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A study of finite symmetrical groups

May Majid

A Study of Finite Symmetrical Groups

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by

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Abstract

We have investigated finite homomorphic images of several progenitors, including $2^{\star 5} : S_5$, $2^{\star 6} : A_6$, and $3^{\star 5} : C_5$. The original symmetric presentation for several important groups such as A_7, $L_2(11)$, $PGL_2(39)$, and $PGL_2(13)$ are discovered. The technique of manual of double coset enumeration is used to construct several groups by hand and computer-based proofs are given for the isomorphism types of the groups that are not constructed.

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Chapter 1

Group Preliminaries

1.1 Groups and Homomorphism

Generalizations of the quadratic formula for cubic and quartic polynomials were discovered in the sixteenth century, and one of the major mathematical problems thereafter was to find analogous formulas for the roots of polynomials of higher degree; all attempts failed. By the middle of the eighteenth century , it was realized that permutations of the roots of a polynomial $f(x)$ were important; for example, it was known that the coefficients of $f(x)$ are "'symmetric functions"' of its roots. In 1770, J.-L Lagrange used permutations to analyze the formulas giving the roots of cubics and quartics, but he could not fully develop this insight because he viewed permutations only as rearrangements, and not as bijections that can be composed (see below). Composition of permutations does appear in work of P. Ruffini and of P. Abbati about 1800, in 1815, A.L. Cauchy established the calculus pf permutations, and this viewpoint was used by N.H. Abel in his proof (1824) that there exist quintic polynomials for which there is no generalization of the quadratic formula. In 1830, E. Galois (only 19 years old at the time) invested groups, associated to each polynomial a group of permutations of its roots, and proved that there is a formula for the roots if and only if the group of permutations has a special property. In one great theorem, Galois founded group theory and used it to solve one of the outstanding problems of his day.

Definition 1. *Let G be a nonempty set together with a binary operation(usually called multiplication) that assigns to each ordered pair (a,b) of elements of G an element in G*

denoted by ab. We say G is a ***group*** *under this operation if the following three properties are satisfied.*

1. *Associativity. The operation is associative; that is,*
   $(ab)c = a(bc)$ *for all a, b, c in G.*

2. *Identity. There is an element e is an element e (called the identity) in G such that* $ae = ea = a$.

3. *Inverses. For each element a in G, there is an element b in G (called an inverse of a) such that* $ab = ba = e$.

Example : The set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$ and the set of real numbers $\mathbb{R}$ are all groups under ordinary addition. In each case, the identity is 0 and the inverse of a is $-a$.

Example : The set of integers under ordinary multiplication is not a group. Since the number 1 is the identity, property 3 fails. For example, there is no integer b such that $5b = 1$.

1.1.1 Finite Groups; Subgroups

Definition 2. ***Order of a Group:*** *The number of elements of a group (finite or infinite) is called its order. We will use* $|G|$ *to denote the order of G.*

Example : The group $\mathbb{Z}$ of integers under addition has infinite order, where as the group $U(10) = \{1, 3, 7, 9\}$ under multiplication modulo 10 has order 4.

Definition 3. ***Order of an Element :*** *The order of an element g in a group G is the smallest positive integer n such that* $g^n = e$. *If no such integer exits, we say that g has infinite order. The order of an element g is denoted by* $|g|$.

Example : Consider $U(15) = \{1, 2, 4, 7, 8, 11, 13, 14\}$ under multiplication modulo 15. This group has order 8. To find the order of the element 7, we compute the sequence $7^1 = 7$, $7^2 = 4$, $7^3 = 13$, $7^4 = 1$, so $|7| = 4$.

Definition 4. *Symmetric Group: Let* $X = \{1, 2, ..., n\}$, S_x *is a group with composition as operation. It is called symmetric group on X, denoted by* S_n.

Definition 5. *Let G will be a group. A subset $H \subseteq G$ is said to be a subgroup if for all $h, k \in H, hk^{-1} \in H$. We write $H \leq G$.*

A subgroup H is said to be proper if H is a proper subset of G (i.e. $H \neq F$) and we write $H < G$.

1.2 Permutations

Definition 6. *A Permutation of a set A is a function from A to A that is both one-to-one and onto. A permutation group of a set A is a set of permutations of A that forms a group under function composition.*

Although groups of permutations of any nonempty set A of objects exist, we will focus on the case where A is finite. Furthermore, it is customary, as well as convenient, to take A to be a set of the form $1, 2, 3, ...n$ for some positive integer n. Unlike in calculus, where most functions are defined on infinite sets and are given by formulas, in algebra, permutations of finite sets are usually given by an explicit listing of each element of the domain and its corresponding functional value.

Example : we define a permutation α of the set $1, 2, 3, 4$ by specifying
$\alpha(1) = 2, \quad \alpha(2) = 3, \quad \alpha(3) = 1, \quad \alpha(4) = 4.$
A more convenient way to express this correspondence is to write α in array form as

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{bmatrix}$$

Here $\alpha(j)$ is placed directly below j for each j. Similarly, the permutation β of the set $1, 2, 3, 4, 5, 6$ given by
$\beta(1) = 5, \quad \beta(2) = 3, \quad \beta(3) = 1, \quad \beta(4) = 6, \quad \beta(5) = 2, \quad \beta(6) = 4$
is expressed in array form as:

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{bmatrix}$$

1.2.1 Cycle Notation

There is another notation commonly used to specify permutations. It is called cycle notation and was first introduced by the great French mathematician Cauchy in

1815. Cycle Notation has theoretical advantages in that certain important properties of the permutation can be readily determined when cycle notation is used.

Definition 7. *If $x \in X$ and $\alpha \in S_x$, then α **fixes** x if $\alpha(x) = x$ and x **moves** x if $\alpha(x) \neq x$.*

Example : In the above example, $\alpha(4) = 4$. We say α fixes 4 and $\alpha(1) = 2$ then α moves 1.

Definition 8. ***Abelian Group:*** *A pair of elements a and b in a semi group commutes if $a * b = b * a$ A group (or a semi group) is abelian if every pair of its elements commutes.*

Example : $\mathbb{Z}_n$ is abelian under addition because for $a, b \in \mathbb{Z}_n$ then $a + b = b + a$.

Definition 9. ***Homomorphisms:*** *let $(G, *)$ and (H, o). be groups. A function $f : G \to H$ is a **homomorphism** if, for all $a, b, \in G$,*

$$f(a * b) = f(a) o f(b)$$

*An **isomorphism** is a homomorphism that is also a bijection. We say that G is **isomorphic** to H, denoted by $G \cong H$, if there exists an isomorphism $f : G \to H$*

1.2.2 Subgroups

Definition 10. *A nonempty subset S of a group G is a **subgroup** of G if $s \in G$ implies $s^{-1} \in G$ and $s, t \in G$ imply $st \in G$.*

If X is a subset of a group G, we write $X \subset G$; if X is a subgroup of G, we write $X \leq G$.

Theorem 11. *If $S \leq G$ (i.e. if S is a subgroup of G), then S is a group in its own right.*

Proof. The hypothesis "$s, t \in S$ imply $st \in S$" shows that S is equipped with an operation (if $\mu : G \times G \to G$ is the given multiplication in G, then its restriction $\mu | S \times S$ has its image contained in S). Since S in nonempty, it contains an element say, s, and the definition of subgroup says that $s^{-1} \in S$; hence, $1 = ss^{-}1 \in S$. Finally the operation on S is associative because $a(bc) = (ab)c$ for every $a, b, c \in G$ implies, in particular that $a(bc) = (ab)c$ for every $a, b, c \in S$ □

1.2.3 Lagrange's Theorem

Definition 12. *If S is a subgroup of G and if $t \in G$, then a* ***right coset*** *of S in G is the subset of G*

$$st = \{st : s \in S\}.$$

a ***left coset*** *is*

$$ts = \{ts : s \in S\}.$$

One calls t a ***representative*** *of st (and also of ts)*

Example : right coset of G in S_3.
Given $S_3 = \{e, (12), (13), (23), (123), (132)\}$ and $t = \{(12)\}$
Let $G = \{e, (12)\}$.
Then $G(123) = \{g(123)|g \in G\} = \{e(123), (12)(123)\} = \{(123), (13)\}$.

Definition 13. *If $S \leq G$, then the* ***index*** *of S in G, denoted by $[G, S]$, is the number of right cosets of S in G.*

Theorem 14. *(Lagrange's theorem) Let G be a group and H a subgroup of G. Then $|G| = |H||G : H|$.*

Example : In the above example
$G(123) = \{g(123)|g \in G\} = \{e(123), (12)(123)\} = \{(123), (13)\}$.
also,
$G(132) = \{e(132), (12)(132)\} = \{(132), (23)\}$.
$G(23) = \{e(23), (12)(23)\} = \{(12), (13)\}$.
Therefore, the number of right cosets of S_3 in G is 3. Thus, $[S_3 : G] = 3$, and $|S_3| = 6$, $|G| = 2$, then $|S_3|/|G| = 3$.

Definition 15. *A subgroup $K \in G$ is a* ***normal subgroup****, denoted by $K \triangleleft G$, if gKg^-1 for every $g \in G$.*

Definition 16. ***Direct Products:*** *if H and K are groups, then their direct products, denoted by $H \times K$, is the group with elements all ordered pairs (h,k) where $h \in H$ and $k \in K$, and with operation*

$$(h,k)(h',k') = (hh',kk')$$

It is easy to check that $H \times K$ is a group: the identity is (1,1); the inverse $(h,k)^{-}1$ is (h^{-1},k^{-1}). Notice that neither H nor K is a subgroup of $H \times K$, but $H \times K$ does contain isomorphic replicas of each, namely, $H \times 1 = (h,1) : h \in H$ and $1 \times K = n(1,K) : k \in K$

Definition 17. ***Symmetric Groups.*** *Two permutations $\alpha, \beta \in S_n$ have the* ***same cycle structure*** *if their complete factorizations into disjoint cycles have the same number of r-cycles for each r.*

Lemma 18. *If $\alpha, \beta \in S_n$ then $\alpha\beta\alpha^{-1}$ is the permutation with the same cycle structure as β which is obtained by applying α to the symbols in β.*

Example : If $\beta = (1 \quad 3)(2 \quad 4 \quad 7)$ and $\alpha = (2 \quad 5 \quad 6)(1 \quad 4 \quad 3)$, then
$\alpha\beta\alpha^{-1} = (\alpha 1 \quad \alpha 3)(\alpha 2 \quad \alpha 4 \quad \alpha 7) = (4 \quad 1)(5 \quad 3 \quad 7)$

Proof. Let π be the permutation defined in the lemma. If β fixes a symbol i, then π fixes $\alpha(i)$, for $\alpha(i)$ resides in a 1-cycle; but $\alpha\beta\alpha^{-1}(\alpha(i)) = \alpha\beta(i) = \alpha(i)$, and so $\alpha\beta\alpha^{-1}$ as well. Assume that β moves i; say, $\beta(i) = j$. Let the complete factorization of β be

$$\beta = \gamma_1\gamma_2...(...ij...)...\gamma_t.$$

□

If $\alpha(i) = K$ and $\alpha(j) = 1$, then $\pi : k \mapsto l$. But $\alpha\beta\alpha^{-1} : k \mapsto i \mapsto j \mapsto l$, and so $\alpha\beta\alpha^{-1} = \pi(k)$. Therefore, π and $\alpha\beta\alpha^{-1}$ agree on all symbols of the form $k = \alpha(i)$; since α is a surjection, it follows that $\pi = \alpha\beta\alpha^{-1}$.

Corollary 19. *A subgroup H of S_n is a normal subgroup if and only if, wherever $\alpha \in H$, then every β having the same cycle structure as α also lies in H.*

Definition 20. *A G-set X is transitive if it has only one orbit; that is, for every $x, y, \in X$, there exists $\sigma \in G$ with $y = \sigma x$.*

- *If X is a G-set, then each of its orbits is a transitive G-set.*
- *If $H \leq G$, then G acts transitively on the set of all conjugates of H .*
- 1. *If $X = x_1, ..., x_n$ is transitive G-set and $H = G_{x1}$, then there are elements $g_1, ..., g_n$ in G with $g_i x_1 = x_i$ such that $g_1H, ..., g_nH$ are the distinct left cosets of H in G.*

2. *The stabilizer H acts on X, and the number of H-orbits of X is the number of (H-H)-double cosets in G.*

- *Let X be a G-set with action $\alpha : G \times X \to X$, and let $\alpha : G \to S_x$ send $g \in G$ into the permutation $x \mapsto gx$.*

  1. *If $K = ker\alpha$, then X is a (G/K)-set if one defines*

$$(gK)x = gx$$

  2. *If X is a transitive G-set, then X is a transitive (G/K)-set.*
  3. *If X is a transitive G-set, then $|ker\alpha| \leq |G|/|X|$. (Hint. If $x \in X$, then $|\theta(x)| = [G : G_x] \leq [G : ker\alpha]$.)*

1.3 Automorphism Groups

Definition 21. *The **automorphism group** of a group G; denoted by Aut(G), is the set of all the automorphisms of G under the operation of composition.*

Example : The identity e maps $G \to G$ is an automorphism.

Definition 22. *An automorphism ϕ of G is **inner** if it is conjugation by some element of G; otherwise it is **outer**. Denote the set of all inner automorphisms of G by Inn(G).*

Definition 23. *Let G be a group and let X be a set of generators of G. The **Cayley graph** $\Gamma = \gamma(G, X)$ is the directed graph with verticies the elements of G and with a directed edge from g to h if $h = gx$ for some $x \in X$*

If coset enumeration of a presentation $(X|\Delta)$ of a group G yields complete relation tables, then one can record the information in these tables as the Cayley graph $\gamma(G, X)$. See next chapters for more examples.

Definition 24. *Let G and H be groups. A group homomorphism from G to H is a function $f : G \to H$ such that $f(g_1g_2) = f(g_1)f(g_2)$ for all $g_1, g_2 \in G$.*
Note that it follows immediately from the definition that $f(e) = e$ and $f(g^{-1}) = f(g)^{-1}$
The kernel of a homomorphism $f : G \to H$ is the set

$$Kerf \overset{def}{=} \{g \in G | f(g) = e_H\}$$

The image of f is the set

$$Imf \overset{def}{=} \{h \in H | \exists g \in G \text{ such that } f(g) = h\}.$$

A (group) homomorphism is called a (group) isomorphism if it is objective. So, a homomorphism $f : G \to H$ is an isomorphism of groups.

Definition 25. *(First Isomorphism Theorem) Let G and H be groups and let $f : G \to H$ be a homomorphism. Then*

1. *the kernel of f,$Kerf$, is a normal subgroup of G,*
2. *the image of f, Imf, is a subgroup of H and*
3. *the quotient group $G/Kerf$ is isomorphic to Imf.*

Theorem 26. *(Second Isomorphism Theorem) Let G be a group, H a subgroup of G and K a normal subgroup of G. Then $HK \overset{def}{=} \{hk | h \in H, k \in K\}$ is a subgroup of G and K is a normal subgroup of HK. Furthermore, $H \cap K$ are isomorphic.*

Theorem 27. *(Third Isomorphism Theorem) Let G be a group and let H and K be normal subgroups such that $H \subseteq K$. Then the quotients $(G/H)/(K/H)$ and (G/K) are isomorphism.*

Definition 28. *Assume that G acts on a set Ω.*

The orbit of an element $\omega \in \Omega$ is the set

$$Orb(\omega) \overset{def}{=} \{\omega.g | g \in G\} \subseteq \Omega.$$

The stabilizer of an element $\omega \in \Omega$ is

$$Stab(\omega) \overset{def}{=} \{g \in G | \omega.g = \omega\} \subseteq G.$$

The stabilizer is a subgroup of G.
An action of a group G on a set Ω is said to be transitive if for any distinct elements $\alpha, \beta \in \Omega$, there exists $g \in G$ such that $\alpha.g = \beta$. Note that the action of G restricted to an orbit is transitive.

Definition 29. *Let G be a group and H a subgroup of G. We define the* ***normalizer*** *of H to be $N_G(H) = \{g \in G | Hg = gH\}$ (so H is normal in its normalizer).*

Definition 30. *Let G be a group. The **centralizer** of the element $h \in G$ is defined to be $C_G(h) \stackrel{def}{=} \{g \in G | hg = gh\}$.*

Both the normalizer and centralizer can be obtained via group actions.

1.4 The (Involuntary) Progenitor

Definition 31. *A progenitor is a semi-direct product of the following form:*

$$P \cong 2^{\star n} : N = \{\pi w | \pi \in N,\ w \text{ a reduced word in the } t_i\},$$

where $2^{\star n}$ denotes a free product of n copies of the cyclic group of order 2 generated by involutions t_i for $i = 1, ..., n$; and N is transitive permutation group of degree n which acts on the free product by permuting the involutory generators.

1.4.1 Free Products of Cyclic Groups of Order 2

We shall be considering a group generated by two elements of order 2, with no further relation holding between them. Thus,

$$G = \langle a, b \quad | \quad a^2 = b^2 = 1 \rangle.$$

Note that the element $x = ab$ has infinite order and, since $\langle ab, a \rangle = \rangle a, b \langle$, we have

$$G = \langle x, a \quad | \quad a^2 = 1, x^a = x^{-1} \rangle.$$

For this reason, we often refer to G *as an infinite dihedral group*; we may write its elements as follows:

$$G = \{1, a, b, ab, ba, aba, ...\},$$

where elements of odd length in a and b are involutions, whilst elements of even length have infinite order. Multiplication of elements of G is achieved by juxtaposition followed by cancellation of any adjacent repetitions, and inversion by reversing the word in a and b. It is intuitively clear from the symmetrical manner in which the group G was defined that interchanging a and b gives rise to an automorphism of G, and we shall verify this assertion in a more general context. We call G, which is generated by two cyclic subgroups of order 2 with no relation between them, a free product of these groups, and write

$$G \cong \langle a \rangle \star \langle b \rangle \cong C_2 * C_2.$$

For convenience, we denote this free product by $2^{\star^2}$.

We can readily extend these ideas to n generators and define a free product if n copies of the cyclic group of order 2 as follows:

$$\begin{aligned} E \cong 2^{\star n} &= \langle \tau_1, \tau_2, ..., \tau_n \mid \tau_1^2 = \tau_2^2 = ... = \tau_n^2 = 1 \rangle \\ &= \langle \tau_1 \rangle \star \langle \tau_2 \rangle \star ... \star \langle \tau_n \rangle \cong \underbrace{C_2 \star C_2 \star ... \star C_2.}_{n \quad times} \end{aligned}$$

So, E consists of all finite products of the elements π_i without adjacent repetitions.

Table 1.1: Examples of groups

Group	Description
C_n	cyclic group of order n, $C_n = \{e, x, x^2, x^3, ..., x^{n-1}\}$
V_4	group of symmetries of a rectangle (Klein's Vierergruppe)
$S_n(n > 2)$	symmetric group of degree n. = group of permutations of n symbols
$A_n(n > 4)$	alternating group of degree n group of even permutations of n symbols
A_f	group of even finitary permutations of $\mathbb{N}$ (σ is finitary if it moves only finitely many points)
D_{2n}	dihedral group of order 2 n = group of symmetries of a regular n-gon
D_∞	infinite dihedral group
$C_\infty = \mathbb{Z}^+$	infinite cyclic group = group of integers under addition
$C_2 \times C_3$	direct product of $C_2 and C_3$
$C_2 \times C_\infty$	direct product of C_2 and C_∞
$C_\infty \times C_\infty$	direct product of C_∞ with itself
Q	quaternion group (or order 8)
GL (n, k), k infinite field	group of $n \times n$ matrices with non-zero determinant, with entries in k
SL (n, k), k infinite field	group of $n \times n$ matrices with determinant 1, with entries in k
{isometries of cube }	
$\mathbb{Q}^+$	group of rational numbers under addition
$\mathbb{Q}^*$	group of non-zero real rational numbers under multiplication
$\mathbb{R}^+$	group of real numbers under addition
$\mathbb{R}^*$	group of non-zero real numbers under multiplication
$\mathbb{C}^+$	group of complex numbers under addition
$\mathbb{C}^*$	group of non-zero complex numbers under multiplication

Chapter 2

Construction of $3^3 : S_3$

2.1 Introduction.

We take the progenitor $G = 3^{*3} : S_3$, where G is the free product of three copies of the cyclic groups of order 3, and N is the group of automorphisms of 3^3 which permutes the three symmetric generators by conjugation and factored it by $t_0t_1 = t_1t_0$.

The double coset enumeration partitions the image of the group G as a union double coset NgN where $g \in 3^3 : N$.

Thus, we can find the set $\{g_1, g_2, ...\}$ of elements of G such that $G = Ng_1N \bigcup Ng_2N \bigcup ...$, and for each i, we have $g_i = p_iw_i$, where $p_i \in N$, and w_i is a word in the t_i's. Hence, the double coset decomposition is given by

$G = Nw_1N \bigcup Nw_2N \bigcup Nw_3N \bigcup .....$

Where $w_1 = e$ (identity).

We perform a double coset enumeration on the group $3^3 : S_3$. Note that the order of each of $3t_i$'s is 3. So, $t^3 = e$ and hence $t^2 = t^{-1}$. The symmetric group representation is given by

$$G \cong \frac{3^{*3}:S_3}{t_0t_1=t_1t_0}.$$

and the symmetric presentation of the progenitor $3^3 : S_3$ is given by:

$$3^3 : S_3 \cong \langle x, y, t | x^3 = y^2 = (xy)^2 = t^3, (t, y), tt^x = t^xt\rangle.$$

Where,

$x \sim (0, 1, 2)(\overline{0}, \overline{1}, \overline{2})$;

$y \sim (0,1)(\overline{0},\overline{1})$.

$xy = (1,2)(\overline{1},\overline{2})$;

The control subgroup is N , is S_3 which is the symmetric group of 3 on three letters and their inverses. $\Rightarrow |N| = 3! = 6$.

$S_3 =< x, y >= \langle (0,1,2)(\overline{0},\overline{1},\overline{2}), (0,1)(\overline{0},\overline{1})\rangle$.

Hence, the set

$N = \{Id, (0,1,2)(\overline{0},\overline{1},\overline{2}), (0,2,1)(\overline{0},\overline{2},\overline{1}), (0,1)(\overline{0},\overline{1}), (0,2)(\overline{0},\overline{2}), (1,2)(\overline{1},\overline{2})\}$.

Using computer-based program - Magma :

1. the order of the group, $|G|$ is at the most equal to 162.
2. there should be 10 double coset in this double coset enumeration of G over N.

2.2 Relations

The factored relation above is $tt^x = t^x t$ where $x \sim (0,1,2)(\overline{0},\overline{1},\overline{2})$.

$t \sim t_0$

$\Rightarrow t_0 t_0^{(0,1,2)(\overline{0},\overline{1},\overline{2})} = t_0^{(0,1,2)(\overline{0},\overline{1},\overline{2})} t_0$.

Therefore, our basic relation will be:

$$t_0 t_1 = t_1 t_0$$

The above relation can be conjugated by each element of the set N,

$\Rightarrow$ we obtain additional and they are :

$t_0 t_1 = t_1 t_0 \qquad \overline{t_0 t_1} = \overline{t_1 t_0}$.

$t_0 t_2 = t_2 t_0 \qquad \overline{t_1 t_2} = t_{\overline{2}} \overline{t_1}$.

$t_1 t_2 = t_2 t_1 \qquad \overline{t_2 t_0} = \overline{t_0 t_2}$.

Also note, to add more relations, we have $t_0 t_1 = t_1 t_0$, so post multiplying both sides by t_1 we get :

$t_0 t_1 \cdot t_1 = t_1 t_0 \cdot t_1$.

$\Rightarrow t_0 t_1^2 = t_1 \underline{t_0 t_1} = t_1 t_1 t_0 = t_1^2 t_0$.

$\Rightarrow t_0 \overline{t_1} = \overline{t_1} t_0$.

$\Rightarrow$ we have $\quad 0\overline{1} = \overline{1}0$. (for simplicity, we omit t sometimes)

Again, we conjugate by the elements of S_3, we get :

$t_0 \overline{t_1} = \overline{t_1} t_0$.

$t_0\overline{t_2} = \overline{t_2}t_0.$
$t_1\overline{t_0} = \overline{t_0}t_1.$
$t_1\overline{t_2} = \overline{t_2}t_1.$
$t_2\overline{t_0} = \overline{t_0}t_2.$
$t_2\overline{t_1} = \overline{t_1}t_2.$

2.3 Double Coset Enumeration of G over S_3

2.3.1 Double Coset [$\star$]:

Note that NeN represents words of length zero, this denote the double coset [$\star$] and it can be represented as:

$$\begin{aligned} NeN &= \{Ne^n : n \in N\} \\ &= \{Ne\} \\ &= \{N\} \end{aligned}$$

We take a representative coset N from [$\star$] and a representative from $\{0, 1, 2, \overline{0}, \overline{1}, \overline{2}\}$, then we need to determine to which double coset Nt_0 and $N\overline{t_0}$ belong.
Hence, [0] and [$\overline{0}$] will be new double cosets that will extend Cayley's graph.

2.3.2 Double Coset [0] and [$\overline{0}$]

The point stabilizer and coset stabilizer for Nt_0, $Nt^{(0)}$ and $N\overline{t_0}$, $N\overline{t^{(0)}}$ are :

- $N^0 = \{Id, (1,2)(\overline{1},\overline{2})\}$, and $N^{(0)} =< (1,2)(\overline{1},\overline{2}) >$, then $N^0 = N^{(0)}$.
- $N^{\overline{0}} = \{Id, (1,2)(\overline{1},\overline{2})\}$, and $N^{(\overline{0})} =< (1,2)(\overline{1},\overline{2}) >$, then $N^{\overline{0}} = N^{(\overline{0})}$.

Therefore, each double coset [0] and [$\overline{0}$] will have 3 distinct equal sets of single cosets as $\frac{|N|}{|N^{(0)}|} = \frac{6}{2} = 3$ and $\frac{|N|}{|N^{(\overline{0})}|} = \frac{6}{2} = 3$.
The orbit of $N^{(0)}$ on $\{0, 1, 2, \overline{0}, \overline{1}, \overline{2}\}$ are $\{0\}$,$\{1, 2\}$, $\{\overline{0}\}$, $\{\overline{1}, \overline{2}\}$.
The orbit of $N^{(\overline{0})}$ on $\{0, 1, 2, \overline{0}, \overline{1}, \overline{2}\}$ are $\{0\}$, $\{1, 2\}$, $\{\overline{0}\}$, $\{\overline{1}, \overline{2}\}$.
Now, we take a representative from each orbit of $N(0)$:

1. $Nt_0t_0 = Nt_0{}^2 = N\bar{t_0} \in [\bar{0}]$.
   Only t_0 will take [0] to $[\bar{0}]$

2. $Nt_0t_1 \in [0,1]$.
   New double coset that will extend the Cayley's graph and hence two t_i's will take [0] to [01].

3. $Nt_0\bar{t_0} = Ne \in [\star]$.
   One t will take [0] to $[\star]$.

4. $Nt_0\bar{t_1} \in [0,\bar{1}]$.
   New double coset that will extend Cayley's graph and hence two t_i's will take [0] to $[0,\bar{1}]$

For $[\bar{0}]$, we take a representative from each orbit of $N(\bar{0})$:

1. $N\bar{t_0}t_0 = Ne \in [\star]$.
   One t will take $[\bar{0}]$ to $[\star]$

2. $N\bar{t_0}t_1 \in [\bar{0},1] \in [0,\bar{1}]$.
   New double coset that will extend the Cayley's graph and hence two t_i's will take [0] to $[0\bar{1}]$.

3. $N\bar{t_0}\bar{t_0} = N\bar{t_0}^2 = Nt_0 \in [0]$.
   One t will take $[\bar{0}]$ to [0]

4. $N\bar{t_0}\bar{t_1} \in [\bar{0},\bar{1}]$.
   New double coset that will extend Cayley's graph, hence two t_i's will take $[\bar{0}]$ to $[\overline{01}]$.

See figure 1

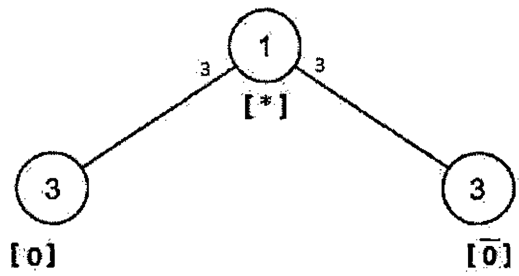

Figure 2.1: Cayley's graph for the double cosets [0] and [$\bar{0}$]

2.3.3 Double Coset [0, 1]

Now at the double coset Nt_0t_1N, [0, 1], we have $N^{01} =< Id >$ and since our relation is
$t_0t_1 = t_1t_0$
$\Rightarrow Nt_0t_1(01)(\overline{01}) = Nt_1t_0 = Nt_0t_1$.
$\Rightarrow (01)(\overline{01}) \in N^{(01)}$,
$\Rightarrow N^{(01)} \geq < Id(N), (01)(\overline{01}) >= S_2$.
Hence, $\frac{|N|}{|N^{(01)}|} = \frac{6}{2} = 3$.
Orbits of $N^{(01)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0, 1\}, \{2\}, \{\bar{0}, \bar{1}\}, \{\bar{2}\}$. Again, we take a representative from each orbit,

- $Nt_0t_1t_1 = Nt_0\overline{t_1} \in [0, \bar{1}]$.
  Since this orbit has two elements, then two t_i's will take [01] to [$0\bar{1}$]

- $Nt_0t_1t_2 \in [0, 1, 2]$.
  New double coset will extend [01] to [012].

- $Nt_0t\overline{t_1} = Nt_0 \in [0]$.
  Since this orbit has two elements, then two t_i's will take [01] to [0].

- $Nt_0\overline{t_2} \in [0, 1, \bar{2}]$.
  New double coset will extend [01] to [$01\bar{2}$].

2.3.4 Double Coset $[0, \bar{1}]$

The double coset $[0, \bar{1}]$, $Nt_0t_{\bar{1}}N$, fixing 0 and $\bar{1}$ implies $\bar{0}$ and 1 are fixed.
Now, we have $N^{0\bar{1}} =< Id >$,
note: $N^{(0\bar{1})} = N^{0\bar{1}}$
Hence, $\frac{|N|}{|N^{(0\bar{1})}|} = \frac{6}{1} = 6$.
Orbits of $N^{(0\bar{1})}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}, \{1\}, \{2\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}$.
We take a representative from each orbit,

- $Nt_0\overline{t_1}t_0 \in [\overline{01}]$
  (Because $\underline{0\bar{1}}0 = \bar{1}00 = \overline{10}$).So t_0 will take $[0\bar{1}]$ to $[\overline{01}]$.

- $Nt_0\overline{t_1}t_1 \in [0]$.
  t_1 will take $[0\bar{1}]$ to $[0]$.

- $Nt_0\overline{t_1}t_2 \in [0, 1, \bar{2}]$.
  One t will take $[0\bar{1}]$ to $[01\bar{2}]$.

- $Nt_0\overline{t_1t_0} \in [\bar{0}]$.
  One t will take $[0\bar{1}]$ to $[\bar{0}]$.

- $Nt_0\overline{t_1t_1} \in [0, 1]$.
  One t will take $[0\bar{1}]$ to $[01]$.

- $Nt_0\overline{t_1t_2} \in [0, \bar{1}, \bar{2}]$.
  New double coset, one t will take $[0\bar{1}]$ to $[0\overline{12}]$.

2.3.5 Double Coset $[\bar{0}, \bar{1}]$

Since, we have $\bar{0}$ and $\bar{1}$ being fixed and permute only $\bar{2}$.
At the same time this means that 0 and 1 also fixed and only 2 can permute,
$\Rightarrow N^{\overline{01}} =< Id >$.
However, we use our basic relation to find more relations that can be added to the stabilising group, we have $t_0t_1 = t_1t_0$, and hence, $\overline{t_0t_1} = \overline{t_1t_0}$, (see previous pages)
$\Rightarrow N\overline{t_0t_1}(01)(\overline{01}) = Nt_{\bar{1}}t_{\bar{0}} = N\overline{t_0t_1}$ (from above).
$\Rightarrow N^{(\overline{01})} \geq < Id, (0, 1)(\bar{0}, \bar{1}) >= S_2$.
$\Rightarrow |N^{(\overline{01})}| = |S_2|$.

Hence, number of single cosets in $[\overline{0},\overline{1}]$ is $\frac{|N|}{|N^{(\overline{01})}|} = \frac{6}{2} = 3$.
Orbits of $N^{(\overline{01})}$ on $\{0,1,2,\overline{0},\overline{1},\overline{2}\}$ are $\{0,1\},\{2\},\{\overline{0},\overline{1}\},\{\overline{2}\}$.
We take a representative from each orbit,

- $N\overline{t_0t_1}t_1 \in [\overline{0}]$,
  (because $\underline{0\overline{1}0} = \overline{1}00 = \overline{10}$).
  Two t_i's will take $[\overline{0},\overline{1}]$ to $[\overline{0}]$.

- $N\overline{t_0t_1}t_2 \in [0,\overline{1},\overline{2}]$.
  One t_i will take $[\overline{0},\overline{1}]$ to $[0,\overline{1},\overline{2}]$.

