2013

Behavior of solutions for Bernoulli initial-value problems

Carlos Marcelo Sardan

Behavior of Solutions for Bernoulli Initial-Value Problems

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

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Abstract

My research will be centered around the Bernoulli Ordinary Differential Equation, named after the Swiss Mathematician James Bernoulli. Bernoulli initial-value problems will be analyzed to find out solution behavior. We are interested in a phenomenon of unbounded growth of solutions in finite time. This phenomenon is called blow-up, and the blow-up time is denoted t_b. A blow-up occurs when the solution $y(t)$ approaches $+\infty$ or $-\infty$. To predict whether a solution would blow up in finite time, certain d-values for the initial condition of the Bernoulli initial-value problem would have to take place, depending on the coefficients and constant n. These d-values will be found for Bernoulli initial-value problems with constant and variable coefficients.

Acknowledgements

I would like to first and foremost thank my family for their endless support over the years. This accomplishment would not have been possible if it wasn't for their help, love, and patience throught my time in grad school. A special thank you goes to my mother who never stops believing in me and who continues to show me that having heart and determination are the keys to succeed in life. I learned in recent years that one can overcome the greatest obstacle in life, and all it takes is just a little bit of hope and self-motivation. This is truly a milestone and I am proud to be the first in my immediate family to graduate with an MA degree in Mathematics. I am proud of my heritage and proud of being a Bolivian mathematics professor, much like my fellow Bolivian, the late Jaime Escalante. I am in deep gratitude and I owe it all to my loving family.

I would also like to thank my thesis adviser, Dr. Nadejda Dyakevich, and committee members, Dr. Zahid Hasan and Dr. Dan Rinne. I am so fortunate to have had them guide my thesis. I am also thankful to have had the opportunity of taking Elementary, Ordinary, and Partial Differential Equations with Dr. Dyakevich as my professor at CSUSB. Her work ethic and teaching skills are admirable and I hope to one day teach Differential Equations as excellent as she does.

My final thanks goes out to all my friends and colleagues, old and new, who I have had the pleasure of taking math courses, teaching math courses, and tutoring mathematics with during my time at CSUSB and Chaffey College.

Table of Contents

Chapter 1

Introduction

1.1 Summary

James Bernoulli (December 27, 1654 - August 16, 1705) is the founder of the solvable first-order nonlinear differential equation which carries his name:

$$y'(t) = a(t)y^n + b(t)y.$$

Though James Bernoulli was not the first to solve the Bernoulli equation, he proposed the equation for solution in 1695.[Sas] Mathematician, Gottfried Leibniz, first solved the Bernoulli equation. The following year, Leibniz reduced the equation into a linear form by making substitutions with y^{n-1} as the dependent variable. That same year, brother and mathematician, John Bernoulli, also solved the Bernoulli equation, indepedently of Leibniz.[Kli72]

The main purpose of this project is to investigate blow-up properties of solutions for specific initial-value problems that involve Bernoulli Ordinary Differential Equations (ODEs). First, I will consider Bernoulli ODEs with constant coefficients. The objective is to find conditions on the coefficients and on the initial-values that lead to unbounded growth of solutions in finite time. This is called finite time blow-up. Then the Comparison Theorem will be used to extend results to Bernoulli initial-value problems with variable coefficients.

1.2 Motivation

It is well known that Bernoulli initial-value problems,

$$\begin{cases} y'(t) = a(t)y^n + b(t)y \\ y(0) = d, \end{cases} \tag{1}$$

where $a(t)$ and $b(t)$ are continuous and differentiable functions, d is a real number and $n > 1$ is a rational number, can be solved explicitly. Using a change of variable,

$$w = \frac{1}{y^{n-1}},$$

Bernoulli equations can be transformed into linear form,

$$w' = (1-n)a(t) + (1-n)b(t)w.$$

Therefore, it is easier to study blow-up properties of the solutions of Bernoulli problems. Since we know $a(t), b(t) \in C^1$, this makes w' Lipschitz continuous on any interval $[0, t]$, where t is less than the blow-up time. We can then apply the Comparison Theorem to extend results to initial-value problems with variable coefficients.

