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Orthogonal polynomials

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Orthogonal Polynomials

A Thesis

Presented to the

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In Partial Fulfillment

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in

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Abstract

This thesis will show work on Orthogonal Polynomials. In mathematics, the type of polynomials that are orthogonal to each other under inner product are called orthogonal polynomials.

Orthogonal polynomials were developed in the late 19th century by P. L. Chebyshev. Furthermore, Gábor Szego, Sergei Bernstein, Naum Akhiezer, Arthur Erdélyi, Yakov Geronimus, Wolfgang Hahn, Theodore Seio Chihara, Mourad Ismail, Waleed Al-Salam, and Richard Askey were mathematicians who at one point in history have extensively studied and worked on orthogonal polynomials.

Jacobi polynomials, Laguerre polynomials, and Hermite polynomials are examples of classical orthogonal polynomials that have been invented in the nineteenth century. The special cases of classical polynomials are Chebyshev polynomial, Legendre polynomials and Gegenbauer polynomials. These polynomials are different because they have different weight functions, w(x). Sometimes they will be classified based on their dimensions such as Gegenbauer polynomials. We will concentrate on Legendre polynomials. The applications are not limited only to mathematics and physics. Scientists found applications for orthogonal polynomials in other subjects, including biology, chemistry, and computer science. The theory of rational approximations is one of the most important applications of orthogonal polynomials.

As another application of Legendre polynomials, we will consider some special values of the Riemann zeta function. The Riemann zeta function is a function of a complex variable s. It is expressed as a continuous sum of the infinite series which converges when s is greater than one. The Riemann zeta function plays a very important role in analytic number theory, in physics, probability theory, and applied statistics.

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I would like to give the most thanks to my loving wife for her patience, understanding, support and commitment in helping me reach my academic and personal goals.

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Chapter 1

Introduction

In mathematics, the type of polynomials that are orthogonal to each other under an inner product are called orthogonal polynomials. Let P_n, and P_m be two polynomials such that $degP_n = n$, and $degP_m = m$, then $\langle P_n, P_m \rangle = 0$, $m \neq n$, and $\langle P_n, P_m \rangle \neq 0$, for $m = n$.

P.L. Chebyshev founded the field of orthogonal polynomials at the end of the 19th century. The study of orthogonal polynomials, after Chebyshev, was continued by A.A. Markov and T.J. Stieltjes. Furthermore, Gábor Szegő, Sergei Bernstein, Naum Akhiezer, Arthur Erdélyi, Yakov Geronimus, Wolfgang Hahn, Theodore Seio Chihara, Mourad Ismail, Waleed Al-Salam, and Richard Askey were mathematicians who at one point in history have extensively studied and worked on orthogonal polynomials [Jac04].

The history of orthogonal polynomials and special functions dates back to the nineteenth century century. Some mathematicians and physicists used orthogonal polynomials and special functions to solve equations in mathematical physics. The applications of orthogonal polynomials are both in mathematics and physics(combinatorics, harmonic analysis, statistics, number theory). The applications are not limited only to mathematics and physics. Scientists have also utilized these studies in the fields of biology, chemistry and computer science. The theory of rational approximations is one of the most important applications of orthogonal polynomials.

Jacobi polynomials, Laguerre polynomials, and Hermite polynomials are examples of classical orthogonal polynomials. The special cases of classical polynomials are Chebyshev polynomial, Legendre polynomials and Gegenbauer polynomials [Jac04].

These polynomials are different because they have different weight functions, $w(x)$. Sometimes they will be classified based on their dimensions such as Gegenbauer polynomials. We will concentrate on Legendre polynomials.

Chapter 2

Legendre Polynomials

An expression in the form of $a_n x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_0$, where a_k are real numbers, and k is a positive integer or zero is called a polynomial. A set of polynomials where the product of any two different ones multiplied by a function $w(x)$ called a weight function and integrated over a certain interval will be equal to zero is called a set of orthogonal polynomials. Legendre polynomials allow us to break down a function $f(x)$ into a sum of different polynomials. Fourier series is similar with the exception that they do not represent orthogonal polynomials. They include $\sin s$ and $\cos s$ but not polynomials.

2.1 Fourier Series

Fourier series breaks down periodic functions or periodic signals into the sum of oscillating functions, such as sines and cosines. The study of Fourier series is a branch of Fourier analysis.

"The Fourier series is named in honour of Joseph Fourier (1768 – 1830), who made important contributions to the study of trigonometric series, after investigations by Leonhard Euler, Jean le Rond d'Alembert, and Daniel Bernoulli" [Tol76]. Joseph Fourier used the Fourier series to solve the heat equation in a metal plate. The idea of decomposing periodic functions into oscillating functions was first used by ancient astronomers in the third century BC to investigate planetary motions.

The subject of Fourier series was invented in the nineteenth century, a time when a precise notion of function and integral was not yet developed. As a result, Fourier's findings are now viewed as informal. With Fourier as their predecessor, Riemann and Dirichlet developed more precise applications of Fourier series.

Even though the intent of the Fourier series was originally to solve the heat equation, it was later used to solve a wide range of mathematical and physical problems, such as those involving linear differential equations with constant coefficients. The Fourier series has many applications in electrical engineering, vibration analysis, acoustics, optics, signal processing, image processing, quantum mechanics, econometrics, and thin-walled shell theory.

2.2 Expansion of Trigonometric Functions

Special attention will be given to expansion of trigonometric functions, cosine and sine. In most cases we will be using complex exponential functions, $e^{i\theta}$ instead of trigonometric functions $\sin x$, and $\cos x$. These trigonometric functions are related by the following formula

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \text{ and } \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2}.$$

To express $e^{i\theta}$ by $\sin\theta$, and $\cos\theta$ we will have

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The advantages of cosine and sine are that they are real-valued functions, and they are even and odd. The advantages of the exponential function $e^{i\theta}$ are that its differentiation formula $(e^{i\theta})' = ie^{i\theta}$ and the addition formula $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$ are much simpler than the corresponding formulas for cosine and sine.

Let the function $f(x)$ be integrable in the interval $[-\pi,\pi]$ and for Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin x), \tag{2.1}$$

where,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \quad (n = 1, 2, 3, ...), \tag{2.2}$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \quad (n = 1, 2, 3, ...). \tag{2.3}$$

We will look at some simple orthogonal systems involving trigonometric functions. Let the first system consist of

$$1, \cos x, \cos 2x, \ldots \cos nx, \ldots \quad .$$

This system is orthogonal in the interval $[0, \pi]$. First, let's investigate the orthogonality of the cosine function and number 1.

$$\int_0^\pi \cos nx dx = \left[\frac{\sin x}{n}\right]_0^\pi = 0 \quad (n = 1, 2, \ldots).$$

This means that the functions $\cos nx$ and 1 are orthogonal. Next, we will look at the combination of two cosine functions in the following integral

$$\begin{aligned}\int_0^\pi \cos nx \cos mx dx &= \frac{1}{2}\int_0^\pi [\cos(n+m)x + \cos(n-m)x]\, dx \\ &= \frac{1}{2}\int_0^\pi \cos(n+m)x + \int_0^\pi \cos(n-m)x \\ &= 0 \quad (n \neq m).\end{aligned}$$

This proves that the system $1, \cos x, \cos 2x, \ldots \cos nx, \ldots$ is orthogonal. The second system is the following set of functions

$$\sin x, \sin 2x, \ldots \sin nx, \ldots$$

We will show that this system is orthogonal also. Lets consider two sine functions first, then

$$\begin{aligned}\int_0^\pi \sin nx \sin mx dx &= \frac{1}{2}\int_0^\pi [\cos(n-m)x - \cos(n+m)x]\, dx \\ &= 0 \quad (n \neq m).\end{aligned}$$

Finally the third system will consist of the following trigonometric functions $\sin x, \sin 3x, \sin 5x \ldots \sin(2n+1)x, \ldots$ We need to show that it is orthogonal on $[0, \pi/2]$.