- $N\overline{t_0t_1}\overline{t_1} \in [0,1]$.
  Two t_i's will take $[\overline{0},\overline{1}]$ to $[0,1]$.

- $N\overline{t_0t_1}\overline{t_2} \in [\overline{0},\overline{1},\overline{2}]$.
  One t_i will take $[\overline{0},\overline{1}]$ to $[\overline{0},\overline{1},\overline{2}]$ which is new double coset.

So far we have the following new double cosets that can be seen in below figure:
$[0,1,2]$, $[0,1,\overline{2}]$, $[0,\overline{1},\overline{2}]$, $[\overline{0},\overline{1},\overline{2}]$.

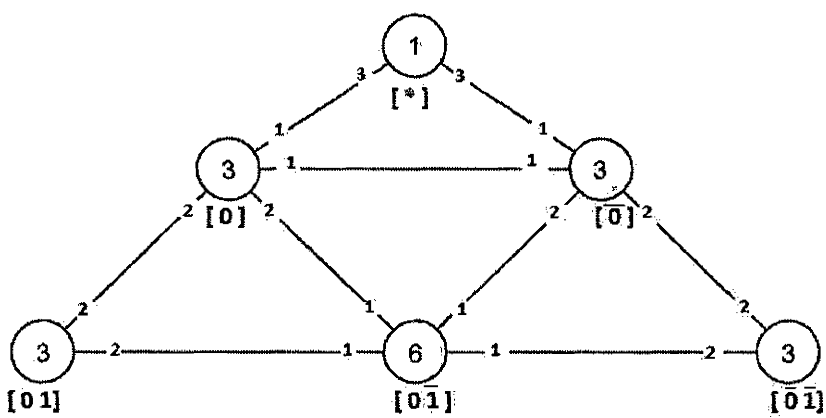

Figure 2.2: Cayley's graph for the extension of double cosets $[0]$ and $[\overline{0}]$

2.3.6 Double Coset $[0, 1, 2]$

We stabilise three elements, $0, 1$, and 2 in N, hence $N^{012} =< e >$.
Now to add more relations to the stabilising set $\{e\}$ we use our defined relation $t_0t_1 = t_1t_0$. By post multiply both sides by t_2.

$\Rightarrow t_0t_1 \cdot t_2 = t_1t_0 \cdot t_2$.

Thus,

$Nt_0t_1t_2 = Nt_1t_0t_2$

$\Rightarrow \quad Nt_0t_1t_2^{(0,1)(\overline{0},\overline{1})} = Nt_1t_0t_2 = Nt_0t_1t_2$.

$\Rightarrow \quad N^{(012)} \geq \langle Id, (0,1)(\overline{0},\overline{1})\rangle$.

$\Rightarrow \quad (0,1)(\overline{0},\overline{1}) \in N^{(012)}$.

Note: we can search for more relations using $012 = 102$

$\Rightarrow 0\underline{12} = \underline{02}1 = 201 \quad \Rightarrow \quad (0,2,1)(\overline{0},\overline{2},\overline{1}) \in N^{(012)}$.

$\Rightarrow N^{(012)} \geq \langle Id, (0,1)(\overline{0},\overline{1}), (0,2,1)(\overline{0},\overline{2},\overline{1})\rangle = S_3$.

Hence, the number of single equal cosets are :

$\frac{|N|}{|N^{(012)}|} = \frac{6}{6} = 1$.

The Orbits of $N^{(012)}$ on $\{0, 1, 2, \overline{0}, \overline{1}, \overline{2}\}$ are $\{0, 1, 2\}, \{\overline{0}, \overline{1}, \overline{2}\}$.

Thus, by taking a representative from each orbit of $N^{(012)}$ and multiply it by $Nt_0t_1t_2$:

- $Nt_0t_1t_2 \cdot t_2 \in [0, 1, \overline{2}]$,

  This is a new double coset that will extend Cayley's graph, and in fact three t_i's will extend [012] to $[01\overline{2}]$.

- $Nt_0t_1t_2 \cdot \overline{t_2} \in [0, 1]$.

  This orbit representative will make Cayley's graph collapse and hence three t_i's will take it back to [01].

2.3.7 Double Coset $[0, 1, \overline{2}]$

At the double coset $Nt_0t_1\overline{t_2}N$, three elements are stabilised 0,1, and $\overline{2}$.
Consequently, $\overline{0}, \overline{1}$, and 2 are also stabilised,

$\Rightarrow N^{01\overline{2}} =< e >$, we use our basic relation $01 = 10$, by post multiply both sides by $\overline{2}$

$\Rightarrow 01\overline{2} = 10\overline{2}$,

i,e $Nt_0t_1\overline{t_2} = Nt_1t_0\overline{t_2}$.

Then, $Nt_0t_1\overline{t_2}(0,1)(\overline{0},\overline{1}) = Nt_1t_0\overline{t_2} = Nt_0t_1\overline{t_2}$.
$\Rightarrow (01)(\overline{01}) \in N^{(01\overline{2})}$.
$\Rightarrow N^{(01\overline{2})} \geq < e, (01)(\overline{01}) >= S_2$.
Thus, the number of single cosets in the double cosets$[0,1,\overline{2}]$ is $\frac{|N|}{|N^{(01\overline{2})}|} = \frac{6}{2} = 3$.
The orbits of $N^{(01\overline{2})}$ on $\{0,1,2,\overline{0},\overline{1},\overline{2}\}$ are $\{0,1\},\{2\},\{\overline{0},\overline{1}\},\{\overline{2}\}$.We take a representative from each orbit

- $Nt_0t_1\overline{t_2}t_1 \in [0,\overline{1},\overline{2}]$.
  Two t_i's will take $[0,1,\overline{2}]$ to $[0,\overline{1},\overline{2}]$.

- $Nt_0t_1\overline{t_2}t_2 \in [0,1]$.
  One t_i will make Cayley's graph collapse, and hence take $[0,1,\overline{2}]$ to $[0,1]$.

- $Nt_0t_1\overline{t_2t_1} \in [0\overline{1}]$.
  Two t_i's will take $[0,1,\overline{2}]$ to $[0,\overline{1}]$.

- $Nt_0t_1\overline{t_2t_2} \in [012]$.
  One t_i will take $[0,1,\overline{2}]$ to $[0,1,2]$.

2.3.8 Double Coset $[0,\overline{1},\overline{2}]$

At this double coset,$Nt_0\overline{t_1t_2}N$, we stabilise $0,\overline{1}$ and $\overline{2}$ then$\overline{0},1$ and 2 are also stabilised.
$\Rightarrow N^{0\overline{12}} =< e >$.
We use our basic relation that is which is $0\overline{1} = \overline{1}0$
by post multiply both sides by $\overline{2}$ then
$\Rightarrow 0\overline{1}\overline{2} = \overline{1}0\overline{2}$.
$\Rightarrow 0\overline{12} = \overline{1}0\overline{2}$.
$\Rightarrow 0\overline{21} = \overline{1}0\overline{2} = 0\overline{12}$.
$\Rightarrow Nt_0\overline{t_2t_1}(1,2)(\overline{1},\overline{2}) = Nt_0\overline{t_1t_2} = Nt_0\overline{t_2t_1}$.
$\Rightarrow (12)(\overline{12}) \in N^{(0\overline{12})}$.
Hence, $N^{(0\overline{12})} \geq < e, (12)(\overline{12}) >= S_2$.
$\Rightarrow$ number of single cosets are in the double coset $[0,\overline{1},\overline{2}]$ is $\frac{|N|}{|N^{(0\overline{12})}|} = \frac{6}{2} = 3$.
The orbits of $N^{(0\overline{12})}$ on $\{0,1,2,\overline{0},\overline{1},\overline{2}\}$ are $\{1,2\},\{0\},\{\overline{1},\overline{2}\},\{\overline{0}\}$. We take a representative from each orbit:

- $Nt_0\overline{t_1t_2}t_2 \in [0\overline{1}]$.
  Two t_i will take $[0, \overline{1}, \overline{2}]$ to $[0, \overline{1}]$.

- $Nt_0\overline{t_1t_2t_2} \in [01\overline{2}]$.
  Two t_i will take $[0, \overline{1}, \overline{2}]$ to $[0, 1, \overline{2}]$.

- $Nt_0\overline{t_1t_2}t_0 \in [\overline{0}, \overline{1}, \overline{2}]$.
  One t_i will take $[0, \overline{1}, \overline{2}]$ to $[0, \overline{1}]$.

- $Nt_0\overline{t_1t_2t_0} \in [\overline{0}, \overline{1}]$.
  One t_i will take $[0, \overline{1}, \overline{2}]$ to $[0, 1, \overline{2}]$.

2.3.9 Double Coset $[\overline{0}, \overline{1}, \overline{2}]$

In the double coset $N\overline{t_0t_1t_2}N$, we fix $\overline{0}, \overline{1}$ and $\overline{2}$. This implies that $1, 2$ and 3 are fixed.

$\Rightarrow N^{\overline{012}} =< e >$,

Since our basic relation is $\overline{01} = \overline{10} \Rightarrow \overline{012} = \overline{102}$

$\Rightarrow N\overline{t_0t_1t_2}(01)(\overline{01}) = N\overline{t_1t_0t_2} = N\overline{t_0t_1t_2}$.

$\Rightarrow (01)(\overline{01}) \in N^{(\overline{012})}$.

$\Rightarrow N^{(\overline{012})} \geq< e, (01)(\overline{01}) >$.

Also note,

$\overline{012} = \overline{102} = \overline{120}$.

$\Rightarrow N^{(\overline{012})} \geq< e, (01)(\overline{01}), (012)(\overline{012}) >= S_3$.

$\Rightarrow$ Number of single cosets exit in the double coset $[\overline{0}, \overline{1}, \overline{2}] = \frac{|N|}{|N^{(\overline{012})}|} = \frac{6}{6} = 1$.

The orbits of $N^{(\overline{012})}$ on $\{0, 1, 2, \overline{0}, \overline{1}, \overline{2}\}$ are $\{0, 1, 2\}, \{\overline{0}, \overline{1}, \overline{2}\}$. We take a representative from each orbit and multiply it by $N\overline{t_0t_1t_2}$, the result is:

- $N\overline{t_0t_1t_2}t_2 \in [\overline{01}]$.
  Three t_i's will make the Cayley's graph collapse.

- $N\overline{t_0t_1t_2t_2} \in [0\overline{12}]$.
  Three t_i's will make the Cayley's graph collapse.

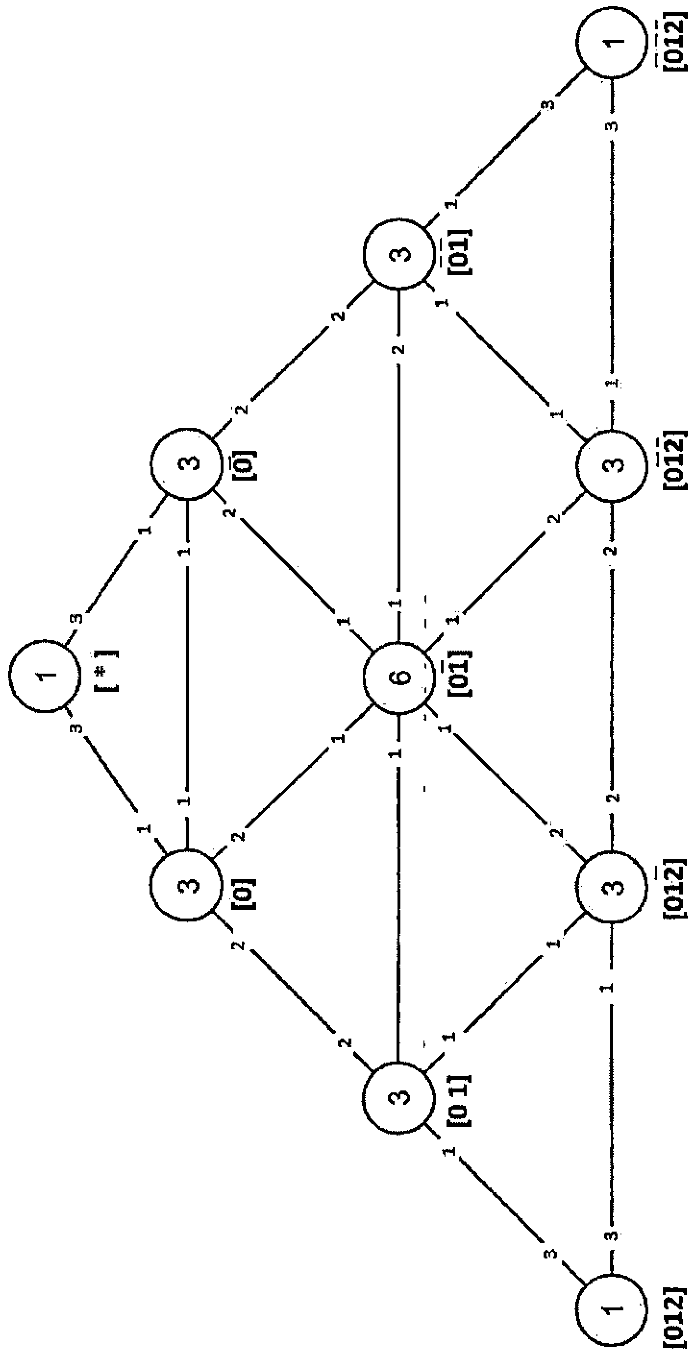

Figure 2.3: Cayley's graph for $3^3 : S_3$

2.3.10 Permutation Labeling

From the above table, we have the following permutations:

$$xx \sim (2,3,4)(5,6,7)(8,10,9)(11,14,15)(13,16,12)(17,19,18)(21,22,23)(24,26,25).$$

$$yy \sim (2,3)(5,6)(9,10)(11,13)(12,14)(15,16)((18,19)(22,23)(24,26).$$

$$tt_0 \sim$$

$$(1,2,5)(3,8,13)(4,9,15)(6,11,17)(7,12,18)(10,20,22)(14,21,25)(16,23,26)(19,24,27).$$

$$xy \sim (3,4)(6,7)(8,9)(11,12)(13,15)(14,16)(17,18)(21,23)(25,26).$$

We define $\widehat{\alpha}$ such that $\widehat{\alpha} : G \to S_{27}$.

Hence,

$\widehat{\alpha}(x) = (2,3,4)(5,6,7)(8,10,9)(11,14,15)(13,16,12)(17,19,18)(21,22,23)(24,26,25)$.

$\widehat{\alpha}(y) = (2,3)(5,6)(9,10)(11,13)(12,14)(15,16)((18,19)(22,23)(24,26)$.

$\widehat{\alpha}(t_0) = (1,2,5)(3,8,13)(4,9,15)(6,11,17)(7,12,18)(10,20,22)(14,21,25)$
$(16,23,26)(19,24,27)$.

To check for homomorphism, we verify the following conditions:

1. From above
   $|xx| = 3, \quad |yy| = 2, \quad |xy| = 2,$
   $G = \langle xx, yy \rangle \cong \langle x, y | x^3, y^2, (xy)^2 \rangle \cong S_3$

2. conjugation of t by $< xx, yy >$ would produce the other $\{t_0, t_1, t_2\}$.
   $tt_0 = (1,2,5)(3,8,13)(4,9,15)(6,11,17)(7,12,18)(10,20,22)$
   $(14,21,25)(16,23,26)(19,24,27)$.
   $tt_0^{xx} = (1,3,6)(4,10,16)(2,8,11)(7,14,19)(5,13,17)(9,20,23)(15,22,24)$
   $(12,21,25)(18,26,27) = tt_1$.
   $tt_1^{xx} = (1,4,7)(2,9,12)(3,10,14)(5,15,18)(6,16,19)(8,20,21)(11,23,26)$
   $(13,22,24)(17,25,27) = tt_2$.

3. conjugation of tt_0, tt_1, tt_2 by xx produces tt_0, tt_1, tt_2 ,
   and conjugation of tt_0, tt_1, tt_2 by yy produces tt_0, tt_1, tt_2.
   $tt_0^{xx} = (1,3,6)(4,10,16)(2,8,11)(7,14,19)(5,13,17)(9,20,23)(15,22,24)$
   $(12,21,25)(18,26,27) = tt_1$.
   $tt_1^{xx} = (1,4,7)(2,9,12)(3,10,14)(5,15,18)(6,16,19)(8,20,21)(11,23,26)$
   $(13,22,24)(17,25,27) = tt_2$.

Table 2.1: Labeling table

Cosets	$x = (0,1,2)(\bar{0},\bar{1},\bar{2})$	$y = (0,1)(\bar{0},\bar{1})$	t_0
(1) N	(1) N	(1) Nt_0	(2) Nt_0
(2) Nt_0	(3) Nt_1	(3) Nt_1	(5) $Nt_{\bar{0}}$
(3) Nt_1	(4) Nt_2	(2) Nt_0	(8) Nt_1t_0
(4) Nt_2	(2) Nt_0	(4) Nt_2	(9) Nt_2t_0
(5) $Nt_{\bar{0}}$	(6) $Nt_{\bar{1}}$	(6) $Nt_{\bar{0}}$	(1) N
(6) $Nt_{\bar{1}}$	(7) $Nt_{\bar{2}}$	(5) $Nt_{\bar{0}}$	(11) $Nt_{\bar{1}}t_0$
(7) $Nt_{\bar{2}}$	(5) $Nt_{\bar{0}}$	(7) $N_{\bar{2}}$	(12) $Nt_{\bar{2}}t_0$
(8) Nt_0t_1	(10) Nt_1t_2	(1) N	(13) $Nt_1t_{\bar{0}}$
(9) Nt_0t_2	(8) Nt_0t_1	(10) Nt_1t_2	(15) $Nt_{\bar{0}}t_2$
(10) Nt_1t_2	(9) Nt_2t_0	(9) N_0t_2	(20) $Nt_1t_2t_0$
(11) $Nt_0t_{\bar{1}}$	(14) $Nt_1t_{\bar{2}}$	(13) $Nt_1t_{\bar{0}}$	(17) $Nt_{\bar{0}}t_{\bar{1}}$
(12) $Nt_0t_{\bar{2}}$	(13) $Nt_1t_{\bar{0}}$	(14) $Nt_1t_{\bar{2}}$	(18) $Nt_{\bar{0}}t_{\bar{2}}$
(13) $Nt_1t_{\bar{0}}$	(16) $Nt_2t_{\bar{1}}$	(11) $Nt_0t_{\bar{1}}$	(3) Nt_1
(14) $Nt_1t_{\bar{2}}$	(15) $Nt_2t_{\bar{0}}$	(12) $Nt_0t_{\bar{2}}$	(21) $Nt_0t_1t_{\bar{2}}$
(15) $Nt_2t_{\bar{0}}$	(11) $Nt_0t_{\bar{1}}$	(16) $Nt_2t_{\bar{1}}$	(4) Nt_2
(16) $Nt_2t_{\bar{1}}$	(12) $Nt_0t_{\bar{2}}$	(15) $N_{\bar{1}}t_{\bar{0}}$	(23) $Nt_0t_2t_{\bar{1}}$
(17) $Nt_{\bar{0}}t_{\bar{1}}$	(19) $Nt_{\bar{1}}t_{\bar{2}}$	(17) $Nt_{\bar{1}}t_{\bar{0}}$	(6) $Nt_{\bar{1}}$
(18) $Nt_{\bar{0}}t_{\bar{2}}$	(17) $Nt_{\bar{1}}t_{\bar{0}}$	(19) $Nt_{\bar{1}}t_{\bar{2}}$	(7) $Nt_{\bar{2}}$
(19) $Nt_{\bar{1}}t_{\bar{2}}$	(18) $Nt_{\bar{2}}t_{\bar{0}}$	(18) $Nt_{\bar{0}}t_{\bar{2}}$	(24) $Nt_0t_{\bar{1}}t_{\bar{2}}$
(20) $Nt_0t_1t_2$	(20) $Nt_1t_2t_0$	(20) $Nt_1t_0t_2$	(22) $Nt_2t_1t_{\bar{0}}$
(21) $Nt_0t_1t_{\bar{2}}$	(22) $Nt_1t_2t_{\bar{0}}$	(21) $Nt_1t_0t_{\bar{2}}$	(25) $Nt_1t_{\bar{0}}t_{\bar{2}}$
(22) $Nt_2t_1t_{\bar{0}}$	(23) $Nt_0t_2t_{\bar{1}}$	(23) $N_2t_0t_{\bar{1}}$	(10) Nt_1t_2
(23) $Nt_0t_2t_{\bar{1}}$	(21) $Nt_1t_0t_{\bar{2}}$	(22) $N_1t_2t_{\bar{0}}$	(26) $Nt_{\bar{0}}t_2t_{\bar{1}}$
(24) $Nt_0t_{\bar{1}}t_{\bar{2}}$	(26) $Nt_1t_{\bar{2}}t_{\bar{0}}$	(26) $N_1t_{\bar{0}}t_{\bar{2}}$	(27) $Nt_{\bar{0}}t_{\bar{1}}t_{\bar{2}}$
(25) $Nt_2t_{\bar{0}}t_{\bar{1}}$	(24) $Nt_0t_{\bar{1}}t_{\bar{2}}$	(25) $Nt_2t_{\bar{1}}t_{\bar{0}}$	(14) $Nt_1t_{\bar{2}}$
(26) $Nt_1t_{\bar{0}}t_{\bar{2}}$	(25) $Nt_2t_{\bar{1}}t_{\bar{0}}$	(24) $N_0t_{\bar{1}}t_{\bar{2}}$	(16) $Nt_2t_{\bar{1}}$
(27) $Nt_{\bar{0}}t_{\bar{1}}t_{\bar{2}}$	(27) $Nt_{\bar{1}}t_{\bar{2}}t_{\bar{0}}$	(27) $Nt_{\bar{1}}t_{\bar{0}}t_{\bar{2}}$	(19) $Nt_{\bar{1}}t_{\bar{2}}$

$tt_2^{xx} = (1,2,5)(3,8,13)(4,9,15)(6,11,17)(7,12,18)(10,20,22)(14,21,25)$
$(16,23,26)(19,24,27) = tt_0.$
$tt_0^{yy} = (1,3,6)(4,10,16)(2,8,11)(7,14,19)(5,13,17)(9,20,23)(15,22,24)$
$(12,21,25)(18,26,27) = tt_1.$
$tt_1^{yy} = (1,2,5)(3,8,13)(4,9,15)(6,11,17)(7,12,18)(10,20,22)(14,21,25)$
$(16,23,26)(19,24,27) = tt_0.$
$tt_2^{yy} = (1,4,7)(2,9,12)(3,10,14)(5,15,18)(6,16,19)(8,20,21)(11,23,26)$
$(13,22,24)(17,25,27) = tt_2.$

Since the above conditions are satisfied, then $< xx, yy, tt_0 >$ is a homomorphic image to G.

2.4 Conclusion.

We have $f(G) =< xx, yy, tt_0 >$.
By Fundamental Theorem,
$G/Kerf \cong f(G)$.
$\Rightarrow G/kerf \cong< xx, yy, tt_0 >$.
$\Rightarrow |G|/|kerf| = | < xx, yy, tt_0 > |$.
$|G| = |kerf| \cdot | < xx, yy, tt_0 > |$.
$|G| = |kerf| \cdot (162) \Rightarrow |G| \geq 162$.
But we have:
$|G| \leq (1+3+3+3+6+3+1+3+3+1)|N| = (27) \cdot 6 = 162$.
So,
$|G| \leq 162$ (from Cayley's graph) and $Kerf = 1$
Therefore: $|G| = 162$.
Figure 3 shows final look of Cayley's graph.

```
> G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,t^3,(t,x*y),
  (t*t^x) = t^x*t>;
> #G;
162
> #DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
10
> N:=Sym(6);
> xx:=N!(1,2,3)(4,5,6);
> yy:=N!(3,1)(4,6);
> N:=sub<N|xx,yy>;
> Index(G,sub<G|x,y>);
27
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> IN:=sub<G1|f(x),f(y)>;
> CompositionFactors(G1);
   G
   |  Cyclic(3)
   *
   |  Cyclic(2)
   *
   |  Cyclic(3)
   *
   |  Cyclic(3)
   *
   |  Cyclic(3)
   1
```

Figure 2.4: Computer-based proof for Composition Factors of G

Chapter 3

Construction of $2^5 : A_5$

In this chapter, we will prove by hand that the group G given by the control subgroup is A_5, given by :

$$G = \frac{2^5 : A_5}{t_2 t_3 = t_3 t_2},$$

is isomorphism to $2^5 : A_5$.

We perform a double coset enumeration of G over N. We call N the control subgroup, ($N = A_5$), which is the symmetric group of degree 5 on six letters, $\{0, 1, 2, 3, 4, 5\}$. N can be generated by x and y, $N =< x, y >$ where $x \sim (1, 2, 3, 4, 5)$ and $y \sim (0, 5, 1)(2, 4, 3)$. The double coset enumeration will enable us find all double cosets $[w]$ that can be represented by:

$$\begin{aligned} NwN &= \{Nwn | n \in N\} \\ &= \left\{Nnn^{-1}wn | n \in N\right\} \\ &= \{Nw^n | n \in N\}. \end{aligned}$$

Now, when all the double cosets of the group are determined, we will be able to determine the number of single cosets of N in G, and hence the process terminates when the set of right cosets is closed under right multiplication by the 6 t_i's where $i \in \{0, 1, 2, 3, 4, 5\}$. The symmetric presentation of G is given by:

$$G\langle x, y, t\rangle = \text{Group}\langle x, y, t | x^5, y^3, (xy)^2, t^2, (t, x), \left(t^{yx^2}, (xy)\right), [xy(t^y)^{x^2}(t^y)^{x^3}]^2\rangle,$$

3.1 Double Coset Enumeration of G over A_5

3.1.1 Relations

- Let $t \sim t_0$ and take $x \sim (1,2,3,4,5)$, $y \sim (0,5,1)(2,4,3)$,
- $t^y = (t_0)^{(0,5,1)(2,4,3)} = t_5$
- Let
  $x = (5,1,2,3,4)$,
  $\Rightarrow x^2 = (5,1,2,3,4)^2 = (5,2,4,1,3)$
  $\Rightarrow x^3 = (5,1,2,3,4)^3 = (5,3,1,4,2)$.
- $(t_0^y)^{x^2} = (t_0^{(0,5,1)(2,4,3)})^{(5,1,2,3,4)^2} = (t_5)^{(5,2,4,1,3)} = t_2$
- $x \cdot y = (5,1,2,3,4)(0,5,1)(2,4,3) = (5,0)(1,4)$
- $(t^y)^{x^3} = (t_5)^{(5,1,2,3,4)^3} = (t_5)^{(5,3,1,4,2)} = t_3$ We have the following relation:

$$\begin{aligned} t_2t_3t_2t_3 &= e \\ t_2t_3t_2t_3 \cdot t_3 &= t_3 \\ t_2t_3t_2 &= t_3 \\ t_2t_3t_2 \cdot t_2 &= t_3t_2 \\ t_2t_3 &= t_3t_2. \end{aligned}$$

- $N = A_5 = \langle x, y \rangle = \langle (1,2,3,4,5), (0,5,1)(2,4,3) \rangle$.
- $|N| = \frac{|S_5|}{2} = \frac{120}{2} = 60$.

3.1.2 Word of Length Zero [$\star$]

We denote the double-coset by [$\star$] and it can be represented as:

$$\begin{aligned} NeN &= \{Ne^n : n \in N\} \\ &= \{Ne\} \\ &= \{N\} \end{aligned}$$

The double coset [$\star$] will have a single coset N. Therefore, $\frac{|N|}{|N|} = \frac{60}{60} = 1$.
Note, since N is transitive on $\{0,1,2,3,4,5\}$, we take a representative coset N from [$\star$]

and a representative from $\{0,1,2,3,4,5\}$ and determine the double coset to which Nt_i belongs, where $i \in \{0,1,2,3,4,5\}$. We consider $i = 0$, so Nt_0 is a representative coset, and hence we will have a new double coset Nt_0N which can denoted by [0]. There will be six possible t_i's in [$\star$] that can advance to the next double coset [0].

3.1.3 Word of Length One [0]

Denoted by $NwN = Nt_0N = \{Nt_0^n | n \in N\}$, or [0]. We need to find the point stabiliser of 0.

$\Rightarrow Nt_0N = [0] = \{Nt_0^n | n \in N\} = \{Nt_0N, Nt_1N, Nt_2N, Nt_3N, Nt_4N, Nt_5N\}$.

Since, the point stabiliser of 0 in the subgroup N is the permutations in N that fixes 0 and permutes the rest of the set $\{1,2,3,4,5\}$.

Note : $N^{(0)} \geq < (1,2,3,4,5), (25)(34) >$.

$\Rightarrow |N^{(0)}| = 10$

$\Rightarrow$ number of single cosets are in [0] is $\frac{|N|}{|N^{(0)}|} = \frac{60}{10} = 6$.

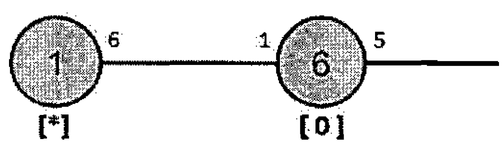

Figure 3.1: Cayley's diagram shows the double cosets [$\star$] and [0]

Now, the orbits of N^0 on $\{0,1,2,3,4,5\}$ are $\{0\}$ and $\{1,2,3,4,5\}$. We choose a representative from each orbit, $\{0\}$ and $\{1,2,3,4,5\}$. If we choose t_0 from the orbit $\{0\}$ and choose t_5 from the orbit $\{1,2,3,4,5\}$, then we notice the following :

- $Nt_0 \cdot t_0 = Nt_0^2 = N \in [\star]$.

  This will collapse and hence it will go back to [$\star$]. This is denoted by number 1 in Cayley's diagram (to the left of the circle containing 6.)

- $Nt_0 \cdot t_5 = Nt_0t_5 \in [05]$.

This is a new double coset, which will extend the Cayley's graph from [0] to [05]. Since there are 5 elements in this orbit, then there will be 5 t_i's that extend [0] to [05] .

3.1.4 Word of Length Two [05]

We are at a new double coset [05], $Nt_0t_5N = \{N(t_0t_5)^n | n \in N\}$. Now, to determine all the single cosets of this double cosets, we need to determine the point stabiliser of N^{05}, this means finding the set of elements that fix 0 and 5 in N and permutes the rest elements of the set $\{1, 2, 3, 4\}$.

Now, using our relation $t_2t_3 = t_3t_2$, We can see that : $2 \to 3$ and $3 \to 2$.

$\Rightarrow Nt_2t_3^{(23)(05)} = Nt_3t_2 = Nt_2t_3$.

$\Rightarrow (23)(05) \in N^{(05)}$.

Also,

$Nt_2t_3^{(14)(05)} = Nt_2t_3$.

$\Rightarrow (14)(05) \in N^{(05)}$.

$\Rightarrow N^{(05)} \geq \langle (14)(05), (23)(05) \rangle$.

Thus, $|N^{(05)}| = 2 \cdot 2 = 4$.

Therefore, the total number of single cosets in [05] is $\frac{|N|}{|N^{(05)}|} = \frac{60}{4} = 15$.

In order to find these 15 single cosets, we need to determine the transversals (right coset representatives) of $N^{(05)}$. Hence, they are:

$$\{e, (1,2,3,4,5), (1,0,5)(2,4,3), (1,3,5,2,4), (1,0)(2,5), (1,4,2,5,3), (1,2,3)(4,0,5), (1,0,2)(3,4,5), (1,5,4,3,2), (1,3,0,5,2), (1,4,5,3,0), (3,5)(4,0), (1,5,2,0,4), (2,0,5,3,4), (1,2,0)(3,5,4)\}$$

Therefore, the different cosets are:

1. $(23 = 32)^e \quad \Rightarrow \quad 23 = 32.$
2. $(23 = 32)^{(1,2,3,4,5)} \quad \Rightarrow \quad 34 = 43.$
3. $(23 = 32)^{(1,0,5)(2,4,3)} \quad \Rightarrow \quad 42 = 24.$
4. $(23 = 32)^{(1,3,5,2,4)} \quad \Rightarrow \quad 45 = 54.$
5. $(23 = 32)^{(1,0)(2,5)} \quad \Rightarrow \quad 53 = 35.$

6. $(23 = 32)^{(1,4,2,5,3)} \Rightarrow 51 = 15.$

7. $(23 = 32)^{(1,2,3)(4,0,5)} \Rightarrow 31 = 13.$

8. $(23 = 32)^{(1,0,2)(3,4,5)} \Rightarrow 14 = 41.$

9. $(23 = 32)^{(1,5,4,3,2)} \Rightarrow 12 = 21.$

10. $(23 = 32)^{(1,3,0,5,2)} \Rightarrow 10 = 01.$

11. $(23 = 32)^{(1,4,5,3,0)} \Rightarrow 20 = 02.$

12. $(23 = 32)^{(3,5)(4,0)} \Rightarrow 25 = 52.$

13. $(23 = 32)^{(1,5,2,0,4)} \Rightarrow 03 = 30.$

14. $(23 = 32)^{(2,0,5,3,4)} \Rightarrow 04 = 40.$

15. $(23 = 32)^{(1,2,0)(3,5,4)} \Rightarrow 05 = 50$

The 15 single distinct cosets of the double coset [05] are:

$$\{Nt_2t_3, Nt_3t_4, Nt_4t_2, Nt_4t_5, Nt_5t_3, Nt_5t_1, Nt_3t_1, Nt_1t_4, Nt_1t_2, Nt_1t_0, Nt_2t_0, Nt_2t_5, Nt_0t_3, Nt_0t_4, Nt_0t_5\}.$$

Now, we need to find the orbits $N^{(05)}$ to advance to the next double coset.
From the generator of $N^{(05)} \geq \langle(14)(05), (23)(05)\rangle$, the orbits of $N^{(05)}$ on $\{0,1,2,3,4,5\}$ are:

$$\{1,4\}, \quad \{0,5\}, \quad \{2,3\}.$$

Considering a representative from each orbit of $N^{(05)}$, we will choose the following representative from each orbit:
t_5 from $\{0,5\}$,
t_4 from $\{1,4\}$,
t_2 from $\{2,3\}$.
Multiply each representative with Nt_0t_5, :

- $Nt_0t_5 \cdot t_5 = Nt_0t_5^2 = Nt_0 \in [0]$.
  Hence two elements will go back to that double coset because $\{0,5\}$ is one orbit.