Definition 1.2.1. Lipschitz Condition: For an equation $u' = F(t, u)$, when F is continuous in some region D, F satisfies the condition that for some constant K

$$|F(t, u_1) - F(t, u_2)| \leq K|u_1 - u_2|,$$

for all points (t, u_1) and (t, u_2) in some D. [Col68][Ost68]

Theorem 1.2.2. ***Comparison Theorem:*** *Let the equations*

$$u'(t) = F(t, u)$$
$$u(\tau) = b_1,$$

and

$$u'(t) = G(t, u)$$
$$u(\tau) = b_2,$$

where F and G are continuous and satisfy Lipschitz conditions on some domain D, have solutions f and g, respectively, and let these solutions satisfy

$$b_1 < b_2.$$

If $F(x,y) \leq G(x,y)$ on D, then

$$f(t) \leq g(t),$$

for $t > \tau$. [Col68]

Chapter 2

Constant Coefficients

Given is a Bernoulli initial-value problem with constant coefficients,

$$\begin{cases} y'(t) = ay^n + by \\ y(0) = d, \end{cases} \tag{2}$$

for some values of a, b, and d, solution $y(t)$ will blow-up, or approach $+\infty$ or $-\infty$. More specifically, the moment of blow-up occurs at some finite time. This is called the blow-up time, denoted as t_b, and the first occurrence of blow-up is considered. Behavior of $y(t)$ for $t > t_b$ will not be considered. I will investigate conditions on a, b, n and d that will lead to unbounded growth of $y(t)$. The Comparison Theorem will be used to extend results to Bernoulli problems with variable coefficients.

Let's apply a change of variable to (2). This would then make (2) become linear and then multiplying by an integrating factor we obtain the solution. The solution to (2) is expressed as

$$y(t) = \left(\frac{b}{-a + \left(a + \frac{b}{d^{n-1}}\right) e^{-b(n-1)t}} \right)^{\frac{1}{n-1}},$$

and the solution $y(t)$ will be analyzed for blow-up in finite time.

To analyze, let

$$I(t) = \left(a + \frac{b}{d^{n-1}}\right) e^{-b(n-1)t},$$

and then

$$I'(t) = -b(n-1)\left(a + \frac{b}{d^{n-1}}\right) e^{-b(n-1)t}.$$

I' will help determine behavior of I and thus determine the behavior of solution $y(t)$.

To find the blow-up time, we must set the denominator equal to zero and solve for t. We'll denote this blow-up time as t_b, where

$$t_b = \frac{1}{b(n-1)} ln\left(1 + \frac{b}{ad^{n-1}}\right).$$

Here t_b is finite and shows $y(t)$ can blow up in finite time.

If $n = 1$, then (2) becomes separable and solution $y(t)$ is an exponential function,

$$y(t) = de^{(a+b)t},$$

which is bounded for all finite time $t \geq 0$. Solution $y(t)$ also does not have blow up in finite time for $n < 1$. In this chapter, we'll only consider Bernoulli ODEs with $n > 1$.

2.1 Case $n = \frac{p}{q}$, Where p is an Even Constant and q is an Odd Constant

Theorem 2.1.1. *The following is true for the solution $y(t)$ of (2) when $n = \frac{p}{q} > 1$, where p is an even number and q is an odd number.*

1. *Let $a > 0$ and $b > 0$. If $d > 0$, then $y(t) \to +\infty$ as $t \to t_b$.*
2. *Let $a > 0$ and $b < 0$. If $d > (\frac{-b}{a})^{\frac{1}{n-1}}$, then $y(t) \to +\infty$ as $t \to t_b$.*
3. *Let $a < 0$ and $b > 0$. If $d < 0$, then $y(t) \to -\infty$ as $t \to t_b$.*
4. *Let $a < 0$ and $b < 0$. If $d < (\frac{-b}{a})^{\frac{1}{n-1}}$, then $y(t) \to -\infty$ as $t \to t_b$.*

Otherwise, $y(t)$ is bounded for all $t \geq 0$.

