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Cassini ovals as elliptic curves

Nozomi Arakaki

CASSINI OVALS AS ELLIPTIC CURVES

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Approved by:

[redacted]

Dr. John Sarli, Committee Chair

November 26, 2012

Date

[redacted]

Dr. Rolland Trapp, Committee Member

[redacted]

Dr. J. Paul Vicknair, Committee Member

[redacted]

Dr. Peter Williams, Chair,
Department of Mathematics

[redacted]

Dr. Charles Stanton
Graduate Coordinator,
Department of Mathematics

Abstract

Cassini ovals, which are considered as plane algebraic curves, are investigated to see if these curves can be elliptic curves. Gian Domenico Cassini, an Italian-born French astronomer, described these ovals as a collection of points where the product of the distances from two fixed points (focal points or foci) is constant. If we represent Cassini ovals as the Cartesian equation $((x-a)^2+y^2)((x+a)^2+y^2)=b^2$, the foci are at $(-a,0)$ and $(a,0)$, $2a$ is the distance between the foci, and b is a constant. Depending on the relationship between a and b, the curves fall into one of the three shapes: one continuous loop, two separate loops, or a lemniscate. Specifically, when $b>a$, the curve will be a continuous loop, but when $b<a$, the curve will be two disconnected loops. Lastly, when $b=a$, the curve will be a lemniscate, which means "pendant ribbon" in Latin. The important basic definitions regarding algebraic curves, such as multiplicity, singularity, inflection, and Hessian curves, will lead us to the computations to find out the genus of the Cassini ovals, which tells us whether the Cassinian ovals are elliptic or not. Then, if the curve is not a lemniscate, a transformation of a Cassini oval to a cubic makes it possible to compute its cross-ratio. Lastly, the J-invariant of the curve is shown to be real in all cases and it will be shown that the ovals are a complete set of representatives for elliptic curves with real J-invariant.

Acknowledgements

I would like to express my sincere gratitude to Dr. Sarli for giving me the opportunity to work on this project and for his guidance, long-term support and great patience. Also, special thanks to Dr. Stanton for his guidance and to Dr. Trapp and Dr. Vicknair for their contribution in my project. Moreover, uncountable thanks to my friends and family, especially my husband, Ky, for their acceptance, cooperation and encouragement. Lastly, I thank God for His steadfast assistance to have me reach this far; and I am looking forward to being lead by Him beyond.

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Chapter 1

Introduction

Gian Domenico Cassini was an Italian-born French astronomer who found significant discoveries about Saturn. Cassini agreed with the solar theory of Nicolaus Copernicus, but he disagreed with the theory of Johannes Kepler that planets trave in ellipses [EB2012]. In 1680, he investigated the relative motions of the Earth and the Sun and proposed that the paths of planets were certain curved ovals. He believed that the Sun was traveling around the Earth on one of the Cassini ovals with the Earth at one focus of the oval.

Cassini described these ovals as a collection of points where the product of the distances from two fixed points (focal points or foci) is constant. The general formula for Cassini ovals is shown below. For example, if the foci are at $(-a, 0)$ and $(a, 0)$ then $2a$ is the distance between the foci. Let b be a constant, then

$$((x-a)^2 + y^2)((x+a)^2 + y^2) = b^2.$$

Depending on the relationship between a and b, the curves fall into one of the three shapes: one continuous loop, two separate loops, or a lemniscate [WCO].

For example, let $a = 1$.

When $b > a$, the curve will be a continuous loop. For instance, let $b = 2$. Then we have the Cassini oval as $\left((x-1)^2 + y^2\right)\left((x+1)^2 + y^2\right) = 4$, which forms a graph in Figure 1.1.

When $b < a$, the curve will be two disconnected loops. For example, let $b = \frac{1}{2}$. Then we have $((x-1)^2 + y^2)\left((x+1)^2 + y^2\right) = \frac{1}{4}$ shown in Figure 1.2.

Lastly, when $b = a$, the curve will be a lemniscate. For example, let $b = 1$, then the Cassini oval turns out to be $((x-1)^2 + y^2)\left((x+1)^2 + y^2\right) = 1$ and it forms a lemniscate

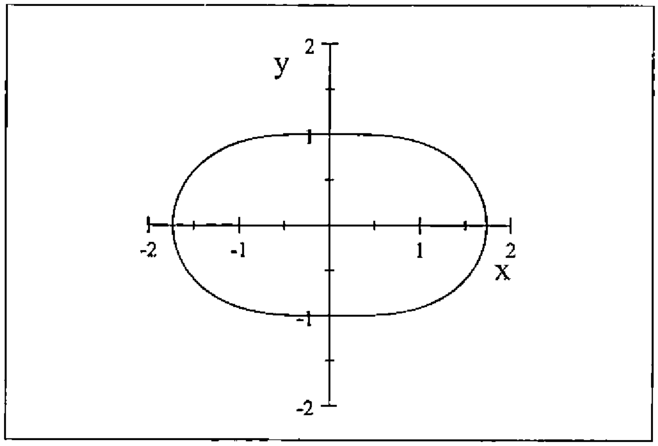

Figure 1.1: A Cassini Oval with $b > a$

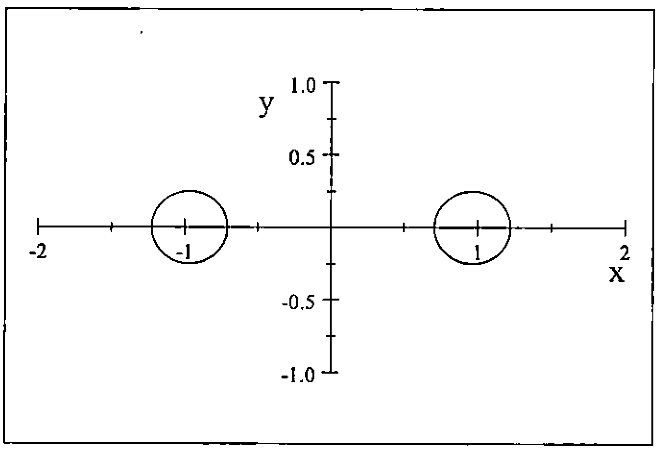

Figure 1.2: A Cassini Oval with $b < a$

shown in Figure 1.3.

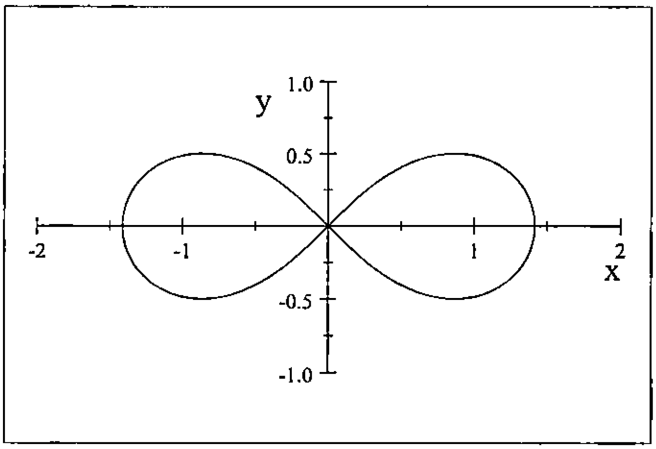

Figure 1.3: A Cassini Oval with $b = a$

As we can see from the figures, depending on the relationship between a and b, the ovals turn into one of these three shapes.

When $b = a$, the oval is called a Lemniscate (which means "pendant ribbon" in Latin) or Lemniscate of Bernoulli. In 1694, Jakob Bernoulli described the lemniscate as a modification of an ellipse which was the locus of points P such that the sum of the distances to each foci was constant. Now, in geometry, the definition of the lemniscate of Bernoulli is a plane curve which is the locus of points P such that $PF_1 * PF_2 = a^2$ where $2a$ is the distance between the foci, F_1 and F_2, and it passes through the point midway between the foci, which is the same definition as one of the Cassini ovals [L1972].

The purpose of this project is to show that Cassini curves that are not lemniscates, when $b \neq 1$, represent elliptic curves. It is also shown that the cross-ratios of these elliptic curves are either real numbers or represented by complex numbers on the unit circle on the complex plane. Since these Cassini ovals are precisely the elliptic curves with a real J-invariant, Cassini ovals can be found to represent the elliptic curves with this property. In order to understand cross-ratio and J-invariant, basic information about algebraic curves will be represented in the next chapter.

Chapter 2

Algebraic Curves

2.1 Algebraic Curves

One of the goals of this project is to show Cassini ovals as elliptic curves. Before going over that further, some basics of algebraic curves need to be informed. The examples of plane algebraic curves are lines, circles, the cuspidal cubic, the Nodal cubic, the Folium of Descartes and hypocycloids. Since lines and circles are well-known, the other three examples are introduced: the cuspidal cubic, the Nodal cubic and the Folium of Descartes.

First of all, the typical cuspidal equation is $x^3 - y^2 = 0$, and the curve looks like the Figure 2.1. The simplest example of a singularity (a point where multiple tangents exist) occurs at the cusp of the cuspidal cubic [F2001].

Secondly, the Nodal cubic is described by the equation $x^2(x+1) - y^2 = 0$, and the curve is shown in the Figure 2.2. Notice that there is an ordinary double point (a point where two distinct tangents locate) at the origin [F2001].

Lastly, the Folium of Descartes is usually described by the equation $x^3 + y^3 - 3xy = 0$. See Figure 2.3. Although the Folium of Descartes is similar to the Nodal cubic, the essential difference between them is that the Folium of Descartes has an asymptote, $x + y + 1 = 0$ [F2001].

2.2 Affine Algebraic Curves

Additionally, some of the important terminologies are restated in this section. An algebraic curve can be imagined as a trajectory of motion.

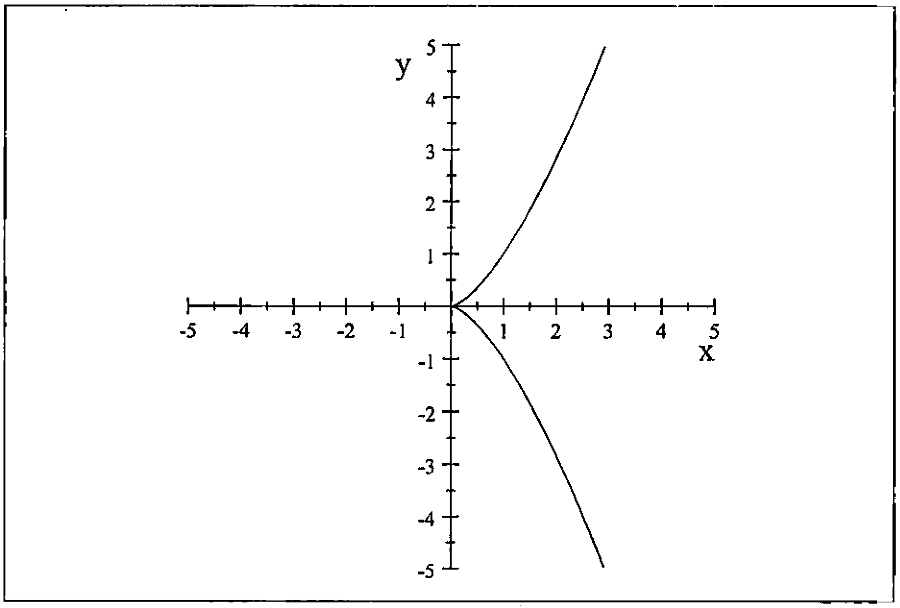

Figure 2.1: The Cuspidal Cubic

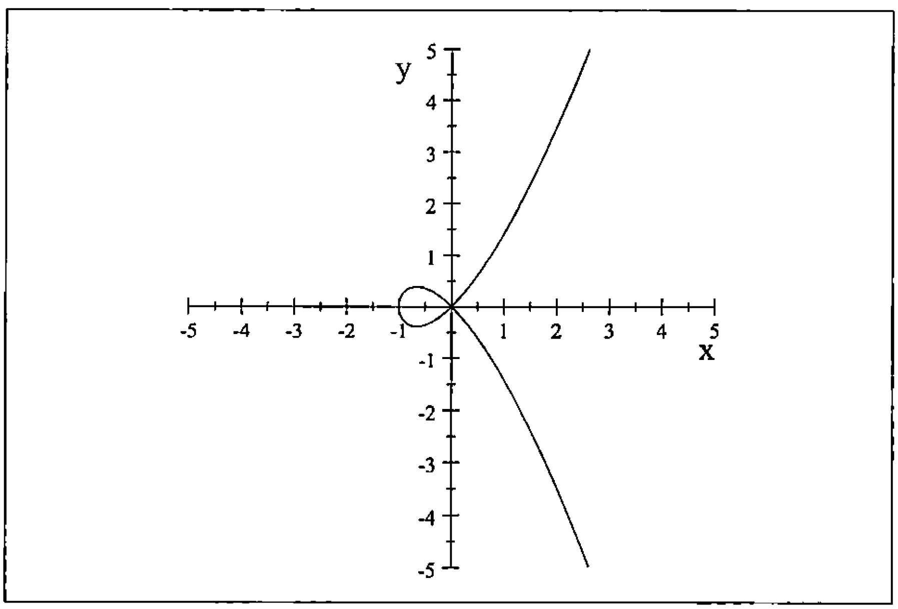

Figure 2.2: The Nodal Cubic

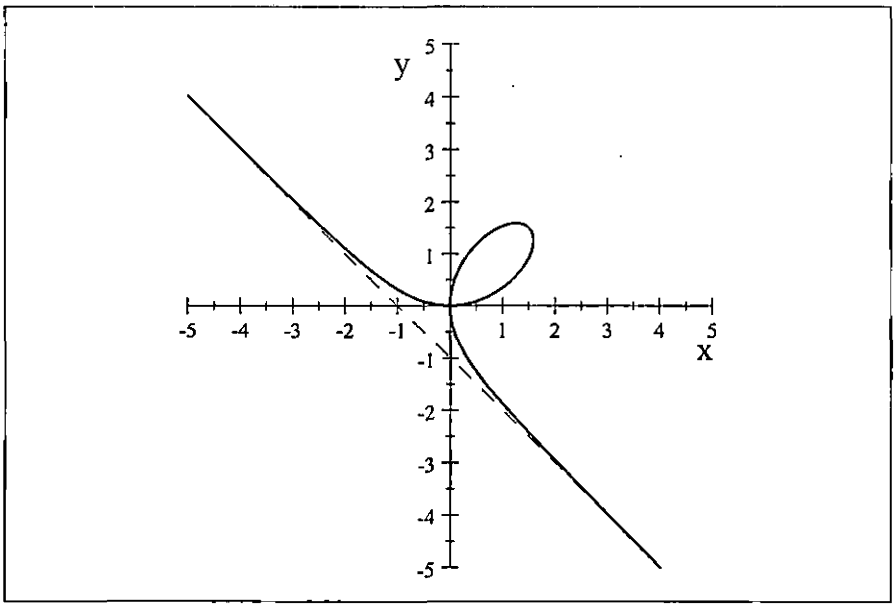

Figure 2.3: The Folium of Descartes

In this project, *plane* means only two-dimensional, and it is the space in which the motion occurs.

