)34\infty0}\ 21\infty0\infty$ (by relation (2) conjugated by $(2,4,3,0,\infty) \in L_2(5)$)

$= (1.3.4.\infty,0)2103\underline{\infty34\infty}021\infty0\infty = (1.3.4.\infty,0)2103\ \underline{(1,2,0)(3,\infty,4)3\infty43}\ 021\infty0\infty$ (by relation (3) conjugated by $(1,4,2,0,3) \in L_2(5)$)

$= (1,\infty)(2,0)021\underline{\infty3\infty}43021\infty0\infty = (1,\infty)(2,0)021\underline{3\infty3}43021\infty0\infty$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,\infty)(2,0)\underline{0213}\infty343021\infty0\infty = (1,\infty)(2,0)\ \underline{(1,0,3,2,\infty)120\infty}\ \infty343021\infty0\infty$ (by relation (1) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (2,3)(0,\infty)120\underline{\infty\infty}343021\underline{\infty0\infty} = (2,3)(0,\infty)1203430210\underline{\infty0}$ (by relation (5))

$= (2,3)(0,\infty)120343\underline{0210}\infty0 = (2,3)(0,\infty)120343\underline{(1,2,0)(3,\infty,4)2012}\infty0$ (by relation (4) conjugated by $(1,4,2,0,3) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)201\underline{\infty 3 \infty}2012\infty 0 = (1,2,\infty)(3,0,4)201\underline{3\infty 3}2012\infty 0$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)2013\underline{\infty 320}12\infty 0 = (1,2,\infty)(3,0,4)2013\ \underline{(2,\infty,0,3,4)23\infty 4}$ $12\infty 0$ (by relation (2) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$)

$= (1,\infty)(2,0)\infty\underline{3142}3\infty 412\infty 0 = (1,\infty)(2,0)\infty\underline{(1,0,4,3,2)4130}3\infty 412\infty 0$ (by relation (2) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)\infty 41\underline{303}\infty 412\infty 0 = (1,\infty,0)(2,4,3)\infty 41\underline{030}\infty 412\infty 0$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)\underline{\infty 410}30\infty 412\infty 0 = (1,\infty,0)(2,4,3)\ \underline{(1,\infty,0,4,2)14\infty 2}$ $30\infty 412\infty 0$ (by relation (1) conjugated by $(1,4,2)(3,0,\infty) \in L_2(5)$)

$= (1,0,\infty,4,3)14\underline{\infty 230}\infty 412\infty 0 = (1,0,\infty,4,3)14\underline{(1,3,\infty,0,2)32\infty 1}\infty 412\infty 0$ (by relation (2) conjugated by $(1,0,\infty,4,3) \in L_2(5)$)

$= (1,2)(4,\infty)3432\infty\underline{1\infty 41}2\infty 0 = (1,2)(4,\infty)3432\infty\underline{(1,4,\infty)(2,3,0)\infty 14\infty}2\infty 0$ (by relation (4) conjugated by $(1,0,2)(3,4,\infty) \in L_2(5)$)

$= (1,3,0,2,4)\underline{0\infty 0}31\infty 14\underline{\infty 2\infty}0 = (1,3,0,2,4)\underline{\infty 0\infty}31\infty 14\underline{2\infty 2}0$ (by relation (5) and relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,3,0,2,4)\infty 0\infty 3\underline{1\infty 1}42\infty 20 = (1,3,0,2,4)\infty 0\infty 3\underline{\infty 1\infty}42\infty 20$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,3,0,2,4)\infty 0\infty 3\infty\underline{1\infty 42}\infty 20 = (1,3,0,2,4)\infty 0\infty 3\infty$ $\underline{(1,2,\infty,3,4)4\infty 13}\ \infty 20$ (by relation (1) conjugated by $(1,\infty,0)(2,4,3) \in L_2(5)$)

$= (1,4,2)(3,0,\infty)3043\underline{44}\infty 13\infty 20 = (1,4,2)(3,0,\infty)304\ \underline{(1,\infty,3)(2,4,0)\infty 31\infty}$ $\infty 20$ (by relation (4) conjugated by $(1,2,3)(4,\infty,0) \in L_2(5)$)

$= (1,0,3,2,\infty)120\infty 31\underline{\infty\infty\infty}20 = (1,0,3,2,\infty)120\infty 3120 = e$ by relation (1) conjugated by $(1,2)(4,\infty) \in L_2(5)$.

t_4 takes $[\infty 0\infty 121]$ to a single coset in $[\infty 01321\infty]$ since $N\infty 0\infty 1214$ $= N(1,\infty)(3,4)3410213 = N3410213 \in [\infty 01321\infty]$. To prove $\infty 0\infty 1214$ $= (1,\infty)(3,4)3410213$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty)(3,4)341\underline{0213}4121\infty 0\infty = (1,\infty)(3,4)341\ \underline{(1,0,3,2,\infty)120\infty}$ $4121\infty 0\infty$ (by relation (1) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (2,\infty,0,3,4)2\underline{4012}0\infty 4121\infty 0\infty = (2,\infty,0,3,4)2\ \underline{(1,4,2,0,3)1043}$ $0\infty 4121\infty 0\infty$ (by relation (1) conjugated by $(1,0,4,3,2) \in L_2(5)$)

$= (1,4,0)(2,\infty,3)01\underline{0430}\infty4121\infty0\infty = (1,4,0)(2,\infty,3)01\ \underline{(1,\infty,2)(3,4,0)4034}$ $\infty4121\infty0\infty$ (by relation (4) conjugated by $(1,\infty)(2,0) \in L_2(5)$)

$= (1,0,\infty,4,3)3\infty4034\infty41\underline{21\infty0}\infty = (1,0,\infty,4,3)3\infty4034\infty41$ $\underline{1,4,\infty,2,0)\infty124\infty}$ (by relation (2) conjugated by $(1,0,2)(3,4,\infty) \in L_2(5)$)

$= (2,0)(3,4)32\infty13\infty2\underline{\infty4\infty}124\infty = (2,0)(3,4)32\infty13\infty2\underline{4\infty4}124\infty$ (by relation (5) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (2,0)(3,4)32\underline{\infty13\infty}24\infty4124\infty = (2,0)(3,4)32\ \underline{(1,\infty,3)(2,4,0)1\infty31}$ $24\infty4124\infty$ (by relation (4) conjugated by $(1,0,2,\infty,4) \in L_2(5)$)

$= (1,\infty,3,0,4)141\underline{\infty312}4\infty4124\infty = (1,\infty,3,0,4)141\ \underline{(1,\infty,2,3,0)13\infty0}$ $4\infty4124\infty$ (by relation (1) conjugated by $(1,3,2)(4,0,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)\infty\underline{4\infty13}\infty04\infty4124\infty = (1,2,3)(4,\infty,0)\infty$ $\underline{(1,4,3,\infty,2)1\infty42}\ \infty04\infty4124\infty$ (by relation (1) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (2,\infty,0,3,4)21\underline{\infty42\infty}04\infty4124\infty = (2,\infty,0,3,4)21\ \underline{(1,0,3)(2,4,\infty)4\infty24}$ $04\infty4124\infty$ (by relation (4) conjugated by $(1,0,3,2,\infty) \in L_2(5)$)

$= (1,0)(3,\infty)\underline{404}\infty2\underline{404}\infty4124\infty = (1,0)(3,\infty)\underline{040}\infty2\underline{040}\infty4124\infty$ (by relation (5) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (1,0)(3,\infty)040\underline{\infty20}40\infty4124\infty = (1,0)(3,\infty)04\ \underline{(1,4,3)(0,2,\infty)\infty02\infty}$ $40\infty4124\infty$ (by relation (4) conjugated by $(2,0,4,\infty,3) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)23\infty02\underline{\infty40\infty}4124\infty = (1,2,\infty)(3,0,4)23\infty02$ $\underline{(1,2,3)(4,\infty,0)4\infty04}\ 4124\infty$ (by relation (3) conjugated by $(1,0,4,3,2) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)3104\underline{34\infty044}124\infty = (1,3,4)(2,0,\infty)3104\ \underline{(1,\infty,3,0,4)\infty431}$ 124∞ (by relation (2) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (1,0,3)(2,4,\infty)0\underline{\infty41\infty}43\underline{11}24\infty = (1,0,3)(2,4,\infty)0\ \underline{(1,4,\infty)(2,3,0)4\infty14}$ $43\underline{11}24\infty$ (by relation (4) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,2,\infty,3,4)24\infty1\underline{44}324\infty = (1,2,\infty,3,4)24\infty1324\infty = e$ by relation (1) conjugated by $(1,4)(0,\infty) \in L_2(5)$.

t_0 takes $[\infty0\infty121]$ to a single coset in $[\infty01\infty201]$ since $N\infty0\infty1210 = N(1,4,\infty)(2,3,0)4124012 = N4124012 \in [\infty01\infty201]$. To prove $\infty0\infty1210 = (1,4,\infty)(2,3,0)4124012$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,\infty)(2,3,0)4124\underline{0120}121\infty0\infty = (1,4,\infty)(2,3,0)4124$ $\underline{(1,0,2)(3,4,\infty)1021}\ 121\infty0\infty$ (by relation (4) conjugated by $(1,\infty,4)(2,0,3) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)\infty 01\infty 102\underline{11}21\underline{\infty 0\infty} = (1,\infty,0)(2,4,3)\infty 01\infty 10\underline{22}1\underline{0\infty 0}$ (by relation (5))

$= (1,\infty,0)(2,4,3)\infty 01\infty\underline{101}0\infty 0 = (1,\infty,0)(2,4,3)\infty 01\infty\underline{010}0\infty 0$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)\infty 01\infty 01\underline{00}\infty 0 = (1,\infty,0)(2,4,3)\infty 01\infty 01\infty 0 = e$ by relation (3) conjugated by $(2,3)(0,\infty) \in L_2(5)$.

t_∞ takes $[\infty 0\infty 121]$ to a single coset in $[\infty 01\infty 201]$ since $N\infty 0\infty 121\infty = N(1,0,\infty,4,3)4124012 = N4124012 \in [\infty 01\infty 201]$. To prove $\infty 0\infty 121\infty = (1,4,\infty)(2,3,0)4124012$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,\infty)(2,3,0)41240\underline{12\infty 1}21\infty 0\infty = (1,4,\infty)(2,3,0)41240$ $\underline{(1,\infty,2)(3,4,0)21\infty 2}\ 21\infty 0\infty$ (by relation (3) conjugated by $(1,\infty)(2,0) \in L_2(5)$)

$= (1,3,\infty,0,2)0\infty 10321\infty\underline{22}1\infty 0\infty = (1,3,\infty,0,2)0\infty 10321\ \underline{\infty 1\infty}\ 0\infty$
$= (1,3,\infty,0,2)0\infty 10321\ \underline{1\infty 1}\ 0\infty$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,3,\infty,0,2)0\underline{\infty 1032}\underline{11}\infty 10\infty = (1,3,\infty,0,2)0\ \underline{(1,2,0,\infty,3)01\infty 2}\ 2\infty 10\infty$ (by relation (2) conjugated by $(1,3,2)(4,0,\infty) \in L_2(5)$)

$= \infty 01\infty\underline{22}\infty 10\infty = \infty 01\underline{\infty\infty}10\infty = \infty 0\underline{11}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

Now consider $[\infty 0\infty 104]$. $N^{(\infty 0\infty 104)}$ has orbits $\{1\}$, $\{4\}$, $\{2,3\}$, and $\{0,\infty\}$. So we need to look at $[\infty 0\infty 1041]$, $[\infty 0\infty 1044]$, $[\infty 0\infty 1043]$, and $[\infty 0\infty 1040]$.

First $[\infty 0\infty 1044] = [\infty 0\infty 10]$. So t_4 takes $[\infty 0\infty 104]$ back to a single coset in $[\infty 0\infty 10]$.

t_1 takes $[\infty 0\infty 104]$ to a single coset in $[\infty 01\infty 414]$ since $N\infty 0\infty 1041 = N(1,\infty,0)(2,4,3)\infty 01\infty 414 = N\infty 01\infty 414 \in [\infty 01\infty 414]$. To prove $\infty 0\infty 1041 = (1,\infty,0)(2,4,3)\infty 01\infty 414$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,0)(2,4,3)\infty 01\infty 4\underline{141}401\infty 0\infty = (1,\infty,0)(2,4,3)\infty 01\infty 4\ \underline{414}$ $401\infty 0\infty$ (by relation (6))

$= (1,\infty,0)(2,4,3)\infty 01\infty\underline{44}1\underline{44}01\infty 0\infty = (1,\infty,0)(2,4,3)\infty 01\infty\underline{101\infty 0\infty}$
$= (1,\infty,0)(2,4,3)\infty 01\infty\underline{0100\infty 0}$ (by relation (6) conjugated by $(2,\infty)(4,0) \in L_2(5)$ and relation (5))

$= (1,\infty,0)(2,4,3)\infty 01\infty 01\underline{00}\infty 0 = (1,\infty,0)(2,4,3)\infty 01\infty 01\infty 0 = e$ by relation (3).

t_3, a representative from one of the 2-orbits, takes $[\infty 0\infty 104]$ to a single coset in $[\infty 024\infty]$ since we previously proved $\infty 024\infty 3 = (1,3,0,2,4)\infty 0\infty 104$. Therefore it follows $N\infty 024\infty = N(1,3,0,2,4)\infty 0\infty 1043 = N\infty 0\infty 1043 \in [\infty 0\infty 1043] = [\infty 024\infty]$.

t_0, a representative of the other 2-orbit, takes $[\infty 0\infty 104]$ to a single coset in $[\infty 010203]$ since $N\infty 0\infty 1040 = N(1,0,4,3,2)421232\infty = N421232\infty \in [\infty 010203]$. To prove $\infty 0\infty 1040 = (1,0,4,3,2)421232\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,4,3,2)421\underline{232}\infty 0401\infty 0\infty = (1,0,4,3,2)421\underline{323}\infty 0401\infty 0\infty$ (by relation (6) conjugated by $(1,3,0)(2,\infty,4) \in L_2(5)$)

$= (1,0,4,3,2)4\underline{21323}\infty 0401\infty 0\infty = (1,0,4,3,2)4\ \underline{(1,2,3)(4,\infty,0)1231}$ $3\infty 0401\infty 0\infty$ (by relation (4) conjugated by $(1,4)(0,\infty) \in L_2(5)$)

$= (1,4)(0,\infty)\infty 12313\infty 0401\underline{\infty 0\infty} = (1,4)(0,\infty)\infty 12313\infty 0401\underline{0\infty 0}$ (by relation (5))

$= (1,4)(0,\infty)\infty 12313\infty 04\underline{010}\infty 0 = (1,4)(0,\infty)\infty 12313\infty 04\underline{101}\infty 0$ (by relation (6) conjugated by $(2,\infty)(4,0) \in L_2(5)$)

$= (1,4)(0,\infty)\infty 123\underline{13\infty 0}4101\infty 0 = (1,4)(0,\infty)\infty 123\ \underline{(1,0,3,2,\infty)\infty 312}$ $4101\infty 0$ (by relation (1) conjugated by $(1,3,0)(2,\infty,4) \in L_2(5)$)

$= (1,4,0)(2,\infty,3)10\infty 2\infty 3\underline{1241}01\infty 0 = (1,4,0)(2,\infty,3)10\infty 2\infty 3$ $\underline{(1,4,2)(3,0,\infty)2142}\ 01\infty 0$ (by relation (3) conjugated by $(1,4,\infty)(2,3,0) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)4\infty 3\underline{1302}14201\infty 0 = (1,2,3)(4,\infty,0)4\infty 3\ \underline{(1,2,3,4,0)0314}$ $14201\infty 0$ (by relation (2) conjugated by $(1,2,3,4,0) \in L_2(5)$)

$= (1,3,2,4,\infty)0\infty 4031\underline{414}201\infty 0 = (1,3,2,4,\infty)0\infty 4031\underline{141}201\infty 0$ (by relation (6))

$= (1,3,2,4,\infty)0\infty 403\underline{11}41201\infty 0 = (1,3,2,4,\infty)0\infty\ \underline{4034}\ 1201\infty 0$

$= (1,3,2,4,\infty)0\infty\ \underline{(1,2,\infty)(3,0,4)0430}\ 1201\infty 0$ (by relation (4) conjugated by $(1,2,4,0,\infty) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)41043\underline{0120}1\infty 0 = (1,0,4)(2,3,\infty)41043\ \underline{(1,0,2)(3,4,\infty)1021}$ $1\infty 0$ (by relation (4) conjugated by $(1,\infty,4)(2,0,3) \in L_2(5)$)

$= (1,2,4,0,\infty)\infty\underline{02\infty 04}102\underline{11}\infty 0 = (1,2,4,0,\infty)\infty\ \underline{(1,\infty,0,4,2)\infty 201}\ 102\infty 0$ (by relation (1) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= 0\infty 20\underline{11}02\infty 0 = 0\infty 2\underline{00}2\infty 0 = 0\infty\underline{22}\infty 0 = 0\underline{\infty\infty}0 = \underline{00} = e.$

Now consider $[\infty 0\infty 214]$. $N^{(\infty 0\infty 214)}$ has orbits $\{1,3,\infty\}$ and $\{2,4,0\}$. So we

need to look at $[\infty 0 \infty 2141]$ and $[\infty 0 \infty 2144]$.

First $[\infty 0 \infty 2144] = [\infty 0 \infty 21]$. So t_4, a representative from one of the 3-orbits, takes $[\infty 0 \infty 214]$ back to a single coset in $[\infty 0 \infty 21]$.

t_∞, a representative of the other 3-orbit, takes $[\infty 0 \infty 214]$ to a single coset in $[\infty 0241]$ since we previously proved $\infty 02414 = (1,0,2)(3,4,\infty)4043\infty 2$. Therefore it follows $N\infty 0241 = N(1,0,2)(3,4,\infty)4043\infty 24 = N4043\infty 24 \in [\infty 0 \infty 214\infty] = [\infty 0241]$.

Now consider $[\infty 0 \infty 232]$. $N^{(\infty 0 \infty 232)}$ has one orbit $\{1,2,3,4,0,\infty\}$. So we need to look at $[\infty 0 \infty 2322]$.

$[\infty 0 \infty 2322] = [\infty 0 \infty 23]$. So t_2, a representative of the 6-orbit, takes $[\infty 0 \infty 232]$ back to a single coset in $[\infty 0 \infty 23]$.

Now consider $[\infty 02431]$. $N^{(\infty 02431)}$ has orbits $\{2\}$ and $\{1,3,4,0,\infty\}$. So we need to look at $[\infty 024311]$ and $[\infty 024312]$.

First $[\infty 024311] = [\infty 0243]$. So t_1, a representative of the 5-orbit, takes $[\infty 02431]$ back to a single coset in $[\infty 0243]$.

t_2 takes $[\infty 02431]$ to a single coset in $[\infty 012\infty 43]$ since $N\infty 024312$ $= N(1,4,\infty)(2,3,0)\infty 012\infty 43 = N\infty 012\infty 43 \in [\infty 012\infty 43]$. To prove $\infty 024312 = (1,4,\infty)(2,3,0)\infty 012\infty 43$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,\infty)(2,3,0)\infty 012\infty\underline{4321}3420\infty = (1,4,\infty)(2,3,0)\infty 012\infty$ $\underline{(1,3,0,2,4)2340}\ 3420\infty$ (by relation (1) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (2,0,4,\infty,3)\infty 234\infty 2\underline{3403}420\infty = (2,0,4,\infty,3)\infty 234\infty 2$ $\underline{(1,2,\infty)(3,0,4)4304}\ 420\infty$ (by relation (4) conjugated by $(1,\infty,2,3,0) \in L_2(5)$)

$= (1,2,4)(3,\infty,0)1\infty 0\underline{31\infty 4}30\underline{44}20\infty = (1,2,4)(3,\infty,0)1\infty 0$ $\underline{(1,2,\infty,3,4)\infty 132}\ 30\underline{44}20\infty$ (by relation (1) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= (1,\infty,0,4,2)230\infty 1\underline{323020}\infty = (1,\infty,0,4,2)230\infty 1\underline{232202}\infty$ (by relation (5) conjugated by $(1,\infty,2)(3,4,0)$ and $(1,3)(2,\infty) \in L_2(5)$)

$= (1,\infty,0,4,2)23\underline{0\infty 12}3\underline{22}02\infty = (1,\infty,0,4,2)23\underline{(1,0,2,\infty,4)1\infty 0}4302\infty$ (by relation (2) conjugated by $(1,2,\infty,3,4) \in L_2(5)$)

$= (1,4,\infty,2,0)\underline{\infty 31\infty}04302\infty = (1,4,\infty,2,0)\underline{(1,3,\infty)(2,0,4)3\infty 1}304302\infty$ (by relation (4) conjugated by $(1,4,3)(2,\infty,0) \in L_2(5)$)

$= (1,2,4)(3,\infty,0)3\infty 1\underline{3043}02\infty = (1,2,4)(3,\infty,0)3\infty 1$ $\underline{(1,\infty,2)(3,4,0)0340}\ 02\infty$ (by relation (4) conjugated by $(1,2,3,4,0) \in L_2(5)$)

$= (2,0,4,\infty,3)42\infty034\underline{00}2\infty = (2,0,4,\infty,3)42\infty0342\infty = e$ by relation (1) conjugated by $(1,2,4,0,\infty) \in L_2(5)$.

Now consider $[\infty02432\infty]$. $N^{(\infty02432\infty)}$ has orbits $\{0\}$, $\{\infty\}$, $\{1,4\}$, and $\{2,3\}$. So we need to look at $[\infty02432\infty0]$, $[\infty02432\infty\infty]$, $[\infty02432\infty1]$, and $[\infty02432\infty2]$.

First $[\infty02432\infty\infty] = [\infty02432]$. So t_∞ takes $[\infty02432\infty]$ back to a single coset in $[\infty02432]$.

t_0 takes $[\infty02432\infty]$ to a single coset in $[\infty0201]$ since we previously proved $\infty0201\infty = (2,0)(3,4)1\infty3431$. Therefore it follows $N\infty0201 = N(2,0)(3,4)1\infty3431\infty = N1\infty3431\infty \in [\infty02432\infty0] = [\infty0201]$.

t_1, a representative from one of the 2-orbits, takes $[\infty02432\infty]$ to a single coset in $[\infty012130]$ since $N\infty0232\infty1 = N(1,0)(4,2)2\infty3034\infty = N2\infty3034\infty \in [\infty012130]$. To prove $\infty02432\infty1 = (1,0)(4,2)2\infty3034\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0)(4,2)2\infty\underline{3034}\underline{\infty1\infty}2320\infty = (1,0)(4,2)2\infty\underline{030}4\underline{1\infty1}2320\infty$ (by relation (5) conjugated by $(2,4)(3,\infty)$ and $(1,0)(2,4) \in L_2(5)$)

$= (1,0)(4,2)2\infty0\underline{3041}\infty12320\infty = (1,0)(4,2)2\infty0\underline{(1,0,\infty,4,3)403\infty}\infty12320\infty$ (by relation (1) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (1,\infty,4,2,3)2\underline{4\infty4}03\underline{\infty\infty}1\underline{232}0\infty = (1,\infty,4,2,3)2\underline{\infty4\infty}031\underline{323}0\infty$ (by relation (5) conjugated by $(1,3)(4,0)$ and $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (1,\infty,4,2,3)2\infty4\infty0\underline{313}230\infty = (1,\infty,4,2,3)2\infty4\infty0\underline{131}230\infty$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,\infty,4,2,3)2\infty\underline{4\infty01}31230\infty = (1,\infty,4,2,3)2\infty\underline{(1,\infty,3,0,4)0\infty43}31230\infty$ (by relation (2) conjugated by $(2,\infty)(4,0) \in L_2(5)$)

$= (1,3,\infty)(2,0,4)230\infty4\underline{33}1230\infty = (1,3,\infty)(2,0,4)230\infty41230\infty = e$ by a previously proved relation $(1,3,0)(2,\infty,4)320\infty\ 14320\infty = e$ conjugated by $(1,4)(2,3) \in L_2(5)$.

t_2, a representative of the other 2-orbit, takes $[\infty02432\infty]$ to a single coset in $[\infty01\infty403]$ since $N\infty02432\infty2 = N(2,0,4,\infty,3)\infty32\infty430 = N\infty32\infty430 \in [\infty01\infty403]$. To prove $\infty02432\infty2 = (2,0,4,\infty,3)\infty32\infty430$, we will move the relation to one side of the equal sign and prove it equals identity.

$(2,0,4,\infty,3)\infty32\infty430\underline{2\infty2}3420\infty = (2,0,4,\infty,3)\infty32\infty430\underline{\infty2\infty}3420\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (2,0,4,\infty,3)\infty32\infty\underline{430\infty}2\infty3420\infty = (2,0,4,\infty,3)\infty32\infty\ \underline{(2,0,4,\infty,3)0342}$ $2\infty3420\infty$ (by relation (1) conjugated by $(1,3,\infty)(2,0,4) \in L_2(5)$)

$= (2,4,3,0,\infty)32\underline{030}34\underline{22}\infty320\infty = (2,4,3,0,\infty)32\underline{30}334\infty320\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (2,4,3,0,\infty)\underline{323}0\underline{33}4\infty320\infty = (2,4,3,0,\infty)\underline{232}04\infty320\infty$ (by relation (5) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (2,4,3,0,\infty)2\underline{3204}\infty320\infty = (2,4,3,0,\infty)2\underline{(2,\infty,0,3,4)023\infty}\infty320\infty$ (by relation (2) conjugated by $(1,4,0,3,\infty) \in L_2(5)$)

$= \infty023\underline{\infty\infty}320\infty = \infty02\underline{33}20\infty = \infty0\underline{22}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

Now consider $[\infty012\infty43]$. $N^{(\infty012\infty43)}$ has orbits $\{2\}$ and $\{1,3,4,0,\infty\}$. So we need to look at $[\infty012\infty432]$ and $[\infty012\infty433]$.

First $[\infty012\infty433] = [\infty012\infty4]$. So t_3, a representative of the 5-orbit, takes $[\infty012\infty43]$ back to a single coset in $[\infty012\infty4]$.

t_2 takes $[\infty012\infty43]$ to a single coset in $[\infty02431]$ since we previously proved $\infty024312 = (1,4,\infty)(2,3,0)\infty012\infty43$. Therefore it follows $N\infty02431$ $= N(1,4,\infty)(2,3,0)\infty012\infty432 = N\infty012\infty432 \in [\infty012\infty432] = [\infty02431]$.

Now consider $[\infty012\infty32]$. $N^{(\infty012\infty32)}$ has one orbit $\{1,2,3,4,0,\infty\}$. So we need to look at $[\infty012\infty322]$.

$[\infty012\infty322] = [\infty012\infty3]$. So t_2, a representative of the 6-orbit, takes $[\infty012\infty32]$ back to a single coset in $[\infty012\infty3]$.

Now consider $[\infty01231\infty]$. $N^{(\infty01231\infty)}$ has orbits $\{1\}$, $\{\infty\}$, $\{3,4\}$, and $\{2,0\}$. So we need to look at $[\infty01231\infty1]$, $[\infty01231\infty\infty]$, $[\infty01231\infty2]$, and $[\infty01231\infty4]$.

First $[\infty01231\infty\infty] = [\infty01231]$, so t_∞ takes $[\infty01231\infty]$ back to a single coset in $[\infty01231]$.

t_1 takes $[\infty01231\infty]$ to a single coset in $[\infty01302]$ since we previously proved $\infty013021 = (2,\infty,0,3,4)4312014$. Therefore it follows $N\infty01302$ $= N(2,\infty,0,3,4)43120141 = N43120141 \in [\infty01231\infty1] = [\infty01302]$.

t_2, a representative from one of the 2-orbits, takes $[\infty01231\infty]$ to a single coset in $[\infty0\infty121]$ since we previously proved $\infty0\infty1214 = (1,0,4,3,2)3014\infty13$. Therefore it follows $N\infty0\infty121 = N(1,0,4,3,2)3014\infty134 = N3014\infty134 \in [\infty01231\infty2] = [\infty0\infty121]$.

t_4, a representative from the other 2-orbit, takes $[\infty01231\infty]$ to a single coset in $[\infty0131\infty]$ since we previously proved $\infty0131\infty0 = (1,2,\infty,3,4)12\infty4321$. Therefore it

follows $N\infty0131\infty = N(1,2,\infty,3,4)12\infty43210 = N12\infty43210 \in [\infty01231\infty4]$ $= [\infty0131\infty]$.

Now consider $[\infty012130]$. $N^{(\infty012130)}$ has orbits $\{1,2,4\}$ and $\{3,0,\infty\}$. So we need to look at $[\infty0121304]$ and $[\infty0121300]$.

First $[\infty0121300] = [\infty01213]$, so t_0, a representative from one of the 3-orbits, takes $[\infty012130]$ back to a single coset in $[\infty01213]$.

t_4, a representative of the other 3-orbit, takes $[\infty012130]$ to a single coset in $[\infty0232\infty]$ since we previously proved $\infty0232\infty1 = (1,0)(4,2)2\infty3034\infty$. Therefore it follows $N\infty0232\infty = N(1,0)(4,2)2\infty3034\infty1 = N2\infty3034\infty1 \in [\infty0121304] = [\infty0232\infty]$.

Now consider $[\infty010203]$. $N^{(\infty010203)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty0102031]$, $[\infty0102032]$, $[\infty0102033]$, $[\infty0102034]$, $[\infty0102030]$, and $[\infty010203\infty]$.

First $[\infty0102033] = [\infty01020]$, so t_3 takes $[\infty010203]$ back to a single coset in $[\infty01020]$.

t_1 takes $[\infty010203]$ to a single coset in $[\infty0103\infty]$ since we previously proved $\infty0103\infty2 = (1,2,4,0,\infty)3\infty2\ \infty0\infty1$. Therefore it follows $N\infty0103\infty$ $= N(1,2,4,0,\infty)3\infty2\ \infty0\infty12 = N3\infty2\infty\ 0\infty12 \in [\infty0102031] = [\infty0103\infty]$.

t_2 takes $[\infty010203]$ to a single coset in $[\infty01031]$ since we previously proved $\infty01031\infty = (1,\infty)(2,0)2414\infty43$. Therefore it follows $N\infty01031$ $= N(1,\infty)(2,0)2414\infty43\infty = N2414\infty43\infty \in [\infty0102032] = [\infty01031]$.

t_0 takes $[\infty010203]$ to a single coset in $[\infty0142\infty]$ since we previously proved $\infty0142\infty1 = (1,2,\infty)(3,0,4)412\ 101\infty$. Therefore it follows $N\infty0142\infty$ $= N(1,2,\infty)(3,0,4)4121\ 01\infty1 = N412101\infty1 \in [\infty0102030] = [\infty0142\infty]$.

t_∞ takes $[\infty010203]$ to a single coset in $[\infty0131\infty]$ since we previously proved $\infty0131\infty3 = (2,0)(3,4)3\infty4\ \infty1\infty0$. Therefore it follows $N\infty0131\infty$ $= N(2,0)(3,4)3\infty4\infty\ 1\infty03 = N3\infty4\infty\ 1\infty03 \in [\infty010203\infty] = [\infty0142\infty]$.

t_4 takes $[\infty010203]$ to a single coset in $[\infty0\infty104]$ since we previously proved $\infty0\infty1040 = (1,0,4,3,2)4212\ 32\infty$. Therefore it follows $N\infty0\infty104 = N(1,0,4,3,2)421$ $232\infty0 = N4212\ 32\infty0 \in [\infty0102034] = [\infty0\infty104]$.

Now consider $[\infty010204]$. $N^{(\infty010204)}$ has orbits $\{1,2,0\}$ and $\{3,4,\infty\}$. So we need to look at $[\infty0102042]$ and $[\infty0102044]$.

First $[\infty0102044] = [\infty01020]$. So t_4, a representative from one of the 3-orbits, takes $[\infty010204]$ back to a single coset in $[\infty01020]$.

t_2, a representative from the other 3-orbit, takes $[\infty010204]$ to $[\infty01430]$ since we previously proved $\infty014302 = (1,3,0)(2,\infty,4)\infty434240$. Therefore it follows $N\infty01430$ $= N(1,3,0)(2,\infty,4)\infty4342402 = N\infty4342402 \in [\infty0102042] = [\infty01430]$.

Now consider $[\infty010232]$. $N^{(\infty010232)}$ has orbits $\{2\}$, $\{\infty\}$, $\{1,3\}$, and $\{4,0\}$. So we need to look at $[\infty0102322]$, $[\infty010232\infty]$, $[\infty0102323]$, and $[\infty0102320]$.

First $[\infty0102322] = [\infty01023]$, so t_2 takes $[\infty010232]$ back to a single coset in $[\infty01023]$.

t_∞ takes $[\infty010232]$ to a single coset in $[\infty0140\infty]$ since we previously proved $\infty0140\infty1 = (1,4)(0,\infty)1303\ \infty2\infty$. Therefore it follows $N\infty0140\infty$ $= N(1,4)(0,\infty)1303\ \infty2\infty1 = N1303\ \infty2\infty1 \in [\infty010232\infty] = [\infty0140\infty]$.

t_3, a representative from one of the 2-orbits, takes $[\infty010232]$ to a single coset in $[\infty01032]$ since we previously proved $\infty010323 = \infty010232$. Therefore it follows $N\infty01032 = N(\infty0102323 \in [\infty0102323] = [\infty01032]$.

t_0, a representative from the other 2-orbit, takes $[\infty010232]$ to a single coset in $[\infty0\infty121]$ since we previously proved $\infty0\infty1213 = 2303414$. Therefore it follows $N\infty0\infty121 = N23034143 \in [\infty0102320] = [\infty0\infty121]$.

Now consider $[\infty01\infty403]$. $N^{(\infty01\infty403)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty01\infty4031]$, $[\infty01\infty4032]$, $[\infty01\infty4033]$, $[\infty01\infty4034]$, $[\infty01\infty4030]$, and $[\infty01\infty403\infty]$.