Proof. For each case below, if $d = 0$, then solution $y(t) = 0$ for all $t \geq 0$.

1. Let $a > 0$ and $b > 0$. If $d > 0$, then $I(t) > 0$ and $I'(t) < 0$ for all $t \geq 0$. I is decreasing for all $t \geq 0$ and $I(t) > a$ on $[0, t_b)$. As $t \to t_b$, $y(t) \to +\infty$.

If $d < 0$, then

$$ln\left(1 + \frac{b}{ad^{n-1}}\right) < 0.$$

Therefore, $t_b > 0$ does not exist and $y(t)$ is bounded for all $t \geq 0$.

2. Let $a > 0$ and $b < 0$. If $d > (\frac{-b}{a})^{\frac{1}{n-1}}$, then $a + \frac{b}{d^{n-1}} > 0$. This means that $I(t) > 0$ and $I'(t) > 0$ for all $t \geq 0$. I is increasing for all $t \geq 0$ and $I(t) < a$ on $[0, t_b)$. As $t \to t_b$, $y(t) \to +\infty$.

If $d = \left(\frac{-b}{a}\right)^{\frac{1}{n-1}}$, then solution $y(t) = \left(\frac{-b}{a}\right)^{\frac{1}{n-1}}$. Here $y(t)$ is bounded for all $t \geq 0$.

If $d < 0$, then

$$ln\left(1 + \frac{b}{ad^{n-1}}\right) > 0.$$

This implies $t_b < 0$. Therefore, $y(t)$ is bounded for all $t \geq 0$. If

$$0 < d < \left(\frac{-b}{a}\right)^{\frac{1}{n-1}},$$

then $ln\left(1 + \frac{b}{ad^{n-1}}\right)$ is undefined, which means $t_b > 0$ does not exist. Therefore, $y(t)$ is bounded for all $t \geq 0$.

3. Let $a < 0$ and $b > 0$. If $d < 0$, then $I(t) < 0$ and $I'(t) > 0$ for all $t \geq 0$. I is increasing for all $t \geq 0$ and $I(t) < a$ on $[0, t_b)$. As $t \to t_b$, $y(t) \to -\infty$.

If $d > 0$, then $\frac{b}{ad^{n-1}} < 0$. This implies

$$ln\left(1 + \frac{b}{ad^{n-1}}\right) < 0.$$

Either way, $t_b > 0$ does not exist. Therefore, solution $y(t)$ is bounded for all $t \geq 0$.

4. Let $a < 0$ and $b < 0$. If $d < \left(\frac{-b}{a}\right)^{\frac{1}{n-1}}$, then $a + \frac{b}{d^{n-1}} < 0$. This implies $I(t) < 0$ and $I'(t) < 0$ for all $t \geq 0$. I is decreasing for all $t \geq 0$ and $I(t) > a$ on $[0, t_b)$. As $t \to t_b$, $y(t) \to -\infty$.

If $d = \left(\frac{-b}{a}\right)^{\frac{1}{n-1}}$, then solution $y(t) = \left(\frac{-b}{a}\right)^{\frac{1}{n-1}}$. Here $y(t)$ is bounded for all $t \geq 0$.

If $d > 0$, then

$$ln(1 + \frac{b}{ad^{n-1}}) > 0.$$

This makes $t_b < 0$. Therefore, $y(t)$ is bounded for all $t \geq 0$.

If

$$\left(\frac{-b}{a}\right)^{\frac{1}{n-1}} < d < 0,$$

then

$$1 + \frac{b}{ad^{n-1}} < 0.$$

This makes $ln\left(1 + \frac{b}{ad^{n-1}}\right)$ undefined. Therefore, $t_b > 0$ does not exist and solution $y(t)$ is bounded for all $t \geq 0$. $\square$

Example Let's consider the following initial-value problem

$$\begin{cases} y'(t) = 6y^{\frac{8}{3}} - 13y \\ y(0) = d \end{cases} \tag{3}$$

with five different values of d as investigated below.