- $Nt_0t_5 \cdot t_4 \in [054]$.
  New double coset, hence two elements can advance from [05] to [054] because $\{1,4\}$ is on same orbit.
- $Nt_0t_5 \cdot t_2 \in [052]$.
  New double coset, hence two elements can advance from [05] to [052] because $\{2,3\}$ is one orbit.

The corresponding Cayley's diagram is:

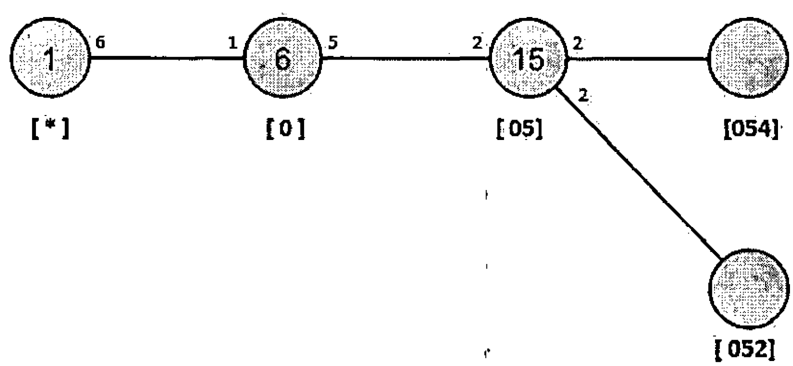

Figure 3.2: Cayley's diagram shows [$\star$], [0], [052] and [054]

Therefore, two new double cosets [054] and [052] will extend the Cayley's diagram.

3.1.5 Word of Length Three [054] and [052]

Word of Length Three [054]

Again we need to find the point stabiliser of 0 , 5 and 4. This is denoted by N^{054}. Therefore, we need to find the permutations in N that fixes 0, 5 and 4 and permutes the rest; 1,2 and 3. The group stabiliser, $N^{(054)} \geq N^{054}$.
Using relation $t_0t_5 = t_5t_0$, if we multiply both sides by t_4,
we will get $t_0t_5t_4 = t_5t_0t_4$
$\Rightarrow (05) \in N^{(054)}$.
Now, $\underline{05}4 \sim 5\underline{04} \sim \underline{54}0 \sim 450$.
and, $0\underline{54} \sim \underline{04}5 \sim 405$.

So, $054 \sim 504 \sim 540 \sim 450 \sim 045 \sim 405$

Thus, $054 \sim 405 \Rightarrow (450) \in N^{(054)}$. $\Rightarrow N(t_0t_5t_4)^{(450)} = Nt_4t_0t_5$.

$\Rightarrow N^{(054)} = \langle(1,3,2)(4,5,0),(2,3)(0,5)\rangle = S_3$

Hence,

$|N^{(054)}| = 3! = 6$, so $\frac{|N|}{|N^{(054)}|} = \frac{60}{6} = 10$.

Therefore, there are 10 distinct coset representatives. The transversals of $N^{(054)}$ in N are:

$$\{e,(1,2,3,4,5),(1,2,4)(3,0,5),(1,3,5,2,4),(1,3,0)(2,5,4),$$
$$(1,4,2,5,3),(1,4,3)(2,0,5),(1,5,4,3,2),(1,3,0,5,2),(1,2,0)(3,5,4)\}$$

Now, we conjugate our relation $054 = 504 = 540 = 450 = 045 = 405$ by the above 10 transversals, we will get :

1. e — $054 = 504 = 540 = 450 = 045 = 405$.
2. $(1,2,3,4,5)$ — $015 = 105 = 150 = 510 = 051 = 501$.
3. $(1,2,4)(3,0,5)$ — $531 = 351 = 315 = 135 = 513 = 153$.
4. $(1,3,5,2,4)$ — $021 = 201 = 210 = 120 = 012 = 102$.
5. $(1,3,0)(2,5,4)$ — $142 = 412 = 421 = 241 = 124 = 214$
6. $(1,4,2,5,3)$ — $032 = 302 = 320 = 230 = 023 = 203$.
7. $(1,4,3)(2,0,5)$ — $523 = 253 = 235 = 325 = 532 = 352$.
8. $(1,5,4,3,2)$ — $043 \sim 403 = 430 = 340 = 034 = 304$.
9. $(1,3,0,5,2)$ — $524 = 254 = 245 = 425 = 542 = 452$.
10. $(1,2,0)(3,5,4)$ — $143 = 413 = 431 = 341 = 134 = 314$.

The 10 single distinct cosets of the double coset [054] are the set

$$\{Nt_0t_5t_4, Nt_0t_1t_5, Nt_5t_3t_1, Nt_0t_2t_1, Nt_1t_4t_2, Nt_0t_3t_2, Nt_5t_2t_3, Nt_0t_4t_3, Nt_5t_2t_4, Nt_1t_4t_3\}$$

The orbits of $N^{(054)}$ on $\{0,1,2,3,4,5\}$ are:

$$\{1, 2, 3\} \quad \text{and} \quad \{0, 4, 5\}.$$

We take a representative from each orbit and multiply it with the coset $Nt_0t_5t_4$. We choose the following representatives:

t_1 from $\{1, 2, 3\}$,

t_4 from $\{0, 4, 5\}$,

Now,

- $Nt_0t_5t_4 \cdot t_4 \in [05]$.
  t_4 will collapse Cayley's graph. Since, this orbit has 0, 5 beside 4, then three t_i's will take [054] to [05].

- $Nt_0t_5t_4 \cdot t_1 \in [0541]$.
  Since, this orbit of length 3, then three t_i's will extend the Cayley graph.

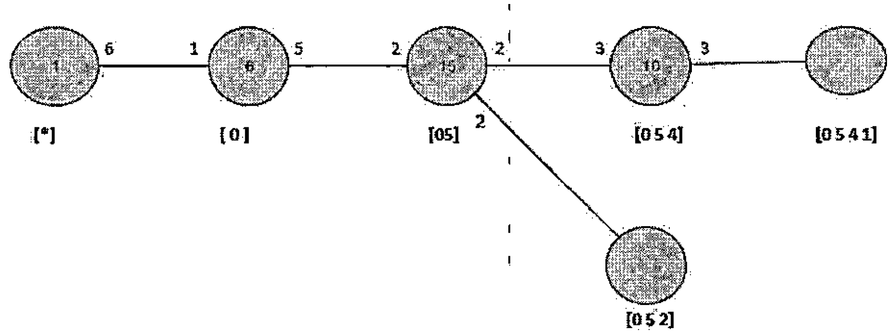

Figure 3.3: Cayley's diagram shows [$\star$],[0],[052],[054] and [0541]

Word of Length Three [052]

Again we need to find the point stabiliser of 0 , 5 and 2, denoted by N^{052}. Therefore, we need to find the permutations in N that fix 0, 5 and 2 and permutes the rest 1,3 and 4.

Hence,

Using relation $t_0t_5 = t_5t_0$,

if we multiply both sides by t_2,

$\Rightarrow t_0t_5t_2 = t_5t_0t_2$

$\Rightarrow (05) \in N^{(052)}$.

Now, $052 = 502$,

Now, $0\underline{52} = \underline{02}5 = 205$. (From above)

$\Rightarrow (025) \in N^{(052)}$.

Thus, $N^{(052)} \geq \langle (05), (025) \rangle = S_3$.

Hence,

$|N^{(052)}| = 3! = 6 \Rightarrow \frac{|N|}{|N^{(052)}|} = \frac{60}{6} = 10.$

So there are 10 distinct coset representatives, the transversals of $N^{(052)}$:

$$\{e, (1,5,0)(2,3,4), (1,0,5)(2,4,3), (1,5,0,4,3), (1,0)(2,5), (2,3)(5,0), (1,0,4,2,3), (1,0,2)(3,4,5), (1,5)(2,4), (1,0,3,5,4)\}$$

Now, we conjugate our relation $t_0t_5t_2 = t_5t_0t_2$ by the above 10 transversals, we will get :

1. $(052 = 502)^e \quad \Rightarrow \quad 052 = 502.$
2. $(052 = 502)^{(1,5,0)(2,3,4)} \quad \Rightarrow \quad 103 = 013.$
3. $(052 = 502)^{(1,0,5)(2,4,3)} \quad \Rightarrow \quad 514 = 154.$
4. $(052 = 502)^{(1,5,0,4,3)} \quad \Rightarrow \quad 402 = 042.$
5. $(052 = 502)^{(1,0)(2,5)} \quad \Rightarrow \quad 125 = 215.$
6. $(052 = 502)^{(2,3)(5,0)} \quad \Rightarrow \quad 503 = 053.$
7. $(052 = 502)^{(1,0,4,2,3)} \quad \Rightarrow \quad 453 = 543.$
8. $(052 = 502)^{(1,0,2)(3,4,5)} \quad \Rightarrow \quad 231 = 321.$
9. $(052 = 502)^{(1,5)(2,4)} \quad \Rightarrow \quad 014 = 104.$
10. $(052 = 502)^{(1,0,3,5,4)} \quad \Rightarrow \quad 342 = 432.$

The 10 single distinct cosets of the double coset [052] are:

$$\{Nt_0t_5t_2, Nt_1t_0t_3, Nt_5t_1t_4, Nt_4t_0t_2, Nt_1t_2t_5, Nt_5t_0t_3, Nt_4t_5t_3, Nt_2t_3t_1, Nt_0t_1t_4, Nt_3t_4t_2\}.$$

Note:

$$\underline{05}2 \sim 5\underline{02} \sim \underline{52}0 \sim 250.$$
$$0\underline{52} \sim \underline{02}5 \sim 205.$$

So

$052 = 502 = 520 = 250 = 025 = 205.$

Therefore, the 10 distinct equal cosets of the double coset [052] are:

1. e	$052 = 502 = 520 = 250 = 025 = 205$
2. $(1,5,0)(2,3,4)$	$103 = 013 = 031 = 301 = 130 = 310$
3. $(1,5,0,4,3)$	$402 = 042 = 024 = 204 = 420 = 240$
4. $(1,0,5)(2,4,3)$	$514 = 154 = 145 = 415 = 541 = 451$
5. $(1,5,0,4,3)$	$402 = 042 = 024 = 204 = 420 = 240$
6. $(1,0)(2,5)$	$125 = 215 = 251 = 521 = 152 = 512$
7. $(2,3)(5,0)$	$503 = 053 = 035 = 305 = 530 = 350$
8. $(1,0,4,2,3)$	$453 = 543 = 534 = 354 = 435 = 345$
9. $(1,5)(2,4)$	$014 = 104 = 140 = 410 = 041 = 401$
10. $(1,0,3,5,4)$	$342 = 432 = 423 = 243 = 324 = 234$

Now, we need to find the orbits of $N^{(052)}$ to advance to the next double coset. From the generator of $N^{(052)} \geq \langle (0,5)(0,2,5) \rangle$, the orbits of $N^{(052)}$ on $\{0,1,2,3,4,5\}$ are:

$$\{0,2,5\}, \{1,3,4\}$$

Taking a representative from each orbit, for example we take:

t_2 from $\{0,2,5\}$, and

t_1 from $\{1,3,4\}$.

We multiply each of the above representatives with $t_0t_5t_2$, we have the following:

$$t_0t_5t_2t_2 \quad \text{and} \quad t_0t_5t_2t_1$$

- $t_0t_5t_2 \cdot t_2 = Nt_0t_5t_2^2 = Nt_0t_5 \in [05]$.
  Hence, three t_i's will make Cayley's graph collapse and go back to the double coset [05] because this orbit is of length 3.

- $t_0t_5t_2 \cdot t_1 = Nt_0t_5t_2t_1 \in [0541]$.
  Since, the orbit $\{1, 3, 4\}$ is of length 3, then three t_i's will take [0521] to [0541]. (See below graph)

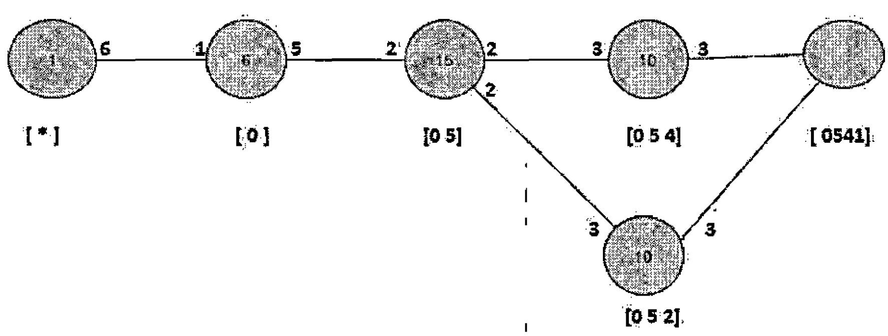

Figure 3.4: Cayley's diagram- extension of [052] to [0541]

3.1.6 Word of Length Four [0541]

We look at the new double coset $Nt_0t_5t_4t_1N$ that is represented by [0541]. We need to determine if it will extend the Cayley's graph further or will collapse it. We have the relation $t_0t_5 \sim t_5t_0$, post multiply both sides by t_4,then followed by t_1,
$\Rightarrow t_0t_5t_4t_1 \sim t_5t_0t_4t_1$.
In order to obtain the elements in the double coset [0541], we have to find the point stabiliser of 0,5,4, and 1 . $N^{0541} =< Id >$
We know

$$\underline{05}41 = 50\underline{41} = 5014$$

So,

$$0541 = 5041 = 5014 = 0514$$

Therefore,

$$N(t_0t_5t_4t_1)^{(05)} = Nt_5t_0t_4t_1 = Nt_0t_5t_4t_1 \Rightarrow (05) \in N^{(0541)}.$$
$$Nt_0t_5t_4t_1^{(05)(14)} = Nt_5t_0t_1t_4 = Nt_0t_5t_4t_1 \Rightarrow (05)(14) \in N^{(0541)}.$$

So, $N^{(0541)} \geq < (05)(14), (05) >$.
These are the generator of $N^{(0541)}$ in N.
Hence, $|N^{(0541)}| = 2 \cdot 2 = 4 \Rightarrow \frac{|N|}{|N^{(0541)}|} = \frac{60}{4} = 15$ single cosets in [0541], and 0541 = 5041 = 5014 = 0514 is one of them. To find the rest of the 15, we need to find the transversals of $N^{(0541)}$ in N. The transversals of $N(0541)$ are :

$$\{e, (1,2,3,4,5), (1,0,5)(2,4,3), (1,3,5,2,4), (1,0)(2,5), (1,4,2,5,3),$$
$$(1,2,3)(4,0,5), (1,0,2)(3,4,5), (1,5,4,3,2), (1,3,0,5,2), (1,4,5,3,0), (3,5)(4,0),$$
$$(1,5,2,0,4), (2,0,5,3,4), (1,2,0)(3,5,4)\}.$$

1. e — 0541 = 5041 = 5014 = 0514
2. $(1,2,3,4,5)$ — 0152 = 1052 = 1025 = 0125
3. $(1,0,5)(2,4,3)$ — 5130 = 1530 = 1503 = 5103
4. $(1,3,5,2,4)$ — 0213 = 2013 = 2031 = 0231
5. $(1,0)(2,5)$ — 1240 = 2140 = 2104 = 1204
6. $(1,4,2,5,3)$ — 0324 = 3024 = 3042 = 0342
7. $(1,2,3)(4,0,5)$ — 5402 = 4502 = 4520 = 5420
8. $(1,0,2)(3,4,5)$ — 2350 = 3250 = 3205 = 2305
9. $(1,5,4,3,2)$ — 0435 = 4035 = 4053 = 0453
10. $(1,3,0,5,2)$ — 5243 = 2543 = 2534 = 5234
11. $(1,4,5,3,0)$ — 1354 = 3154 = 3145 = 1345
12. $(3,5)(4,0)$ — 4301 = 3401 = 3410 = 4310
13. $(1,5,2,0,4)$ — 4215 = 2415 = 2451 = 4251
14. $(2,0,5,3,4)$ — 5321 = 3521 = 3512 = 5312
15. $(1,2,0)(3,5,4)$ — 1432 = 4132 = 4123 = 1423

Note: the orbit of $N^{(0541)}$ on $\{0,1,2,3,4,5\}$ are:

$$\{1,4\}, \quad \{2,3\}, \quad \{0,5\}.$$

We take a representative from each orbit of $N^{(0541)}$ and we multiply it with $Nt_0t_5t_4t_1$ and determine if it will extend the Cayley's graph or collapse it.

- $Nt_0t_5t_4t_1 \cdot t_1 = Nt_0t_5t_4(t_1)^2 = Nt_0t_5t_4 \in [054]$.
  Since, the orbit $\{1,4\}$ of length 2, then two t_i's will collapse the Cayley's graph to [054]

- $Nt_0t_5t_4t_1 \cdot t_2 = Nt_0t_5t_4t_1t_2 \in [05412]$.
  Since, the orbit $\{2,3\}$ of length 2, then two t_i's will extend the Cayley's graph from [0541] to [05412].

- $Nt_0t_5t_4t_1 \cdot t_0 = Nt_0t_5t_4t_0 = Nt_0\underline{t_5t_4t_1}t_0 = Nt_0\underline{t_5t_1t_4}t_0 = Nt_0t_0t_5t_2t_0 = Nt_0t_5t_2 \in$ [052].
  So, two t_i's will collapse the Cayley's graph and will take [05410] to [052].

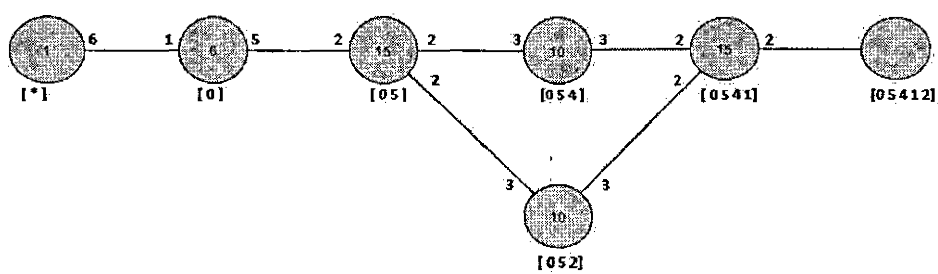

Figure 3.5: Extension of Cayley's graph by [05412]

3.1.7 Word of Length Five [05412]

The new double coset $Nt_0t_5t_4t_1t_2N$ which is given by [05412]. The point stabiliser of $0,5,4,1$ and 2, N^{05421} in N.
Thus, $N^{05421} =< e > \quad \Rightarrow \quad$ the coset stabiliser $N^{(05421)} \geq N^{05421}$.
The equal single cosets are:
$05412 = 01254 = 50142 = 10524 = 21045 = 54201 = 24510 = 45021 = 42150 = 12405.$

So, we note $05412 = 42150 \Rightarrow (04152) \in N^{(05421)}$ because:
$N(t_0t_5t_4t_1t_2)^{(04152)} = Nt_4t_2t_1t_5t_0 = Nt_0t_5t_4t_1t_2$.
Also, $05412 = 01254 \Rightarrow (51)(24) \in N^{(05421)}$ because:
$N(t_0t_5t_4t_1t_2)^{(51)(24)} = Nt_0t_1t_2t_5t_4 = Nt_0t_5t_4t_1t_2$.
Thus, $N^{(05421)} \geq < (51)(24), (04152) >$.
Therefore, $|N^{(05412)}| = 2 \cdot 5 = 10 \Rightarrow \frac{|N|}{|N^{(05412)}|} = \frac{60}{10} = 6$ single cosets in [05412].
The transversals of $N^{(05421)}$ in $N = \{e, (1,5,0)(2,3,4), (1,0,5)(2,4,3), (2,4,3,5,0), (1,0,4,2,3), (1,5)(3,0)\}$.
If we conjugate $(05412 = 01254 = 50142 = 10524 = 21045 = 54201 = 24510 = 45021 = 42150 = 12405)$ by the above transversals, we will get the following distinct cosets.

1. $(05412 = 01254 = 50142 = 10524 = 21045 = 54201 = 24510 = 45021 = 42150 = 12405)^e$
$\Rightarrow 05412 = 01254 = 50142 = 10524 = 21045 = 54201 = 24510 = 45021 = 42150 = 12405.$

2. $(05412 = 01254 = 50142 = 10524 = 21045 = 54201 = 24510 = 45021 = 42150 = 12405)^{(1,5,0)(2,3,4)}$
$\Rightarrow 10253 = 15302 = 01523 = 51032 = 35120 = 02315 = 32051 = 20135 = 23501 = 53210.$

3. $(05412 = 01254 = 50142 = 10524 = 21045 = 54201 = 24510 = 45021 = 42150 = 12405)^{(1,0,5)(2,4,3)}$
$\Rightarrow 51304 = 50413 = 15034 = 05143 = 40531 = 13450 = 43105 = 31540 = 34015 = 04351.$

4. $(05412 = 01254 = 50142 = 10524 = 21045 = 54201 = 24510 = 45021 = 42150 = 12405)^{(2,4,3,5,0)}$
$\Rightarrow 20314 = 21403 = 02134 = 12034 = 41230 = 63421 = 43612 = 36241 = 34162 = 14326.$

5. $(05412 = 01254 = 50142 = 10524 = 21045 = 54201 = 24510 = 45021 = 42150 = 12405)^{(1,0,4,2,3)}$
$\Rightarrow 45203 = 40352 = 54023 = 04532 = 30425 = 52340 = 32504 = 25430 = 23054 = 03245.$

6. $(05412 = 01254 = 50142 = 10524 = 21045 = 54201 = 24510 = 45021 =$
   $42150 = 12405)^{(1,5)(3,0)}$
   $\Rightarrow 31452 = 35214 = 13542 = 53124 = 25341 = 14235 = 24153 = 41325 =$
   $42513 = 52431.$

Therefore, The distinct single cosets of the double coset [05412] are:
$[05412] = \{05412, 10253, 51304, 20314, 45203, 31452\}$.
Hence, the orbits of $N^{(05421)}$ on $\{0, 1, 2, 3, 4, 5\}$ are :

$$\{3\}, \qquad \{0, 1, 2, 4, 5\}$$

We now take a representative from each orbit, say t_3 from $\{3\}$ and t_2 from $\{0, 1, 2, 4, 5\}$. So, we have

$$t_0t_5t_4t_1t_2t_3 \qquad \text{and} \qquad t_0t_5t_4t_1t_2t_2$$

- $t_0t_5t_4t_1t_2t_2 \in [0541]$.
  Hence, 5 t_i's will make Cayley's graph collapse and therefore any of the following: t_0,t_1, t_2,t_4 or t_5 will take [05412] to [0541].

- $t_0t_5t_4t_1t_2t_3 \in [054123]$.
  This represented by 1 in Cayley's graph. (see figure 6).

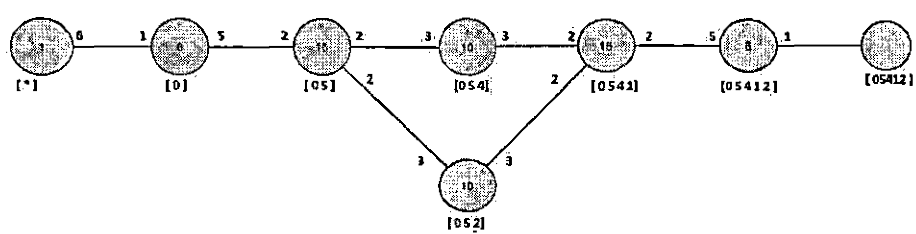

Figure 3.6: Cayley's graph with extension to [054123]

3.1.8 Word of Length Six [054123]

At the double coset $Nt_0t_5t_4t_1t_2t_3N$, we will have all elements being fixed in N. $\Rightarrow N^{054123} = < e >$.

We already know that $N^{(05412)} \geq <(51)(24),(04152)>=<(051)(243),(12345)>= A_5$.

Now,

$05\underline{41}23 = 0514\underline{23} = 051\underline{43}2 = 051342$.

$\Rightarrow 051423 = 051342 \quad \Rightarrow \quad (243) \in N^{(054123)}$.

$\Rightarrow N^{(05412)} \geq <(51)(24),(04152),(243)>= A_5$.

From the generators of $N^{(054123)}$, there will be one single orbit generated which is $\{0,1,2,3,4,5\}$.

Thus, $|N^{(054123)}| = A_5 = 60$.

$\Rightarrow \frac{|N|}{|N^{(054123)}|} = \frac{60}{60} = 1$ single coset in [054123]. The equal single cosets of the double coset [054123] are:

$$(051423 = 524316 = 413250 = 210453 = 352140 =$$
$$203145 = 143205 = 532104 = 351204 = 235014 =$$
$$230541 = 153024 = 134502 = 425130 = 531240 =$$
$$135420 = 320514 = 504132 = 105243 = 450213 =$$
$$421503 = 314520 = 102534 = 023154 = 214035 =$$
$$431052 = 341025 = 032451 = 043512 = 241530 =$$
$$021345 = 403521 = 325041 = 405312 = 054123 =$$
$$124053 = 045321 = 015234 = 302415 = 201354 =$$
$$253410 = 412305 = 510324 = 452031 = 430125 =$$
$$315402 = 142350 = 542013 = 034215 = 245103 =$$
$$340152 = 501423 = 523401 = 304251 = 150342)^{Id(N)}$$

$$\Downarrow$$

$$051423 = 524316 = 413250 = 210453 = 352140 =$$
$$203145 = 143205 = 532104 = 351204 = 235014 =$$
$$230541 = 153024 = 134502 = 425130 = 531240 =$$
$$135420 = 320514 = 504132 = 105243 = 450213 =$$
$$421503 = 314520 = 102534 = 023154 = 214035 =$$
$$431052 = 341025 = 032451 = 043512 = 241530 =$$
$$021345 = 403521 = 325041 = 405312 = 054123 =$$
$$124053 = 045321 = 015234 = 302415 = 201354 =$$
$$253410 = 412305 = 510324 = 452031 = 430125 =$$
$$315402 = 142350 = 542013 = 034215 = 245103 =$$
$$340152 = 501423 = 523401 = 304251 = 150342$$

3.1.9 Conclusion

Therefore: the double coset enumeration gives that
$|G| \leq (\frac{|N|}{|N|} + \frac{|N|}{|N^{(0)}|} + \frac{|N|}{|N^{(05)}|} + \frac{|N|}{|N^{(054)}|} + \frac{|N|}{|N^{(052)}|} + \frac{|N|}{|N^{(0541)}|} + \frac{|N|}{|N^{(05412)}|} + \frac{|N|}{|N^{(054123)}|}) \cdot |N|.$
$= \{1 + 6 + 15 + 10 + 10 + 15 + 6 + 1\} \cdot |N| = 64 \cdot 60 = 3840$.(from Cayley's graph).
Note,

- $|N| = 60$.
- $|G| \geq 3840$ (from Magma) .

Therefore,
$|G| = 3840$.
See final Cayley's graph next page.

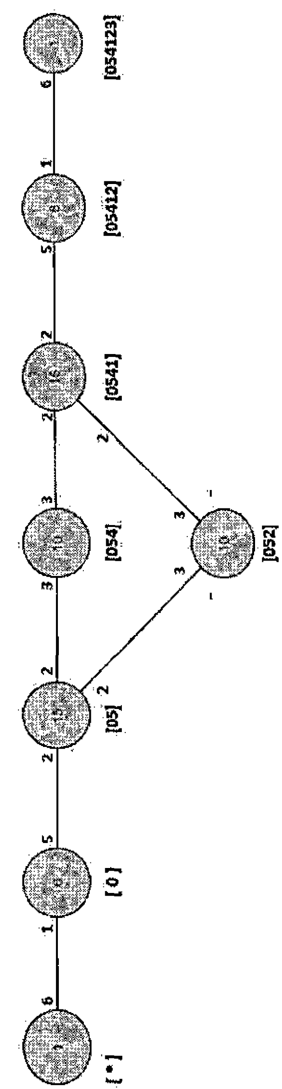

Figure 3.7: Cayley graph of G over A_5

Chapter 4

Double Coset Enumeration of $L_2(7)$ Over A_4

In this chapter, we want to do a double coset enumeration for:

$$G \cong \frac{3^{*6} :_m A_4}{[(1,2,3)(\overline{1},\overline{2},\overline{3})t_3]^5}$$

Where,

$x \sim (1,2,3)(\overline{1},\overline{2},\overline{3})$,

$y \sim (1,\overline{1})(2,\overline{2})$.

$xy = (1,\overline{2},\overline{3})(2,3,\overline{1})$.

The abstract presentation of the progenitor $3^{*6} :_m A_4$ is:

$$G\langle x,y,t\rangle \cong Group\langle x,y,t|x^3,y^2,(x*y)^3,t^3,(t,y),t*t^{(y^x)},(x*t)^5\rangle.$$

The control subgroup, N, is S_6 which is the symmetric group of degree 6 on six letters 1, 2, 3, $\overline{1}$, $\overline{2}$, and $\overline{3}$. N can be generated by x and y (above). Thus, $N =< x, y >$.
Note: 3^{*6} means $t_1^3 = t_2^3 = t_3^3 = \overline{t_1}^3 = \overline{t_2}^3 = \overline{t_3}^3 =$ Identity.

4.1 Relations

$t_1^3 = e$.

$t_i^3\overline{t_i} = e \cdot \overline{t_i}$

$t_i^2 = \overline{t_i}$

Therefore,

$$t_1^2 = \overline{t_1}$$
$$t_2^2 = \overline{t_2}$$
$$t_3^2 = \overline{t_3}$$

so far, we have the following relations

$t_i^3 = e$ and $t_i^2 = \overline{t_i}$ where $i = 1, 2, 3$.

given $t \sim t_3$. and we have the relation $[(1,2,3)(\overline{1},\overline{2},\overline{3})t_3]^5 = Id$

Now let

$$\pi = (1,2,3)(\overline{1},\overline{2},\overline{3}).$$
$$\pi^2 = (1,2,3)(\overline{1},\overline{2},\overline{3}) \cdot (1,2,3)(\overline{1},\overline{2},\overline{3}) = (1,3,2)(\overline{1},\overline{3},\overline{2}).$$
$$\pi^3 = (1,3,2)((\overline{1},\overline{3},\overline{2}) \cdot (1,2,3)(\overline{1},\overline{2},\overline{3}) = Id.$$
$$\pi^4 = e \cdot (1,2,3)(\overline{1},\overline{2},\overline{3}) = \pi.$$
$$\pi^5 = \pi^2.$$

Thus, our given relation can be simplified to

$$\pi t_3 \pi t_3 \pi t_3 \pi \underbrace{t_3 \pi t_3} = Id.$$
$$\Rightarrow \quad \pi \pi^4 \pi^{-4} t_3 \pi \pi^3 \pi^{-3} t_3 \pi \pi^2 \pi^{-2} t_3 \pi \pi^1 \pi^{-1} t_3 = Id.$$
$$\Rightarrow \quad \pi^5 (\pi^{-4} t_3 \pi^4)(\pi^{-3} t_3 \pi^3)(\pi^{-2} t_3 \pi^2)(\pi^{-1} t_3 \pi^1) t_3 = Id$$

we know that $\pi^{-1} t_i \pi = t_i^{\pi}$.