First $[\infty01\infty4033] = [\infty01\infty40]$, so t_3 takes $[\infty01\infty403]$ back to a single coset in $[\infty01\infty40]$.

t_0 takes $[\infty01\infty403]$ to a single coset in $[\infty012\infty4]$ since we previously proved $\infty012\infty40 = (2,3,\infty,4,0)2032401$. Therefore it follows $N\infty012\infty4$ $= N(2,3,\infty,4,0)2032\ 4010 = N2032\ 4010 \in [\infty01\infty4030] = [\infty012\infty4]$.

t_2 takes $[\infty01\infty403]$ to a single coset in $[\infty01430]$ since we previously proved $\infty014303 = (1,2,4,0,\infty)4214\ 02\infty$. Therefore it follows $N\infty01430 = N(1,2,4,0,\infty)4214$ $02\infty3 = N4214\ 02\infty3 \in [\infty01\infty4032] = [\infty01430]$.

t_4 takes $[\infty01\infty403]$ to a single coset in $[\infty01403]$ since we previously proved $\infty014030 = (1,2,4,0,\infty)421\ 402\infty$. Therefore it follows $N\infty01403 = N(1,2,4,0,\infty)4214$ $02\infty0 = N4214\ 02\infty0 \in [\infty01\infty4034] = [\infty01403]$.

t_1 takes $[\infty 01\infty 403]$ to a single coset in $[\infty 0232\infty]$ since we previously proved $\infty 0232\infty 2 = (2,0,4.\infty,3)\infty 32\ \infty 430$. Therefore it follows $N\infty 0232\infty$
$= N(2,0,4.\infty,3)\infty 32\infty\ 4302 = N\infty 32\infty\ 4302 \in [\infty 01\infty 4031] = [\infty 0232\infty]$.

t_∞ takes $[\infty 01\infty 403]$ to a single coset in $[\infty 01032]$ since we previously proved $\infty 010321 = (1,3,4)(2,0.\infty)1\infty 01\ 2\infty 4$. Therefore it follows $N\infty 01032$
$= N(1,3,4)(2,0.\infty)1\infty 01\ 2\infty 41 = N1\infty 01\ 2\infty 41 \in [\infty 01\infty 403\infty] = [\infty 01032]$.

Now consider $[\infty 01\infty 414]$. $N^{(\infty 01\infty 414)}$ has orbits $\{4\}$ and $\{1,2,3,0,\infty\}$. So we need to look at $[\infty 01\infty 4144]$ and $[\infty 01\infty 4141]$.

First $[\infty 01\infty 4144] = [\infty 01\infty 41]$, so t_4 takes $[\infty 01\infty 414]$ back to a single coset in $[\infty 01\infty 41]$.

t_1, a representative of the 5-orbit, take $[\infty 01\infty 414]$ to a single coset in $[\infty 0\infty 104]$ since we previously proved $\infty 0\infty 1041 = (1,\infty,0)(2,4,3)\infty 01\infty 414$. Therefore it follows $N\infty 0\infty 104 = N(1,\infty,0)(2,4,3)\infty 01\infty 4141 = N\infty 01\infty 4141 \in [\infty 01\infty 4141] = [\infty 0\infty 104]$.

Now consider $[\infty 01\infty 20\infty]$. $N^{(\infty 01\infty 20\infty 3)}$ has orbits $\{1,2,\infty\}$ and $\{3,4,0\}$. So we need to look at $[\infty 01\infty 20\infty\infty]$ and $[\infty 01\infty 20\infty 3]$.

First $[\infty 01\infty 20\infty\infty] = [\infty 01\infty 20]$, so t_∞, a representative from one of the 3-orbits, takes $[\infty 01\infty 20\infty]$ back to a single coset in $[\infty 01\infty 20]$.

t_3, a representative of the other 3-orbit, takes $[\infty 01\infty 20\infty]$ to a single coset in $[\infty 0103\infty]$ since we previously proved $\infty 0103\infty 1 = (1,4,\infty,2,0)2402\ 342$. Therefore it follows $N\infty 0103\infty = N(1,4,\infty,2,0)2402\ 3421 = N24023421 \in [\infty 01\infty 20\infty 3]$
$= [\infty 0103\infty]$.

Now consider $[\infty 01\infty 241]$. $N^{(\infty 01\infty 241)}$ has one orbit $\{1,2,3,4,0,\infty\}$, so we need to look at $[\infty 01\infty 2411]$.

$[\infty 01\infty 2411] = [\infty 01\infty 24]$, so t_1, a representative of the 6-orbit, takes $[\infty 01\infty 241]$ back to a single coset in $[\infty 01\infty 24]$.

Now consider $[\infty 01\infty 212]$. $N^{(\infty 01\infty 212)}$ has orbits $\{2\}$, $\{3\}$, $\{1,4\}$, and $\{0,\infty\}$. So we need to look at $[\infty 01\infty 2122]$, $[\infty 01\infty 2123]$, $[\infty 01\infty 2121]$, and $[\infty 01\infty 212\infty]$.

First $[\infty 01\infty 2122] = [\infty 01\infty 21]$, so t_2 takes $[\infty 01\infty 212]$ back to a single coset in $[\infty 01\infty 21]$.

t_3 takes $[\infty 01\infty 212]$ to a single coset in $[\infty 01\infty 2123]$. $N\infty 01\infty 2123 = N4314$

$2120 = N34\infty\ 32\infty20 = N10\infty\ 12\infty24 = N3103\ 202\infty = N0130\ 2324 = N4\infty\ 042021 = N\infty43\infty\ 2321 = N13412\ 42\infty = N0\infty40\ 3423$ since we previously proved $N\infty01\ \infty212 = N0\infty40242$. Therefore it follows that $N\infty01\infty\ 2123 = N0\infty40\ 2423$. If we conjugate this equation by $(3,0)(4,\infty)$, $(1,\infty)(3,4)$, $(1,0)(3,\infty)$ and $(1,0,\infty,4,3) \in L_2(5)$ we find $N4314\ 2120 = N34\infty3\ 2\infty20$, $N10\infty1\ 2\infty24 = N0130\ 2324$, $N3103\ 202\infty = N1341\ 242\infty$, and $N4\infty04\ 2021 = N\infty43\infty\ 2321$. To show that all of these single cosets are equal we will prove $\infty01\infty\ 2123 = (1,4,\infty)(2,3,0)4314\ 2120$, $\infty01\infty\ 2123 = (2,4)(3,\infty)3103\ 202\infty$, $\infty01\infty\ 2123 = (1,0,2)(3,4,\infty)10\infty1\ 2\infty24$, and $\infty01\infty\ 2123 = (1,3)(2,\infty)\infty43\infty\ 2321$.

To prove $\infty01\infty2123 = (1,4,\infty)(2,3,0)43142120$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,\infty)(2,3,0)431421\underline{2032}12\infty10\infty = (1,4,\infty)(2,3,0)431421$ $\underline{(1,4,\infty)(2,3,0)0230}\ 12\infty10\infty$ (by relation (3) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,\infty,4)(2,0,3)\underline{\infty04\infty}34023012\infty10\infty = (1,\infty,4)(2,0,3)$ $\underline{(1,3,2)(4,0,\infty)0\infty40}\ 34023012\ \infty10\infty$ (by relation (3) conjugated by $(1,4)(2,3) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)0\infty4\underline{0340}23012\infty10\infty = (1,4,3)(2,\infty,0)0\infty4$ $\underline{(1,2,\infty)(3,0,4)3043}\ 23012\ \infty10\infty$ (by relation (4) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)41\underline{33}04\underline{323}012\infty10\infty = (1,3,2)(4,0,\infty)4104\ \underline{232}\ 012\infty10\infty$ (by relation (5) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)410423\underline{2012}\infty10\infty = (1,3,2)(4,0,\infty)410423$ $\underline{(1,0,2)(3,4,\infty)0210}\ \infty10\infty$ (by relation (4) conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (1,4,2,0,3)\infty02\infty14021\underline{0\infty10}\infty = (1,4,2,0,3)\infty02\infty\ 14021$ $\underline{(1,\infty,0)(2,4,3)\infty01\infty}\infty\infty$ (by relation (3))

$= (1,3,\infty,0,2)\underline{0140}\infty314\underline{\infty\infty\infty}\ 01\underline{\infty\infty\infty} = (1,3,\infty,0,2)\ \underline{((1,0,4)(2,3,\infty)1041}$ $\infty31401$ (by relation (4) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$)

$= (1,\infty,4)(2,0,3)1\underline{041\infty}31401 = (1,\infty,4)(2,0,3)1\underline{(1,0,\infty,4,3)}140331401$ (by relation (1) conjugated by $(1,4,3,\infty,2) \in L_2(5)$)

$= (1,4,0)(2,\infty,3)0140\underline{33}1401 = (1,4,0)(2,\infty,3)01401401 = e$ by relation (4) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$.

To prove $\infty01\infty2123 = (2,4)(3,\infty)3103202\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(2,4)(3,\infty)31\underline{03202}\infty3212\infty10\infty = (2,4)(3,\infty)31\ \underline{(1,4,\infty)(2,3,0)3023}$

$2\infty3212\ \infty10\infty$ (by relation (3) conjugated by $(1,2,4)(3,\infty,0) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)0430\underline{232\infty3}212\infty10\infty = (1,4,3)(2,\infty,0)04302$ $\underline{(1,0,4)(2,3,\infty)23\infty2}\ 212\infty10\infty$ (by relation (3) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (3,0)(4,\infty)41\infty4323\infty\underline{22}12\infty10\infty = (3,0)(4,\infty)41\infty\ 4323\ \underline{\infty12\infty}\ 10\infty =$ $(3,0)(4,\infty)41\infty\ 4323\ \underline{(1,\infty,2)(3,4,0)1\infty21}\ 10\infty$ (by relation (3) conjugated by $(1,2,3,4,0) \in L_2(5)$)

$= (1,\infty,0,4,2)0\infty204\underline{141}\infty2\underline{11}0\infty = (1,\infty,0,4,2)0\infty204\underline{414}\infty20\infty$ (by relation (6))

$= (1,\infty,0,4,2)0\infty20\underline{44}14\infty20\infty = (1,\infty,0,4,2)0\ \underline{\infty201}\ 4\infty20\infty$ $= (1,\infty,0,4,2)0\ \underline{(1,2,4,0,\infty)02\infty4}\ 4\infty20\infty$ (by relation (1) conjugated by $(1,2,0,\infty,3)$ $\in L_2(5)$)

$= \infty02\infty\underline{44}\infty20\infty = \infty02\underline{\infty\infty}20\infty = \infty0\underline{22}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

To prove $\infty01\infty2123 = (1,3,4,\infty,0)10\infty12\infty24$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,2)(3,4,\infty)10\infty12\underline{\infty243}212\infty10\infty = (1,0,2)(3,4,\infty)10\infty12$ $\underline{(2,0,4,\infty,3)42\infty0}\ 212\infty10\infty$ (by relation (1) conjugated by $(1,2,4,0,\infty) \in L_2(5)$)

$= (1,4,3,\infty,2)\underline{1431}042\infty0212\infty10\infty = (1,4,3,\infty,2)\ \underline{(1,3,4))(2,0,\infty)4134}$ $042\infty\ 0212\ \infty10\infty$ (by relation (3) conjugated by $(1,3,\infty)(2,0,4) \in L_2(5)$)

$= (2,3)(0,\infty)413\underline{4042}\infty0212\infty10\infty = (2,3)(0,\infty)413\underline{0402}\infty0212\infty10\infty$ (by relation (5) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (2,3)(0,\infty)41304\underline{02\infty0}212\infty10\infty = (2,3)(0,\infty)41304\ \underline{(1,3,4)(2,0,\infty)20\infty2}$ $212\infty10\infty$ (by relation (4) conjugated by $(1,3,\infty)(2,0,4) \in L_2(5)$)

$= (1,3,0,2,4)134\infty120\infty\underline{22}12\infty10\infty = (1,3,0,2,4)134\infty120\ \underline{\infty12\infty}\ 10\infty =$ $(1,3,0,2,4)134\infty\ 120\ \underline{(1,\infty,2)(3,4,0)1\infty21}\ 10\infty$ (by relation (3) conjugated by $(1,2,3,4,0) \in L_2(5)$)

$= (1,4,\infty,2,0)\infty402\infty131\infty2\underline{11}0\infty = (1,4,\infty,2,0)\infty40\ 2\infty131\ \underline{\infty20\infty}$ $= (1,4,\infty,2,0)\infty40\ 2\infty131\ \underline{(1,4,3)(2,\infty,0)2\infty02}$ (by relation (4) conjugated by $(1,3,0)(2,\infty,4) \in L_2(5)$

$= (1,3)(4,0)032\infty0\underline{414}2\infty02 = (1,3)(4,0)032\infty0\underline{141}2\infty02$ (by relation (6))

$= (1,3)(4,0)032\infty01\underline{412\infty}02 = (1,3)(4,0)032\infty01\underline{(1,3,2,4,\infty)214}302$ (by relation (2) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (1,2,4,0,\infty)\underline{0241}03214302 = (1,2,4,0,\infty)\underline{(1,2,3,4,0)}420303214302$ (by rela-

tion (2))

$= (1,3,4)(2,0,\infty)42\underline{0303}214302 = (1,3,4)(2,0,\infty)42\underline{303}3214302$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)42303\underline{3}3214302 = (1,3,4)(2,0,\infty)4\ \underline{2302}\ 14302$

$= (1,3,4)(2,0,\infty)4\ \underline{(1,\infty,3,0,4)3202}\ 14302$ (by relation (3) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (1,0,3)(2,4,\infty)13\underline{2031}4302 = (1,0,3)(2,4,\infty)13\underline{(1,0,4,3,2)3024}4302$ (by relation (2) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,4,\infty)(2,3,0)02302\underline{44}302 = (1,4,\infty)(2,3,0)02302302 = e$ by relation (3) conjugated by $(1,3)(2,\infty) \in L_2(5)$.

To prove $\infty 01\infty 2123 = (1,3)(2,\infty)\infty 43\infty 2321$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3)(2,\infty)\infty 43\infty 23\underline{2132}12\infty 10\infty = (1,3)(2,\infty)\infty 43\infty 23$ $\underline{(1,2,3)(4,\infty,0)1231}\ 12\infty 10\infty$ (by relation (4) conjugated by $(1,4)(0,\infty) \in L_2(5)$)

$= (2,0,4,\infty,3)0\infty 103\underline{11}23\underline{11}2\infty 10\infty = (2,0,4,\infty,3)0\infty 103\ \underline{232}\ \infty 10\infty$

$= (2,0,4,\infty,3)0\infty 103\ \underline{323}\ \infty 10\infty$ (by relation (5) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (2,0,4,\infty,3)\underline{0\infty 1033}23\infty 10\infty = (2,0,4,\infty,3)\ \underline{(1,\infty,0)(2,4,3)\infty 01\infty}$ $23\infty 10\infty$ (by relation (3))

$= (1,\infty,2)(3,4,0)\infty\underline{01\infty 2}3\infty 10\infty = (1,\infty,2)(3,4,0)\infty\ \underline{(1,3,\infty,0,2)\infty 103}$ $3\infty 10\infty$ (by relation (2) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (1,0,\infty)(2,3,4)0\infty 10\underline{33}\infty 10\infty = (1,0,\infty)(2,3,4)0\infty 10\infty 10\infty = e$ by relation 3.

t_1, a representative from one of the 2-orbits, takes $[\infty 01\infty 212]$ to a single coset in $[\infty 01031]$ since we previously proved $\infty 010312 = (2,\infty,0,3,4)1\infty 21424$. Therefore it follows $N\infty 01031 = N(2,\infty,0,3,4)1\infty 214242 = N1\infty 214242 \in [\infty 01\infty 2121] = [\infty 01031]$.

t_∞, a representative of the other 2-orbit, takes $[\infty 01\infty 212]$ to a single coset in $[\infty 01403]$ since we previously proved $\infty 014034 = (1,3,4)(2,0,\infty)4204\infty 0\infty$. Therefore it follows $N\infty 01403 = N(1,3,4)(2,0,\infty)4204\infty 0\infty 4 = N4204\infty 0\infty 4 \in [\infty 01\infty 212\infty] = [\infty 01403]$.

Now consider $[\infty 01\infty 201]$. $N^{(\infty 01\infty 201)}$ has orbits $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty 01\infty 2011]$, $[\infty 01\infty 2012]$, $[\infty 01\infty 2013]$, $[\infty 01\infty 2014]$, $[\infty 01\infty 2010]$, and $[\infty 01\infty 201\infty]$.

First $[\infty 01\infty 2011] = [\infty 01\infty 20]$, so t_1 takes $[\infty 01\infty 201]$ back to a single coset in $[\infty 01\infty 20]$.

t_2 takes $[\infty 01\infty 201]$ to a single coset in $[\infty 0\infty 121]$ since we previously proved $\infty 0\infty 1210 = (1,4,\infty)(2,3,0)4124012$. Therefore it follows $N\infty 0\infty 121$ $= N(1,4,\infty)(2,3,0)4124\ 0120 = N41240120 \in [\infty 01\infty 2012] = [\infty 0\infty 121]$.

t_∞ takes $[\infty 01\infty 201]$ to a single coset in $[\infty 01310]$ since we previously proved $\infty 013104 = (1,\infty,0,4,2)4204\ \infty 20$. Therefore it follows $N\infty 01310 = N(1,\infty,0,4,2)4204$ $\infty 204 = N4204\ \infty 204 \in [\infty 01\infty 201\infty] = [\infty 01310]$.

t_0 takes $[\infty 01\infty 201]$ to a single coset in $[\infty 01213]$ since we previously proved $\infty 01213\infty = (1,3)(4,0)1\infty 21\ 4\infty 2$. Therefore it follows $N\infty 01213 = N(1,3)(4,0)1\infty 21$ $4\infty 2\infty = N1\infty 214\ \infty 2\infty \in [\infty 01\infty 2010] = [\infty 01213]$.

t_4 takes $[\infty 01\infty 201]$ to a single coset in $[\infty 010\infty 2]$ since we previously proved $\infty 010\infty 21 = (1,3)(4,0)0320432$. Therefore it follows $N\infty 010\infty 2 = N(1,3)(4,0)0320$ $4321 = N0320\ 4321 \in [\infty 01\infty 2014] = [\infty 010\infty 2]$.

t_3 takes $[\infty 01\infty 201]$ to a single coset in $[\infty 0\infty 121]$ since we previously proved $\infty 0\infty 121\infty = (1,0,\infty,4,3)4124012$. Therefore it follows $N\infty 0\infty 121$ $= N(1,0,\infty,4,3)4124\ 012\infty = N4124012\infty \in [\infty 01\infty 2013] = [\infty 0\infty 121]$.

Now consider $[\infty 01\infty 2123]$. $N^{(\infty 01\infty 2123)}$ has orbits $\{2\}$ and $\{1,3,4,0,\infty\}$. So we need to look at $[\infty 01\infty 21232]$ and $[\infty 01\infty 21233]$.

First $[\infty 01\infty 21233] = [\infty 01\infty 212]$, so t_3, a representative of the 5-orbit, takes $[\infty 01\infty 2123]$ back to a single coset in $[\infty 01\infty 212]$.

t_2 takes $[\infty 01\infty 2123]$ to a single coset in $[\infty 01\infty 21232]$. All of the single cosets in $[\infty 01\infty 21232]$ are equal since we previously proved $N\infty 01\infty\ 2123 = N4314\ 2120$ $= N34\infty 3\ 2\infty 20 = N10\infty 1\ 2\infty 24 = N3103\ 202\infty = N0130\ 2324 = N4\infty 04\ 2021 =$ $N\infty 43\infty\ 2321 = N1341\ 242\infty = N0\infty 40\ 3423$. Therefore it follows that $N\infty 01\infty\ 21232$ $= N4314\ 21202 = N34\infty 3\ 2\infty 202 = N10\infty 1\ 2\infty 242 = N3103\ 202\infty 2 = N0130\ 23242 =$ $N4\infty 04\ 20212 = N\infty 43\infty\ 23212 = N1341\ 242\infty 2 = N0\infty 40\ 34232$. If we conjugate this equation by $(1,0,4,3,2)$, $(1,0,2,\infty,4)$, $(1,3,4)(2,0,\infty)$, $(2,3)(0,\infty)$, and $(1,3,2,4,\infty) \in$ $L_2(5)$ we obtain $N\infty 40\infty\ 10121 = N3203\ 12141 = N23\infty 2\ 1\infty 141 = N04\infty 0\ 1\infty 131$ $= N20421\ 41\infty 1 = N4024\ 12131 = N3\infty 43\ 14101 = N\infty 32\infty\ 12101 = N02301\ 31\infty 1$ $= N4\infty 34\ 13121$, $N4204\infty\ 0\infty 3\infty = N1301\infty\ 0\infty 2\infty = N3143\infty\ 4\infty 2\infty = N0240\infty$ $4\infty 1\infty = N3023\infty\ 2\infty 4\infty = N2032\infty\ 3\infty 1\infty = N1421\infty\ 2\infty 0\infty = N4134\infty\ 3\infty 0\infty$

$= N0310\infty\ 1\infty4\infty = N2412\infty\ 1\infty3\infty$, $N2\infty320\ 3040 = N1431\ 030\infty0 = N4124\ 020\infty0$ $= N3\infty23\ 02010 = N43\infty40\ \infty020 = N\infty34\infty\ 04010 = N12\infty10\ \infty030 = N21420\ 4030$ $= N3413\ 01020 = N\infty21\infty\ 01040$, $N0\infty10\ 31323 = N4214\ 313\infty3 = N2402\ 303\infty3$ $= N1\infty01\ 30343 = N21\infty2\ 3\infty303 = N\infty12\infty\ 32343 = N40\infty4\ 3\infty313 = N04203\ 2313$ $= N12413\ 4303 = N\infty04\infty\ 34323$, and $N1031\ 43424 = N\infty23\infty\ 43404 = N2\infty12\ 41404$ $= N30134\ 14\infty4 = N2302\ 40414 = N0320\ 424\infty4 = N\infty10\infty\ 40434 = N1\infty21\ 42434$ $= N32\infty34\ \infty414 = N01\infty04\ \infty424$. To prove that all these single cosets are equal we will prove the following relations: $\infty01\infty\ 21232 = (1,0,\infty)(2,3,4)\infty40\infty\ 10121$, $\infty01\infty\ 21232 = (1,3,\infty,0,2)4204\infty\ 0\infty3\infty$, $\infty01\infty\ 21232 = (1,0)(2,4)2\infty32\ 03040$, $\infty01\infty\ 21232 = (1,3,\infty,0,2)0\infty103\ 1323$, and $\infty01\infty\ 21232 = (2,4,3,0,\infty)10314\ 3424$.

To prove $\infty01\infty21232 = (1,0,\infty)(2,3,4)\infty40\infty10121$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,\infty)(2,3,4)\infty40\infty1012123\underline{212}\infty10\infty = (1,0,\infty)(2,3,4)\infty40\infty\ 1012123\ \underline{121}$ $\infty10\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,0,\infty)(2,3,4)\infty40\infty1012\underline{1231}21\infty10\infty = (1,0,\infty)(2,3,4)\infty40\infty\ 1012$ $\underline{(1,3,2)(4,0,\infty)2132}$

$21\infty10\infty$ (by relation (4) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (1,\infty,3,0,4)40\infty43\infty31213\underline{22}1\infty10\infty = (1,\infty,3,0,4)40\infty\ 43\infty\ 312\ \underline{131}$ $\infty10\infty = (1,\infty,3,0,4)40\infty\ 43\infty\ 312\ \underline{313}$

$\infty10\infty$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,\infty,3,0,4)40\infty43\infty\underline{3123}13\infty10\infty = (1,\infty,3,0,4)40\infty\ 43\infty$ $\underline{(1,3,2)(4,0,\infty)1321}\ 13\infty10\infty$ (by relation (4) conjugated by $(1,4)(2,3) \in L_2(5)$)

$= (1,4,3,\infty,2)0\infty4\underline{0241}32\underline{11}3\infty10\infty = (1,4,3,\infty,2)0\infty4\ \underline{(1,2,3,4,0)4203}$ $323\infty10\infty$ (by relation (2))

$= (1,0)(3,\infty)\underline{1\infty04}20\underline{33}23\infty10\infty = (1,0)(3,\infty)\underline{(1,4,\infty,2,0)0\infty1}22023\infty10\infty$ (by relation (2) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$)

$= (2,0,4,\infty,3)0\infty1\underline{22}023\infty10\infty = (2,0,4,\infty,3)\underline{0\infty10}\ 23\infty10\infty = (2,0,4,\infty,3)$ $\underline{(1,\infty,0)(2,4,3)\infty01\infty}\ 23\infty10\infty$ (by relation (3))

$= (1,\infty,2)(3,4,0)\infty\underline{01\infty2}3\infty10\infty = (1,\infty,2)(3,4,0)\infty\ \underline{(1,3,\infty,0,2)\infty103}$ $3\infty10\infty$ (by relation (2) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= (1,0,\infty)(2,3,4)0\infty10\underline{33}\infty10\infty = (1,0,\infty)(2,3,4)0\infty10\infty10\infty = e$ by relation (3).

To prove $\infty 01\infty 21232 = (1,3,\infty,0,2)4204\infty 0\infty 3\infty$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,\infty,0,2)4204\infty 0\underline{\infty 3\infty}\ \underline{232}\ 12\infty 10\infty = (1,3,\infty,0,2)4204\infty 0\ \underline{3\infty 3}\ \underline{323}\ 12\infty 10\infty$ (by relation (5) conjugated by $(1,2)(3,0)$ and $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,3,\infty,0,2)4204\infty 03\infty\underline{33}2312\infty 10\infty = (1,3,\infty,0,2)4204\infty 0\ \underline{3\infty 23}\ 12\infty 10\infty = (1,3,\infty,0,2)4204\infty 0\ \underline{(1,4,0)(2,\infty,3)\infty 32\infty}\ 12\infty 10\infty$ (by relation (3) conjugated by $(1,2,0,\infty,3) \in L_2(5)$)

$= (1,2,4,0,\infty)0\infty 1031\infty 3\underline{2\infty 12}\infty 10\infty = (1,2,4,0,\infty)0\infty\ 1031\infty 3\ \underline{(1,\infty,2)(3,4,0)\infty 21\infty}\ \infty 10\infty$ (by relation (3) conjugated by $(2,4,3,0,\infty) \in L_2(5)$)

$= (2,0)(3,4)32\infty 3\underline{4\infty 24}\infty 21\underline{\infty\infty}10\infty = (2,0)(3,4)32\infty 3\ \underline{(1,3,0)(2,\infty,4)\infty 42\infty}\ \infty 2110\infty$ (by relation (4) conjugated by $(1,3,2,4,\infty) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)0\infty 40\infty 42\underline{\infty\infty}2\underline{11}0\infty = (1,3,2)(4,0,\infty)0\infty\ 40\infty 4\ \underline{22}0\infty = (1,3,2)(4,0,\infty)0\infty\ 40\infty\ 40\infty = e$ by relation (3) conjugated by $(1,4)(2,3) \in L_{(}5)$.

To prove $\infty 01\infty 21232 = (1,0)(2,4)2\infty 3203040$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0)(2,4)2\infty 32030\underline{4023}212\infty 10\infty = (1,0)(2,4)2\infty\ 32030\ \underline{(2,4,3,0,\infty)204\infty}\ 212\infty 10\infty$ (by relation (2) conjugated by $(1,3,\infty)(2,0,4) \in L_2(5)$)

$= (1,\infty,2,3,0)4204\underline{\infty 0\infty}204\infty 212\infty 10\infty = (1,\infty,2,3,0)4204\ \underline{0\infty 0}\ 204\infty\ 212\infty\ 10\infty$ (by relation (5))

$= (1,\infty,2,3,0)42040\infty\underline{020}4\infty 212\infty 10\infty = (1,\infty,2,3,0)42040\infty\ \underline{202}\ 4\infty 212\infty\ 10\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,\infty,2,3,0)\underline{4204}0\infty 2024\infty 212\infty 10\infty = (1,\infty,2,3,0)\underline{(1,\infty,3)(2,4,0)2402}\ 0\infty 2024\ \infty 212\ \infty 10\infty$ (by relation (3) conjugated by $(1,0,2,\infty,4) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)240\underline{20\infty 2}024\infty 212\infty 10\infty = (1,3,2)(4,0,\infty)240\ \underline{(1,4,3)(2,\infty,0)02\infty 0}\ 024\infty\ 212\infty\ 10\infty$ (by relation (4) conjugated by $(1,4,0,3,\infty) \in L_2(5)$)

$= (2,4)(3,\infty)\infty 3202\infty\underline{00}24\infty 212\infty 10\infty = (2,4)(3,\infty)\infty 3202\ \underline{\infty 24\infty}\ 212\infty\ 10\infty = (2,4)(3,\infty)\infty 3202\ \underline{(1,3,0)(2,\infty,4)2\infty 42}\ 212\infty\ 10\infty$ (by relation (4) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= (1,3,4,\infty,0)40\infty 1\underline{\infty 2\infty}4\underline{22}12\infty 10\infty = (1,3,4,\infty,0)40\infty 1\ \underline{2\infty 2}\ 412\infty 10\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,3,4,\infty,0)40\underline{\infty 12\infty}2412\infty 10\infty = (1,3,4,\infty,0)40\ \underline{(1,\infty,2)(3,4,0)1\infty 21}$

$2412\ \infty 10\infty$ (by relation (3) conjugated by $(1,2,3,4,0) \in L_2(5)$)

$= (1,4,2)(3,0,\infty)031\infty 2\underline{1241}2\infty 10\infty = (1,4,2)(3,0,\infty)031\ \infty 2$ $\underline{(1,4,2)(3,0,\infty)2142}\ 2\infty 10\infty$ (by relation (3) conjugated by $(1,4,\infty)(2,3,0) \in L_2(5)$)

$= (1,2,4)(3,\infty,0)\infty 043\underline{121}4\underline{22}\infty 10\infty = (1,2,4)(3,\infty,0)\infty 043\underline{2124}\infty 10\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,2,4)(3,\infty,0)\infty\underline{0432}124\infty 10\infty = (1,2,4)(3,\infty,0)\infty\ \underline{(1,3,0,2,4)3401}\ 124\infty\ 10\infty$ (by relation (1) conjugated by $(1,4)(2,3) \in L_2(5)$)

$= (1,4,3,\infty,2)\infty 340\underline{1124\infty 1}0\infty = (1,4,3,\infty,2)\infty 340\ \underline{(1,4,3,\infty,2)\infty 423}\ 0\infty$ (by relation (1) conjugated by $(1,4,3)(2,\infty,0) \in L_2(5)$)

$= (1,3,2,4,\infty)2\underline{\infty 30\infty}4230\infty = (1,3,2,4,\infty)2\underline{(1,2,4)(3,\infty,0)3\infty}034230\infty$ (by relation (4) conjugated by $(1,2,\infty)(3,0,4) \in L_2(5)$)

$= (1,\infty,2)(3,4,0)43\underline{\infty 034}230\infty = (1,\infty,2)(3,4,0)43\underline{(2,3,\infty,4,0)30\infty}2230\infty$ (by relation (1) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$)

$= (1,4,2)(3,0,\infty)0\infty 30\infty\underline{22}30\infty = (1,4,2)(3,0,\infty)0\infty 30\infty 30\infty = e$ by relation (4) conjugated by $(2,\infty)(4,0) \in L_2(5)$.

To prove $\infty 01\infty 21232 = (1,3,\infty,0,2)0\infty 1031323$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,\infty,0,2)0\infty 103132\underline{323}212\infty 10\infty = (1,3,\infty,0,2)0\infty 10\ 3132\ \underline{232}$ $212\infty 10\infty$ (by relation (5) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (1,3,\infty,0,2)0\infty 10313\underline{22}3\underline{22}12\infty 10\infty = (1,3,\infty,0,2)0\infty\ 1031\ \underline{33}\ 12\infty\ 10\infty$ $= (1,3,\infty,0,2)0\infty\ 103\ \underline{11}\ 2\infty\ 10\infty = (1,3,\infty,0,2)0\ \underline{\infty 103}\ 2\infty 10\infty = (1,3,\infty,0,2)0$ $\underline{(1,2,0,\infty,3)01\infty 2}\ 2\infty 10\infty$ (by relation (2) conjugated by $(1,3,2)(4,0,\infty) \in L_2(5)$)

$= \infty 01\infty\underline{22}\infty 10\infty = \infty 01\underline{\infty\infty}10\infty = \infty 0\underline{11}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

To prove $\infty 01\infty 21232 = (2,4,3,0,\infty)103143424$, we will move the relation to one side of the equal sign and prove it equals identity.

$(2,4,3,0,\infty)103143\underline{424}23212\infty 10\infty = (2,4,3,0,\infty)103143\underline{24}223212\infty 10\infty$ (by relation (6) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (2,4,3,0,\infty)1031\underline{432422}3212\infty 10\infty = (2,4,3,0,\infty)1031\ \underline{(1,0,\infty)(2,3,4)3423}$ $3212\ \infty 10\infty$ (by relation (4) conjugated by $(1,\infty,0)(2,4,3) \in L_2(5)$)

$= (1,0)(3,\infty)0\infty 40342\underline{33}212\infty 10\infty = (1,0)(3,\infty)0\infty 4034\ \underline{22}\ 12\infty\ 10\infty$ $= (1,0)(3,\infty)0\infty 40\ \underline{3412}\ \infty 10\infty = (1,0)(3,\infty)0\infty 40\ \underline{(1,3,2,4,\infty)142\infty}\ \infty 10\infty$ (by relation (2) conjugated by $(1,2,4)(3,\infty,0) \in L_2(5)$)

$= (1,0,3)(2,4,\infty)\underline{01\infty0}143\underline{\infty\infty}10\infty = (1,0,3)(2,4,\infty)\ \underline{(1,0,\infty)(2,3,4)10\infty1}$ 14310∞ (by relation (3) conjugated by $(1,\infty,0)(2,4,3) \in L_2(5)$)

$= (1,\infty,3,0,4)10\infty\underline{11}4310\infty = (1,\infty,3,0,4)10\infty4310\infty = e$ by relation (2) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$.

There is one distinct single coset in $[\infty01\infty21232]$.

Finally, consider $[\infty01\infty21232]$. $N^{(\infty01\infty21232)}$ has one orbit $\{1,2,3,4,0,\infty\}$, so we need to look at $[\infty01\infty212322]$.

$[\infty01\infty212322] = [\infty01\infty2123]$ so t_2, a representative of the 6-orbit, takes $[\infty01\infty21232]$ back to a single coset in $[\infty01\infty2123]$.

Our double coset enumeration must be complete since the set of right cosets is closed under right multiplication by the symmetric generators.

Thus we have the Cayley diagram that is shown in Figure 6.1.

The maximum possible index of N in $G \cong \frac{2^{*6}:L_2(5)}{[(0,1,2,3,4)t_0]^8,[(\infty,0,1)(2,4,3)t_\infty]^8,[(\infty,0)(1,4)t_1]^6}$ is $\frac{|N|}{|N|} + \frac{|N|}{|N^{(\infty)}|} + \frac{|N|}{|N^{(\infty0)}|} + \frac{|N|}{|N^{(\infty0\infty)}|} + \frac{|N|}{|N^{(\infty02)}|} + \cdots + \frac{|N|}{|N^{(\infty01\infty21232)}|} = 1 + 6 + 30 + 60 + 60 + \cdots + 1 = 3168$. Thus $|G| \leq 3168 \times |N| = 3168 \times 60 = 190,080$. In order to show $|G| = 190,080$, we consider G as a subgroup of S_{3168} acting on 3168 cosets that we have found, and labeled according to the MAGMA segment, "for i in [1..3168] do print i, cst[i]; end for;".

For this purpose we compute the action of the control group N as well as the action of $t_\infty, t_0, t_1, t_2, t_3$, and t_4 on the 3168 cosets. These permutations can be obtained by the following MAGMA segment "f(x); f(y); f(t);".