Notice that $a = 6 > 0$ and $b = -13 < 0$ imply that

$$\left(\frac{-b}{a}\right)^{\frac{1}{n-1}} = (13/6)^{3/5} \approx 1.590286093.$$

According to Theorem 2.1.1, the solution to problem (3) should blow up in finite time for any $d > (13/6)^{3/5}$, otherwise $y(t)$ is bounded. The solution to problem (3) is

$$y(t) = \left(\frac{-13}{-6 + (6 - 13d^{-\frac{5}{3}})e^{\frac{65t}{3}}}\right)^{\frac{3}{5}}.$$

If $d = -1$, then

$$y(t) = \left(\frac{-13}{-6 + 19e^{\frac{65t}{3}}}\right)^{\frac{3}{5}},$$

which is bounded for all $t \geq 0$.

If $d = 0$, then

$$y(t) = 0,$$

considering

$$y(t) = \left(\frac{-13d^{\frac{5}{3}}}{-6d^{\frac{5}{3}} + (6d^{\frac{5}{3}} - 13)e^{\frac{65t}{3}}}\right)^{\frac{3}{5}}$$

as the equivalent to the solution of (3).

If $d = 1$, then

$$y(t) = \left(\frac{13}{6 + 7e^{\frac{65t}{3}}}\right)^{\frac{3}{5}},$$

which is bounded for all $t \geq 0$.

If $d = (13/6)^{3/5}$, then

$$y(t) = \left(\frac{13}{6}\right)^{\frac{3}{5}},$$

which is bounded for all $t \geq 0$.

If $d = 8$, then

$$y(t) = \left(\frac{-416}{-192 + 179e^{\frac{65t}{3}}}\right)^{\frac{3}{5}}.$$

Here $y(t) \to +\infty$ when $t \to t_b = -\frac{3}{65} ln\left(\frac{179}{192}\right)$.

2.2 Case $n = \frac{p}{q}$, Where p is an Odd Constant and q is an Odd Constant

In this section, we'll continue our investigation of the behavior of $y(t)$ for a different n. If we let $n = \frac{p}{q}$, where p is an odd number and q is an odd number, then $\frac{1}{n-1}$ has an even denominator. This makes solution $y(t)$ an even root function. Therefore, the solution $y(t)$ of (2) is

$$y(t) = \pm\left(\frac{b}{-a + \left(a + \frac{b}{d^{n-1}}\right)e^{-b(n-1)t}}\right)^{\frac{1}{n-1}}.$$

Since the initial condition is

$$y(0) = d,$$

d will influence $y(t)$. If d is positive, then $y(t)$ is positive. If d is negative, then $y(t)$ is negative for all $t \geq 0$.

Theorem 2.2.1. *The following is true for the solution $y(t)$ of (2) when $n = \frac{p}{q} > 1$, where p is an odd number and q is an odd number.*

1. *Let $a > 0$ and $b > 0$. If $d > 0$, then $y(t) \to +\infty$ as $t \to t_b$. If $d < 0$, then $y(t) \to -\infty$ as $t \to t_b$.*

2. *Let $a > 0$ and $b < 0$. If $d > \left(\frac{-b}{a}\right)^{\frac{1}{n-1}}$, then $y(t) \to +\infty$ as $t \to t_b$. If $d < -\left(\frac{-b}{a}\right)^{\frac{1}{n-1}}$, then $y(t) \to -\infty$ as $t \to t_b$. Otherwise, $y(t)$ is bounded for all $t \geq 0$.*

3. *If $a \leq 0$, then $y(t)$ is bounded for all $t \geq 0$.*

Proof. For each case below, if $d = 0$, then solution $y(t) = 0$ for all $t \geq 0$.