For $(n \neq m)$ and $n, m = 1, 2, 3, ...$we will have

$$
\begin{aligned}
& \int_0^{\pi/2} \sin(2n+1)x \sin(2m+1)x dx \\
= & \ \frac{1}{2}\int_0^{\pi/2} \left[\cos 2(n-m)x - \cos(n+m+1)x\right] dx \\
= & \ \frac{1}{2}\left[\frac{\sin 2(n-m)x}{2(n-m)}\right]_0^{\pi/2} - \frac{1}{2}\left[\frac{\sin 2(n+m+1)x}{2(n+m+1)}\right]_0^{\pi/2} \\
= & \ 0.
\end{aligned}
$$

2.3 Orthogonal Polynomials

Orthogonal polynomials can be used in specifying basic states in quantum mechanics. The orthogonality of polynomials is determined by their inner product formula, $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)w(x)dx = 0$. Here $w(x)$ is a nonnegative weight function as illustrated in the definition of the inner product.

The Legendre polynomials $P_n(x), n = 0, 1, 2...$are orthogonal in the interval from -1 to 1 with weight function $w(x) \equiv 1$. Here n represents the degree of the polynomial $P_n(x)$. These polynomials are expressed by the following integral:

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \delta_{mn}\frac{2}{2n+1}. \tag{2.4}$$

The Kronecker delta is zero if $n \neq m$, and unity if $n = m$. In most cases with applications, $x = \cos\theta$, where θ is from 0 to π. The derivative is $dx = \sin\theta d\theta$. The Legendre polynomials are a special case of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ that are orthogonal on $(-1, 1)$. By changing the variable we can change the interval from $(-1, 1)$ to (a, b).

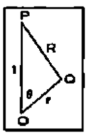

Figure 2.1: Charge q at Q.

In the figure 2.1 let q be an electric charge at point Q. We want the potential at some point P. The distance PO is considered as unity for convenience. The potential to this charge is q/R. We can determine R as a function of r and also θ can be found using Law of Cosines: $R^2 = 1+r^2-2r\cos\theta = 1-2rx+r^2$,$(x = \cos\theta)$. Next we can expand $1/R$ in powers of r as $1/R = \sum P_n(x)\, r^n$. $1/R$ is called the generating function of the Legendre polynomials. Generating functions are possible for most orthogonal polynomials.

If we make $x = 1$, it is easy to see that $P_n(1) = 1$, and $P_n(-1) = (-1)^n$. By taking partial derivative of $1/R$ with respect to x and r, and then considering the coefficients of individual powers of r, we can find the number relations between the polynomials and their derivatives. We can use this to find the recursion relation:

$$(n+1)\,P_{(n+1)}(x) = (2n+1)\,xP_n(x) - nP_{(n-1)}(x)\,. \tag{2.5}$$

It also can be used to find the differential equation satisfied by the polynomials:

$$\left(1-x^2\right)P_n''(x) - 2xP_n'(x) + n\,(n+1)\,P_n(x) = 0. \tag{2.6}$$

The in-detail proofs of (2.5) and (2.6) are as follows. Let the generating function of Legendre polynomials be

$$g(r,x) = \frac{1}{\sqrt{1-2rx+r^2}} = \sum_{n=0}^{\infty} P_n(x)r^n. \tag{2.7}$$

We need to show that $P_n(x)$ satisfies equation (2.5). We will start by generating a recurrence relation between Legendre polynomials of different order. First we are going to differentiate the equation (2.7), $g(r,x) = \frac{1}{\sqrt{1-2rx+r^2}} = \sum\limits_{n=0}^{\infty} P_n(x)r^n$ with respect to r.

$$\frac{dg(r,x)}{dr} = \frac{x-r}{(1-2xr+r^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n(x)r^{n-1}. \tag{2.8}$$

To simplify, we will multiply two sides of this equation by $1-2xr+r^2$ and making the equation equal zero we will have

$$(1-2xr+r^2)\sum_{n=0}^{\infty} nP_n(x)r^{n-1} + \frac{r-x}{\sqrt{1-2xr+r^2}} = 0, \tag{2.9}$$

which, using the generation function, equation (2.7) becomes

$$(1-2xr+r^2)\sum_{n=0}^{\infty} nP_n(x)r^{n-1} + (r-x)\sum_{n=0}^{\infty} P_n(x)r^n = 0. \tag{2.10}$$

Therefore, by distributing both summations and simplifying we will get

$$\begin{aligned}
&\sum_{n=0}^{\infty} nP_n(x)r^{n-1} - 2xr\sum_{n=0}^{\infty} nP_n(x)r^{n-1} + r^2\sum_{n=0}^{\infty} nP_n(x)r^{n-1} \\
&+r\sum_{n=0}^{\infty} nP_n(x)r^{n} - x\sum_{n=0}^{\infty} P_n(x)r^{n} \\
=\ &\sum_{n=0}^{\infty} nP_n(x)r^{n-1} - 2x\sum_{n=0}^{\infty} nP_n(x)r^{n} + \sum_{n=0}^{\infty} nP_n(x)r^{n+1} \\
&+\sum_{n=0}^{\infty} nP_n(x)r^{n+1} - \sum_{n=0}^{\infty} xP_n(x)r^{n} \\
=\ &\sum_{m} mP_m(x)r^{m-1} - 2x\sum_{n} nP_n(x)r^{n} + \sum_{s} sP_s(x)r^{s+1} \\
&+\sum_{s} P_n(x)r^{s+1} - \sum_{n=0}^{\infty} xP_n(x)r^{n} \\
=\ &0.
\end{aligned}$$

Let $m = n+1$, and $s = n-1$, then this equation will become

$$\sum_{n=0}^{\infty} \left[(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x)\right] r^n = 0. \tag{2.11}$$

Since $r^n \neq 0$, for $n = 1, 2, 3...$ we will have the following

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0. \tag{2.12}$$

This is the recurrence relation between Legendre polynomials of different order. Using this equation we can generate Legendre polynomials of higher order. In finding the recurrence relation we differentiated equation (2.7) with respect to r. Now we will differentiate equation (2.7) with respect to x.

$$\frac{dg(r,x)}{dx} = \frac{r}{(1-2xr+r^2)^{3/2}} = \sum_{n=0}^{\infty} nP_n'(x)r^n.$$

We can write this as

$$(1-2xr+r^2)\sum_{n=0}^{\infty} P_n'(x)r^n - \frac{r}{\sqrt{1-2xr+r^2}} = 0,$$

and by using the generating function we will have

$$(1-2xr+r^2)\sum_{n=0}^{\infty} P_n'(x)r^n - r\sum_{n=0}^{\infty} P_n(x)r^n = 0, \text{ for all } r.$$