It readily checks that the order of $\langle x, y, t\rangle$, a subgroup of the symmetric group S_{3168} acting on the 3168 right cosets of N in G, is 190,080. Visibly $|x| = 5$, $|y| = 3$, and $|xy| = 2$, hence $\langle x, y\rangle \cong L_2(5)$. If we conjugate t by $L_2(5)$ we see that t has exactly six conjugates. We conclude that $\langle x, y, t\rangle$ is a homomorphic image of the progenitor $2^{*6} : L_2(5)$.

Thus if the original six relations hold in $\langle x, y, t\rangle$, then $\langle x, y, t\rangle$ is a homomorphic image of G and this will give $|G| \geq |\langle x, y, t\rangle| = 190,080$.

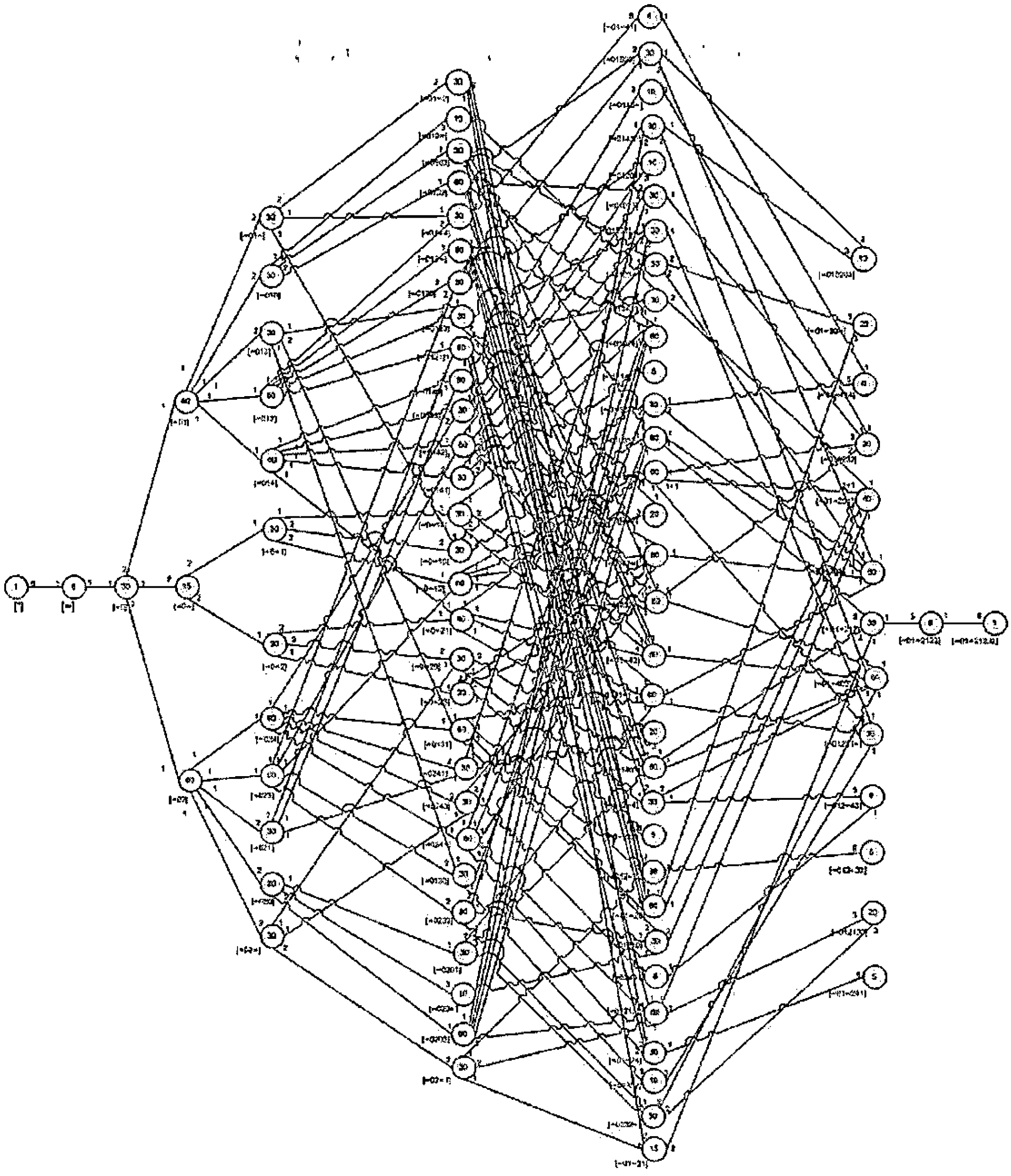

Figure 6.1: Cayley Diagram of of $2 \times M_{12}$ over $L_2(5)$

Verify relation (1) $t_2t_1t_0t_4t_3t_2t_1t_0 = (1,3,0,2,4)$ by conjugating the six symmetric generators by $t_2t_1t_0t_4t_3t_2t_1t_0$. By multiplying the permutations found using the MAGMA segments previously listed, we find $t_2t_1t_0t_4t_3t_2t_1t_0 = (1,3,0,2,4)$. Thus when

we conjugate $t_\infty, t_0, t_1, t_2, t_3$ and t_4 by $t_2t_1t_0t_4t_3t_2t_1t_0$ we obtain:

$t_\infty^{t_2t_1t_0t_4t_3t_2t_1t_0} = t_\infty$ $\quad t_0^{t_2t_1t_0t_4t_3t_2t_1t_0} = t_2$
$t_1^{t_2t_1t_0t_4t_3t_2t_1t_0} = t_3$ $\quad t_2^{t_2t_1t_0t_4t_3t_2t_1t_0} = t_4$
$t_3^{t_2t_1t_0t_4t_3t_2t_1t_0} = t_0$ $\quad t_4^{t_2t_1t_0t_4t_3t_2t_1t_0} = t_1$

So $t_2t_1t_0t_4t_3t_2t_1t_0$ acts as $(1,3,0,2,4)$.

Verify relation (2) $t_4t_2t_0t_3t_1t_4t_2t_0 = (1,0,4,3,2)$ by conjugating the six symmetric generators by $t_4t_2t_0t_3t_1t_4t_2t_0$. By multiplying the permutations found using the MAGMA segments previously listed, we find $t_4t_2t_0t_3t_1t_4t_2t_0 = (1,0,4,3,2)$. Thus when we conjugate $t_\infty, t_0, t_1, t_2, t_3$ and t_4 by $t_4t_2t_0t_3t_1t_4t_2t_0$ we obtain:

$t_\infty^{t_4t_2t_0t_3t_1t_4t_2t_0} = t_\infty$ $\quad t_0^{t_4t_2t_0t_3t_1t_4t_2t_0} = t_4$
$t_1^{t_4t_2t_0t_3t_1t_4t_2t_0} = t_0$ $\quad t_2^{t_4t_2t_0t_3t_1t_4t_2t_0} = t_1$
$t_3^{t_4t_2t_0t_3t_1t_4t_2t_0} = t_2$ $\quad t_4^{t_4t_2t_0t_3t_1t_4t_2t_0} = t_3$

So $t_4t_2t_0t_3t_1t_4t_2t_0$ acts as $(1,0,4,3,2)$.

Verify relation (3) $t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty = (1,\infty,0)(2,4,3)$ by conjugating the six symmetric generators by $t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty$. By multiplying the permutations found using the MAGMA segments previously listed, we find $t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty = (1,\infty,0)(2,4,3)$. Thus when we conjugate $t_\infty, t_0, t_1, t_2, t_3$ and t_4 by $t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty$ we obtain:

$t_\infty^{t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty} = t_0$ $\quad t_0^{t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty} = t_1$
$t_1^{t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty} = t_\infty$ $\quad t_2^{t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty} = t_4$
$t_3^{t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty} = t_2$ $\quad t_4^{t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty} = t_3$

So $t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty$ acts as $(1,\infty,0)(2,4,3)$.

Verify relation (4) $t_4t_2t_3t_4t_2t_3t_4t_2 = (1,\infty,0)(2,4,3)$ by conjugating the six symmetric generators by $t_4t_2t_3t_4t_2t_3t_4t_2$. By multiplying the permutations found using the MAGMA segments previously listed, we find $t_4t_2t_3t_4t_2t_3t_4t_2 = (1,\infty,0)(2,4,3)$. Thus when we conjugate $t_\infty, t_0, t_1, t_2, t_3$ and t_4 by $t_4t_2t_3t_4t_2t_3t_4t_2$ we obtain the following results:

$t_\infty^{t_4t_2t_3t_4t_2t_3t_4t_2} = t_0$ $\quad t_0^{t_4t_2t_3t_4t_2t_3t_4t_2} = t_1$
$t_1^{t_4t_2t_3t_4t_2t_3t_4t_2} = t_\infty$ $\quad t_2^{t_4t_2t_3t_4t_2t_3t_4t_2} = t_4$
$t_3^{t_4t_2t_3t_4t_2t_3t_4t_2} = t_2$ $\quad t_4^{t_4t_2t_3t_4t_2t_3t_4t_2} = t_3$

So $t_4t_2t_3t_4t_2t_3t_4t_2$ acts as $(1,\infty,0)(2,4,3)$.

Verify relation (5) $t_0t_\infty t_0t_\infty t_0t_\infty = 1$ by conjugating the six symmetric generators by $t_0t_\infty t_0t_\infty t_0t_\infty$. By multiplying the permutations found using the MAGMA segments previously listed, we find $t_0t_\infty t_0t_\infty t_0t_\infty$ equals identity. Thus when we conjugate $t_\infty, t_0, t_1, t_2, t_3$ and t_4 by $t_0t_\infty t_0t_\infty t_0t_\infty$ we obtain the following results:

$t_{\infty}^{t_0t_\infty t_0t_\infty t_0t_\infty} = t_\infty$ $\quad$ $t_0^{t_0t_\infty t_0t_\infty t_0t_\infty} = t_0$
$t_1^{t_0t_\infty t_0t_\infty t_0t_\infty} = t_1$ $\quad$ $t_2^{t_0t_\infty t_0t_\infty t_0t_\infty} = t_2$
$t_3^{t_0t_\infty t_0t_\infty t_0t_\infty} = t_3$ $\quad$ $t_4^{t_0t_\infty t_0t_\infty t_0t_\infty} = t_4$

So $t_0t_\infty t_0t_\infty t_0t_\infty$ acts as the identity.

Verify relation (6) $t_4t_1t_4t_1t_4t_1 = 1$ by conjugating the six symmetric generators by $t_4t_1t_4t_1t_4t_1$. By multiplying the permutations found using the MAGMA segments previously listed, we find $t_4t_1t_4t_1t_4t_1$ equals identity. Thus when we conjugate $t_\infty, t_0, t_1, t_2, t_3$ and t_4 by $t_4t_1t_4t_1t_4t_1$ we obtain the following results:

$t_{\infty}^{t_4t_1t_4t_1t_4t_1} = t_\infty$ $\quad$ $t_0^{t_4t_1t_4t_1t_4t_1} = t_0$
$t_1^{t_4t_1t_4t_1t_4t_1} = t_1$ $\quad$ $t_2^{t_4t_1t_4t_1t_4t_1} = t_2$
$t_3^{t_4t_1t_4t_1t_4t_1} = t_3$ $\quad$ $t_4^{t_4t_1t_4t_1t_4t_1} = t_4$

So $t_4t_1t_4t_1t_4t_1$ acts as the identity.

Thus $G/ker\phi \cong \langle x, y, t\rangle$ and $|G| \geq |\langle x, y, t\rangle| = 190080$. As shown earlier, $|G| \leq 190080$. Hence $|G| = 190080$.

Moreover, $b = (1,3,\infty,0,2)t_\infty t_0t_1t_3t_0t_2 = bb = x^{-2}yt_\infty t_0t_1t_3t_0t_2$, $c = (1,2,4)(3,\infty,0)t_1t_2t_4t_1t_\infty t_2 = cc = yx^{-1}yx^2t_1t_2t_4t_1t_\infty t_2$, and $a = (1,4,\infty)(2,3,0)t_\infty t_0t_1t_\infty t_2t_1t_2t_3t_2 = aa = yx^{-2}yx^{-1}t_\infty t_0t_1t_\infty t_2t_1t_2t_3t_2$ are in G with $2XM_{12} \cong \langle b, c, a | b^2 = c^3 = (bc)^{11} = [b,c]^6 = (bcbcbc^{-1})^6 = [b, cbc]^5 = 1 = a^2 = [b,a] = [c,a]\rangle$. So $\langle b,c,a\rangle \leq G$, but $|\langle b,c,a\rangle| = |G|$, therefore $G = \langle b,c,a\rangle \cong 2XM_{12}$.

Chapter 7

Construction of M_{12}

It is known the group constructed in Chapter 6 factored by its center, $Z(G)$, is isomporphic to M_{12}. We will construct by hand M_{12} using the technique of manual double coset enumeration of $G_1 \cong \frac{2^{*6}:L_2(5)}{[(0,1,2,3,4)t_0]^5,[(\infty,0,1)(2,4,3)t_\infty]^8,[(\infty,0)(1,4)t_1]^6,Z(G)}$ over $L_2(5)$.

We found the center using MAGMA (see Appendix B and [C+05]). Simplifying this expression we obtained the following relation: $(2,3,\infty,4,0)t_2t_1t_0t_1t_3t_2t_\infty t_1t_0t_3t_\infty$ $t_0t_4t_0t_3t_2t_3t_4t_3t_2t_3t_0t_4 = 1$. We can further simplify this relation as follows:

$(2,3,\infty,4,0)2101\underline{32\infty1}03\infty040323432304 = (2,3,\infty,4,0)2101$ $\underline{(3,1,2,0,\infty)\infty230}\ 03\infty040323\ 432304$ (by relation (2) conjugated by $(2,0)(4,\infty) \in L_2(5)$)

$= (1,2)(4,\infty)02\infty2\infty23\underline{00}3\infty040323432304 = (1,2)(4,\infty)02\ \underline{\infty2\infty}\ 2\underline{33}\ \infty04032$ $3432304 = (1,2)(4,\infty)02\ \underline{2\infty2}\ 2\infty040323\ 432304$ (by relation (5) conjugated by $(2,0)(3,4)$ $\in L_2(5)$)

$= (1,2)(4,\infty)0\underline{22}\infty\underline{22}\infty040323432304 = (1,2)(4,\infty)0\ \underline{\infty\infty}\ 04032\ 3432304 =$ $(1,2)(4,\infty)\underline{00}\ 40\underline{323}\ 432304 = (1,2)(4,\infty)40\ \underline{232}\ 432304$ (by relation (5) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (1,2)(4,\infty)\underline{4023}2432304 = (1,2)(4,\infty)\underline{(4,3,0,\infty,2)204\infty}2432304$ (by relation (2) conjugated by $(1,3,\infty)(2,0,4) \in L_2(5)$)

$= (1,4,2)(3,0,\infty)204\infty24\underline{323}04 = (1,4,2)(3,0,\infty)204\infty24\underline{232}04$ (by relation (5) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (1,4,2)(3,0,\infty)204\infty242\underline{3204} = (1,4,2)(3,0,\infty)204\infty242\underline{(3,4,2,\infty,0)023\infty}$ (by relation (2) conjugated by $(1,4,0,3,\infty) \in L_2(5)$)

$= (1,2)(4,\infty)\infty320\infty\underline{2\infty02}3\infty = (1,2)(4,\infty)\infty320\infty\underline{(1,3,4)(2,0,\infty)\infty20}\infty3\infty$

(by relation (4) conjugated by $(3,0)(4,\infty) \in L_2(5)$)

$= (1,0,\infty)(2,3,4)240\underline{\infty 2\infty}20\infty 3\infty = (1,0,\infty)(2,3,4)2402\underline{\infty 22}0\infty 3\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,0,\infty)(2,3,4)2402\infty\underline{22}0\infty 3\infty = (1,0,\infty)(2,3,4)2402\infty 0\infty 3\infty$.

Hence $Z(G) = \langle (1,0,\infty)(2,3,4)t_2t_4t_0t_2t_\infty t_0t_\infty t_3t_\infty \rangle$.

So now we factor $2^{*6} : L_2(5)$ by the six relations of Chapter 6 as well as the center. So now our relations are as follows:

1. $(0,3,1,4,2)t_2t_1t_0t_4t_3t_2t_1t_0 = 1$
2. $(0,1,2,3,4)t_4t_2t_0t_3t_1t_4t_2t_0 = 1$
3. $(\infty,1,0)(2,3,4)t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty = 1$
4. $(\infty,1,0)(2,3,4)t_4t_2t_3t_4t_2t_3t_4t_2 = 1$
5. $t_0t_\infty t_0t_\infty t_0t_\infty = 1$
6. $t_4t_1t_4t_1t_4t_1 = 1$
7. $(1,0,\infty)(2,3,4)t_2t_4t_0t_2t_\infty t_0t_\infty t_3t_\infty = 1$

Equivalenty, relation (7) can be expressed as $\infty 3\infty 0\infty = (1,0,\infty)(2,3,4)2402$ or $2402\infty = (1,\infty,0)(2,4,3)\infty 3\infty 0$.

We now perform the manual double coset enumeration of
$G_1 = \frac{2^{*6}:L_2(5)}{[(0,1,2,3,4)t_0]^8,[(\infty,0,1)(2,4,3)t_\infty]^8,[(\infty,0)(1,4)t_1]^6,Z(G)}$ over $L_2(5)$

The double coset enumeration of this group is identical to the double coset enumeration of Chapter 6, up through the double cosets of word length three. So I will begin my double coset enumeration by considering $[\infty 0\infty 1]$. The orbits of $N^{(\infty 0\infty 1)}$ are $\{1\}$, $\{4\}$, $\{2,3\}$, and $\{0,\infty\}$. So we need to look at $[\infty 0\infty 11]$, $[\infty 0\infty 14]$, $[\infty 0\infty 12]$, and $[\infty 0\infty 10]$.

First, $[\infty 0\infty 11] = [\infty 0\infty]$ so t_1 takes $N\infty 0\infty 1 \in [\infty 0\infty 1]$ back to $N\infty 0\infty \in [\infty 0\infty]$.

t_4 takes $N\infty 0\infty 1 \in [\infty 0\infty 1]$ to $N\infty 0\infty 14 \in [\infty 0\infty 14]$. There are 30 distinct single cosets in $[\infty 0\infty 14]$ as was proved in Chapter 6.

t_2, a representative from one of the 2-orbits, takes $[\infty 0 \infty 1]$ to a single coset in $[\infty 0 \infty 12]$. There are 60 distinct single cosets in $[\infty 0 \infty 12]$.

t_∞, a representative from the other 2-orbit, takes $[\infty 0 \infty 1]$ to a single coset in $[\infty 02 \infty]$ since $N\infty 0 \infty 10 = N(1,0,4)(2,3,\infty)3213 = N3213 \in [\infty 02\infty]$. To prove $\infty 0\infty 10 = (1,0,4)(2,3,\infty)3213$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,4)(2,3,\infty)321301\underline{\infty 0 \infty} = (1,0,4)(2,3,\infty)3213010\underline{\infty}0$ (by relation (5))

$= (1,0,4)(2,3,\infty)3213\underline{010}\infty 0 = (1,0,4)(2,3,\infty)3213\underline{101}\infty 0$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)32\underline{131}01\infty 0 = (1,0,4)(2,3,\infty)32\underline{313}01\infty 0$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)\underline{32313}01\infty 0 = (1,0,4)(2,3,\infty)\ \underline{(1,3,4)(2,0,\infty)\infty 01\infty}\ 01\infty 0$ (by relation (7) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)\underline{\infty 01\infty}01\infty 0 = (1,\infty,0)(2,4,3)\ \underline{(1,0,\infty)(2,3,4)0\infty 10}\ 01\infty 0$ (by relation (3))

$= 0\infty 1\underline{00}1\infty 0 = 0\infty\underline{11}\infty 0 = 0\underline{\infty\infty}0 = 00 = e.$

Now consider $[\infty 0\infty 2]$. The orbits of $N^{(\infty 0 \infty 2)}$ are $\{2\}$, $\{3\}$, $\{1,4\}$, and $\{0,\infty\}$. So we need to look at $[\infty 0\infty 22]$, $[\infty 0\infty 23]$, $[\infty 0\infty 21]$, and $[\infty 0\infty 20]$.

First, $[\infty 0\infty 22] = [\infty 0 \infty]$ so t_2 takes $[\infty 0\infty 2]$ back to a single coset in $[\infty 0\infty]$.

t_3 takes $[\infty 0\infty 2]$ to a single coset in $[\infty 0\infty 23]$. There are 30 distinct single cosets in $[\infty 0\infty 23]$ as was proved in Chapter 6.

t_1, a representative from one of the 2-orbits, takes $[\infty 0\infty 2]$ to a single coset in $[\infty 0\infty 21]$. There are 60 distinct single cosets in $[\infty 0\infty 23]$.

t_0, a representative from the other 2-orbit, takes $[\infty 0\infty 2]$ to a single coset in $[\infty 01\infty]$ since $N\infty 0\infty 20 = N(1,0,\infty)(2,3,4)2402 = N2402 \in [\infty 01\infty]$. To prove $\infty 0\infty 20 = (1,0,\infty)(2,3,4)2402$, we will start with relation (7) as follows:

$\underline{\infty 3\infty}0\infty = (1,0,\infty)(2,3,4)2402$

$\Longrightarrow \underline{3\infty 3}0\infty = (1,0,\infty)(2,3,4)\underline{2402}$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$\Longrightarrow 3\infty 30\infty = (1,0,\infty)(2,3,4)\underline{(1,3,\infty)(2,0,4)4204}$ (by relation (3) conjugated by $(1,0,4)(2,3,\infty) \in L_2(5)$)

$\Longrightarrow 3\infty 30\infty = (1,0,\infty)(2,3,4)4124.$

Now consider $[\infty 012]$. The orbits of $N^{(\infty 012)}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty 0121]$, $[\infty 0122]$, $[\infty 0123]$, $[\infty 0124]$, $[\infty 0120]$, and $[\infty 0 \infty 2 \infty]$.

First, $[\infty 0122] = [\infty 01]$ so t_2 takes $[\infty 012]$ back to a single coset in $[\infty 01]$.

t_1 takes $[\infty 012]$ to a single coset in $[\infty 0121]$. There are 60 distinct single cosets in $[\infty 0121]$.

t_3 takes $[\infty 012]$ to a single coset in $[\infty 0123]$. There are 30 distinct single cosets in $[\infty 0123]$ as was proved in Chapter 6.

t_4 takes $[\infty 012]$ to a single coset in $[\infty 0124]$. There are 30 distinct single cosets in $[\infty 0124]$ as was proved in Chapter 6.

t_0 takes $[\infty 012]$ to a single coset in $[\infty 0120]$. There are 30 distinct single cosets in $[\infty 0123]$ as was proved in Chapter 6.

t_∞ takes $[\infty 012]$ to a single coset in $[\infty 012\infty]$. There are 60 distinct single cosets in $[\infty 012\infty]$.

Now consider $[\infty 023]$. The orbits of $N^{(\infty 023)}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty 0231]$, $[\infty 0232]$, $[\infty 0233]$, $[\infty 0234]$, $[\infty 0230]$, and $[\infty 023\infty]$.

First, $[\infty 0233] = [\infty 02]$, so t_3 takes $[\infty 023]$ back to a single coset in $[\infty 02]$.

t_1 takes $[\infty 023]$ to a single coset in $[\infty 0120]$ as was proved in Chapter 6.

t_2 takes $[\infty 023]$ to a single coset in $[\infty 0232]$. There are 30 distinct single cosets in $[\infty 0232]$ as was proved in Chapter 6.

t_4 takes $[\infty 023]$ to a single coset in $[\infty 0234]$. There are 60 distinct single cosets in $[\infty 0234]$.

t_0 takes $[\infty 023]$ to a single coset in $[\infty 020]$ since $N\infty 0230$ $= N(1,2,\infty)(3,0,4)\infty 131 = N\infty 131 \in [\infty 020]$. To prove $\infty 0230$ $= (1,2,\infty)(3,0,4)\infty 131$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,\infty)(3,0,4)\infty\underline{131}0320\infty = (1,2,\infty)(3,0,4)\infty\underline{313}0320\infty$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)\infty 31\underline{303}20\infty = (1,2,\infty)(3,0,4)\infty 31\underline{030}20\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)\infty 3103\underline{020}\infty = (1,2,\infty)(3,0,4)\infty 3103\underline{202}\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)\infty\underline{3103}202\infty = (1,2,\infty)(3,0,4)\infty\underline{(1,3,0)(2,\infty,4)1301}202\infty$ (by relation (3) conjugated by $(1,0)(3,\infty) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)4\underline{1301}202\infty = (1,\infty,3)(2,4,0)4\underline{(1,3,\infty)(2,0,4)2\infty20}02\infty$ (by relation (7) conjugated by $(1,4,3,\infty,2) \in L_2(5)$)

$= \underline{22}\infty2\underline{00}2\infty = \infty\underline{22}\infty = \underline{\infty\infty} = e.$

t_∞ takes $[\infty023]$ to a single coset in $[\infty0\infty12]$ as was proved in Chapter 6.

Now consider $[\infty021]$. The orbits of $N^{(\infty021)}$ are $\{4\}$, $\{0\}$, $\{1,3\}$, and $\{2,\infty\}$. So we need to look at $[\infty0214]$, $[\infty0210]$, $[\infty0211]$, and $[\infty0212]$.

First, $[\infty0211] = [\infty02]$ so t_1, a representative from one of the 2-orbits, takes $[\infty021]$ back to a single coset in $[\infty02]$.

t_4 takes $[\infty021]$ to a single coset in $[\infty0214]$. There are 30 distinct single cosets in $[\infty0214]$ as was proved in Chapter 6.

t_0 takes $[\infty021]$ to a single coset in $[\infty010]$ since $N\infty0210 = N(1,\infty,3)(2,4,0)4313 = N4313 \in [\infty010]$. To prove $\infty0210 = (1,\infty,3)(2,4,0)4313$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,3)(2,4,0)43\underline{13012}0\infty = (1,\infty,3)(2,4,0)43\underline{(1,3,\infty)(2,0,4)2\infty20}0\infty$ (by relation (7) conjugated by $(1,4,3,\infty,2) \in L_2(5)$)

$= \underline{2\infty2}\infty2\underline{00}\infty = \underline{\infty2\infty}\infty2\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= \infty2\underline{\infty\infty}2\infty = \infty\underline{22}\infty = \underline{\infty\infty} = e.$

t_2, a representative from the other 2-orbit, takes $[\infty021]$ to a single coset in $[\infty0121]$ as was proved in Chapter 6.

Now consider $[\infty024]$. The orbits of $N^{(\infty024)}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty0241]$, $[\infty0242]$, $[\infty0243]$, $[\infty0244]$, $[\infty0240]$, and $[\infty024\infty]$.

First, $[\infty0244] = [\infty02]$ so t_4 takes $[\infty024]$ back to a single coset in $[\infty02]$.

t_1 takes $[\infty024]$ to a single coset in $[\infty0214]$ as was proved in Chapter 6.

t_2 takes $[\infty024]$ to a single coset in $[\infty0242]$. There are 60 distinct single cosets in $[\infty0242]$.

t_3 takes $[\infty024]$ to a single coset in $[\infty0243]$. There are 30 distinct single cosets in $[\infty0243]$ as was proved in Chapter 6.

t_0 takes $[\infty024]$ to a single coset in $[\infty0240]$. There are 30 distinct single cosets in $[\infty0240]$ as was proved in Chapter 6.

t_∞ takes $[\infty 024]$ to a single coset in $[\infty 024\infty]$. There are 60 distinct single cosets in $[\infty 024\infty]$.

Now consider $[\infty 013]$. The orbits of $N^{(\infty 013)}$ are $\{2\}$, $\{0\}$, $\{1,\infty\}$, and $\{3,4\}$. So we need to look at $[\infty 0132]$, $[\infty 0130]$, $[\infty 0131]$, and $[\infty 0133]$.

First, $[\infty 0133] = [\infty 01]$ so t_4, a representative from one of the 2-orbits, takes $[\infty 013]$ back to a single coset in $[\infty 01]$.

t_2 takes $[\infty 013]$ to a single coset in $[\infty 0123]$ as was proved in Chapter 6.

t_0 takes $[\infty 013]$ to a single coset in $[\infty 020]$ since $N\infty 0130$ $= N(1,2,0)(3,\infty,4)2434 = N2434 \in [\infty 020]$. To prove $\infty 0130 = (1,2,0)(3,\infty,4)2434$, we will start with relation (7).

$\infty 3\infty 0\infty = (1,0,\infty)(2,3,4)2402$

$\Longrightarrow \underline{\infty}\infty 3\infty 0\infty = \underline{\infty}(1,0,\infty)(2,3,4)2402$

$\Longrightarrow 3\infty 0\infty = (1,0,\infty)(2,3,4)12402$

Conjugated this equation by $(1,\infty,4)(2,0,3) \in L_2(5)$,

$\Longrightarrow 2434 = (1,0,2)(3,4,\infty)\infty 0130$

Multiply both sides of this equation by $(1,2,0)(3,\infty,4) \in L_2(5)$,

$\Longrightarrow \infty 0130 = (1,2,0)(3,\infty,4)2434.$

t_1, a representative from the other 2-orbit, takes $[\infty 013]$ to a single coset in $[\infty 0242]$ as was proved in Chapter 6.

Now consider $[\infty 014]$. The orbits of $N^{(\infty 014)}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty 0141]$, $[\infty 0142]$, $[\infty 0143]$, $[\infty 0144]$, $[\infty 0140]$, and $[\infty 014\infty]$.

First, $[\infty 0144] = [\infty 01]$, so t_4 takes $[\infty 014]$ back to a single coset in $[\infty 01]$.

t_1 takes $[\infty 014]$ to a single coset in $[\infty 0141]$. There are 30 distinct single cosets in $[\infty 0141]$ as was proved in Chapter 6.

t_2 takes $[\infty 014]$ to a single coset in $[\infty 0234]$ as was proved in Chapter 6.

t_3 takes $[\infty 014]$ to a single coset in $[\infty 0240]$ as was proved in Chapter 6.

t_0 takes $[\infty 014]$ to a single coset in $[\infty 010]$ since $N\infty 0140$ $= N(1,\infty,3)(2,4,0)\infty 343 = N\infty 343 \in [\infty 010]$. To prove $\infty 0140$ $= (1,\infty,3)(2,4,0)\infty 343$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,3)(2,4,0)\infty\underline{343}0410\infty = (1,\infty,3)(2,4,0)\infty\underline{434}0410\infty$ (by relation (6) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)\infty\underline{4340}410\infty = (1,\infty,3)(2,4,0)\infty\underline{(1,3,\infty)(2,0,4)1\infty011}0\infty$ (by relation (7) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= \underline{11}\infty0\underline{11}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

t_∞ takes $[\infty014]$ to a single coset in $[\infty0\infty21]$ as was proved in Chapter 6.

Now consider $[\infty010]$. The orbits of $N^{(\infty010)}$ are $\{3\}$, $\{\infty\}$, $\{1,0\}$, and $\{2,4\}$. So we need to look at $[\infty0103]$, $[\infty010\infty]$, $[\infty0100]$, and $[\infty0102]$.

First, $[\infty0100] = [\infty01]$ so t_0, a representative from one of the 2-orbits, takes $[\infty010]$ back to a single coset in $[\infty01]$.

t_3 takes $[\infty010]$ to a single coset in $[\infty021]$ since $N\infty0103$ $= N(1,0,4)(2,3,\infty)4321 = N4321 \in [\infty021]$. To prove $\infty0103 = (1,0,4)(2,3,\infty)4321$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,4)(2,3,\infty)4\underline{3213}010\infty = (1,0,4)(2,3,\infty)4\ \underline{(1,4,0)(2,\infty,3)\infty0\infty10}$ 010∞ (by a previously proved relation)

$= \underline{0\infty0}\infty1\underline{00}10\infty = \underline{\infty0\infty}\infty\underline{11}0\infty$ (by relation (5))

$= \infty0\underline{\infty\infty\infty}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

t_∞ takes $[\infty010]$ to a single coset in $[\infty010\infty]$. There are 10 distinct single cosets in $[\infty010\infty]$ as was proved in Chapter 6.

t_2, a representative from the other 2-orbit, takes $[\infty010]$ to a single coset in $[\infty014]$ since $N\infty0102 = N(1,4,2)(3,0,\infty)\infty231 = N\infty231 \in [\infty014]$. To prove $\infty0102 = (1,4,2)(3,0,\infty)\infty231$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,2)(3,0,\infty)\infty2312\underline{010}\infty = (1,4,2)(3,0,\infty)\infty2312\underline{101}\infty$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (1,4,2)(3,0,\infty)\infty23\underline{121}01\infty = (1,4,2)(3,0,\infty)\infty23\underline{212}01\infty$ (By relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,4,2)(3,0,\infty)\infty\underline{23212}01\infty = (1,4,2)(3,0,\infty)\infty\underline{(1,2,4)(3,\infty,0)0\infty1}001\infty$ (by relation (7) conjugated by $(1,4,\infty,2,0) \in L_2(5)$)

$= \underline{00}\infty1\underline{00}1\infty = \infty\underline{11}\infty = \underline{\infty\infty} = e.$

Now consider $[\infty020]$. The orbits of $N^{(\infty020)}$ are $\{1\}$, $\{\infty\}$, $\{2,0\}$, and $\{3,4\}$. So we need to look at $[\infty0201]$, $[\infty020\infty]$, $[\infty0200]$, and $[\infty0203]$.

First, $[\infty0200] = [\infty02]$ so t_0, a representative from one of the 2-orbits, takes $[\infty020]$ back to a single coset in $[\infty02]$.

t_∞ takes $[\infty 020]$ to a single coset in $[\infty 020\infty]$. There are 10 distinct single cosets in $[\infty 020\infty]$ as was proved in Chapter 6.

t_1 takes $[\infty 020]$ to a single coset in $[\infty 013]$ since $N\infty 0201$ $= N(1,\infty,3)(2,4,0)4130 = N4130 \in [\infty 013]$. To prove $\infty 0201 = (1,\infty,3)(2,4,0)4130$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,3)(2,4,0)41301\underline{020}\infty = (1,\infty,3)(2,4,0)41301\underline{202}\infty$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)4\underline{1301}202\infty = (1,\infty,3)(2,4,0)4\underline{(1,3,\infty)(2,0,4)2\infty 20}02\infty$ (by relation (7) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= \underline{22}\infty 2\underline{00}2\infty = \infty\underline{22}\infty = \underline{\infty\infty} = e.$

t_3, a representative from the other 2-orbit, takes $[\infty 020]$ to a single coset in $[\infty 023]$ since $N\infty 0203 = N(1,2,\infty)(3,0,4)\infty 310 = N\infty 310 \in [\infty 023]$. To prove $\infty 0203 = (1,2,\infty)(2,0,3)2312$, we will start with a previously proven relation, $\infty 0230$ $= (1,2,\infty)(3,0,4)\infty\underline{131}$.