1. Let $a > 0$ and $b > 0$. If $d > 0$, then

$$a + \frac{b}{d^{n-1}} > 0.$$

This implies $I(t) > 0$ and $I'(t) < 0$ for all $t \geq 0$. I is decreasing for all $t \geq 0$ and $I(t) > a$ on $[0, t_b)$. As $t \to t_b$, $y(t) \to +\infty$.

If $d < 0$, then

$$a + \frac{b}{d^{n-1}} > 0.$$

This implies $I(t) > 0$ and $I'(t) < 0$ for all $t \geq 0$. I is decreasing for all $t \geq 0$ and $I(t) > a$ on $[0, t_b)$. As $t \to t_b$, $y(t) \to -\infty$.

2. Let $a > 0$ and $b < 0$. If $d > \left(\frac{-b}{a}\right)^{\frac{1}{n-1}}$, then

$$a + \frac{b}{d^{n-1}} > 0.$$

This implies $I(t) > 0$ and $I'(t) > 0$ for all $t \geq 0$. I is increasing for all $t \geq 0$ and $I(t) < a$ on $[0, t_b)$. As $t \to t_b$, $y(t) \to +\infty$.

If $d < -\left(\frac{-b}{a}\right)^{\frac{1}{n-1}}$, then

$$a + \frac{b}{d^{n-1}} > 0.$$

This implies $I(t) > 0$ and $I'(t) > 0$ for all $t \geq 0$. I is increasing for all $t \geq 0$ and $I(t) < a$ on $[0, t_b)$. As $t \to t_b$, $y(t) \to -\infty$.

If

$$-\left(\frac{-b}{a}\right)^{\frac{1}{n-1}} < d < \left(\frac{-b}{a}\right)^{\frac{1}{n-1}},$$

then

$$\left(1 + \frac{b}{ad^{n-1}}\right) < 0.$$

This implies $ln\left(1 + \frac{b}{ad^{n-1}}\right)$ is undefined and so $t_b > 0$ does not exist. Solution $y(t)$ is bounded for all $t \geq 0$.

If $d = \pm\left(\frac{-b}{a}\right)^{\frac{1}{n-1}}$, then solution $y(t) = \pm\left(\frac{-b}{a}\right)^{\frac{1}{n-1}}$, respectively. Therefore, $y(t)$ is bounded for all $t \geq 0$.

3. Let $a < 0$. For any non-zero d, if $b > 0$, then $ln\left(1 + \frac{b}{ad^{n-1}}\right)$ in t_b is undefined. Therefore, $t_b > 0$ does not exist and $y(t)$ is bounded for all $t \geq 0$.

If $b < 0$, then $ln\left(1 + \frac{b}{ad^{n-1}}\right)$ is positive for any non-zero d. However, $t_b < 0$ since $b < 0$. Therefore, $t_b > 0$ does not exist and solution $y(t)$ is bounded for all $t \geq 0$.

And finally, let $a = 0$. For any non-zero b, the solution $y(t) = de^{bt}$. Therefore, $y(t)$ is bounded for all finite $t \geq 0$. □

Example Let's consider the following initial-value problem,

$$\begin{cases} y'(t) = 5y^9 + 10y \\ y(0) = d, \end{cases} \tag{4}$$

with three different values of d as investigated below.

Notice that $a = 5 > 0$ and $b = 10 > 0$. According to Theorem 2.2.1, the solution to problem (5) should blow up in finite time for any $d > 0$ and any $d < 0$, otherwise $y(t)$ is bounded. The solution to problem (4) is

$$y(t) = \left(\frac{10}{-5 + (5 + 10d^{-8})e^{-80t}}\right)^{\frac{1}{8}}.$$

If $d = 2$, then

$$y(t) = \left(\frac{1280}{-640 + 645e^{-80t}}\right)^{\frac{1}{8}},$$

where $y(t) \to +\infty$ as $t \to t_b = \frac{1}{80}ln\left(\frac{129}{128}\right)$.