Hence, our given relation is:

$$\pi^5 t_3^{\pi^4} t_3^{\pi^3} t_3^{\pi^2} t_3^{\pi^1} t_3 = Id.$$
$$\Rightarrow \pi^2 t_3^{(1,2,3)(\overline{1},\overline{2},\overline{3})} t_3^{Id} t_3^{(1,3,2)(\overline{1},\overline{3},\overline{2})} t_3^{(1,2,3)(\overline{1},\overline{2},\overline{3})} t_3 = Id.$$
$$\Rightarrow \pi^2 t_1 t_3 t_2 t_1 t_3 = Id. \qquad \Rightarrow \pi^2 t_1 t_2 t_3 = \overline{t_3 t_1}.$$

4.2 Double Coset Enumeration of G over A_4

4.2.1 Double Coset $[\star]$:

NeN denote the double coset $[\star]$. Since N is transitive on $\{1,2,3,\overline{1},\overline{2},\overline{3}\}$, we take a representative coset N from $[\star]$, and a representative from $\{1,2,3,\overline{1},\overline{2},\overline{3}\}$, let it be

t_3 and determine if the new double coset will extend Cayley's graph or collapse it.

$$NeN = \{NeN|n \in N\}.$$
$$= \{Nn|n \in N\}.$$
$$= \{N\}.$$

$[\star]$ will have $\frac{|N|}{|N|}$ single coset which is $\frac{12}{12} = 1$ because $|N| = |A_4| = 12$.

$$N \cdot t_3 = Nt_3 \in [3].$$

Now, since t_3 is among other 6 elements of the orbits, this indicates 6 elements will extend Cayley's graph from $[\star]$ to $[3]$. See figure 1.

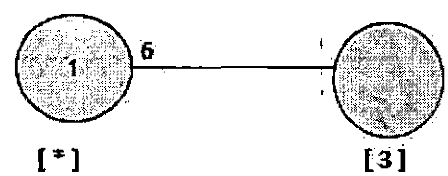

Figure 4.1: Extension of Cayley's graph from $[\star]$

4.2.2 Double Coset [3]:

Secondly, we are at the double coset $Nt_3N = [3]$. This is a word of length one. To find the number of single cosets, we need the point stabiliser of 3, N^3 in N. $N^{(3)} =< (1, \bar{1}), (2, \bar{2}) >= S_2$

Thus,

$$\frac{|N|}{|N^{(3)}|} = \frac{12}{2} = 6.$$

So there are 6 single cosets of length one word in the double coset [3]. Next we need to determine the orbits of $N^{(3)}$. $N^{(3)}$ orbits can be obtained from the generators of $N^{(3)}$. We will have the following orbits:

$$\{3\}, \{\bar{3}\}, \{1, \bar{1}\}, \{2, \bar{2}\}.$$

To find all the elements in this double coset, we need to find the transversals of $N^{(3)}$ in N and then we conjugate Nt_3 by them. Using Magma program, the transversals (right coset representatives) of Nt_3 are:

$$\{Id, (1,2,3)(\overline{1},\overline{2},\overline{3}), (1,3,2)(\overline{1},\overline{3},\overline{2}), (1,\overline{2},\overline{3})(2,3,\overline{1}), (1,3,\overline{2})(2,\overline{1},\overline{3}), (1,\overline{1})(3,\overline{3})\}$$

Therefore,

$Nt_3^{(Id)} = Nt_3$.
$Nt_3^{(1,2,3)(\overline{1},\overline{2},\overline{3})} = Nt_1$.
$Nt_3^{(1,3,2)(\overline{1},\overline{3},\overline{2})} = Nt_2$.
$Nt_3^{(1,3,\overline{2})(2,\overline{1},\overline{3})} = N\overline{t_2}$.
$Nt_3^{(1,\overline{2},\overline{3})(2,3,\overline{1})} = N\overline{t_1}$.
$Nt_3^{(1,\overline{1})(3,\overline{3})} = N\overline{t_3}$.

Thus,

$$Nt_3N = [3] = \{Nt_1N, Nt_2N, Nt_3N, N\overline{t_1}N, N\overline{t_2}N, N\overline{t_3}N\}.$$

Now, we go back to our orbits of $N^{(3)}$ and take a representative from each orbit and multiply it by Nt_3
to determine if Cayley's graph extends.

- $Nt_3 \cdot t_3 = Nt_3^2 = N\overline{t_3} \in [3]$
  One element will go to the same double coset [3]
- $Nt_3 \cdot \overline{t_3} = N \in [\star]$.
  One element will go back to $[\star]$.
- $Nt_3 \cdot t_1 \in [31]$.
  Since $1 \in \{1, \overline{1}\}$ then two elements will extend Cayley's graph.
- $Nt_3 \cdot t_2 \in [32]$.
  Since $2 \in \{2, \overline{2}\}$ then two elements will extend Cayley's graph.

Therefore, the double cosets [31] and [32] are new double cosets that extends Cayley's graph. The new Cayley's graph would be:

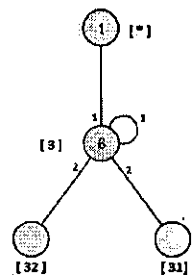

Figure 4.2: Extension of Cayley's graph from [3]

4.2.3 Double Coset [31]:

We look at the double coset Nt_3t_1N which denote [31]. To find its single cosets, we need to determine its point stabiliser. Therefore, we fix 3 and 1, N^{31}, and permute the rest letters. We know that $N^{(31)} \geq N^{31}$.

$N(t_3t_1)^{Id} = Nt_3t_1$.

$\Rightarrow N^{(31)} \geq \langle Id \rangle$.

Thus,

$$|N^{(31)}| = 1.$$

Hence, the number of single cosets in the double coset [31] is $\frac{|N|}{|N^{(31)}|} = \frac{12}{1} = 12$.

The orbits of $N^{(31)}$ are $\{1\}, \{\bar{1}\}, \{2\}, \{\bar{2}\}, \{3\}, \{\bar{3}\}$. Taking a representative from each orbit and multiply it by Nt_3t_1 and see if it will expand Cayley's graph or the t_i will collapse it.

Now,

- $Nt_3t_1 \cdot t_1 = Nt_3t_1^2 = Nt_3\overline{t_1} \in [31]$.
- $Nt_3t_1 \cdot t_2 \in [312]$.
- $Nt_3t_1 \cdot t_3 \in [313]$.
- $Nt_3t_1 \cdot \overline{t_1} = Nt_3 \in [3]$.
- $Nt_3t_1 \cdot \overline{t_2} \in Nt_3t_2 \in [32]$.

  Because $31\bar{2} = 32\bar{2}1\bar{2} = 32\overline{21}\underline{\overline{12}} = 32\overline{21}231 = 32\bar{2}1\underline{123}1 = 32\underline{\bar{2}1\bar{3}}\bar{1}1 = \underline{323}2 = 32$ since our relation is $123 = \overline{31}$.

- $Nt_3t_1\overline{t_3} \in [31\overline{3}]$.

So Cayley's graph would be:

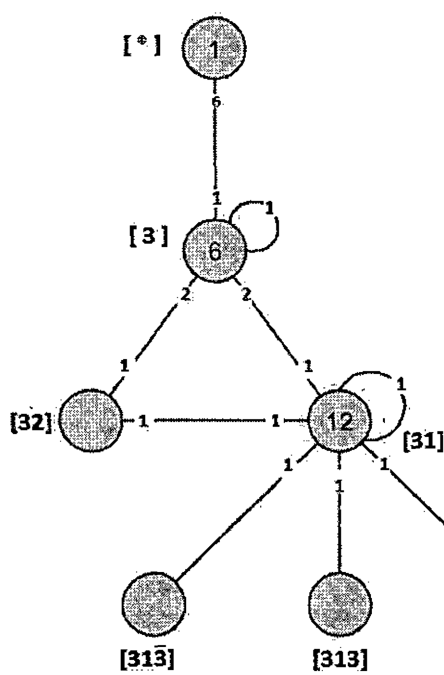

Figure 4.3: Extension of Cayley's graph from [31]

$[31] = Nt_3t_1N = \{N(t_3t_1)^n | n \in N\}$.
The transversals of $N^{(31)}$ are:

$$\{Id(N), (1,2,3)(\overline{1},\overline{2},\overline{3}), (1,\overline{1})(2,\overline{2}), (1,3,2)(\overline{1},\overline{3},\overline{2}), (1,\overline{2},\overline{3})(2,3,\overline{1}), (1,\overline{2},3)(2,\overline{3},\overline{1}), (1,3,\overline{2})(2,\overline{1},\overline{3}), (1,\overline{3},2)(3,\overline{2},\overline{1}), (1,\overline{3},\overline{2})(2,\overline{1},3), (1,2,\overline{3})(3,\overline{1},\overline{2}), (2,\overline{2})(3,\overline{3}), (1,\overline{1})(3,\overline{3})\}$$

- $Nt_3t_1^{(123)(\overline{123})} = Nt_1t_2$.
- $Nt_3t_1^{(132)(\overline{132})} = Nt_2t_3$.
- $Nt_3t_1^{(1\overline{23})(\overline{1}23)} = N\overline{t_1t_2}$.
- $Nt_3t_1^{(13\overline{2})(\overline{13}2)} = N\overline{t_2}t_3$.
- $Nt_3t_1^{(1\overline{1})(3\overline{3})} = N\overline{t_3t_1}$.

- $Nt_3t_1^{(1\bar{1})(2\bar{2})} = Nt_3\bar{t_1}$.
- $Nt_3t_1^{(1\bar{2}3)(2\bar{3}\bar{1})} = Nt_1\bar{t_2}$.
- $Nt_3t_1^{(1\bar{3}\bar{2})(2\bar{1}3)} = Nt_2\bar{t_3}$.
- $Nt_3t_1^{(12\bar{3})(3\bar{1}\bar{2})} = N\bar{t_1}t_2$.
- $Nt_3t_1^{(1\bar{3}2)(3\bar{2}\bar{1})} = N\overline{t_2t_3}$.
- $Nt_3t_1^{(2\bar{2})(3\bar{3})} = N\bar{t_3}t_1$.
- $Nt_3t_1^{(Id)} = Nt_3t_1$.

The 12 distinct single cosets of Nt_3t_1N are
$[31] = \{12, 23, \overline{12}, \bar{2}3, \overline{31}, 3\bar{1}, 1\bar{2}, 2\bar{3}, \bar{1}2, \overline{23}, \bar{3}1, 31\}$.
The new double cosets are $[312], [313], [31\bar{3}]$. We will discuss these cosets later.
Now we go back to determine the double coset [32].

4.2.4 Double Coset [32]:

The double coset [32] is denoted by Nt_3t_2N. To find its cosets, we again need to determine the point stabiliser of 3 and 2, N^{32}. It is the permutations in N which fix 3,2 and permutes the rest.
Thus,

$$N^{32} = Id$$
$$\Rightarrow |N^{(32)}| = 1.$$

We have,

$$N^{(32)} \geq N^{32}.$$
$$N(t_3t_2)^{(Id)} = Nt_3t_2.$$
$$\Rightarrow Id \in N^{(32)} \Rightarrow N^{(32)} \geq \langle Id \rangle.$$

Thus,
the total number of single cosets in [32] is $\frac{|N|}{|N^{(32)}|} = \frac{12}{1} = 12$.
The orbits of $N^{(32)}$ are $\{1\}, \{2\}, \{3\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}$.
To determine the elements of [32], we conjugate $Nt_3t_2$by 12 transversals which are:

$$\{Id(N), (1,2,3)(\bar{1},\bar{2},\bar{3}), (1,\bar{1})(2,\bar{2}), (1,3,2)(\bar{1},\bar{3},\bar{2}), (1,\bar{2},\bar{3})(2,3,\bar{1}), (1,\bar{2},3)(2,\bar{3},\bar{1}),$$
$$(1,3,\bar{2})(2,\bar{1},\bar{3}), (1,\bar{3},2)(3,\bar{2},\bar{1}), (1,\bar{3},\bar{2})(2,\bar{1},3), (1,2,\bar{3})(3,\bar{1},\bar{2}), (2,\bar{2})(3,\bar{3}), (1,\bar{1})(3,\bar{3})\}$$

Therefore,

- $Nt_3t_2^{(123)(\overline{123})} = Nt_1t_3$.
- $Nt_3t_2^{(1\bar{1})(2\bar{2})} = Nt_3\overline{t_2}$.
- $Nt_3t_2^{(132)(\overline{132})} = Nt_2t_1$.
- $Nt_3t_2^{(1\overline{23})(23\bar{1})} = N\overline{t_1}t_3$.
- $Nt_3t_2^{(1\bar{2}3)(2\overline{31})} = Nt_1\overline{t_3}$.
- $Nt_3t_2^{(13\bar{2})(2\overline{13})} = N\overline{t_2t_1}$.
- $Nt_3t_2^{(1\bar{3}2)(3\overline{21})} = N\overline{t_2}t_1$.
- $Nt_3t_2^{(1\overline{32})(2\bar{1}3)} = Nt_2\overline{t_1}$.
- $Nt_3t_2^{(12\bar{3})(3\overline{12})} = N\overline{t_1t_3}$.
- $Nt_3t_2^{(2\bar{2})(3\bar{3})} = N\overline{t_3t_2}$.
- $Nt_3t_2^{(1\bar{1})(3\bar{3})} = N\overline{t_3}t_2$.
- $Nt_3t_2^{Id} = Nt_3t_2$.

$[32] = \{13, 3\bar{2}, 21, \bar{1}3, 1\bar{3}, \overline{21}, \bar{2}1, 2\bar{1}, \overline{13}, \overline{32}, \bar{3}2, 32\}$.
The orbits of $N^{(32)}$ have been previously determined, so we consider a representative from each orbit and multiply it by Nt_3t_2.

- $Nt_3t_2t \cdot t_1 \in [31]$.
  $N\overline{t_3\underline{t_3t_2}t_2}t_1 = N\overline{t_3t_3}t_2\overline{t_2}t_1 = Nt_3t_1$.
  Hence, one element will go back to [31].

- $Nt_3t_2t \cdot t_2 = Nt_3\overline{t_2} \in [32]$.
  One element will go to [32].

- $Nt_3t_2 \cdot t_3 \in [313]$.
  $3\overline{22}3 = 3\underline{\overline{21}1\bar{2}}3 = 3\bar{1}3\overline{31}3 = 313$.

- $Nt_3t_2 \cdot \overline{t_1} \in [312]$.
  $32\bar{1} = 3\bar{1}1\underline{21}1 = 3\bar{1}12\bar{1}1 = 3\bar{1}\underline{12} = 3\overline{\bar{1}1}2 = 312$

- $Nt_3t_2 \cdot \overline{t_2} \in [3]$.
- $Nt_3t_2\overline{t_3} \in [32\overline{3}]$
  This is a new double coset that will extend Cayley's graph.

The new extended Cayley's graph will be as shown in figure 4,

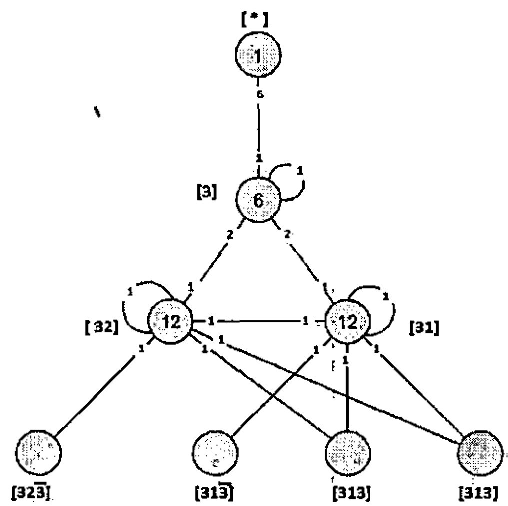

Figure 4.4: Extension of Cayley's graph from [32]

4.2.5 Double Coset [312]:

Now, we look at the double coset $Nt_3t_1t_2N$, which denotes [312]. To determine how many elements are in this double coset, we need to find the point stabiliser of 3,1 and 2, N^{312}.

$\Rightarrow N^{312} = e$.

We need to add relations to the stabilsing set.

Now

$(14)(25)t_3t_1t_2 = t_1\overline{t_2t_3}$.

$\Rightarrow (31\overline{2})(2\overline{3}) \in N^{(312)}$.

Similarly if we consider the double cosets [312] and $[\overline{2}3\overline{1}]$.

Since,

$(3\overline{3})(1\overline{1})t_3t_1t_2 = \overline{t_2}t_3\overline{t_1}$.

$\Rightarrow (13\overline{2})(\overline{1}2) \in N^{(312)}$,

$\Rightarrow N^{(312)} \geq <(31\overline{2})(2\overline{3}), (13\overline{2})(\overline{1}2)>$.

Thus

$$|N^{(312)}| = 3.$$

Therefore, number of single cosets in [312] is $\frac{|N|}{|N^{(312)}|} = \frac{12}{3} = 4$.

Note: $((13\overline{2})(2\overline{1}\overline{3}))^{-1} = (\overline{2}31)(\overline{3}\overline{1}2) \in N^{(312)}$.

Hence, the equal cosets are : $312 = \overline{2}3\overline{1} = 1\overline{2}\overline{3}$.

By conjugating these equal cosets by four transversals which are:

$$\{Id, (123)(\overline{1}\overline{2}\overline{3}), (132)(\overline{1}\overline{3}\overline{2}), (1\overline{2}\overline{3})(23\overline{1})\}.$$

Thus,

1. $(312 = \overline{2}3\overline{1} = 1\overline{2}\overline{3})^{Id} \Rightarrow 312 = \overline{2}3\overline{1} = 1\overline{2}\overline{3}$.
2. $(312 = \overline{2}3\overline{1} = 1\overline{2}\overline{3})^{(123)(\overline{1}\overline{2}\overline{3})} \Rightarrow 123 = \overline{3}1\overline{2} = 2\overline{3}\overline{1}$.
3. $(312 = \overline{2}3\overline{1} = 1\overline{2}\overline{3})^{(132)(\overline{1}\overline{3}\overline{2})} \Rightarrow 231 = \overline{1}2\overline{3} = 3\overline{1}\overline{2}$.
4. $(312 = \overline{2}3\overline{1} = 1\overline{2}\overline{3})^{(1\overline{2}\overline{3})(23\overline{1})} \Rightarrow \overline{1}\overline{2}3 = \overline{3}\overline{1}2 = \overline{2}\overline{3}1$.

Therefore,

The different single cosets of the double coset [312] are:

$$\{312, 123, 231, \overline{1}\overline{2}3\}.$$

The orbits of $N^{(312)}$ on $\{1, 2, 3, \overline{1}, \overline{2}, \overline{3}\}$ are $\{1, 3, \overline{2}\}$and $\{2, \overline{1}, \overline{3}\}$.

Taking a representative from each orbit and multiply it by $Nt_3t_1t_2$, we get the following:

- $Nt_3t_1t_2 \cdot \overline{t_2} = N_3t_1 \in [31]$.
  Since this orbit contains 3 elements, then three $t_i's$
  will take [312] to [31],
  Thus, Cayley's graph collapse.

- $Nt_3t_1t_2 \cdot t_2 = N_3\underline{t_1t_2}t_2 = Nt_3t_2t_3t_2$.
  $= Nt_3t_2\overline{t_3}\underline{\overline{t_3}t_2} = Nt_3t_2\overline{t_3}t_3t_2 = Nt_3t_2 \in [32]$.
  Thus 3 t_i's will make Cayley's graph collapse.
  because this orbit of length 3 so $[312] \to [32]$.
  Therefore, at the double coset [312], 3 t_i's will go back to [31]
  and 3 t_i's will return to [32].
  Thus, at [312] there is no further extension to Caylay's graph.

4.2.6 Double Coset [313]:

To determine the double coset $Nt_3t_1t_3N$, [313], again we need to find the single cosets number.

Thus, we need to find the point stabiliser of 3,1 and 3, N^{313} in N.

$\Rightarrow N^{313} = Id.$

$N^{(313)} \geq N^{313}.$

$|N^{(313)}| = 1.$

The total number of elements in this double coset is $\frac{|N|}{|N^{(313)}|} = \frac{12}{1} = 12.$

Thus,

The distinct cosets of $Nt_3t_1t_3N$ will be:

$[313] = Nt_3t_1t_3N = \{N(t_3t_1t_3)^n | n\ inN\},$

The 12 distinct cosets are:

$[313] = \{Nt_1t_2t_1, Nt_3\overline{t_1}t_3, Nt_2t_3t_2, N\overline{t_1t_2t_1}, Nt_1\overline{t_2}t_1, N\overline{t_2}t_3\overline{t_2},$
$N\overline{t_2t_3t_2}, Nt_2\overline{t_3}t_2, N\overline{t_1}t_2\overline{t_1}, N\overline{t_3}t_1\overline{t_3}, N\overline{t_3t_1t_3}, Nt_3t_1t_3\}.$

The orbits of $Nt_3t_1t_3N$ on $\{1,2,3,\overline{1},\overline{2},\overline{3}\}$ are $\{1\},\{2\},\{3\},\{\overline{t_1}\},\{\overline{t_2}\},\{\overline{t_3}\}.$

The same procedure, we take a representative from each orbit and multiply by $Nt_3t_1t_3$.

- $Nt_3t_1t_3 \cdot t_1 = Nt_3t_1\overline{t_3\underline{t_3t_1}t_1} = Nt_3t_1\overline{t_3t_3}t_1\overline{t_1} = Nt_3t_1t_3 \in [313].$

  One element will go back to [313].
- $Nt_3t_1t_3 \cdot t_2 = Nt_3t_1\underline{t_3\overline{t_2}}\overline{t_2} = Nt_3t_1\overline{t_3}t_2\overline{t_2} = Nt_3t_1\overline{t_3} \in [31\overline{3}].$

  One element will go back to $[31\overline{3}]$.
- $Nt_3t_1t_3 \cdot t_3 = Nt_3t_1\overline{t_3} \in [31\overline{3}].$

  One element will go back to $[31\overline{3}]$.
- $Nt_3t_1t_3 \cdot \overline{t_1} = Nt_3t_1\underline{t_3t_1}t_1 = Nt_3t_1t_3\overline{t_1}t_1 = Nt_3t_1t_3 \in [313].$

  One element will go back to [313].
- $Nt_3t_1t_3 \cdot \overline{t_2} = Nt_3\overline{t_1\underline{t_1t_3}t_3t_2} = Nt_3\overline{\underline{t_1t_3}t_2t_3t_2} = Nt_3t_2\overline{\underline{t_1t_2t_3t_2}}$
  $= Nt_3t_2t_3t_1\overline{t_1t_3} \in [32].$

  One element will go back to [32].
- $Nt_3t_1t_3 \cdot \overline{t_3} = Nt_3t_1 \in [31].$

  One element will go back to [31].

Therefore, the double coset [313] will not extend Cayley's graph any further.(Figure 5 shows the collapsing of [313]).

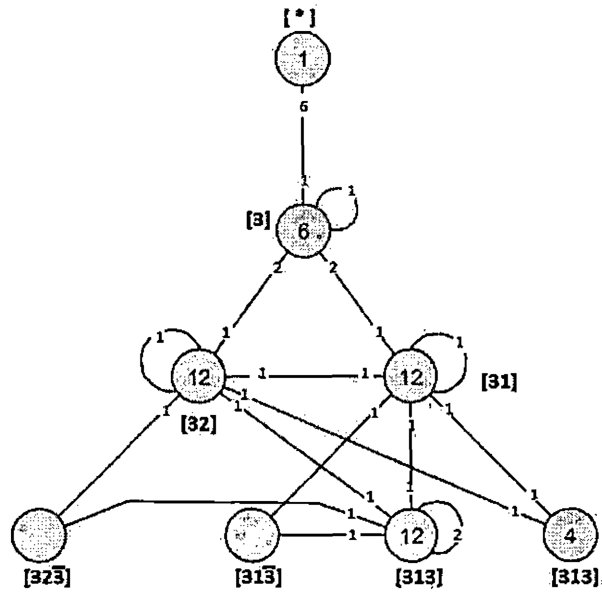

Figure 4.5: Collapsing of the double coset [313]

4.2.7 Double Coset [31$\overline{3}$]:

The double coset , $Nt_3t_1\overline{t_3}$, denotes [31$\overline{3}$]. Thus, we need to determine the point stabiliser for 3,1 and $\overline{3}$.

$\Rightarrow N^{31\overline{3}} = Id.$

Now,

$n\ t_3t_1t_{\overline{3}} = t_{\overline{1}}t_{\overline{2}}t_1.$

$\Rightarrow n = (3\overline{12})(\overline{3}12).$

$\Rightarrow (3\overline{12})(\overline{3}12)t_3t_1t_{\overline{3}} = t_{\overline{1}}t_{\overline{2}}t_1.$

$\Rightarrow (\overline{3}1\overline{2})(3\overline{1}) \in N^{(31\overline{3})}.$

and

$n\ t_3t_1t_{\overline{3}} = t_2t_{\overline{3}}t_{\overline{2}}.$

$\Rightarrow n = (321)(\overline{321})t_3t_1t_{\overline{3}} = t_2t_{\overline{3}}t_{\overline{2}}.$

$\Rightarrow (1\overline{32})(32) \in N^{(31\overline{3})}.$

Thus,

$N^{(31\overline{3})} \geq <(\overline{3}1\overline{2})(3\overline{1}), (1\overline{32})(32).$

Therfore, $|N^{(31\overline{3})}| = 3.$

The total number of equal single cosets are :

$$\frac{|N|}{|N^{(31\bar{3})}|} = \frac{12}{3} = 4.$$

$$((3\bar{1}2)(1\bar{2}\bar{3}))^{-1} = (2\bar{1}3)(\bar{3}\bar{2}\bar{1}) \in N^{(31\bar{3})}$$

Thus,

$$31\bar{3}^{(ID)} = 31\bar{3}.$$

$$31\bar{3}^{(3\bar{1}2)(1\bar{2}\bar{3})} = \bar{1}\bar{2}1.$$

$$31\bar{3}^{(2\bar{1}3)(\bar{3}\bar{2}\bar{1})} = 2\bar{3}\bar{2}.$$

Therefore, the equal cosets are $31\bar{3} = \bar{1}\bar{2}1 = 2\bar{3}\bar{2}$

To find the other 3 distinct cosets of $[31\bar{3}]$, we conjugate the above equal coset by the following transversals set:

$$\{Id, (123)(\bar{1}\bar{2}\bar{3}), (132)(\bar{1}\bar{3}\bar{2}), ((13\bar{2})(2\bar{1}\bar{3})\}.$$

- $(31\bar{3} = \bar{1}\bar{2}1 = 2\bar{3}\bar{2})^{(Id)} \quad \Rightarrow \quad 31\bar{3} = \bar{1}\bar{2}1 = 2\bar{3}\bar{2}.$
- $(31\bar{3} = \bar{1}\bar{2}1 = 2\bar{3}\bar{2})^{(123)(\bar{1}\bar{2}\bar{3})} \quad \Rightarrow \quad 12\bar{1} = \bar{2}\bar{3}2 = 3\bar{1}\bar{3}.$
- $(31\bar{3} = \bar{1}\bar{2}1 = 2\bar{3}\bar{2})^{(13\bar{2})(2\bar{1}\bar{3})} \quad \Rightarrow \quad \bar{2}32 = \bar{3}13 = \bar{1}21.$
- $(31\bar{3} = \bar{1}\bar{2}1 = 2\bar{3}\bar{2})^{(132)(\bar{1}\bar{3}\bar{2})} \quad \Rightarrow \quad 23\bar{2} = \bar{3}\bar{1}3 = 1\bar{2}\bar{1}.$

These are the 4 distinct equal cosets of the double coset $[31\bar{3}]$.

The orbits of $N^{(31\bar{3})}$ on $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$ are $\{3, \bar{1}, 2\}$, and $\{1, \bar{2}, \bar{3}\}$. We take a representative from each orbit, and multiply it by $Nt_3t_1\overline{t_3}$, we get :

1. $Nt_3t_1\overline{t_3}t_3 = Nt_3t_1 \in [31]$.
   3 t_i's will make $[31\bar{3}]$ collapse and go back to $[31]$.
2. $Nt_3t_1\overline{t_3}\overline{t_3} = Nt_3t_1t_3 \in [313]$.
   3 t_i's will make $[31\bar{3}]$ collapse and go back to $[313]$.

4.2.8 Double Coset $[32\bar{3}]$

The point stabiliser of $Nt_3t_2\overline{t_3}N$, is where the points 3, 2 and $\bar{3}$ being fixed in N and the rest of elements are permuted.

$\Rightarrow N^{32\bar{3}} = Id.$

We know,

$\Rightarrow nt_3t_2\overline{t_3} = \overline{t_1t_3}t_1.$

$\Rightarrow n = (3\bar{1}2)(\bar{3}1\bar{2})$.
$\Rightarrow (3\bar{1}2)(\bar{3}1\bar{2})t_3t_2\overline{t_3} = \overline{t_1}\overline{t_3}t_1$.
$\Rightarrow (1\bar{3}2)(\bar{1}3) \in N^{(32\bar{3})}$.
and
$nt_3t_2\overline{t_3} = \overline{t_2}t_1\overline{t_2}$.
$\Rightarrow n = (3\bar{2}1)(\bar{3}2\bar{1})$.
$\Rightarrow (3\bar{2}1)(\bar{3}2\bar{1})t_3t_2\overline{t_3} = \overline{t_2}t_1\overline{t_2}$.
$\Rightarrow (12\bar{3})(3\bar{2}) \in N^{(32\bar{3})}$.
$\Rightarrow N^{(32\bar{3})} \geq < (1\bar{3}2)(\bar{1}3), (12\bar{3})(3\bar{2}) >$.
So the orbits of $N^{(32\bar{3})}$ on $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$ are $\{1, \bar{3}, 2\}$ and $\{\bar{1}, \bar{2}, 3\}$. Furthermore, the total number of single cosets in $[32\bar{3}]$ is :

$$\frac{|N|}{|N^{(32\bar{3})}|} = \frac{12}{3} = 4.$$

Now by taking a representation from each orbit and multiply it by $Nt_3t_1\overline{t_3}$, we get

- $Nt_3t_2\overline{t_3}t_3 = Nt_3t_2 \in [32]$.
  Three t_i's will make Cayley's graph collapse and go back to [32].

- $Nt_3t_2\overline{t_3}\overline{t_3} = N_3t_2t_3 \in [323] \in [313]$.
  Three t_i's will make Cayley's graph collapse and go back to [313].

Cayley's graph would look like this.

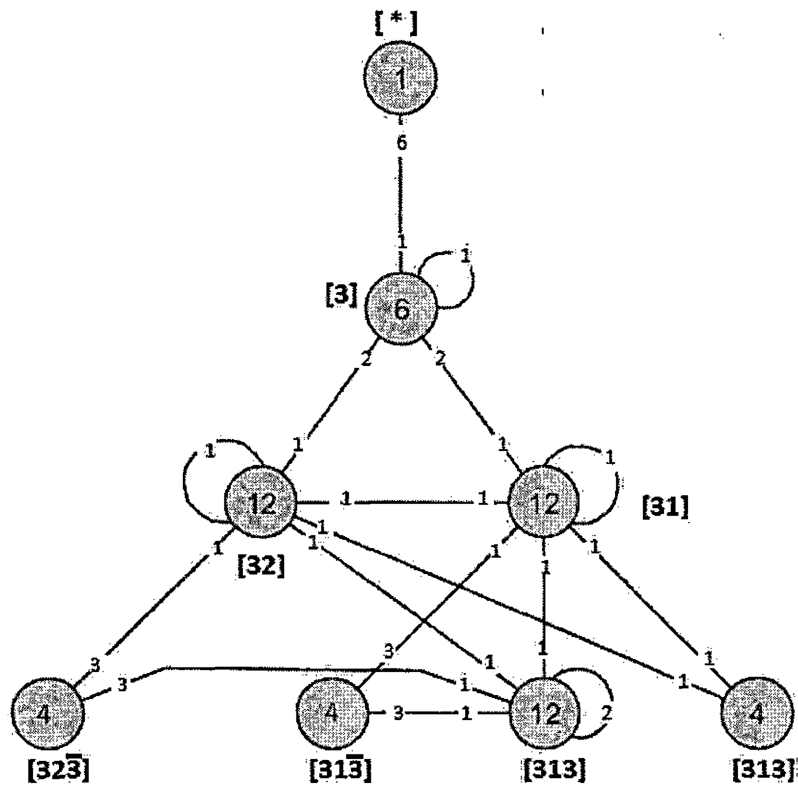

Figure 4.6: Cayley's graph for $L_2(7)$ over A_4

Chapter 5

Construction of Finite Homomorphic Images of $2^{\star 5} : S_5$

In this chapter, we will find and construct finite homomorphic images of $2^{\star 5} : S_5$. Given,

$$G < x, y, t > \cong \text{Group } \langle x, y, t | x^5, y^2, (xy)^4, t^2, (x^{-1}(xy)^1)^2, (x^{-2}(xy)^2)^3,$$
$$(x^{-3}(xy)^3)^4, (x^{-4}(xy)^4)^5, (x^{-2}yx^2y)^2, (t, x^2yx^{-1}), (t, y^x), (t, y),$$
$$(xyt)^a, ((xy)^{-1}t)^b, (xt^{(xy)^2})^c, (x^{(yt)^d}, y^{(xy)}xt^{(xy)^2})^e, (t^{(xy)}t^{(xy)^2})^f \rangle;$$

We will give range of values for each of the following parameters a, b, c, d, e, f. The smaller range, the less time takes the computer to generate subgroups. The chosen interval for the parameters a, b, c, d, e, f are as follow:

$$a \in [0..50], \quad b \in [0..20], \quad c \in [0..20],$$
$$d \in [0..20], \quad e \in [0..10], \quad f \in [0..10],$$

We let the code to run in the background without interaction. Each change of parameter will produce new subgroup. Note that some of the generated homomorphic image of $2^5 : S_5$ will be repeated more than once. For example, the parameters $(0, 0, 0, 0, 0, \mathbf{2})$ is isomorphic to the parameters $(0, 0, 0, 0, 4, \mathbf{2})$. Both of them generate the same group order,32. However, the bold face numbers that in the above parameters are the keys to generate the group, and the rest of parameters will enable us to add more factors to the group. For information, when we run this code, the process took couple of days to complete and there were about 100 pages of subgroups of $2^{\star 5} : S_5$. Please see table 1 for sample of generated subgroups of $2^{*5}:S_5$. If we consider the first entries of parameters in table 1, where

$a=0, b=0, c=0, d=0, e=0, f=2$, order of the group is 32

and substitute them in the above general form of the group G. We will obtain the following symmetric presentation:

$$G < x,y,t > \cong \text{Group } \langle x,y,t | x^5, y^2, (xy)^4, t^2, (x^{-1}(xy)^1)^2, (x^{-2}(xy)^2)^3, (x^{-3}(xy)^3)^4,$$
$$(x^{-4}(xy)^4)^5, (x^{-2}yx^2y)^2, (t, x^2yx^{-1}), (t, y^x), (t, y), (t^{(xy)}t^{(xy)^2})^2\rangle;$$

The control subgroup N, is S_5 which is the symmetric group of degree 5 on five letters $1,2,3,4$ and 5. N can be generated by x and y, where $x \sim (1,2,3,4,5)$ and $y \sim (1,5)$ and has order of $5! = 120$. Thus, $N = \langle x, y\rangle$.