Also, the moving object is assumed to be a *point* and its motion in the plane is described by a map

$$\varphi : I \to \mathbb{R}^2, t \to \varphi(t) = (x(t), y(t)),$$

where $I \subset \mathbb{R}$ denotes an interval. The parameter t can be viewed as time. In this project, a point in a plane will be denoted by a coordinate pair (x, y).

A *line* can be described by

$$\varphi(t) = v + tw,$$

where vectors $v, w \in \mathbb{R}^2$, the direction vector $w \neq 0$.

Before stating the property of an algebraic curve, the following notations lead to the definition of an affine algebraic curve.

For a function $f \in \mathbb{R}[X, Y]$, the *variety* of f is noted as the set

$$V(f) := \left\{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\right\}.$$

For example, in $\mathbb{R}^2$,

$V(X^2 + Y^2 - 1)$ is a circle,

$V(X^2 + Y^2)$ is a point, and

$V(X^2 + Y^2 + 1)$ is the empty set.

However, notice in $\mathbb{C}^2$,

$V(X^2 + Y^2) = V(X + iY) \cup V(X - iY) \quad \because X^2 + Y^2 = (X + iY)(X - iY)$

This means that the zero set consists two lines that intersect at the origin.

Now the fundamental definition of an *affine algebraic curve* is stated below.

Definition: A subset $C \subset \mathbb{C}^2$ is called an *affine algebraic curve* if there exists a polynomial $f \in \mathbb{C}[X, Y]$ such that $\deg f \geq 1$ and

$$C = V(f) = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\}$$

[F2001].

In other words, an affine algebraic curve is a curve that locates in a complex coordinate plane and consists of the zeros of a polynomial in two complex variables.

2.3 Points at Infinity

In some cases, it can be useful to extend the affine space K^n over a field K to the projective space $P_n(K)$. Note that only the plane (i.e. $n = 2$) occurs in this discussion though the field K can be arbitrary. Also let K^* be the non-zero members of field.

The *projective plane over* K is denoted as $P_2(K)$, that is, the set of all lines through the origin in K^3.

If $0 \neq a = (x, y, z) \in K^3$, then

$$[x, y, z] = K \cdot (x, y, z)$$

denotes the *line* through a which is a *point* in $P_2(K)$.

For computations with these *homogeneous coordinates*, the rule is

$$[x, y, z] = [x_1, y_1, z_1] \Longleftrightarrow (x, y, z) = \lambda(x_1, y_1, z_1),$$

where $\lambda \in K^*$.

The *canonical embedding* of the affine plane is given by

$$\iota : K^2 \longrightarrow P_2(K), (x, y) \longmapsto [x, y, 1].$$

The set of *points at infinity* of K^2 is

$$P_2(K) \setminus \iota(K^2) = \{[x, y, z] \in P_2(K) : z = 0\},$$

a *projective line* $P_1(K)$.

Every point at infinity $[x, y, 0]$ has a corresponding direction $[x, y]$ in K^2. Instead of the line $z = 0$, any other line in $L \subset P_2(K)$ can take on the role of the points at infinity of the affine plane $P_2(K) \setminus L$ [F2001].

2.4 Projective Algebraic Curve

Now, an affine curve $C \subset K^2$ is extended to a projective curve $\overline{C} \subset P_2(K)$.

If $F \in K[X, Y, Z]$ is a homogeneous polynomial, then

$$V(F) := \{[x, y, z] \in P_2(K) : F(x, y, z) = 0\}$$

is called the ***variety*** of F; more precisely, $V(F)$ is the set of lines through 0 in the *affine cone*

$$\{(x, y, z) \in K^3 : F(x, y, z) = 0\}$$

corresponding to $V(F)$.

For $f \in K[X, Y]$, a homogeneous $F \in K[X, Y, Z]$, the homogenization of f, is constructed as follows: if $n = \deg f$ and

$$f(X, Y) = f_{(0)} + f_{(1)} + \cdots + f_{(n)}$$

is its decomposition into homogeneous parts, with graded degrees, where $f_{(n)} \neq 0$, then

$$F(X, Y, Z) := Z^n f_{(0)} + \cdots + Z f_{(n-1)} + f_{(n)}.$$

Clearly, $F = Z^n f(\frac{X}{Z}, \frac{Y}{Z})$ and $f = F(X, Y, 1)$.

In this project, "curves" are to be in $K = \mathbb{C}$ even though these calculations are valid for arbitrary fields [F2001].

Finally, the definition of a *projective algebraic curve* is stated below.

Definition: A subset $\overline{C} \subset P_2(\mathbb{C})$ is called a *projective algebraic curve* if there is a homogeneous $F \in \mathbb{C}[X, Y, Z,]$ such that $\deg F \geq 1$ and $\overline{C} = V(F)$.

If $C = V(F) \subset \mathbb{C}^2$ is an affine algebraic curve and F is the homogenization of f, then $\overline{C} = V(F) \subset P_2\,(\mathbb{C})$ is called the projective closure of C [F2001].

Extension of an affine curve to a projective curve will be practiced in Chapter 3 to find the singularities as explained in the next section.

2.5 Singularities and Tangents

After the mathematical definition of an algebraic curve, additional important definitions need to be covered for this project.

First of all, the definition of *singularity* is a point on an algebraic curve with multiple tangents. In order to understand a singular point in detail, let us go over a few more definitions.

Definition: Let the algebraic curve $C = V(f) \subset \mathbb{C}^2$ be described by a minimal polynomial $f \in \mathbb{C}[X, Y]$. We say that C is *smooth* at a point $P \in C$ if

$$\text{grad}_P f := (\frac{\partial f}{\partial X}(P), \frac{\partial f}{\partial Y}(P)) \neq (0, 0).$$

Instead of *not smooth*, we say *singular* [F2001].

Also an *ordinary singularity* is a singularity where the tangents are distinct [W1962]. So singular points, P, can be found by setting the gradient of a function to zero, and the tangents at singular points are figured out in this project.

In algebraic curves, *tangents* are defined as below.

Definition: If C is smooth at P, the line

$$T_P C := \{(x, y) \in \mathbb{C}^2 : \frac{\partial f}{\partial X}(P) \cdot x + \frac{\partial f}{\partial Y}(P) \cdot y = c\}$$

is called the *tangent* to C at P. Here $c \in C$ is chosen so that $P \in T_P C$. The tangent is the line through P that, near P, passes as close to C as possible [F2001].

In this project, we the tangents for algebraic curves are determined repeatedly, and the following is the summary of this strategy. As an example, the hypocycloid of three cusps is examined here. A hypocycloid is the path traced by a point on a circle of radius r rolling along the inside of a circle of radius $R > r$. When r/R is rational, it is closed.

Let $R = 1$ and $r = 1/3$, then we can achieve the equation of the hypocycloid of three cusps as

$$3(x^2 + y^2)^2 + 8x(3y^2 - x^2) + 6(x^2 + y^2) = 1$$

[F2001].

So the affine curve is described as

$$f(x,y,z) = 3(x^2+y^2)^2 + 8x(3y^2-x^2) + 6(x^2+y^2) - 1 = 0,$$

and it is shown in the Figure 2.4. The hypocycloid is in solid thin, and the dashed-circle is the circle with the radius $R = 1$. The dotted-circle is the circle with the radius $r = \frac{1}{3}$, and the point that traces the hypocycloid is at $(1,0)$ in the figure.

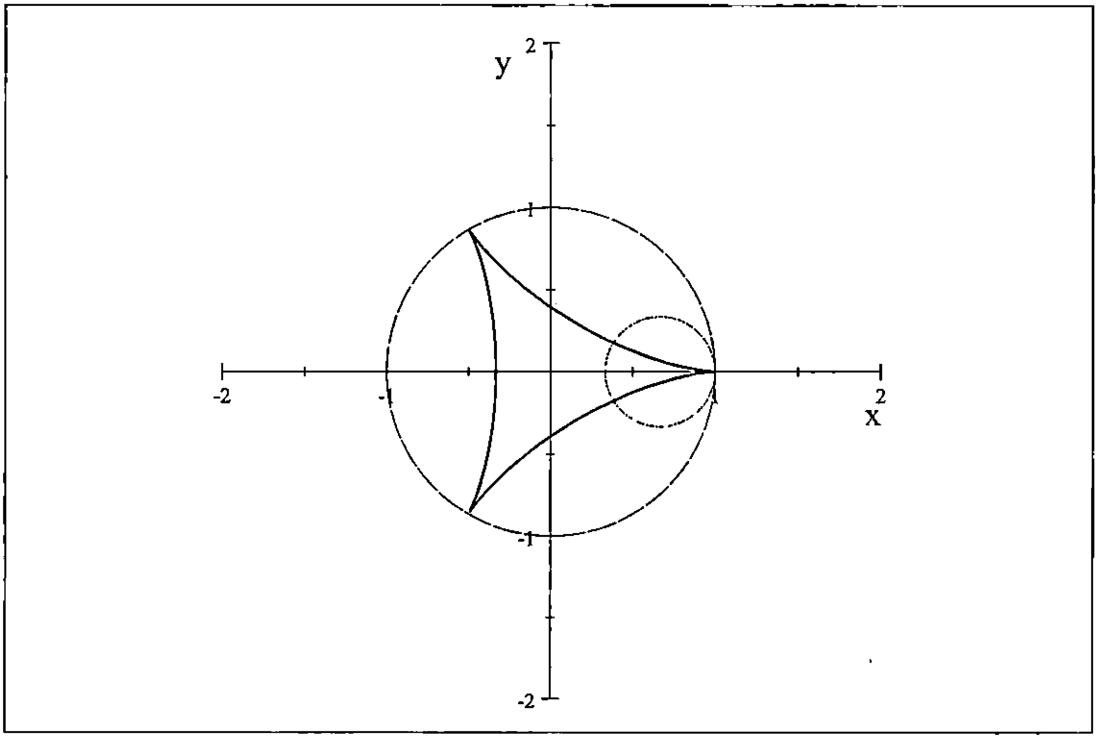

Figure 2.4: A Hypocycloid Inside a Circle with $R = 1$ and an Inner Circle with $r = \frac{1}{3}$

Since this polynomial is a quartic (degree four), the corresponding homogeneous polynomial is

$$F(x,y,z) = 3x^4 - 8x^3z + 6x^2y^2 + 6x^2z^2 + 24xy^2z + 3y^4 + 6y^2z^2 - z^4.$$

Therefore, the gradient of $F(x,y,z)$ is

$$\nabla F = \begin{bmatrix} 12\left(x^3 - 2x^2z + 2y^2z + xy^2 + xz^2\right) \\ 12y\left(x^2+y^2+z^2+4xz\right) \\ -4\left(2x+z\right)\left(x^2-3y^2+z^2-2xz\right) \end{bmatrix}.$$

Now, the solutions for $\nabla F = (0,0,0)$ are the singular points of $f(x,y,z)$. So the singularities are at $[1,0,1]$, $[-1,\sqrt{3},2]$ and $[-1,-\sqrt{3},2]$.

Now, the tangents at each singularity need to be found.
For $[1,0,1]$, this point has coordinate $(1,0)$ in the affine plane $z=1$ and

$$f(x,y) = 3(x^2+y^2)^2 + 8x(3y^2-x^2) + 6(x^2+y^2) - 1 = 0$$

is the corresponding affine curve.

Before continuing the calculation, the intersection of curve and line needs to be explained.

Let $C = f(x,y)$ be an algebraic curve with a homogeneous polynomial of degree n, such that $F(x,y,z)=0$. In order to investigate the intersections of L and C at a particular point P of C, choose an affine coordinate system in which P has coordinate (a,b) and C is the affine curve $f(x,y)=0$. The parametric equations of L can be put in the forms

$$\begin{aligned} x &= a+\lambda t \\ y &= b+\mu t \end{aligned}$$

and L can be determined by the ratio $\lambda : \mu$.

Then the intersections of L and C are determined by the roots of

$$f(a+\lambda t, b+\mu t) = 0.$$

Since $f(a,b)=0$, by expanding $f(a+\lambda t, b+\mu t)$ in a Taylor series in t, we have

$$f_x(\lambda + f_y\mu)t + \frac{1}{2!}(f_{xx}\lambda^2 + 2f_{xy}\lambda_\mu + f_{yy}\mu^2)t^2 + \cdots = 0,$$

where f_x, f_y, $\cdots$ are the values at P of the derivatives of f.