Then,

$$\sum_{n=0}^{\infty} P_n'(x)r^n - 2xr\sum_{n=0}^{\infty} P_n'(x)r^n + r^2\sum_{n=0}^{\infty} P_n'(x)r^n - r\sum_{n=0}^{\infty} P_n(x)r^n = 0,$$

and

$$\sum_{m} P_m'(x)r^m - 2xr\sum_{n} P_s'(x)r^s + r^2\sum_{s} P_n'(x)r^n - r\sum_{n=0}^{\infty} P_n(x)r^n = 0.$$

Like in the previous problem we will set $m = n+1$ and $s = n-1$, and this equation will become

$$\sum_{n+1} P_{n+1}'(x)r^{n+1} - 2xr\sum_{n-1} P_{n-1}'(x)r^{n-1} + r^2\sum_{s}^{\infty} P_n'(x)r^n - r\sum_{n=0}^{\infty} P_n(x)r^n = 0,$$

or,

$$\sum_{n=0}^{\infty}\left[P_{n+1}'(x) + P_{n-1}'(x) - 2xP_n'(x) + P_n(x)\right]r^n = 0,$$

and by setting to zero each power of r we will have

$$P_{n+1}'(x) + P_{n-1}'(x) - 2xP_n'(x) + P_n(x) = 0. \tag{2.13}$$

If we differentiate equation (2.13) we will have

$$(n+1)P_{n+1}'(x) + nP_{n-1}'(x) = (2n+1)\left[P_n(x) + xP_n'(x)\right]. \tag{2.14}$$

We will use equations (2.14) and (2.15) to eliminate $P_n'(x)$

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x). \tag{2.15}$$

Next we will subtract (2.16) from (2.14)

$$P_{n+1}'(x) + P_{n-1}'(x) - 2xP_n'(x) + P_n(x) - P_{n+1}'(x) - P_{n-1}'(x) - (2n+1)P_n(x) = 0.$$

By doing this we will eliminate $P_{n+1}'(x)$ and get

$$P_{n-1}'(x) = -nP_n(x) + xP_n'(x). \tag{2.16}$$

In the next step instead of subtracting, we will add (2.16) and (2.14). The purpose of doing this is to eliminate $P_{n-1}'(x)$.

$$P_{n+1}'(x) = (n+1)P_n(x) + xP_n'(x). \tag{2.17}$$

In the next step, by replacing n with $n-1$ and adding it to x times the equation (18) we will get

$$P_n'(x) - nP_{n-1}(x) + xP_{n-1}'(x) + x(P_{n-1}'(x) + nP_n(x) + xP_n'(x)) = 0.$$

By combining like terms and factoring $(1-x^2)$ we will have

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x). \tag{2.18}$$

Finally, if we differentiate (2.19) we will have

$$(1-x^2)P_n''(x) - 2xP_n'(x) + nP_n(x) + n\left[xP_n'(x) - (P_{n-1}'(x)\right] = 0.$$

Notice that, based on (2.17),

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x),$$

therefore, the final result is

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0. \tag{2.19}$$

The recurrence relation allows us to find all the polynomials, since it is easy to find $P_0(x) = 1$, $P_1(x) = x$, directly from the generating function. The differential equations allows us to apply the polynomials to problems in mathematics and physics, among which is the important problem of the solution of Laplace's equation and spherical harmonics.

The recurrence relation shows that the coefficient A_n of the highest power of x satisfies the relation $A_{(n+1)} = (2k+1)/(k+1)\,A_n$. Hence from the known coefficients for $n = 0, 1$ we can find the coefficient of the highest power of x in P_n is $1 \cdot 3 \cdot 5 \ldots (2n-1)/n!$.

The polynomials can also be found by solving the differential equation by differentiating the coefficients of a power series substituted in the equation. This method often used in quantum mechanics texts. It is also important to indicate that this method does not allow us to investigate the properties of polynomials. However it describes only the individual polynomials themselves. Consider the polynomials

$$G_n(x) = \frac{d^n}{dx^n}\left(x^2-1\right)^n.$$

When differentiating the equation (2.20) n times, we will get a polynomial of order n. They will consist of all odd or all even powers of x, as n is odd or even. The coefficient of the highest power of x is $2n\,(2n-1)\,(2n-2)\ldots(n+1)$. Therefore, the first two polynomials will be 1 and $2x$. If $G\,(x)$ is substituted in the recurrence relation for the Legendre polynomials, it is found to satisfy it if we substitute $G\,(x)$ by the constant $2^n n!$, then the first two polynomials are 1 and x. Because of this,

$$P_n\,(x) = \left(\frac{1}{2^n n!}\right) \frac{d^n}{dx^n}\,(x^2-1)^n . \qquad (2.20)$$

This formula is called Rodriguez's formula. The advantage of Rodriguez's formula is that it is in n^{th} derivative form. The orthogonality of Legendre polynomials follows when Rodriguez's formula is used. The Rodriguez's formula is used to find the recurrence relation, the differential equation, and many other properties.

2.4 Special Cases

We will calculate values of $P_n(x)$ for $n = 2$, and $n = 3$.

If $n = 2$, then

$$\begin{aligned} P_2(x) &= \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 \\ &= \frac{1}{4\cdot 2}\frac{d^2}{dx^2}(x^4-2x^2+1) \\ &= \frac{1}{8}(4x^3-4x)' \\ &= \frac{1}{8}(12x^2-4) \\ &= \frac{1}{2}(3x^2-1). \end{aligned}$$

If $n = 3$, then

$$\begin{aligned} P_3(x) &= \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3 \\ &= \frac{1}{48}(x^6-3x^4+3x^2-1)''' \\ &= \frac{1}{48}(6x^5-12x^3+6x)'' \\ &= \frac{1}{48}(30x^4-36x^2+6)' \\ &= \frac{1}{48}(120x^3-72x) \\ &= \frac{120}{48}x^3-\frac{72}{48}x \\ &= \frac{1}{2}(5x^3-3x). \end{aligned}$$

2.5 Associated Legendre Functions

For finding solutions to Laplace's equation in spherical coordinates, Legendre polynomials are sufficient for problems axially symmetric, in which there is no φ-dependence. The more general problem requires the introduction of related function called the associated Legendre functions that are constructed from Jacobi polynomials, and can be expressed in terms of Legendre polynomials [Sze75].

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal on the interval $(-1,1)$ with respect the weight function $w(x)=(1-x)^{\alpha}(1+x)^{\beta}$, for $\alpha,\beta>-1$.

The Rodriguez's formula for the Jacobi polynomials is[Sze75]:

$$P_n^{(\alpha,\beta)}(x)=\left[\frac{(-1)^n}{2^n n!}\right](1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}(1-x)^{\alpha+n}(1+x)^{\beta+n}. \tag{2.21}$$

The ordinary Legendre polynomial $P^n(x)=P_n^{(0,0)}(x)$. They will satisfy the differential equation:

$$\left(1-x^2\right)P_n''^{(\alpha,\beta)}+[\beta-\alpha-(\alpha+\beta+2)x]P_n'^{(\alpha,\beta)}+n(\alpha+\beta+n+1)P_n^{(\alpha,\beta)}(x)=0. \tag{2.22}$$

To solve Laplace's equation by using method of separating variables we can obtain θ dependence $T(x), x=\cos\theta$, the differential equation:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dT}{dx}\right]=\left[l\left(l+1\right)-\frac{m^2}{\left(1-x^2\right)}\right]T=0.$$

The substitution $T\left(x\right)=\left(1-x^2\right)^{\frac{m}{2}}y\left(x\right)$ will produce the following equation:

$$\left(1-x^2\right)y''-2\left(m+1\right)xy'+\left[l\left(l+1\right)-m\left(m+1\right)\right]y=0. \qquad (2.23)$$

We recognize this as satisfied by the Jacobi polynomial $P_{1-m}^{(m,m)}\left(x\right)$. Hence, $T\left(x\right)=\left(1-x^2\right)^{\frac{m}{2}}y\left(x\right)P_{l-m}^{(m,m)}\left(x\right)$. This is an associated Legendre function, often denoted $P_l^{(m)}\left(x\right)$ in physics texts. It is defined as $\left(-1\right)^m\left(1-x^2\right)^{\frac{m}{2}}\frac{d^m}{dx^m}P_l$. Here the subscript is not the degree of the polynomial.