$\infty 0230 = (1,2,\infty)(3,0,4)\infty\underline{313}$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$\Longrightarrow \infty 0230\underline{3} = (1,2,\infty)(3,0,4)\infty 31\underline{33}$

$\Longrightarrow \infty 023\underline{030} = (1,2,\infty)(3,0,4)\infty 31\underline{0}$

$\Longrightarrow \infty 023\underline{303} = (1,2,\infty)(3,0,4)\infty 310$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$\Longrightarrow \infty 02\underline{33}03 = (1,2,\infty)(3,0,4)\infty 310$

$\Longrightarrow \infty 0203 = (1,2,\infty)(3,0,4)\infty 310.$

Now consider $[\infty 01\infty]$. The orbits of $N^{(\infty 01\infty)}$ are $\{1\}$, $\{4\}$, $\{2,3\}$, and $\{0,\infty\}$. So we need to look at $[\infty 01\infty 1]$, $[\infty 01\infty 4]$, $[\infty 01\infty 2]$, and $[\infty 01\infty\infty]$.

First, $[\infty 01\infty\infty] = [\infty 01]$ so t_∞, a representative from one of the 2-orbits, takes $[\infty 01\infty]$ back to a single coset in $[\infty 01]$.

t_4 takes $[\infty 01\infty]$ to a single coset in $[\infty 0124]$ as was proved in Chapter 6.

t_1 takes $[\infty 01\infty]$ to a single coset in $[\infty 02\infty]$ since $N\infty 01\infty 1$ $= N(1,\infty,4)(2,0,3)2312 = N2312 \in [\infty 02\infty]$. To prove $\infty 01\infty 1 = (1,\infty,4)(2,0,3)2312$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,4)(2,0,3)23\underline{121}\infty 10\infty = (1,\infty,4)(2,0,3)23\underline{212}\infty 10\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,\infty,4)(2,0,3)\underline{23}212\infty10\infty = (1,\infty,4)(2,0,3)\underline{32}312\infty10\infty$ (by relation (5) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,\infty,4)(2,0,3)32312\underline{\infty10\infty} = (1,\infty,4)(2,0,3)32312\ \underline{(1,\infty,0)(2,4,3)1\infty01}$ (by relation (3) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)242\underline{\infty41\infty}01 = (1,0,2)(3,4,\infty)242\ \underline{(1,4,\infty)(2,3,0)4\infty14}01$ (by relation (4) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,2,4)(3,\infty,0)3\underline{\infty34\infty1}401 = (1,2,4)(3,\infty,0)3\underline{(1,4,2)(3,0,\infty)1014}401$ (by relation (7) conjugated by $(1,2,\infty)(3,0,4) \in L_2(5)$)

$= \underline{010}1\underline{44}01 = \underline{101}101$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= 10\underline{11}01 = 1\underline{00}1 = \underline{11} = e.$

t_2, a representative from the other 2-orbit, takes $[\infty01\infty]$ to a single coset in $[\infty0\infty2]$ since $N\infty01\infty2 = N(1,\infty,4)(2,0,4)2321 = N2321 \in [\infty0\infty2]$. To prove $\infty01\infty2 = (1,\infty,4)(2,0,4)2321$, we will start with a previously proven relation,

$\infty01\infty1 = (1,\infty,4)(2,0,3)2312$

$\infty01\infty\underline{11} = (1,\infty,4)(2,0,3)231\underline{21}$

$\Longrightarrow \infty01\infty = (1,\infty,4)(2,0,3)23\underline{212}$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$\Longrightarrow \infty01\infty\underline{2} = (1,\infty,4)(2,0,3)23212\underline{2}$

$\Longrightarrow \infty01\infty2 = (1,\infty,4)(2,0,3)2321.$

Now consider $[\infty02\infty]$. The orbits of $N^{(\infty02\infty)}$ are $\{2\}$, $\{3\}$, $\{1,4\}$, and $\{0,\infty\}$. So we need to look at $[\infty02\infty2]$, $[\infty02\infty3]$, $[\infty02\infty1]$, and $[\infty02\infty\infty]$.

First, $[\infty02\infty\infty] = [\infty02]$ so t_∞, a representative from one of the 2-orbits, takes $[\infty02\infty]$ back to a single coset in $[\infty02]$.

t_2 takes $[\infty02\infty]$ to a single coset in $[\infty01\infty]$ since $N\infty02\infty2 = N(1,2,3)(4,\infty,0)1421 = N1421 \in [\infty01\infty]$. To prove $\infty02\infty2 = (1,2,3)(4,\infty,0)1421$, we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,3)(4,\infty,0)1421\underline{2\infty2}0\infty = (1,2,3)(4,\infty,0)1421\underline{\infty2\infty}0\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)1421\infty2\underline{\infty0\infty} = (1,2,3)(4,\infty,0)1421\infty2\ \underline{0\infty0}$ (by relation (5))

$= (1,2,3)(4,\infty,0)14\underline{21\infty20}\infty0 = (1,2,3)(4,\infty,0)14\ \underline{(1,3,2)(4,0,\infty)030\infty}\ \infty0$ (by relation (7) conjugated by $(1,4)(0,\infty) \in L_2(5)$)

$= 3\underline{00}30\underline{\infty\infty}0 = \underline{3300} = e.$

t_3 takes $[\infty02\infty]$ to a single coset in $[\infty0243]$ as was proved in Chapter 6.

t_1, a representative from the other 2-orbit, takes $[\infty02\infty]$ to a single coset in $[\infty0\infty1]$ since $N\infty02\infty1 = N(1,2,3)(4,\infty,0)1412 = N1412 \in [\infty0\infty1]$. To prove $\infty02\infty1 = (1,2,3)(4,\infty,0)1412$, we will start with a previously proven relation, $\infty02\infty2 = (1,2,3)(4,\infty,0)1421$.

$\infty02\infty\underline{2\underline{2}} = (1,2,3)(4,\infty,0)14\underline{21\underline{2}}$

$\Longrightarrow \infty02\infty = (1,2,3)(4,\infty,0)14\underline{121}$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$\Longrightarrow \infty02\infty\underline{1} = (1,2,3)(4,\infty,0)1412\underline{1\underline{1}}$

$\Longrightarrow \infty02\infty1 = (1,2,3)(4,\infty,0)1412.$

Now consider $[\infty0\infty21]$. The orbits of $N^{(\infty0\infty21)}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty0\infty211]$, $[\infty0\infty212]$, $[\infty0\infty213]$, $[\infty0\infty214]$, $[\infty0\infty210]$, and $[\infty0\infty21\infty]$.

First, $[\infty0\infty211] = [\infty0\infty2]$ so t_1, takes $[\infty0\infty21]$ back to a single coset in $[\infty0\infty2]$.

t_2 takes $[\infty0\infty21]$ to a single coset in $[[\infty0\infty212]$. There are 60 distinct single cosets in $[\infty0\infty212]$.

t_3 takes $[\infty0\infty21]$ to a single coset in $[\infty014]$ as was proved in Chapter 6.

t_4 takes $[\infty0\infty21]$ to a single coset in $[\infty0\infty214]$. There are 20 distinct single cosets in $[\infty0\infty214]$ as was proved in Chapter 6.

t_0 takes $[\infty0\infty21]$ to a single coset in $[\infty0242]$ since $N\infty0\infty210 = N(1,\infty,3)(2,4,0)24313 = N24313 \in [\infty0242]$. To prove $\infty0\infty210 = (1,\infty,3)(2,4,0)24313$, we will start with a previously proven relation, $\infty0210 = (1,\infty,3)(2,4,0)4313$.

$\Longrightarrow \underline{\underline{0}\infty}0210 = \underline{0}(1,\infty,3)(2,4,0)4313.$

$\Longrightarrow \underline{\infty0\infty}210 = (1,\infty,3)(2,4,0)24313.$(by relation (5))

t_∞ takes $[\infty0\infty21]$ to a single coset in $[\infty012\infty]$ since $N\infty0\infty21\infty = N(1,2,3)(4,\infty,0)142\infty1 = N142\infty1 \in [\infty012\infty]$. To prove $\infty0\infty21\infty = (1,2,3)(4,\infty,0)142\infty1$, we will start with a previously proven relation, $\infty02\infty2 = (1,2,3)(4,\infty,0)1421$.

$\Longrightarrow \underline{\underline{0}\infty}02\infty\underline{2\underline{2}} = \underline{0}(1,2,3)(4,\infty,0)1421\underline{2}.$

$\Longrightarrow \underline{\infty 0 \infty} 2\infty = (1,2,3)(4,\infty,0)\underline{414}212$. (by relation (5))

$\Longrightarrow \infty 0 \infty 2\underline{\infty\underline{\infty}} = (1,2,3)(4,\infty,0)\underline{141}212\underline{\infty}$.(by relation (6))

$\Longrightarrow \infty 0 \infty 2 = (1,2,3)(4,\infty,0)141\underline{212}\infty$.(by relation (6))

$\Longrightarrow \infty 0 \infty 2 = (1,2,3)(4,\infty,0)141\underline{121}\infty$.(by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$\Longrightarrow \infty 0 \infty 2\underline{1\infty} = (1,2,3)(4,\infty,0)14\underline{1}121\infty\underline{1\infty}$.

$\Longrightarrow \infty 0 \infty 21\infty = (1,2,3)(4,\infty,0)142\underline{1\infty 1}\infty$.

$\Longrightarrow \infty 0 \infty 21\infty = (1,2,3)(4,\infty,0)142\underline{\infty 1\infty}\infty$.(by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$\Longrightarrow \infty 0 \infty 21\infty = (1,2,3)(4,\infty,0)142\infty 1\underline{\infty\infty}$.

$\Longrightarrow \infty 0 \infty 21\infty = (1,2,3)(4,\infty,0)142\infty 1$.

Now consider $[\infty 0 \infty 23]$. The orbits of $N^{(\infty 0 \infty 23)}$ are $\{2\}$, $\{3\}$, $\{1,4\}$, and $\{0,\infty\}$. So we need to look at $[\infty 0 \infty 232]$, $[\infty 0 \infty 233]$, $[\infty 0 \infty 231]$, and $[\infty 0 \infty 230]$.

First, $[\infty 0 \infty 233] = [\infty 0 \infty 2]$ so t_3 takes $[\infty 0 \infty 23]$ back to a single coset in $[\infty 0 \infty 2]$.

t_2 takes $[\infty 0 \infty 23]$ to a single coset in $[\infty 0 \infty 232]$. There are 5 distinct single cosets in $[\infty 0 \infty 232]$ as was proved in Chapter 6.

t_1, a representative from one of the 2-orbits, takes $[\infty 0 \infty 23]$ to a single coset in $[\infty 024\infty]$ since $N\infty 0 \infty 231 = N(1,\infty,2,3,0)03\infty 20 = N03\infty 20 \in [\infty 024\infty]$. To prove $\infty 0 \infty 231 = (1,\infty,2,3,0)03\infty 202$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,2,3,0)03\underline{\infty 201}32\infty 0\infty = (1,\infty,2,3,0)03\underline{(1,2,4,0,\infty)02\infty 432}\infty 0\infty$ (by relation (1) conjugated by $(1,2,0,\infty,3) \in L_2(5)$)

$= (2,3,\infty,4,0)\infty 302\infty 432\underline{\infty 0\infty} = (2,3,\infty,4,0)\infty 302\infty 432\underline{0\infty 0}$ (by relation (5))

$= (2,3,\infty,4,0)\infty 302\underline{\infty 4320}\infty 0 = (2,3,\infty,4,0)\infty 302\ \underline{(1,0,\infty,4,3)\infty 21\infty 3}\ \infty 0$ (by a previously proved relation)

$= (1,0,2)(3,4,\infty)41\underline{\infty 2\infty}21\infty 3\infty 0 = (1,0,2)(3,4,\infty)41\underline{2\infty 2}21\infty 3\infty 0$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)412\infty\underline{22}1\infty 3\infty 0 = (1,0,2)(3,4,\infty)412\ \underline{\infty 1\infty 3\infty}0$

$= (1,0,2)(3,4,\infty)412\ \underline{(1,2,0)(3,\infty,4)02300}$ (by relation (7) conjugated by $(1,4,2,0,3) \in L_2(5)$)

$= 32\underline{00}23\underline{00} = 3\underline{22}3 = \underline{33} = e.$

t_0, a representative from the other 2-orbit, takes $[\infty 0 \infty 23]$ to a single coset in $[\infty 0121]$ since $N\infty 0\infty 230 = N(1,2,\infty)(3,0,4)4\infty 313 = N4\infty 313 \in [\infty 0121]$. To prove $\infty 0\infty 230 = (1,2,\infty)(3,0,4)4\infty 313$, we will start with a previously proven relation, $\infty 0230 = (1,2,\infty)(3,0,4)\infty 131$.

$\Longrightarrow \underline{0}\infty 0230 = \underline{0}(1,2,\infty)(3,0,4)\infty 131.$

$\Longrightarrow \underline{0\infty 0}230 = (1,2,\infty)(3,0,4)4\infty 131.$

$\Longrightarrow \underline{\infty 0\infty}230 = (1,2,\infty)(3,0,4)4\infty 131.$(by relation (5))

$\Longrightarrow \infty 0\infty 230 = (1,2,\infty)(3,0,4)4\infty\underline{131}$

$\Longrightarrow \infty 0\infty 230 = (1,2,\infty)(3,0,4)4\infty\underline{313}$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$).

Now consider $[\infty 0\infty 14]$. The orbits of $N^{(\infty 0\infty 14)}$ are $\{1\}$, $\{4\}$, $\{2,3\}$, and $\{0,\infty\}$. So we need to look at $[\infty 0\infty 141]$, $[\infty 0\infty 144]$, $[\infty 0\infty 142]$, and $[\infty 0\infty 140]$.

First, $[\infty 0\infty 144] = [\infty 0\infty 1]$ so t_4 takes $[\infty 0\infty 14]$ back to a single coset in $[\infty 0\infty 1]$.

t_1 takes $[\infty 0\infty 14]$ to a single coset in $[\infty 0\infty 141]$. There are 5 distinct single cosets in $[\infty 0\infty 141]$ as was proved in Chapter 6.

t_2, a representative from one of the 2-orbits, takes $[\infty 0\infty 14]$ to a single coset in $[\infty 012\infty]$ since $N\infty 0\infty 142 = N(1,4,\infty,2,0)\infty 401\infty = N\infty 401\infty \in [\infty 012\infty]$. To prove $\infty 0\infty 142 = (1,4,\infty,2,0)\infty 401\infty$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,\infty,2,0)\infty 401\infty\underline{241\infty 0\infty}\infty = (1,4,\infty,2,0)\infty 401\infty\ \underline{(1,3,0,2,4)2\infty 321}\infty$ (by a previously proved relation $\infty 0124 = (1,3,4,\infty,0)\infty 23\infty 1$ conjugated by $(2,\infty)(4,0) \in L_2(5)$)

$= (3,0)(4,\infty)\infty 1\underline{23\infty 2}\infty 321\infty = (3,0)(4,\infty)\infty 1\underline{(1,4,0)(2,\infty,3)32\infty 3}\infty 321\infty$ (by relation (3) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)3432\infty\underline{3\infty 3}21\infty = (1,4,3)(2,\infty,0)3432\infty\underline{\infty 3\infty}21\infty$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)3432\underline{\infty\infty\infty}3\infty 21\infty = (1,4,3)(2,\infty,0)\underline{34323}\ \infty 21\infty$

$= (1,4,3)(2,\infty,0)\ \underline{(1,4,\infty)(2,3,0)1\infty 21}\ \infty 21\infty$ (by relation (7) conjugated by $(1,0,2)(3,4,\infty) \in L_2(5)$)

$= ((1,\infty,2)(3,4,0)1\underline{\infty 21\infty}21\infty = ((1,\infty,2)(3,4,0)1\underline{(1,2,\infty)(3,0,4)2\infty 122}1\infty$

(by relation (3) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= \underline{22}\infty 1\underline{22}1\infty = \infty\underline{11}\infty = \underline{\infty\infty} = e.$

t_0, a representative of the other 2-orbit, takes $[\infty 0\infty 14]$ to a single coset in $[\infty 0242]$ since $N\infty 0\infty 140 = N(1,\infty,3)(2,4,0)2\infty 434 = N2\infty 434 \in [\infty 0242]$. To prove $\infty 0\infty 140 = (1,\infty,3)(2,4,0)2\infty 434$, we will start with a previously proven relation, $\infty 0140 = (1,\infty,3)(2,4,0)\infty 343$.

$\Longrightarrow \underline{0}\infty 0140 = \underline{0}(1,\infty,3)(2,4,0)\infty 343$

$\Longrightarrow \underline{0\infty 0}140 = (1,\infty,3)(2,4,0)2\infty 343$

$\Longrightarrow \underline{\infty 0\infty}140 = (1,\infty,3)(2,4,0)2\infty\underline{343}$ (by relation (5))

$\Longrightarrow \infty 0\infty 140 = (1,\infty,3)(2,4,0)2\infty\ \underline{434}$ (by relation (6) conjugated by $(1,3)(2,\infty) \in L_2(5)$).

Now consider $[\infty 0\infty 12]$. The orbits of $N^{(\infty 0\infty 12)}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty 0\infty 121]$, $[\infty 0\infty 122]$, $[\infty 0\infty 123]$, $[\infty 0\infty 124]$, $[\infty 0\infty 120]$, and $[\infty 0\infty 12\infty]$.

First, $[\infty 0\infty 122] = [\infty 0\infty 1]$ so t_2 takes $[\infty 0\infty 12]$ back to a single coset in $[\infty 0\infty 1]$.

t_1 takes $[\infty 0\infty 12]$ to a single coset in $[\infty 0\infty 212]$ as was proved in Chapter 6.

t_3 takes $[\infty 0\infty 12]$ to a single coset in $[\infty 0\infty 123]$. There are 20 distinct single cosets in $[\infty 0\infty 123]$ as was proved in Chapter 6.

t_4 takes $[\infty 0\infty 12]$ to a single coset in $[\infty 023]$ as was proved in Chapter 6.

t_0 takes $[\infty 0\infty 12]$ to a single coset in $[\infty 024\infty]$ since $N\infty 0\infty 120$ $= N(1,4,2)(3,0,\infty)23102 = N23102 \in [\infty 024\infty]$. To prove $\infty 0\infty 120$ $= (1,4,2)(3,0,\infty)23102$, we will start with a previously proven relation, $\infty 0102$ $= (1,4,2)(3,0,\infty)\infty 231$, which gives

$\underline{0}\infty 0102 = \underline{0}(1,4,2)(3,0,\infty)\infty 231.$

$\Longrightarrow \underline{0\infty 0}102 = (1,4,2)(3,0,\infty)\underline{\infty\infty}231$

$\Longrightarrow \underline{\infty 0\infty}102\underline{02} = (1,4,2)(3,0,\infty)231\underline{02}$ (by relation (5))

$\Longrightarrow \infty 0\infty 10\underline{202} = (1,4,2)(3,0,\infty)23102$

$\Longrightarrow \infty 0\infty 10\underline{020} = (1,4,2)(3,0,\infty)23102$ (by relation (5) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$\Longrightarrow \infty 0\infty 1\underline{00}20 = (1,4,2)(3,0,\infty)23102$

$\Longrightarrow \infty 0\infty 120 = (1,4,2)(3,0,\infty)23102.$

t_∞ take $[\infty0\infty12]$ to a single coset in $[\infty0121]$ since $N\infty0\infty12\infty = N(1,3,\infty)(2,0,4)13424 = N13424 \in [\infty0121]$. To prove $\infty0\infty12\infty = (1,3,\infty)(2,0,4)13424$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,\infty)(2,0,4)13424\underline{\infty21\infty}0\infty = (1,3,\infty)(2,0,4)13424$ $\underline{(1,2,\infty)(3,0,4)2\infty12}0\infty$ (by relation (3) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,0,3)(2,4,\infty)203\infty3\underline{2\infty120}\infty = (1,0,3)(2,4,\infty)203\infty3$ $\underline{(1,3,0)(2,\infty,4)0401}\infty$ (by relation (7) conjugated by $(1,3,4,\infty,0) \in L_2(5)$)

$= \infty104\underline{00}401\infty = \infty10\underline{44}01\infty = \infty1\underline{00}1\infty = \infty\underline{11}\infty = \underline{\infty\infty} = e.$

Now consider $[\infty0121]$. The orbits of $N^{(\infty0121)}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty01211]$, $[\infty01212]$, $[\infty01213]$, $[\infty01214]$, $[\infty01210]$, and $[\infty0121\infty]$.

First, $[\infty01211] = [\infty012]$ so t_1 takes $[\infty0121]$ back to a single coset in $[\infty012]$.

t_2 takes $[\infty0121]$ to a single coset in $[\infty021]$ as was proved in Chapter 6.

t_3 takes $[\infty0121]$ to a single coset in $[\infty0\infty12]$ since $N\infty01213 = N(1,4,2)(3,0,\infty)343\infty2 = N343\infty2 \in [\infty0\infty12]$. To prove $\infty01213 = (1,4,2)(3,0,\infty)343\infty2$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,2)(3,0,\infty)34\underline{3\infty231}210\infty = (1,4,2)(3,0,\infty)34\ \underline{(1,2,4)(3,\infty,0)1012}\ 210\infty$ (by relation (7) conjugated by $(1,4,\infty)(2,3,0) \in L_2(5)$)

$= \infty\underline{11}01\underline{22}10\infty = \infty0\underline{11}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

t_4 takes $[\infty0121]$ to a single coset in $[\infty0\infty23]$ since $N\infty01214 = N(1,\infty,4)(2,0,3)40431 = N40431 \in [\infty0\infty23]$. To prove $\infty01214 = (1,\infty,4)(2,0,3)40431$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,4)(2,0,3)40\underline{4314}1210\infty = (1,\infty,4)(2,0,3)40\ \underline{(1,3,4)(2,0,\infty)3413}$ 1210∞ (by relation (3) conjugated by $(3,0)(4,\infty) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)1\infty34\underline{131121}0\infty = (1,2,\infty)(3,0,4)1\infty34\ \underline{(1,\infty,2)(3,4,0)0420}$ 0∞ (by relation (7) conjugated by $(1,\infty)(2,0) \in L_2(5)$)

$= \infty24\underline{00}42\underline{00}\infty = \infty2\underline{44}2\infty = \infty\underline{22}\infty = \underline{\infty\infty} = e.$

t_0 takes $[\infty0121]$ to a single coset in $[\infty01210]$. There are 30 distinct single cosets in $[\infty01210]$ as was proved in Chapter 6.

t_∞ takes $[\infty0121]$ back to a single coset in $[\infty0121]$ since $N\infty0121\infty$ $= N(2,4)(3,\infty)\infty0121 = N\infty0121 \in [\infty0121]$. To prove $\infty0121\infty = (2,4)(3,\infty)\infty0121$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(2,4)(3,\infty)\underline{\infty0121\infty}1210\infty = (2,4)(3,\infty)\underline{(1,\infty)(2,0)431302}1210\infty$ (by a relation previously proved in Chapter 6)

$= (1,\infty,3)(2,4,0)43130\underline{212}10\infty = (1,\infty,3)(2,4,0)43130\underline{121}10\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)43\underline{1301211}0\infty = (1,\infty,3)(2,4,0)43\underline{(1,3,\infty)(2,0,4)2\infty200}\infty$ (by relation (7) conjugated by $(1,4,3,\infty,2) \in L_2(5)$)

$= 2\underline{\infty2\infty}2\underline{00}\infty = 2\underline{2\infty2}2\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= \underline{22}\infty\underline{22}\infty = \underline{\infty\infty} = e.$

Now consider $[\infty0123]$. The orbits of $N^{(\infty0123)}$ are $\{1\}$, $\{\infty\}$, $\{2,0\}$, and $\{3,4\}$. So we need to look at $[\infty01231]$, $[\infty0123\infty]$, $[\infty01232]$, and $[\infty01233]$.

First, $[\infty01233] = [\infty012]$ so t_3, a representative from one of the 2-orbits, takes $[\infty0123]$ back to a single coset in $[\infty012]$.

t_1 takes $[\infty0123]$ to a single coset in $[\infty0232]$ since $N\infty01231$ $= N(1,2,\infty)(3,0,4)1\infty343 = N1\infty343 \in [\infty0232]$. To prove $\infty01231$ $= (1,2,\infty)(3,0,4)1\infty343$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,\infty)(3,0,4)1\infty343\underline{1321}0\infty = (1,2,\infty)(3,0,4)1\infty343$ $\underline{(1,2,3)(4,\infty,0)3123}\ 0\infty$ (by relation (4) conjugated by $(1,0,4,3,2) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)20\underline{1\infty131}230\infty = (1,3,4)(2,0,\infty)20\ \underline{(1,4,3)(2,\infty.0)2032}$ 230∞ (by relation (7) conjugated by $(1,4,0,3,\infty) \in L_2(5)$)

$= \infty\underline{22}032\underline{22}30\infty = \infty0\underline{33}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

t_∞ takes $[\infty0123]$ to a single coset in $[\infty013]$ as was proved in Chapter 6.

t_2, a representative from the other 2-orbit, takes $[\infty0123]$ to a single coset in $[\infty0\infty123]$ as was proved in Chapter 6.

Now consider $[\infty0124]$. The orbits of $N^{(\infty0124)}$ are $\{1\}$, $\{3\}$, $\{2,\infty\}$, and $\{4,0\}$. So we need to look at $[\infty01241]$, $[\infty01243]$, $[\infty01242]$, and $[\infty01244]$.

First, $[\infty01244] = [\infty012]$ so t_4, a representative from one of the 2-orbits, takes $[\infty0124]$ back to $[\infty012]$.

t_1 takes $[\infty0124]$ to a single coset in $[\infty01\infty]$ as was proved in Chapter 6.

t_3 takes $[\infty 0124]$ to a single coset in $[\infty 01243]$. There are 6 distinct single cosets in $[\infty 01243]$ as was proved in Chapter 6.

t_2, a representative from the other 2-orbit, takes $[\infty 0124]$ to a single coset in $[\infty 012\infty]$ since $N\infty 01242 = N(1,4,\infty,2,0)04\infty 10 = N04\infty 10 \in [\infty 012\infty]$. To prove $\infty 01242 = (1,4,\infty,2,0)04\infty 10$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,\infty,2,0)04\infty 1\underline{02421}0\infty = (1,4,\infty,2,0)04\infty 1\ \underline{(1,0,\infty)(2,3,4)31\infty 4}0\infty$ (by a previously proved relation, $\infty 0103 = (1,0,4)(2,3,\infty)4321$, conjugated by $(1,4,3)(2,\infty,4) \in L_2(5)$)

$= (1,2,\infty,3,4)\infty 210\underline{31\infty 4}0\infty = (1,2,\infty,3,4)\infty 210\underline{(1,2,\infty,3,4)\infty 132}0\infty$ (by relation (1) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= (1,\infty,4,2,3)3\underline{\infty 20\infty}1320\infty = (1,\infty,4,2,3)3\underline{(1,4,3)(2,\infty,0)2\infty 02}1320\infty$ (by relation (4) conjugated by $(1,3,0)(2,\infty,4) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)12\underline{\infty 021}320\infty = (1,0,2)(3,4,\infty)12\underline{(1,0,3,2,\infty)20\infty 3}320\infty$ (by relation (1) conjugated by $(1,0,\infty,4,3) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)0\infty 20\infty\underline{33}20\infty = (1,3,4)(2,0,\infty)0\infty 20\infty 20\infty = e$ by relation (4) conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$.

Now consider $[\infty 0232]$. The orbits of $N^{(\infty 0232)}$ are $\{0\}$, $\{\infty\}$, $\{1,4\}$, and $\{2,3\}$. So we need to look at $[\infty 02320]$, $[\infty 0232\infty]$, $[\infty 02321]$, and $[\infty 02322]$.

First, $[\infty 02322] = [\infty 023]$ so t_2, a representative from one of the 2-orbits, takes $[\infty 0232]$ back to a single coset in $[\infty 023]$.

t_0 takes $[\infty 0232]$ to a single coset in $[\infty 02320]$. There are 30 distinct single cosets in $[\infty 02320]$ as was proved in Chapter 6.

t_∞ takes $[\infty 0232]$ to a single coset in $[\infty 0123]$ since $N\infty 0232\infty$ $= N(1,\infty,0)(2,4,3)04\infty 13 = N04\infty 13 \in [\infty 0123]$. To prove $\infty 0232\infty$ $= (1,\infty,0)(2,4,3)04\infty 13$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,0)(2,4,3)04\infty 13\infty\underline{232}0\infty = (1,\infty,0)(2,4,3)04\infty 13\infty\underline{323}0\infty$ (by relation (5) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)04\infty 1\underline{3\infty 3}230\infty = (1,\infty,0)(2,4,3)04\infty 1\underline{\infty 3\infty}230\infty$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)04\underline{\infty 1\infty 3\infty}230\infty = (1,\infty,0)(2,4,3)04\ \underline{(1,2,0)(3,\infty,4)0230}$

230∞ (by relation (7) conjugated by $(1,4,2,0,3) \in L_2(5)$)

$= (1,4,\infty)(2,3,0)13\underline{0230}230\infty = (1,4,\infty)(2,3,0)13\underline{(1,\infty,4)(2,0,3)2032}230\infty$ (by relation (3) conjugated by $(1,3,\infty,0,2) \in L_2(5)$)

$= \infty\underline{22}032\underline{22}30\infty = \infty0\underline{33}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

t_1, a representative from the other 2-orbit, takes $[\infty0232]$ to a single coset in $[\infty0234]$ since $N\infty02321 = N(1,3,4)(2,0,\infty)201\infty3 = N201\infty3 \in [\infty0234]$. To prove $\infty02321 = (1,3,4)(2,0,\infty)201\infty3$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,4)(2,0,\infty)201\infty31\underline{232}0\infty = (1,3,4)(2,0,\infty)201\infty31\underline{323}0\infty$ (by relation (5) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (1,3,4)(2,0,\infty)201\infty\underline{31323}0\infty = (1,3,4)(2,0,\infty)201\infty \underline{(1,0,4)(2,3,\infty)4024}0\infty$ (by relation (7) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)3\underline{4024}0240\infty = (1,\infty,3)(2,4,0)3\underline{(1,3,\infty)(2,0,4)0420}0240\infty$ (by relation (3) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= \infty042\underline{00}240\infty = \infty04\underline{22}40\infty = \infty0\underline{44}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

Now consider $[\infty0234]$. The orbits of $N^{(\infty0234)}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty02341]$, $[\infty02342]$, $[\infty02343]$, $[\infty02344]$, $[\infty02340]$, and $[\infty0234\infty]$.

First, $[\infty02344] = [\infty023]$ so t_4 takes $[\infty0234]$ back to a single coset in $[\infty023]$. + t_1 takes $[\infty0234]$ to a single coset in $[\infty014]$ as was proved in Chapter 6.

t_2 takes $[\infty0234]$ to a single coset in $[\infty0232]$ since $N\infty02342 = N(1,4,2)(3,0,\infty)30\infty4\infty = N30\infty4\infty \in [\infty0232]$. To prove $\infty02342 = (1,4,2)(3,0,\infty)30\infty4\infty$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,2)(3,0,\infty)30\infty\underline{4\infty24}320\infty = (1,4,2)(3,0,\infty)30\infty \underline{(1,3,0)(2,\infty,4)\infty42\infty}\ 320\infty$ (by relation (4) conjugated by $(1,3,2,4,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)01\underline{4\infty04}2\infty320\infty = (1,2,3)(4,\infty,0)01\underline{\infty4\infty}2\infty320\infty$ (by relation (6) conjugated by $(1,\infty)(2,0) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)01\underline{\infty4\infty2\infty}320\infty = (1,2,3)(4,\infty,0)01\ \underline{(1,2,\infty)(3,0,4)0320}\ 320\infty$ (by relation (7) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,\infty,4)(2,0,3)42\underline{0320}320\infty = (1,\infty,4)(2,0,3)42\underline{(1,4,\infty)(2,3,0)3023}320\infty$ (by relation (3) conjugated by $(1,2,4)(3,\infty,0) \in L_2(5)$)

$= \infty\underline{33}02\underline{33}20\infty = \infty0\underline{22}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

t_3 takes $[\infty0234]$ to a single coset in $[\infty024\infty]$ since $N\infty02343$ $= N(2,3,\infty,4,0)03\infty20 = N03\infty20 \in [\infty024\infty]$. To prove $\infty02343$ $= (2,3,\infty,4,0)03\infty20$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(2,3,\infty,4,0)03\infty20\underline{343}20\infty = (2,3,\infty,4,0)03\infty20\underline{434}20\infty$ (by relation (6) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (2,3,\infty,4,0)03\underline{\infty2043}420\infty = (2,3,\infty,4,0)03\ \underline{(1,4,2)(3,0,\infty)402\infty1}\ 420\infty$ (by relation previously proved in Chapter 6, $\infty0234 = (1,3,0)(2,\infty,4)320\infty1$, conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,4,\infty,2,0)\infty0402\infty\underline{1420}\infty = (1,4,\infty,2,0)\infty0402\infty\ \underline{(1,0,4,3,\infty)2413}\infty$ (by relation (2) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,3,2,4,\infty)\infty4341\infty\underline{2413\infty} = (1,3,2,4,\infty)\infty4341\infty\underline{(1,0,4,3,2)\infty3401}$ (by a relation previously proved in Chapter 6, $\infty0132 = (1,4,0,3,\infty)23041$, conjugated by $(2,\infty)(4,0) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)\infty\underline{3230}\underline{\infty\infty}3401 = (1,2,3)(4,\infty,0)\infty\underline{23}203401$ (by relation (5) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)\infty2\underline{3203}401 = (1,2,3)(4,\infty,0)\infty2\underline{(1,3,2)(4,0,\infty)4140}01$ (by relation (7) conjugated by $(1,\infty,4,2,3) \in L_2(5)$)

$= \underline{41}414\underline{00}1 = \underline{141}141$ (by relation (6))

$= 14\underline{11}41 = 1\underline{44}1 = \underline{11} = e.$

t_0 takes $[\infty0234]$ to a single coset in $[\infty0141]$ since $N\infty02340$ $= N(1,\infty)(3,4)\infty2313 = N\infty2313 \in [\infty0141]$. To prove $\infty02340 = (1,\infty)(3,4)\infty2313$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty)(3,4)\infty2\underline{313}04320\infty = (1,\infty)(3,4)\infty2\underline{131}04320\infty$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,\infty)(3,4)\infty213\underline{1043}20\infty = (1,\infty)(3,4)\infty213\underline{(1,3,0,2,4)4012}20\infty$ (by relation (1) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)\infty430401\underline{22}0\infty = (1,\infty,3)(2,4,0)\infty43\ \underline{04010}\infty$ $= (1,\infty,3)(2,4,0)\infty43\ \underline{(1,0,3)(2,4,\infty)2\infty12}\infty$ (by relation (7) conjugated by $(1,3,4,\infty,0) \in L_2(5)$)

$= (1,2,\infty)(3,0,4)2\underline{\infty12\infty}12\infty = (1,2,\infty)(3,0,4)2\ \underline{(1,\infty,2)(3,4,0)1\infty21}\ 12\infty$

(by relation (3) conjugated by $(1,2,3,4,0) \in L_2(5)$)

$= \underline{11}\infty 2\underline{11}2\infty = \infty\underline{22}\infty = \underline{\infty\infty} = e.$

t_∞ takes $[\infty 0234]$ to a single coset in $[\infty 012\infty]$ since $N\infty 0234\infty$ $= N(2,3,\infty,4,0)2\infty 302 = N2\infty 302 \in [\infty 012\infty]$. To prove $\infty 0234\infty$ $= (2,3,\infty,4,0)2\infty 302$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$\Longrightarrow (2,3,\infty,4,0)2\infty 302\underline{\infty 4320}\infty = (2,3,\infty,4,0)2\infty 302\,\underline{(1,0,\infty,4,3)\infty 21\infty 3\infty}$ (by relation proved in Chapter 6, $\infty 0124 = (1,3,4,\infty,0)\infty 23\infty 1$, conjugated by $(1,3)(4,0) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)241\underline{\infty 2\infty}21\infty 3\infty = (1,0,2)(3,4,\infty)2412\underline{\infty\infty 2}21\infty 3\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$= (1,0,2)(3,4,\infty)2412\infty\underline{22}1\infty 3\infty = (1,0,2)(3,4,\infty)2412\,\underline{\infty 1\infty 3\infty}$ $= (1,0,2)(3,4,\infty)2412\,\underline{(1,2,0)(3,\infty,4)0230}$ (by relation (7) conjugated by $(1,4,2,0,3) \in L_2(5)$)

$= 032\underline{00}230 = 03\underline{22}30 = 0\underline{33}0 = \underline{00} = e.$

Now consider $[\infty 0214]$. The orbits of $N^{(\infty 0214)}$ are $\{4\}$, $\{0\}$, $\{1,3\}$, and $\{2,\infty\}$. So we need to look at $[\infty 02144]$, $[\infty 02140]$, $[\infty 02141]$, and $[\infty 02142]$.