5.1 Relation

$2^5 =< t_1 > \star < t_2 > \star < t_3 > \star < t_4 > \star < t_5 >$,

in which, each t_i is of order 2.

We let $t = t_4$ and take $x \sim (1,2,3,4,5)$, $y \sim (1,5)$.

The given relation is:

$(t^{(xy)}t^{(xy)^2})^2 = e$,

$xy = (1,2,3,4,5)(1,5) = (1,2,3,4)$

$(xy)^2 = (1,2,3,4)^2 = (1,3)(2,4)$.

Thus, our relation is $(t_4^{(xy)}t_4^{(xy)^2})^2 = (t_4^{(1,2,3,4)}t_4^{(1,3)(2,4)})^2 = e$,

$\Rightarrow (t_1t_2)^2 = e$.

$\Rightarrow t_1t_2t_1t_2 = e$.

Thus the final relation is:

$$t_1t_2 = t_2t_1.$$

5.2 Double Coset Enumeration of G over N

We will perform a double coset enumeration of G over N. This means, we determine all the double cosets $[w]$ until the set of right cosets is closed under right multiplication by the 5 t_is. Furthermore, we will determine the number of single cosets in each double coset.

It suffices to determine for the double coset $[w]$, the double coset to which Nwt_i belongs for one t_i from each orbit of the coset stabiliser $N^{(w)}$ of the coset on $\{1,2,3,4,5\}$.
We denote the first double coset NeN by $[\star]$. Since N is transitive on $\{1,2,3,4,5\}$, we take a representative coset N from $[\star]$, and a representative from $\{1,2,3,4,5\}$ and determine the double coset to which Nt_4 belongs.

$$Nt_4N = [4] = \{Nt_4^n | n \in N\} = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5\}$$

Therefore, the double coset [4] will have 5 single cosets. Since $|N^{(4)}| = 4! = 24$
The number of single cosets in [4] is given by $\frac{|N|}{|N^{(4)}|} = \frac{5!}{4!} = 5$
$N^{(0)} \geq <(1,2,3),(1,5)>$.
The orbits of $N^{(4)}$ on $\{1,2,3,4,5\}$ are $\{4\}$ and $\{1,2,3,5\}$. We take a representative from each orbit and multiply it by Nt_4, we get:

- $Nt_4 \cdot t_4 \sim Nt_4^2 \sim Ne = N \in [\star]$.
  Since, $t_4t_4 = t_4^2 = e \Rightarrow Nt_4t_4 = Ne = N$.
  t_4 will collapse Cayley's graph, and hence will take [4] to $[\star]$
- $Nt_4 \cdot t_2 \in [42]$.
  Four t_i's will take [4] to [42], and hence it will extend Cayley's graph.

In order to find the number of the single cosets in [42], we need to determine the point stabiliser of 4 and 2, N^{42}. Furthermore, in N, fixing 4 and 2 will result in permuting only three elements of N which will be 1, 3 and 5.
Thus,

$N^{42} = S_3 = <(1,3,5),(1,3)>$.

Note that;

$N^{(42)} \geq N^{42}$.

$Nt_4t_2N = \{N(t_4t_2)^n | n \in N\}$.

$N(t_4t_2)^{(42)} = Nt_2t_4 = Nt_4t_2$. (by relation $t_1t_2 = t_2t_1 \Rightarrow t_2t_4 = t_4t_2$).

$\Rightarrow (42) \in N^{(42)}$.

$\Rightarrow N^{(42)} \geq <N^{42},(42)> = <(135),(35),(24)>$.

This means that $|N^{(42)}| = |S_3| \cdot 2 = 3! \cdot 2 = 12$.

$\Rightarrow$ the number of single cosets in $[42] = \frac{|N|}{|N^{(42)}|} = \frac{120}{12} = 10$.

The transversals (right coset representatives) of $N^{(42)}$ are:

$\{Id(N), (1,2,3,4,5), (1,3,5,2,4), (1,5,4,3,2), (1,4,2,5,3), (1,3)(2,4,5), (1,5,4),$
$(1,3,4,5,2), (1,4,2,3), (1,2,3)\}$.

Therefore, different cosets are:

$(42 \sim 24)^{Id(N)} \Rightarrow 42 = 24.$

$(42 \sim 24)^{(1,2,3,4,5)} \Rightarrow 53 = 35.$

$(42 \sim 24)^{(1,3,5,2,4)} \Rightarrow 14 = 41.$

$(42 \sim 24)^{(1,5,4,3,2)} \Rightarrow 31 = 13.$

$(42 \sim 24)^{(1,4,2,5,3)} \Rightarrow 25 = 52.$

$(42 \sim 24)^{(1,3)(2,4,5)} \Rightarrow 54 = 45.$

$(42 \sim 24)^{(1,5,4)} \Rightarrow 12 = 21.$

$(42 \sim 24)^{(1,3,4,5,2)} \Rightarrow 51 = 51.$

$(42 \sim 24)^{(1,4,2,3)} \Rightarrow 23 = 32.$

$(42 \sim 24)^{(1,2,3)} \Rightarrow 43 = 34.$

Hence, the 10 distinct single cosets are:

$42, 53, 14, 31, 25, 54, 12, 51, 23, 43$

The orbits of $N^{(42)}$ on $\{1,2,3,4,5\}$ are $\{2,4\}$ and $\{1,3,5\}$.

Taking a representative from each orbit and multiplying with Nt_4t_2 and see if this multiplication will extend the Cayley's graph or collapse it.

- $Nt_4t_2 \cdot t_2 = Nt_4t_2t_2 \in [4]$.
  Since, $t_1t_2 = t_2t_1 \Rightarrow t_2t_4 = t_4t_2$, t_2 will take [42] to [4],

- $Nt_4t_2 \cdot t_3 \in [423]$.
  The three t_i's, where $i \in \{1,3,5\}$ will extend Cayley's graph from [42] to [423].

Now, for the double coset $Nt_4t_2t_3N$, denoted by [423], need to determine how many single cosets are in the double coset and which t_i's will extend Cayley's graph.

Therefore, we must first determine the point stabilsier N^{423}. This means we fix 4,2 and 3 in N, and permute the other two elements.

Thus, $N^{423} =< (1,5) > \quad \Rightarrow |N^{(423)}| = 2!$.

We know that $N^{(423)} \geq N^{423}$.

Now, $Nt_4t_2t_3N = \{N(t_4t_2t_3)^n | n \in N\}$.

We have $N(t_4t_2t_3)^{(423)} = Nt_2t_3t_4 = Nt_4t_2t_3$. $\qquad$ (since $t_4t_2 = t_2t_4$)

$\Rightarrow (423) \in N^{(423)}$.

$\Rightarrow N^{(423)} \geq < N^{423}, (423) >$.

we look for more relations,

$N(t_4t_2t_3)^{(23)} = Nt_4t_3t_2$.

$\Rightarrow (23) \in N^{(423)}$.

$\Rightarrow N^{(423)} \geq < N^{423}, (423), (23) > \cong 2 \times S_3$.

This means that $|N^{(423)}| = 2! \cdot 3 \cdot 2 = 12$.

$\Rightarrow$ the number of single cosets in $[423] = \frac{|N|}{|N^{(423)}|} = \frac{120}{12} = 10$.

$4\underline{23} = \underline{43}2 = 3\underline{42} = \underline{32}4 = 2\underline{34} = 243$.

Thus;

$423 = 432 = 342 = 324 = 234 = 243$.

We conjugate the above equal cosets by 10 transversals which are:

$\{Id(N), (1,2,3,4,5), (1,3,5,2,4), (1,2,3,4), (1,4,2,5,3), (1,3,4,5), (1,5,4,3,2),$
$(1,4)(2,3,5), (1,5,3,2), (1,4,5)(2,3)\}$.

Conjugating $(423 = 432 = 342 = 324 = 234 = 243)$ with transversals:

$(423 = 432 = 342 = 324 = 234 = 243)^{Id(N)}$

$\Rightarrow 423 = 432 = 342 = 324 = 234 = 243$.

$(423 = 432 = 342 = 324 = 234 = 243)^{(1,2,3,4,5)}$

$\Rightarrow 534 = 543 = 453 = 435 = 345 = 354$.

$(423 = 432 = 342 = 324 = 234 = 243)^{(1,3,5,2,4)}$

$\Rightarrow 145 = 154 = 514 = 541 = 451 = 415$.

$(423 = 432 = 342 = 324 = 234 = 243)^{(1,2,3,4)}$

$\Rightarrow 134 = 143 = 413 = 431 = 341 = 314$.

$(423 = 432 = 342 = 324 = 234 = 243)^{(1,4,2,5,3)}$

$\Rightarrow 251 = 215 = 125 = 152 = 512 = 521$.

$(423 = 432 = 342 = 324 = 234 = 243)^{(1,3,4,5)}$

$\Rightarrow 524 = 542 = 452 = 425 = 245 = 254$.

$(423 = 432 = 342 = 324 = 234 = 243)^{(1,5,4,3,2)}$

$\Rightarrow 312 = 321 = 231 = 213 = 123 = 132$.

$(423 = 432 = 342 = 324 = 234 = 243)^{(1,4)(2,3,5)}$

$\Rightarrow 135 = 153 = 513 = 531 = 351 = 315$.

$(423 = 432 = 342 = 324 = 234 = 243)^{(1,5,3,2)}$

$\Rightarrow 412 = 421 = 241 = 214 = 124 = 142$.

$(423 = 432 = 342 = 324 = 234 = 243)^{(1,4,5)(2,3)}$

$$\Rightarrow 532 = 523 = 253 = 235 = 325 = 352.$$

The 10 distinct single cosets of $Nt_4t_2t_3N$ are given below:

$[423] = \{423, 534, 145, 134, 251, 524, 312, 135, 412, 532\}$.

The orbits of $N^{(423)}$ on $\{1,2,3,4,5\}$ are $\{2,3,4\}$ and $\{1,5\}$.

We take a representative from each orbit and multiply it by $Nt_4t_2t_3$,we will have the following:

- $Nt_4t_2t_3 \cdot t_2 = Nt_4t_2t_3t_2 = Nt_4t_2t_2t_3 = Nt_4t_3 \in [42]$.
- $Nt_4t_2t_3 \cdot t_5 \in [4235]$.

  New double coset that will extend Cayley's graph from [423] is [4235]. Hence, t_1 and t_5 will extend it.

The double coset $Nt_4t_2t_3t_5N$, which is denoted by [4235], has point stabiliser of $N^{4235} =< e >$, since four points are fixed and only one can permute.

However, using the above relations, we can see that $N^{(4235)} \geq N^{4235}$.

Now, we have

$423 \sim 243$

$\Rightarrow 4235 \sim 2435$

$42\underline{35} = \underline{42}53 = 2\underline{45}3 = \underline{25}43 = 5243 = 2435.$

$\Rightarrow (5243) \in N^{(4235)}$

$\Rightarrow N^{(4235)} \geq < e, (24), (423), (4235) > = < (2345), (23) >.$

$\Rightarrow |N^{(4235)}| = 2 \cdot 3 \cdot 4 = 24.$

$\Rightarrow$ number of single cosets are $\frac{|N|}{|N^{(4235)}|} = \frac{120}{24} = 5.$

The transversals of $N^{(4235)}$ are:

$$\{Id(N), (1,2,3,4,5), (1,5,2,4), (1,3,5,2,4), (1,4,2,5,3)\}.$$

We have the equal cosets:

$4352 = 2354 = 5234 = 3425 = 3524 = 2534 = 3245 = 2345 = 5423 = 2543 = 4235$
$= 4325 = 2435 = 5243 = 4523 = 4532 = 3542 = 5342 = 3254 = 3452 = 5324 = 4253 =$
$2453 = 5432.$

If we conjugate the above equal cosets by each of the transversals, we will find:

1. $(4352 = 2354 = 5234 = 3425 = 3524 = 2534 = 3245 = 2345 = 5423 = 2543 = 4235 = 4325 = 2435 = 5243 = 4523 = 4532 = 3542 = 5342 = 3254 = 3452 =$

$5324 = 4253 = 2453 = 5432)^{Id(N)}$

$\Rightarrow (4352 = 2354 = 5234 = 3425 = 3524 = 2534 = 3245 = 2345 = 5423 = 2543 = 4235 = 4325 = 2435 = 5243 = 4523 = 4532 = 3542 = 5342 = 3254 = 3452 = 5324 = 4253 = 2453 = 5432.$

2. $(4352 = 2354 = 5234 = 3425 = 3524 = 2534 = 3245 = 2345 = 5423 = 2543 = 4235 = 4325 = 2435 = 5243 = 4523 = 4532 = 3542 = 5342 = 3254 = 3452 = 5324 = 4253 = 2453 = 5432)^{(1,2,3,4,5)}$

   $\Rightarrow 5431 = 4351 = 1345 = 3154 = 5134 = 3514 = 3145 = 1543 = 1453 = 4513 = 1534 = 4315 = 3415 = 4531 = 4135 = 3541 = 4153 = 5143 = 1354 = 5341 = 5413 = 5314 = 3451 = 1435.$

3. $(4352 = 2354 = 5234 = 3425 = 3524 = 2534 = 3245 = 2345 = 5423 = 2543 = 4235 = 4325 = 2435 = 5243 = 4523 = 4532 = 3542 = 5342 = 3254 = 3452 = 5324 = 4253 = 2453 = 5432)^{(1,5,2,4)}$

   $\Rightarrow 1243 = 3241 = 2341 = 1324 = 1342 = 3421 = 2431 = 1423 = 4123 = 3142 = 2413 = 4231 = 4321 = 4312 = 3412 = 4132 = 3214 = 3124 = 4213 = 2134 = 1432 = 1234 = 2314 = 2143.$

4. $(4352 = 2354 = 5234 = 3425 = 3524 = 2534 = 3245 = 2345 = 5423 = 2543 = 4235 = 4325 = 2435 = 5243 = 4523 = 4532 = 3542 = 5342 = 3254 = 3452 = 5324 = 4253 = 2453 = 5432)^{(1,3,5,2,4)}$

   $\Rightarrow 2451 = 1245 = 2514 = 1425 = 1254 = 5421 = 2541 = 4125 = 1524 = 5241 = 2415 = 1542 = 5124 = 1452 = 4512 = 4521 = 5214 = 4152 = 4215 = 5142 = 5412 = 2145 = 2154 = 4251.$

5. $(4352 = 2354 = 5234 = 3425 = 3524 = 2534 = 3245 = 2345 = 5423 = 2543 = 4235 = 4325 = 2435 = 5243 = 4523 = 4532 = 3542 = 5342 = 3254 = 3452 = 5324 = 4253 = 2453 = 5432)^{(1,4,2,5,3)}$

   $\Rightarrow 2531 = 2513 = 1325 = 3521 = 5231 = 1253 = 1523 = 5123 = 3152 = 5132 = 3512 = 1532 = 5213 = 3215 = 5312 = 3251 = 2153 = 1352 = 2135 = 1235 = 2315 = 5312 = 2351 = 5321.$

Hence, the distinct cosets of $Nt_4t_2t_3t_5N$ will be :

$$[4235] = \{4235, 4352, 5431, 1243, 2451, 2531\}$$

The orbits of $N^{(4235)}$ on $\{1,2,3,4,5\}$ are $\{1\}$, $\{2,3,4,5\}$.

We take a representative from each orbit and multiply by $Nt_4t_2t_3t_5$, we get:

- $Nt_4t_2t_3t_5 \cdot t_1 \in [42351]$.
  New double coset that will extend Cayley's graph.
- $Nt_4t_2t_3t_5 \cdot t_2 = Nt_4t_3t_5 \in [435] \in [423]$.

At the double coset $Nt_4t_2t_3t_5t_1N$, [42351], we will have all elements being fixed in N,
$\Rightarrow N^{42351} =< e >$;
we already know that $N^{(4235)} \geq < (42), (4235) > \cong S_4$.
Now,

$42351 = 42315 \rightarrow (15) \in N^{(42351)}$.

$\Rightarrow N^{(42351)} \geq < (15), (42), (42351 >$.
Since one of the generators of $N^{(42351)}$ is (42351), this would indicate that the orbit of $N^{(42351)}$ on $\{1,2,3,4,5\}$ is the single orbit, which is $\{1,2,3,4,5\}$.
The transversal of $N^{(42351)} = Id(N)$. Therefore,
$(12543 = 51342 = 45312 = 43125 = 25143 = 45321 = 42153 = 25134 = 14253 = 51324 = 32514 = 31542 = 25314 = 13524 = 42351 = 32415 = 23145 = 3412,5 = 41532 = 31245 = 42513 = 41352 = 15423 = 54231 = 51423 = 43251 = 42135 = 25413 = 15342 = 41235 = 15243 = 34521 = 23154 = 53421 = 15234 = 42531 = 21543 = 24513 = 45132 = 45123 = 52413 = 32541 = 34152 = 32154 = 31254 = 21435 = 12435 = 35214 = 41325 = 35412 = 15432 = 12534 = 53124 = 14352 = 12453 = 123,54 = 13245 = 24531 = 23451 = 52431 = 15324 = 54312 = 53412 = 43521 = 35241 = 41523 = 52134 = 21453 = 52314 = 52341 = 14532 = 52143 = 35124 = 45231 = 43152 = 35142 = 21345 = 53214 = 35421 = 24315 = 13452 = 43215 = 24135 = 24351 = 31425 = 12345 = 31524 = 54321 = 34251 = 45213 = 42315 = 13425 = 21534 = 14523 = 31452 = 14235 = 41253 = 51432 = 54132 = 34215 = 51234 = 13542 = 23415 = 43512 = 24153 = 53142 = 54123 = 34512 = 23514 = 32451 = 32145 = 51243 = 25431 = 21354 = 13254 = 54213 = 14325 = 23541 = 25341 = 53241)^{Id(N)}$.

$$\Downarrow$$

$12543 = 51342 = 45312 = 43125 = 25143 = 45321 = 42153 = 2,5,134 = 14253 = 51324 = 32514 = 31542 = 25314 = 13524 = 42351 = 32415 = 23145 = 34125 = 41532 =$

31245 = 42513 = 41352 = 15423 = 54231 = 51423 = 43251 = 42135 = 25413 = 15342 = 41235 = 15243 = 34521 = 23154 = 53421 = 15234 = 42531 = 21543 = 24513 = 45132 = 45123 = 52413 = 32541 = 34152 = 32154 = 31254 = 21435 = 12435 = 35214 = 41325 = 35412 = 15432 = 12534 = 53124 = 14352 = 12453 = 12354 = 13245 = 24531 = 23451 = 52431 = 15324 = 54312 = 53412 = 43521 = 35241 = 41523 = 52134 = 21453 = 52314 = 52341 = 14532 = 52143 = 35124 = 45231 = 43152 = 35142 = 21345 = 53214 = 35421 = 24315 = 13452 = 43215 = 24135 = 24351 = 31425 = 12345 = 31524 = 54321 = 34251 = 45213 = 42315 = 13425 = 21534 = 14523 = 31452 = 14235 = 41253 = 51432 = 54132 = 34215 = 51234 = 13542 = 23415 = 43512 = 24153 = 53142 = 54123 = 34512 = 23514 = 32451 = 32145 = 51243 = 25431 = 21354 = 13254 = 54213 = 14325 = 23541 = 25341 = 53241.

This step would terminate the double coset enumeration.

Hence, all the double coset along with the single cosets they contain have been found.

1. There are six double cosets which are $[\star], [4], [42], [423], [4235], [42351]$.

2. There are a total of $1, 5, 10, 10, 5, 1$ single cosets respectively for the above double cosets.

3. $G = NeN \cup Nt_4N \cup Nt_4t_2N \cup Nt_4t_2t_3N \cup Nt_4t_2t_3t_5N \cup Nt_4t_2t_3t_5t_1N$.

4. $|G| = (\frac{|N|}{|N|} + \frac{|N|}{|N^{(4)}|} + \frac{|N|}{|N^{(42)}|} + \frac{|N|}{|N^{(423)}|} + \frac{|N|}{|N^{(4235)}|} + \frac{|N|}{|N^{(42351)}|}) \cdot |N|$.

5. $|G| = (1 + 5 + 10 + 10 + 5 + 1) \cdot 120 = 3840$.

Now according to Magma, the order of G is 3840, this results will confirm that our double coset enumeration of G over our control group $N = S_5$ is correct.

Table 5.1: Some finite subgroups of $2^{*5} : S_5$

a	b	c	d	e	f	Order of G
0	0	0	0	0	2	32
0	0	0	0	4	0	1042
0	0	0	0	4	2	32
0	0	0	0	4	4	1024
0	0	0	0	4	6	32
0	0	0	0	4	8	1024
0	0	0	0	4	10	32
0	0	0	0	8	2	32
0	0	0	1	0	2	32
0	0	0	1	6	2	32
0	0	0	2	0	2	32
0	0	0	2	4	0	1024
0	0	0	2	4	2	32
0	0	0	2	4	4	1024
0	0	0	2	4	6	32
0	0	0	2	4	8	1024
0	0	0	2	4	10	32
0	0	0	2	8	2	32
0	0	0	3	0	2	32
0	0	0	3	3	3	720
0	0	0	3	6	2	32
0	0	0	4	0	2	32
0	0	0	4	4	0	1024
0	0	0	4	4	2	32
0	0	0	4	4	4	1024
0	0	0	4	4	6	32
0	0	0	4	4	8	1024
0	0	0	4	4	10	32
0	0	0	4	8	2	32
0	0	0	5	0	2	32
0	0	0	5	3	3	720
0	0	0	5	6	2	32
0	0	0	6	0	2	32
0	0	0	6	4	0	1024

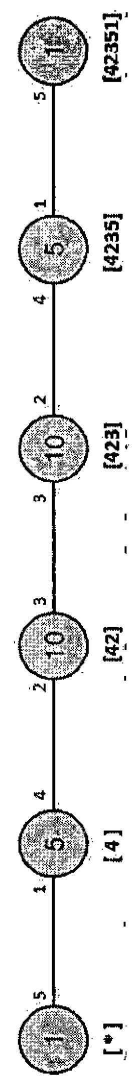

Figure 5.1: Cayley's graph

Table 5.2: Some discovered finite subgroups of $2^{*5} : S_5$

a	b	c	d	e	f	Order of G	Group Name
0	0	0	0	0	2	32	$2^5 : S_5$
0	8	10	2	4	4	1024	$2^{10} : S_5$
0	0	0	3	3	3	720	$S_6 : S_5$
0	0	5	0	0	0	16	$S_5 \times 2^4$
0	10	6	1	3	3	12	$S_6 \times 2$
0	0	8	1	0	4	1960	$PGL_2(49) \times 2$
0	0	10	0	0	3	162	$(3^4 : 2) : S_5$
0	0	10	1	0	7	4802	$(7^4 : 2) : S_5$
0	10	18	0	5	3	236196	$(3^9 : 2) : S_6$
0	0	10	3	0	9	13122	$(3^8 : 2) : S_5$
0	0	10	0	0	5	1250	$(5^4 : 2) : S_5$
0	0	12	0	0	3	6144	$(2^{10} : S_6)$
0	5	0	0	0	0	6	S_6

Chapter 6

Construction of $2^{\star 5} : S_5$ in MAGMA

In this chapter we will write the MAGMA code for $\frac{2^{\star 5}:S_5}{t_1 t_2 = t_2 t_1}$ and give a compute-based proof that $G \cong 2^{\star 5} : S_5$.

We have,

G $\langle x, y, t \rangle \cong$ Group $\langle x, y, t | x^5, y^2, (xy)^2, t^2, (x^{-1}(xy)^1)^2, (x^{-2}(xy)^2)^2, (x^{-3}(xy)^3)^4,$
$(x^{-4}(xy)^4)^5, (x^{-2}yx^2y)^2, (t, x^2yx^{-1}), (t, y^x), (t, y), (t^{(xy)}t^{(xy)^2})^2 \rangle;$

The group input into MAGMA as below:

```
> N:=Sym(5);
> xx:=N!(1,2,3,4,5);
> yy:=N!(1,5);
> N:=sub<N|xx,yy>;
> G<x,y,t>:= Group<x,y,t|x^5, y^2,(x*y)^4,  (x^-1 *(x*y)^1)^2,
 (x^-2*(x*y) ^2)^3, (x^-3 *(x*y)^3)^4, (x^-4 *(x*y)^4)^5,
 (x^-2 *y *x^2*y)^2,t^2, (t,x^2*y*x^-1),
 (t,y^x),(t,y),(t^(x*y)*t^((x*y)^2))^2>;
> Index(G,sub<G|x,y>);
32
>
```

Figure 6.1: Magma input

Note: Index will tell us how many single cosets are there in the above defined group.

6.1 Relations

We know that,

$$(t_1t_2)^2 = e;$$
$$\text{OR}$$
$$t_1t_2 = t_2t_1.$$

In order to start the double coset enumeration for G over N using computer based, MAGMA, we must define the entries of x in terms of the stabilised element, t_4.
We must store each entry of x under a unique labeling called $ts[i]$ where $i \in \{1,2,3,4,5\}$. (See figure 2)
We have $t \sim t_4 \quad \Rightarrow \quad ts[4] = f(t);$
Now, we define x in respect to the above expression.

1. $t_5 = t^{4^x}$
   $\Rightarrow ts[5] = f(t_4^x).$

2. $t_1 = t^{4^{x^2}}$
   $\Rightarrow ts[1] = f(t_4^{x^2}).$

3. $t_2 = t_4^{x^3}.$
   $\Rightarrow ts[2] = f(t_4^{x^3}).$

4. $t_3 = t_4^{x^4}.$
   $\Rightarrow ts[3] = f(t_4^{x^4}).$

```
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> IN:=sub<G1|f(x),f(y)>;
> ts:=[Id(G1) : i in [1..5]];
> ts[4]:=f(t); ts[5]:=f(t^x); ts[1]:=f(t^(x^2)); ts[2]:=f(t^(x^3));
 ts[3]:=f(t^(x^4));
```

Figure 6.2: Defining ts's

According to figure 5.2 output, there exist 32 single cosets. Each single coset is stored in a **cst**. The number of how many cst's exist in a double coset, is denoted by m. Thus, each new double coset we determine, there exist a counter called, m, $m = [1..32]$ that is created. If the double coset is new and therefore will extend Cayley's graph, then m will increase. Otherwise, m remains unchanged.

Figure 5.3 will show how cst's are stored and labeled [1..32].

```
> for i in [1..31] do i, cst[i];end for;
1 []
2 [ 4 ]
3 [ 5 ]
4 [ 3 ]
5 [ 1 ]
6 [ 5, 4 ]
7 [ 4, 3 ]
8 [ 2 ]
9 [ 1, 4 ]
10 [ 1, 5 ]
11 [ 3, 2 ]
12 [ 4, 2 ]
13 [ 2, 5 ]
14 [ 5, 3 ]
15 [ 1, 2 ]
16 [ 5, 4, 1 ]
17 [ 3, 2, 4 ]
18 [ 3, 1 ]
19 [ 5, 4, 2 ]
20 [ 4, 3, 5 ]
21 [ 1, 4, 2 ]
22 [ 2, 1, 5 ]
23 [ 3, 2, 1 ]
24 [ 4, 3, 1 ]
25 [ 3, 1, 5 ]
26 [ 3, 2, 5 ]
27 [ 2, 1, 4, 5 ]
28 [ 1, 4, 2, 3 ]
29 [ 1, 5, 3, 4 ]
30 [ 4, 3, 5, 2 ]
31 [ 5, 3, 1, 2 ]
```

Figure 6.3: Show how cst's are stored and labeled [1..32].

The corresponding Cayley's diagram as illustrated earlier is as below:

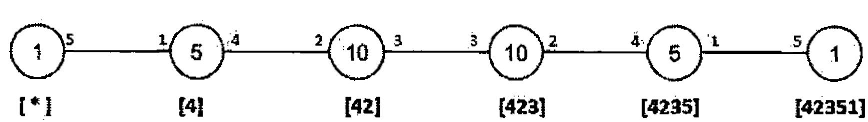

Table 1 below, expresses the content of each double cosets that is determined in the group with the value of corresponding m.
Note that, in MAGMA,

1. m does not count the first double coset, [$\star$], and therefore, 1 must be added each time the codes run.

2. Each new double coset will increase the value of m.
   For example; the double coset [4] will have 5 single cosets, and the double coset [42] will have 10. MAGMA will produce the value of $m = 15$. Since, m counts the most updated total of single cosets and not individually computed.

Table 1, defines the double cosets of $2^{\star 5} : S_5$ with corresponding single cosets;

Double Cosets	Single Cosets #
[$\star$]	1
[4]	5
[42]	10
[423]	10
[4235]	5
[42351]	1

6.2 Composition Factors

When we type the command of composition factors in Magma, the following result will be: (see Figure 5.4)

```
> f1,G1,k1:=CosetAction(G,sub<G|x,y>);
> CompositionFactors(G1);
  G
  |  Cyclic(2)
  *
  |  Alternating(5)
  *
  |  Cyclic(2)
  *
  |  Cyclic(2)
  *
  |  Cyclic(2)
  *
  |  Cyclic(2)
  *
  |  Cyclic(2)
  1
```

Figure 6.4: Composition factors of $2^{*5} : S_5$

To discuss the above further, we name each subgroup from bottom to top with the following label,

$\star \rightarrow G_6 \rightarrow G_6/1 = C_2 \Rightarrow G_6 \cong C_2$, and hence $G_6 \trianglelefteq G_5$

$\star \rightarrow G_5 \rightarrow G5/G_6 = C_2$

$\star \rightarrow G_4 \rightarrow G_4/G_5 = C_2$

$\star \rightarrow G_3 \rightarrow G_3/G_4 = C_2$

$\star \rightarrow G_2 \rightarrow G_2/G_3 = C_2$

$\star \rightarrow G_1 \rightarrow G_1/G_2 = A_5$

$\star \rightarrow G \rightarrow G/G_2 = C_2$

$\Rightarrow$

$G_6/1 = C_2 \Rightarrow G_6 = C_2.$

$G_5/G_6 = C_2 \Rightarrow G_5 = C_2 \times G_6 = C_2 \times C_2.$

$G_4/G_5 = C_2 \Rightarrow G_4 = C_2 \times C_2 \times C_2.$

$G_3/G_4 = C_2 \Rightarrow G_3 = C_2 \times C_2 \times C_2 \times C_2.$

$G_2/G_3 = C_2 \Rightarrow G_2 = C_2 \times C_2 \times C_2 \times C_2 \times C_2.$

$G_1/G_2 = A_5 \Rightarrow G_1 = A_1 : C_2 \times C_2 \times C_2 \times C_2 \times C_2.$

$G/G_2 = C_2 \Rightarrow G = C_2 : A_1 : C_2 \times C_2 \times C_2 \times C_2 \times C_2.$

$$\Rightarrow G = \underbrace{C_2 : A_5}_{S_5} : \underbrace{C_2 \times C_2 \times C_2 \times C_2 \times C_2}_{2^5}.$$

⇒ our group is isomorphic to $2^{\star 5} : S_5$.