The above formulas and expressions are for positive m. Due to the equation containing m^2, the solution for negative m is essentially the same, except for a multiplicative factor. It is applicable for spherical harmonics, where relative phases matter.

"In mathematics, spherical harmonics are the angular portion of a set of solutions to Laplace's equation. Represented in a system of spherical coordinates, Laplace's spherical harmonics are a specific set of spherical harmonics that forms an orthogonal system, first introduced by Pierre Simon de Laplace in 1782. Spherical harmonics are important in many theoretical and practical applications, particularly in the computation of atomic orbital electron configurations, representation of gravitational fields, geoids, and the magnetic fields of planetary bodies and stars, and characterization of the cosmic microwave background radiation. In three dimensional computer graphics, spherical harmonics play a special role in a wide variety of topics including indirect lighting and recognition of three dimensional shapes"[Mac67].

In physics it is as follows: $P_l^{-m}\left(x\right)=\left(-1\right)^m\left[\left(1-m\right)!/\left(1+m\right)!\right]P_l^m\left(x\right)$, where m is always positive on the right. If we work explicitly with the function for $+m$ and $-m$ are essentially the same, and differ at most by a factor of -1.

For the same m, $P_1^m\left(x\right)$ are orthogonal, and the integral of the square of $P_l^m\left(x\right)$ is the same as for $P_l\left(x\right)$, multiplied by $\left(1-m\right)!/\left(1+m\right)!$. The functions are not orthogonal for different values of m; orthogonality of spherical harmonics in this case depends on the φ function.

Chapter 3

Spherical Coordinates and Legendre Function

In this section we will work on deriving the formula for the Laplace operator in the polar coordinate systems in the plane and in three dimensional space. We will also investigate some insights of integration in polar coordinates in $n-$space. First, the relation between polar coordinates (r, θ) in $\mathbb{R}^2$ and Cartesian coordinates (x, y) will be expressed as the following:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned}$$

Let u be a function of class C^2 in $\mathbb{R}^2$. By differentiating and applying the chain rule we will have

$$\begin{aligned} u_r &= \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = u_x\cos\theta + u_y\sin\theta, \\ u_\theta &= \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -u_x r\sin\theta + u_y r\cos\theta. \end{aligned} \tag{3.1}$$

Using the equations for u_r and u_θ we can derive an expression for u_y

$$u_y = u_r\sin\theta + r^{-1}u_\theta\cos\theta\ . \tag{3.2}$$

Our next step is to find the derivatives of u_r, and u_θ by using a chain rule

$$\begin{aligned} u_{rr} &= (u_{xx}\cos\theta + u_{xy}\sin\theta)\cos\theta + (u_{xy}\cos\theta + u_{yy}\sin\theta) \\ &= u_{xx}\cos^2\theta + 2u_{xy}\sin\theta\cos\theta + u_{yy}\sin^2\theta, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} u_{\theta\theta} &= -(u_{xx}r\sin\theta + u_{xy}r\cos\theta)r\sin\theta - u_x r\cos\theta \\ &\quad +(-u_{xy}r\sin\theta + u_{yy}r\cos\theta)r\cos\theta - u_y r\sin\theta \\ &= r^2(u_{xx}\sin^2\theta - 2u_{xy}\sin\theta\cos\theta + u_{yy}\cos^2\theta) - r(u_x\cos\theta + u_y\sin\theta). \end{aligned} \tag{3.4}$$

By dividing both sides by r^2 and replacing $u_x\cos\theta + u_y\sin\theta$ by u_r we will have the following expression

$$r^{-2}u_{\theta\theta} = u_{xx}\sin^2\theta - 2u_{xy}\sin\theta\cos\theta + u_{yy}\cos^2\theta - r^{-1}u_r. \tag{3.5}$$

By adding equations (2), and (4) we will have the following

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = u_{xx} + u_{yy}. \tag{3.6}$$

Therefore, the Laplacian of u, which is $\nabla^2 u = u_{xx} + u_{yy}$ in rectangular coordinates, in polar coordinates will be written as

$$\nabla^2 u = u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}. \tag{3.7}$$

3.1 The Laplacian in Cylindrical and Spherical Coordinates

Let (ρ, θ, z) in $\mathbb{R}^3$ be polar coordinates by using (ρ, θ) in the $xy-$plane and having z fixed. In Cartesian coordinates (xyz) we will have

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = z.$$

In cylindrical and spherical coordinates we will use ρ for r. Let ρ and θ be fixed (similar to holding x, and y fixed in Cartesian coordinates) in cylindrical and spherical coordinates and we will find a partial derivative with respect to z using (3.7) . The Laplacian in cylindrical coordinates will be

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = u_{\rho\rho} + \rho^{-1}u_\rho + \rho^{-2}u_{\theta\theta} + u_{zz}. \tag{3.8}$$

Next we will consider spherical coordinates (r, θ, z) in R^3 and in Cartesian coordinates (x, y, z) will be by

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi.$$

Accordingly, for cylindrical coordinates (ρ, θ, z) we will have

$$\rho = r\sin\phi, \quad \theta = \theta, \quad z = r\cos\phi.$$

In a sphere θ is a longitude, and ϕ is the angle from the north pole. θ is in radians in the interval of length $2\pi, (-\pi, \pi]$, or $[0, 2\pi)$. Also, ϕ is restricted to $[0, \pi]$.

To convert cylindrical coordinates to spherical coordinates longitude θ will be unchanged, and the variables (z, ρ) are going to be related to spherical variables (r, ϕ) by the following equations

$$z = r\cos\phi, \qquad \rho = r\sin\phi.$$

In this two equations the variables have different names, although they represent the same values. Thus, based on the formulas (3.2) and (3.7) and relabeling the variables we will have

$$u_\rho = u_r\sin\phi + r^{-1}u_\phi\cos\phi$$

and

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\phi\phi} = u_{zz} + u_{\rho\rho}.$$

By substituting u_p and $u_{zz} + u_{pp}$ in formula (3.8),we will have

$$\begin{aligned}\nabla^2 u &= u_{\rho\rho} + u_{zz} + \rho^{-1}u_\rho + \rho^{-2}u_{\theta\theta} \\ &= u_{rr} + r^{-1}u_r + r^{-2}u_{\phi\phi} + \rho^{-1}u_r\sin\phi + (r\rho)^{-1}u_\phi\cos\phi + \rho^{-2}u_{\theta\theta}.\end{aligned}$$

Finally by considering $\rho = r\sin\phi$ we will get

$$\nabla^2 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2\tan\phi}u_\phi + \frac{1}{r^2}u_{\phi\phi} + \frac{1}{r^2\sin^2\phi}u_{\phi\phi}. \tag{3.9}$$