If $d = -1$, then

$$y(t) = -\left(\frac{10}{-5 + 15e^{-80t}}\right)^{\frac{1}{8}},$$

where $y(t) \to -\infty$ as $t \to t_b = \frac{1}{80}ln(3)$.

If $d = 0$, then

$$y(t) = 0,$$

which is bounded for all $t \geq 0$.

2.3 Case $n = \frac{p}{q}$, Where p is an Odd Constant and q is an Even Constant

In this final section for the constant coefficients chapter, we consider $n = \frac{p}{q} > 1$, where p is an odd number and q is an even number. Here, we will notice similarities with the solution behavior as with the case of n from the previous section. It is important to see the effect n will have in this section. Since $n = \frac{p}{q} > 1$, where p is an odd number and q is an even number, the expression $n-1$ becomes a rational number with an even denominator. This means d must be positive for the real solution $y(t)$ to exist. If $d < 0$, then d^{n-1} isn't a real number and t_b will not exist. Solution of (2) is

$$y(t) = \left(\frac{b}{-a + \left(a + \frac{b}{d^{n-1}}\right) e^{-b(n-1)t}} \right)^{\frac{1}{n-1}}.$$

Theorem 2.3.1. *The following is true for the solution $y(t)$ of (2) when $n = \frac{p}{q} > 1$, where p is an odd number and q is an even number.*

1. *Let $a > 0$ and $b > 0$. If $d > 0$, then $y(t) \to +\infty$ as $t \to t_b$. Otherwise, $y(t)$ is bounded for all $t \geq 0$.*
2. *Let $a > 0$ and $b < 0$. If $d > \left(\frac{-b}{a}\right)^{\frac{1}{n-1}}$, then $y(t) \to +\infty$ as $t \to t_b$. Otherwise, $y(t)$ is bounded for all $t \geq 0$.*
3. *If $a \leq 0$, then $y(t)$ is bounded for all $t \geq 0$.*

Proof. For each case below, If $d = 0$, then solution $y(t) = 0$ for all $t \geq 0$.

1. Let $a > 0$, $b > 0$ and $d > 0$. Proof is the same as in *Theorem 2.2.1* part 1.

2. Let $a > 0$, $b < 0$ and $d > \left(\frac{-b}{a}\right)^{\frac{1}{n-1}}$. Proof is same as in *Theorem 2.2.1* part 2.

3. Let $a \leq 0$ and consider any positive d. Proof is same as in *Theorem 2.2.1* part 3. □

Example Let's consider the following initial-value problem,

$$\begin{cases} y'(t) = 5y^{\frac{5}{2}} - 4y \\ y(0) = d, \end{cases} \tag{5}$$

with two different values of d as investigated below.

Notice that $a = 5 > 0$ and $b = -4 < 0$ imply that

$$\left(\frac{-b}{a}\right)^{\frac{1}{n-1}} = (4/5)^{2/3} \approx 0.861773876.$$

According to Theorem 2.3.1, the solution to problem (5) should blow up in finite time for any $d > (4/5)^{2/3}$. The solution to problem (5) is

$$y(t) = \left(\frac{-4}{-5 + (5 - 4d^{-\frac{3}{2}})e^{6t}} \right)^{\frac{2}{3}}.$$

If $d = \frac{9}{4}$, then

$$y(t) = \left(\frac{-108}{-135 + 103e^{6t}} \right)^{\frac{2}{3}},$$

where $y(t) \to +\infty$ as $t \to t_b = -1/6ln\left(\frac{103}{135}\right)$.

If $d = \frac{1}{4}$, then

$$y(t) = \left(\frac{4}{5 + 27e^{6t}} \right)^{\frac{2}{3}},$$

which is bounded for all $t \geq 0$.