First, $[\infty 02144] = [\infty 021]$ so t_4 takes $[\infty 0214]$ back to a single coset in $[\infty 021]$.

t_0 takes $[\infty 0214]$ to a single coset in $[\infty 0141]$ since $N\infty 02140$ $= N(1,2,4,0,\infty)042\infty 2 = N042\infty 2 \in [\infty 0141]$. To prove $\infty 02140$ $= (1,2,4,0,\infty)042\infty 2$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,4,0,\infty)042\infty 204\underline{120\infty} = (1,2,4,0,\infty)042\infty 204\underline{(1,\infty,2,3,0)0213}$ (by relation (1) conjugated by $(1,2,0)(3,\infty,4) \in L_2(5)$)

$= (1,3,0,2,4)1432\underline{3140}213 = (1,3,0,2,4)1432\underline{(1,0,4,3,2)41320}13$ (by a relation proved in Chapter 6, $\infty 0143 = (1,\infty,3,0,4)10\infty 34$, conjugated by $(1,4,0)(2,\infty,3) \in L_2(5)$)

$= (1,2,3,4,0)032\underline{14}132013 = (1,2,3,4,0)032\underline{41}432013$ (by relation (6))

$= (1,2,3,4,0)\underline{03241}432013 = (1,2,3,4,0)\underline{(1,0,3,2,\infty)04\infty 02}\ 432013$ (by a relation proved in Chapter 6, $\infty 0124 = (1,3,4,\infty,0)\infty 23\infty 1$, conjugated by $(1,2,4)(3,\infty,0)$ $\in L_2(5)$)

$= (1,\infty)(3,4)\underline{04\infty 0}2432013 = (1,\infty)(3,4)\underline{(1,3,2)(4,0,\infty)40\infty 4}2432013$ (by re-

lation (3) conjugated by $(1,\infty,0,4,2) \in L_2(5))$

$= (1,4,2)(3,0,\infty)40\infty\underline{424}32013 = (1,4,2)(3,0,\infty)40\infty\underline{242}32013$ (by relation (6) conjugated by $(1,2)(3,0) \in L_2(5))$

$= (1,4,2)(3,0,\infty)40\infty\underline{24232}013 = (1,4,2)(3,0,\infty)40\infty\underline{(1,4,0)(2,\infty,3)1031}013$ (by relation (7) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5))$

$= (1,0,3)(2,4,\infty)01\underline{31031}013 = (1,0,3)(2,4,\infty)01\underline{(1,3,0)(2,\infty,4)1301}1013$ (by relation (3) conjugated by $(1,0)(3,\infty) \in L_2(5))$

$= \underline{131}30\underline{11}013 = \underline{313}3\underline{00}13$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5))$

$= 31\underline{33}13 = 3\underline{11}3 = \underline{33} = e.$

t_1, a representative from one of the 2-orbits, takes $[\infty0214]$ to a single coset in $[\infty0\infty214]$ as was proved in Chapter 6.

t_2, a representative from the other 2-orbit, takes $[\infty0214]$ to $[\infty024]$ as was proved in Chapter 6.

Now consider $[\infty0242]$. The orbits of $N^{(\infty0242)}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty02421]$, $[\infty02422]$, $[\infty02423]$, $[\infty02424]$, $[\infty02420]$, and $[\infty0242\infty]$.

First, $[\infty02422] = [\infty024]$ so t_2 takes $[\infty0234]$ back to a single coset in $[\infty023]$.

t_1 takes $[\infty0242]$ to a single coset in $[\infty0\infty21]$ since $N\infty02421$ $= N(1,0,\infty)(2,3,4)313\infty4 = N313\infty4 \in [\infty0\infty21]$. To prove $\infty02421$ $= (1,0,\infty)(2,3,4)313\infty4$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,\infty)(2,3,4)\underline{313}\infty412420\infty = (1,0,\infty)(2,3,4)\underline{131}\infty412420\infty$ (by relation (6) conjugated by $(2,0)(3,4) \in L_2(5))$

$= (1,0,\infty)(2,3,4)131\infty\underline{4124}20\infty = (1,0,\infty)(2,3,4)131\infty\ \underline{(1,4,2)(3,0,\infty)1421}$ 20∞ (by relation (3) conjugated by $(1,2,0)(3,\infty,4) \in L_2(5))$

$= (1,\infty,4)(2,0,3)404314\underline{212}0\infty = (1,\infty,4)(2,0,3)404314\underline{121}0\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5))$

$= (1,\infty,4)(2,0,3)4043\underline{141}210\infty = (1,\infty,4)(2,0,3)4043\underline{414}210\infty$ (by relation (6))

$= (1,\infty,4)(2,0,3)\underline{40434}14210\infty = (1,\infty,4)(2,0,3)\ \underline{(1,3,4)(2,0,\infty)2\infty32}$ 14210∞ (by relation (7) conjugated by $(3,0)(4,\infty) \in L_2(5))$

$= (1,2,\infty)(3,0,4)2\infty3\underline{214}210\infty = (1,2,\infty)(3,0,4)2\infty3$

$\underline{(1,2,4)(3,\infty,0)1241}10\infty$ (by relation (3) conjugated by $(1,4,\infty,2,0) \in L_2(5)$)

$= (1,4,\infty,2,0)40\infty124\underline{11}0\infty = (1,4,\infty,2,0)40\infty1240\infty = e$ by relation (2) conjugated by $(1,2,0,\infty,3) \in L_2(5)$.

t_3 takes $[\infty0242]$ to a single coset in $[\infty0\infty14]$ since $N\infty02423$ $= N(1,4,0)(2,\infty,3)03012 = N03012 \in [\infty0\infty14]$. To prove $\infty02423$ $= (1,4,0)(2,\infty,3)03012$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,0)(2,\infty,3)0301\underline{232420}\infty = (1,4,0)(2,\infty,3)0301$ $\underline{(1,4,2)(3,0,\infty)\infty04\infty0\infty}$ (by relation (7) conjugated by $(2,\infty)(4,0) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)\infty0\infty\underline{4\infty04}\infty0\infty = (1,2,3)(4,\infty,0)\infty0\infty$ $\underline{(1,3,2)(4,0,\infty)\infty40\infty}\infty\infty0\infty$

$= 4\infty\underline{4\infty40}\underline{\infty\infty\infty}0\infty = 4\infty\underline{\infty4\infty}\ \underline{00}\infty$ (by relation (4) conjugated by $(1,\infty)(2,0)$ $\in L_2(5)$)

$= 4\underline{\infty\infty}4\underline{\infty\infty} = \underline{44} = e$.

t_4 takes $[\infty0242]$ to a single coset in $[\infty013]$ as was proved in Chapter 6.

t_0 takes $[\infty0242]$ to a single coset in $[\infty02320]$ as was proved in Chapter 6.

t_∞ takes $[\infty0242]$ back to a single coset in $[\infty0242]$ since $N\infty0242\infty$ $= N(1,\infty)(3,4)\infty0242 = N\infty0242 \in [\infty0242]$. To prove $\infty0242\infty$ $= (1,\infty)(3,4)\infty0242$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty)(3,4)\infty024\underline{2\infty242}0\infty = (1,\infty)(3,4)\infty024\underline{(1,3,\infty)(2,0,4)3143}0\infty$ (by relation (7) conjugated by $(1,0,4)(2,3,\infty) \in L_2(5)$)

$= (2,0,4,\infty,3)1\underline{4023}1430\infty = (2,0,4,\infty,3)1\underline{(2,4,3,0,\infty)204\infty}1430\infty$ (by relation (2) conjugated by $(1,3,\infty)(2,0,4) \in L_2(5)$)

$= (2,\infty,0,3,4)120\underline{4\infty14}30\infty = (2,\infty,0,3,4)120\underline{(1,\infty,4)(2,0,3)\infty41\infty3}0\infty$ (by relation (4) conjugated by $(1,0,3)(2,4,\infty) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)\infty03\infty\underline{41\infty3}0\infty = (1,\infty,3)(2,4,0)\infty03\infty$ $\underline{(1,0,\infty,4,3)\infty140}0\infty$ (by relation (1) conjugated by $(2,\infty)(4,0) \in L_2(5)$)

$= (1,4,\infty)(2,3,0)4\infty14\infty14\underline{00}\infty = (1,4,\infty)(2,3,0)4\infty14\infty14\infty = e$ by relation (4) conjugated by $(1,3)(2,\infty) \in L_2(5)$.

Now consider $[\infty0243]$. The orbits of $N^{(\infty0243)}$ are $\{1\}$, $\{2\}$, $\{3,0\}$, and $\{4,\infty\}$. So we need to look at $[\infty02431]$, $[\infty02432]$, $[\infty02433]$, and $[\infty02434]$.

First, $[\infty 02433] = [\infty 024]$ so t_3, a representative from one of the 2-orbit, takes $[\infty 0243]$ back to a single coset in $[\infty 024]$.

t_1 takes $[\infty 0243]$ to a single coset in $[\infty 02431]$. There are 6 distinct single cosets in $[\infty 02431]$ as was proved in Chapter 6.

t_2 takes $[\infty 0243]$ to a single coset in $[\infty 02\infty]$ as was proved in Chapter 6.

t_4, a representative from the other 2-orbit, takes $[\infty 0243]$ to a single coset in $[\infty 024\infty]$ since $N\infty 02434 = N(2,3,\infty,4,0)03\infty 20 = N03\infty 20 \in [\infty 024\infty]$. To prove $\infty 02434 = (2,3,\infty,4,0)03\infty 20$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(2,3,\infty,4,0)0\underline{3\infty 204}3420\infty = (2,3,\infty,4,0)0\ \underline{(1,4,3,\infty,2)30132}\ 3420\infty$ (by a relation proved in Chapter 6, $\infty 0124 = (1,3,4,\infty,0)\infty 23\infty 1$, conjugated by $(1,2,0,\infty,3) \in L_2(5)$)

$= (1,4,0)(2,\infty,3)\underline{030}1323420\infty = (1,4,0)(2,\infty,3)\underline{303}1323420\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,4,0)(2,\infty,3)30\underline{31323}420\infty = (1,4,0)(2,\infty,3)30\underline{(1,0,4)(2,3,\infty)4024}420\infty$ (by relation (7) conjugated by $(1,\infty,3)(2,4,0) \in L_2(5)$)

$= \infty\underline{44}02\underline{44}20\infty = \infty 0\underline{22}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

Now consider $[\infty 0120]$. The orbits of $N^{(\infty 0120)}$ are $\{2\}$, $\{4\}$, $\{1,0\}$, and $\{3,\infty\}$. So we need to look at $[\infty 01202]$, $[\infty 01204]$, $[\infty 01200]$, and $[\infty 01203]$.

First, $[\infty 01200] = [\infty 012]$ so t_0, a representative from one of the 2-orbits, takes $[\infty 0120]$ back to a single coset in $[\infty 012]$.

t_2 takes $[\infty 0120]$ to a single coset in $[\infty 0240]$ since $N\infty 01202$ $= N(1,\infty,0)(2,4,3)03\infty 23 = N03\infty 23 \in [\infty 0240]$. To prove $\infty 01202$ $= (1,\infty,0)(2,4,3)03\infty 23$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,0)(2,4,3)0\underline{3\infty 23}20210\infty = (1,\infty,0)(2,4,3)0\ \underline{(1,4,0)(2,\infty,3)\infty 32\infty}$ 20210∞ (by relation (3) conjugated by $(1,2,0,\infty,3) \in L_2(5)$)

$= (1,3,\infty)(2,0,4)1\infty 3\underline{2\infty 202}10\infty = (1,3,\infty)(2,0,4)1\infty 3$ $\underline{(1,\infty,3)(2,4,0)1301}10\infty$ (by relation (7) conjugated by $(1,4,3,\infty,2) \in L_2(5)$)

$= \infty 3\underline{11}30\underline{11}0\infty = \infty\underline{3300}\infty = \underline{\infty\infty} = e.$

t_4 takes $[\infty 0120]$ to a single coset in $[\infty 01204]$. There are 10 distinct single cosets in $[\infty 01204]$ as was proved in Chapter 6.

t_3, a representative from the other 2-orbit, takes $[\infty0120]$ to a single coset in $[\infty023]$ as was proved in Chapter 6.

Now consider $[\infty0240]$. The orbits of $N^{(\infty0240)}$ are $\{3\}$, $\{4\}$, $\{1, \infty\}$, and $\{2, 0\}$. So we need to look at $[\infty02403]$, $[\infty02404]$, $[\infty02401]$, and $[\infty02400]$.

First, $[\infty02400] = [\infty024]$ so t_0, a representative from one of the 2-orbits, takes $[\infty0240]$ back to a single coset in $[\infty024]$.

t_3 takes $[\infty0240]$ to a single coset in $[\infty01204]$ as was proved in Chapter 6.

t_4 takes $[\infty0240]$ to a single coset in $[\infty0120]$ since $N\infty02404 = N(1, \infty, 2)(3, 4, 0)2\infty14\infty = N2\infty14\infty \in [\infty0120]$. To prove $\infty02404 = (1, \infty, 2)(3, 4, 0)2\infty14\infty$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1, \infty, 2)(3, 4, 0)2\underline{\infty14\infty}40420\infty = (1, \infty, 2)(3, 4, 0)2\ \underline{(1, \infty, 4)(2, 0, 3)1\infty41}$ 40420∞ (by relation (4) conjugated by $(1, 2, infty, 3, 4) \in L_2(5)$)

$= (1, 4, 3)(2, \infty, 0)01\infty\underline{41404}20\infty = (1, 4, 3)(2, \infty, 0)01\infty\ \underline{(1, 2, 3)(4, \infty, 0)3203}$ 20∞ (by relation (7) conjugated by $(1, \infty, 4, 2, 3) \in L_2(5)$)

$= (1, \infty, 4)(2, 0, 3)420\underline{320}320\infty = (1, \infty, 4)(2, 0, 3)420\underline{(1, 4, \infty)(2, 3, 0)230}220\infty$ (by relation (3) conjugated by $(1, 0, 2)(3, 4, \infty) \in L_2(5)$)

$= \infty3\underline{22}230\underline{22}0\infty = \infty\underline{3300}\infty = \underline{\infty\infty} = e.$

t_1, a representative from the other 2-orbit, takes $[\infty0240]$ to a single coset in $[\infty014]$ as was proved in Chapter 6.

Now consider $[\infty012\infty]$. The orbits of $N^{(\infty012\infty)}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty012\infty1]$, $[\infty012\infty2]$, $[\infty012\infty3]$, $[\infty012\infty4]$, $[\infty012\infty0]$, and $[\infty012\infty\infty]$.

First, $[\infty012\infty\infty] = [\infty012]$ so t_∞ takes $[\infty012\infty]$ back to a single coset in $[\infty012]$.

t_1 takes $[\infty012\infty]$ back to a single coset in $[\infty012\infty]$ since $N\infty012\infty1 = N(2, 4)(3, \infty)\infty012\infty = N\infty012\infty \in [\infty012\infty]$. To prove $\infty012\infty1 = (2, 4)(3, \infty)\infty012\infty$, we will start with a previously proven relation, $\infty0121\infty = (2, 4)(3, \infty)\infty0121$.

$\Longrightarrow \infty012\underline{1\infty\underline{1}} = (2, 4)(3, \infty)\infty012\underline{1\underline{1}}.$

$\Longrightarrow \infty012\underline{\infty1\infty} = (2, 4)(3, \infty)\infty012$ (by relation (5) conjugated by $(2, 0)(3, 4) \in L_2(5)$)

$\Longrightarrow \infty 012\infty 1\underline{\infty\infty} = (2,4)(3,\infty)\infty 012\underline{\infty}$

$\Longrightarrow \infty 012\infty 1 = (2,4)(3,\infty)\infty 012\infty$.

t_2 takes $[\infty 012\infty]$ to a single coset in $[\infty 0\infty 21]$ since $N\infty 012\infty 2 = N(1,\infty,4)(2,0,3)2321\infty = N2321\infty \in [\infty 0\infty 21]$. To prove $\infty 012\infty 2 = (1,\infty,4)(2,0,3)2321\infty$, we will start with a previously proven relation, $\infty 01\infty 2 = (1,\infty,4)(2,0,3)2321$, which gives

$\infty 01\underline{\infty 2\infty} = (1,\infty,4)(2,0,3)2321\underline{\infty}$

$\Longrightarrow .\infty 01\underline{2\infty 2} = (1,\infty,4)(2,0,3)2321\infty$ (by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$).

t_3 takes $[\infty 012\infty]$ to a single coset in $[\infty 0\infty 14]$ since $N\infty 012\infty 3 = N(1,3,\infty,0,2)\infty 1\infty 20 = N\infty 1\infty 20 \in [\infty 0\infty 14]$. To prove $\infty 012\infty 3 = (1,3,\infty,0,2)\infty 1\infty 20$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,\infty,0,2)\infty 1\infty 20\underline{3\infty 21}0\infty = (1,3,\infty,0,2)\infty 1\infty 20$ $\underline{(1,\infty,4,2,3)2\infty 340}\infty$ (by relation (2) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= (2,\infty,0,3,4)4\infty 430\underline{2\infty 340}\infty = (2,\infty,0,3,4)4\infty 430$ $\underline{(1,\infty,4,2,3)04\infty 13}\infty$ (by a relation proved in Chapter 6, $\infty 0132 = (1,4,0,3,\infty)23041$, conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)2421\underline{00}4\infty 13\infty = (1,\infty,0)(2,4,3)2$ $\underline{4214\infty}13\infty = (1,\infty,0)(2,4,3)2$ $\underline{(1,0,\infty)(2,3,4)\infty 3\infty 1}$ 13∞ (by relation (7) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= \underline{3\infty 3}\infty\underline{11}3\infty = \underline{\infty 3\infty}\infty 3\infty$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= \infty 3\underline{\infty\infty}3\infty = \infty\underline{33}\infty = \underline{\infty\infty} = e$.

t_4 takes $[\infty 012\infty]$ to a single coset in $[\infty 0124]$ since $N\infty 012\infty 4 = N(1,0,2,\infty,4)1\infty 240 = N1\infty 240 \in [\infty 0124]$. To prove $\infty 012\infty 4 = (1,0,2,\infty,4)1\infty 240$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,0,2,\infty,4)\underline{1\infty 240}4\infty 210\infty = (1,0,2,\infty,4)$ $\underline{(1,\infty,2,3,0)14312}$ $4\infty 210\infty$ (by a relation proved in Chapter 6, $\infty 0124 = (1,3,4,\infty,0)\infty 23\infty 1$, conjugated by $(1,2,4,0,\infty) \in L_2(5)$)

$= (3,0)(4,\infty)\underline{1431}24\infty 210\infty = (3,0)(4,\infty)\underline{(1,3,4)(2,0,\infty)4134}24\infty 210\infty$ (by relation (3) conjugated by $(1,3,\infty)(2,0,4) \in L_2(5)$)

$= (1,3,\infty)(2,0,4)413\underline{424}\infty210\infty = (1,3,\infty)(2,0,4)413\underline{242}\infty210\infty$ (by relation (6) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,3,\infty)(2,0,4)413\underline{242\infty2}10\infty = (1,3,\infty)(2,0,4)413$ $\underline{(1,0,4)(2,3,\infty)01\infty0}10\infty$ (by relation (7) conjugated by $(1,3,4)(2,0,\infty) \in L_2(5)$)

$= (1,\infty,0)(2,4,3)10\underline{\infty01\infty}010\infty = (1,\infty,0)(2,4,3)10$ $\underline{(1,0,\infty)(2,3,4)0\infty10}\ 010\infty$ (by relation (3))

$= \underline{0\infty0}\infty1\underline{00}10\infty = \underline{\infty0\infty}\infty\underline{11}0\infty$ (by relation (5))

$= \infty0\underline{\infty\infty}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

t_0 takes $[\infty012\infty]$ to a single coset in $[\infty0234]$ since $N\infty012\infty0 = N(1,\infty,2,3,0)02\infty13 = N02\infty13 \in [\infty0234]$. To prove $\infty012\infty0 = (1,\infty,2,3,0)02\infty13$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,2,3,0)02\infty1\underline{30\infty2}10\infty = (1,\infty,2,3,0)02\infty1\underline{(2,0,4,\infty,3)\infty034}10\infty$ (by relation (1) conjugated by $(1,0,3,2,\infty) \in L_2(5)$)

$= (1,3,4,\infty,0)4031\infty0\underline{3410\infty} = (1,3,4,\infty,0)4031\infty0\ \underline{(1,2,\infty,3,4)30231}$ (by a relation proved in Chapter 6, $\infty0124 = (1,3,4,\infty,0)\infty23\infty1$, conjugated by $(2,0,4,\infty,3) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)10423\underline{030}231 = (1,4,3)(2,\infty,0)10423\underline{303}231$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,4,3)(2,\infty,0)10423\underline{33}03231 = (1,4,3)(2,\infty,0)1\ \underline{0420}3231$ $= (1,4,3)(2,\infty,0)1\ \underline{(1,\infty,3)(2,4,0)4024}\ 3231$ (by relation (3) conjugated by $(1,2,3)(4,\infty,0) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)\infty4024\underline{3231} = (1,0,4)(2,3,\infty)\infty4024\underline{232}1$ (by relation (5) conjugated by $(1,0,2)(3,4,\infty) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)\infty40\underline{242321} = (1,0,4)(2,3,\infty)\infty40\underline{(1,4,0)(2,\infty,3)10311}$ (by relation (7) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= 30\underline{11}03\underline{11} = 3\underline{00}3 = \underline{33} = e.$

Now consider $[\infty024\infty]$. The orbits of $N^{(\infty024\infty)}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty024\infty1]$, $[\infty024\infty2]$, $[\infty024\infty3]$, $[\infty024\infty4]$, $[\infty024\infty0]$, and $[\infty024\infty\infty]$.

First, $[\infty024\infty\infty] = [\infty024]$ so t_∞ takes $[\infty024\infty]$ back to a single coset in $[\infty024]$.

t_1 takes $[\infty024\infty]$ to a single coset in $[\infty0\infty23]$ since $N\infty024\infty1$ $= N(1,\infty,0,4,2)2\infty240 = N2\infty240 \in [\infty0\infty23]$. To prove $\infty024\infty1$ $= (1,\infty,0,4,2)2\infty240$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,0,4,2)2\infty240\underline{1\infty42}0\infty = (1,\infty,0,4,2)2\infty240\underline{(1,2,\infty,3,4)4\infty13}0\infty$ (by relation (1) conjugated by $(1,\infty,0)(2,4,3) \in L_2(5)$)

$= (1,3,4,\infty,0)\underline{\infty3\infty}104\infty130\infty = (1,3,4,\infty,0)\underline{3\infty3}104\infty130\infty$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,3,4,\infty,0)3\infty\underline{3104\infty}130\infty = (1,3,4,\infty,0)3\infty\ \underline{(1,4,0,3,\infty)013\infty4}\ 130\infty$ (by a relation proved in Chapter 6, $\infty0143 = (1,\infty,3,0,4)10\infty34$, conjugated by $(1,0)(3,\infty) \in L_2(5)$)

$= (1,\infty,3,0,4)\infty1013\infty\underline{4130}\infty = (1,\infty,3,0,4)\infty1013\infty\underline{(1,2,3,4,0)3142}\infty$ (by relation (2) conjugated by $(1,0,4,3,2) \in L_2(5)$)

$= (1,\infty,4,2,3)\infty21\underline{24\infty31}42\infty = (1,\infty,4,2,3)\infty21\ \underline{(1,2,0,\infty,3)32\infty42}42\infty$ (by a relation proved in Chapter 6, $\infty0231 = (1,\infty,4,2,3)3\infty20\infty$, conjugated by $(2,\infty)(4,0) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)30\underline{232}\infty4\underline{242}\infty = (1,3,2)(4,0,\infty)30\underline{323}\infty4\underline{424}\infty$ (by relation (5) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$ and relation (6) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,3,2)(4,0,\infty)30323\infty\underline{44}24\infty = (1,3,2)(4,0,\infty)303\ \underline{23\infty24}\infty$ $= (1,3,2)(4,0,\infty)303\ \underline{(1,2,3)(4,\infty,0)414\infty\infty}\infty$ (by relation (7) conjugated by $(1,0,\infty,4,3) \in L_2(5)$)

$= \underline{141}414\underline{\infty\infty} = \underline{414}414$ (by relation (6))

$= 41\underline{44}14 = 4\underline{11}4 = \underline{44} = e$.

t_2 takes $[\infty024\infty]$ back to a single coset in $[\infty024\infty]$ since $N\infty024\infty2$ $= N(1,\infty)(3,4)\infty024\infty = N\infty024\infty \in [\infty024\infty]$. To prove $\infty024\infty2$ $= (1,\infty)(3,4)\infty024\infty$, we will start with a previously proven relation, $\infty0242\infty = (1,\infty)(3,4)\infty0242$.

$\Longrightarrow \infty0242\underline{\infty2} = (1,\infty)(3,4)\infty0242\underline{2}$.

$\Longrightarrow \infty024\underline{\infty2\infty\infty\infty} = (1,\infty)(3,4)\infty024\ \underline{\infty}$.(by relation (5) conjugated by $(2,0)(3,4) \in L_2(5)$)

$\Longrightarrow \infty024\infty2 = (1,\infty)(3,4)\infty024\infty$.

t_3 takes $[\infty 024\infty]$ to a single coset in $[\infty 0243]$ since $N\infty 024\infty 3$ $= N(2,4,3,0,\infty)3042\infty = N3042\infty \in [\infty 0243]$. To prove $\infty 024\infty 3$ $= (2,4,3,0,\infty)3042\infty$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(2,4,3,0,\infty)3042\underline{\infty 3\infty}420\infty = (2,4,3,0,\infty)3042\underline{3\infty 3}420\infty$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (2,4,3,0,\infty)3\underline{0423\infty}3420\infty = (2,4,3,0,\infty)3\ \underline{(1,\infty,0,4,2)03102}\ 3420\infty$ (by a relation proved in Chapter 6, $\infty 0124 = (1,3,4,\infty,0)\infty 23\infty 1$, conjugated by $(1,2,3)(4,\infty,0) \in L_2(5)$)

$= (1,\infty)(3,4)30310\underline{2342}0\infty = (1,\infty)(3,4)30310\underline{(1,\infty,0)(2,4,3)3243}0\infty$ (by relation (4) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= (1,0)(2,4)2\underline{12\infty 13}2430\infty = (1,0)(2,4)2\underline{(1,2,0)(3,\infty,4)303\infty}2430\infty$ (by relation (7) conjugated by $(1,4,2)(3,0,\infty) \in L_2(5)$)

$= (2,3,\infty,4,0)0\underline{303}\infty 2430\infty = (2,3,\infty,4,0)0\underline{030}\infty 2430\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (2,3,\infty,4,0)\underline{00}30\infty 2430\infty = (2,3,\infty,4,0)30\infty 2430\infty = e$ by relation (2) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$.

t_4 takes $[\infty 024\infty]$ to a single coset in $[\infty 0\infty 12]$ since $N\infty 024\infty 4$ $= N(1,4,0)(2,\infty,3)1412\infty = N3042\infty \in [\infty 0\infty 12]$. To prove $\infty 024\infty 4$ $= (1,4,0)(2,\infty,3)1412\infty$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,4,0)(2,\infty,3)1412\infty\underline{4\infty 4}20\infty = (1,4,0)(2,\infty,3)1412\infty\underline{\infty 4\infty}20\infty$ (by relation (5) conjugated by $(1,3)(4.0) \in L_2(5)$)

$= (1,4,0)(2,\infty,3)1412\underline{\infty\infty}4\infty 20\infty = (1,4,0)(2,\infty,3)1\ \underline{4124\infty}\ 20\infty$ $= (1,4,0)(2,\infty,3)1\underline{(1,0,4)(2,3,\infty)\infty 0\infty 2}\ 20\infty$ (by relation (7) conjugated by $(1,3,0,2,4) \in L_2(5)$)

$= \underline{0\infty 0}\infty\underline{22}0\infty = \underline{\infty 0\infty}\infty 0\infty$ (by relation (5))

$= \infty 0\underline{\infty\infty}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

t_0 takes $[\infty 024\infty]$ to a single coset in $[\infty 0234]$ since $N\infty 024\infty 0$ $= N(2,0,4,\infty,3)2\infty 403 = N2\infty 403 \in [\infty 0234]$. To prove $\infty 024\infty 0$ $= (2,0,4,\infty,3)2\infty 403$, we will start with a previously proven relation, $\infty 02434$ $= (2,3,\infty,4,0)03\infty 20$.

$\Longrightarrow \underline{(2,0,4,\infty,3)}\infty02434 = \underline{(2,0,4,\infty,3)(2,3,\infty,4,0)}03\infty20$

$\Longrightarrow (2,0,4,\infty,3)\infty02\underline{434} = 03\infty20$

$\Longrightarrow (2,0,4,\infty,3)\infty02\underline{343} = 03\infty20$ (by relation (6) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$\Longrightarrow (2,0,4,\infty,3)\infty023433 = 03\infty20\underline{3}$

$\Longrightarrow (2,0,4,\infty,3)\infty0234 = 03\infty203$

Conjugate the above equation by $(2,4,3,0,\infty) \in L_2(5)$.

$\Longrightarrow (2,0,4,\infty,3)2\infty403 = \infty024\infty0$.

Now consider $[\infty0141]$. The orbits of $N^{(\infty0141)}$ are $\{0\}$, $\{\infty\}$, $\{1,4\}$, and $\{2,3\}$. So we need to look at $[\infty01410]$, $[\infty0141\infty]$, $[\infty01411]$, and $[\infty01412]$.

First, $[\infty01411] = [\infty014]$ so t_1, a representative from one of the 2-orbits, takes $[\infty0141]$ back to a single coset in $[\infty014]$.

t_0 takes $[\infty0141]$ to a single coset in $[\infty01210]$ as was proved in Chapter 6.

t_∞ takes $[\infty0141]$ to a single coset in $[\infty0214]$ since $N\infty0141\infty$ $= N(1,3,4,\infty,0)4\infty130 = N4\infty130 \in [\infty0214]$. To prove $\infty0141\infty$ $= (1,3,4,\infty,0)4\infty130$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3,4,\infty,0)\underline{4\infty13}0\infty1410\infty = (1,3,4,\infty,0)\underline{(1,4,3,\infty,2)1\infty42}0\infty1410\infty$ (by relation (1) conjugated by $(1,\infty,0,4,2) \in L_2(5)$)

$= (1,\infty,0,4,2)1\infty\underline{420\infty1}410\infty = (1,\infty,0,4,2)1\infty\ \underline{(1,4,2,0,3)4\infty340}\ 410\infty$ (by a relation proved in Chapter 6, $\infty0124 = (1,3,4,\infty,0)\infty23\infty1$, conjugated by $(1,0,2,\infty,4) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)\underline{4\infty4}\infty340410\infty = (1,\infty,3)(2,4,0)\underline{\infty4\infty}\infty340410\infty$ (by relation (6) conjugated by $(1,\infty)(2,0) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)\infty4\underline{\infty\infty}340410\infty = (1,\infty,3)(2,4,0)\infty\ \underline{43404}10\infty$
$= (1,\infty,3)(2,4,0)\infty\underline{(1,3,\infty)(2,0,4)1\infty01}10\infty$ (by relation (7) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= \underline{11}\infty0\underline{11}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

t_2, a representative from the other 2-orbit, takes $[\infty0141]$ to a single coset in $[\infty0234]$ since $N\infty01412 = N(1,\infty)(3,4)\infty2043 = N\infty2043 \in [\infty0234]$. To prove $\infty01412 = (1,\infty)(3,4)\infty2043$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty)(3,4)\infty 2\underline{0432}1410\infty = (1,\infty)(3,4)\infty 2\underline{(1,3,0,2,4)3401}1410\infty$ (by relation (1) conjugated by $(1,4)(2,3) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)\infty 4340\underline{11}410\infty = (1,\infty,3)(2,4,0)\infty\ \underline{4340}410\infty$

$= (1,\infty,3)(2,4,0)\infty\underline{(1,3,\infty)(2,0,4)1\infty 01}\ 10\infty$ (by relation (7) conjugated by $(1,2)(4,\infty) \in L_2(5)$)

$= \underline{11}\infty 0\underline{11}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e.$

Now consider $[\infty 010\infty]$. The orbits of $N^{(\infty 010\infty)}$ are $\{1,0,\infty\}$ and $\{2,3,4\}$. So we need to look at $[\infty 010\infty\infty]$ and $[\infty 010\infty 2]$.

First, $[\infty 010\infty\infty] = [\infty 010]$ so t_∞, a representative from one of the 3-orbits, takes $[\infty 010\infty]$ back to a single coset in $[\infty 010]$.

t_2, a representative from the other 3-orbit, takes $[\infty 010\infty]$ to a single coset in $[\infty 01210]$ as was proved in Chapter 6.

Now consider $[\infty 020\infty]$. The orbits of $N^{(\infty 020\infty)}$ are $\{2,0,\infty\}$ and $\{1,3,4\}$. So we need to look at $[\infty 020\infty\infty]$ and $[\infty 020\infty 1]$.