Formula (3.9) can be simplified by using the following to facts

$$\begin{aligned} u_{rr} + \frac{2}{r}u_r &= \frac{1}{r}(ru)_{rr} \\ &= \frac{1}{r^2}(r^2 u_r)_r, \end{aligned}$$

and

$$\frac{1}{\tan\phi} + u^{\phi\phi} = \frac{1}{\sin\phi}(u_\phi \sin\phi)_\phi.$$

Therefore the Laplacian in spherical coordinates will be the following:

$$\begin{aligned} \nabla^2 u &= \frac{1}{r}(ru)_{rr} + \frac{1}{r^2\sin^2\phi}(u_\phi \sin\phi)_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta} \\ &= \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2\sin^2\phi}(u_\phi \sin\phi)_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}\ . \end{aligned} \tag{3.10}$$

By using the Laplacian in spherical coordinates formula we will solve some boundary value problems. First we will consider the Dirichlet problem for the unit ball in $\mathbb{R}^3$:

$$\nabla^2 u(r,\theta,\phi) = 0 \quad \text{for } r < 1, \qquad u(1,\theta,\phi) = f(\theta,\phi). \tag{3.11}$$

To find solutions of $\nabla^2 u = 0$ in the form of $u = R(r)\Theta(\theta)\Phi(\theta)$ we will use the method of separation of variables. By substituting $u = R(r)\Theta(\theta)\Phi(\theta)$ Φ in to $\nabla^2 u = 0$ we will have the following

$$r^2\sin^2\phi\left[\frac{R''}{R} + \frac{2R'}{rR}\right] + \sin\phi\frac{(\Phi'\sin\phi)'}{\Phi} = -\frac{\Theta''}{\Theta}. \tag{3.12}$$

Both sides of this equation have to be equal to a constant m^2, hence

$$m^2\Theta'' + m^2\Theta = 0.$$

Therefore,

$$\Theta(\theta) = ae^{im\theta} + be^{-im\theta}.$$

In spherical coordinates θ represents the longitude; therefore, the period of Θ must be also 2π, and m needs to be a nonnegative integer. Hence we will make the left side of the equation (3.12) equal to m^2 and separate r and ϕ :

$$\frac{r^2R'' + 2rR'}{R} = \frac{m^2}{\sin^2\phi} - \frac{(\Phi'\sin\phi)'}{\Phi\sin\phi}. \tag{3.13}$$

Both sides of the equation (3.13) must be equal to λ. Hence Φ and R will be expressed as

$$\frac{(\Phi' \sin\phi)'}{\Phi \sin\phi} - \frac{m^2\Phi}{\sin^2\phi} + \lambda\Phi = 0, \tag{3.14}$$

and

$$r^2R'' + 2rR' - \lambda R = 0. \tag{3.15}$$

Now, we are going to transform the equation (3.15) in to a close relative of the Legendre equation. The Legendre equation is

$$\left[(1-x^2)y'\right]' + \lambda y = 0. \tag{3.16}$$

Let $s = \cos\phi$,where ϕ has a range equal to $[0, 2\pi]$. The transformation $\phi \to s = \cos\phi$ is a one-to-one function between $[0,\pi]$ and $[-1,1]$. Let q be dependent on s. By differentiating q in respect to ϕ we will have

$$\frac{dq}{d\phi} = \frac{dq}{ds}\frac{ds}{d\phi} = -\sin\phi\frac{dq}{ds},$$

or

$$\frac{1}{\sin\phi}\frac{dq}{d\phi} = -\frac{dq}{ds}.$$

Let $s = \cos\phi$, and $S(s) = S(\cos\phi) = \Phi(\phi)$. Then $\sin^2\phi = 1 - s^2$. Substituting in to the previous equation we will have

$$\frac{1}{\sin\phi}\frac{d}{d\phi}(\sin\phi\frac{d\Phi}{d\phi}) = -\frac{d}{ds}((1-s^2)\frac{dS}{ds}).$$

Hence, $\Phi(\phi)$ satisfies the equation (13) if and only if $S(s)$=$\Phi(ar\cos s)$ satisfies

$$\left[(1-s^2)S'\right]' - \frac{m^2}{1-s^2} + \lambda s = 0. \tag{3.17}$$

We can see that (3.17) will become the Legendre equation for the values of $m = 0$. Because of this it is called **associated Legendre equation of order m.** We can find solutions for (3.17) if m is a positive integer, and the equation becomes an ordinary Legendre equation

$$\left[(1-s^2)w'\right]' + \lambda w = 0. \tag{3.18}$$

Let f, and g be functions, and we will apply the product rule for $(m+1)$th order derivatives,

$$(fg)^{(m+1)} = \sum_{0}^{m+1} \frac{(m+1)!}{k!(m+1-k)!} f^{(k)} g^{(m+1-k)}.$$

By substituting $f(s) = 1 - s^2$, and $g(s)$=$w'(s)$ we will get the following

$$\left[(1-s^2)w'\right]^{(m+1)} = (1-s^2)w^{(m+2)} - 2(m+1)sw^{(m+1)} - m(m+1)w^{(m)}.$$

By differentiating (3.18) m times we will get the following

$$(1-s^2)w^{(m+2)} - 2(m+1)sw^{(m+1)} - m(m+1)w^{(m)} + \lambda w^{(m)} = 0. \tag{3.19}$$

Let $(1-s^2)^{m/2}w^{(m)} = S$, then we will have

$$(1-s^2)S' = -ms(1-s^2)^{m/2}w^{(m)} + (1-s^2)^{m/2}w^{(m+1)}.$$

By differentiating both sides of this equation we will have

$$\begin{aligned}\left[(1-s^2)S'\right]' &= (1-s^2)^{m/2} \qquad (3.20)\\ &\quad * \left[(1-s^2)w^{(m+2)} - 2(m+1)sw^{(m+1)} + \frac{m^2 w^{(m)}}{1-s^2} - m(m+1)w^{(m)}\right].\end{aligned}$$

It follows from the equations (3.19) and (3.20) that

$$\left[(1-s^2)S'\right]' = \frac{m^2 S}{1-s^2} - \lambda S.$$

This shows that if w satisfies the equation $\left[(1-s^2)w'\right]' + \lambda w = 0$, then $S = (1-s^2)^{m/2}$ satisfies the equation $\left[(1-s^2)S'\right]' - \frac{m^2}{1-s^2} + \lambda s = 0$.

Let $\lambda = n(n+1)$, and let w be the Legendre polynomial P_n, then the **associated Legendre function** P_n^m will be

$$P_n^m(s) = (1-s^2)^{m/2}\frac{d^m P_n(s)}{ds^m} = \frac{(1-s^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{ds^{n+m}}(s^2-1).^n \tag{3.21}$$

We can see from the formula (3.21) that $P_n^m(s) = 0$ when $m > n$ because P_n is a polynomial of degree n, therefore P_n^m is important for $n \geq m$. Hence for $m = 1, 2, 3, ...$and $n \geq m$, P_n^m will be a solution of the boundary problem

$$\left[(1-x^2)y'\right] + \frac{m^2 y}{1-x^2} + n(n+1)y = 0. \tag{3.22}$$

Chapter 4

Riemann Zeta Function

As another application of Legendre polynomials, we will consider some special values of the Riemann zeta function.