Chapter 3

Variable Coefficients

We have completed our investigation of the behavior of solutions for Bernoulli IVPs with constant coefficients. We know now when solution $y(t)$ of (2) will have unbounded growth in finite time. We should now focus our attention on IVP (1) and investigate the behavior of $y(t)$ for general Bernoulli IVPs with variable coefficients, where $a(t)$ and $b(t)$ are continuous and differentiable functions for all $t \geq 0$. With the Comparison Theorem, we'll extend the results of Chapter 2 to better see the behavior of $y(t)$ for IVP (1).

The solution $y(t)$ of (1) is

$$y(t) = \frac{e^{\int_0^t b(\hat{t})d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t a(\tilde{t})e^{(n-1)\int_0^{\tilde{t}} b(\hat{t})d\hat{t}}d\tilde{t}\right]^{\frac{1}{n-1}}}.$$

From the Comparison Theorem, we'll show and compare solution $y(t)$ to a function, $\bar{y}(t)$, that is a solution to some other Bernoulli IVP. This will help analyze the behavior of $y(t)$. Solution $\bar{y}(t)$ itself will be a function that has unbounded growth in finite time, $t_{\bar{b}} > 0$. It is important to see that for $y(t)$ to blow up, $y(t) \leq \bar{y}(t) < 0$ or $y(t) \geq \bar{y}(t) > 0$. This will imply that $t_b \leq t_{\bar{b}}$.

Unbounded growth for $y(t)$ will depend on $a(t)$, $b(t)$, d and n. In the next few sections, we'll again consider $n > 1$.

3.1 Case $n = \frac{p}{q}$, Where p is an Even Constant and q is an Odd Constant

Theorem 3.1.1. *The following is true for the solution $y(t)$ of (1) when $n = \frac{p}{q} > 1$, where p is an even number and q is an odd number.*

1. *Let $a(t) \geq a > 0$ and $b(t) \geq b > 0$. If $d > 0$, then solution $y(t) \to +\infty$ as $t \to t_b$.*
2. *Let $a_1 \geq a(t) \geq a > 0$ and $b_1 \leq b(t) \leq b < 0$. If $d > \left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}}$, then solution $y(t) \to +\infty$ as $t \to t_b$. If $d \leq \left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$, then solution $y(t)$ is bounded for all $t \geq 0$.*
3. *Let $a(t) \leq a < 0$ and $b(t) \geq b > 0$. If $d < 0$, then solution $y(t) \to -\infty$ as $t \to t_b$.*
4. *Let $a_1 \leq a(t) \leq a < 0$ and $b_1 \leq b(t) \leq b < 0$. If $d < \left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}}$, then solution $y(t) \to -\infty$ as $t \to t_b$. If $d \geq \left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$, then solution $y(t)$ is bounded for all $t \geq 0$.*

Proof. For each case, if $d = 0$, then solution $y(t) = 0$ and $y(t)$ is bounded for all $t \geq 0$.

1. Let $a(t) \geq a > 0$ and $b(t) \geq b > 0$. Let's construct a function $\bar{y}(t) \leq y(t)$ to prove $y(t) \to +\infty$ for some finite $t_b > 0$.

$$
\begin{aligned}
y(t) &\geq \frac{e^{\int_0^t b d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t a e^{(n-1)\int_0^{\tilde{t}} b d\hat{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\
&= \frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - a(n-1)\int_0^t e^{b(n-1)\tilde{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\
&= \frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - \frac{a}{b}\left(e^{b(n-1)t} - 1\right)\right]^{\frac{1}{n-1}}} \\
&= \bar{y}(t).
\end{aligned}
$$

Let

$$\bar{I}(t) = \frac{a}{b}\left(e^{b(n-1)t} - 1\right),$$

and then

$$\bar{I}'(t) = a(n-1)e^{b(n-1)t}.$$

From above, $\bar{I} \geq 0$ and $\bar{I}' > 0$ for all $t \geq 0$. $\bar{I}$ is increasing for all $t \geq 0$. If $d > 0$, then

$$\bar{I}(t) < \left(\frac{1}{d^{n-1}}\right)$$

on $[0, t_{\bar{b}})$, where

$$t_{\bar{b}} = \frac{1}{b(n-1)} ln \left(1 + \frac{b}{ad^{n-1}}\right).$$

Therefore, $\bar{y}(t) \to +\infty$ as $t \to t_{\bar{b}}$. Since $\bar{y}(t) \leq y(t)$, $y(t) \to +\infty$ as $t \to t_b \leq t_{\bar{b}}$.