Case 1: If f_x and f_y are not both zero, then every line through P has a single intersection with C at P with one exception corresponding to the value of $\lambda : \mu$ which makes $f_x\lambda + f_y\mu = 0$. This exceptional line is the tangent to C at P.

Case 2: If $f_x = f_y = 0$ but not all of f_{xx}, f_{xy}, f_{yy} are zero, then every line through P has at least two intersections at P and at most two lines corresponding to the roots of

$$f_{xx}\lambda^2 + 2f_{xy}\lambda_\mu + f_{yy}\mu^2 = 0$$

has more than two intersections. These exceptional lines are called tangents to C at P. If the equation above has a double root, then there are two coincident tangents.

Case 3: If all derivatives of f up to and including the $(r-1)$-*th* vanish at P but at least one r-*th* derivative does not vanish at P, then every line through P has at least r intersections with C at P. The exceptional lines, the tangents to C at P, corresponding to the roots of

$$f_{x^r}\lambda^r + \binom{r}{1} f_{x^{r-1}y}\lambda^{r-1}\mu + \cdots + \binom{r}{r} f_{y^r}\mu^r = 0,$$

are counted with multiplicities equal to the multiplicities of the corresponding roots of this equation. Thus P is said to be a point of C of multiplicity r, or an r-*fold* point. Notice since $f(x,y)$ is not identically zero, there must be some derivative of order less than or equal to n which does not vanish at P. Hence Case 3 occurs for some r with $1 \leq r \leq n$. Also note that a point of C of multiplicity one is called a single point of C, and one of multiplicity two is called a double point. If the r tangents at the point are distinct, a point of multiplicity r is ordinary. A point of multiplicity greater than one is called singular, and a point (a,b) is singular if $f(a,b) = f_x(a,b) = f_y(a,b) = 0$ [W1962].

Going back to the example, now any line in the affine plane that contains $(1,0)$ can be parameterized by

$$\begin{aligned} x &= 1 + \lambda t \\ y &= \mu t. \end{aligned}$$

Consider $f(x,y) = 0$,

$$f(x,y) = 3\left(\lambda^2+\mu^2\right)^2 t^4 - \left(8\lambda\left(\lambda^2 - 3\mu^2\right)t^3 - 12\lambda\left(\lambda^2+\mu^2\right)\right) + 36\mu^2 t^2 = 0,$$

then find the value of λ and μ such that $36\mu^2 = 0$, which is the coefficient of t^2.
So, $\mu = 0$.
Since $t = \frac{x-1}{\lambda} = \frac{y}{\mu}$, then $y = \frac{\mu(x-1)}{\lambda}$.
Therefore, the tangent is

$$y = 0$$

which is shown as the x-axis and the thick solid line in the Figure 2.5.

Similarly, the singular point at $[-\frac{1}{2}, \frac{\sqrt{3}}{2}, 1]$ has coordinate $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ in the affine plane $z = 1$, and it is parameterized as

$$\begin{aligned} x &= -\frac{1}{2} + \lambda t \\ y &= \frac{\sqrt{3}}{2} + \mu t. \end{aligned}$$

Thus the curve has a form

$$f(x,y) = 3\left(\lambda^2 + \mu^2\right)^2 t^4 - 2\left(7\lambda^2 + 3\mu^2 + 4\sqrt{3}\lambda\mu\right)\left(\lambda - \sqrt{3}\mu\right)t^3 + 3\left(3\lambda + \sqrt{3}\mu\right)^2 t^2 = 0.$$

So now the coefficient of t^2 needs to be set equal to zero,

$$3\left(3\lambda + \sqrt{3}\mu\right)^2 = 0.$$

Then

$$\lambda = -\frac{\sqrt{3}}{3}\mu.$$

Since $t = \frac{2x+1}{\lambda} = \frac{2y-\sqrt{3}}{\mu}$, the tangent is

$$3x + \sqrt{3}y = 0$$

which is shown as the thick dashed line in the Figure 2.5.

Since the last singular point has a similar coordinate with another one, it can be assumed that the tangent for the last singularity is

$$3x - \sqrt{3}y = 0$$

which is the thin dashed line in the Figure 2.5. This assumption is easily verified.

All the tangents for the three singularities are plotted in the Figure 2.5, and these singularities are not ordinary since the tangents are not distinct at a singular point.

This method will also be used to find tangents in Chapter 3, where the singularities of Cassini ovals are determined ordinary.

2.6 Genus and Birational Equivalence

Genus is an invariant associated with any irreducible curve. It will not be explained in detail, but the formula of it is used to compute genus in this project.

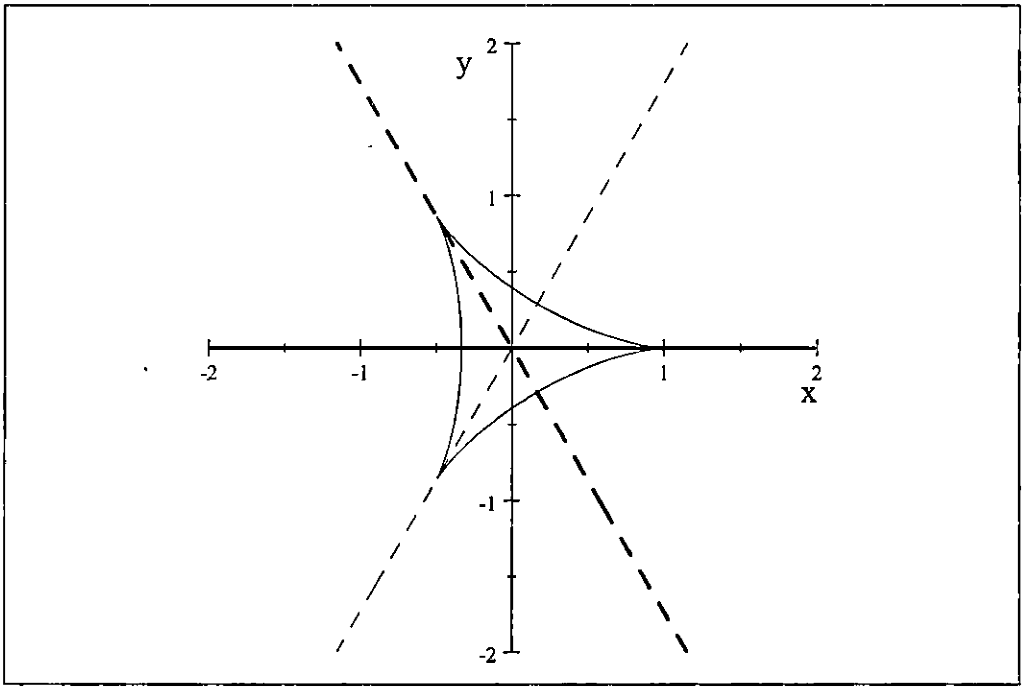

Figure 2.5: The Three Tangents of A Hypocycloid at The Three Singularities

Formula: If the curve has only ordinary singularities, then its genus p is computed by using

$$p = \frac{(n-1)(n-2)}{2} - \sum \frac{r_i(r_i-1)}{2}$$

where n is the order of the curve and the r_i are the multiplicities of the singularities [W1962].

For instance, the order of the Folium of Descartes, $f(x, y, z) = x^3 + y^3 - 3xyz$, is 3, and the multiplicity of its singularity $[0, 0, 1]$ is 2. So the genus p is computed as

$$p = \frac{(3-1)(3-2)}{2} - \frac{2(2-1)}{2} = 1 - 1 = 0.$$

The singularity has distinct multiple tangents; therefore, the singularity is ordinary.

One of the properties of elliptic curves is that they are algebraic curves of genus 1. Also, there is a theorem that describes the property of elliptic curves.

Theorem: Any elliptic curve is birationally equivalent to a non-singular plane cubic [W1962].

Any elliptic curve is birationally equivalent to a cubic of the form: $y^2 = x^3 + ax + b$ which is non-singular; that is, its graph has no cusps or self-intersections, provided $4a^3 + 27b^2 \neq 0$ [MM99].

Now, in order to understand the theorem above, the meaning of birationally equivalent is explained next.

Two curves are *birationally equivalent* if their associated function fields are isomorphic. The function field of an irreducible curve is the quotient field obtained by factoring the polynomial ring by the principal ideal generated by the curve. Two birationally equivalent curves are related by a birational transformation that is induced by an isomorphism of the function fields. Typically, birational transformations are obtained by rational function changes of variable. Thus, any projective transformation is birational but, in general, such transformations will change the degree of the curve. When the real points of a plane curve are represented as points in the complex plane, any linear fractional transformation of the corresponding complex variable induces a birational transformation of the curve [S2012]. This follows from the theorem that says, each birational transformation of the plane is a product of quadratic transformations (**M. Noether's Theorem**) [BK86].

Since two curves that are birationally equivalent have the same genus, genus is a birational invariant. However, two curves that have the same genus are not necessarily birationally equivalent. We will show this with the ovals of Cassini later. Generally, an exception (two curves with the same genus are birationally equivalent) is curves of genus 0 as we have seen in an example of the Folium of Descartes. It can be shown that a curve is rational if and only if it is of genus 0 [W1962]; these exceptional curves are all birationally equivalent. For example, a projective line (algebraic curve of degree 1) has genus 0, and, as the genus formula shows, so does any conic. Thus conics and lines are birationally equivalent. For genus 1, the elliptic curves, the birational equivalence classes are determined by a stronger invariant, the cross-ratio [S2012].

Keeping those definitions and theorems in mind, as the first step, the Cassini ovals are shown to be elliptic in Chapter 3.

Chapter 3

Genus of Cassini Ovals

3.1 Lemniscate

In this chapter, Cassini ovals are examined as elliptic curves. In algebraic curves, an *elliptic curve* is defined as an algebraic curve of genus 1. Recalling some definitions, a *singularity* is a point on an algebraic curve with multiple tangents. An *ordinary singularity* is a singularity where the tangents are distinct. Additionally, *genus* is a birational invariant associated with any irreducible curve. If the curve has only ordinary singularities (i.e. distinct tangents) then its genus is computed using the equation from Chapter 2,

$$p = \frac{(n-1)(n-2)}{2} - \sum \frac{r_i(r_i-1)}{2}$$

where n is the order (or degree) of the curve and the r_i are the multiplicities at the respective singularities. For example, a non-singular cubic curve has genus 1, whereas a cubic with a single ordinary singularity of multiplicity two has genus 0.

First consider the lemniscate, $a = b = 1$. The goal here is to find singularities of the curve and find tangents at those points to see if the tangents are distinct or not. By following the strategy explained in Chapter 2, the tangents of an affine curve of Cassini ovals are obtained. From Chapter 1, the affine Cassini Oval has a form

$$f(x,y) = (x^2+y^2)^2 - 2(x^2-y^2) = x^4 + y^4 + 2x^2y^2 - 2x^2 + 2y^2 = 0.$$

Now let

$$F(x,y,z) = (x^2+y^2)^2 - 2(x^2-y^2)z^2 = x^4 + y^4 + 2x^2y^2 - 2x^2z^2 + 2y^2z^2$$

be the corresponding homogeneous polynomial.

In order to find its singularities, the gradient of the polynomial, ∇F is computed. Then by solving $\nabla F(x,y,z) = (0,0,0)$, the singular points $[x,y,z]$ can be found. So

$$\nabla F = \begin{bmatrix} \frac{dF}{dx} \\ \frac{dF}{dy} \\ \frac{dF}{dz} \end{bmatrix} = \begin{bmatrix} 4x\left(x^2+y^2-z^2\right) \\ 4y\left(x^2+y^2+z^2\right) \\ -4z\left(x-y\right)\left(x+y\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now as the first step to find the solutions to the equations, consider the case when $z = 0$, the line at infinity.

Then ∇F leaves

$$\begin{aligned} x\left(x^2+y^2\right) &= 0 \\ y\left(x^2+y^2\right) &= 0. \end{aligned}$$

Thus

$$x^2 + y^2 = 0.$$

Since x and y cannot both be 0, the singularities at infinity are $[1,i,0]$, and $[1,-i,0]$.

Next, consider the case when $z \neq 0$.

Suppose $z = 1$, then

$$\begin{aligned} x\left(x^2+y^2-1\right) &= 0 \\ y\left(x^2+y^2+1\right) &= 0 \\ (x-y)\left(x+y\right) &= 0 \end{aligned}$$

which has the unique solution $x = y = 0$.

Therefore, these three are the singular points of the projective curve F: $[0,0,1]$, $[1,i,0]$ and $[1,-i,0]$.

Next, the tangents at the singular points are looked for. Just like the example of the hypocycloid of three cusps in Chapter 2, the tangents for each singular point can be determined by following the same strategy.

At $[0,0,1]$, this point has coordinate $[0,0]$ in the affine plane $z = 1$ and $f(x,y) = x^4 + y^4 + 2x^2y^2 - 2x^2 + 2y^2 = 0$ is the corresponding affine curve. So in this case, the line,

L, that contains $P = (0,0)$ can be parameterized as follows:

$$\begin{aligned} x &= \lambda t \\ y &= \mu t. \end{aligned}$$

Now we look at $f(\lambda t, \mu t) = 0$.

$$f(\lambda t, \mu t) = \left(\lambda^2 + \mu^2\right)^2 t^4 - 2(\lambda + \mu)(\lambda - \mu)t^2 = 0$$

Then find the values of λ, μ such that $(\lambda + \mu)(\lambda - \mu) = 0$ (the coefficient of t^2).
So $\lambda = \mu$ or $\lambda = -\mu$.
Then $\frac{x}{\lambda} = \frac{y}{\mu} = t$.
Thus $x = y$ or $x = -y$.
Projectively, the tangents to the singular point $[0, 0, 1]$ are the lines

$$\begin{aligned} x - y &= 0 \\ x + y &= 0. \end{aligned}$$

By following the same process, the tangents for the other two singular points, $[1, i, 0]$ and $[1, -i, 0]$, can be found.