"The Riemann zeta function $\zeta(s)$, is a function of a complex variable s. It is expressed as a continuous sum of the infinite series which converges when s is greater than one" [Tit86]. The representations of $\zeta(s)$ for all s will be given later in this section. The Riemann zeta function plays a very important role in analytic number theory, in physics, probability theory, and applied statistics.

The Riemann zeta function is also called Euler-Riemann zeta function. It was introduced and studied by Leonhard Euler in the first half of the eighteenth century. In 1859 Bernhard Riemann published *"On the Number of Primes Less Than a Given Magnitude"* where he extended the Euler definition to a complex variable. He established a relation between zeros and the distribution of prime numbers [Edw01].

4.1 History

The zeta function was discovered by Swiss mathematician Leonhard Euler in 1737, but it was first studied extensively by the German mathematician Bernhard Riemann.

In 1859 Riemann discovered explicit formula for the number of primes up to any preassigned limit. Riemann's formula depended on knowing the values at which the zeta function equals zero. The Riemann zeta function is defined for all complex numbers of the form $\sigma + it$, where σ and t are real and $i = \sqrt{(-1)}$. The function equals zero for all negative even integers $-2, -4, -6, \ldots$ which he called trivial zeros. There are infinite number of zeros in the interval of $\sigma = 0$ and $\sigma = 1$. Riemann concluded that all of the nontrivial zeros are on the critical line $t = \frac{1}{2}$, which later became known as the Riemann hypothesis.

According to David Hilbert, a German mathematician, the Riemann hypothesis is one of the most important studies in mathematics. Shortly after David Hilberts study of the Riemann hypothesis, in 1915 the English mathematician Godfrey Hardy proved that an infinite number of zeros occur on the critical line. In1986 the first $1,500,000,001$ nontrivial zeros were all shown to be on the critical line. Even though this hypothesis may or may not be false, the study of the matter has deeply increased the understanding of complex numbers [Edw01].

4.2 Definition

The Riemann zeta function $\zeta(s)$ is a function of a complex variable $s = \sigma + it$ where, s, σ and t are real notations associated with the $\zeta-$function. The following infinite series converges for all complex numbers s with real part greater than 1, and is defined $\zeta(s)$ as;

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots \tag{4.1}$$

The Riemann zeta function is defined as the continuation of the function defined for $\sigma > 1$ by the sum of the series (4.1).

Leonhard Euler discovered the above series in 1740 for positive integer values of s, and later Chebyshev extended the definition to real $s > 1$.

The above series converges for $s > 1$ and diverges for all other values of s. Riemann showed that the function defined by the series on the half-plane of convergence can be continued analytically to all complex values $s \neq 1$. For $s = 1$ the series is the harmonic series which diverges to $+\infty$.

Thus the Riemann zeta function is a convergent function on the whole complex $s-$plane, which is convergent everywhere except for a simple pole at $s = 1$.
computed the values of the Riemann zeta function for even positive integers. The first of them, $\zeta(2)$, provides a solution to the Basel problem. In 1979 Apery proved the irrationality of $\zeta(3)$. The values at negative integer points, also found by Euler, are rational

numbers and play an important role in the theory of modular forms.

4.3 Specific Values

For a positive even number $2n$,

$$\zeta(2n) = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!}. \tag{4.2}$$

In this expression B_{2n} is a Bernoulli number. This formula can be proved by several means. We will show it below for $n = 2$, and $n = 3$. In mathematics, the Bernoulli numbers B_n are a sequence of rational numbers with deep connections to number theory. The values of the first few Bernoulli numbers are

$B_0 = 1, B_1 = \pm 1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42, B_7 = 0, B_8 = -1/30$.

If the condition $B_1 = -1/2$ is used, this sequence is also known as the first Bernoulli numbers; with the condition $B_1 = +1/2$ is is known as the second Bernoulli numbers. Except for this one difference, the first and second Bernoulli numbers agree. Since $B_n = 0$ for all odd $n > 1$, and many formulas only involve even-index Bernoulli numbers, some authors write B_n instead of B_{2n}[Tit86].

From equation (4.1), if n is a positive integer then

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}. \tag{4.3}$$

We can see that for all $n \geq 1$ ζ approaches zero at all negative integers since $B_m = 0$ for all odd $m \neq 1$.

The most commonly used values of zeta function are;

$$\begin{aligned}
\zeta(0) &= -\frac{1}{2}. \qquad (4.4)\\
\zeta(1) &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+... = \infty.\\
\zeta(2) &= 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+... = \frac{\pi^2}{6} \cong 1.645.\\
\zeta(3) &= 1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+... \cong 1.202.\\
\zeta(4) &= 1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+... = \frac{\pi^4}{90} \cong 1.0823.
\end{aligned}$$

4.4 Finding $\zeta(2)$

In this section we will find $\zeta(2)$ using the Fourier series for $f(\theta) = \theta^2$, in the interval $(-\pi < \theta < \pi)$. This series is

$$\theta^2 = \frac{\pi^3}{3} + 4\sum_{1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta. \tag{4.5}$$

We will show that $\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

To solve this problem we will choose a specific value for θ. Let $\theta = \pi$, then

$$\pi^2 = \frac{\pi^3}{3} + 4\sum_{1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi.$$

By subtracting $\frac{\pi^3}{3}$ from two sides we will the following,

$$\pi^2 - \frac{\pi^3}{3} = 4\sum_{1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi,$$

and

$$\frac{3\pi^2 - \pi^2}{3} = 4\sum_{1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi.$$

Considering the fact that $\cos n\pi = (-1)^n$, $\frac{2\pi^2}{3} = 4\sum_{1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n$.

Since $(-1)^n(-1)^n = (-1)^{2n} = 1$, we will have $\frac{2\pi^2}{12} = \sum_{1}^{\infty} \frac{1}{n^2}$, or

$$\frac{\pi^2}{6} = \sum_{1}^{\infty} \frac{1}{n^2}. \tag{4.6}$$

⊠

4.5 Finding $\zeta(4)$

To calculate $\zeta(4)$ we will work on some supporting steps. First, by referring to (4.5) we will show that $\theta^3 - \pi^2\theta = 12\sum_{1}^{\infty} \frac{(-1)^n \sin n\theta}{n^3}$. By integrating both two sides of the equation $\theta^2 = \frac{\pi^3}{3} + 4\sum_{1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta$ we will have

$$\frac{\theta^2}{3} = \frac{\pi^3}{3}\theta + 4\sum_1^\infty \frac{(-1)^n}{n^2}\sin n\theta + C.$$

Notice that $C = 0$ for $\theta = 0$, therefore;

$$\frac{\theta^3 - \pi^2}{3} = 4\sum_1^\infty \frac{(-1)^n}{n^2}, \tag{4.7}$$

and finally

$$\theta^3 - \pi^2\theta = 12\sum_1^\infty \frac{(-1)^n}{n^3}\sin n\theta. \tag{4.8}$$

Next, by using the equation (4.8) we will show that

$$\theta^4 - \pi^2\theta^2 = 48\sum_1^\infty \frac{(-1)^n \cos n\theta}{n^4} - \frac{7\pi^4}{15}$$

By integrating the equation (4.8) in the first section,

$$\theta^3 - \pi^2\theta = 12\sum_1^\infty \frac{(-1)^n}{n^3}\sin n\theta,$$

and so we get

$$\theta^4 - 2\pi^2\theta^2 = 48\sum_1^\infty \frac{(-1)^{n+1}\cos n\theta}{n^4} + C. \tag{4.9}$$