The normal subgroup lattice will be:

```
Normal subgroup lattice
-----------------------

[1]  Order 1     Length 1  Maximal Subgroups:
---
[2]  Order 2     Length 1  Maximal Subgroups: 1
---
[3]  Order 16    Length 1  Maximal Subgroups: 1
---
[4]  Order 32    Length 1  Maximal Subgroups: 2 3
---
[5]  Order 960   Length 1  Maximal Subgroups: 3
---
[6]  Order 1920  Length 1  Maximal Subgroups: 5
[7]  Order 1920  Length 1  Maximal Subgroups: 5
[8]  Order 1920  Length 1  Maximal Subgroups: 4 5
---
[9]  Order 3840  Length 1  Maximal Subgroups: 6 7 8
```

Figure 6.5: Normal subgroup lattice

The corresponding Normal Lattice is :

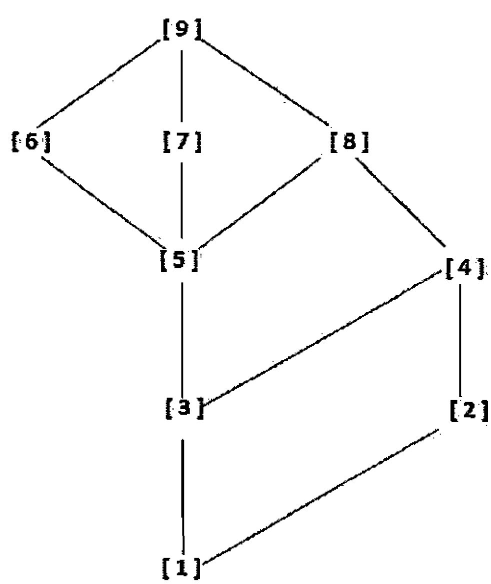

Figure 6.6: Normal subgroup lattice diagram

Now, to check if our above assumption is true and hence $G = 2^{*5} : S_5$,

$\Rightarrow G_6 \trianglelefteq G_5 \trianglelefteq G_4 \trianglelefteq G_3 \trianglelefteq G_2$.

The corresponding command for the above is:

```
> H<a,b,c,d,e>:= Group <a,b,c,d,e|a^2,b^2,c^2,d^2,e^2,
  (a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),
  (c,e),(d,e)>;
> f4,H4,k4:= CosetAction(H,sub<H|Id(H)>);
> r:=IsIsomorphic(H4,NL[4]);
> r;
true
```

Figure 6.7: MAGMA code to check for isomorphism against $NL[4]$

Therefore, G_4 is isomorphic to $NL[4]$ and hence we can write:

$G_6 \trianglelefteq G_5 \trianglelefteq G_4 \trianglelefteq G_3 \trianglelefteq G_2$.

We continue the process to the next level up in the Normal Lattice.

Now,

According to the Normal Lattice diagram, $NL[8]$ is the next possible isomorphic group that we can check against. It turned to be true as stated below.

```
> H<a,b,c,d,e,f,g,h>:= Group <a,b,c,d,e,f,g,h|a^2,b^2,c^2,d^2,
  e^2,(a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),
  (d,e),f^3,g^3,h^3,(f*g)^2,(f*h)^2,(g*h)^2,a^f=d,b^f=b,c^f=c,
  d^f=e,e^f=a,a^g=a,b^g=d,c^g=c,d^g=e,e^g=b,a^h=a,b^h=b,
  c^h=d,d^h=e,e^h=c >;
> f4,H5,k5:= CosetAction(H,sub<H|Id(H)>);
> r:=IsIsomorphic(H5,NL[8]);
> r;
true
```

Figure 6.8: MAGMA code to check for isomorphism against $NL[8]$

Finally, we check if G is isomorphic with $NL[9]$, MAGMA answer's:

```
> H<a,b,c,d,e,f,g,h,i>:= Group <a,b,c,d,e,f,g,h,i|a^2,b^2,c^2,d^2,
  e^2,(a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e),
  f^3,g^3,h^3,(f*g)^2,(f*h)^2,(g*h)^2,a^f=d,b^f=b,c^f=c,d^f=e,
  e^f=a,a^g=a,b^g=d,c^g=c,d^g=e,e^g=b,a^h=a,b^h=b,c^h=d,d^h=e,
  e^h=c,i^2,a^i=b,a^b=a,c^i=c,d^i=d,e^i=e,f^i=g,g^i=f,h^i=h >;
 > f6,H6,k6:= CosetAction(H,sub<H|Id(H)>);
 > r:=IsIsomorphic(H6,NL[9]);
 > r;
true
```

Figure 6.9: MAGMA code to check for isomorphism against $NL[9]$

Therefore, From figure 6.8 we conclude that G is isomorphic to $2^{*5} : S_5$. The complete MAGMA code is as below:

```
N:=Sym(5);
xx:=N!(1,2,3,4,5);
yy:=N!(1,5);
N:=sub<N|xx,yy>;
G<x,y,t>:= Group<x,y,t|x^5, y^2,(x*y)^4, t^2, (x^-1 *(x*y)^1)^2,
 (x^-2*(x*y)^2)^3, (x^-3 *(x*y)^3)^4, (x^-4 *(x*y)^4)^5,
(x^-2 *y *x^2*y)^2, (t,x^2*y*x^-1), (t,y^x),(t,y);
(t^(x*y)*t^((x*y)^2))^2>;
Index(G,sub<G|x,y>);

f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
ts:=[Id(G1) : i in [1..5]];
ts[4]:=f(t); ts[5]:=f(t^x); ts[1]:=f(t^(x^2)); ts[2]:=f(t^(x^3));
ts[3]:=f(t^(x^4));

prodim := function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
cst := [null : i in [1 .. 32]] where null is [Integers() | ];
for i := 1 to 5 do
cst[prodim(1, ts, [i])] := [i];
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
end for; m;

N42:=Stabiliser(N,[4,2]); N42s:=N42;
for g in N do if 4^g eq 2 and 2^g eq 4 then N42s:=sub<N|N42s,g>;
end if; end for;
T:=Transversal (N,N42s);
for i := 1 to #T do
```

```
ss := [4,2]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
end for; m;


N423:=Stabiliser(N,[4,2,3]); N423s:=N423;
for g in N do if 4^g eq 3 and 3^g eq 2 and 2^g eq 4 then
N423s:=sub<N|N423s,g>; end if; end for;
T:=Transversal (N,N423s);
for i := 1 to #T do
ss := [4,2,3]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
N4235:=Stabiliser(N,[4,2,3,5]); N4235s:=N4235;
for g in N do if 4^g eq 4 and 5^g eq 2 and 3^g eq 3
and 2^g eq 5 then
N4235s:=sub<N|N4235s,g>; end if; end for;
T:=Transversal (N,N4235s);
for i := 1 to #T do
ss := [4,2,3,5]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1;
 end if; end for; m;
N42351:=Stabiliser(N,[4,2,3,5,1]); N42351s:=N42351;
for g in N do if 3^g eq 4 and 5^g eq 5 and 2^g eq 1 and 1^g eq 3
and 4^g eq 2 then N42351s:=sub<N|N42351s,g>; end if; end for;
T:=Transversal (N,N42351s);
for i := 1 to #T do
ss := [4,2,3,5,1]^T[i];
cst[prodim(1, ts, ss)] := ss;
```

```
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

 f1,G1,k1:=CosetAction(G,sub<G|x,y>);
 CompositionFactors(G1);
  CompositionFactors(G1);
  NL:=NormalLattice(G1);
  NL;

H<a,b,c,d,e>:= Group <a,b,c,d,e|a^2,b^2,c^2,d^2,e^2,(a,b),(a,c),
(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e)>;
f4,H4,k4:= CosetAction(H,sub<H|Id(H)>);
r:=IsIsomorphic(H4,NL[4]);
r;

H<a,b,c,d,e,f,g,h>:= Group <a,b,c,d,e,f,g,h|a^2,b^2,c^2,d^2,e^2,
(a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e),f^3,
g^3,h^3,(f*g)^2,(f*h)^2,(g*h)^2,a^f=d,b^f=b,c^f=c,d^f=e,e^f=a,
a^g=a,b^g=d,c^g=c,d^g=e,e^g=b,a^h=a,b^h=b,c^h=d,d^h=e,e^h=c >;
f4,H5,k5:= CosetAction(H,sub<H|Id(H)>);
r:=IsIsomorphic(H5,NL[8]);
r;
H<a,b,c,d,e,f,g,h,i>:= Group <a,b,c,d,e,f,g,h,i|a^2,b^2,c^2,d^2,
e^2,(a,b),(a,c),(a,d),(a,e),(b,c),(b,d),(b,e),(c,d),(c,e),(d,e),
f^3,g^3,h^3,(f*g)^2,(f*h)^2,(g*h)^2,a^f=d,b^f=b,c^f=c,d^f=e,e^f=a,
a^g=a,b^g=d,c^g=c,d^g=e,e^g=b,a^h=a,b^h=b,c^h=d,d^h=e,e^h=c,i^2,
a^i=b,a^b=a,c^i=c,d^i=d,e^i=e,f^i=g,g^i=f,h^i=h >;
f6,H6,k6:= CosetAction(H,sub<H|Id(H)>);
r:=IsIsomorphic(H6,NL[9]);
r;
```

Chapter 7

Wreath Product of Permutation Groups

7.1 Definition

Definition 32. *Let H and K be permutation groups acting on sets X and Y respectively . We shall describe a very important way of constructing a new permutation group called the wreath product of H and K. This is to act on the set product $Z = X \times Y$.*

There are two types of the wreath product:

1. The unrestricted wreath product of X and Y, given by X Wr Y or in symbol $X \wr Y$.
2. The restricted wreath product of X and Y. Given by X wr Y.

Define $H \leq S_X$ and $K \leq S_Y$.
Let,

$$Z = X \times Y, \text{ such that } X \cap Y = \phi.$$

Define permutation group on Z, and let $\gamma \in H$, $y \in Y$ and $k \in K$.
now let,

$$\gamma(y) = \begin{cases} (x, y) & \mapsto (x\gamma, y) \\ (x, y_1) & \mapsto (x, y_1), \quad y \neq y_1. \end{cases}$$

$\gamma(y) \in S_Z$ since $(\gamma(y))^{-1} = \gamma^{-1}(y)$.
Then,

$$\phi : H \xrightarrow{1-1} S_Z \qquad \Longrightarrow \qquad H = \{\gamma(y) | \gamma \in H\} = H(y).$$

$\gamma \longmapsto \gamma(y)$

$$< H(y)|y \in Y >= Dr_{y\in Y} H(y)$$

Note : $\gamma(y)$ and $\gamma(y_1)$, do not move the same element of Z. $\quad (y \neq 1)$

$H(y_i) \prod < H(y)|y \in Y, y \neq y_i > \quad = 1$ and

$H(y_i) \trianglelefteq < H(y)|y \in >$.

Define,

$k \in K$, define $\quad k^\star(x, y) \longmapsto (x, yk)$.

Since

$(k^\star)^{-1} = (k^{-1})^\star, \quad k^\star \in S_Z$.

So given,

$\psi : K \longrightarrow S_Z$

$k \overset{1-1}{\longmapsto} k^\star$

$\Rightarrow K \cong \{k^\star | k \in K\} = k^\star$.

Therefore, the functions $\gamma \longmapsto \gamma(y)$ (y is a fixed element of Y with image $H(y)$), and $k \longmapsto k^\star$ with image $K^\star$ are monomorphism from H and K to Sym Z. This is written $H \wr K =< H(y), k^\star y \in Y >$, this is called the **Base**,B.

B $= Dr_{y\in Y} H(y) : K^\star$. Note:$(k^\star)^{-1} H(y) k^\star$ maps:

1. $(x, yk) \longmapsto (x\gamma, yk)$.

2. $(x, y_1) \longmapsto (x, y_1), \quad y_1 \neq yk$.

Hence, by definition

$$(k^\star)^{-1}\gamma(y)k^\star = \gamma(yx) \quad \text{and} \quad (k^\star)^{-1}H(y)k^\star = H(yk).$$

For more details see reference [Rob96].

7.2 Presentation of $2^{\star 9} : \mathbb{Z}_3 \wr S_3$

For better understanding the wreath products, we will consider the following example. We will obtain a presentation for $2^{\star 9} : \mathbb{Z}_3 \wr S_3$.

Permutation Wreath Products

$$\begin{array}{ccc} & 2^{\star 9} : \mathbb{Z}_3 \wr S_3. & \\ \swarrow & & \searrow \\ 3^3 & & 3!. \end{array}$$

$3^3 \cdot 3! = 27 \cdot 6.$
$X = \{1, 2, 3\},$
$Y = \{4, 5, 6\},$
$\mathbb{Z}_3 = \{e, (123)\},$
$S_3 =< (123) >.$
$\gamma = (123).$
$Z = X \times Y = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}.$
Label the elements of Z by order $1 - 9$ as follow:

Table 7.1: Labeling Z elements set

(1)	$(1,4)$
(2)	$(1,5)$
(3)	$(1,6)$
(4)	$(2,4)$
(5)	$(2,5)$
(6)	$(2,6)$
(7)	$(3,4)$
(8)	$(3,5)$
(9)	$(3,6)$

To find $\gamma(4), \gamma(5), \gamma(6)$, where 4, 5 and 6 are elements of Y. We compute them using the above defined relation, which is :

$$\gamma(y) = \begin{cases} (x, y) & \mapsto (x\gamma, y) \\ (x, y_1) & \mapsto (x, y_1), \quad y \neq y_1. \end{cases}$$

Table 7.2: Substituting for $y = 4$ and labeling permutations

1. (1,4)	(2,4) . 4
2. (1,5)	(1,5) . 2
3. (1,6)	(1,6) . 3
4. (2,4)	(3,4) . 7
5. (2,5)	(2,5) . 5
6. (2,6)	(2,6) . 6
7. (3,4)	(1,4) . 1
8. (3,5)	(3,5) . 8
9. (3,6)	(3,6) . 9

So from the above table,

$(1,4) \to (2,4)$, and

$(2,4) \to (3,4)$, and

$(3,4) \to (1,4)$.

$\Rightarrow \quad H(4) = (1 \quad 4 \quad 7)$.

Now, computing $y = 5$

Table 7.3: Substituting for $y = 5$ and labeling permutations

1. (1,4)	(1,4) . 1
2. (1,5)	(2,5) . 5
3. (1,6)	(1,6) . 3
4. (2,4)	(2,4) . 4
5. (2,5)	(3,5) . 8
6. (2,6)	(2,6) . 6
7. (3,4)	(3,4) . 1
8. (3,5)	(1,5) . 2
9. (3,6)	(3,6) . 9

Therefore,

$(1,5) \to (2,5)$, and

$(2,5) \to (3,5)$, and

$(3,5) \to (1,5)$.

$\Rightarrow \quad H(5) = (2 \quad 5 \quad 8)$. (from the above table).

Also, (from table 7.4) we have

$(1,6) \to (2,6)$, $(2,6) \to (3,6)$, and $(3,6) \to (1,6)$.

$\Rightarrow \quad H(6) = (3 \quad 6 \quad 9)$

Table 7.4: Substituting for $y = 6$ and labeling permutations

1. (1,4)	(1,4) . 1
2. (1,5)	(1,5) . 2
3. (1,6)	(2,6) . 6
4. (2,4)	(2,4) . 4
5. (2,5)	(2,5) . 5
6. (2,6)	(3,6) . 9
7. (3,4)	(3,4) . 7
8. (3,5)	(3,5) . 8
9. (3,6)	(1,6) . 3

Now, we compute the Base, B;

1. We computed
$H(4) \times H(5) \times H(6) =< (1 \quad 4 \quad 6) \times (2 \quad 5 \quad 8) \times (3 \quad 6 \quad 9) >$.

2. We need to compute
$k^{\star}$ and $k_1^{\star}$.

Now, given $k = (4 \quad 5 \quad 6)$,
and the formula for $k^{\star}$ is : $(x, y) \longmapsto (x, yk)$
the corresponding table will be:

Table 7.5: $k^{\star}$ permutations

1. (1,4)	(1,5) .2
2. (1,5)	(1,6) .3
3. (1,6)	(1,4) .1
4. (2,4)	(2,5) .5
5. (2,5)	(2,6) .6
6. (2,6)	(2,4) .4
7. (3,4)	(3,5) .8
8. (3,5)	(3,6) .9
9. (3,6)	(3,4) .7

the permutation will be: $k^{\star} = (1 \quad 2 \quad 3)(4 \quad 5 \quad 6)(7 \quad 8 \quad 9)$.
Similarly, for $k_1 := (4, 5)$
$k_1^{\star} = (1 \quad 2)(4 \quad 5)(7 \quad 8)$.(from table 6)
Therefore, the generators of the group, N, have been established :

Table 7.6: k_1 permutations

1. (1,4)	(1,5) . 2
2. (1,5)	(1,4) . 1
3. (1,6)	(1,6) . 3
4. (2,4)	(2,5) . 5
5. (2,5)	(2,4) . 4
6. (2,6)	(2,6) . 6
7. (3,4)	(3,5) . 8
8. (3,5)	(3,4) . 7
9. (3,6)	(3,6) . 9

$N =< (147)(258)(369), (123)(456)(78), (12)(45)(78) >$.

Hence, using program computer, Magma, to find the size of the group N, the result is 162.

Now, we are given $2^{*6} : \mathbb{Z}_3 \wr S_3$

Which is Wreath Product (cyclic group(3),sym(3)).

We need to find a presentation for $\mathbb{Z}_3 \wr S_3$.

The general presentation for the above group is:

$$N =< z, t, u, x, y | z^3, t^3, u^3, (z,t), (z,u), (t,u), x^3, y^2, (x,y)^2,$$
$$z^x = \quad , t^x = \quad , u^x = \quad , z^y = \quad , t^y = \quad , u^y = \quad >.$$

Determining $z^x, t^x, u^x, z^y, t^y, u^y$

We have:

$z \sim (1,4,7), \quad t \sim (2,5,8), \quad u \sim (3,6,9),$

$x \sim (1,2,3)(4,5,6)(7,8,9)$, and $y \sim (1,2)(4,5)(7,8)$.

Now,

$z^x = (2,5,8) = t, \quad t^x = (3,6,9) = u, \quad u^x = (1,4,7) = z,$

$z^y = (2,5,8) = t, \quad t^y = (1,4,7) = z, \quad u^y = (3,6,9) = u.$

Therefore, a presentation for $\mathbb{Z}_3 \wr S_3$ would be:

$$N =< z, t, u, x, y | z^3, t^3, u^3, (z,t), (z,u), (t,u), x^3, y^2, (x,y)^2,$$
$$z^x = t, t^x = u, u^x = z, z^y = t, t^y = z, u^y = u >.$$

We use Magma to check if the above results are corrects. We define $Sym(9)$ to be symmetric group with 9 letters and N, the obtained above group of order 162, Magma indicates that W, $\mathbb{Z}_3 \wr S_3$, is isomorphic to N. (See next pages)

```
> S:= Sym(9);
> N:=sub<S|S!(1,4,7),S!(2,5,8),S!(3,6,9),
          S!(1,2,3)(4,5,6)(7,8,9),
          S!(1,2)(4,5)(7,8)>;
> W:= WreathProduct(CyclicGroup(3),Sym(3));
> r:=IsIsomorphic(N,W);
> r;
true
>
> G<z,t,u,x,y>:=Group<z,t,u,x,y|z^3,t^3,u^3,(z,t),
  (z,u),(t,u),x^3,y^2,(x*y)^2, z^x=t, t^x=u, u^x=z,
    z^y=t, t^y=z, u^y= u>;
> f,G1,k:=CosetAction(G,sub<G|Id(G)>);
> s:=IsIsomorphic(G1,N);
> s:=IsIsomorphic(G1,N);
> s;
> zz:=S!(1,4,7);
> tt:=S!(2,5,8);
> uu:=S!(3,6,9);
> xx:=S!(1,2,3)(4,5,6)(7,8,9);
> yy:=S!(1,2)(4,5)(7,8);
> N:=sub<S|zz,tt,uu,xx,yy>;
> NN<a,b,c,d,e>:=Group<a,b,c,d,e|a^3,b^3,c^3,(a,b),
 (a,c),(b,c),d^3,e^2,(d*e)^2, a^d=b, b^d=c, c^d=a,
  a^e=b, b^e=a, c^e= c>;
> #G;
  162

> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..162]];
> for i in [2..162] do
for>  P:=[Id(N): 1 in [1..#Sch[i]]];
for> for j in [1..#Sch[i]] do
for|for>  if Eltseq(Sch[i])[j] eq 1 then P[j]:=zz; end if;
```

```
for|for> if Eltseq(Sch[i])[j] eq -1 then P[j]:=zz^-1; end if;
for|for> if Eltseq(Sch[i])[j] eq 2 then P[j]:=tt; end if;
for|for> if Eltseq(Sch[i])[j] eq -2  then P[j]:=tt^-1; end if;|
for|for> if Eltseq(Sch[i])[j] eq 3 then P[j]:=uu; end if;
for|for> if Eltseq(Sch[i])[j] eq -3  then P[j]:=uu^-1; end if;
for|for> if Eltseq(Sch[i])[j] eq 4 then P[j]:=xx; end if;
for|for> if Eltseq(Sch[i])[j] eq -4  then P[j]:=xx^-1; end if;
for|for> if Eltseq(Sch[i])[j] eq 5 then P[j]:=yy; end if;
for|for>  end for;
for>
>  Sch;
{@ Id(NN), a, b, c, d, e, a^-1, b^-1, c^-1, d^-1, a * b,
  a * c, a * d, a * e, a * b^-1, a * c^-1, a * d^-1, b * c,
  b * d, b * e, b * a^-1, b * c^-1, b * d^-1, c * d, c * e,
  c * a^-1, c * b^-1, c * d^-1, d * e, d * a^-1, d * b^-1,
  d * c^-1, e * d, e * a^-1, e * b^-1, e * c^-1, a^-1 * b^-1,
  a^-1 * c^-1, a^-1 * d^-1, b^-1 * c^-1, b^-1 * d^-1, c^-1 * d^-1,
  a * b * c, a * b * d, a * b * e, a * b * c^-1, a * b * d^-1,
  a * c * d, a * c * e, a * c * b^-1, a * c * d^-1, a * d * e,
  a * d * a^-1, a * d * c^-1, a * e * d, a * e * a^-1, a * e * c^-1,
  a * b^-1 * c^-1, a * b^-1 * d^-1, a * c^-1 * d^-1, b * c * d,
  b * c * e, b * c * a^-1, b * c * d^-1, b * d * e, b * d * a^-1,
  b * d * b^-1, b * e * d, b * e * b^-1, b * e * c^-1, b * a^-1 * c^-1,
  b * a^-1 * d^-1, b * c^-1 * d^-1, c * d * e, c * d * b^-1,
  c * d * c^-1, c * e * d, c * e * a^-1, c * e * b^-1, c * a^-1 * b^-1,
  c * a^-1 * d^-1, c * b^-1 * d^-1, d * e * a^-1, d * e * b^-1,
  d * e * c^-1, d * a^-1 * b^-1, d * a^-1 * c^-1, d * b^-1 * c^-1,
  e * d * a^-1, e * d * b^-1, e * d * c^-1, e * a^-1* b^-1,
  e * a^-1 * c^-1, e * b^-1 * c^-1, a^-1 * b^-1 * c^-1, a^-1 * b^-1 *
  d^-1, a^-1 * c^-1 * d^-1, b^-1 * c^-1 * d^-1, a * b * c * d,
  a * b * c * e, a * b * c * d^-1, a * b * d * e, a * b * d * a^-1,
  a * b * e * d, a * b * e * c^-1, a * b * c^-1 * d^-1, a * c * d * e,
  a * c * d * c^-1, a * c * e * d, a * c * e * a^-1,
  a * c * b^-1 * d^-1, a * d * e * b^-1, a * d * e * c^-1,
```

```
a * d * a^-1 * c^-1, a * e * d * a^-1, a * e * d * b^-1,
a * e * a^-1 * c^-1, a * b^-1 * c^-1 * d^-1, b * c * d * e,
b * c * d * b^-1, b * c * e * d, b * c * e * b^-1,
b * c * a^-1 * d^-1,  b * d * e * a^-1, b * d * e * b^-1,
b * d * a^-1 * b^-1, b * e * d * a^-1, b * e * d * c^-1,
b * e * b^-1 * c^-1, b * a^-1 * c^-1 * d^-1, c * d * e * a^-1,
c * d * e * c^-1, c * d * b^-1 * c^-1, c * e * d * b^-1,
c * e * d *c^-1, c * e * a^-1 * b^-1, c * a^-1 * b^-1 * d^-1,
d * e * a^-1 * b^-1, d * e * a^-1 * c^-1, d * e * b^-1 * c^-1,
d * a^-1 * b^-1 * c^-1, e * d * a^-1 * b^-1, e * d * a^-1 * c^-1,
e * d * b^-1 * c^-1, e * a^-1 * b^-1 * c^-1, a^-1 * b^-1 * c^-1
* d^-1, a * b * c * d * e, a * b * c * e * d, a * b * d * e * b^-1,
a * b *e * d * a^-1, a * c * d * e * c^-1, a * c * e * d * b^-1,
a * d * e * b^-1 * c^-1, a * e * d * a^-1 * b^-1, b * c * d *
e * a^-1, b * c * e * d * c^-1, b * d* e * a^-1 * b^-1, b * e *
d * a^-1 * c^-1, c * d * e * a^-1 * c^-1, c * e * d * b^-1 * c^-1,
d * e * a^-1 * b^-1 * c^-1, e * d * a^-1 * b^-1 * c^-1 @}
> N1:=Stabiliser(N,1);
> N1;
Permutation group N1 acting on a set of cardinality 9
Order = 18 = 2 * 3^2
  (2, 5, 8)
  (3, 6, 9)
  (2, 3, 8, 9, 5, 6)
> for i in [1..162] do if ArrayP[i] eq N!(2,9,8,6,5,3)
  then print Sch[i]; end if; end for;
> for i in [1..162] do if ArrayP[i] eq N!(2,5,8)
  then print Sch[i]; end if; end for;
> for i in [1..162] do if ArrayP[i] eq N!(3,6,9)
  then print Sch[i]; end if; end for;
> G<a,b,c,d,e,t>:=Group<a,b,c,d,e,t|a^3,b^3,c^3,(a,b),(a,c),
  (b,c),d^3,e^2,(d*e)^2, a^d=b, b^d=c, c^d=a, a^e=b, b^e=a,
  c^e= c,t^2,(t,d * e * c^-1),(t,b),(t,c)>
>
```

Chapter 8

Construction of $S_6 \times S_5$

In this chapter, we will find and construct finite homomorphic images of $S_6 \times S_5$, Given,

$$G < x, y, t > \cong$$
$$Group\langle x, y, t | x^5, y^2, (xy)^4, (x^{-1}(xy)^1)^2, (x^{-2}(xy)^2)^3, (x^{-3}(xy)^3)^4, (x^{-4}(xy)^4)^5,$$
$$(x^{-2}yx^2y)^2, t^2, (t, x^2yx^{-1}), (t, y^x), (t, y), (x^{(yt)^3}, (y^{(xy)}xt^{(xy)^2})^3, (t^{(xy)}t^{(xy)^2})^3\rangle;$$

Note: this group is one of the groups that was discovered in chapter 5. The control subgroup is N, is S_5 which is the symmetric group of degree 5 on five letters1, 2, 3, 4, and 5. N can be generated by x and y, where $x \sim (1,2,3,4,5)$ and $y \sim (1,5)$. Thus $N = \langle x, y \rangle$.

8.1 Relations

We have $(t^{(xy)}t^{(xy)^2})^3 = e$,

where,

$t \sim t_4$,

$x \sim (1,2,3,4,5)$,

$y \sim (1,5)$

$xy = (1,2,3,4,5)(1,5) = (1,2,3,4)$,

$(xy)^2 = (1,2,3,4)^2 = (1,3)(2,4)$.

The relation will be

$t_4^{(1,2,3,4)}t_4^{(1,3)(2,4)})^3 = e$.

$\Rightarrow \quad (t_1t_2)^3 = e.$

$\Rightarrow \quad t_1t_2t_1t_2t_1t_2 = e,$

$\Rightarrow \quad t_1t_2t_1 = t_2t_1t_2.$

8.2 Double Coset Enumeration of G over S_5

8.2.1 Double Coset [$\star$]

NeN, since N is transitive on $\{1, 2, 3, 4, 5\}$ we take a representative coset N from [$\star$] and a representative from $\{1, 2, 3, 4, 5\}$ and determine the double coset Nt_4 belongs.

8.2.2 Double Coset [4]

Nt_4N, point of stabiliser of 4 in N is the permutations in N that fixes 4,
Thus,
$N^{(4)} \geq< (1, 2, 3, 5), (1, 5) >= S_4$.
$\Rightarrow |N^{(4)}| = 4!$.
$\Rightarrow$ number of cosets are in [4] is $\frac{|N|}{|N^{(4)}|} = \frac{5!}{4!} = 5$.
which are:
$Nt_4N = [4] = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5\}$. Therefore, the double coset [4] will have 5 single cosets.
Now, $N^{(4)} \geq< (1, 2, 3), (1, 5) >$.
The orbits of $N^{(4)}$ on N are {4} and {1,2,3,5}. We take a representative from each orbit and multiply it by Nt_4

- $Nt_4 \cdot t_4 = Nt_4^2 = N \in [\star]$.

  Therefore, t_4 will collapse Cayley's graph.

- $Nt_4 \cdot t_2 \in [42]$.

  Since this orbit is of order of 4, then four t_i's will extend Cayley's.

The process will continue to determine all the double cosets $[w]$, and hence all the set of right cosets is closed under right multiplication by the 5 t_i's. Next graph will show all the determined double cosets for this constructed group.

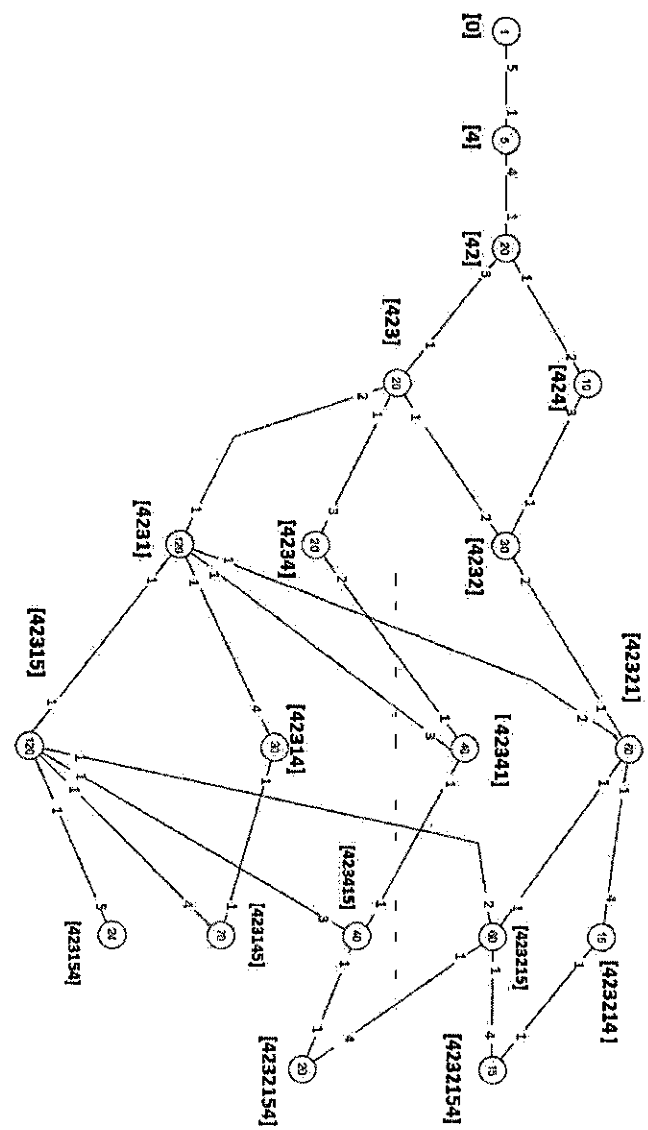

Figure 8.1: Cayley's graph for $S_6 \times S_5$

Chapter 9

Construction of Finite Homomorphic Images of $3^{\star 5} : C_5$

9.1 Introduction

In this chapter, we will search for the homomorphic images of $3^{*5} : C_5$. In order to find some of the subgroups we run the following codes with parameters a, b, c, and d as below:

Given

for a,b,c,d in [0..10] do

$$G < x, t > := Group < x, t | x^5, t^3, (xt)^a, (xtt^x)^a, (tt^x t)^c, (txt)^d >;;$$

The above range of parameters have generated some new groups that can be constructed. See table below:

We choose the parameters $(0, 2, 0, 5)$ with group order of 660 to be our group that will perform a double coset enumeration on it. Hence, the symmetric presentation of the progenitor is given by:

$$G < x, t > := Group < x, t | x^5, t^3, (xtt^x)^2, (txt)^5 >;$$

Where $C_5 =< x >$ and $x \sim (1, 2, 3, 4, 5)$.

Table 9.1: Some finite subgroups of $3^{*5} : C_5$

a	b	c	d	Order of G	Group Name
0	0	0	2	60	A_5
0	0	10	0	15	$\mathbb{Z}_{15}$
0	0	2	4	360	A_6
0	2	0	5	660	$L(2,11)$
0	2	0	7	161280	$2^5 : S_7$
0	2	0	8	14880	$L_2(31)$
0	2	3	7	2520	A_7
3	3	9	5	233280	
3	5	0	5	62400	$U(3,4)$
3	6	3	0	92160	
3	9	3	0	699840	
5	3	4	4	737280	$2^{10} : S_6$
5	6	2	8	184320	$2^8 : S_6$
8	3	2	0	5760	
8	6	2	5	184320	$2^8 : S_6$

9.2 Relation

We have:

$t \sim t_5$.