So parameterize the line that goes through $[1, i, 0]$ as

$$\begin{aligned} x &= 1 + \lambda t \\ y &= i + \mu t \\ z &= \nu t. \end{aligned}$$

Then the homogeneous equation is

$$\begin{aligned} F(x,y,z) &= \left(\left(\lambda^2 + \mu^2\right)^2 - 2\nu^2\left(\lambda^2 - \mu^2\right)\right) t^4 \\ &\quad -4(\nu^2\left(\lambda - i\mu\right) - \left(\lambda + i\mu\right)\left(\lambda^2 + \mu^2\right))t^3 \\ &\quad -4\left(\nu^2 - \left(\lambda + i\mu\right)^2\right) t^2. \end{aligned}$$

Then the singularity $[1, i, 0]$ has multiplicity of 2, because $\lambda + i\mu - \nu = 0$ and $\lambda + i\mu + \nu = 0$. Therefore, the tangents are

$$\begin{aligned} x + iy + z &= 0 \\ x + iy - z &= 0. \end{aligned}$$

Similarly, parameterize the line that goes through $[1, -i, 0]$.

$$\begin{aligned} x &= 1 + \lambda t \\ y &= -i + \mu t \\ z &= \nu t. \end{aligned}$$

Then we have

$$\begin{aligned} F(x, y, z) &= \left(\left(\lambda^2 + \mu^2\right)^2 - 2\nu^2\left(\lambda^2 - \mu^2\right)\right) t^4 \\ &\quad -4(\nu^2\left(\lambda + i\mu\right) + \left(\lambda - i\mu\right)\left(\lambda^2 + \mu^2\right))t^3 \\ &\quad -4\left(\nu^2 - \left(\lambda - i\mu\right)^2\right) t^2. \end{aligned}$$

So the singularity $[1, -i, 0]$ has the multiplicity of 2 as well, because $\lambda - i\mu\lambda - i\mu + \nu = 0$ and $\lambda - i\mu - \nu = 0$. Thus the tangents at $[1, -i, 0]$ are

$$\begin{aligned} -x + iy - z &= 0 \\ x - iy - z &= 0. \end{aligned}$$

3.2 Non-Lemniscate

Second, consider the ovals that are not lemniscates, that is $a \neq b$ for $f(x, y) = x^4 + y^4 + 2x^2y^2 - 2a^2x^2 + 2a^2y^2 + a^2 - b^2 = 0$

Assume that $a = 1$ then consider two cases: $0 < b < 1$ and $b > 1$ in

$$f(x, y) = 1 - b^2 + x^4 + y^4 - 2x^2 + 2y^2 + 2x^2y^2 = 0.$$

As a curve in the affine plane $z = 1$, the homogeneous projective curve is obtained,

$$F(x, y, z) = (1 - b^2)z^4 + x^4 + y^4 - 2x^2z^2 + 2y^2z^2 + 2x^2y^2.$$

Now the gradient is

$$\nabla F(x, y, z) = \begin{bmatrix} 4x\left(x^2 + y^2 - z^2\right) \\ 4y\left(x^2 + y^2 + z^2\right) \\ -4z\left(b^2z^2 + x^2 - y^2 - z^2\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus the singularities at infinity, $z = 0$, are $[1, i, 0]$ and $[1, -i, 0]$.

To find the tangents at these points. Just like the previous calculation for $[1, \pm i, 0]$, the lines can be parameterized as

$$\begin{aligned} x &= 1 + \lambda t \\ y &= \pm i + \mu t \\ z &= \nu t. \end{aligned}$$

Then

$$\begin{aligned} F(x, y, z) &= \left((1-b^2)\nu^4 + \lambda^4 + \mu^4 + 2(\lambda^2\mu^2 - \lambda^2\nu^2 + \mu^2\nu^2)\right) t^4 \\ &\quad +4\left(\lambda \mp i\mu\right)\left(\lambda \pm i\mu + \nu\right)\left(\lambda \pm i\mu - \nu\right) t^3 \\ &\quad +4\left(\lambda \pm i\mu \pm \nu\right)\left(\lambda \pm i\mu \mp \nu\right) t^2. \end{aligned}$$

So solve $(\lambda \pm i\mu \pm \nu)(\lambda \pm i\mu \mp \nu) = 0$ to obtain the tangents at $[1, i, 0]$;

$$\begin{aligned} x + iy + z &= 0 \\ x + iy - z &= 0, \end{aligned}$$

and at $[1, -i, 0]$;

$$\begin{aligned} x - iy + z &= 0 \\ x - iy - z &= 0. \end{aligned}$$

Now consider the case when $z = 1$. The gradient leaves

$$\begin{aligned} x^2 + y^2 - 1 &= 0 \\ x^2 + y^2 + 1 &= 0 \\ x^2 - y^2 + b^2 - 1 &= 0. \end{aligned}$$

From the third equation, $y^2 = x^2 + b^2 - 1$. Then plug this equation into y^2 of the first two equations to solve for x:

$$\begin{aligned} 2x^2 + b^2 &= 2 \\ 2x^2 + b^2 &= 0. \end{aligned}$$

However, because there is no x that satisfies both equations, there is no solution for x in this case. Thus there are no other singularities in the affine plane in cases of $0 < b < 1$ and $b > 1$.

In conclusion, there are two ordinary singularities for the non-lemniscates at infinity, but there are no singularities in the affine plane.

3.3 Computing Genus

Additionally, computing the genus of Cassini curves will show if they are elliptic or not. Since a curve of genus 1 is elliptic, the Cassini Curves that have genus 1, $p = 1$, are searched.

First, consider the lemniscate, $b = a$. From Chapter 2, the formula can be used to determine its genus,

$$p = \frac{(n-1)(n-2)}{2} - \sum \frac{r_i(r_i - 1)}{2},$$

where n is the order of the curve and r_i is the multiplicities at each ordinary singularity. From 3.1, there are three ordinary singularities and each of them has multiplicity of two. So the genus of the lemniscate is

$$p = \frac{(4-1)(4-2)}{2} - (\frac{2(2-1)}{2} + \frac{2(2-1)}{2} + \frac{2(2-1)}{2}) = 0.$$

Thus the lemniscate is not elliptic, but it is a rational curve.

Next, consider the non-lemniscate, $b \neq a$. From 3.2, there are two singularities and the multiplicities of those points are both two. So we can compute the genus as

$$p = \frac{(4-1)(4-2)}{2} - (\frac{2(2-1)}{2} + \frac{2(2-1)}{2}) = 1.$$

Therefore the non-lemniscates are elliptic.

Overall, Cassini ovals are elliptic if they are not lemniscates, in other words, an affine algebraic curve

$$f(x, y) = x^4 + y^4 + 2x^2y^2 - 2a^2x^2 + 2a^2y^2 + a^4 - b^2 = 0$$

is elliptic if $b \neq a$.

Chapter 4

Transforming Cassini Ovals to Cubics

4.1 Linear Fractional Transformation

In this chapter, elliptic curves of Cassini ovals are transformed into cubics. Remember that elliptic curves are the ones with genus 1, so two cases of the Cassini ovals are to be considered, $b < a$ and $b > a$ with $f(x, y) = x^4+y^4+2x^2y^2-2a^2x^2+2a^2y^2+a^4-b^2 = 0$.

In order to transform Cassini ovals to cubics, represent an oval that is not a lemniscate as an affine curve in the plane $z = 1$. Then view this affine curve as a locus in the plane of complex numbers Z. Then use linear fractional transformation (LFT) to find the transformations of Z and apply the transformations of Z to obtain transformations of the ovals.

First of all, recall the definition of linear fractional transformation (LFT).

Definition: *Linear fractional transformation* is a conformal (angle-preserving) mapping transformation of the form $T(Z) = \frac{aZ+b}{cZ+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ [WLFT].

LFTs are one-to-one and onto mappings of the completed plane $C \cup \{\infty\}$ to itself. Note that if $T(Z) = \frac{aZ+b}{cZ+d}$ is represented by the complex projective matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then LFTs compose by matrix multiplication and therefore T^{-1} is represented by

$$\begin{bmatrix} a & -b \\ -c & d \end{bmatrix}.$$

LFTs are used to induce birational transformations on affine curves. In general, a suitable LFT is applied to a Cassini curve to transform it to a cubic and then this cubic is used to compute the cross-ratio in case the curve is elliptic. The fact that LFTs induce birational transformations is a consequence of a theorem discovered independently by Clifford and M. Noether.

Theorem: Each birational transformation of the plane is a product of quadratic transformations [BK86].

How LFT can be factored into quadratic transformations in general will not be discussed here, but the effect on Cassini curves is demonstrated beginning with the lemniscate as an example. Again, let $a = 1$ and let b be replaced by k (just because it makes it simple and convenient in notations later).

So the Cassini ovals can be rewritten as

$$x^4 + y^4 + 2x^2y^2 - 2x^2 + 2y^2 + 1 - k^2 = 0.$$

The lemniscate occurs when $k = 1$, so

$$x^4 + y^4 + 2x^2y^2 - 2x^2 + 2y^2 = 0.$$

Now consider the LFT

$$\begin{aligned} T &= \begin{bmatrix} \sqrt{2} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 1 \end{bmatrix} \\ T^{-1} &= \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & \sqrt{2} \end{bmatrix} \end{aligned}$$

and let $Z = X + iY$ with $X, Y \in \mathbb{R}$. Then

$$\begin{aligned} T(Z) &= \frac{\sqrt{2}Z - \frac{1}{2}}{-\frac{1}{\sqrt{2}}Z + 1} \\ T^{-1}(Z) &= -\frac{\sqrt{2}\,(X + iY) - \frac{1}{2}}{\frac{1}{2}\sqrt{2}\,(X + iY) - 1}, \end{aligned}$$

and so the lemniscate is transformed by substituting the real and imaginary parts of this expression for x and y, respectively. That is, the affine quartic $(x^2+y^2)^2+2(y^2-x^2)=0$ transforms into the cubic

$$(4X+5)(2X+1)^2 = 4(1-4X)Y^2$$

shown in the Figure 4.1.

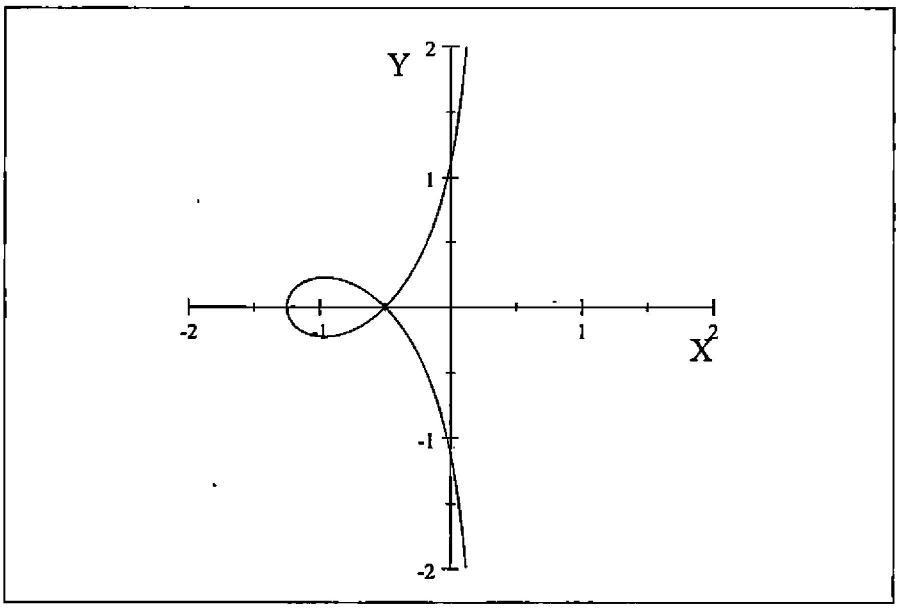

Figure 4.1: Transformed Cubic Lemniscate

This is the Nodal cubic that was introduced in Chapter 2. The point $(-\frac{1}{2},0)$ is an ordinary double point. This is the only singularity, so the genus of this curve is 0, as expected, since the genus of the lemniscate is invariant under a birational transformation.

4.2 Transformation of Cassini Ovals to Cubics

Since the focus is only on elliptic curves, let $k \neq 1$ in this case. Additionally, let $Z = X + iY$ (Note that this Z is a complex variable and $X, Y \in \mathbb{R}$).