Notice that $(-1)^{n+1}\cos n\theta = (-1)^{2n+1} = -1$, since $2n+1$ is an odd number. To determine the value of C we will use the following integration formula,

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\theta^4 - 2\pi^2\theta^2)d\theta = \frac{-7\pi^4}{15},$$

and $\frac{a_0}{2} = \frac{-7\pi^4}{15}$, therefore,

$$\theta^4 - \pi^2\theta^2 = 48\sum_1^\infty \frac{(-1)^n \cos n\theta}{n^4} - \frac{7\pi^4}{15} \tag{4.10}$$

Finally we will use the information in previous two steps to prove that

$$\sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Considering the formula (4.10),

$$\theta^4 - \pi^2\theta^2 = 48\sum_1^\infty \frac{(-1)^n \cos n\theta}{n^4} - \frac{7\pi^4}{15},$$

and let $\theta = \pi$, we will have

$$\pi^4 - 2\pi^2\pi^2 = 48\sum_1^\infty \frac{(-1)^n \cos n\theta}{n^4} - \frac{7\pi^4}{15},$$

and

$$15\pi^4 - 30\pi^4 = -720\sum_1^\infty \frac{(-1)^{n+1}}{n^4}(-1)^n - 7\pi^4.$$

Then,

$$15\pi^4 - 30\pi^4 = -720\sum_1^\infty \frac{(-1)^{n+1}}{n^4}(-1)^n - 7\pi^4,$$

and

$$-15\pi^4 + 7\pi^4 = -720\sum_1^\infty \frac{(-1)^{n+1}}{n^4}(-1)^n.$$

Since $2n+1$ is an odd number$(-1)^{2n+1} = -1$, therefore;

$$-8\pi^4 = -720\sum_1^\infty \frac{1}{n^4},$$

and,

$$\frac{\pi^4}{90} = \sum_1^\infty \frac{1}{n^4}.$$

⊠

Chapter 5

Some Notes on $\zeta(3)$

The first historical result obtained on values of $\zeta(s)$ at odd positive integers is due to Apery who proved in 1978 that $\zeta(3)$ is irrational (for that reason, $\zeta(3)$ is now sometimes called the Apery constant). It is not known if $\zeta(3)$ is transcendental. Apery's proof for $\zeta(3)$ does not generalize for $\zeta(5)$, $\zeta(7)$, . . ., and it is not known if any of these constants are irrational or not [Tit86].

A transcendental number is defined as a number that is not the root of any integer polynomial. It is not an algebraic number. Rational numbers are solutions of $bx + a = 0$, where a, b are integers. Algebraic numbers are solutions to an n^{th}degree polynomials in the form of $a_n x^n + a_n x^n + ... + a_0 = 0$ where $a_0, a_1, ...a_n$ are integers. Therefore, by definition, a rational number is an algebraic number of degree one, and every real transcendental number must also be irrational.

5.1 Liouville Criterion for Rationality

The following lemmas are exercises for Fourier Analysis by T.W Korner by a citation to the bibliography [Kör89].

Lemma 1. *Let p and q be integers with no common factor and $q \geq 1$.Then, if A and B are integers either $A + \frac{Bp}{q} = 0$ or $\left|A + \frac{Bp}{q}\right| \geq q^{-1}$. Then, if x is real we can find A_n, B_n integers with $0 < |A_n + B_n x| \to 0$, as $n \to \infty$, x is irrational.*

To prove the first part of the lemma we will rewrite $A+\frac{Bp}{q}$ as

$$\frac{Aq+Bp}{q}=(Aq+Bp)\frac{1}{q}.$$

Since q is an integer with $q\geq 1$ it follows that $\frac{1}{q}\neq 0$. Therefore,

$$\begin{aligned} & Aq+Bp \\ = \; & A+\frac{Bp}{q} \\ = \; & \frac{Aq+Bp}{q} \\ = \; & (Aq+Bp)\frac{1}{q} \\ = \; & 0. \end{aligned}$$

For part two, we will assume that x is rational. Then for $x=\frac{p}{q}$, for some integers p, and q that satisfy part one we will have

$$|A_n+B_nx|\geq\frac{1}{q} \text{ for all } n.$$

This is a contradiction, since $|A_n+B_nx|\to 0$, as $n\to\infty$. Therefore, x is irrational.

⊠

Lemma 2. *Let d_n be the lowest common multiple of $1,2,...,n$. We need to show that $d_n=\Pi_{p\leq n}p^{[\log n/\log p]}\leq n^{\pi(n)}$ and deduce that $d_n\leq 3^n$.*

Considering the fact that $[x]$ is the greatest integer function of x, and $\pi(n)$ =number of primes$\leq n$ we will have the following

$$d_n=\Pi_{p\leq n}p^{[\log n/\log p]}\leq\Pi_{p\leq n}p^{\frac{\log n}{\log p}}.$$

By applying the logarithm to two sides of this expression we will have

$$\begin{aligned} \log d_n &\leq \sum_{p\leq n}\log(p^{[\log n/\log p]}) \\ &= \sum_{p\leq n}\frac{\log n}{\log p}\log p \\ &= \sum_{p\leq n}\log n \\ &= (\log n)\pi(n). \end{aligned}$$

By using exponential function we will get

$$d_n \leq n^{\pi(n)}.$$

Since $n^{\pi(n)} \approx n^{\frac{n}{\log n}}$ we can derive that for $\log d_n \to n$, and $d_n \to e^n$. Since $e < 3$, we will have $d_n \leq 3^n$. ⊠

Lemma 3. *We introduce the polynomial* $P_n(x) = (\frac{1}{n!})\frac{d^n}{dx^n}\left[x^n(1-x)^n\right]$.

Then P_n *will be a polynomial of degree* n, *and* $\int\limits_0^1 P_n(x)P_m(x)dx = 0$.

If $n \neq m$, *so the* P_n *are scalar multiples of the Legendre polynomials for* $[0,1]$.

The coefficients of P_n *are integers.*

To prove that the coefficients of P_n are integers we will consider the k_{th} term in the equation (5.1) and have the following

$$\begin{aligned}
& \frac{1}{n!}\sum_{k=1}^{n}\binom{n}{k}\frac{d^k}{dx^n}x^n\frac{d^{n-k}}{dx^{n-k}}(1-x)^n \\
= \quad & \frac{1}{n!}\sum_{k=1}^{n}\frac{n!}{k!(n-k)!}\frac{n!}{(n-k)!}x^{n-k}\frac{n!}{k!}(-1)^{n-k}(1-x)^k \\
= \quad & \frac{n!}{n!}\sum_{k=1}^{n}\binom{n}{k}x^{n-k}\binom{n}{k}(-1)^{n-k}(1-x)^k, \text{ for all } k \leq n.
\end{aligned}$$

Notice that $\frac{n!}{n!} = 1$, and $\binom{n}{k}$, $\binom{n}{k}(-1)^{n-k}$ are integers. ⊠

Lemma 4. *Let*

$$I_n = \int\limits_0^1\int\limits_0^1 \frac{-\log xy}{1-xy}P_n(x)P_n(y)dxdy, \tag{5.1}$$

then by using the Rodriguez formula we will have

$$I_n = A_n d_n^{-3} + B_n\zeta(3), \tag{5.2}$$

where A_n, *and* B_n *are integers.*

To prove this we will start from the following

$$\begin{aligned} I_n &= \int_0^1\int_0^1 \frac{-\log xy}{1-xy} P_n(x)P_n(y)dxdy \\ &= \int_0^1\int_0^1 \frac{-\log xy}{1-xy} \sum_{k=1}^{\infty} a_k x^k \sum_{j=1}^{\infty} a_j x^j dxdy \\ &= \int_0^1\int_0^1 \frac{-\log xy}{1-xy} \sum_{0\le k\le n}^{\infty} a_k a_j x^k x^j dxdy. \end{aligned} \tag{5.3}$$