First, $[\infty 020\infty\infty] = [\infty 020]$ so t_∞, a representative from one of the 3-orbits, takes $[\infty 020\infty]$ back to a single coset in $[\infty 020]$.

t_1, a representative from the other 3-orbit, takes $[\infty 020\infty]$ to a single coset in $[\infty 02320]$ as was proved in Chapter 6.

Now consider $[\infty 0\infty 212]$. The orbits of $N^{(\infty 0\infty 212)}$ are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{0\}$, and $\{\infty\}$. So we need to look at $[\infty 0\infty 2121]$, $[\infty 0\infty 2122]$, $[\infty 0\infty 2123]$, $[\infty 0\infty 2124]$, $[\infty 0\infty 2120]$, and $[\infty 0\infty 212\infty]$.

First, $[\infty 0\infty 2122] = [\infty 0\infty 21]$ so t_2 takes $[\infty 0\infty 212]$ back to a single coset in $[\infty 0\infty 21]$.

t_1 takes $[\infty 0\infty 212]$ to a single coset in $[\infty 0\infty 12]$ as was proved in Chapter 6.

t_3 takes $[\infty 0\infty 212]$ to a single coset in $[\infty 01210]$ since $N\infty 0\infty 2123$ $= N(1,\infty)(2,0)301410 = N301410 \in [\infty 01210]$. To prove $\infty 0\infty 2123$ $= (1,\infty)(2,0)301410$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty)(2,0)3014103\underline{212}\infty 0\infty = (1,\infty)(2,0)3014103\underline{121}\infty 0\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,\infty)(2,0)3014\underline{103121}\infty 0\infty = (1,\infty)(2,0)3014\underline{(1,0,4)(2,3,\infty)24231}\infty 0\infty$ (by relation (7) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1, 2, 4)(3, \infty, 0)\infty4012\underline{4231}\infty0\infty = (1, 2, 4)(3, \infty, 0)\infty4012$
$\underline{(1, 2, \infty, 3, 4)324\infty}\ \infty0\infty$ (by relation (1) conjugated by $(1, 2, 3)(4, \infty, 0) \in L_2(5)$)

$= (1, \infty, 0, 4, 2)3102\underline{\infty32}4\underline{\infty\infty}0\infty = (1, \infty, 0, 4, 2)310$
$\underline{(1, 0, 4)(2, 3, \infty)\infty23\infty}40\infty$ (by relation (3) conjugated by $(1, 3, 4)(2, 0, \infty) \in L_2(5)$)

$= (1, 2, 0)(3, \infty, 4)\infty04\infty\underline{23\infty4}0\infty = (1, 2, 0)(3, \infty, 4)\infty04\infty$
$\underline{(2, 4, 3, 0, \infty)\infty320}0\infty$ (by relation (2) conjugated by $(1, 4, \infty)(2, 3, 0) \in L_2(5)$)

$= (1, 4, 0)(2, \infty, 3)2\infty32\infty32\underline{00}\infty = (1, 4, 0)(2, \infty, 3)2\infty32\infty32\infty = e$ by relation (3) conjugated by $(1, 3, 0, 2, 4) \in L_2(5)$.

t_4 takes $[\infty0\infty212]$ to a single coset in $[\infty02320]$ since $N\infty0\infty2124$ $= N(1, 2, 4)(3, \infty, 0)132\infty23 = N132\infty23 \in [\infty02320]$. To prove $\infty0\infty2124$ $= (1, 2, 4)(3, \infty, 0)132\infty23$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1, 2, 4)(3, \infty, 0)132\infty234\underline{212}\infty0\infty = (1, 2, 4)(3, \infty, 0)132\infty234\underline{121}\infty0\infty$ (by relation (6) conjugated by $(2, 4)(3, \infty) \in L_2(5)$)

$= (1, 2, 4)(3, \infty, 0)132\infty2341\underline{21\infty0}\infty = (1, 2, 4)(3, \infty, 0)132\infty2341$
$\underline{(1, 4, \infty, 2, 0)\infty124}\infty$ (by relation (2) conjugated by $(1, 0, 2)(3, 4, \infty) \in L_2(5)$)

$= (1, 0, 3, 2, \infty)430203\underline{\infty4\infty}124\infty = (1, 0, 3, 2, \infty)4302034\underline{\infty4}124\infty$ (by relation (5) conjugated by $(1, 3)(4, 0) \in L_2(5)$)

$= (1, 0, 3, 2, \infty)4302034\infty\underline{4124\infty} = (1, 0, 3, 2, \infty)4302034\infty$
$\underline{(1, 0, 4)(2, 3, \infty)\infty0\infty2}$ (by relation (7) conjugated by $(1, 3, 0, 2, 4) \in L_2(5)$)

$= (1, 4)(0, \infty)1\infty434\infty12\underline{\infty0\infty}2 = (1, 4)(0, \infty)1\infty434\infty120\underline{\infty0}2$ (by relation (5))

$= (1, 4)(0, \infty)1\infty434\infty\underline{120\infty}02 = (1, 4)(0, \infty)1\infty434\infty\underline{(1, \infty, 2, 3, 0)0213}02$ (by relation (1) conjugated by $(1, 2, 0)(3, \infty, 4) \in L_2(5)$)

$= (1, 4, \infty)(2, 3, 0)\infty2\underline{40420}21302 = (1, 4, \infty)(2, 3, 0)\infty2\underline{04002}01302$ (by relation (5) conjugated by $(1, 2)(4, \infty)$ and $(1, 3)(2, \infty) \in L_2(5)$)

$= (1, 4, \infty)(2, 3, 0)\infty204\underline{00}201302 = (1, 4, \infty)(2, 3, 0)\infty\ \underline{204}201302$
$= (1, 4, \infty)(2, 3, 0)\infty\underline{(1, \infty, 3)(2, 4, 0)0240}\ 01302$ (by relation (3) conjugated by $(1, 4, 3, \infty, 2) \in L_2(5)$)

$= (1, 0, 4, 3, 2)3024\underline{00}1302 = (1, 0, 4, 3, 2)30241302 = e$ by relation (2) conjugated by $(2, 0)(3, 4) \in L_2(5)$.

t_0 takes $[\infty0\infty212]$ to a single coset in $[\infty0\infty212]$ since $N\infty0\infty2120$

$= N(1,3)(4,0)\infty0\infty212 = N\infty0\infty212 \in [\infty0\infty212]$. To prove $\infty0\infty2120 = (1,3)(4,0)\infty0\infty212$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,3)(4,0)\infty0\infty\underline{2120}212\infty0\infty = (1,3)(4,0)\infty0\infty\,\underline{(1,4,\infty)(2,3,0)\infty40\infty}\,12\infty0\infty$ (by relation (7) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,0,\infty)(2,3,4)121\underline{\infty40\infty}12\infty0\infty = (1,0,\infty)(2,3,4)121\,\underline{(1,2,3)(4,\infty,0)4\infty04}\,12\infty0\infty$ (by relation (3) conjugated by $(1,0,4,3,2) \in L_2(5)$)

$= (1,4,3,\infty,2)232\underline{4\infty041}2\infty0\infty = (1,4,3,\infty,2)232\underline{(1,0,3)(2,4,\infty)121}02\infty0\infty$ (by relation (7) conjugated by $(1,3,2,4,\infty) \in L_2(5)$)

$= (1,\infty,4)(2,0,3)\underline{414}12102\infty0\infty = (1,\infty,4)(2,0,3)\underline{141}12102\infty0\infty$ (by relation (6))

$= (1,\infty,4)(2,0,3)14\underline{11}2102\infty0\infty = (1,\infty,4)(2,0,3)\underline{14210}\,2\infty0\infty$ $= (1,\infty,4)(2,0,3)\underline{(1,4,\infty)(2,3,0)0\infty02}\,2\infty0\infty$ (by relation (7) conjugated by $(1,3,\infty,0,2) \in L_2(5)$)

$= \underline{0\infty022}\infty0\infty = \underline{\infty0\infty}\infty0\infty$ (by relation (5))

$= \infty0\underline{\infty\infty}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e$.

t_∞ takes $[\infty0\infty212]$ to a single coset in $[\infty0\infty212]$ since $N\infty0\infty212\infty = N(3,\infty)(2,4)\infty0\infty212 = N\infty0\infty212 \in [\infty0\infty212]$. To prove $\infty0\infty212\infty = (3,\infty)(2,4)\infty0\infty212$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(3,\infty)(2,4)\infty0\infty\underline{212}\infty212\infty0\infty = (3,\infty)(2,4)\infty0\infty\underline{121}\infty212\infty0\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (3,\infty)(2,4)\infty0\infty1\underline{21\infty2}12\infty0\infty = (3,\infty)(2,4)\infty0\infty1\,\underline{(1,2,\infty)(3,0,4)12\infty1}\,12\infty0\infty$ (by relation (3) conjugated by $(1,\infty,2,3,0) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)141\underline{212}\infty\underline{11}2\infty0\infty = (1,2,3)(4,\infty,0)141\underline{121}\infty2\infty0\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,2,3)(4,\infty,0)14\underline{11}21\infty2\infty0\infty = (1,2,3)(4,\infty,0)\underline{1421}\,\infty2\infty0\infty$ $= (1,2,3)(4,\infty,0)\,\underline{(1,2,4)(3,\infty,0)4124}\,\infty2\infty0\infty$ (by relation (3) conjugated by $(1,2,\infty)(3,0,4) \in L_2(5)$)

$= (1,4,0)(2,\infty,3)\underline{4124\infty2}\infty0\infty = (1,4,0)(2,\infty,3)\,\underline{(1,0,4)(2,3,\infty)\infty0\infty2}\,2\infty0\infty$ (by relation (7) conjugated by $(1,3,0,2,4) \in L_2(5)$)

$= \infty0\infty\underline{22}\infty0\infty = \infty0\underline{\infty\infty}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e$.

Now consider $[\infty 01204]$. The orbits of $N^{(\infty 01204)}$ are $\{1, 4, 0\}$ and $\{2, 3, \infty\}$. So we need to look at $[\infty 012044]$ and $[\infty 012042]$.

First, $[\infty 012044] = [\infty 0120]$ so t_4, a representative from one of the 3-orbits, takes $[\infty 01204]$ back to a single coset in $[\infty 0120]$.

t_2, a representative from the other 3-orbit, takes $[\infty 01204]$ to $[\infty 0240]$ as was proved in Chapter 6.

Now consider $[\infty 0\infty 141]$. $N^{(\infty 0\infty 141)}$ has just one orbit $\{1, 2, 3, 4, 0, \infty\}$. So we need to look at $[\infty 0\infty 1411]$.

$[\infty 0\infty 1411] = [\infty 0\infty 14]$ so t_1, a representative from the 6-orbit, takes $[\infty 0\infty 141]$ back to single coset in $[\infty 0\infty 14]$.

Now consider $[\infty 0\infty 232]$. $N^{(\infty 0\infty 232)}$ has just one orbit $\{1, 2, 3, 4, 0, \infty\}$. So we need to look at $[\infty 0\infty 2322]$.

$[\infty 0\infty 2322] = [\infty 0\infty 23]$ so t_2, a representative from the 6-orbit, takes $[\infty 0\infty 232]$ back to a single coset in $[\infty 0\infty 23]$.

Now consider $[\infty 0\infty 123]$. The orbits of $N^{(\infty 0\infty 123)}$ are $\{1, 3, \infty\}$ and $\{2, 4, 0\}$. So we need to look at $[\infty 0\infty 1233]$ and $[\infty 0\infty 1232]$.

First, $[\infty 0\infty 1233] = [\infty 0\infty 12]$ so t_3, a representative from one of the 3-orbits, takes $[\infty 0\infty 123]$ back to a single coset in $[\infty 0\infty 12]$.

t_2, a representative from the other 3-orbit, takes $[\infty 0\infty 123]$ to a single coset in $[\infty 0123]$ as was proved in Chapter 6.

Now consider $[\infty 0\infty 214]$. The orbits of $N^{(\infty 0\infty 214)}$ are $\{1, 3, \infty\}$ and $\{2, 4, 0\}$. So we need to look at $[\infty 0\infty 2141]$ and $[\infty 0\infty 2144]$.

First, $[\infty 0\infty 2144] = [\infty 0\infty 21]$ so t_4, a representative from one of the 3-orbits, takes $[\infty 0\infty 214]$ back to a single coset in $[\infty 0\infty 21]$.

t_1, a representative from the other 3-orbit, takes $[\infty 0\infty 214]$ to a single coset in $[\infty 0214]$ as was proved in Chapter 6.

Now consider $[\infty 01210]$. The orbits of $N^{(\infty 01210)}$ are $\{2\}$, $\{4\}$, $\{1, 0\}$, and $\{3, \infty\}$. So we need to look at $[\infty 012102]$, $[\infty 012104]$, $[\infty 012100]$, and $[\infty 012103]$.

First, $[\infty 012100] = [\infty 0121]$ so t_0, a representative from one of the 2-orbits, takes $[\infty 01210]$ back to a single coset in $[\infty 0121]$.

t_2 takes $[\infty 01210]$ to a single coset in $[\infty 0141]$ as was proved in Chapter 6.

t_4 takes $[\infty 01210]$ to a single coset in $[\infty 010\infty]$ as was proved in Chapter 6.

t_3, a representative from the other 2-orbit, takes $[\infty 01210]$ to a single coset in $[\infty 0\infty 212]$ since $N\infty 012103 = N(1,\infty,3)(2,4,0)\infty 1\infty 404 = N\infty 1\infty 404 \in [\infty 0\infty 212]$. To prove $\infty 012103 = (1,\infty,3)(2,4,0)\infty 1\infty 404$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,\infty,3)(2,4,0)\underline{\infty 1\infty}404301210\infty = (1,\infty,3)(2,4,0)\underline{1\infty 1}404301210\infty$ (by relation (5) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (1,\infty,3)(2,4,0)1\underline{\infty 140}4301210\infty = (1,\infty,3)(2,4,0)1\ \underline{(1,3,4,\infty,0)41\infty 3}$ 4301210∞ (by relation (1) conjugated by $(2,4,3,0,\infty) \in L_2(5)$)

$= (1,0,2,\infty,4)341\infty\underline{343}01210\infty = (1,0,2,\infty,4)341\infty\underline{434}01210\infty$ (by relation (6) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,0,2,\infty,4)341\infty 4\underline{3401}210\infty = (1,0,2,\infty,4)341\infty 4\ \underline{(1,4,2,0,3)0432}\ 210\infty$ (by relation (1) conjugated by $(1,4,2,0,3) \in L_2(5)$)

$= (1,3)(2,\infty)1\underline{24\infty 2}043\underline{22}10\infty = (1,3)(2,\infty)1\ \underline{(1,3,0)(2,\infty,4)42\infty 4}\ 04310\infty$ (by relation (4) conjugated by $(1,0)(3,\infty) \in L_2(5)$)

$= (1,0)(2,4)342\infty\underline{404}310\infty = (1,0)(2,4)342\infty\underline{040}310\infty$ (by relation (6) conjugated by $(1,0)(3,\infty) \in L_2(5)$)

$= (1,0)(2,4)342\infty 04\underline{0310}\infty = (1,0)(2,4)342\infty 04\underline{(1,3,0)(2,\infty,4)3013}\infty$ (by relation (3) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= (3,0)(4,\infty)02\infty 412\underline{3013\infty} = (3,0)(4,\infty)02\infty 412\underline{(1,2,\infty)(3,0,4)\infty 4\infty 1}$ (by relation (7) conjugated by $(1,2,3,4,0) \in L_2(5)$)

$= (1,2,\infty,3,4)4\infty 132\underline{\infty\infty}4\infty 1 = (1,2,\infty,3,4)4\infty 1324\infty 1 = e$ by relation (1) conjugated by $(1,\infty,0)(2,4,3) \in L_2(5)$.

Now consider $[\infty 02320]$. The orbits of $N^{(\infty 02320)}$ are $\{0\}$, $\{\infty\}$, $\{2,3\}$, and $\{1,4\}$. So we need to look at $[\infty 012102]$, $[\infty 012104]$, $[\infty 012100]$, and $[\infty 012103]$.

First, $[\infty 023200] = [\infty 0232]$ so t_0 takes $[\infty 02320]$ back to a single coset in $[\infty 0232]$.

t_∞ takes $[\infty 02320]$ to a single coset in $[\infty 020\infty]$ as was proved in Chapter 6.

t_2, a representative from one of the 2-orbits, takes $[\infty 02320]$ to a single coset in $[\infty 0242]$ as was proved in Chapter 6.

t_1, a representative from the other 2-orbit, takes $[\infty 02320]$ to a single coset in $[\infty 0\infty 212]$ since $N\infty 023201 = N(1,2,\infty)(3,0,4)3432\infty 2 = N3432\infty 2 \in [\infty 0\infty 212]$. To

prove $\infty 023201 = (1,2,\infty)(3,0,4)3432\infty 2$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,\infty)(3,0,4)3432\infty 210\underline{232}0\infty = (1,2,\infty)(3,0,4)3432\infty 210\underline{323}0\infty$ (by relation (5) conjugated by $(1,\infty,3)(2,4,0) \in L_5(5)$)

$= (1,2,\infty)(3,0,4)3432\infty 2\underline{1032}30\infty = (1,2,\infty)(3,0,4)3432\infty 2$
$\underline{(1,2,0,\infty,3)301\infty}\ 30\infty$ (by relation (2) conjugated by $(1,2,0)(3,\infty,4) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)1410\underline{30}301\infty 30\infty = (1,0,4)(2,3,\infty)1410\underline{03}001\infty 30\infty$ (by relation (5) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (1,0,4)(2,3,\infty)141\underline{00}3\underline{00}1\infty 30\infty = (1,0,4)(2,3,\infty)14131\ \underline{\infty 30\infty}$
$= (1,0,4)(2,3,\infty)14131\underline{(1,2,4)(3,\infty,0)3\infty 03}$ (by relation (4) conjugated by $(1,2,\infty)(3,0,4) \in L_2(5)$)

$= (1,3,0)(2,\infty,4)212\underline{\infty 023\infty 0}3 = (1,3,0)(2,\infty,4)212\underline{(1,0,3)(2,4,\infty)0403}3$ (by relation (7) conjugated by $(2,\infty,0,3,4) \in L_2(5)$)

$= \underline{404}040\underline{33} = \underline{040}040$ (by relation (6) conjugated by $(1,0)(3,\infty) \in L_2(5)$)

$= 04\underline{00}40 = 0\underline{44}0 = \underline{00} = e.$

Now consider $[\infty 01243]$. The orbits of $N^{(\infty 01243)}$ are $\{1\}$ and $\{2,3,4,0,\infty\}$. So we need to look at $[\infty 012431]$ and $[\infty 012433]$.

First, $[\infty 012433] = [\infty 0124]$ so t_3, a representative from the 5-orbit, takes $[\infty 01243]$ back to a single coset in $[\infty 0124]$.

t_1 takes $[\infty 01243]$ to a single coset in $[\infty 02431]$ since $N\infty 012431$
$= N(1,2,3,4,0)\infty 41320 = N\infty 41320 \in [\infty 02431]$. To prove $\infty 012431$
$= (1,2,3,4,0)\infty 41320$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(1,2,3,4,0)\infty 4132\underline{0134}210\infty = (1,2,3,4,0)\infty 4132\underline{(1,\infty,3,0,4)310\infty}210\infty$ (by relation (2) conjugated by $(1,4,3,\infty,2) \in L_2(5)$)

$= (1,2,0,\infty,3)31\underline{\infty 02310}\infty 210\infty = (1,2,0,\infty,3)31\ \underline{(1,\infty,4,2,3)3\infty 20\infty}$
$0\infty 210\infty$ (by a relation proved in Chapter 6)

$= (1,3,\infty)(2,0,4)1\infty 3\infty 20\underline{\infty 0\infty}210\infty = (1,3,\infty)(2,0,4)1\infty\ 3\infty 20\ \underline{0\infty 0}\ 210\infty$ (by relation (5))

$= (1,3,\infty)(2,0,4)1\underline{\infty 3\infty}2\underline{00}\infty 0210\infty = (1,3,\infty)(2,0,4)1\underline{3\infty 3}2\infty 0210\infty$ (by relation (5) conjugated by $(1,2)(3,0) \in L_2(5)$)

$= (1,3,\infty)(2,0,4)13\underline{\infty 32\infty 0}210\infty = (1,3,\infty)(2,0,4)13\ \underline{(1,\infty,3)(2,4,0)0102}$

210∞ (by relation (7) conjugated by $(1,4,3)(2,\infty,0) \in L_2(5)$)

$= \infty\underline{101}0\underline{22}10\infty = \infty\underline{010}010\infty$ (by relation (5) conjugated by $(1,\infty)(3,4) \in L_2(5)$)

$= \infty010\underline{01}0\infty = \infty0\underline{11}0\infty = \infty\underline{00}\infty = \underline{\infty\infty} = e$.

Now consider $[\infty02431]$. The orbits of $N^{(\infty02431)}$ are $\{2\}$ and $\{1,3,4,0,\infty\}$. So we need to look at $[\infty024312]$ and $[\infty024311]$.

First, $[\infty024311] = [\infty0243]$ so t_1, a representative from the 5-orbit, takes $[\infty02431]$ back to a single coset in $[\infty0243]$.

t_2 takes $[\infty02431]$ to a single coset in $[\infty01243]$ since $N\infty024312$ $= N(2,4,3,0,\infty)0423\infty1 = N0423\infty1 \in [\infty01243]$. To prove $\infty024312$ $= (2,4,3,0,\infty)0423\infty1$, we will we will move the relation to one side of the equal sign and prove it equals identity.

$(2,4,3,0,\infty)0423\infty\underline{121}3420\infty = (2,4,3,0,\infty)0423\infty\underline{212}3420\infty$ (by relation (6) conjugated by $(2,4)(3,\infty) \in L_2(5)$)

$= (2,4,3,0,\infty)04\underline{23\infty2}123420\infty = (2,4,3,0,\infty)04\ \underline{(1,4,0)(2,\infty,3)32\infty3}$ 123420∞ (by relation (3) conjugated by $(1,\infty,2)(3,4,0) \in L_2(5)$)

$= (1,4,2,0,3)10\underline{32\infty31}23420\infty = (1,4,2,0,3)10\underline{(1,\infty,0)(2,4,3)141}\infty23420\infty$ (by relation (7) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$)

$= (1,3,\infty,0,2)\infty\underline{11}41\infty23420\infty = (1,3,\infty,0,2)\underline{\infty41\infty}\ 23420\infty = (1,3,\infty,0,2)$ $\underline{(1,4,\infty)(2,3,0)4\infty14}$ 23420∞ (by relation (4) conjugated by $(1,3)(2,\infty) \in L_2(5)$)

$= (1,0,3)(2,4,\infty)4\infty1\underline{4234}20\infty = (1,0,3)(2,4,\infty)4\infty1$ $\underline{(1,\infty,0)(2,4,3)2432}20\infty$ (by relation (4) conjugated by $(1,0)(2,4) \in L_2(5)$)

$= (2,3,\infty,4,0)30\infty243\underline{22}0\infty = (2,3,\infty,4,0)30\infty2430\infty = e$ by relation (1) conjugated by $(1,0,\infty)(2,3,4) \in L_2(5)$.

Our double coset enumeration must be complete since the set of right cosets is closed under right multiplication by the symmetric generators.

Thus we have the Cayley diagram that is shown in Figure 7.1.

The maximum possible index of N in $G_1 \cong \frac{2^{*6}:L_2(5)}{[(0,1,2,3,4)t_0]^8,[(\infty,0,1)(2,4,3)t_\infty]^8,[(\infty,0)(1,4)t_1]^6,Z(G)}$ is $\frac{|N|}{|N|} + \frac{|N|}{|N^{(\infty)}|} + \frac{|N|}{|N^{(\infty0)}|} + \frac{|N|}{|N^{(\infty0\infty)}|} + \frac{|N|}{|N^{(\infty02)}|} + \cdots + \frac{|N|}{|N^{(\infty02431)}|} = 1 + 6 + 30 + 60 + 60 + \cdots + 6 = 1584$. Thus

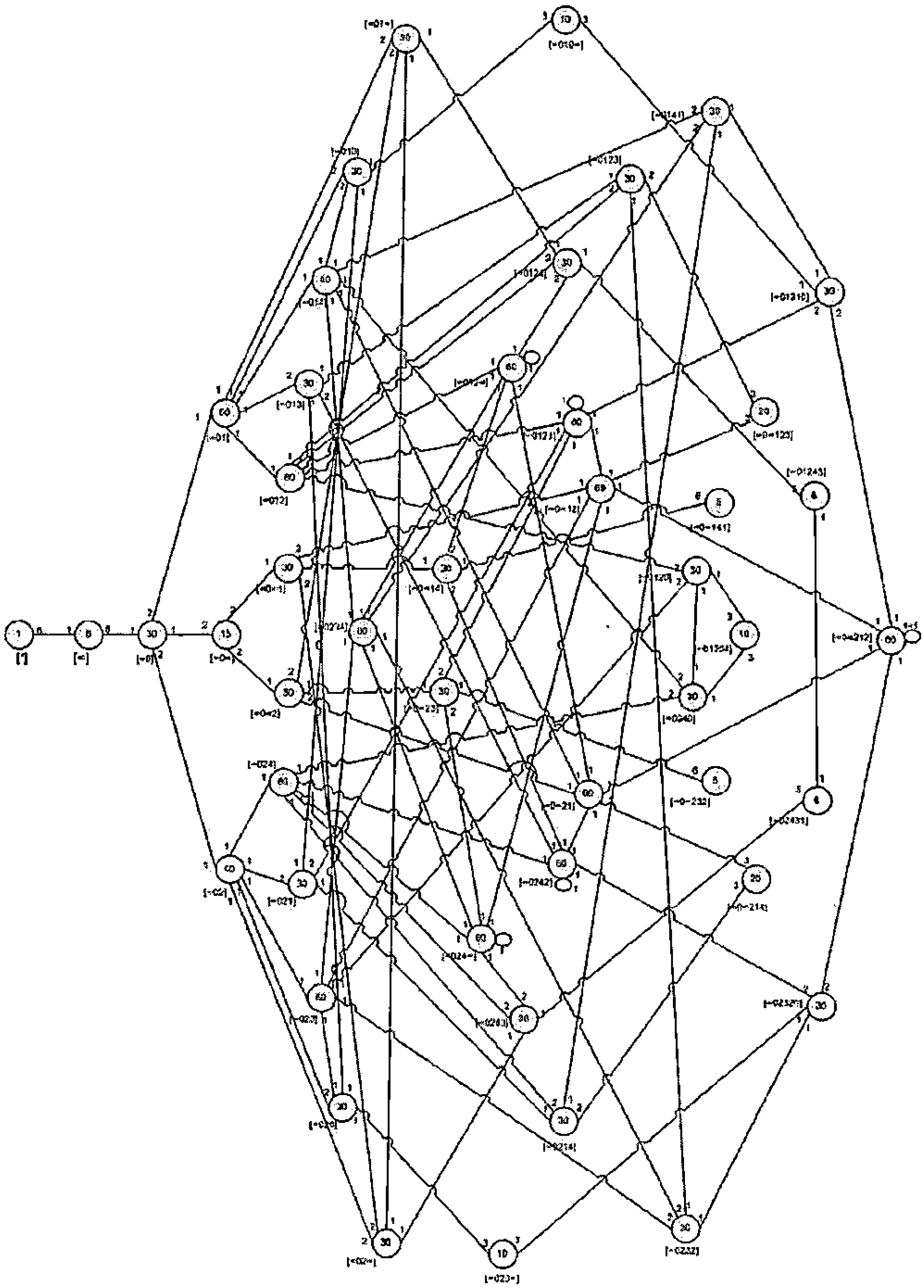

Figure 7.1: Cayley Diagram of M_{12} over $L_2(5)$

$|G_1| \leq 1584 \times |N| = 1584 \times 60 = 95,040$. In order to show $|G_1| = 95,040$, we consider G_1 as a subgroup of S_{1584} acting on 3168 cosets that we have found, and labeled according to the MAGMA segment, "for i in [1..1584] do print i, cst[i]; end for;"..

For this purpose we compute the action of the control group N as well as the action of $t_\infty, t_0, t_1, t_2, t_3$, and t_4 on the 1584 cosets. These permutations can be obtained by the following MAGMA segment "f(x); f(y); f(t);".

It readily checks that the order of $\langle x, y, t\rangle$, a subgroup of the symmetric group S_{1584} acting on the 1584 right cosets of N in G_1, is 95,040. Visibly $|x| = 5$, $|y| = 3$, and $|xy| = 2$, hence $\langle x, y\rangle \cong L_2(5)$. If we conjugate t by $L_2(5)$ we see that t has exactly six conjugates. We conclude that $\langle x, y, t\rangle$ is a homomorphic image of the progenitor $2^{*6} : L_2(5)$.

Thus if the original six relations of Chapter 6, plus the seventh relation we found for the center of $2 \times M_{12}$, hold in $\langle x, y, t\rangle$, then $\langle x, y, t\rangle$ is a homomorphic image of G_1 and this will give $|G_1| \geq |\langle x, y, t\rangle| = 95,040$.

Verify relation (1) $t_2t_1t_0t_4t_3t_2t_1t_0 = (1,3,0,2,4)$ by conjugating the six symmetric generators by $t_2t_1t_0t_4t_3t_2t_1t_0$. By multiplying the permutations found using the MAGMA segments previously listed, we find $t_2t_1t_0t_4t_3t_2t_1t_0 = (1,3,0,2,4)$. Thus when we conjugate $t_\infty, t_0, t_1, t_2, t_3$ and t_4 by $t_2t_1t_0t_4t_3t_2t_1t_0$ we obtain:

$t_\infty^{t_2t_1t_0t_4t_3t_2t_1t_0} = t_\infty$ $\quad$ $t_0^{t_2t_1t_0t_4t_3t_2t_1t_0} = t_2$
$t_1^{t_2t_1t_0t_4t_3t_2t_1t_0} = t_3$ $\quad$ $t_2^{t_2t_1t_0t_4t_3t_2t_1t_0} = t_4$
$t_3^{t_2t_1t_0t_4t_3t_2t_1t_0} = t_0$ $\quad$ $t_4^{t_2t_1t_0t_4t_3t_2t_1t_0} = t_1$

So $t_2t_1t_0t_4t_3t_2t_1t_0$ acts as $(1,3,0,2,4)$.

Verify relation (2) $t_4t_2t_0t_3t_1t_4t_2t_0 = (1,0,4,3,2)$ by conjugating the six symmetric generators by $t_4t_2t_0t_3t_1t_4t_2t_0$. By multiplying the permutations found using the MAGMA segments previously listed, we find $t_4t_2t_0t_3t_1t_4t_2t_0 = (1,0,4,3,2)$. Thus when we conjugate $t_\infty, t_0, t_1, t_2, t_3$ and t_4 by $t_4t_2t_0t_3t_1t_4t_2t_0$ we obtain:

$t_\infty^{t_4t_2t_0t_3t_1t_4t_2t_0} = t_\infty$ $\quad$ $t_0^{t_4t_2t_0t_3t_1t_4t_2t_0} = t_4$
$t_1^{t_4t_2t_0t_3t_1t_4t_2t_0} = t_0$ $\quad$ $t_2^{t_4t_2t_0t_3t_1t_4t_2t_0} = t_1$
$t_3^{t_4t_2t_0t_3t_1t_4t_2t_0} = t_2$ $\quad$ $t_4^{t_4t_2t_0t_3t_1t_4t_2t_0} = t_3$

So $t_4t_2t_0t_3t_1t_4t_2t_0$ acts as $(1,0,4,3,2)$.

Verify relation (3) $t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty = (1,\infty,0)(2,4,3)$ by conjugating the six symmetric generators by $t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty$. By multiplying the permutations found using

the MAGMA segments previously listed, we find $t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty = (1,\infty,0)(2,4,3)$. Thus when we conjugate $t_\infty, t_0, t_1, t_2, t_3$ and t_4 by $t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty$ we obtain:

$$\begin{array}{ll} t_\infty^{t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty} = t_0 & t_0^{t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty} = t_1 \\ t_1^{t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty} = t_\infty & t_2^{t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty} = t_4 \\ t_3^{t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty} = t_2 & t_4^{t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty} = t_3 \end{array}$$

So $t_0t_\infty t_1t_0t_\infty t_1t_0t_\infty$ acts as $(1,\infty,0)(2,4,3)$.

Verify relation (4) $t_4t_2t_3t_4t_2t_3t_4t_2 = (1,\infty,0)(2,4,3)$ by conjugating the six symmetric generators by $t_4t_2t_3t_4t_2t_3t_4t_2$. By multiplying the permutations found using the MAGMA segments previously listed, we find $t_4t_2t_3t_4t_2t_3t_4t_2 = (1,\infty,0)(2,4,3)$. Thus when we conjugate $t_\infty, t_0, t_1, t_2, t_3$ and t_4 by $t_4t_2t_3t_4t_2t_3t_4t_2$ we obtain:

$$\begin{array}{ll} t_\infty^{t_4t_2t_3t_4t_2t_3t_4t_2} = t_0 & t_0^{t_4t_2t_3t_4t_2t_3t_4t_2} = t_1 \\ t_1^{t_4t_2t_3t_4t_2t_3t_4t_2} = t_\infty & t_2^{t_4t_2t_3t_4t_2t_3t_4t_2} = t_4 \\ t_3^{t_4t_2t_3t_4t_2t_3t_4t_2} = t_2 & t_4^{t_4t_2t_3t_4t_2t_3t_4t_2} = t_3 \end{array}$$

So $t_4t_2t_3t_4t_2t_3t_4t_2$ acts as $(1,\infty,0)(2,4,3)$.

Verify relation (5) $t_0t_\infty t_0t_\infty t_0t_\infty = 1$ by conjugating the six symmetric generators by $t_0t_\infty t_0t_\infty t_0t_\infty$. By multiplying the permutations found using the MAGMA segments previously listed, we find $t_0t_\infty t_0t_\infty t_0t_\infty$ equals identity. Thus when we conjugate $t_\infty, t_0, t_1, t_2, t_3$ and t_4 by $t_0t_\infty t_0t_\infty t_0t_\infty$ we obtain:

$$\begin{array}{ll} t_\infty^{t_0t_\infty t_0t_\infty t_0t_\infty} = t_\infty & t_0^{t_0t_\infty t_0t_\infty t_0t_\infty} = t_0 \\ t_1^{t_0t_\infty t_0t_\infty t_0t_\infty} = t_1 & t_2^{t_0t_\infty t_0t_\infty t_0t_\infty} = t_2 \\ t_3^{t_0t_\infty t_0t_\infty t_0t_\infty} = t_3 & t_4^{t_0t_\infty t_0t_\infty t_0t_\infty} = t_4 \end{array}$$

So $t_0t_\infty t_0t_\infty t_0t_\infty$ acts as the identity.