Now, let the LFT be set as $T(Z) = \frac{aZ+b}{cZ+d}$, which takes the oval to a cubic curve. Notice that T is invertible because $ad - bc \neq 0$. This means that the inverse of the LFT,

T^{-1}, will transform the cubic to the oval. So the transformation can be written as

$$\begin{aligned} T(Z) &= \left((b+Xa)\frac{d+Xc}{(d+Xc)^2+Y^2c^2}+Y^2a\frac{c}{(d+Xc)^2+Y^2c^2}\right) \\ &\quad +i\left(-Yc\frac{b+Xa}{(d+Xc)^2+Y^2c^2}+Ya\frac{d+Xc}{(d+Xc)^2+Y^2c^2}\right) \\ &= W. \end{aligned}$$

This means that the W has its general form as $W = x + iy$ where

$$\begin{aligned} x &= (b+Xa)\frac{d+Xc}{(d+Xc)^2+Y^2c^2}+Y^2a\frac{c}{(d+Xc)^2+Y^2c^2} \\ y &= -Yc\frac{b+Xa}{(d+Xc)^2+Y^2c^2}+Ya\frac{d+Xc}{(d+Xc)^2+Y^2c^2}. \end{aligned}$$

Now, plug the x and y obtained above into the equation $f(x,y) = (x^2+y^2)^2 + 2(y^2-x^2)+1-k^2 = 0$. So, after removing the insignificant factor, $\frac{1}{((Xc+d)^2+Y^2c^2)^2}$, $f(x,y)$ has a form of

$$\begin{aligned} f(X,Y) &= \left(\left(a^2+(k-1)c^2\right)\left(a^2-(k+1)c^2\right)\left(X^4+Y^4+2X^2Y^2\right)\right. \\ &\quad +4\left(a\left(a^2b-bc^2-acd\right)-(k-1)(k+1)c^3d\right)\left(X^3+XY^2\right) \\ &\quad +2(3a^2b^2-a^2d^2-b^2c^2-4abcd-3(k-1)(k+1)c^2d^2)X^2 \\ &\quad +2\left(a^2b^2+a^2d^2+b^2c^2-4abcd-(k-1)(k+1)c^2d^2\right)Y^2 \\ &\quad +4(ab^3-abd^2-b^2cd-(k-1)(k+1)cd^3)X \\ &\quad \left.+\left(b^2+(k-1)d^2\right)\left(b^2-(k+1)d^2\right)\right). \end{aligned}$$

Since the goal is to make the equation cubic, the values; a, b, c and d, are to be found in a manner that leaves the variables of degree three or less. In other words, the coefficients of degree four variables need to be zero to eliminate X^4, Y^4 and X^2Y^2. Therefore, the values of a, b, c and d need to satisfy the condition below:

$$\left(a^2+(k-1)c^2\right)\left(a^2-(k+1)c^2\right) = 0.$$

So the values of a, b, c and d are to satisfy $a^2+(k-1)c^2 = 0$ or $a^2-(k+1)c^2 = 0$. Then

$$a = \pm c\sqrt{1-k}$$

or

$$a = \pm c\sqrt{1+k}.$$

By picking a and c as described above, the following equation is obtained.

$$\begin{aligned} f(X,Y) &= 4\left(a(a^2b - acd - bc^2) + (1-k^2)c^3d\right)X^3 \\ &\quad +2\left(3a^2b^2 - a^2d^2 - b^2c^2 - 4abcd + 3(1-k^2)c^2d^2\right)X^2 \\ &\quad +4\left(a(a^2b - acd - bc^2) + (1-k^2)c^3d\right)XY^2 \\ &\quad +4\left(b(ab^2 - bcd - ad^2) + (1-k^2)cd^3\right)X^2 \\ &\quad +2\left(a^2b^2 + a^2d^2 + b^2c^2 - 4abcd + (1-k^2)c^2d^2\right)Y^2 \\ &\quad +\left(b^2 - d^2(k-1)\right)\left(b^2 + d^2(k-1)\right) \end{aligned}$$

The real values for a and c can be found by choosing $a = \pm c\sqrt{1+k}$ and letting $a = 1$. Then $c = \pm\frac{1}{\sqrt{1+k}}$, but for convenience, pick the positive value. Then the equation with only b and d is left.

$$\begin{aligned} f(X,Y) &= \frac{1}{(k+1)^{\frac{3}{2}}}(4\left(dk(k+1) - b\sqrt{k+1}\left(\sqrt{k+1}-1\right)\left(\sqrt{k+1}+1\right)\right)X^3 \\ &\quad -2\left(\sqrt{k+1}\left(b^2(2b+3k+2bk+2) - d^2(k+1)(2b+3k-2)\right) - 2d(k+1)\left(2b + d^2k^2 + b^2 - d^2\right)\right)X^2 \\ &\quad +4\left(dk(k+1) - b\sqrt{k+1}\left(\sqrt{k+1}-1\right)\left(\sqrt{k+1}+1\right)\right)XY^2 \\ &\quad -2\left(\sqrt{k+1}\left(b^2(k+1) + d^2(k+1) + b^2 + d^2(1-k^2)\right) - 4bd(k+1)\right)Y^2 \\ &\quad -\left(\sqrt{k+1}\right)^3(b^2 + d^2(k-1))\left(b^2 - d^2(k+1)\right)) \end{aligned}$$

Thus, at this point, b and d are independent from a and c.

In order to make this equation simpler, pick $b = \frac{1}{k+1}$ and $d = \sqrt{k+1}$.

$$\begin{aligned} f(X,Y) &= \frac{k^2}{(k+1)^4}((k+2)(4(k+1)^2X^3 \\ &\quad +2k(2k+3)(k+2)(k+1)X^2 \\ &\quad +4(k+1)^2XY^2 + 2(k+1)\left(k^2-2\right)Y^2 \\ &\quad +\left(k^2+2k+2\right)\left(k^3+2k^2-2\right)) \end{aligned}$$

Just to check if $ad - bc \neq 0$, solve for k for the following equation

$$ad - bc = \sqrt{k+1} - \frac{1}{(k+1)^{\frac{3}{2}}} = 0,$$

which cannot be satisfied since $k > 0$. Therefore, this combination satisfies the condition.

Now, a possible combination of values for a, b, c, and d is determined:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{k+1} \\ \frac{1}{\sqrt{k+1}} & \sqrt{k+1} \end{bmatrix} = \begin{bmatrix} k+1 & 1 \\ \sqrt{k+1} & (k+1)^{\frac{3}{2}} \end{bmatrix}.$$

So a transformation $T(Z)$ with the values a, b, c, and d can be formed.

As a next step, find the inverse of $T(Z)$ because if an affine curve $f(x, y)$ is transformed by $(x, y) \longmapsto T(x, y)$, then $f(T^{-1}(x, y))$ is the transformed curve. So if f is an oval and $T(f) = \tilde{f}$ is a cubic then $f(T^{-1}(x, y)) = \tilde{f}$ [S2012]. Therefore, the inverse of the T is

$$T^{-1} = \begin{bmatrix} (k+1)^{\frac{3}{2}} & -1 \\ -\sqrt{k+1} & k+1 \end{bmatrix}.$$

So the LFT that takes the ovals to cubic is

$$T_k(Z) = \frac{(k+1)^{\frac{3}{2}}\, Z - 1}{-\sqrt{k+1}Z + (k+1)}.$$

Thus the transformed cubic of the Cassini oval is

$$\begin{aligned} f(X, Y) &= 4\,(k+1)^2\, X^3 \\ &\quad +2\,(k+1)\left(3k^2 + 4k + 2\right) X^2 \\ &\quad +4\,(k+1)\left(k^3 + 2k^2 + k - 1\right) X \\ &\quad +(k+1)XY^2 + 2\,(k+1)\left(k^2 - 2\right) Y^2 \\ &\quad +\left(k^2 + 2k + 2\right)\left(k^3 + 2k^2 - 2\right). \end{aligned}$$

Just to see the results of the obtained cubic, plug in a certain value for k. For the case of $0 < k < 1$, let $k = \frac{1}{2}$,

$$f(X, Y) = 9X^3 + 9XY^2 + \frac{57}{4}X^2 - \frac{21}{4}Y^2 + \frac{3}{4}X - \frac{143}{32} = 0$$

which gives two separate ovals. The Figure 4.2 shows one disconnected loop and one unbounded curve.

Similarly, for the case $k > 1$, let $k = 2$, which forms a continuous loop:

$$f(X, Y) = 4\left(9X^3 + 9XY^2 + 33X^2 + 3Y^2 + 51X + 35\right) = 0$$

In Figure 4.3, an unbounded cubic curve is formed.

Since the degree four Cassini oval equation is transformed into the degree three equation, it is now possible to find its cross-ratio which will be computed in the next chapter.

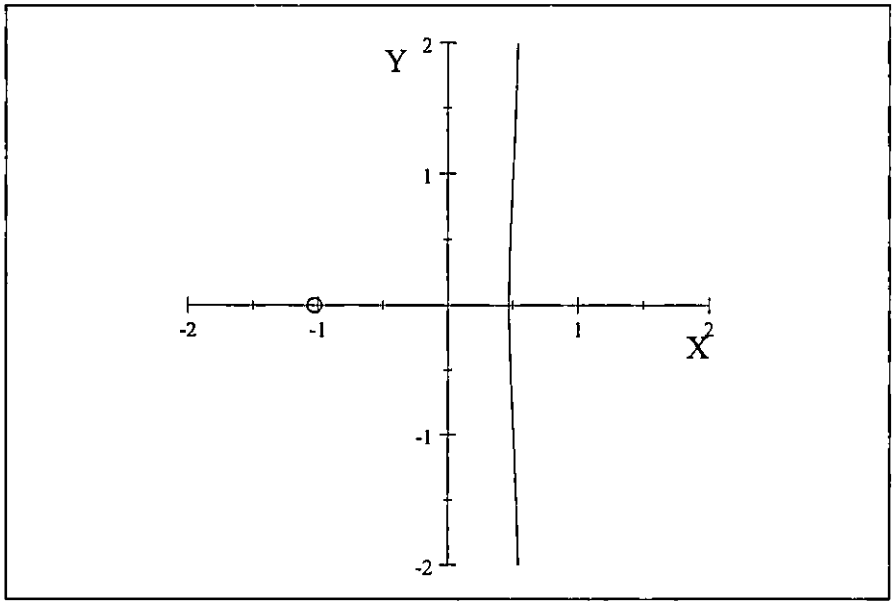

Figure 4.2: Cassini Cubic with $k = \frac{1}{2}$

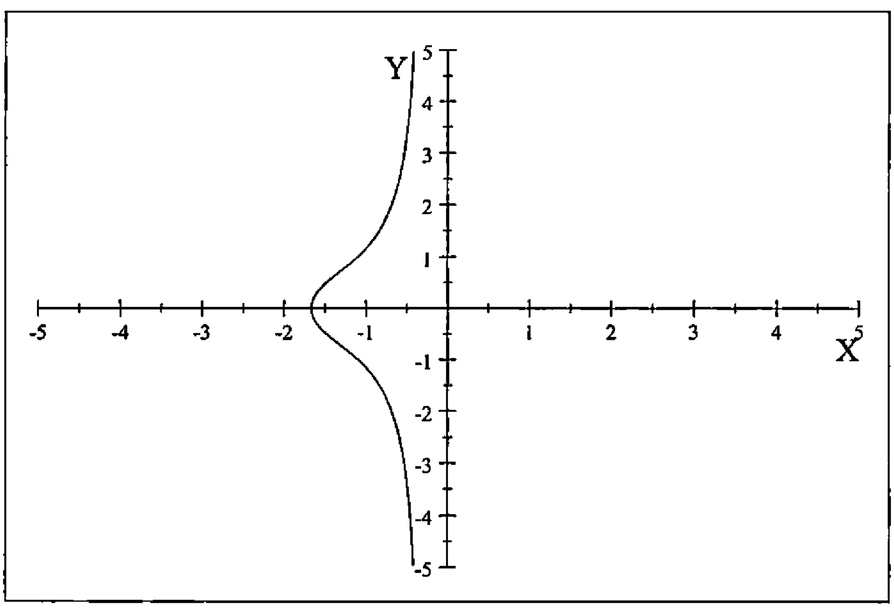

Figure 4.3: Cassini Cubic with $k = 2$

Chapter 5

Computing Tangents and Cross-ratio

5.1 Computing Tangents

In order to find the cross-ratio, the tangents to the cubic need to be found. The affine curve that has been obtained in the last chapter is

$$\begin{aligned} f(x,y) &= 4(k+1)^2 x^3 \\ &+(k+1)xy^2 \\ &+2(k+1)\left(4k+3k^2+2\right)x^2 \\ &+2(k+1)\left(k^2-2\right)y^2 \\ &+4(k+1)\left(k+2k^2+k^3-1\right)x \\ &+\left(2k+k^2+2\right)\left(2k^2+k^3-2\right). \end{aligned}$$

So the homogeneous equation of the cubic is the following:

$$\begin{aligned} F(x,y,z) &= 4(k+1)^2 x^3 \\ &+2(k+1)\left(4k+3k^2+2\right)x^2 z \\ &+4(k+1)\left(k+2k^2+k^3-1\right)xz^2 \\ &+4(k+1)^2 xy^2 + 2(k+1)\left(k^2-2\right)y^2 z \\ &+\left(2k+k^2+2\right)\left(2k^2+k^3-2\right)z^3. \end{aligned}$$

Then the gradient of F is shown below.

$$\nabla F = \begin{bmatrix} 4\,(k+1)\,(3x^2\,(k+1) + y^2\,(k+1) + z^2\left(k^3 + 2k^2 + k - 1\right) + xz\left(3k^2 + 4k + 2\right)) \\ 4y\,(k+1)\left(2x(k+1) + z(k^2-2)\right) \\ 3z^2\left(k^3 + 2k^2 - 2\right)\left(k^2 + 2k + 2\right) + 2x^2\,(k+1)\left(3k^2 + 4k + 2\right) + 2y^2\left(k^2 - 2\right)(k+1) + 8xz\,(k+1)\left(k^3 + 2k^2 + k - 1\right) \end{bmatrix}$$

Now, recall the definition of *Hessian matrix* of F.

If $F \in \mathbb{C}[X_0, X_1, X_2]$ is homogeneous of degree ≥ 2, then the symmetric 3×3 matrix

$$H_F := \left(\frac{\partial^2 F}{\partial X_i \partial X_j}\right)_{0 \leq i,j \leq 2}$$

is called the *Hessian matrix* of F [F2001].

So, in this case, the Hessian matrix, H_F, turns out to be as follows.

$$H_F = \begin{bmatrix} 4(k+1)\left(6x+2z+3k^2z+6kx+4kz\right) & 8y(k+1)^2 & 4(k+1)\left(2x-2z+3k^2x+4k^2z+2k^3z+4kx+2kz\right) \\ 8y(k+1)^2 & 4(k+1)\left(2x-2z+k^2z+2kx\right) & 4y\left(k^2-2\right)(k+1) \\ 4(k+1)\left(2x-2z+3k^2x+4k^2z+2k^3z+4kx+2kz\right) & 4y\left(k^2-2\right)(k+1) & 6z\left(k^3+2k^2-2\right)\left(k^2+2k+2\right)+8x(k+1)\left(k^3+2k^2+k-1\right) \end{bmatrix}$$

The determinant of H_F is called the *Hessian* of the curve, which is denoted by H.