Relation 1 : $(xtt^x)^2 = e$.

Relation 2 :$(txt)^5 = e$.

Relation One, $(xtt^x)^2$ **:**

$(xtt^x)^2 \sim (xt_5t_5^x)^2$.

$(xt_5t_5^x)^2 = (xt_5t_5^{(12345)})^2$.

$(xt_5t_5^{(12345)})^2 = (xt_5t_1)^2 = xt_5t_1xt_5t_1 = x^2 \underbrace{x^{-1}t_5t_1x}_{(t_5t_1)^x} t_5t_1.$

$= x^2(t_5t_1)^x t_5t_1$.

$= x^2t_1t_2t_5t_1$.

Hence, the final relation is:

$x^2t_1t_2 = \overline{t_1t_5}$.

Relation Two, $(t * x * t)^5 = e$.

$(txt)(txt)(txt)(txt)(txt) = e$.

$txt^2xt^2xt^2xt^2xt = 2$.

$tx^5x^{-4}t^2x^4x^{-3}t^2x^3x^{-2}t^2x^{-1}t^2xt = e$.

$$tx^5 \underbrace{x^{-4}t^2x^4}_{t^{2^{x^{-4}}}} \underbrace{x^{-3}t^2x^3}_{t^{2^{x^{-3}}}} \underbrace{x^{-2}t^2x^2}_{t^{2^{x^{-2}}}} \underbrace{x^{-1}t^2x^1}_{t^{2^{x^{-1}}}} t = e.$$

$$tx^5 \underbrace{x^{-4}t^2x^4}_{t^{2^{x^1}}} \underbrace{x^{-3}t^2x^3}_{t^{2^{x^2}}} \underbrace{x^{-2}t^2x^2}_{t^{2^{x^3}}} \underbrace{x^{-1}t^2x^1}_{t^{2^{x^4}}} t = e.$$

$$t_5 t_5^{2^{x^1}} t_5^{2^{x^2}} t_5^{2^{x^3}} t_5^{2^{x^4}} t_5 = e$$

We know that

$x \sim (12345)$.

$x^2 \sim (13524)$.

$x^3 \sim (14253)$.

$x^4 \sim (54321)$.

$x^5 \sim Id$.

Thus,

$$t_5 t_1^2 t_2^2 t_3^2 t_4^2 t_5 = e$$

Therefore, the final relation is:

$$t_5 \overline{t_1 t_2 t_3 t_4} t_5 = e.$$

hence,

$$t_0 \overline{t_1 t_2} = \overline{t_5} t_4 t_3.$$

9.3 Double Coset Enumeration

9.3.1 Double Coset [⋆]

The first double coset [⋆]. N is transitive on $\{1, 2, 3, 4, 5, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$.
We take a representative coset N from [⋆] and a representative from $\{1, 2, 3, 4, 5, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$, like 5 and determine the double coset to which Nt_5 belongs.

$$Nt_5N = [5] = \{Nt_5^n | n \in N\} = \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5\}.$$
$$N\overline{t_5}N = [\bar{5}] = \{N\overline{t_5}^n | n \in N\} = \{N\overline{t_1}, N\overline{t_2}, N\overline{t_3}, N\overline{t_4}, N\overline{t_5}\}.$$

9.3.2 Double Coset [5]

Therefore, stabilising 5, N^5, then $N^5 = e$,
Orbits N^5 are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}, \{\bar{4}\}, \{\bar{5}\}$.
From above, the double coset [5] will have 5 single cosets, we take a representative from each orbit, and multiply it by Nt_5,

1. $Nt_5 \cdot t_1 \in [51]$.
   Extends Cayley's graph.

2. $Nt_5 \cdot t_2 \in [52]$.
   Extends Cayley's graph.

3. $Nt_5 \cdot t_3 \in [53]$.
   Extends Cayley's graph.

4. $Nt_5 \cdot t_4 \in [54]$.
   Extends Cayley's graph.

5. $Nt_5 \cdot t_5 = N\overline{t_5} \in [\overline{5}]$.
   [55] will go back to $[\overline{5}]$

6. $Nt_5 \cdot \overline{t_1} \in [5\overline{1}]$. Extends Cayley's graph.

7. $Nt_5 \cdot \overline{t_2} \in [5\overline{2}]$. Extends Cayley's graph.

8. $Nt_5 \cdot \overline{t_3} \in [5\overline{3}] \in [5\overline{1}]$.
   Because $t_5\overline{t_3} = (54321)(\overline{54321})(t_5t_{\overline{1}})^{(51234)(\overline{51234})}$.
   Thus, $t_5t_{\overline{3}} = (54321)(\overline{54321})t_1\overline{t_2} = x^4t_1t_{\overline{2}}$.
   $\Rightarrow [5\overline{3}] = [1\overline{2}]$. Collapse Cayley's graph

9. $Nt_5 \cdot \overline{t_4} \in [5\overline{2}]$.
   Now, $t_5\overline{t_2} = g(t_5\overline{t_4})^h$.
   where $g = (51234)(\overline{51234}) = x$ and $h = Id$.
   $\Rightarrow [5\overline{2}] = [5\overline{4}]$. Collapse Cayley's graph.

10. $Nt_5 \cdot \overline{t_5} = Ne \in [\star]$.
    $[5\overline{5}]$ will go back to $[\star]$.

9.3.3 Double Coset $[\overline{5}]$

Stabilising $\overline{5}$, $N^{\overline{5}}$, then $N^{\overline{5}} = e$,
Orbits $N^{\overline{5}}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{\overline{1}\}, \{\overline{2}\}, \{\overline{3}\}, \{\overline{4}\}, \{\overline{5}\}$.
From above, the double coset $[\overline{5}]$ will have 5 single cosets, we take a representative from each orbit, and multiply it by $N\overline{t_5}$,

1. $N\overline{t_5}t_1 \in [\overline{5}1]$.

   Extends Cayley's graph.

2. $N\overline{t_5}t_2 \in [\overline{5}2]$.

   Extends Cayley's graph.

3. $N\overline{t_5}t_3 \in [\overline{5}3] \in [\overline{5}1]$.

   Now $\overline{t_5}t_1 = (51234)(\overline{51234})(\overline{t_5}t_3)^{Id}$.

   $\Rightarrow [\overline{5}1] = [\overline{5}3]$.

   Collapse Cayley's graph

4. $Nt_{\overline{5}}t_4 \in [\overline{5}2]$.

   Since, $N\overline{t_5}t_2 = g(N\overline{t_5}t_4)^h$ where $g = (51234)(\overline{51234})$ and $h = (54321)(\overline{54321})$

   $\Rightarrow [\overline{1}02] = g[\overline{4}3]$

5. $N\overline{t_5}t_5 \in [\star]$.

6. $N\overline{t_5 t_1} \in [\overline{5}\overline{1}]$.

   Extends Cayley's graph.

7. $N\overline{t_5}\overline{t_2} \in [53]$

   $[53] = g[\overline{5}2]^h$, where $g = (53142)(\overline{53142}) = x^3$, and $h = (52413)(\overline{52413})$.

   $\Rightarrow [53] = x^3[\overline{24}]$.

8. $N\overline{t_5 t_3} \in [\overline{33}]$

   Extends Cayley's graph.

9. $N\overline{t_5 t_4} \in [51]$

   $[51] = g[\overline{54}]^h$ where $g = x^3$ and $h = e$.

   $\Rightarrow [51] = x^3[\overline{54}]$.

10. $N\overline{t_5 t_5} \in [5]$

9.3.4 Double Coset [51]

Stabilising 5 and 1, N^{51}, then $N^{51} = e$,
Orbits N^{51} are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{\overline{1}\}, \{\overline{2}\}, \{\overline{3}\}, \{\overline{4}\}, \{\overline{5}\}$.
the double coset [51] will have 5 single cosets, we take a representative from each orbit, and multiply it by Nt_5t_1,

1. $Nt_5t_1t_1 = Nt_5t_{\bar{1}} \in [5\bar{1}]$.

2. $Nt_5t_1t_2 \in [\overline{54}]$.
   Because $[\bar{5},4] = g[512]^h$ where $g = e$, $h = x$.
   $\Rightarrow [\bar{5},4] = [123]$.

3. $Nt_5t_1t_3 \in [513]$.
   Extends Cayley's graph.

4. $Nt_5t_1t_4 \in [\bar{5}]$.
   Since,$[\bar{5}] = g[514]^h$ where $g = x^2$ and $h = e$.
   $\Rightarrow [\bar{5}] = x^2[514]$.

5. $Nt_5t_1t_5 \in [515]$.
   Extends Cayley's graph.

6. $N_5t_1\overline{t_1} = Nt_5 \in [5]$.

7. $Nt_5t_1\overline{t_2} \in [12\bar{3}]$.
   Since, $[513] = g[51\bar{2}]^h$ where $g = x^3$ and $h = x$
   $\Rightarrow [513] = x^3[12\bar{3}]$.

8. $Nt_5t_1\overline{t_3} \in [51]$.
   Since, $[51] = g[51\bar{3}]$, where $x = x^2$, and $h = x^4$.
   $\Rightarrow [51] = x^2[45\bar{2}]$.

9. $Nt_5t_1\overline{t_4} \in [\bar{5}2]$.
   Since, $[\bar{5}2] = x^3[51\bar{4}]^{x^4}$
   $\Rightarrow [\bar{5}2] = x^3[45\bar{3}]$.

10. $Nt_5t_1\overline{t_5} \in [51\bar{5}]$.
    Extends Cayley's graph.

Due to limited time, I was unable to finish this chapter. However, I have attached all my computer-based proof, in which all double cosets were determined. (Please see Appendix C and below the corresponding Cayley's graph.)

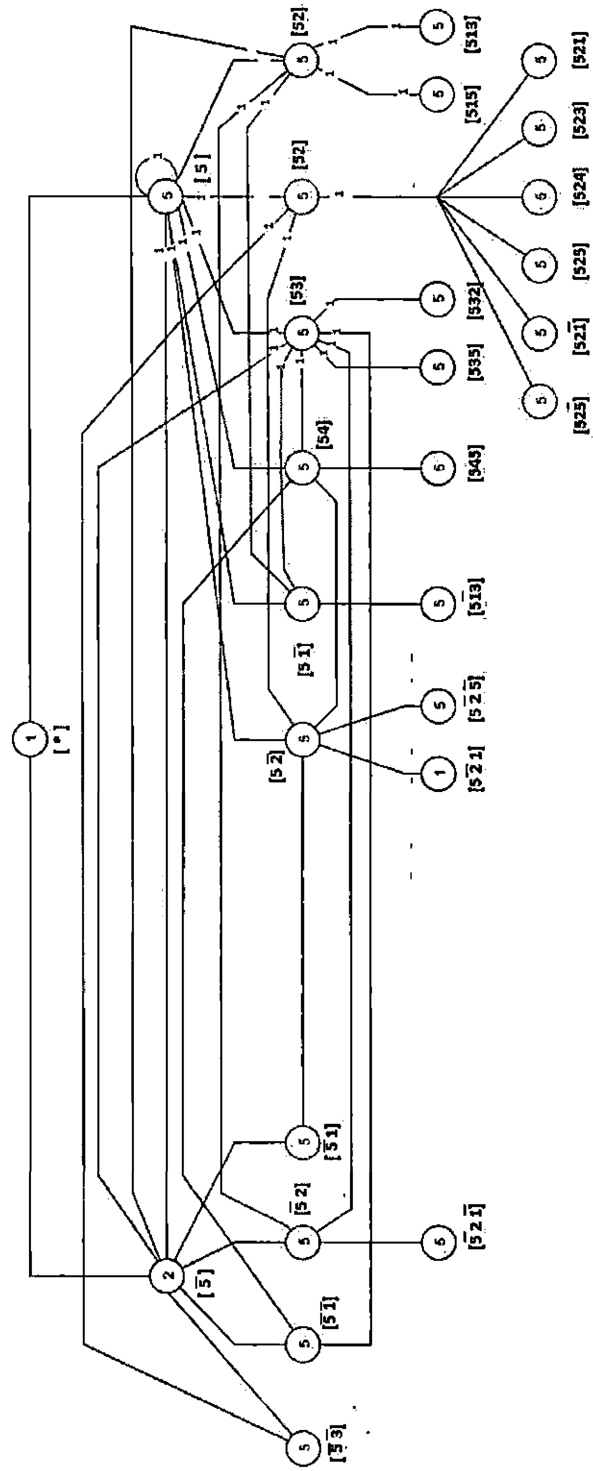

Figure 9.1: Cayley's graph for G over $L_2(11)$

Chapter 10

Finite Homomorphic Images of Some Progenitors

In this chapter, we will search for the homomorphic images of some progenitors.

10.1 The Homomorphic Images of Progenitor $2^{*6} : A_6$

The following parameters are chosen: for a in [0..50] do for b in [0..20] do for c in [0..20] do for d in [0..20] do for e in [0..10] do for f in [0..10] do

$G < x, y, t > \cong Group < x, y, t | x^4, y^3, (xy)^5, (y^{-1}x^{-1}yx)^2, t^2, (t, y),$

$(t, (x^{y^2})^{-1}(xyt)^a, ((xy)^{-1}t)^b, (xt^{(xy)^2})^c, (x^{yt})^d, (y^{xy}xt^{(xy)^2})^e, (t^{xy}t^{(x*y)^2})^f >;$

$A :=$ *ToddCoxeter*$(G, sub < G|x, y >$: *CosetLimit* $:= 10000000)$;

if A ge 4 then a, b, c, d, e, f, A; end if;

The table below shows some of the discovered groups.

Table 10.1: Some finite subgroups of $2^{*6} : A_6$

a	b	c	d	e	f	Order of G	Group Name
0	0	0	0	0	2	64	$2^5 : S_6$
0	0	0	0	6	0	14	S_7
0	0	0	0	10	3	486	$3^5 : S_6$
0	0	0	0	10	4	65536	$3^5 : S_6$
0	0	8	0	0	0	128	$2^6 : S_6$

10.2 $PGL_2(13)$ is an Image of the Progenitor $2^{\star 7} : S_3$

given by

$G < x, y, t > \cong Group < x, y, t | x^3, y^2, (xy)^2, t^7, (t, y), (xt)^7, (y^{x^2} t)^4 >;$

The order of the group, $|G|$ is equal to 2184, and the composition factor for the group is:

Composition Factors($G1$);

G

| Cyclic(2)

$\star$

| $A(1, 13)$ $= L(2, 13)$

1

10.3 $S_4 \times 7^2$ is an Image of the Progenitor $2^{\star 7} : S_4$

given by

$G < x, y, t > \cong Group < x, y, t | x^4, y^2, (xy)^3, t^7, (t, y), (t^x, y), (xt)^4 >;$

The order of $|G|$ is equal to 8232.

The composition factors is:

G

| Cyclic(2)

$\star$

| Cyclic(3)

$\star$

| Cyclic(2)

$\star$

| Cyclic(2)

$\star$

| Cyclic(7)

$\star$

| Cyclic(7)

$\star$

1

Appendix A: MAGMA Code for $S_6 \times S_5$

```
N:=Sym(5);
xx:=N!(1,2,3,4,5);
yy:=N!(1,5);
N:=sub<N|xx,yy>;
G<x,y,t>:= Group<x,y,t|x^5, y^2,(x*y)^4, (x^-1 *(x*y)^1)^2,
(x^-2*(x*y)^2)^3,(x^-3 *(x*y)^3)^4, (x^-4 *(x*y)^4)^5,
(x^-2 *y *x^2*y)^2,t^2, (t,x^2*y*x^-1), (t,y^x),(t,y),
(x^(y*t)^3,(y^(x*y)*x*t^((x*y)^2)))^3,(t^(x*y)*t^((x*y)^2))^3>;
Index(G,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
ts:=[Id(G1) : i in [1..5]];
ts[4]:=f(t);
ts[5]:=f(t^x);
ts[1]:=f(t^(x^2));
ts[2]:=f(t^(x^3));
ts[3]:=f(t^(x^4));
N4:=Stabiliser(N,[4]); N4s:=N4;
S:={[4]};
SS:=S^N;
#SS;
```

```
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4] eq g*ts[Rep(SSS[i])[1]]
then print SSS[i]; end if; end for; end for;
N4:=Stabiliser(N,[4]);
#N4; N4;
N4s:=N4;
N4s;  #N4s;Orbits (N4s);
[4]^N4s;
T:=Transversal (N,N4s);
#T,T;
for  i in [1..#T] do ([4]^N4s)^T[i];end for;
for i := 1 to #T do
ss := [4]^T[i];
end for;
prodim := function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
cst := [null : i in [1 .. 720]] where null is [Integers() | ];
for i := 1 to 5 do
cst[prodim(1, ts, [i])] := [i];
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
end for; m;
N42:=Stabiliser(N,[4,2]); N42s:=N42;
S:={[4,2]};
SS:=S^N;
#SS;
```

```
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2] eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2] eq g*( ts[4])^k then g,k;
 break; end if; end for;end for;
#N42; N42;
N42s:=N42;
N42s;  #N42s;Orbits (N42s);
[4,2]^N42s;
T:=Transversal (N,N42s);
#T,T;
for  i in [1..#T] do ([4,2]^N42s)^T[i];end for;
for i := 1 to #T do
ss := [4,2]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if; end for;
 m;
N423:=Stabiliser(N,[4,2,3]); N423s:=N423;
S:={[4,2,3]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3] eq g*ts[Rep(SSS[i])[1]]
*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3] eq
g*( ts[4])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3] eq
 g*( ts[4]*ts[2])^k then g,k; break; end if; end for;end for;
```

```
#N423; N423;
N423s:=N423;
N423s;  #N423s;Orbits (N423s);
[4,2,3]^N42s;
T:=Transversal (N,N423s);
#T,T;
for  i in [1..#T] do ([4,2,3]^N423s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,3]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
 end for; m;
N424:=Stabiliser(N,[4,2,4]); N424s:=N424;
S:={[4,2,4]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[4] eq
 g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[4] eq g*( ts[4])^k
then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[4] eq g*( ts[4]*ts[2])^k
 then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[4]
eq g*( ts[4]*ts[2]*ts[3])^k
then g,k; break; end if; end for;end for;
#N424; N424;
N424s:=N424;
N424s;  #N424s;Orbits (N424s);
```

```
[4,2,4]^N424s;
T:=Transversal (N,N424s);
#T,T;
for  i in [1..#T] do ([4,2,4]^N424s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,4]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
N4231:=Stabiliser(N,[4,2,3,1]); N4231s:=N4231;
S:={[4,2,3,1]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3] *ts[1]eq g*ts[Rep(SSS[i])[1]]
*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])[4]]
then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1]
eq g*( ts[4])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3] *ts[1]
 eq g*( ts[4]*ts[2])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1]
eq g*( ts[4]*ts[2]*ts[3])^k then g,k; break; end if; end for;end for;
#N4231; N4231;
N4231s:=N4231;
N4231s;  #N4231s;Orbits (N4231s);
[4,2,3,1]^N4231s;
T:=Transversal (N,N4231s);
#T,T;
for  i in [1..#T] do ([4,2,3,1]^N4231s)^T[i];end for;
```

```
for i := 1 to #T do
ss := [4,2,3,1]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
 end for; m;
N4234:=Stabiliser(N,[4,2,3,4]); N4234s:=N4234;
S:={[4,2,3,4]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3]*ts[4] eq
g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
*ts[Rep(SSS[i])[4]]
then print SSS[i]; end if; end for; end for;
for g in N do if 4^g eq 3 and 2^g eq 4 and 3^g eq 2 and 4^g eq 3
then N4234s:=sub<N|N4234s,g>; end if; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4]
eq g*( ts[4])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3] *ts[4]
eq g*( ts[4]*ts[2])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4]
eq g*( ts[4]*ts[2]*ts[3])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4]
eq g*( ts[4]*ts[2]*ts[3]*ts[1])^k then g,k; break; end if;
 end for;end for;
#N4234; N4234;
N4234s:=N4234;
N4234s;  #N4234s;Orbits (N4234s);
[4,2,3,4]^N4234s;
T:=Transversal (N,N4234s);
```

```
#T,T;
for  i in [1..#T] do ([4,2,3,4]^N4234s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,3,4]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1;
 end if; end for; m;
N4232:=Stabiliser(N,[4,2,3,2]); N4232s:=N4232;
S:={[4,2,3,2]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3]*ts[2]eq g*ts[Rep(SSS[i])[1]]
*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])[4]]
then print SSS[i]; end if; end for; end for;
for g in N do if 4^g eq 4 and 2^g eq 3 and 3^g eq 2
 and 2^g eq 3
then N4232s:=sub<N|N4232s,g>; end if; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2]
eq g*( ts[4])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3] *ts[2]
 eq g*( ts[4]*ts[2])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2]
eq g*( ts[4]*ts[2]*ts[3])^k then g,k; break; end if;
end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2]
eq g*( ts[4]*ts[2]*ts[4])^k then g,k; break; end if;
end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2]
 eq g*( ts[4]*ts[2]*ts[3]*ts[1])^k then g,k; break;
```

```
end if; end for;end for;
#N4232; N4232;
N4232s:=N4232;
N4232s;  #N4232s;Orbits (N4232s);
[4,2,3,2]^N4232s;
T:=Transversal (N,N4232s);
#T,T;
for  i in [1..#T] do ([4,2,3,2]^N4232s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,3,2]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
N42321:=Stabiliser(N,[4,2,3,2,1]); N42321s:=N42321;
S:={[4,2,3,2,1]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3] *ts[2]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
*ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])[4]]*ts[Rep(SSS[i])[5]]
then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2]
*ts[3]*ts[2] *ts[1] eq g*( ts[4])^k
 then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2]
*ts[1] eq g*( ts[4]*ts[2])^k
 then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2]
*ts[1] eq g*( ts[4]*ts[2]*ts[3])^k
```

```
 then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2]
*ts[1] eq g*( ts[4]*ts[2]*ts[4])^k
 then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2]
*ts[1] eq g*( ts[4]*ts[2]*ts[3]*ts[4])^k
 then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2]
 *ts[1] eq g*( ts[4]*ts[2]*ts[3]*ts[1])^k
 then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2]
*ts[1] eq g*( ts[4]*ts[2]*ts[3]*ts[2])^k
 then g,k; break; end if; end for;end for;
for g in N do if 4^g eq 4 and 3^g eq 2 and
 2^g eq 3 and 3^g eq 2 and 1^g eq 1
 then N42321s:=sub<N|N42321s,g>; end if; end for;
#N42321; N42321;
N42321s:=N42321;
N42321s;  #N42321s;Orbits (N42321s);
[4,2,3,2,1]^N42321s;
T:=Transversal (N,N42321s);
#T,T;
for  i in [1..#T] do ([4,2,3,2,1]^N42321s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,3,2,1]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
end for; m;
N42341:=Stabiliser(N,[4,2,3,4,1]); N42341s:=N42341;
S:={[4,2,3,4,1]};
SS:=S^N;
```

```
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3] *ts[4]*ts[1]
eq g*ts[Rep(SSS[i])[1]]
*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])[4]]
*ts[Rep(SSS[i])[5]]
then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4]
*ts[1] eq g*( ts[4])^k
 then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4]
 *ts[1] eq g*( ts[4]*ts[2])^k
 then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4]
*ts[1] eq g*( ts[4]*ts[2]*ts[3])^k
then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4]
*ts[1] eq g*( ts[4]*ts[2]*ts[4])^k
 then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4]
*ts[1] eq g*( ts[4]*ts[2]*ts[3]*ts[2])^k
 then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4]
*ts[1] eq g*( ts[4]*ts[2]*ts[3]*ts[1])^k
then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4]
 *ts[1] eq g*( ts[4]*ts[2]*ts[3]*ts[4])^k
 then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4]
 *ts[1] eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1])^k
 then g,k; break; end if; end for;end for;
```

```
for g in N do if 4^g eq 3 and 2^g eq 4 and 3^g eq 2
 and 4^g eq 3 and 1^g eq 1
 then N42341s:=sub<N|N42341s,g>; end if; end for;
#N42341; N42341;
N42341s:=N42341;
N42341s;  #N42341s;Orbits (N42341s);
[4,2,3,4,1]^N42341s;
T:=Transversal (N,N42341s);
#T,T;
for  i in [1..#T] do ([4,2,3,4,1]^N42341s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,3,4,1]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
end for; m;
N42314:=Stabiliser(N,[4,2,3,1,4]); N42314s:=N42314;
S:={[4,2,3,1,4]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3] *ts[1]*ts[4]eq g*ts[Rep(SSS[i])[1]
]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])[4]]
*ts[Rep(SSS[i])[5]]
then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2])^k then g,k;
break; end if; end for;end for;
```

```
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3]*ts[2])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3]*ts[4]*ts[1])^k then g,k;
break; end if; end for;end for;
#N42314; N42314;
N42314s:=N42314;
N42314s;  #N42314s;Orbits (N42314s);
[4,2,3,1,4]^N4214s;
T:=Transversal (N,N42314s);
#T,T;
for  i in [1..#T] do ([4,2,3,1,4]^N42314s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,3,1,4]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
```

```
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
end for; m;
N42315:=Stabiliser(N,[4,2,3,1,5]); N42315s:=N42315;
S:={[4,2,3,1,5]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3] *ts[1]*ts[5]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
*ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])[4]]
*ts[Rep(SSS[i])[5]]then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
 eq g*( ts[4]*ts[2]*ts[4])^k then g,k; break; end if;
 end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3]*ts[2])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
```

```
eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3]*ts[4]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[4])^k then g,k;
break; end if; end for;end for;
#N42315; N42315;
N42315s:=N42315;
N42315s;  #N42315s;Orbits (N42315s);
[4,2,3,1,5]^N4215s;
T:=Transversal (N,N42315s);
#T,T;
for  i in [1..#T] do ([4,2,3,1,5]^N42315s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,3,1,5]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
 end for; m;
N423214:=Stabiliser(N,[4,2,3,1,4]); N423214s:=N423214;
S:={[4,2,3,2,1,4]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3] *ts[2]*ts[1]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]
]*ts[Rep(SSS[i])[4]]*ts[Rep(SSS[i])[5]]*ts[Rep(SSS[i])[6]]
then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
```

```
 eq g*( ts[4])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 eq g*( ts[4]*ts[2])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 eq g*( ts[4]*ts[2]*ts[3])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 eq g*( ts[4]*ts[2]*ts[4])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
eq g*( ts[4]*ts[2]*ts[3]*ts[2])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 eq g*( ts[4]*ts[2]*ts[3]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 eq g*( ts[4]*ts[2]*ts[3]*ts[4])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 eq g*( ts[4]*ts[2]*ts[3]*ts[4]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[5])^k then g,k;
 break; end if; end for;end for;
#N423214; N423214;
N423214s:=N423214;
N423214s;  #N423214s;Orbits (N423214s);
[4,2,3,2,1,4]^N423214s;
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```
T:=Transversal (N,N423214s);
#T,T;
for  i in [1..#T] do ([4,2,3,2,1,4]^N423214s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,3,2,1,4]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
 end for; m;
N423215:=Stabiliser(N,[4,2,3,1,5]); N423215s:=N423215;
S:={[4,2,3,2,1,5]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3] *ts[2]*ts[1]*ts[5]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
*ts[Rep(SSS[i])[4]]*ts[Rep(SSS[i])[5]]*ts[Rep(SSS[i])[6]]
then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
*ts[5] eq g*( ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
*ts[5] eq g*( ts[4]*ts[2])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
*ts[5] eq g*( ts[4]*ts[2]*ts[3])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
*ts[5] eq g*( ts[4]*ts[2]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
```

```
*ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[2])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
*ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
*ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
 *ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
*ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[4]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
 *ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
*ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[5])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
*ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1]*ts[4])^k then g,k;
 break; end if; end for;end for;
#N423215; N423215;
N423215s:=N423215;
N423215s;  #N423215s;Orbits (N423215s);
[4,2,3,2,1,5]^N423215s;
T:=Transversal (N,N423215s);
#T,T;
for  i in [1..#T] do ([4,2,3,2,1,5]^N423214s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,3,2,1,5]^T[i];
```

```
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
N423415:=Stabiliser(N,[4,2,3,4,1,5]); N423415s:=N423415;
S:={[4,2,3,4,1,5]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3] *ts[4]*ts[1]*ts[5]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
*ts[Rep(SSS[i])[4]]*ts[Rep(SSS[i])[5]]*ts[Rep(SSS[i])[6]]
then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4] *ts[1] *ts[5]
 eq g*( ts[4])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4] *ts[1] *ts[5]
 eq g*( ts[4]*ts[2])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4] *ts[1] *ts[5]
 eq g*( ts[4]*ts[2]*ts[3])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4] *ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[4])^k then g,k; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4] *ts[1] *ts[5]
 eq g*( ts[4]*ts[2]*ts[3]*ts[2])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4] *ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4] *ts[1] *ts[5]
 eq g*( ts[4]*ts[2]*ts[3]*ts[4])^k then g,k;
 break; end if; end for;end for;
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for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4] *ts[1] *ts[5]
 eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4] *ts[1] *ts[5]
 eq g*( ts[4]*ts[2]*ts[3]*ts[4]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4] *ts[1] *ts[5]
 eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4] *ts[1] *ts[5]
 eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[5])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4] *ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[4] *ts[1] *ts[5]
eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1]*ts[5])^k then g,k;
break; end if; end for;end for;
#N423415; N423415;
N423415s:=N423415;
N423415s;  #N423415s;Orbits (N423415s);
[4,2,3,4,1,5]^N423415s;
T:=Transversal (N,N423415s);
#T,T;
for  i in [1..#T] do ([4,2,3,4,1,5]^N423414s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,3,4,1,5]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
 end for; m;
N423154:=Stabiliser(N,[4,2,3,1,5,4]); N423154s:=N423154;
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S:={[4,2,3,1,5,4]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3] *ts[1]*ts[5]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
*ts[Rep(SSS[i])[4]]*ts[Rep(SSS[i])[5]]*ts[Rep(SSS[i])[6]]
then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5] *ts[4]
 eq g*( ts[4]*ts[2]*ts[3]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5] *ts[4]
eq g*( ts[4]*ts[2]*ts[3]*ts[2])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5] *ts[4]
eq g*( ts[4]*ts[2]*ts[3]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5] *ts[4]
eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5] *ts[4]
eq g*( ts[4]*ts[2]*ts[3]*ts[4]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5] *ts[4]
eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[4])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5] *ts[4]
eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[5])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5] *ts[4]
 eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1]*ts[4])^k then g,k;
```

```
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5] *ts[4]
eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1]*ts[5])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5] *ts[4]
 eq g*( ts[4]*ts[2]*ts[3]*ts[4]*ts[1]*ts[5])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[1] *ts[5] *ts[4]
eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[4]*ts[5])^k then g,k;
 break; end if; end for;end for;
for g in N do if 4^g eq 5 and 2^g eq 4 and 3^g eq 2 and 1^g eq 3
 and 5^g eq 1 and 4^g eq 5 then N423154s:=sub<N|N423154s,g>;
 end if; end for;
#N423154; N423154;
N423154s:=N423154;
N423154s;  #N423154s;Orbits (N423154s);
[4,2,3,1,5,4]^N42154s;
T:=Transversal (N,N423154s);
#T,T;
for  i in [1..#T] do ([4,2,3,1,5,4]^N423154s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,3,1,5,4]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
N4232145:=Stabiliser(N,[4,2,3,2,1,4,5]); N4232145s:=N4232145;
S:={[4,2,3,2,1,4,5]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3] *ts[2]*ts[1]*ts[4] *ts[5]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
```

```
*ts[Rep(SSS[i])[4]]*ts[Rep(SSS[i])[5]]*ts[Rep(SSS[i])[6]]
*ts[Rep(SSS[i])[7]]
then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
 *ts[4] *ts[5] eq g*( ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
*ts[4] *ts[5] eq g*( ts[4]*ts[2])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
 *ts[4] *ts[5] eq g*( ts[4]*ts[2]*ts[3])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 *ts[5] eq g*( ts[4]*ts[2]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 *ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 *ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[4]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 *ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[4])^k then g,k
; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4
] *ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[5])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 *ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4
] *ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1]*ts[5])^k then g,k
```

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; break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4
] *ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[4]*ts[1]*ts[5])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[4]
 *ts[5] eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[5]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in N do if 4^g eq 1 and 2^g eq 2 and 3^g eq 3
and 2^g eq 2 and 1^g eq 4 and 4^g eq 1 and 5^g eq 5
then N4232145s:=sub<N|N4232145s,g>; end if; end for;
#N4232145; N4232145;
N4232145s:=N4232145;
N4232145s;  #N4232145s;Orbits (N4232145s);
[4,2,3,2,1,4,5]^N4232145s;
T:=Transversal (N,N4232145s);
#T,T;
for  i in [1..#T] do ([4,2,3,2,1,4,5]^N4232145s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,3,2,1,4,5]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1;
 end if; end for; m;
N4232154:=Stabiliser(N,[4,2,3,2,1,5,4]);
N4232154s:=N4232154;
S:={[4,2,3,2,1,5,4]};
SS:=S^N;
#SS;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[2]*ts[3] *ts[2]*ts[1]*ts[5] *ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
```