Verify relation (6) $t_4t_1t_4t_1t_4t_1 = 1$ by conjugating the six symmetric generators by $t_4t_1t_4t_1t_4t_1$. By multiplying the permutations found using the MAGMA segments previously listed, we find $t_4t_1t_4t_1t_4t_1$ equals identity. Thus when we conjugate $t_\infty, t_0, t_1, t_2, t_3$ and t_4 by $t_4t_1t_4t_1t_4t_1$ we obtain:

$$\begin{array}{ll} t_\infty^{t_4t_1t_4t_1t_4t_1} = t_\infty & t_0^{t_4t_1t_4t_1t_4t_1} = t_0 \\ t_1^{t_4t_1t_4t_1t_4t_1} = t_1 & t_2^{t_4t_1t_4t_1t_4t_1} = t_2 \\ t_3^{t_4t_1t_4t_1t_4t_1} = t_3 & t_4^{t_4t_1t_4t_1t_4t_1} = t_4 \end{array}$$

So $t_4t_1t_4t_1t_4t_1$ acts as the identity.

Verify relation (7) $t_2t_4t_0t_2t_\infty t_0t_\infty t_3t_\infty = (1,\infty,0)(2,4,3)$ by conjugating the six symmetric generators by $t_2t_4t_0t_2t_\infty t_0t_\infty t_3t_\infty$. By multiplying the permutations found using the MAGMA segments previously listed, we find $t_2t_4t_0t_2t_\infty t_0t_\infty t_3t_\infty =$ $(1,\infty,0)(2,4,3)$. Thus when we conjugate $t_\infty, t_0, t_1, t_2, t_3$ and t_4 by $t_2t_4t_0t_2t_\infty t_0t_\infty t_3t_\infty$ we obtain:

$$t_\infty^{t_2t_4t_0t_2t_\infty t_0t_\infty t_3t_\infty} = t_0 \qquad t_0^{t_2t_4t_0t_2t_\infty t_0t_\infty t_3t_\infty} = t_1$$
$$t_1^{t_2t_4t_0t_2t_\infty t_0t_\infty t_3t_\infty} = t_\infty \qquad t_2^{t_2t_4t_0t_2t_\infty t_0t_\infty t_3t_\infty} = t_4$$
$$t_3^{t_2t_4t_0t_2t_\infty t_0t_\infty t_3t_\infty} = t_2 \qquad t_4^{t_2t_4t_0t_2t_\infty t_0t_\infty t_3t_\infty} = t_3$$

Thus $G_1/ker\phi \cong \langle x, y, t\rangle$ and $|G_1| \geq |\langle x, y, t\rangle| = 95040$. As shown earlier, $|G_1| \leq 95040$. Hence $|G_1| = 95040$.

Moreover, $b = (1, 3, \infty, 0, 2)t_\infty t_0 t_1 t_3 t_0 t_2 = bb = x^{-2}yt_\infty t_0 t_1 t_3 t_0 t_2$ and $c = (1, 2, 4)(3, \infty, 0)t_1 t_2 t_4 t_1 t_\infty t_2 = cc = yx^{-1}yx^2 t_1 t_2 t_4 t_1 t_\infty t_2$,are in G_1 with $M_{12} \cong \langle b, c | b^2 = c^3 = (bc)^{11} = [b, c]^6 = (bcbcbc^{-1})^6 = [b, cbc]^5 = 1\rangle$. So $\langle b, c\rangle \leq G_1$, but $|\langle b, c\rangle| = |G_1|$, therefore $G_1 = \langle b, c\rangle \cong M_{12}$.

Appendix A

MAGMA Code for Construction of $2 \times M_{12}$

```
S6:=Sym(6)
xx:=S6!(5,1,2,3,4);
yy:=S6!(6,5,1)(2,4,3);
N:=sub<S6|xx,yy>;

G<x,y,t>:=Group<x,y,t|x^5,y^3,(y*x)^2,t^2,(t,x),(t^(y*x^2),x*y),
(x*t^y)^8,(y*t)^8,(t^(y*x^4)*t^(y^2))^3>;

f,G1,k:=CosetAction(G,sub<G|x,y>);
N6:=Stabiliser(N,6);
R:=[Id(G1): i in [1..6]];
R[6]:=f(t);
R[5]:=f(t^y);
R[1]:=f(t^(y^2));
R[2]:=f(t^(y*x^2));
R[3]:=f(t^(y*x^3));
R[4]:=f(t^(y*x^4));
S:={[6,5]};
N65:=Stabilizer(N6,5);
T6:=Transversal(N,N6);
T65:=Transversal(N,N65);
N656:=Stabilizer(N65,6);

for g in N do if (6^g eq 5 and 5^g eq 6) then
N656s:=sub<N|N656,g>;
```

```
end if; end for;
T656:=Transversal(N,N656s);

ts := R;
cst := [null : i in [1 .. 3168]] where null is [Integers() | ];
for i := 1 to 6 do
 prodim := function(pt, Q, I)
  v := pt;
  for i in I do
    v := v^(Q[i]);
  end for;
  return v;
 end function;
end for;

for i in [1..#T6] do
  ss := [6]^T6[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

for i in [1..#T65] do
  ss := [6,5]^T65[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
for i in [1..#T656] do
 ss := [6,5,6]^T656[i];
 cst[prodim(1, ts, ss)] := ss;
end for;

N651:=Stabilizer(N65,1);
N652:=Stabilizer(N65,2);
T651:= Transversal(N,N651);
T652:= Transversal(N,N652);
for i in [1..#T651] do
 ss := [6,5,1]^T651[i];
 cst[prodim(1, ts, ss)] := ss;
end for;

for i in [1..#T652] do
  ss := [6,5,2]^T652[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6561:=Stabilizer(N656,1);
```

```
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1) then
  N6561s:=sub<N|N6561,g>;
end if; end for;

T6561:= Transversal(N,N6561s);
for i in [1..#T6561] do
  ss := [6,5,6,1]^T6561[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6562:=Stabilizer(N656,2);
for g in N do if (6^g eq 5 and 5^g eq 6 and 2^g eq 2) then
 N6562s:=sub<N|N6562,g>;
end if; end for;

T6562:= Transversal(N,N6562s);
for i in [1..#T6562] do
  ss := [6,5,6,2]^T6562[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651:=Stabilizer(N65,1);
N6516:=Stabilizer(N651,6);
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1) then
  N6516s:=sub<N|N6516,g>;
end if; end for;

T6516:= Transversal(N,N6516s);
for i in [1..#T6516] do
  ss := [6,5,1,6]^T6516[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6515:=Stabilizer(N651,5);
for g in N do if (6^g eq 6 and 5^g eq 1 and 1^g eq 5) then
  N6515s:=sub<N|N6515,g>;
end if; end for;

T6515:= Transversal(N,N6515s);
for i in [1..#T6515] do
  ss := [6,5,1,5]^T6515[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
```

```
N6512:=Stabilizer(N651,2);
T6512:= Transversal(N,N6512);
for i in [1..#T6512] do
  ss := [6,5,1,2]^T6512[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6513:=Stabilizer(N651,3);
for g in N do if (6^g eq 1 and 5^g eq 5 and 1^g eq 6
and 3^g eq 4) then
  N6513s:=sub<N|N6513,g>;
end if; end for;

T6513:= Transversal(N,N6513s);
for i in [1..#T6513] do
  ss := [6,5,1,3]^T6513[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6514:=Stabilizer(N651,4);
T6514:= Transversal(N,N6514);
for i in [1..#T6514] do
  ss := [6,5,1,4]^T6514[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6526:=Stabilizer(N652,6);
for g in N do if (6^g eq 5 and 5^g eq 6 and 2^g eq 2) then
  N6526s:=sub<N|N6526,g>;
end if; end for;

T6526:= Transversal(N,N6526s);
for i in [1..#T6526] do
  ss := [6,5,2,6]^T6526[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6525:=Stabilizer(N652,5);
for g in N do if (6^g eq 6 and 5^g eq 2 and 2^g eq 5) then
  N6525s:=sub<N|N6525,g>;
end if; end for;

T6525:= Transversal(N,N6525s);
for i in [1..#T6525] do
```

```
  ss := [6,5,2,5]^T6525[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6521:=Stabilizer(N652,1);
for g in N do if (6^g eq 2 and 5^g eq 5 and 2^g eq 6 and 1^g eq 3) then
  N6521s:=sub<N|N6521,g>;
end if; end for;

T6521:= Transversal(N,N6521s);
for i in [1..#T6521] do
  ss := [6,5,2,1]^T6521[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6523:=Stabilizer(N652,3);
N6523:=Stabilizer(N652,3);
T6523:= Transversal(N,N6523);
for i in [1..#T6523] do
  ss := [6,5,2,3]^T6523[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6524:=Stabilizer(N652,4);
T6524:= Transversal(N,N6524);
for i in [1..#T6524] do
  ss := [6,5,2,4]^T6524[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65614:=Stabilizer(N6561,4);
for g in N do if (6^g eq 5 and 5^g eq 6 and 4^g eq 4) then
  N65614s:=sub<N|N65614,g>;
end if; end for;

T65614:= Transversal(N,N65614s);
for i in [1..#T65614] do
  ss := [6,5,6,1,4]^T65614[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65612:=Stabilizer(N6561,2);
T65612:= Transversal(N,N65612);
for i in [1..#T65612] do
```

```
  ss := [6,5,6,1,2]^T65612[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65615:=Stabilizer(N6561,5);
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1) then
  N65615s:=sub<N|N65615,g>;
end if; end for;

T65615:= Transversal(N,N65615s);
for i in [1..#T65615] do
  ss := [6,5,6,1,5]^T65615[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65623:=Stabilizer(N6562,3);
for g in N do if (6^g eq 5 and 5^g eq 6 and 2^g eq 2 and 3^g eq 3) then
  N65623s:=sub<N|N65623,g>;
end if; end for;

T65623:= Transversal(N,N65623s);
for i in [1..#T65623] do
  ss := [6,5,6,2,3]^T65623[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65621:=Stabilizer(N6562,1);
T65621:= Transversal(N,N65621);
for i in [1..#T65621] do
  ss := [6,5,6,2,1]^T65621[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65625:=Stabilizer(N6562,5);
for g in N do if (6^g eq 5 and 5^g eq 6 and 2^g eq 2) then
  N65625s:=sub<N|N65625,g>;
end if; end for;

T65625:= Transversal(N,N65625s);
for i in [1..#T65625] do
  ss := [6,5,6,2,5]^T65625[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
```

```
N65164:=Stabilizer(N6516,4);
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1 and 4^g eq 4) then
  N65164s:=sub<N|N65164,g>;
end if; end for;

T65164:= Transversal(N,N65164s);
for i in [1..#T65164] do
  ss := [6,5,1,6,4]^T65164[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65162:=Stabilizer(N6516,2);
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1 and 2^g eq 3) then
  N65162s:=sub<N|N65162,g>;
end if; end for;

T65162:= Transversal(N,N65162s);
for i in [1..#T65162] do
  ss := [6,5,1,6,2]^T65162[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65152:=Stabilizer(N6515,2);
T65152:= Transversal(N,N65152);
for i in [1..#T65152] do
  ss := [6,5,1,5,2]^T65152[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65153:=Stabilizer(N6515,3);
for g in N do if (6^g eq 6 and 5^g eq 1 and 1^g eq 5 and 3^g eq 3) then
  N65153s:=sub<N|N65153,g>;
end if; end for;

T65153:= Transversal(N,N65153s);
for i in [1..#T65153] do
  ss := [6,5,1,5,3]^T65153[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65121:=Stabilizer(N6512,1);
T65121:= Transversal(N,N65121);
for i in [1..#T65121] do
  ss := [6,5,1,2,1]^T65121[i];
```

```
  cst[prodim(1, ts, ss)] := ss;
end for;

N65156:=Stabilizer(N6515,6);
N65156s:=sub<N|N65156>;
for g in N do if ((6^g eq 1 and (5^g eq 5 or 5^g eq 6) and
(1^g eq 6 or 1^g eq 5)) or (6^g eq 5 and (5^g eq 1 or 5^g eq 6)
and (1^g eq 6 or 1^g eq 1)) or (6^g eq 6 and 5^g eq 1 and 1^g eq 5)) then
  N65156s:=sub<N|N65156s,g>;
end if; end for;

T65156:= Transversal(N,N65156s);
for i in [1..#T65156] do
  ss := [6,5,1,5,6]^T65156[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65123:=Stabilizer(N6512,3);
N65123s:=sub<N|N65123>;
for g in N do if (6^g eq 6 and 5^g eq 2 and 1^g eq 1
and 2^g eq 5 and 3^g eq 4) then
  N65123s:=sub<N|N65123s,g>;
end if; end for;

T65123:= Transversal(N,N65123s);
for i in [1..#T65123] do
  ss := [6,5,1,2,3]^T65123[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65125:=Stabilizer(N6512,5);
N65125s:=sub<N|N65125>;
for g in N do if (6^g eq 3 and 5^g eq 1 and 1^g eq 5 and 2^g eq 2) then
  N65125s:=sub<N|N65125s,g>;
end if; end for;

T65125:= Transversal(N,N65125s);
for i in [1..#T65125] do
  ss := [6,5,1,2,5]^T65125[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65126:=Stabilizer(N6512,6);
T65126:= Transversal(N,N65126);
```

```
for i in [1..#T65126] do
  ss := [6,5,1,2,6]^T65126[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65135:=Stabilizer(N6513,5);
N65135s:=sub<N|N65135>;
for g in N do if (6^g eq 1 and 5^g eq 5 and 1^g eq 6 and 3^g eq 4) then
  N65135s:=sub<N|N65135s,g>;
end if; end for;

T65135:= Transversal(N,N65135s);
for i in [1..#T65135] do
  ss := [6,5,1,3,5]^T65135[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65131:=Stabilizer(N6513,1);
T65131:= Transversal(N,N65131);
for i in [1..#T65131] do
  ss := [6,5,1,3,1]^T65131[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65141:=Stabilizer(N6514,1);
N65141s:=sub<N|N65141>;
for g in N do if (6^g eq 6 and 5^g eq 5 and 4^g eq 1 and 1^g eq 4) then
  N65141s:=sub<N|N65141s,g>;
end if; end for;

T65141:= Transversal(N,N65141s);
for i in [1..#T65141] do
  ss := [6,5,1,4,1]^T65141[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65142:=Stabilizer(N6514,2);
T65142:= Transversal(N,N65142);
for i in [1..#T65142] do
  ss := [6,5,1,4,2]^T65142[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65143:=Stabilizer(N6514,3);
```

```
N65143s:=sub<N|N65143>;
for g in N do if (6^g eq 1 and 5^g eq 5 and 1^g eq 6 and
4^g eq 3 and 3^g eq 4) then
  N65143s:=sub<N|N65143s,g>;
end if; end for;

T65143:= Transversal(N,N65143s);
for i in [1..#T65143] do
  ss := [6,5,1,4,3]^T65143[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65145:=Stabilizer(N6514,5);
N65145s:=sub<N|N65145>;
for g in N do if (6^g eq 6 and 2^g eq 5 and 1^g eq 1 and 3^g eq 4) then
  N65145s:=sub<N|N65145s,g>;
end if; end for;

T65145:= Transversal(N,N65145s);
for i in [1..#T65145] do
  ss := [6,5,1,4,5]^T65145[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65241:=Stabilizer(N6524,1);
N65241s:=sub<N|N65241>;
for g in N do if (6^g eq 6 and 5^g eq 4 and 2^g eq 2 and
4^g eq 5 and 1^g eq 3) then
  N65241s:=sub<N|N65241s,g>;
end if; end for;

T65241:= Transversal(N,N65241s);
for i in [1..#T65241] do
  ss := [6,5,2,4,1]^T65241[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65243:=Stabilizer(N6524,3);
N65243s:=sub<N|N65243>;
for g in N do if (6^g eq 4 and 5^g eq 3 and 2^g eq 2 and
4^g eq 6 and 3^g eq 5) then
  N65243s:=sub<N|N65243s,g>;
end if; end for;
```

```
T65243:= Transversal(N,N65243s);
for i in [1..#T65243] do
  ss := [6,5,2,4,3]^T65243[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65246:=Stabilizer(N6524,6);
T65246:= Transversal(N,N65246);
for i in [1..#T65246] do
  ss := [6,5,2,4,6]^T65246[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65232:=Stabilizer(N6523,2);
N65232s:=sub<N|N65232>;
for g in N do if (6^g eq 6 and 5^g eq 5 and 2^g eq 3 and 3^g eq 2) then
  N65232s:=sub<N|N65232s,g>;
end if; end for;

T65232:= Transversal(N,N65232s);
for i in [1..#T65232] do
  ss := [6,5,2,3,2]^T65232[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65251:=Stabilizer(N6525,1);
N65251s:=sub<N|N65251>;
for g in N do if (6^g eq 6 and 5^g eq 2 and 2^g eq 5 and 1^g eq 1) then
  N65251s:=sub<N|N65251s,g>;
end if; end for;

T65251:= Transversal(N,N65251s);
for i in [1..#T65251] do
  ss := [6,5,2,5,1]^T65251[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65256:=Stabilizer(N6525,6);
N65256s:=sub<N|N65256>;
for g in N do if ((6^g eq 2 and 5^g eq 5 and 2^g eq 6) or
(6^g eq 5 and 5^g eq 2 and 2^g eq 6) or (6^g eq 5 and 5^g eq 6
and 2^g eq 2) or (6^g eq 2 and 5^g eq 6 and 2^g eq 5) or
(6^g eq 6 and 5^g eq 2 and 2^g eq 5)) then
  N65256s:=sub<N|N65256s,g>;
```

```
end if; end for;

T65256:= Transversal(N,N65256s);
for i in [1..#T65256] do
  ss := [6,5,2,5,6]^T65256[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65253:=Stabilizer(N6525,3);
T65253:= Transversal(N,N65253);
for i in [1..#T65253] do
  ss := [6,5,2,5,3]^T65253[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65261:=Stabilizer(N6526,1);
N65261s:=sub<N|N65261>;
for g in N do if (6^g eq 5 and 5^g eq 6 and 2^g eq 2 and 1^g eq 4) then
  N65261s:=sub<N|N65261s,g>;
end if; end for;

T65261:= Transversal(N,N65261s);
for i in [1..#T65261] do
  ss := [6,5,2,6,1]^T65261[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651621:=Stabilizer(N65162,1);
N651621s:=sub<N|N651621>;
for g in N do if ((6^g eq 5 and 5^g eq 6 and 1^g eq 4 and 2^g eq 2)
 or (6^g eq 6 and 5^g eq 5 and 1^g eq 4 and 2^g eq 3) or
(6^g eq 5 and 5^g eq 6 and 1^g eq 1 and 2^g eq 3)) then
  N651621s:=sub<N|N651621s,g>;
end if; end for;

T651621:= Transversal(N,N651621s);
for i in [1..#T651621] do
  ss := [6,5,1,6,2,1]^T651621[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651624:=Stabilizer(N65162,4);
N651624s:=sub<N|N651624>;
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1 and
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2^g eq 3 and 4^g eq 4) then
  N651624s:=sub<N|N651624s,g>;
end if; end for;

T651624:= Transversal(N,N651624s);
for i in [1..#T651624] do
  ss := [6,5,1,6,2,4]^T651624[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651625:=Stabilizer(N65162,5);
T651625:= Transversal(N,N651625);
for i in [1..#T651625] do
  ss := [6,5,1,6,2,5]^T651625[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651641:=Stabilizer(N65164,1);
N651641s:=sub<N|N651641>;
for g in N do if (6^g eq 2 and 5^g eq 1 and 1^g eq 6 and 4^g eq 4)
or (6^g eq 3 and 5^g eq 6 and 1^g eq 2 and 4^g eq 4) or
(6^g eq 2 and 5^g eq 5 and 1^g eq 3 and 4^g eq 4) or
(6^g eq 1 and 5^g eq 2 and 1^g eq 6 and 4^g eq 4) or
(6^g eq 3 and 5^g eq 1 and 1^g eq 5 and 4^g eq 4) or
(6^g eq 6 and 5^g eq 3 and 1^g eq 2 and 4^g eq 4) or
(6^g eq 5 and 5^g eq 6 and 1^g eq 1 and 4^g eq 4) or
(6^g eq 5 and 5^g eq 2 and 1^g eq 3 and 4^g eq 4) or
(6^g eq 1 and 5^g eq 3 and 1^g eq 5 and 4^g eq 4) then
  N651641s:=sub<N|N651641s,g>;
end if; end for;

T651641:= Transversal(N,N651641s);
for i in [1..#T651641] do
  ss := [6,5,1,6,4,1]^T651641[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651645:=Stabilizer(N65164,5);
T651645:= Transversal(N,N651645);
for i in [1..#T651645] do
  ss := [6,5,1,6,4,5]^T651645[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
```

```
N651523:=Stabilizer(N65152,3);
N651523s:=sub<N|N651523>;
for g in N do if (6^g eq 6 and 5^g eq 4 and 1^g eq 3
and 2^g eq 2 and 3^g eq 1) then
  N651523s:=sub<N|N651523s,g>;
end if; end for;

T651523:= Transversal(N,N651523s);
for i in [1..#T651523] do
  ss := [6,5,1,5,2,3]^T651523[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651524:=Stabilizer(N65152,4);
T651524:= Transversal(N,N651524);
for i in [1..#T651524] do
  ss := [6,5,1,5,2,4]^T651524[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651525:=Stabilizer(N65152,5);
N651525s:=sub<N|N651525>;
for g in N do if (6^g eq 3 and 5^g eq 1 and 1^g eq 5
and 2^g eq 2) then
  N651525s:=sub<N|N651525s,g>;
end if; end for;

T651525:= Transversal(N,N651525s);
for i in [1..#T651525] do
  ss := [6,5,1,5,2,5]^T651525[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651536:=Stabilizer(N65153,6);
N651536s:=sub<N|N651536>;
for g in N do if (6^g eq 6 and 5^g eq 1 and 1^g eq 5 and 3^g eq 3) then
  N651536s:=sub<N|N651536s,g>;
end if; end for;

T651536:= Transversal(N,N651536s);
for i in [1..#T651536] do
  ss := [6,5,1,5,3,6]^T651536[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
```

```
N651531:=Stabilizer(N65153,1);
T651531:= Transversal(N,N651531);
for i in [1..#T651531] do
  ss := [6,5,1,5,3,1]^T651531[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651532:=Stabilizer(N65153,2);
T651532:= Transversal(N,N651532);
for i in [1..#T651532] do
  ss := [6,5,1,5,3,2]^T651532[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651562:=Stabilizer(N65156,2);
N651562s:=sub<N|N651562>;
for g in N do if (6^g eq 1 and 5^g eq 5 and 1^g eq 6 and 2^g eq 2) then
  N651562s:=sub<N|N651562s,g>;
end if; end for;

T651562:= Transversal(N,N651562s);
for i in [1..#T651562] do
  ss := [6,5,1,5,6,2]^T651562[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651213:=Stabilizer(N65121,3);
T651213:= Transversal(N,N651213);
for i in [1..#T651213] do
  ss := [6,5,1,2,1,3]^T651213[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651231:=Stabilizer(N65123,1);
N651231s:=sub<N|N651231>;
for g in N do if (6^g eq 6 and 5^g eq 2 and 1^g eq 1 and
2^g eq 5 and 3^g eq 4) then
  N651231s:=sub<N|N651231s,g>;
end if; end for;

T651231:= Transversal(N,N651231s);
for i in [1..#T651231] do
  ss := [6,5,1,2,3,1]^T651231[i];
```

```
  cst[prodim(1, ts, ss)] := ss;
end for;

N651232:=Stabilizer(N65123,2);
N651232s:=sub<N|N651232>;
for g in N do if (6^g eq 3 and 5^g eq 1 and 1^g eq 4 and 2^g eq 6
and 3^g eq 2) or (6^g eq 2 and 5^g eq 4 and 1^g eq 5 and 2^g eq 3
and 3^g eq 6) then
  N651232s:=sub<N|N651232s,g>;
end if; end for;

T651232:= Transversal(N,N651232s);
for i in [1..#T651232] do
  ss := [6,5,1,2,3,2]^T651232[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651254:=Stabilizer(N65125,4);
N651254s:=sub<N|N651254>;
for g in N do if (6^g eq 6 and 5^g eq 5 and 1^g eq 4 and 2^g eq 3
and 4^g eq 1) or (6^g eq 2 and 5^g eq 4 and 1^g eq 1 and 2^g eq 6
and 4^g eq 5) or (6^g eq 3 and 5^g eq 1 and 1^g eq 5 and 2^g eq 2
and 4^g eq 4) or (6^g eq 3 and 5^g eq 1 and 1^g eq 4 and 2^g eq 6
and 4^g eq 5) or (6^g eq 2 and 5^g eq 4 and 1^g eq 5 and 2^g eq 3
and 4^g eq 1) then
  N651254s:=sub<N|N651254s,g>;
end if; end for;

T651254:= Transversal(N,N651254s);
for i in [1..#T651254] do
  ss := [6,5,1,2,5,4]^T651254[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651263:=Stabilizer(N65126,3);
N651263s:=sub<N|N651263>;
for g in N do if (6^g eq 1 and 5^g eq 5 and 1^g eq 6 and
2^g eq 2 and 3^g eq 4) then
  N651263s:=sub<N|N651263s,g>;
end if; end for;

T651263:= Transversal(N,N651263s);
for i in [1..#T651263] do
  ss := [6,5,1,2,6,3]^T651263[i];
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  cst[prodim(1, ts, ss)] := ss;
end for;

N651264:=Stabilizer(N65126,4);
N651264s:=sub<N|N651264>;
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 4 and
2^g eq 2 and 4^g eq 1) then
  N651264s:=sub<N|N651264s,g>;
end if; end for;

T651264:= Transversal(N,N651264s);
for i in [1..#T651264] do
  ss := [6,5,1,2,6,4]^T651264[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651352:=Stabilizer(N65135,2);
N651352s:=sub<N|N651352>;
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1 and 3^g eq 2
and 2^g eq 3) or (6^g eq 6 and 5^g eq 1 and 1^g eq 5 and 3^g eq 3
and 2^g eq 4) or (6^g eq 1 and 5^g eq 5 and 1^g eq 6 and 3^g eq 4
and 2^g eq 2) or (6^g eq 5 and 5^g eq 1 and 1^g eq 6 and 3^g eq 2
and 2^g eq 4) or (6^g eq 1 and 5^g eq 6 and 1^g eq 5 and 3^g eq 4
and 2^g eq 3) then
  N651352s:=sub<N|N651352s,g>;
end if; end for;

T651352:= Transversal(N,N651352s);
for i in [1..#T651352] do
  ss := [6,5,1,3,5,2]^T651352[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651315:=Stabilizer(N65131,5);
N651315s:=sub<N|N651315>;
for g in N do if (6^g eq 4 and 5^g eq 3 and 1^g eq 1 and 3^g eq 5) then
  N651315s:=sub<N|N651315s,g>;
end if; end for;

T651315:= Transversal(N,N651315s);
for i in [1..#T651315] do
  ss := [6,5,1,3,1,5]^T651315[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
```

```
N651316:=Stabilizer(N65131,6);
T651316:= Transversal(N,N651316);
for i in [1..#T651316] do
  ss := [6,5,1,3,1,6]^T651316[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

for i in [1..#T651645] do
  ss := [6,5,1,6,4,5]^T651645[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651426:=Stabilizer(N65142,6);
T651426:= Transversal(N,N651426);
for i in [1..#T651426] do
  ss := [6,5,1,4,2,6]^T651426[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651435:=Stabilizer(N65143,5);
N651435s:=sub<N|N651435>;
for g in N do if (6^g eq 1 and 5^g eq 5 and 1^g eq 6 and
4^g eq 3 and 3^g eq 4) then
  N651435s:=sub<N|N651435s,g>;
end if; end for;

T651435:= Transversal(N,N651435s);
for i in [1..#T651435] do
  ss := [6,5,1,4,3,5]^T651435[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651456:=Stabilizer(N65145,6);
N651456s:=sub<N|N651456>;
for g in N do if (6^g eq 3 and 5^g eq 2 and 1^g eq 5 and 4^g eq 6)
or (6^g eq 6 and 5^g eq 2 and 1^g eq 1 and 4^g eq 3) or (6^g eq 3
and 5^g eq 1 and 1^g eq 5 and 4^g eq 4) or (6^g eq 4 and 5^g eq 5
and 1^g eq 2 and 4^g eq 6) or (6^g eq 4 and 5^g eq 1 and 1^g eq 2
and 4^g eq 3) then
  N651456s:=sub<N|N651456s,g>;
end if; end for;

T651456:= Transversal(N,N651456s);
```

```
for i in [1..#T651456] do
  ss := [6,5,1,4,5,6]^T651456[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651453:=Stabilizer(N65145,3);
T651453:= Transversal(N,N651453);
for i in [1..#T651453] do
  ss := [6,5,1,4,5,3]^T651453[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N656141:=Stabilizer(N65614,1);
N656141s:=sub<N|N656141>;
for g in N do if (6^g eq 6 and 5^g eq 5 and 1^g eq 4 and 4^g eq 1)
or (6^g eq 4 and 5^g eq 1 and 1^g eq 2 and 4^g eq 3) or
(6^g eq 5 and 5^g eq 6 and 1^g eq 4 and 4^g eq 1) or
(6^g eq 4 and 5^g eq 1 and 1^g eq 3 and 4^g eq 2) or
(6^g eq 2 and 5^g eq 3 and 1^g eq 5 and 4^g eq 6) or
(6^g eq 2 and 5^g eq 3 and 1^g eq 6 and 4^g eq 5) or
(6^g eq 3 and 5^g eq 2 and 1^g eq 5 and 4^g eq 6) then
  N656141s:=sub<N|N656141s,g>;
end if; end for;

T656141:= Transversal(N,N656141s);
for i in [1..#T656141] do
  ss := [6,5,6,1,4,1]^T656141[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N656121:=Stabilizer(N65612,1);
T656121:= Transversal(N,N656121);
for i in [1..#T656121] do
  ss := [6,5,6,1,2,1]^T656121[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N656154:=Stabilizer(N65615,4);
N656154s:=sub<N|N656154>;
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1 and 4^g eq 4) then
  N656154s:=sub<N|N656154s,g>;
end if; end for;

T656154:= Transversal(N,N656154s);
```

```
for i in [1..#T656154] do
  ss := [6,5,6,1,5,4]^T656154[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N656214:=Stabilizer(N65621,4);
N656214s:=sub<N|N656214>;
for g in N do if ((6^g eq 3 and 5^g eq 2 and 2^g eq 4 and
1^g eq 6 and 4^g eq 5) or (6^g eq 1 and 5^g eq 4 and
 2^g eq 5 and 1^g eq 3 and 4^g eq 2)) then
  N656214s:=sub<N|N656214s,g>;
end if; end for;

T656214:= Transversal(N,N656214s);
for i in [1..#T656214] do
  ss := [6,5,6,2,1,4]^T656214[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N656232:=Stabilizer(N65623,2);
N656232s:=sub<N|N656232>;
for g in N do if ((6^g eq 6 and 5^g eq 5 and 2^g eq 3 and 3^g eq 2)
or (6^g eq 5 and 5^g eq 6 and 2^g eq 3 and 3^g eq 2) or
(6^g eq 4 and 5^g eq 1 and 2^g eq 6 and 3^g eq 5) or
(6^g eq 3 and 5^g eq 2 and 2^g eq 1 and 3^g eq 4) or
(6^g eq 3 and 5^g eq 2 and 2^g eq 4 and 3^g eq 1) or
(6^g eq 2 and 5^g eq 3 and 2^g eq 1 and 3^g eq 4) or
(6^g eq 1 and 5^g eq 4 and 2^g eq 5 and 3^g eq 6) or
(6^g eq 1 and 5^g eq 4 and 2^g eq 6 and 3^g eq 5) or
(6^g eq 4 and 5^g eq 1 and 2^g eq 5 and 3^g eq 6) or
(6^g eq 5 and 5^g eq 6 and 2^g eq 2 and 3^g eq 3) or
(6^g eq 2 and 5^g eq 3 and 2^g eq 4 and 3^g eq 1)) then
  N656232s:=sub<N|N656232s,g>;
end if; end for;

T656232:= Transversal(N,N656232s);
for i in [1..#T656232] do
  ss := [6,5,6,2,3,2]^T656232[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N652431:=Stabilizer(N65243,1);
N652431s:=sub<N|N652431>;
for g in N do if ((6^g eq 1 and 5^g eq 5 and 2^g eq 2 and 4^g eq 3
```

```
and 3^g eq 4 and 1^g eq 6) or (6^g eq 4 and 5^g eq 3 and 2^g eq 2
and 4^g eq 6 and 3^g eq 5 and 1^g eq 1) or (6^g eq 4 and 5^g eq 6
and 2^g eq 2 and 4^g eq 3 and 3^g eq 1 and 1^g eq 5) or (6^g eq 1
and 5^g eq 3 and 2^g eq 2 and 4^g eq 5 and 3^g eq 6 and 1^g eq 4)
or (6^g eq 3 and 5^g eq 1 and 2^g eq 2 and 4^g eq 4 and 3^g eq 6
and 1^g eq 5) or (6^g eq 3 and 5^g eq 4 and 2^g eq 2 and 4^g eq 1
and 3^g eq 5 and 1^g eq 6) or (6^g eq 6 and 5^g eq 4 and 2^g eq 2
and 4^g eq 5 and 3^g eq 1 and 1^g eq 3) or (6^g eq 5 and 5^g eq 1
and 2^g eq 2 and 4^g eq 6 and 3^g eq 4 and 1^g eq 3) or (6^g eq 5
and 5^g eq 6 and 2^g eq 2 and 4^g eq 1 and 3^g eq 3 and 1^g eq 4)) then
  N652431s:=sub<N|N652431s,g>;
end if; end for;

T652431:= Transversal(N,N652431s);
for i in [1..#T652431] do
  ss := [6,5,2,4,3,1]^T652431[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N652326:=Stabilizer(N65232,6);
N652326s:=sub<N|N652326>;
for g in N do if (6^g eq 6 and 5^g eq 5 and 2^g eq 3 and 3^g eq 2) then
  N652326s:=sub<N|N652326s,g>;
end if; end for;

T652326:= Transversal(N,N652326s);
for i in [1..#T652326] do
  ss := [6,5,2,3,2,6]^T652326[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6516212:=Stabilizer(N651621,2);
N6516212s:=sub<N|N6516212>;
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 4 and 2^g eq 2) then
  N6516212s:=sub<N|N6516212s,g>;
end if; end for;

T6516212:= Transversal(N,N6516212s);
for i in [1..#T6516212] do
  ss := [6,5,1,6,2,1,2]^T6516212[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
```

```
N6516241:=Stabilizer(N651624,1);
N6516241s:=sub<N|N6516241>;
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1 and 2^g eq 3
and 4^g eq 4) or (6^g eq 6 and 5^g eq 5 and 1^g eq 4 and 2^g eq 3
and 4^g eq 1) or (6^g eq 2 and 5^g eq 3 and 1^g eq 5 and 2^g eq 4
and 4^g eq 6) or (6^g eq 2 and 5^g eq 3 and 1^g eq 6 and 2^g eq 1
and 4^g eq 5) or (6^g eq 1 and 5^g eq 4 and 1^g eq 2 and 2^g eq 6
and 4^g eq 3) or (6^g eq 1 and 5^g eq 4 and 1^g eq 3 and 2^g eq 5
and 4^g eq 2) or (6^g eq 4 and 5^g eq 1 and 1^g eq 2 and 2^g eq 5
and 4^g eq 3) or (6^g eq 5 and 5^g eq 6 and 1^g eq 4 and 2^g eq 2
and 4^g eq 1) or (6^g eq 3 and 5^g eq 2 and 1^g eq 6 and 2^g eq 4
and 4^g eq 5) or (6^g eq 4 and 5^g eq 1 and 1^g eq 3 and 2^g eq 6
and 4^g eq 2) or (6^g eq 3 and 5^g eq 2 and 1^g eq 5 and 2^g eq 1
and 4^g eq 6) then
  N6516241s:=sub<N|N6516241s,g>;
end if; end for;

T6516241:= Transversal(N,N6516241s);
for i in [1..#T6516241] do
  ss := [6,5,1,6,2,4,1]^T6516241[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6516251:=Stabilizer(N651625,1);
T6516251:= Transversal(N,N6516251);
for i in [1..#T6516251] do
  ss := [6,5,1,6,2,5,1]^T6516251[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6516256:=Stabilizer(N651625,6);
N6516256s:=sub<N|N6516256>;
for g in N do if (6^g eq 1 and 5^g eq 4 and 1^g eq 2 and 2^g eq 6)
or (6^g eq 2 and 5^g eq 3 and 1^g eq 6 and 2^g eq 1) then
  N6516256s:=sub<N|N6516256s,g>;
end if; end for;

T6516256:= Transversal(N,N6516256s);
for i in [1..#T6516256] do
  ss := [6,5,1,6,2,5,6]^T6516256[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6516414:=Stabilizer(N651641,4);
```

```
N6516414s:=sub<N|N6516414>;
for g in N do if (6^g eq 2 and 5^g eq 5 and 1^g eq 3 and 4^g eq 4)
or (6^g eq 3 and 5^g eq 6 and 1^g eq 2 and 4^g eq 4) or (6^g eq 1
and 5^g eq 2 and 1^g eq 6 and 4^g eq 4) or (6^g eq 5 and 5^g eq 6
and 1^g eq 1 and 4^g eq 4) or (6^g eq 2 and 5^g eq 1 and 1^g eq 6
and 4^g eq 4) or (6^g eq 5 and 5^g eq 2 and 1^g eq 3 and 4^g eq 4)
or (6^g eq 6 and 5^g eq 3 and 1^g eq 2 and 4^g eq 4) or (6^g eq 3
and 5^g eq 1 and 1^g eq 5 and 4^g eq 4) or (6^g eq 1 and 5^g eq 3
and 1^g eq 5 and 4^g eq 4) then
  N6516414s:=sub<N|N6516414s,g>;
end if; end for;

T6516414:= Transversal(N,N6516414s);
for i in [1..#T6516414] do
  ss := [6,5,1,6,4,1,4]^T6516414[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6516453:=Stabilizer(N651645,3);
T6516453:= Transversal(N,N6516453);
for i in [1..#T6516453] do
  ss := [6,5,1,6,4,5,3]^T6516453[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6515232:=Stabilizer(N651523,2);
N6515232s:=sub<N|N6515232>;
for g in N do if (6^g eq 6 and 5^g eq 4 and 1^g eq 3 and
2^g eq 2 and 3^g eq 1) then
  N6515232s:=sub<N|N6515232s,g>;
end if; end for;

T6515232:= Transversal(N,N6515232s);
for i in [1..#T6515232] do
  ss := [6,5,1,5,2,3,2]^T6515232[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6515254:=Stabilizer(N651525,4);
N6515254s:=sub<N|N6515254>;
for g in N do if (6^g eq 3 and 5^g eq 1 and 1^g eq 5 and 2^g eq 2
and 4^g eq 4) or (6^g eq 4 and 5^g eq 1 and 1^g eq 2 and 2^g eq 5
and 4^g eq 3) or (6^g eq 4 and 5^g eq 5 and 1^g eq 2 and 2^g eq 1
and 4^g eq 6) or (6^g eq 3 and 5^g eq 2 and 1^g eq 5 and 2^g eq 1
```

```
and 4^g eq 6) or (6^g eq 6 and 5^g eq 2 and 1^g eq 1 and 2^g eq 5
and 4^g eq 3) then
  N6515254s:=sub<N|N6515254s,g>;
end if; end for;

T6515254:= Transversal(N,N6515254s);
for i in [1..#T6515254] do
  ss := [6,5,1,5,2,5,4]^T6515254[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6515253:=Stabilizer(N651525,3);
T6515253:= Transversal(N,N6515253);
for i in [1..#T6515253] do
  ss := [6,5,1,5,2,5,3]^T6515253[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6512135:=Stabilizer(N651213,5);
N6512135s:=sub<N|N6512135>;
for g in N do if (6^g eq 5 and 5^g eq 3 and 1^g eq 2 and 2^g eq 4
and 3^g eq 6) or (6^g eq 3 and 5^g eq 6 and 1^g eq 4 and 2^g eq 1
and 3^g eq 5) then
  N6512135s:=sub<N|N6512135s,g>;
end if; end for;

T6512135:= Transversal(N,N6512135s);
for i in [1..#T6512135] do
  ss := [6,5,1,2,1,3,5]^T6512135[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6512316:=Stabilizer(N651231,6);
N6512316s:=sub<N|N6512316>;
for g in N do if (6^g eq 6 and 5^g eq 2 and 1^g eq 1 and 2^g eq 5
and 3^g eq 4) then
  N6512316s:=sub<N|N6512316s,g>;
end if; end for;

T6512316:= Transversal(N,N6512316s);
for i in [1..#T6512316] do
  ss := [6,5,1,2,3,1,6]^T6512316[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
```

```
N6512632:=Stabilizer(N651263,2);
N6512632s:=sub<N|N6512632>;
for g in N do if (6^g eq 3 and 5^g eq 6 and 1^g eq 4 and 2^g eq 1
and 3^g eq 5) or (6^g eq 4 and 5^g eq 6 and 1^g eq 3 and 2^g eq 1
and 3^g eq 2) or (6^g eq 5 and 5^g eq 3 and 1^g eq 2 and 2^g eq 4
and 3^g eq 6) or (6^g eq 2 and 5^g eq 3 and 1^g eq 5 and 2^g eq 4
and 3^g eq 1) or (6^g eq 1 and 5^g eq 5 and 1^g eq 6 and 2^g eq 2
and 3^g eq 4) or (6^g eq 1 and 5^g eq 2 and 1^g eq 6 and 2^g eq 5
and 3^g eq 3) or (6^g eq 3 and 5^g eq 1 and 1^g eq 4 and 2^g eq 6
and 3^g eq 2) or (6^g eq 5 and 5^g eq 4 and 1^g eq 2 and 2^g eq 3
and 3^g eq 1) or (6^g eq 2 and 5^g eq 4 and 1^g eq 5 and 2^g eq 3
and 3^g eq 6) or (6^g eq 4 and 5^g eq 1 and 1^g eq 3 and 2^g eq 6
and 3^g eq 5) or (6^g eq 6 and 5^g eq 2 and 1^g eq 1 and 2^g eq 5
and 3^g eq 4) then
  N6512632s:=sub<N|N6512632s,g>;
end if; end for;

T6512632:= Transversal(N,N6512632s);
for i in [1..#T6512632] do
  ss := [6,5,1,2,6,3,2]^T6512632[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6512643:=Stabilizer(N651264,3);
N6512643s:=sub<N|N6512643>;
for g in N do if (6^g eq 3 and 5^g eq 1 and 1^g eq 5 and 2^g eq 2
and 4^g eq 4 and 3^g eq 6) or (6^g eq 5 and 5^g eq 6 and 1^g eq 4
and 2^g eq 2 and 4^g eq 1 and 3^g eq 3) or (6^g eq 4 and 5^g eq 6
and 1^g eq 5 and 2^g eq 2 and 4^g eq 3 and 3^g eq 1) or (6^g eq 6
and 5^g eq 4 and 1^g eq 3 and 2^g eq 2 and 4^g eq 5 and 3^g eq 1)
or (6^g eq 1 and 5^g eq 3 and 1^g eq 4 and 2^g eq 2 and 4^g eq 5
and 3^g eq 6) or (6^g eq 5 and 5^g eq 1 and 1^g eq 3 and 2^g eq 2
and 4^g eq 6 and 3^g eq 4) or (6^g eq 3 and 5^g eq 4 and 1^g eq 6
and 2^g eq 2 and 4^g eq 1 and 3^g eq 5) or (6^g eq 4 and 5^g eq 3
and 1^g eq 1 and 2^g eq 2 and 4^g eq 6 and 3^g eq 5) or (6^g eq 1
and 5^g eq 5 and 1^g eq 6 and 2^g eq 2 and 4^g eq 3 and 3^g eq 4) then
  N6512643s:=sub<N|N6512643s,g>;
end if; end for;

T6512643:= Transversal(N,N6512643s);
for i in [1..#T6512643] do
  ss := [6,5,1,2,6,4,3]^T6512643[i];
  cst[prodim(1, ts, ss)] := ss;
```

```
end for;

N65162123:=Stabilizer(N6516212,3);
N65162123s:=sub<N|N65162123>;
for g in N do if (6^g eq 4 and 5^g eq 3 and 1^g eq 1 and 2^g eq 2
and 3^g eq 5) or (6^g eq 3 and 5^g eq 4 and 1^g eq 6 and 2^g eq 2
and 3^g eq 5) or (6^g eq 1 and 5^g eq 5 and 1^g eq 6 and 2^g eq 2
and 3^g eq 4) or (6^g eq 3 and 5^g eq 1 and 1^g eq 5 and 2^g eq 2
and 3^g eq 6) or (6^g eq 5 and 5^g eq 1 and 1^g eq 3 and 2^g eq 2
and 3^g eq 4) or (6^g eq 4 and 5^g eq 6 and 1^g eq 5 and 2^g eq 2
and 3^g eq 1) or (6^g eq 6 and 5^g eq 4 and 1^g eq 3 and 2^g eq 2
and 3^g eq 1) or (6^g eq 1 and 5^g eq 3 and 1^g eq 4 and 2^g eq 2
and 3^g eq 6) or (6^g eq 5 and 5^g eq 6 and 1^g eq 4 and 2^g eq 2
and 3^g eq 3) then
  N65162123s:=sub<N|N65162123s,g>;
end if; end for;

T65162123:= Transversal(N,N65162123s);
for i in [1..#T65162123] do
  ss := [6,5,1,6,2,1,2,3]^T65162123[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651621232:=Stabilizer(N65162123,2);
N651621232s:=sub<N|N651621232>;
for g in N do if (6^g eq 6 and 5^g eq 4 and 1^g eq 5 and 2^g eq 1
and 3^g eq 2) or (6^g eq 4 and 5^g eq 2 and 1^g eq 5 and 2^g eq 6
and 3^g eq 3) or (6^g eq 6 and 5^g eq 4 and 1^g eq 3 and 2^g eq 2
and 3^g eq 1) or (6^g eq 3 and 5^g eq 4 and 1^g eq 6 and 2^g eq 2
and 3^g eq 5) or (6^g eq 2 and 5^g eq 6 and 1^g eq 3 and 2^g eq 5
and 3^g eq 4) or (6^g eq 5 and 5^g eq 2 and 1^g eq 4 and 2^g eq 6
and 3^g eq 1) or (6^g eq 5 and 5^g eq 3 and 1^g eq 1 and 2^g eq 6
and 3^g eq 4) or (6^g eq 5 and 5^g eq 6 and 1^g eq 1 and 2^g eq 3
and 3^g eq 2) then
  N651621232s:=sub<N|N651621232s,g>;
end if; end for;

T651621232:= Transversal(N,N651621232s);
for i in [1..#T651621232] do
  ss := [6,5,1,6,2,1,2,3,2]^T651621232[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
```