In this case, its determinant is

$$\begin{aligned} H(x,y,z) &= z^3k^7 \\ &+4z^2(2x+z)k^6 \\ &+2z\left(9x^2-y^2+3z^2+12xz\right)k^5 \\ &+2\left(6x^3-3z^3+23x^2z-2xy^2+10xz^2-7y^2z\right)k^4 \\ &+8\left(3x^3-3z^3+3x^2z-xy^2-3xz^2-5y^2z\right)k^3 \\ &-4\left(x^3+3z^3+9x^2z+5xy^2+11xz^2+7y^2z\right)k^2 \\ &-16(2x-z)\left(x^2+y^2+z^2+2xz\right)k \\ &-16(x-z)\left(x^2+y^2+z^2+2xz\right). \end{aligned}$$

The calculation of cross-ratio is made simpler if the tangents from a flex point of the cubic are found. The *flexes of* F are its non-singular points which are intersections with the curve $H=0$ [W1962]. Thus, the points $[x,y,z]$ ought to meet $F(x,y,z)=0$ and $H(x,y,z)=0$, but $\nabla F(x,y,z)\neq 0$.

First of all, let $z=0$ then

$$\nabla F = \begin{bmatrix} 2(k+1)\left(3x^2+y^2\right) \\ 4(k+1)xy \\ \left(3k^2+4k+2\right)x^2+\left(k^2-2\right)y^2 \end{bmatrix} \neq 0.$$

From the last two equations, say $x=0$ and $y\neq 0$.

Then a flex is to be $O=[0,1,0]$ since

$$\nabla F(0,1,0) = \begin{bmatrix} 4(k+1)^2 \\ 0 \\ 2\left(k^2-2\right)(k+1) \end{bmatrix} \neq 0.$$

So the tangent at the flex $O=[0,1,0]$ is the line

$$\left\langle 2(k+1)^2, 0, \left(k^2-2\right)(k+1)\right\rangle,$$

i.e., $2(k+1)^2x+\left(k^2-2\right)(k+1)z=0$.

In addition to this tangent at O, there are three distinct tangents to F passing through O [W1962].

Next, the other three tangents to F that pass through O are figured out. Let P be on the curve F, $P = [a, b, c]$ so that $F(a, b, c) = 0$, where $x = a$, $y = b$, and $z = c$. Then

$$\nabla F(a,b,c) = \begin{bmatrix} 4(k+1)\left(c^2\left(k^3+2k^2+k-1\right)+3a^2(k+1)+b^2(k+1)+ac\left(3k^2+4k+2\right)\right) \\ 4b\left(2a(k+1)+c\left(k^2-2\right)\right)(k+1) \\ 3c^2\left(k^3+2k^2-2\right)\left(k^2+2k+2\right)+2a^2(k+1)\left(3k^2+4k+2\right)+2b^2\left(k^2-2\right)(k+1)+8ac(k+1)\left(k^3+2k^2+k-1\right) \end{bmatrix}.$$

The tangent at P is $T_PF = 0$, where

$$\begin{aligned} T_PF = {} & 4(k+1)\left(c^2\left(k^3+2k^2+k-1\right)+3a^2(k+1)+b^2(k+1)+ac\left(3k^2+4k+2\right)\right)x \\ & +4b\left(2a(k+1)+c\left(k^2-2\right)\right)(k+1)y \\ & +(3c^2\left(k^3+2k^2-2\right)\left(k^2+2k+2\right)+2a^2(k+1)\left(3k^2+4k+2\right)+2b^2\left(k^2-2\right)(k+1)+8ac(k+1)\left(k^3+2k^2+k-1\right))z. \end{aligned}$$

For this line to contain O it must have

$$b\left(2a(k+1)+c\left(k^2-2\right)\right) = 0.$$

First suppose $b = 0$, then

$$F(a,0,c) = \left(2a(k+1)+c\left(k^2+2k+2\right)\right)\left(2a(k+1)(a+ck)+c^2\left(k^3+2k^2-2\right)\right) = 0.$$

So now there are two cases to solve: $2a(k+1)+c\left(k^2+2k+2\right) = 0$ and $2a(k+1)(a+ck)+c^2\left(k^3+2k^2-2\right) = 0$.

Case 1: $2a(k+1)+c\left(k^2+2k+2\right) = 0$. Then

$$a = -\frac{\left(k^2+2k+2\right)}{2(k+1)}c.$$

So

$$P = [k^2+2k+2, 0, -2k-2].$$

Then $a = k^2+2k+2$, $b = 0$, and $c = -2k-2$, and $\nabla F(a,b,c) = \begin{bmatrix} 2(k+1) \\ 0 \\ k^2+2k+2 \end{bmatrix}$.

So the tangent at P is $T_PF = 0$, where

$$T_PF = 2(k+1)x + (k^2+2k+2)z.$$

As a check, this line passes through $O = [0, 1, 0]$ and meets the condition $F(a, 0, c) = 0$.

Case 2: $2a\,(k+1)\,(a+ck)+c^2(k^3+2k^2-2)=0$. Then

$$a=-\frac{k\,(k+1)\pm(k+2)\,\sqrt{1-k^2}}{2k+2}c.$$

Notice that there are two solutions in this case.

For $a=-\frac{k(k+1)+(k+2)\sqrt{1-k^2}}{2k+2}c$,

$$P=[k+(2+k)\sqrt{1-k^2}+k^2,0,-2k-2]$$

and

$$\nabla F(a,b,c)=\begin{bmatrix} 2\left(\sqrt{1-k^2}+k^2-1\right) \\ 0 \\ (k^2+2k-2)\sqrt{1-k^2}+\left(k^2-2\right)(k-1) \end{bmatrix}.$$

Thus one of the tangents is

$$2\left(\sqrt{1-k^2}+k^2-1\right)x+\left((k^2+2k-2)\sqrt{1-k^2}+\left(k^2-2\right)(k-1)\right)z=0.$$

Similarly, for $a=-\frac{k(k+1)-(k+2)\sqrt{1-k^2}}{2k+2}c$,

$$P=[k-(2+k)\sqrt{1-k^2}+k^2,0,-2k-2].$$

Thus the other tangent is

$$2\left(\sqrt{1-k^2}-k^2+1\right)x+\left((k^2+2k-2)\sqrt{1-k^2}-\left(k^2-2\right)(k-1)\right)z=0.$$

Now suppose $2a\,(k+1)+c\left(k^2-2\right)=0$. If $a=c=0$ then $b\neq 0$. But this is the same case as $P=O$, which already has been considered. So assume $c=1$ and $a=-\frac{k^2-2}{2k+2}$. Thus

$$P=[k^2-2,0,-2k-2].$$

However, in this case $F(a,0,c)=-8k^2\,(k+1)^2\,(k+2)^3$, which cannot be 0, so P is not on F for any value of k.

Therefore, the four tangents that pass through $O=[0,1,0]$ are at these four points on F;

$$\begin{aligned} A &= O=[0,1,0] \\ B &= [k^2+2k+2,0,-2k-2] \\ C &= [k+(k+2)\sqrt{1-k^2}+k^2,0,-2k-2] \\ D &= [k-(k+2)\sqrt{1-k^2}+k^2,0,-2k-2]. \end{aligned}$$

The corresponding tangents to the points above are spanned by the four position vectors which are denoted as follows;

$$\begin{aligned} a &= \left\langle 2\left(k+1\right), 0, k^2-2\right\rangle \\ b &= \left\langle 2\left(k+1\right), 0, k^2+2k+2\right\rangle \\ c &= \left\langle 2\left(\sqrt{1-k^2}+k^2-1\right), 0, \left((k^2+2k-2)\sqrt{1-k^2}+\left(k^2-2\right)\left(k-1\right)\right)\right\rangle \\ d &= \left\langle 2\left(\sqrt{1-k^2}-k^2+1\right), 0, \left((k^2+2k-2)\sqrt{1-k^2}-\left(k^2-2\right)\left(k-1\right)\right)\right\rangle. \end{aligned}$$

In the next section, we now can compute the cross-ratio $(abcd)$.

5.2 Computing Cross-ratio

Before computing the cross-ratio, review its definition.

Let a, b, c, and d be four concurrent lines in projective space represented by the position vectors a, b, c, and d, and let

$$\begin{aligned} c &= \alpha a + \beta b \\ d &= \gamma a + \delta b. \end{aligned}$$

Then the *cross-ratio* is

$$(abcd) = \frac{\beta}{\alpha} / \frac{\delta}{\gamma} = \frac{\beta\gamma}{\alpha\delta}$$

[BEG99].

The cross-ratio of F is computed as follows.
From the previous section, 5.3, $c = \alpha a + \beta b$ is

$$\left\langle 2\left(\sqrt{1-k^2}+k^2-1\right), 0, \left((k^2+2k-2)\sqrt{1-k^2}+\left(k^2-2\right)(k-1)\right)\right\rangle = \alpha\left\langle 2\left(k+1\right), 0, k^2-2\right\rangle + \beta\left\langle 2\left(k+1\right), 0, k^2+2k+2\right\rangle,$$

which obtains the two relations

$$\begin{aligned} 2\left(\sqrt{1-k^2}+k^2-1\right) &= 2\left(k+1\right)\alpha + 2\left(k+1\right)\beta \\ (k^2+2k-2)\sqrt{1-k^2}+\left(k^2-2\right)\left(k-1\right) &= (k^2-2)\alpha + (k^2+2k+2)\beta. \end{aligned}$$

These two equations can be solved to obtain

$$\begin{aligned} \alpha &= -\frac{(k^2-2)\sqrt{1-k^2}-2k^2+2}{2(k+1)} \\ \beta &= \frac{k^2\sqrt{1-k^2}}{2(k+1)}. \end{aligned}$$

So the ratio of α and β is

$$\frac{\beta}{\alpha} = -\frac{k^2\sqrt{1-k^2}}{\sqrt{1-k^2}\,(k^2-2) - 2k^2 + 2}.$$

Similarly, $d = \gamma a + \delta b$ can be rewritten as

$$\left\langle 2\left(\sqrt{1-k^2} - k^2 + 1\right), 0, \left((k^2+2k-2)\sqrt{1-k^2} - \left(k^2-2\right)(k-1)\right)\right\rangle = \gamma\left\langle 2\,(k+1), 0, k^2-2\right\rangle + \delta\left\langle 2\,(k+1), 0, k^2+2k+2\right\rangle$$

which yields

$$\begin{aligned} 2\left(\sqrt{1-k^2} - k^2 + 1\right) &= 2\,(k+1)\,\gamma + 2\,(k+1)\,\delta \\ (k^2+2k-2)\sqrt{1-k^2} - \left(k^2-2\right)(k-1) &= \left(k^2-2\right)\gamma + \left(k^2+2k+2\right)\delta. \end{aligned}$$

Thus γ and δ are solved as

$$\begin{aligned} \gamma &= -\frac{(k^2-2)\sqrt{1-k^2} + 2k^2 - 2}{2\,(k+1)} \\ \delta &= \frac{k^2\sqrt{1-k^2}}{2k+2} \end{aligned}$$

and the ratio is

$$\frac{\delta}{\gamma} = -\frac{k^2\sqrt{1-k^2}}{\sqrt{1-k^2}\,(k^2-2) + 2k^2 - 2}.$$

Therefore the cross-ratio, χ, is

$$\chi = \frac{\beta}{\alpha}\Big/\frac{\delta}{\gamma} = \frac{\left(\sqrt{1-k^2}+1\right)^2}{\left(\sqrt{1-k^2}-1\right)^2}.$$

The value of χ depends on the order that has been chosen for the position vectors representing the tangent lines. Recall that if $(abcd) = \chi$, then

$$(bacd) = (abdc) = 1/\chi$$

and

$$(acbd) = (dbca) = 1 - \chi$$

[BEG99].

Hence

$$\begin{aligned}\frac{1}{\chi} &= \frac{\left(\sqrt{1-k^2}-1\right)^2}{\left(\sqrt{1-k^2}+1\right)^2}\\ 1-\chi &= \frac{4\sqrt{1-k^2}}{2\sqrt{1-k^2}+k^2-2}.\end{aligned}$$

Also, compositions of these two involutions yield three more possible cross-ratios.

$$\begin{aligned}1-\frac{1}{\chi} &= \frac{4\sqrt{1-k^2}}{2\sqrt{1-k^2}+2-k^2}\\ \frac{1}{1-\chi} &= \frac{2\sqrt{1-k^2}+k^2-2}{4\sqrt{-k^2+1}}\\ \frac{\chi}{\chi-1} &= -\frac{1}{4}\frac{k^4}{k^2\sqrt{1-k^2}-2\sqrt{1-k^2}-2k^2+2}.\end{aligned}$$

So there are six possible cross-ratios. The equivalence class under these permutations is denoted $[\chi]$.

5.3 The Range of χ

Now the range of χ is examined. Note that if $0 < k < 1$, then χ is real. Choose $k = \cos\theta$ with $0 < \theta < \frac{\pi}{2}$, so that k is naturally greater than zero and less than 1. Then the cross-ratio turns into a function of $\cos\theta$, which is simplified in terms of $\sin\theta$ below.

$$\chi = \frac{\left(\sqrt{1-\cos^2\theta}+1\right)^2}{\left(\sqrt{1-\cos^2\theta}-1\right)^2} = \frac{(\sin\theta+1)^2}{(\sin\theta-1)^2}.$$

Now find all the possible ranges for each of the six values of the cross-ratio.