```
*ts[Rep(SSS[i])[4]]*ts[Rep(SSS[i])[5]]*ts[Rep(SSS[i])[6]]
*ts[Rep(SSS[i])[7]]
then print SSS[i]; end if; end for; end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
*ts[5] *ts[4] eq g*( ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
 *ts[5] *ts[4] eq g*( ts[4]*ts[2])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[5]
 *ts[4] eq g*( ts[4]*ts[2]*ts[3])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[5]
 *ts[4] eq g*( ts[4]*ts[2]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[5]
 *ts[4] eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[5
] *ts[4]eq g*( ts[4]*ts[2]*ts[3]*ts[4]*ts[1])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[5
] *ts[4] eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[5]
 *ts[4] eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[5])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[5]
 *ts[4] eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1]*ts[4])^k then g,k;
 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[5]
 *ts[4] eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1]*ts[5])^k then g,k;
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 break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[5]
 *ts[4] eq g*( ts[4]*ts[2]*ts[3]*ts[4]*ts[1]*ts[5])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[5]
 *ts[4] eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[5]*ts[4])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1] *ts[5]
 *ts[4] eq g*( ts[4]*ts[2]*ts[3]*ts[1]*ts[4]*ts[5])^k then g,k;
break; end if; end for;end for;
for g in IN do for k in IN do if ts[4]*ts[2] *ts[3]*ts[2] *ts[1]
*ts[5] *ts[4] eq g*( ts[4]*ts[2]*ts[3]*ts[2]*ts[1]*ts[4]*ts[5])^k
 then g,k; break; end if; end for;end for;
for g in N do if 4^g eq 4 and 2^g eq 3 and 3^g eq 2 and 2^g
eq 3 and 1^g eq 1 and 5^g eq 5 and 4^g eq 4
 then N4232154s:=sub<N|N4232154s,g>; end if; end for;
#N4232154; N4232154;
N4232154s:=N4232154;
N4232154s;  #N4232154s;Orbits (N4232154s);
[4,2,3,2,1,5,4]^N4232154s;
T:=Transversal (N,N4232154s);
#T,T;
for  i in [1..#T] do ([4,2,3,2,1,5,4]^N4232154s)^T[i];end for;
for i := 1 to #T do
ss := [4,2,3,2,1,5,4]^T[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
end for; m;
```

Appendix B: MAGMA code for $3^{\star 5} : C_5$

```
MAGMA OURPUT
 S:=Sym(10);

>  xx:=S!(1,2,3,4,5)(6,7,8,9,10);
>  N:=sub<S|xx>;
> G<x,t>:=Group<x,t|x^5,t^3,(x*t*t^x)^2,(t*x*t)^5>;
>  Index(G,sub<G|x>);
132
>  f,G1,k:=CosetAction(G,sub<G|x>);
>  IN:=sub<G1|f(x)>;
>  CompositionFactors(G1);
    G
    |  A(1, 11)                = L(2, 11)
    1
>   #DoubleCosets(G,sub<G|x>,sub<G|x>);
28
>  prodim := function(pt, Q, I)
function>  /*
function>  Return the image of pt under permutations Q[I]
applied sequentially\
.\
```

```
function>
function> */
function>   v := pt;
function>     for i in I do
function|for>         v := v^(Q[i]);
function|for>     end for;
function>  return v;
function>  end function;
>  ts := [Id(G1): i in [1 .. 10] ];
>  ts[5]:=f(t); ts[1]:=f(t^x); ts[2]:=f(t^(x^2));
ts[3]:=f(t^(x^3));ts[4]:=f(t^(x^4));
>  ts[10]:=(f(t))^2; ts[6]:=(f(t^x))^2;
ts[7]:=(f(t^(x^2)))^2;
 ts[8]:=(f(t^(x^3)))^2;ts[9]:=(f(t^(x^4)))^2;
>  cst := [null : i in [1 .. 132]] where null
is [Integers() | ];
>     for i := 1 to 5 do
for>          cst[prodim(1, ts, [i])] := [i];
for>     end for;
>  m:=0;
>  for i in [1..132] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
5
>
> for i := 6 to 10 do
for>          cst[prodim(1, ts, [i])] := [i];
for>     end for;
>  m:=0;
>  for i in [1..132] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
10
-------------------------------------------------------
```

```
> N51:=Stabiliser (N,[5,1]);
> SSS:={[5,1]}; SSS:=SSS^N;
> SSS;
GSet{{[ 1, 2 ]},{[ 2, 3 ]},{[ 4, 5 ]},
    {[ 3, 4 ]},{[ 5, 1 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[ {[ 1, 2 ]}, {[ 2, 3 ]},{[ 4, 5 ]},
  {[ 3, 4 ]}, {[ 5, 1 ] }]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[1] eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*ts[Rep(Seqq[i])[2]]
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 5, 1 ],[ 5, 1 ],[ 5, 1 ],[ 5, 1 ],[ 5, 1 ]
> N51; #N51;Orbits(N51s);
Permutation group N51 acting on a set of
cardinality 10
Order = 1
1
 N51; #N51;
> N51s:=N51;
>   #N51s;
1
> [5,1]^N51s;
GSet{[ 5, 1 ]}
> T51:=Transversal(N,N51);
> for i in [1..#T51] do
```

```
for> ss:=[5,1]^T51[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
15
-----------------------------------------------------
> N52:=Stabiliser (N,[5,2]);
> SSS:={[5,2]}; SSS:=SSS^N;
> SSS;
GSet{{[ 1, 3 ]}, {[ 3, 5 ]},
     {[ 5, 2 ]}, {[ 4, 1 ]},
    {[ 2, 4 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 1, 3 ]},{[ 3, 5 ]},{[ 5, 2 ]},
    {[ 4, 1 ]},{[ 2, 4 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[2] eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for;
end for;end for;
[ 5, 2 ],[ 5, 2 ],[ 5, 2 ],[ 5, 2 ],[ 5, 2 ]
> N52; #N52;Orbits(N52s);
Permutation group N52 acting on a set of
cardinality 10
```

```
Order = 1
1
> N52s:=N52;
>   #N52s;
1
> [5,2]^N52s;
GSet{[ 5, 2 ]}
> T52:=Transversal(N,N52);
> for i in [1..#T52] do
for> ss:=[5,2]^T52[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
20
--------------------------------------------------
 N53:=Stabiliser (N,[5,3]);
> SSS:={[5,3]}; SSS:=SSS^N;
> SSS;
GSet{{ [ 4, 2 ]}, {[ 3, 1 ]}, {[ 1, 4 ]},
    {[ 5, 3 ]}, { [ 2, 5 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 4, 2 ]}, {[ 3, 1 ]}, {[ 1, 4 ]},
    {[ 5, 3 ]}, {[ 2, 5 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[3] eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
```

```
ts[Rep(Seqq[i])[2]]
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for;
end for;end for;
[ 5, 3 ],[ 5, 3 ],[ 5, 3 ],[ 5, 3 ],[ 5, 3 ]
> N53; #N53;Orbits(N53s);
Permutation group N53 acting on a set of
 cardinality 10
Order = 1
1
> N53; #N53;
> N53s:=N53;
>  #N53s;
1
> [5,3]^N53s;
GSet{[ 5, 3 ]}
> T53:=Transversal(N,N53);
> for i in [1..#T53] do
for> ss:=[5,3]^T53[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
25
------------------------------------------------
> N54:=Stabiliser (N,[5,4]);
> SSS:={[5,4]}; SSS:=SSS^N;
> SSS;
GSet{{[ 5, 4 ] }, {[ 2, 1 ]},{[ 4, 3 ]},
    {[ 1, 5 ] }, { [ 3, 2 ] }}
> #(SSS);
5
```

```
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 5, 4 ]}, {[ 2, 1 ]}, {[ 4, 3 ]},
{[ 1, 5 ]}, {[ 3, 2 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[4] eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for;
end for;end for;
[ 5, 4 ],[ 5, 4 ],[ 5, 4 ],[ 5, 4 ],[ 5, 4 ]
> N54; #N54;Orbits(N54s);
Permutation group N54 acting on a set of
 cardinality 10
Order = 1
1
>> N54; #N54;
> N54s:=N54;
>   #N54s;
1
> [5,4]^N54s;
GSet{[ 5, 4 ]}
> T54:=Transversal(N,N54);
> for i in [1..#T54] do
for> ss:=[5,4]^T54[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
```

30

```
N56:=Stabiliser (N,[5,6]);
> SSS:={[5,6]}; SSS:=SSS^N;
> SSS;
GSet{{[ 3, 9 ]}, {[ 5, 6 ] }, {[ 2, 8 ]},
    {[ 4, 10 ]}, {[ 1, 7 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 3, 9 ]}, {[ 5, 6 ]},{[ 2, 8 ]},{[ 4, 10 ]},
    {[ 1, 7 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[6] eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for;
end for;end for;
[ 5, 6 ],[ 5, 6 ],[ 5, 6 ],[ 5, 6 ],[ 5, 6 ]
> N56; #N56;Orbits(N56s);
Permutation group N56 acting on a set of
cardinality 10
Order = 1
1
> N56; #N56;
> N56s:=N56;
>  #N56s;
1
```

```
> [5,6]^N56s;
GSet{ [ 5, 6 ]}
> T56:=Transversal(N,N56);
> for i in [1..#T56] do
for> ss:=[5,6]^T56[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
35
-----------------------------------------------
 N57:=Stabiliser (N,[5,7]);
> SSS:={[5,7]}; SSS:=SSS^N;
> SSS;
GSet{{[ 5, 7 ]}, {[ 3, 10 ] }, {[ 1, 8 ]},
    {[ 2, 9 ]},{[ 4, 6 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 5, 7 ]},{[ 3, 10 ]}, { [ 1, 8 ]},{[ 2, 9 ] },
{[ 4, 6 ] }]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[7] eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for;
end for;end for;
[ 5, 7 ],[ 5, 7 ],[ 5, 7 ],[ 5, 7 ],[ 5, 7 ]
```

```
> N57; #N57;Orbits(N57s);
Permutation group N57 acting on a set of
cardinality 10
Order = 1
1
> N57; #N57;
> N57s:=N57;
>  #N57s;
1
> [5,7]^N57s;
GSet{[ 5, 7 ]
> T57:=Transversal(N,N57);
> for i in [1..#T57] do
for> ss:=[5,7]^T57[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
40
-----------------------------------------------
 N101:=Stabiliser (N,[10,1]);
> SSS:={[10,1]}; SSS:=SSS^N;
> SSS;
GSet{{[ 6, 2 ]},{[ 9, 5 ]},{[ 8, 4 ]},{[ 10, 1 ]},
    {[ 7, 3 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 6, 2 ]},{[ 9, 5 ]}, {[ 8, 4 ]}, {[ 10, 1 ]},
    {[ 7, 3 ]}]
> for i in [1..#SSS] do
```

```
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[10]*ts[1] eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 10, 1 ], 10, 1 ],[ 10, 1 ],[ 10, 1 ],[ 10, 1 ]
> N101; #N101;Orbits(N101s);
Permutation group N101 acting on a set
of cardinality 10
Order = 1
1
> N101; #N101;
> N101s:=N101;
>   #N101s;
1
> [10,1]^N101s;
GSet{[ 10, 1 ]}
> T101:=Transversal(N,N101);
> for i in [1..#T101] do
for> ss:=[10,1]^T101[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
45
----------------------------------------------

N102:=Stabiliser (N,[10,2]);
> SSS:={[10,2]}; SSS:=SSS^N;
> SSS;
```

```
GSet{{[ 6, 3 ]},{[ 8, 5 ]},{[ 10, 2 ]},{[ 7, 4 ]},
    {[ 9, 1 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 6, 3 ]},{[ 8, 5 ]},{[ 10, 2 ]},
    {[ 7, 4 ]},{[ 9, 1 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[10]*ts[2] eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for;
end for;end for;
[ 10, 2 ],[ 10, 2 ],[ 10, 2 ],
[ 10, 2 ],[ 10, 2 ]
> N102; #N102;Orbits(N102s);
Permutation group N102 acting on a set of
 cardinality 10
Order = 1
1
> N102; #N102;
> N102s:=N102;
>   #N102s;
1
> [10,2]^N102s;
GSet{[ 10, 2 ]}
> T102:=Transversal(N,N102);
> for i in [1..#T102] do
```

```
for> ss:=[10,2]^T102[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
50
-------------------------------------------------
 N106:=Stabiliser (N,[10,6]);
> SSS:={[10,6]}; SSS:=SSS^N;
> SSS;
GSet{{[ 10, 6 ]},{[ 8, 9 ]},{[ 6, 7 ]},
    {[ 7, 8 ]},{[ 9, 10 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 10, 6 ]},{[ 8, 9 ]},{[ 6, 7 ]},
    {[ 7, 8 ]},{[ 9, 10 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[10]*ts[6] eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for;
end for;end for;
[ 10, 6 ],[ 10, 6 ],[ 10, 6 ],
[ 10, 6 ],[ 10, 6 ]
> N106; #N106;Orbits(N106s);
Permutation group N106 acting on a set of
cardinality 10
```

```
Order = 1
1
> N106; #N106;
> N106s:=N106;
>  #N106s;
1
> [10,6]^N105s;
> [10,6]^N105s;
> T106:=Transversal(N,N106);
> for i in [1..#T106] do
for> ss:=[10,6]^T106[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
55
---------------------------------------------
 N108:=Stabiliser (N,[10,8]);
> SSS:={[10,8]}; SSS:=SSS^N;
> SSS;
GSet{{[ 7, 10 ]},{[ 9, 7 ]},{[ 10, 8 ]},
    {[ 6, 9 ]}, {[ 8, 6 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 7, 10 ]},{[ 9, 7 ]},{[ 10, 8 ]},
    {[ 6, 9 ]},{[ 8, 6 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[10]*ts[8] eq
```

```
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 10, 8 ],[ 10, 8 ],[ 10, 8 ],
[ 10, 8 ],[ 10, 8 ]
> N108; #N108;Orbits(N108s);
Permutation group N108 acting on a set
of cardinality 10
Order = 1
1
> N108; #N108;
> N108s:=N108;
>  #N108s;
1
> [10,8]^N108s;
GSet{[ 10, 8 ]}
> T108:=Transversal(N,N108);
> for i in [1..#T108] do
for> ss:=[10,8]^T108[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
60
---------------------------------------------
N513:=Stabiliser (N,[5,1,3]);
> SSS:={[5,1,3]}; SSS:=SSS^N;
> SSS;
GSet{{[ 5, 1, 3 ]},{[ 2, 3, 5 ]}, {[ 1, 2, 4 ]},
    {[ 4, 5, 2 ]},{[ 3, 4, 1 ]}}
> #(SSS);
```

```
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 5, 1, 3 ]},{[ 2, 3, 5 ]},{[ 1, 2, 4 ]},
    {[ 4, 5, 2 ]},{ [ 3, 4, 1 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[1] *ts[3]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]]
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for;
end for;end for;
[ 5, 1, 3 ],[ 5, 1, 3 ],[ 5, 1, 3 ],
[ 5, 1, 3 ],[ 5, 1, 3 ]
> N513; #N513;Orbits(N513s);
Permutation group N513 acting on a set of
cardinality 10
Order = 1
1
> N513; #N513;
> N513s:=N513;
>   #N513s;
1
> [5,1,3]^N513s;
GSet{[ 5, 1, 3 ]}
> T513:=Transversal(N,N513);
> for i in [1..#T513] do
for> ss:=[5,1,3]^T513[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
```

```
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
65
 N515:=Stabiliser (N,[5,1,5]);
> SSS:={[5,1,5]}; SSS:=SSS^N;
> SSS;
GSet{{[ 1, 2, 1 ]},{[ 5, 1, 5 ]},{[ 3, 4, 3 ]},
    {[ 2, 3, 2 ]},{[ 4, 5, 4 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 1, 2, 1 ]},{[ 5, 1, 5 ]},{ [ 3, 4, 3 ]},
    {[ 2, 3, 2 ]},{[ 4, 5, 4 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[1] *ts[5]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]] \

for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for;
end for;end for;
[ 5, 1, 5 ],[ 5, 1, 5 ],[ 5, 1, 5 ],
[ 5, 1, 5 ],[ 5, 1, 5 ]
> N515; #N515;Orbits(N515s);
Permutation group N515 acting on a set of
cardinality 10
Order = 1
1
> N515; #N515;
```

```
> N515s:=N515;
>   #N515s;
1
> [5,1,5]^N515s;
GSet{
    [ 5, 1, 5 ]
}
> T515:=Transversal(N,N515);
> for i in [1..#T515] do
for> ss:=[5,1,5]^T515[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
70
-----------------------------------------------

 N526:=Stabiliser (N,[5,2,6]);
> SSS:={[5,2,6]}; SSS:=SSS^N;
> SSS;
GSet{{ [ 2, 4, 8 ]},{[ 3, 5, 9 ]},{ [ 5, 2, 6 ]},
    {[ 4, 1, 10 ]}, {[ 1, 3, 7 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 2, 4, 8 ]},{[ 3, 5, 9 ]}, { [ 5, 2, 6 ]},
    {[ 4, 1, 10 ]},{[ 1, 3, 7 ] }]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[2] *ts[6]eq
```

```
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]] \
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for;
end for;end for;
[ 5, 2, 6 ],[ 5, 2, 6 ],[ 5, 2, 6 ],
[ 5, 2, 6 ],[ 5, 2, 6 ]
> N526; #N526;Orbits(N526s);
Permutation group N526 acting on a set of
 cardinality 10
Order = 1
1
> N526; #N526;
> N526s:=N526;
>   #N526s;
1
> [5,2,6]^N526s;
GSet{ [ 5, 2, 6 ]}
> T526:=Transversal(N,N526);
> for i in [1..#T526] do
for> ss:=[5,2,6]^T526[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
75
-----------------------------------------------
 N521:=Stabiliser (N,[5,2,1]);
> SSS:={[5,2,1]}; SSS:=SSS^N;
> SSS;
GSet{ {[ 3, 5, 4 ]},{[ 1, 3, 2 ]},{[ 4, 1, 5 ]},
    {[ 5, 2, 1 ]},{[ 2, 4, 3 ]}}
```

```
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 3, 5, 4 ]},{[ 1, 3, 2 ]},{[ 4, 1, 5 ]},
{[ 5, 2, 1 ]},{[ 2, 4, 3 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[2] *ts[1]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]] \
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 5, 2, 1 ],[5, 2, 1 ],[ 5, 2, 1 ],
[ 5, 2, 1 ],[ 5, 2, 1 ]
> N521; #N521;Orbits(N521s);
Permutation group N521 acting on a set of
cardinality 10
Order = 1
1
> N521; #N521;
> N521s:=N521;
>   #N521s;
1
> [5,2,1]^N521s;
GSet{[ 5, 2, 1 ]}
> T521:=Transversal(N,N521);
> for i in [1..#T521] do
for> ss:=[5,2,1]^T521[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
```

```
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
80
-------------------------------------------------
 N523:=Stabiliser (N,[5,2,3]);
> SSS:={[5,2,3]}; SSS:=SSS^N;
> SSS;
GSet{{ [ 5, 2, 3 ]},{[ 3, 5, 1 ]},{[ 4, 1, 2 ]},
    {[ 2, 4, 5 ]},{[ 1, 3, 4 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 5, 2, 3 ]},{[ 3, 5, 1 ]},{[ 4, 1, 2 ]},
    {[ 2, 4, 5 ]},{[ 1, 3, 4 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[2] *ts[3]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]] \
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 5, 2, 3 ],[ 5, 2, 3 ],[ 5, 2, 3 ],
[ 5, 2, 3 ],[ 5, 2, 3 ]
> N523; #N523;Orbits(N523s);
Permutation group N523 acting on a set of
cardinality 10
Order = 1
1
> N523; #N523;
> N523s:=N523;
```

```
>  #N523s;
1
> [5,2,3]^N523s;
GSet{
    [ 5, 2, 3 ]
}
> T523:=Transversal(N,N523);
> for i in [1..#T523] do
for> ss:=[5,2,3]^T523[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
85
--------------------------------------------------
 N524:=Stabiliser (N,[5,2,4]);
> SSS:={[5,2,4]}; SSS:=SSS^N;
> SSS;
GSet{{[ 4, 1, 3 ]}, {[ 2, 4, 1 ]},{ [ 1, 3, 5 ]},
    { [ 3, 5, 2 ]}, {[ 5, 2, 4 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[ { [ 4, 1, 3 ]}, {[ 2, 4, 1 ]},{[ 1, 3, 5 ]},
    {[ 3, 5, 2 ]}, {[ 5, 2, 4 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[2] *ts[4]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]] \
```

```
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 5, 2, 4 ],[ 5, 2, 4 ],[ 5, 2, 4 ],
[ 5, 2, 4 ],[ 5, 2, 4 ]
> N524; #N524;Orbits(N524s);
Permutation group N524 acting on a set of
cardinality 10
Order = 1
1
> N524; #N524;
> N524s:=N524;
>   #N524s;
1
> [5,2,4]^N524s;
GSet{[ 5, 2, 4 ]}
> T524:=Transversal(N,N524);
> for i in [1..#T524] do
for> ss:=[5,2,4]^T524[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
90
-------------------------------------------------
 N525:=Stabiliser (N,[5,2,5]);
> SSS:={[5,2,5]}; SSS:=SSS^N;
> SSS;
GSet{{[ 5, 2, 5 ] }, {[ 1, 3, 1 ] },{ [ 4, 1, 4 ]},
    {[ 3, 5, 3 ]}, {[ 2, 4, 2 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
```

```
> Seqq;
[{[ 5, 2, 5 ]},{[ 1, 3, 1 ]},{[ 4, 1, 4 ]},
    {[ 3, 5, 3 ]}, { [ 2, 4, 2 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[2] *ts[5]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]] \
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 5, 2, 5 ],[ 5, 2, 5 ],[ 5, 2, 5 ],
[ 5, 2, 5 ],[ 5, 2, 5 ]
> N525; #N525;Orbits(N525s);
Permutation group N525 acting on a set of
cardinality 10
Order = 1
1
> N525; #N525;
> N525s:=N525;
>   #N525s;
1
> [5,2,5]^N525s;
GSet{[ 5, 2, 5 ]}
> T525:=Transversal(N,N525);
> for i in [1..#T525] do
for> ss:=[5,2,5]^T525[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
95
```

```
-----------------------------------------------
 N5210:=Stabiliser (N,[5,2,10]);
> SSS:={[5,2,10]}; SSS:=SSS^N;
> SSS;
GSet{{[ 3, 5, 8 ]},{[ 1, 3, 6 ]},{[ 2, 4, 7 ]},
    {[ 5, 2, 10 ]},{ [ 4, 1, 9 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 3, 5, 8 ]}, {[ 1, 3, 6 ]},{[ 2, 4, 7 ]},
    {[ 5, 2, 10 ]},{[ 4, 1, 9 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[2] *ts[10]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]] \
 for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 5, 2, 10 ],[ 5, 2, 10 ],[ 5, 2, 10 ],[ 5, 2, 10 ]
[ 5, 2, 10 ]
> N5210; #N5210;Orbits(N5210s);
Permutation group N5210 acting on a set of
cardinality 10
Order = 1
1
> N5210; #N5210;
> N5210s:=N5210;
>   #N5210s;
1
> [5,2,10]^N5210s;
```

```
GSet{[ 5, 2, 10 ]}
> T5210:=Transversal(N,N5210);
> for i in [1..#T5210] do
for> ss:=[5,2,10]^T5210[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
100
---------------------------------------------
N532:=Stabiliser (N,[5,3,2]);
> SSS:={[5,3,2]}; SSS:=SSS^N;
> SSS;
GSet{{ [ 3, 1, 5 ]},{[ 1, 4, 3 ]},{[ 4, 2, 1 ]},
    {[ 5, 3, 2 ]},{[ 2, 5, 4 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 3, 1, 5 ]},{[ 1, 4, 3 ]},{[ 4, 2, 1 ]},
    {[ 5, 3, 2 ]},{[ 2, 5, 4 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[3] *ts[2]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]]
*ts[Rep(Seqq[i])[3]] \
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 5, 3, 2 ],[ 5, 3, 2 ],[ 5, 3, 2 ],[ 5, 3, 2 ]
[ 5, 3, 2 ]
```

```
> N532; #N532;Orbits(N532s);
Permutation group N532 acting on a set of
 cardinality 10
Order = 1
1
> N532; #N532;
> N532s:=N532;
>   #N532s;
1
> [5,3,2]^N532s;
GSet{[ 5, 3, 2 ]}
> T532:=Transversal(N,N532);
> for i in [1..#T532] do
for> ss:=[5,3,2]^T532[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for; m;
105
--------------------------------------------
 N535:=Stabiliser (N,[5,3,5]);
> SSS:={[5,3,5]}; SSS:=SSS^N;
> SSS;
GSet{{[ 5, 3, 5 ]},{ [ 4, 2, 4 ]},{ [ 3, 1, 3 ]},
{[ 1, 4, 1 ]},{ [ 2, 5, 2 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 5, 3, 5 ]},{ [ 4, 2, 4 ]},{[ 3, 1, 3 ]},
{[ 1, 4, 1 ]},{[ 2, 5, 2 ]}]
> for i in [1..#SSS] do
```

```
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[3] *ts[5]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]] \

for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 5, 3, 5 ],[ 5, 3, 5 ],[ 5, 3, 5 ],
[ 5, 3, 5 ],[ 5, 3, 5 ]
> N535; #N535;Orbits(N535s);
Permutation group N535 acting on a set of
cardinality 10
Order = 1
1
> N535; #N535;
> N535s:=N535;
>   #N535s;
1
> [5,3,5]^N535s;
GSet{     [ 5, 3, 5 ]}
> T535:=Transversal(N,N535);
> for i in [1..#T535] do
for> ss:=[5,3,5]^T535[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
110
------------------------------------------------
 N563:=Stabiliser (N,[5,6,3]);
> SSS:={[5,6,3]}; SSS:=SSS^N;
```

```
> SSS;
GSet{{[ 1, 7, 4 ]},{[ 4, 10, 2 ]}, {[ 3, 9, 1 ]},
    { [ 2, 8, 5 ]},{[ 5, 6, 3 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{ [ 1, 7, 4 ]},{ [ 4, 10, 2 ]},{[ 3, 9, 1 ]},
    { [ 2, 8, 5 ]},{[ 5, 6, 3 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[6] *ts[3]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]] \
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 5, 6, 3 ],[ 5, 6, 3 ],[ 5, 6, 3 ],
[ 5, 6, 3 ],[ 5, 6, 3 ]
> N563; #N563;Orbits(N563s);
Permutation group N563 acting on a set of
 cardinality 10
Order = 1
1
> N563; #N563;
> N563s:=N563;
>   #N563s;
1
> [5,6,3]^N563s;
GSet{ [ 5, 6, 3 ]}
> T563:=Transversal(N,N563);
> for i in [1..#T563] do
```

```
for> ss:=[5,6,3]^T563[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
115
-----------------------------------------------
 N5710:=Stabiliser (N,[5,7,10]);
> SSS:={[5,7,10]}; SSS:=SSS^N;
> SSS;
GSet{{[ 5, 7, 10 ]},{[ 2, 9, 7 ]},{[ 4, 6, 9 ]},
    {[ 3, 10, 8 ]},{[ 1, 8, 6 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 5, 7, 10 ]},{[ 2, 9, 7 ]},{[ 4, 6, 9 ]},
    {[ 3, 10, 8 ]}, {[ 1, 8, 6 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[7] *ts[10]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]] \
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 5, 7, 10 ],[ 5, 7, 10 ],[ 5, 7, 10 ],
[ 5, 7, 10 ],[ 5, 7, 10 ]
> N5710; #N5710;Orbits(N5710s);
Permutation group N5710 acting on a set of
 cardinality 10
Order = 1
```

```
1
> N5710; #N5710;
> N5710s:=N5710;
>   #N5710s;
1
> [5,7,10]^N5710s;
GSet{
    [ 5, 7, 10 ]
}
> T5710:=Transversal(N,N5710);
> for i in [1..#T5710] do
for> ss:=[5,7,10]^T5710[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
120
---------------------------------------------

 N571:=Stabiliser (N,[5,7,1]);
> SSS:={[5,7,1]}; SSS:=SSS^N;
> SSS;
GSet{{ [ 2, 9, 3 ] },{[ 4, 6, 5 ]},{[ 3, 10, 4 ]},
{ [ 1, 8, 2 ]},{ [ 5, 7, 1 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{ [ 2, 9, 3 ]},{[ 4, 6, 5 ]},{[ 3, 10, 4 ]},
{[ 1, 8, 2 ]},{[ 5, 7, 1 ]}]
> for i in [1..#SSS] do
for> for n in IN do
```

```
for|for> for n in IN do
for|for|for> if ts[5]*ts[7] *ts[1]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]] \
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 2, 9, 3 ],[ 2, 9, 3 ],[ 2, 9, 3 ],
[ 2, 9, 3 ],[ 2, 9, 3 ],[ 4, 6, 5 ],
[ 4, 6, 5 ],[ 4, 6, 5 ],[ 4, 6, 5 ],
[ 4, 6, 5 ],[ 3, 10, 4 ],[ 3, 10, 4 ],
[ 3, 10, 4 ],[ 3, 10, 4 ],[ 3, 10, 4 ],
[ 1, 8, 2 ],[ 1, 8, 2 ],[ 1, 8, 2 ],
[ 1, 8, 2 ],[ 1, 8, 2 ],[ 5, 7, 1 ],
[ 5, 7, 1 ],[ 5, 7, 1 ],[ 5, 7, 1 ],
[ 5, 7, 1 ]
> N571; #N571;Orbits(N571s);
Permutation group N571 acting on a set of
 cardinality 10
Order = 1
1
> N571; #N571;
> N571s:=N571;
>   #N571s;
1
> [5,7,1]^N571s;
GSet{[ 5, 7, 1 ]}
> T571:=Transversal(N,N571);
> for i in [1..#T571] do
for> ss:=[5,7,1]^T571[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
```

```
for|if> then m:=m+1; end if; end for;  m;
121
 N1026:=Stabiliser (N,[10,2,6]);
> SSS:={[10,2,6]}; SSS:=SSS^N;
> SSS;
GSet{{[ 8, 5, 9 ]},{[ 10, 2, 6 ]},{[ 6, 3, 7 ]},
    {[ 9, 1, 10 ]},{[ 7, 4, 8 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 8, 5, 9 ]},{[ 10, 2, 6 ]},{[ 6, 3, 7 ]},
{[ 9, 1, 10 ] },{[ 7, 4, 8 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[10]*ts[2] *ts[6]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]] \
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for;
end for;end for;
[ 10, 2, 6 ],[ 10, 2, 6 ],[ 10, 2, 6 ],
[ 10, 2, 6 ],[ 10, 2, 6 ]
> N1026; #N1026;Orbits(N1026s);
Permutation group N1026 acting on a set of
 cardinality 10
Order = 1
1
> N1026; #N1026;
> N1026s:=N1026;
>   #N1026s;
```

```
1
> [10,2,6]^N1026s;
GSet{[ 10, 2, 6 ]}
> T1026:=Transversal(N,N1026);
> for i in [1..#T1026] do
for> ss:=[10,2,6]^T1026[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
126
------------------------------------------------
> N545:=Stabiliser (N,[5,4,5]);
> SSS:={[5,4,5]}; SSS:=SSS^N;
> SSS;
GSet{{[ 2, 1, 2 ]},{[ 4, 3, 4 ]},{[ 5, 4, 5 ]},
 {[ 1, 5, 1 ]},{[ 3, 2, 3 ]}}
> #(SSS);
5
> Seqq:=Setseq(SSS);
> Seqq;
[{[ 2, 1, 2 ]},{[ 4, 3, 4 ]},{[ 5, 4, 5 ]},
    {[ 1, 5, 1 ]}, {[ 3, 2, 3 ]}]
> for i in [1..#SSS] do
for> for n in IN do
for|for> for n in IN do
for|for|for> if ts[5]*ts[4] *ts[5]eq
for|for|for|if> n*ts[Rep(Seqq[i])[1]]
*ts[Rep(Seqq[i])[2]]*ts[Rep(Seqq[i])[3]] \
for|for|for|if> then print Rep(Seqq[i]);
for|for|for|if> end if; end for; end for;end for;
[ 5, 4, 5 ],[ 5, 4, 5 ],[ 5, 4, 5 ],
```

```
[ 5, 4, 5 ],[ 5, 4, 5 ]
> N545; #N545;Orbits(N545s);
Permutation group N545 acting on a set of
 cardinality 10
Order = 1
1
> N545; #N545;
> N545s:=N545;
>   #N545s;
1
> [5,4,5]^N545s;
GSet{ [ 5, 4, 5 ]}
> T545:=Transversal(N,N545);
> for i in [1..#T545] do
for> ss:=[5,4,5]^T545[i];
for> cst[prodim(1, ts, ss)] := ss;
for> end for;
> m:=0; for i in [1..132] do if cst[i] ne []
for|if> then m:=m+1; end if; end for;  m;
131
-------------------------------------------------
```

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