Considering the fact that $a_k a_j \in \mathbb{R}$, it follows that $\sum_{0\le k\le n}^{\infty} a_k a_j$ is an integer. In the exercises textbook of [Kör93] on page 174 we proved that

$$\int_0^1\int_0^1 \frac{x^r y^s}{1-xy} \log xydxdy = -2\zeta(3) + \frac{a_r}{d_r^3}. \tag{5.4}$$

We can see that $-2\zeta(3) + \frac{a_r}{d_r^3}$ corresponds to $A_n d_n^{-3} + B_n\zeta(3)$, since $A_n, B_n, 2$, and a_r are integers, therefore,

$$\begin{aligned} I_n &= \int_0^1\int_0^1 \frac{-\log xy}{1-xy} \sum_{0\le j,k\le n} a_k a_j x^k x^j dxdy \\ &= \sum_{0\le j,k\le n} a_k a_j \int_0^1\int_0^1 \frac{-\log xy}{1-xy} x^k x^j dxdy \\ &= \sum_{0\le j,k\le n} a_k a_j \left[-2\zeta(3) + \frac{a_r}{d_r^3}\right] \\ &= A_n d_n^{-3} + B_n\zeta(3). \end{aligned} \tag{5.5}$$

⊠

And finally, we need to show that there are integers A_n, and C_n such that

$$\begin{aligned} 0 &< |A_n + C_n\zeta(3)| = 2\zeta(3)d_n^3(\sqrt{2}-1)^{4n} \\ &< 2\zeta(3)3^{3n}(\sqrt{2}-1)^{4n} \\ &< (\frac{4}{5})^n\zeta(3). \end{aligned} \tag{5.6}$$

5.2 Problems on $\zeta(3)$

First we need to show that

$$-\int_0^1\int_0^1 \frac{x^r y^s}{1-xy}\log xy\,dxdy = \sum_{n=0}^{\infty}\frac{1}{(n+r+1)(n+s+1)}\left(\frac{1}{n+r+1}+\frac{1}{n+s+1}\right),$$

and then we will show that

$$\int_0^1\int_0^1 \frac{1}{1-xy}\log xy\,dxdy = -2\zeta(3).$$

To begin we will consider the first equation;

$$\begin{aligned}\int_0^1\int_0^1 x^r y^s \log xy\,dxdy &= \int_0^1\int_0^1 x^r y^s(\log x+\log y)dxdy\\ &= \int_0^1 y^s\int_0^1 x^r\log x\,dxdy + \int_0^1 x^r\int_0^1 y^s\log y\,dxdy.\end{aligned}$$

By using the integrating by parts method for $u=\log x$, and $dv = x^r dx$ with $du = \frac{1}{x}dx$, and $v = \frac{1}{r+1}x^{r+1}dx$ we will have

$$\begin{aligned}&\left[\frac{1}{r+1}x^{r+1}\log x\right]_0^1 - \frac{1}{r+1}\int_0^1 x^r dx\\ =\ &\frac{-1}{(r+1)^2}\frac{1}{(s+1)}\end{aligned}$$

Since $\frac{1}{r+1}x^{r+1}\log x = 0$, for both, $x=1$, and $x=0$. ⊠

Next, we will prove the second equation;

$$
\begin{aligned}
& -\int_0^1\int_0^1 \frac{x^r y^s}{1-xy}\log xy dx dy \\
= & \sum_{n=0}^{\infty}\int_0^1\int_0^1 x^{(n+r)}y^{(n+s)}\log xy dx dy \\
= & -1\sum_{n=0}^{\infty}\left[\frac{-1}{(n+s+1)}\frac{1}{(n+r+1)^2}-\frac{-1}{(n+s+1)^2}\frac{1}{(n+r+1)}\right] \\
= & \sum_{n=0}^{\infty}\left[\frac{1}{(n+r+1)^2(n+s+1)}+\frac{1}{(n+s+1)^2(n+r+1)}\right] \\
= & \sum_{n=0}^{\infty}\frac{1}{(n+r+1)(n+s+1)}\left[\frac{1}{(n+r+1)}+\frac{1}{(n+s+1)}\right].
\end{aligned}
$$

Let $r = s = 0$, then

$$
\begin{aligned}
& \lim_{N\to\infty}\sum_{n=0}^{\infty}\frac{1}{(n+r+1)(n+s+1)}\left[\frac{1}{(n+r+1)}+\frac{1}{(n+s+1)}\right] \\
= & \sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\left[\frac{1}{n+1}+\frac{1}{n+1}\right] \\
= & \sum_{n=0}^{\infty}\frac{2}{(n+1)^3} \\
= & 2\zeta(3),
\end{aligned}
$$

since $\sum_{n=0}^{\infty}\frac{1}{(n+1)^3} = \zeta(3)$.

Theorem 5. *$\zeta(3)$ is irrational.*

From lemma 4 we can see that for all $n \to \infty$, $(\frac{4}{5})^n \to 0$. Hence, based on Lemma 1 where $0 < |A_n + C_n x| \to 0$, x is irrational, we can conclude that $\zeta(3)$ is irrational.

⊠

Bibliography

[AS92] Milton Abramowitz and Irene A. Stegun, editors. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Dover Publications Inc., New York, 1992. Reprint of the 1972 edition.

[Edw01] H. M. Edwards. *Riemann's zeta function.* Dover Publications Inc., Mineola, NY, 2001. Reprint of the 1974 original [Academic Press, New York].

[Fol92] Gerald B. Folland. *Fourier analysis and its applications.* The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1992.

[Jac04] Dunham Jackson. *Fourier series and orthogonal polynomials.* Dover Publications Inc., Mineola, NY, 2004. Reprint of the 1941 original.

[Kel67] Oliver Dimon Kellogg. *Foundations of potential theory.* Reprint from the first edition of 1929. Die Grundlehren der Mathematischen Wissenschaften, Band 31. Springer-Verlag, Berlin, 1967.

[Kör89] T. W. Körner. *Fourier analysis.* Cambridge University Press, Cambridge, second edition, 1989.

[Kör93] T. W. Körner. *Exercises for Fourier analysis.* Cambridge University Press, Cambridge, 1993.

[Mac67] T. M. MacRobert. *Spherical harmonics. An elementary treatise on harmonic functions with applications.* Third edition revised with the assistance of I. N. Sneddon. International Series of Monographs in Pure and Applied Mathematics, Vol. 98. Pergamon Press, Oxford, 1967.

[Sze75] Gábor Szegő. *Orthogonal polynomials.* American Mathematical Society, Providence, R.I., fourth edition, 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.

[Tit86] E. C. Titchmarsh. *The theory of the Riemann zeta-function.* The Clarendon Press Oxford University Press, New York, second edition, 1986. Edited and with a preface by D. R. Heath-Brown.

[Tol76] Georgi P. Tolstov. *Fourier series.* Dover Publications Inc., New York, 1976. Second English translation, Translated from the Russian and with a preface by Richard A. Silverman.