The function $f(\theta) = \frac{(\sin\theta+1)^2}{(\sin\theta-1)^2}$ has critical points only when $\cos\theta = 0$, so if $0 < \theta < \frac{\pi}{2}$ the range of χ is $(1,\infty)$. Consequently, the range of $\frac{1}{\chi}$ or of $1-\frac{1}{\chi}$ is $(0,1)$, the range of $1-\chi$ or of $\frac{1}{1-\chi}$ is $(-\infty, 0)$, and the range of $\frac{\chi}{1-\chi}$ is also $(1,\infty)$. From the ranges that have been obtained, it is concluded that the range of the cross-ratio of F for $0 < k < 1$ is $\mathbb{R}\backslash\{0,1\}$, a real number other than 0 or 1, because the four tangents are distinct for any non-singular cubic.

Similarly, consider the case when $k > 1$. A function that is naturally greater than 1 is the secant function, so let $k = \sec\theta$ with $0 < \theta < \frac{\pi}{2}$. Since

$$1 - \sec^2\theta = -\tan^2\theta,$$

then

$$\sqrt{1-\sec^2\theta} = i\tan\theta.$$

By computing $\chi(k)$, the range of cross-ratio can be determined. Thus, after a considerable amount of simplification, χ is obtain obtained to be

$$\begin{aligned}\chi &= \left(8\cos^4\theta - 8\cos^2\theta + 1\right) + i\left(8\cos^3\theta\sin\theta - 4\cos\theta\sin\theta\right)\\ &= \cos 4\theta + i\sin 4\theta.\end{aligned}$$

Thus

$$\chi = e^{i4\theta}.$$

Therefore, when $k > 1$, the cross-ratio of F, for the chosen order of the tangents, is always on the unit circle and every complex number on the unit circle (other than 1) occurs for some k since $0 < 4\theta < 2\pi$. In Chapter 6, the ranges for the other possible values of the cross-ratio are found and the properties of the J-invariant is used to obtain our main result.

Chapter 6

J-Invariant

6.1 The J-Invariant for Cassini Cross-Ratios

Finally it is time to obtain our main result that the non-lemniscate Cassini curves are a complete set of representatives for the birational equivalence classes of elliptic curves with real J-invariant. This result will follow from the following theorem.

Salmon's Theorem: Two non-singular cubics have the same cross ratio if and only if they are birationally equivalent [W1962].

Recall that a non-singular affine cubic can always be put in the form $y^2 = x^3 + ax + b$ (as noted in Chapter 2), provided $4a^3 + 27b^2 \neq 0$. In this form, the J-invariant can be expressed in terms of coefficients as

$$J(a,b) = \frac{4a^3}{4a^3 + 27b^2}.$$

For the purposes of this project, a form of the J-invariant in terms of a single parameter, the cross-ratio χ, is used. The cross-ratio χ for each elliptic Cassini curve has been computed after transforming these curves into non-singular cubics. However, depending on the chosen order of tangents, χ can be up to six distinct values. It is useful to have a function that is constant on the equivalence class $[\chi]$. Therefore, we use the form

$$J(\chi) = \frac{4}{27}\frac{(1-\chi+\chi^2)^3}{\chi^2(1-\chi)^2}$$

which derives from the theory of elliptic integrals [MM99]. The reader can easily verify that $J(\chi) = J\left(\frac{1}{\chi}\right) = J\left(1-\chi\right) = J\left(1-\frac{1}{\chi}\right) = J\left(\frac{1}{1-\chi}\right) = J\left(\frac{\chi}{\chi-1}\right)$.

Before taking advantage of the property of J-invariant, that $J(\chi)$ remains the same as long as χ is any permutation of cross-ratio, consider all six possible cross-ratios of a Cassini oval: χ, $\frac{1}{\chi}$, $1-\chi$, $1-\frac{1}{\chi}$, $\frac{1}{1-\chi}$ and $\frac{\chi}{\chi-1}$. Suppose $\chi \in \mathbb{C}$ and $Z = x+iy$ where $x, y \in \mathbb{R}$. If $\chi = e^{4i\theta}$ with $0 < \theta < \frac{\pi}{2}$, χ belongs to $|Z| = 1$, the unit circle on a complex plane. $\frac{1}{\chi}$ also can be expressed as $|Z| = 1$ because $\frac{1}{Z} = \frac{1}{x+iy} = x - iy$ on the unit circle. But $1-\chi$ as well as $1-\frac{1}{\chi}$ shift $|Z| = 1$ one unit to the right along the real axis which turns out to be $|Z-1| = 1$. Also $\frac{1}{1-\chi} = \frac{1}{2} - \frac{\sin 4\theta}{2\cos 4\theta - 2} i$ can be described as $x = \frac{1}{2}$, and, similarly, $\frac{\chi}{\chi-1}$ overlaps on $x = \frac{1}{2}$. In addition to the real numbers, the complex numbers that are the cross-ratio of Cassini curves are

$$\begin{aligned} |Z| &= 1 \\ |Z-1| &= 1 \\ \mathrm{Re}(Z) &= \frac{1}{2} \end{aligned}$$

excluding 0 and 1. See Figure 6.1. This information will be used later to show that $J(\chi)$ is a real number if and only if χ belongs this figure.

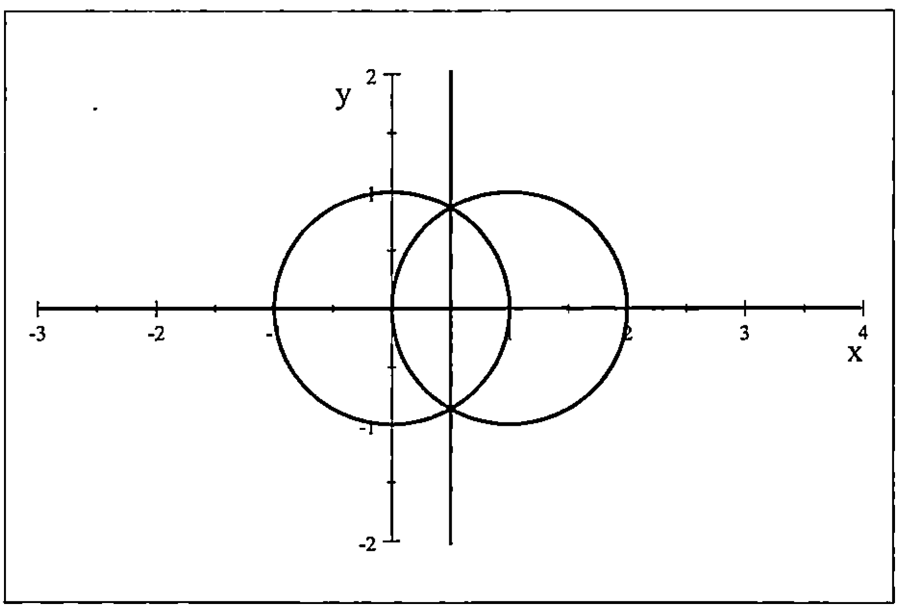

Figure 6.1: The Range of All Possible Cross-Ratios

Now it is necessary to know the range of J-invariant. As mentioned in Chapter 5, two cases need to be considered: when $\chi \in \mathbb{R}$ and when χ is on the unit circle.

6.2 The Case When χ is a Real Number

First of all, consider the case when $\chi \in \mathbb{R}$. Then the range of $J(\chi)$ is found by the following steps. Suppose $\chi \in \mathbb{R}$, then

$$J(\chi) = \frac{4}{27}\frac{(\chi^2 - \chi + 1)^3}{\chi^2(\chi - 1)^2}$$

is a real function. By using the basic calculus, the extrema are to be at $(-1, 1)$, $(\frac{1}{2}, 1)$ and $(2, 1)$, and the range of $J(\chi)$ is $[1, \infty)$.

Interestingly, if $J(-1) = J(\frac{1}{2}) = J(2) = 1$, the curve with $\chi = [-1]$ is called *harmonic* according to Walker [W1962]. See the Figure 6.2, the graph of J for $\chi = r \in \mathbb{R}$.

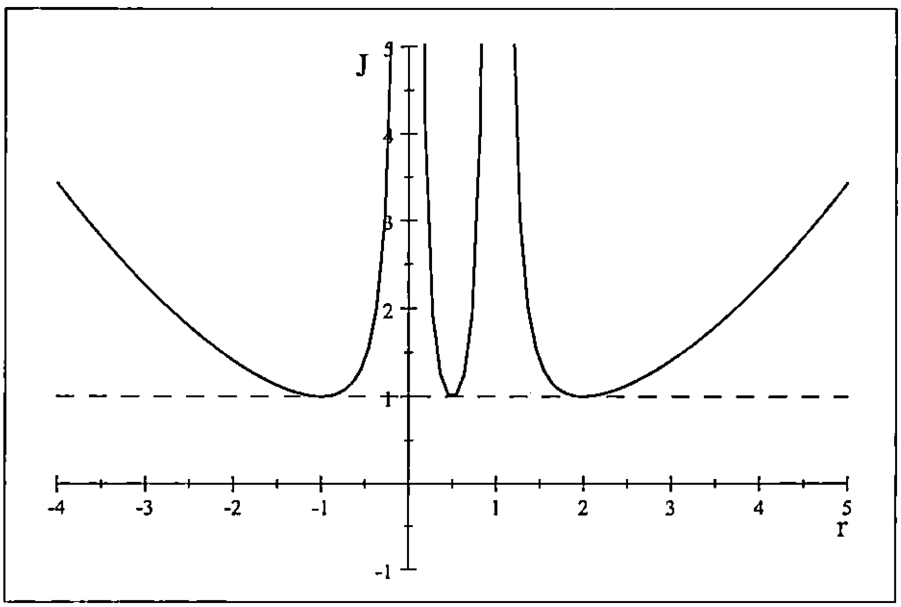

Figure 6.2: The Range of J-Invariant When $\chi \in \mathbb{R}$

Notice that the full range $[1, \infty)$ of $J(\chi)$ is covered by Cassini curves with $0 < k < 1$ because $\chi = \frac{(\sin\theta+1)^2}{(\sin\theta-1)^2}$ where $0 < \theta < \frac{\pi}{2}$ with $k = \cos\theta$. In particular, for χ in $(0, 1)$, as we can see from the figure, the full range is attained by Cassini curves.

Now, consider $\chi(k) = \frac{(\sqrt{1-k^2}+1)^2}{(\sqrt{1-k^2}-1)^2}$ where $0 < k < 1$ from Chapter 5. The range of $J(k)$ is examined in order to find the particular k value that causes harmonic Cassini curves. So $J(k)$ has a form

$$J(k) = \frac{1}{108}\frac{(k^4 - 16k^2 + 16)^3}{k^8(1 - k^2)},$$

and the extremum of $J(k)$ is at $(2\sqrt{3\sqrt{2}-4}, 1)$ and $J(k) \in [1, \infty)$.

So when $k = 2\sqrt{3\sqrt{2}-4} \approx 0.985\,17$, the Cassini curve will be harmonic:

$$\left((x-1)^2 + y^2\right)\left((x+1)^2 + y^2\right) = 12\sqrt{2} - 16$$

or

$$x^4 + y^4 + 2x^2y^2 - 2x^2 + 2y^2 - 12\sqrt{2} + 17 = 0$$

and the affine curve looks like the Figure 6.3.

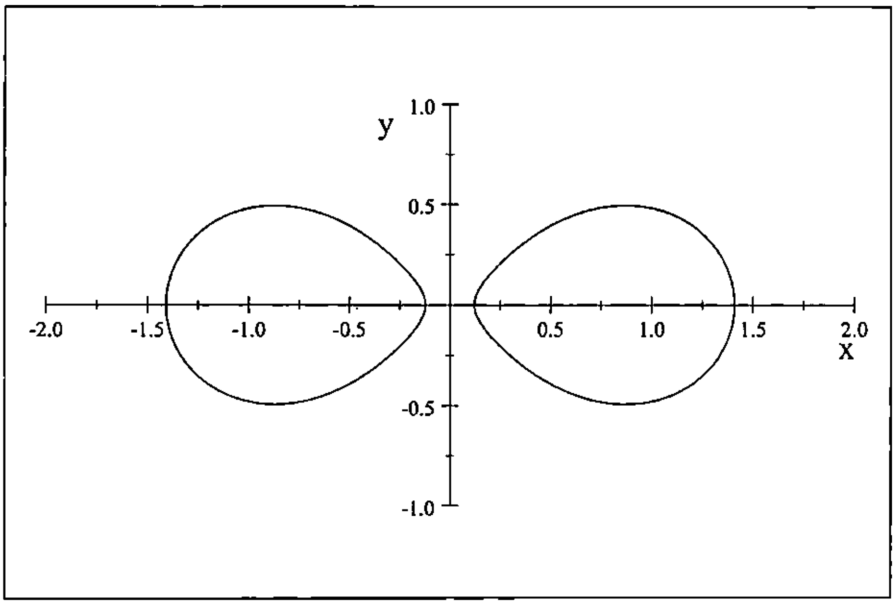

Figure 6.3: The Harmonic Cassini Oval

6.3 The Case When χ is Not Real

Next, suppose $k > 1$. In the previous chapter, it was figured out that when $k > 1$, it can be written as $\chi = e^{4i\theta}$, where $k = \sec\theta$ with $0 < \theta < \frac{\pi}{2}$, which is a complex number on the unit circle.