Appendix B

Center of

$$\frac{2^{*6}:L_2(5)}{[(0,1,2,3,4)t_0]^8,[(\infty,0,1)(2,4,3)t_\infty]^8,[(\infty,0)(1,4)t_1]^6}$$

```
(y^2 * x )* t *( y * x )* t *( y * x^2 * y * x^2 * y * x * y * x^2 * y *
x^2 )* t *( y * x^2 * y * x * y * x^2 )* t * (y * x^2 * y * x * y * x^2 *
y * x^2 )* t * (y * x^2 * y * x * y * x^2 * y * x^2 * y * x * y * x * y *
x^2 * y * x * y * x^2 )* t * (y * x^2 * y * x * y * x^2 * y * x^2 * y *
x * y * x * y * x^2 )* t *( y * x^2 * y * x * y * x^2 * y * x^2 * y * x
* y * x * y * x * y * x^2) * t * (y * x^2 * y^-1 )* t^-1 *( x^-2 * y^-1)
* t^-1 *( y^-1 * x * y^-1) * t^-1 * (y^-1 )* t^-1 *( x^-1 * y^2 * x^-2
* y^-1 * x^-2 * y^-1 * x^-2 * y^-1 * x^-1 * y^-1 * x^-2 * y^-1 )* t^-1
*( x^-2 * y^-1 * x^-1 * y^-1 * x^-2 * y^-1 * x^-1 * y^-1 * x^-1 * y^-1
* x^-2 * y^-1 * x^-2 * y^-1 * x^-1 * y^-1 * x^-2 * y^-1) * t^-1 * (x^-2
* y^-1 * x^-2 * y^-1 * x^-1 * y^-1 * x^-2 * y^-1) * t^-1 * (x^-2 * y^-1
* x^-1 * y^-1 * x^-2 * y^-1 )* t^-1 * (x^-2 * y^-1 * x^-2 * y^-1 * x^-1
* y^-1 * x^-2 * y^-1 * x^-2 * y^-1 )* t^-1 *( x^-1 * y^-1 )* t^-1 *
(x^-1 * y^-3 * t * y^3 * x )* t * (y * x )* t * (y * x^2 * y * x^2 * y
* x * y * x^2 * y * x^2) * t * (y * x^2 * y * x * y * x^2 )* t * (y *
x^2 * y * x * y * x^2 * y * x^2) * t * (y * x^2 * y * x * y * x^2 *
y * x^2 * y * x * y * x * y * x^2 * y * x * y * x^2 )* t * (y * x^2 *
y * x * y * x^2 * y * x^2 * y * x)
```

Appendix C

MAGMA Code for Construction of M_{12}

```
G<x,y,t>:=Group<x,y,t|x^5,y^3,(y*x)^2,t^2,(t,x),(t^(y*x^2),
x*y),(x*t^y)^8,(y*t)^8,(t^(y*x^4)*t^(y^2))^3,y^-1*t^(y*x^2)*
t^(y*x^4)*t^y*t^(y*x^2)*t*t^y*t*t^(y*x^3)*t>;

S6:=Sym(6);
xx:=S6!(5,1,2,3,4);
yy:=S6!(6,5,1)(2,4,3);
N:=sub<S6|xx,yy>;
Order(N);

f,G1,k:=CosetAction(G,sub<G|x,y>);

R:=[Id(G1): i in [1..6]];
R[6]:=f(t);
R[5]:=f(t^y);
R[1]:=f(t^(y^2));
R[2]:=f(t^(y*x^2));
R[3]:=f(t^(y*x^3));
R[4]:=f(t^(y*x^4));

ts := R;
 cst := [null : i in [1 .. 3168]] where null is [Integers() | ];
 for i := 1 to 6 do
  prodim := function(pt, Q, I)
   v := pt;
```

```
  for i in I do
       v := v^(Q[i]);
  end for;
 return v;
 end function;
 end for;

N6:=Stabiliser(N,6);
T6:=Transversal(N,N6);
for i in [1..#T6] do
  ss := [6]^T6[i];
cst[prodim(1, ts, ss)] := ss;
  end for;

N65:=Stabilizer(N6,5);
T65:=Transversal(N,N65);
for i in [1..#T65] do
  ss := [6,5]^T65[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N656:=Stabilizer(N65,6);
 for g in N do if (6^g eq 5 and 5^g eq 6) then
  N656s:=sub<N|N656,g>;
  end if; end for;

T656:=Transversal(N,N656s);
for i in [1..#T656] do
  ss := [6,5,6]^T656[i];
cst[prodim(1, ts, ss)] := ss;
 end for;

N651:=Stabilizer(N65,1);
T651:=Transversal(N,N651);
for i in [1..#T651] do
  ss := [6,5,1]^T651[i];
cst[prodim(1, ts, ss)] := ss;
 end for;

N652:=Stabilizer(N65,2);
T652:=Transversal(N,N652);
for i in [1..#T652] do
  ss := [6,5,2]^T652[i];
cst[prodim(1, ts, ss)] := ss;
```

```
 end for;

N6561:=Stabiliser(N656,1);
 for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1) then
  N6561s:=sub<N|N6561,g>;
 end if; end for;

T6561:=Transversal(N,N6561s);
for i in [1..#T6561] do
  ss := [6,5,6,1]^T6561[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6562:=Stabiliser(N656,2);
 for g in N do if (6^g eq 5 and 5^g eq 6 and 2^g eq 2) then
  N6562s:=sub<N|N6562,g>;
 end if; end for;

T6562:=Transversal(N,N6562s);
for i in [1..#T6562] do
  ss := [6,5,6,2]^T6562[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6512:=Stabiliser(N651,2);
T6512:=Transversal(N,N6512);
for i in [1..#T6512] do
  ss := [6,5,1,2]^T6512[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6513:=Stabiliser(N651,3);
for g in N do if (6^g eq 1 and 5^g eq 5 and 1^g eq 6
and 3^g eq 4) then
  N6513s:=sub<N|N6513,g>;
 end if; end for;

T6513:=Transversal(N,N6513s);
for i in [1..#T6513] do
  ss := [6,5,1,3]^T6513[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6514:=Stabiliser(N651,4);
```

```
T6514:=Transversal(N,N6514);
for i in [1..#T6514] do
  ss := [6,5,1,4]^T6514[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6515:=Stabiliser(N651,5);
for g in N do if (6^g eq 6 and 5^g eq 1 and 1^g eq 5) then
  N6515s:=sub<N|N6515,g>;
end if; end for;

T6515:=Transversal(N,N6515s);
for i in [1..#T6515] do
  ss := [6,5,1,5]^T6515[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6516:=Stabiliser(N651,6);
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1) then
  N6516s:=sub<N|N6516,g>;
end if; end for;

T6516:=Transversal(N,N6516s);
for i in [1..#T6516] do
  ss := [6,5,1,6]^T6516[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6521:=Stabiliser(N652,1);
for g in N do if (6^g eq 2 and 5^g eq 5 and 2^g eq 6
and 1^g eq 3) then
  N6521s:=sub<N|N6521,g>;
end if; end for;

T6521:=Transversal(N,N6521s);
for i in [1..#T6521] do
  ss := [6,5,2,1]^T6521[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6523:=Stabiliser(N652,3);
T6523:=Transversal(N,N6523);
for i in [1..#T6523] do
  ss := [6,5,2,3]^T6523[i];
```

```
  cst[prodim(1, ts, ss)] := ss;
end for;

N6524:=Stabiliser(N652,4);
T6524:=Transversal(N,N6524);
for i in [1..#T6524] do
  ss := [6,5,2,4]^T6524[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6525:=Stabiliser(N652,5);
for g in N do if (6^g eq 6 and 5^g eq 2 and 2^g eq 5) then
  N6525s:=sub<N|N6525,g>;
end if; end for;

T6525:=Transversal(N,N6525s);
for i in [1..#T6525] do
  ss := [6,5,2,5]^T6525[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N6526:=Stabiliser(N652,6);
for g in N do if (6^g eq 5 and 5^g eq 6 and 2^g eq 2) then
  N6526s:=sub<N|N6526,g>;
end if; end for;

T6526:=Transversal(N,N6526s);
for i in [1..#T6526] do
  ss := [6,5,2,6]^T6526[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65614:=Stabiliser(N6561,4);
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1
and 4^g eq 4) then
  N65614s:=sub<N|N65614,g>;
end if; end for;

T65614:=Transversal(N,N65614s);
for i in [1..#T65614] do
  ss := [6,5,6,1,4]^T65614[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
```

```
N65612:=Stabiliser(N6561,2);
T65612:=Transversal(N,N65612);
for i in [1..#T65612] do
  ss := [6,5,6,1,2]^T65612[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65615:=Stabiliser(N6561,5);
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1) then
  N65615s:=sub<N|N65615,g>;
end if; end for;

T65615:=Transversal(N,N65615s);
N65615:=Stabiliser(N6561,5);
N65623:=Stabiliser(N6562,3);
for g in N do if (6^g eq 5 and 5^g eq 6 and 2^g eq 2
and 3^g eq 3) then
  N65623s:=sub<N|N65623,g>;
end if; end for;

T65623:=Transversal(N,N65623s);
for i in [1..#T65623] do
  ss := [6,5,6,2,3]^T65623[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65621:=Stabiliser(N6562,1);
T65621:=Transversal(N,N65621);
for i in [1..#T65621] do
  ss := [6,5,6,2,1]^T65621[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65121:=Stabiliser(N6512,1);
T65121:=Transversal(N,N65121);
for i in [1..#T65121] do
  ss := [6,5,1,2,1]^T65121[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65123:=Stabiliser(N6512,3);
for g in N do if (6^g eq 6 and 5^g eq 2 and 1^g eq 1
and 2^g eq 5 and 3^g eq 4) then
  N65123s:=sub<N|N65123,g>;
```

```
end if; end for;

T65123:=Transversal(N,N65123s);
for i in [1..#T65123] do
  ss := [6,5,1,2,3]^T65123[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65124:=Stabiliser(N6512,4);
for g in N do if (6^g eq 2 and 5^g eq 4 and 1^g eq 1
and 2^g eq 6 and 4^g eq 5) then
  N65124s:=sub<N|N65124,g>;
end if; end for;

T65124:=Transversal(N,N65124s);
for i in [1..#T65124] do
  ss := [6,5,1,2,4]^T65124[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65125:=Stabiliser(N6512,5);
for g in N do if (6^g eq 3 and 5^g eq 1 and 1^g eq 5
and 2^g eq 2) then
  N65125s:=sub<N|N65125,g>;
end if; end for;

T65125:=Transversal(N,N65125s);
for i in [1..#T65125] do
  ss := [6,5,1,2,5]^T65125[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65126:=Stabiliser(N6512,6);
T65126:=Transversal(N,N65126);
for i in [1..#T65126] do
  ss := [6,5,1,2,6]^T65126[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65231:=Stabiliser(N6523,1);
for g in N do if (6^g eq 2 and 5^g eq 5 and 2^g eq 6
and 3^g eq 1 and 1^g eq 3) then
  N65231s:=sub<N|N65231,g>;
end if; end for;
```

```
T65231:=Transversal(N,N65231s);
N65232:=Stabiliser(N6523,2);
for g in N do if (6^g eq 6 and 5^g eq 5 and 2^g eq 3
and 3^g eq 2) then
  N65232s:=sub<N|N65232,g>;
end if; end for;

T65232:=Transversal(N,N65232s);
for i in [1..#T65232] do
  ss := [6,5,2,3,2]^T65232[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65234:=Stabiliser(N6523,4);
T65234:=Transversal(N,N65234);
for i in [1..#T65234] do
  ss := [6,5,2,3,4]^T65234[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65235:=Stabiliser(N6523,5);
for i in [1..#T65235] do
  ss := [6,5,2,3,5]^T65235[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65214:=Stabiliser(N6521,4);
for g in N do if (6^g eq 2 and 5^g eq 5 and 2^g eq 6
and 1^g eq 3 and 4^g eq 4) then
  N65214s:=sub<N|N65214,g>;
end if; end for;

T65214:=Transversal(N,N65214s);
for i in [1..#T65214] do
  ss := [6,5,2,1,4]^T65214[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65242:=Stabiliser(N6524,2);
T65242:=Transversal(N,N65242);
for i in [1..#T65242] do
  ss := [6,5,2,4,2]^T65242[i];
  cst[prodim(1, ts, ss)] := ss;
```

```
end for;

N65243:=Stabiliser(N6524,3);
for g in N do if (6^g eq 4 and 5^g eq 3 and 2^g eq 2
and 4^g eq 6 and 3^g eq 5) then
  N65243s:=sub<N|N65243,g>;
end if; end for;

T65243:=Transversal(N,N65243s);
for i in [1..#T65243] do
  ss := [6,5,2,4,3]^T65243[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65245:=Stabiliser(N6524,5);
for g in N do if (6^g eq 1 and 5^g eq 2 and 2^g eq 5
and 4^g eq 4) then
  N65245s:=sub<N|N65245,g>;
end if; end for;

T65245:=Transversal(N,N65245s);
for i in [1..#T65245] do
  ss := [6,5,2,4,5]^T65245[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65246:=Stabiliser(N6524,6);
T65246:=Transversal(N,N65246);
for i in [1..#T65246] do
  ss := [6,5,2,4,6]^T65246[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65141:=Stabiliser(N6514,1);
for g in N do if (6^g eq 6 and 5^g eq 5 and 1^g eq 4
and 4^g eq 1) then
  N65141s:=sub<N|N65141,g>;
end if; end for;

T65141:=Transversal(N,N65141s);
for i in [1..#T65141] do
  ss := [6,5,1,4,1]^T65141[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
```

```
N65156:=Stabiliser(N6515,6);
N65156s:=sub<N|N65156>;
for g in N do if ((6^g eq 6 and 5^g eq 1 and 1^g eq 5) or
(6^g eq 5 and 5^g eq 6 and 1^g eq 1) or (6^g eq 5 and 5^g eq 1
and 1^g eq 6) or (6^g eq 1 and 5^g eq 6 and 1^g eq 5) or
(6^g eq 1 and 5^g eq 5 and 1^g eq 6)) then
  N65156s:=sub<N|N65156s,g>;
end if; end for;

T65156:=Transversal(N,N65156s);
for i in [1..#T65156] do
  ss := [6,5,1,5,6]^T65156[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65256:=Stabiliser(N6525,6);
N65256s:=sub<N|N65256>;
for g in N do if ((6^g eq 6 and 5^g eq 2 and 2^g eq 5)
or (6^g eq 5 and 5^g eq 6 and 2^g eq 2) or (6^g eq 5 and
5^g eq 2 and 2^g eq 6) or (6^g eq 2 and 5^g eq 6 and 2^g eq 5)
or (6^g eq 2 and 5^g eq 5 and 2^g eq 6)) then
  N65256s:=sub<N|N65256s,g>;
end if; end for;

T65256:=Transversal(N,N65256s);
for i in [1..#T65256] do
  ss := [6,5,2,5,6]^T65256[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N65161:=Stabiliser(N6516,1);
N65161s:=sub<N|N65161>;
for g in N do if (6^g eq 5 and 5^g eq 6 and 1^g eq 1) then
  N65161s:=sub<N|N65161s,g>;
end if; end for;

T65161:=Transversal(N,N65161s);
for i in [1..#T65256] do
  ss := [6,5,2,5,6]^T65256[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N656212:=Stabiliser(N65621,2);
```

```
T656212:=Transversal(N,N656212);
for i in [1..#T656212] do
  ss := [6,5,6,2,1,2]^T656212[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N656214:=Stabiliser(N65621,4);
N656214s:=sub<N|N656214>;
for g in N do if (6^g eq 3 and 5^g eq 2 and 2^g eq 4 and
1^g eq 6 and 4^g eq 5) or (6^g eq 1 and 5^g eq 4 and 2^g eq 5
and 1^g eq 3 and 4^g eq 2) then
  N656214s:=sub<N|N656214s,g>;
end if; end for;

T656214:=Transversal(N,N656214s);
for i in [1..#T656214] do
  ss := [6,5,6,2,1,4]^T656214[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N656232:=Stabiliser(N65623,2);
N656232s:=sub<N|N656232>;
for g in N do if (6^g eq 6 and 5^g eq 5 and 2^g eq 3 and 3^g eq 2)
or (6^g eq 5 and 5^g eq 6 and 2^g eq 3 and 3^g eq 2) or (6^g eq 5
and 5^g eq 6 and 2^g eq 2 and 3^g eq 3) or (6^g eq 4 and 5^g eq 1
and 2^g eq 6 and 3^g eq 5) or (6^g eq 4 and 5^g eq 1 and 2^g eq 5
and 3^g eq 6) or (6^g eq 3 and 5^g eq 2 and 2^g eq 1 and 3^g eq 4)
or (6^g eq 3 and 5^g eq 2 and 2^g eq 4 and 3^g eq 1) or (6^g eq 2
and 5^g eq 3 and 2^g eq 1 and 3^g eq 4) or (6^g eq 2 and 5^g eq 3
and 2^g eq 4 and 3^g eq 1) or (6^g eq 1 and 5^g eq 4 and 2^g eq 5
and 3^g eq 6) or (6^g eq 1 and 5^g eq 4 and 2^g eq 6 and 3^g eq 5) then
  N656232s:=sub<N|N656232s,g>;
end if; end for;

T656232:=Transversal(N,N656232s);
for i in [1..#T656232] do
  ss := [6,5,6,2,3,2]^T656232[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N656141:=Stabiliser(N65614,1);
N656141s:=sub<N|N656141>;
for g in N do if (6^g eq 6 and 5^g eq 5 and 1^g eq 4 and 4^g eq 1)
or (6^g eq 5 and 5^g eq 6 and 1^g eq 4 and 4^g eq 1) or (6^g eq 5
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and 5^g eq 6 and 1^g eq 1 and 4^g eq 4) or (6^g eq 4 and 5^g eq 1
and 1^g eq 2 and 4^g eq 3) or (6^g eq 4 and 5^g eq 1 and 1^g eq 3
and 4^g eq 2) or (6^g eq 3 and 5^g eq 2 and 1^g eq 5 and 4^g eq 6)
or (6^g eq 3 and 5^g eq 2 and 1^g eq 6 and 4^g eq 5) or (6^g eq 2
and 5^g eq 3 and 1^g eq 5 and 4^g eq 6) or (6^g eq 2 and 5^g eq 3
and 1^g eq 6 and 4^g eq 5) or (6^g eq 1 and 5^g eq 4 and 1^g eq 2
and 4^g eq 3) or (6^g eq 1 and 5^g eq 4 and 1^g eq 3 and 4^g eq 2) then
  N656141s:=sub<N|N656141s,g>;
end if; end for;

T656141:=Transversal(N,N656141s);
for i in [1..#T656141] do
  ss := [6,5,6,1,4,1]^T656141[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N656123:=Stabiliser(N65612,3);
N656123s:=sub<N|N656123>;
for g in N do if (6^g eq 1 and 5^g eq 4 and 1^g eq 3 and
2^g eq 5 and 3^g eq 6) or (6^g eq 3 and 5^g eq 2 and 1^g eq 6
and 2^g eq 4 and 3^g eq 1) then
  N656123s:=sub<N|N656123s,g>;
end if; end for;

T656123:=Transversal(N,N656123s);
for i in [1..#T656123] do
  ss := [6,5,6,1,2,3]^T656123[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651215:=Stabiliser(N65121,5);
N651215s:=sub<N|N651215>;
for g in N do if (6^g eq 3 and 5^g eq 1 and 1^g eq 5 and
2^g eq 2) then
  N651215s:=sub<N|N651215s,g>;
end if; end for;

T651215:=Transversal(N,N651215s);
for i in [1..#T651215] do
  ss := [6,5,1,2,1,5]^T651215[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N651243:=Stabiliser(N65124,3);
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```
N651243s:=sub<N|N651243>;
for g in N do if (6^g eq 6 and 5^g eq 2 and 1^g eq 1 and
2^g eq 5 and 4^g eq 3 and 3^g eq 4) or (6^g eq 5 and 5^g eq 3
and 1^g eq 6 and 2^g eq 5 and 4^g eq 2 and 3^g eq 4) or (6^g eq 5
and 5^g eq 6 and 1^g eq 1 and 2^g eq 3 and 4^g eq 4 and 3^g eq 2)
or (6^g eq 4 and 5^g eq 2 and 1^g eq 1 and 2^g eq 3 and 4^g eq 5
and 3^g eq 6)  then
  N651243s:=sub<N|N651243s,g>;
end if; end for;

T651243:=Transversal(N,N651243s);
for i in [1..#T651243] do
  ss := [6,5,1,2,4,3]^T651243[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N652325:=Stabiliser(N65232,5);
N652325s:=sub<N|N652325>;
for g in N do if (6^g eq 6 and 5^g eq 5 and 2^g eq 3
and 3^g eq 2) then
  N652325s:=sub<N|N652325s,g>;
end if; end for;

T652325:=Transversal(N,N652325s);
for i in [1..#T652325] do
  ss := [6,5,2,3,2,5]^T652325[i];
  cst[prodim(1, ts, ss)] := ss;
end for;

N652431:=Stabiliser(N65243,1);
N652431s:=sub<N|N652431>;
for g in N do if (6^g eq 6 and 5^g eq 4 and 2^g eq 2 and
4^g eq 5 and 3^g eq 1 and 1^g eq 3) or (6^g eq 5 and
5^g eq 1 and 2^g eq 2 and 4^g eq 6 and 3^g eq 4 and 1^g eq 3)
or (6^g eq 5 and 5^g eq 6 and 2^g eq 2 and 4^g eq 1 and
3^g eq 3 and 1^g eq 4) or (6^g eq 4 and 5^g eq 3 and 2^g eq 2
and 4^g eq 6 and 3^g eq 5 and 1^g eq 1) or (6^g eq 4 and 5^g eq 6
and 2^g eq 2 and 4^g eq 3 and 3^g eq 1 and 1^g eq 5) then
  N652431s:=sub<N|N652431s,g>;
end if; end for;

T652431:=Transversal(N,N652431s);
for i in [1..#T652431] do
  ss := [6,5,2,4,3,1]^T652431[i];
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  cst[prodim(1, ts, ss)] := ss;
end for;

N651254:=Stabiliser(N65125,4);
N651254s:=sub<N|N651254>;
for g in N do if (6^g eq 6 and 5^g eq 5 and 1^g eq 4 and
2^g eq 3 and 4^g eq 1) or (6^g eq 3 and 5^g eq 1 and 1^g eq 5
and 2^g eq 2 and 4^g eq 4) or  (6^g eq 2 and 5^g eq 4 and 1^g eq 1
and 2^g eq 6 and 4^g eq 5) or (6^g eq 3 and 5^g eq 1 and 1^g eq 4
and 2^g eq 6 and 4^g eq 5) or (6^g eq 2 and 5^g eq 4 and 1^g eq 5
and 2^g eq 3 and 4^g eq 1) then
  N651254s:=sub<N|N651254s,g>;
end if; end for;

T651254:=Transversal(N,N651254s);
for i in [1..#T651254] do
  ss := [6,5,1,2,5,4]^T651254[i];
  cst[prodim(1, ts, ss)] := ss;
end for;
```

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