Suppose $\chi = e^{it}$ where $0 < t < 2\pi$. Then the J-invariant has a form

$$J(t) = \frac{4}{27}\frac{\left(e^{2(it)} - e^{it} + 1\right)^3}{e^{2(it)}\left(e^{it} - 1\right)^2}.$$

$J(t)$ can be simplified, and $J'(t)$ and $J''(t)$ are found below.

$$\begin{aligned}
\frac{27}{4}(\cos t-1)\,J(t) &= (2\cos t-1)^3\\
\frac{27}{4}(\cos t-1)^2\,J'(t) &= (2\cos t-1)^2\,(5-4\cos t)(\sin t)\\
\frac{27}{4}(\cos t-1)^2\,J''(t) &= (20\cos^3 t-18\cos^2 t-4\cos t+7)(\cos t+2)(1-2\cos t)
\end{aligned}$$

In particular, it has been shown that $J(\chi)\in\mathbb{R}$ for any elliptic Cassini curve. Now the critical points of $J(t)$ are found from $J'(t)=0$. So $t=\frac{1}{3}\pi$, $t=\pi$ and $t=\frac{5}{3}\pi$, and the critical points are $(\pi,1)$, $(\frac{1}{3}\pi,0)$ and $(\frac{5}{3}\pi,0)$. Hence, $J(t)$ has two distinct cross-ratios in the equivalent class, and $(\frac{1}{3}\pi,0)$ and $(\frac{5}{3}\pi,0)$ are both critical and inflection points because $J'(\frac{1}{3}\pi)=J'(\frac{5}{3}\pi)=0$ and $J''(\frac{1}{3}\pi)=J''(\frac{5}{3}\pi)=0$. Note also that $e^{\pm i\frac{\pi}{3}}$ are the roots of $\chi^2-\chi+1$, so these are precisely the zeroes of J.

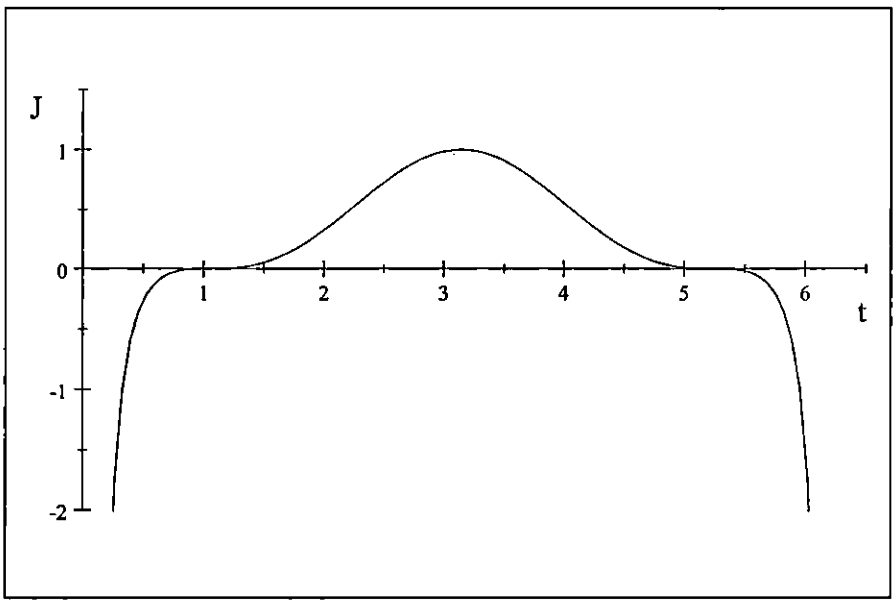

Figure 6.4: The Range of J-Invariant When $\chi\notin\mathbb{R}$

It is clear from the Figure 6.4 and the calculation of critical points that the range of $J(\chi)$ is $(-\infty,1]$ if $k>1$.

Just like the previous section, 6.2, the k value ($k>1$) that gives a harmonic Cassini curve, where $J(\chi)=1$ can be found. Then

$$J(k)=\frac{1}{108}\frac{\left(k^4-16k^2+16\right)^3}{k^8\,(1-k^2)}=1,$$

so $k=\sqrt{2}\approx 1.4142$.

By definition, k that gives an *equianharmonic* Cassini curve is at $J(\chi)=0$ [W1962], so

$$J(k)=\frac{1}{108}\frac{\left(k^4-16k^2+16\right)^3}{k^8\left(1-k^2\right)}=0.$$

Since $k^8\left(1-k^2\right)\neq 0$, it has to be $k^4-16k^2+16=0$. Thus the qualified k are

$$\begin{aligned} k &= \sqrt{2}\left(\sqrt{3}-1\right)\approx 1.0353 \\ k &= \sqrt{2}\left(\sqrt{3}+1\right)\approx 3.8637. \end{aligned}$$

Thus the harmonic Cassini curve is

$$x^4+y^4+2x^2y^2-2x^2+2y^2-1=0$$

which is shown as a thick-lined loop in the Figure 6.5. By Salmon's Theorem, this curve is birationally equivalent to the curve in the Figure 6.2 in the section 6.2.

Also the equianharmonic Cassini curves are

$$\begin{aligned} x^4+y^4+2x^2y^2-2x^2+2y^2+4\sqrt{3}-7 &= 0 \\ x^4+y^4+2x^2y^2-2x^2+2y^2-4\sqrt{3}-7 &= 0 \end{aligned}$$

which are shown as thin-lined loops in the Figure 6.5. The equianharmonic curves describe these with $[\chi]$ consisting of precisely two values, the primitive cube roots of -1.

6.4 The Real J-Invariant

In this section, the domain of $\chi\in\mathbb{C}$ when the J-invariant $J(\chi)$ is real is investigated. So let $\chi\in\mathbb{C}$ and $\chi=x+iy$ with $x,y\in\mathbb{R}$. Then the imaginal part of $J(\chi)$ turns out to be

$$\operatorname{Im}J(\chi)=\frac{4}{27}y\left(2x-1\right)\left(x^2+y^2-1\right)\left(x^2-2x+y^2\right)\frac{\left(x^2-x+1\right)^2+y^2\left(2x^2-2x+y^2+3\right)}{\left(-2x+x^2+y^2+1\right)^2\left(x^2+y^2\right)^2}.$$

For $J(\chi)\in\mathbb{R}$, it must have $\operatorname{Im}J(\chi)=0$. Luckily, there are four convenient factors that can be zero in $\operatorname{Im}J(\chi)=0$ above:

$$\begin{aligned} &y \\ &2x-1 \\ &x^2+y^2-1 \\ &x^2-2x+y^2. \end{aligned}$$

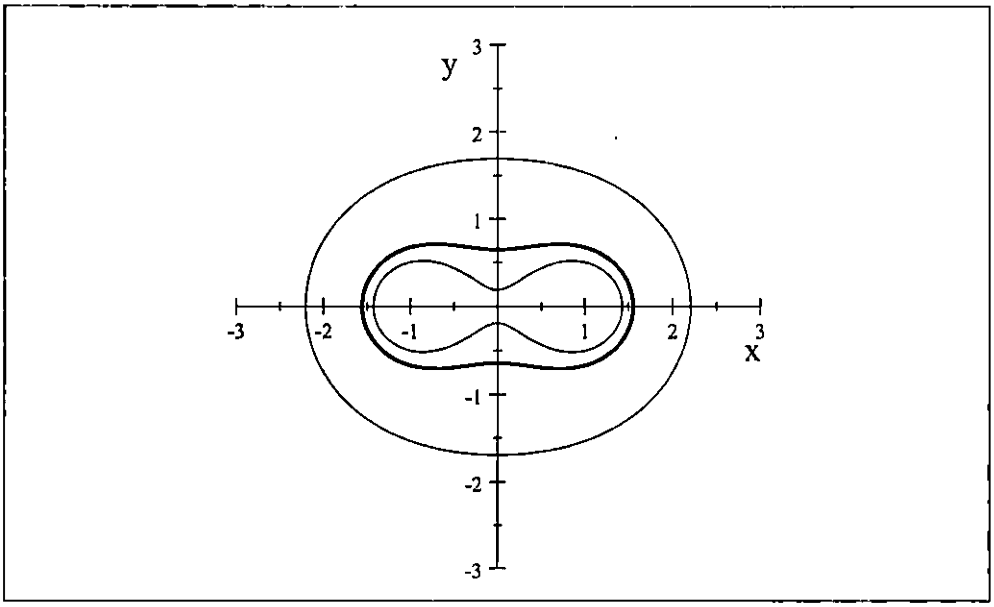

Figure 6.5: The Harmonic and Equianharmonic Cassini Ovals

But first consider the last factor $\left(x^2-x+1\right)^2+y^2\left(y^2+2x^2-2x+3\right)$.

Let $u=y^2$, which cannot be negative because the y is squared. So if $u^2+(2x^2-2x+3)u+\left(x^2-x+1\right)^2=0$, then $u^2+Bu+C=0$, where $B=2x^2-2x+3$ and $C=\left(x^2-x+1\right)^2$. However, B and C are positive for all values of x, so $u<0$ for any solution u. Thus the factor is not sufficient to have it to be zero. Also it is obvious that the factors in the denominator cannot be zero. Therefore, the domain of χ for $J(\chi)\in\mathbb{R}$ consists of the four sets of complex numbers $x+iy$:

$$
\begin{aligned}
&y=0\\
&2x-1=0\\
&x^2+y^2-1=0\\
&x^2-2x+y^2=0,
\end{aligned}
$$

and the figure of these equations is shown in the Figure 6.6.

As was shown earlier, the domain of χ for $J(\chi)\in\mathbb{R}$ is the same as the ranges of cross-ratios of Cassini ovals.

Lastly, from the two cases above, It is clear that $J(\chi)\in\mathbb{R}$, obviously when $\chi\in\mathbb{R}\setminus\{0,1\}$ and particularly when $\chi\in|Z|=1$, $\chi\in|Z-1|=1$, or $\chi\in\frac{1}{2}$ where $Z=x+iy$

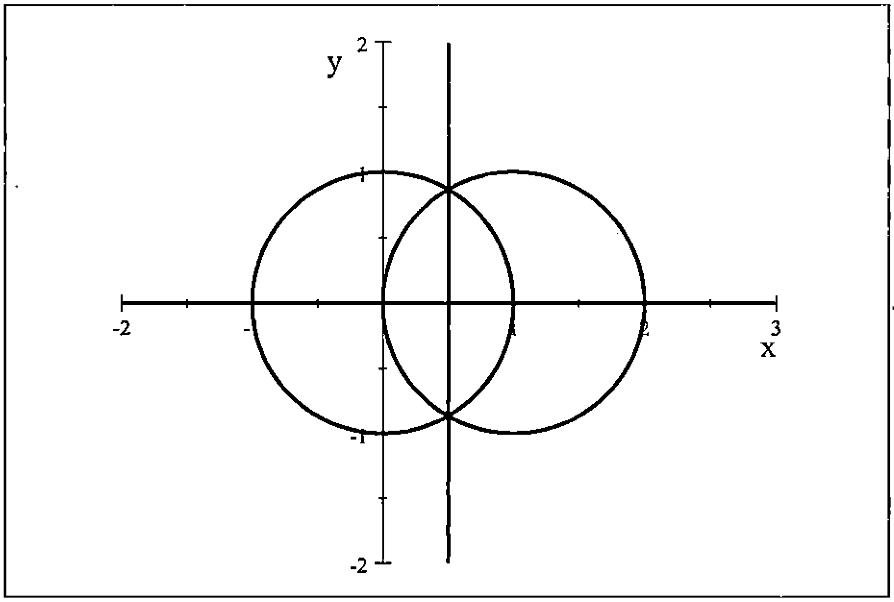

Figure 6.6: The Domain of χ for $J(\chi) \in \mathbb{R}$

with $x, y \in \mathbb{R}$. Also, by assuming $J(\chi) \in \mathbb{R}$, the range of χ has found out to be the cross-ratios of Cassini ovals.

6.5 Conclusion

In this project, the focus was on the Cassini Ovals, which have been described in affine form as $f(x,y) = 0$, where

$$f(x,y) = ((x-a)^2 + y^2)((x+a)^2 + y^2) - b^2.$$

Three main results have been shown:

1) Every Cassini curve that is not a lemniscate is an elliptic curve.

2) Birational transformation of an elliptic Cassini curve produces a non-singular cubic with $[\chi]$ represented by either a real number or a number on the unit circle.

3) The complex numbers χ for which $J(\chi) \in \mathbb{R}$ are precisely those that occur as cross-ratios of Cassini elliptic curves [S2012].

Setting $a = 1$, the three cases of Cassini curves were studied based on whether $0 < b < 1$, $b = 1$, or $b > 1$. In Chapter 3, the multiplicities of singular points for all three cases were found. From those results, the genus of the three cases were computed, and the Cassini Ovals were determined to be elliptic curves except the lemniscate (the case of $b = 1$). Then, in Chapter 4, by using the M. Noether Theorem, the linear fractional transformation $T(Z)$ was discovered to transform the quartic non-lemniscate Cassini curve to a cubic curve. This process was beneficial to compute the cross-ratios of Cassini ovals because it is hard to compute cross-ratios of a degree-four polynomial. In order to compute the cross-ratios of non-lemniscate Cassini curves, the tangents that pass through a flex point, O, were figured out algebraically in position vectors. From the information obtained, the cross-ratio was computed as $\chi(b) = \frac{(\sqrt{1-b^2}+1)^2}{(\sqrt{1-b^2}-1)^2}$, and its range was discovered to be $\mathbb{R}\backslash\{1,0\}$ when $0 < b < 1$ and on the unit circle when $b > 1$ in Chapter 5. Lastly, by using the property of J-invariant, $J(\chi)$ was shown to be real for $[\chi]$ within the range of $\chi(b)$. Also, when $J(\chi)$ is real, the domain of χ was shown to be the same as the range of $\chi(b)$. The non-lemniscate Cassini ovals are non-singular cubics with $J(\chi) \in \mathbb{R}$ for $[\chi]$; therefore, using Salmon's Theorem, it is concluded that the theorem below is true.

Theorem: The non-lemniscate Cassini ovals are a complete set of representatives for the birational equivalence classes of elliptic curves with real J-invariant.

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