If $d < 0$, then $\left(\frac{1}{d^{n-1}}\right) < 0$ and the denominator of solution $y(t)$ is negative for all $t \geq 0$. Therefore, $y(t)$ will remain bounded for all $t \geq 0$.

2. Let $a_1 \geq a(t) \geq a > 0$ and $b_1 \leq b(t) \leq b < 0$. Let's construct a function $\bar{y}(t) \to +\infty$ such that $y(t) \geq \bar{y}(t)$.

$$\begin{aligned} y(t) &\geq \frac{e^{\int_0^t b_1 d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t a e^{(n-1)\int_0^{\tilde{t}} b_1 d\hat{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= \frac{e^{b_1 t}}{\left[\left(\frac{1}{d^{n-1}}\right) - a(n-1)\int_0^t e^{b_1(n-1)\tilde{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= \frac{e^{b_1 t}}{\left[\left(\frac{1}{d^{n-1}}\right) - \frac{a}{b_1}\left(e^{b_1(n-1)t} - 1\right)\right]^{\frac{1}{n-1}}} \\ &= \bar{y}(t). \end{aligned}$$

Let

$$\bar{I}(t) = \frac{a}{b_1}\left(e^{b_1(n-1)t} - 1\right),$$

and then

$$\bar{I}'(t) = a(n-1)e^{b_1(n-1)t}.$$

From above, $0 \leq \bar{I} < \frac{-a}{b_1}$ and $\bar{I}' > 0$ for all $t \geq 0$. $\bar{I}$ is increasing for all $t \geq 0$. If $d > \left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}}$, then

$$\bar{I}(t) < \left(\frac{1}{d^{n-1}}\right)$$

on $[0, t_{\bar{b}})$, where

$$t_{\bar{b}} = \frac{1}{b_1(n-1)} ln \left(1 + \frac{b_1}{ad^{n-1}}\right).$$

Therefore, $\bar{y}(t) \to +\infty$ as $t \to t_{\bar{b}}$. Since $y(t) \geq \bar{y}(t)$, $y(t) \to +\infty$ as $t \to t_b \leq t_{\bar{b}}$.

To see boundedness for solution $y(t)$, let's construct a bounded function, $\tilde{y}(t)$, such that $\tilde{y}(t) \geq y(t)$.

$$\begin{aligned}
y(t) &\leq \frac{e^{\int_0^t b d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t a_1 e^{(n-1)\int_0^{\tilde{t}} b d\hat{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\
&= \frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - a_1(n-1)\int_0^t e^{b(n-1)\tilde{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\
&= \frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - \frac{a_1}{b}\left(e^{b(n-1)t} - 1\right)\right]^{\frac{1}{n-1}}} \\
&= \tilde{y}(t).
\end{aligned}$$

Let

$$\tilde{I}(t) = \frac{a_1}{b}\left(e^{b(n-1)t} - 1\right),$$

and then

$$\tilde{I}'(t) = a_1(n-1)e^{b(n-1)t}.$$

From above, $0 \leq \tilde{I} < \frac{-a_1}{b}$ and $\tilde{I}' > 0$ for all $t \geq 0$. $\tilde{I}$ is increasing for all $t \geq 0$.

If $d \leq \left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$, then

$$\tilde{I}(t) < \left(\frac{1}{d^{n-1}}\right)$$

for all $t \geq 0$. Therefore, If $d \leq \left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$, then $\tilde{y}(t)$ is bounded for all $t \geq 0$. Since $\tilde{y}(t) \geq y(t)$, $y(t)$ is bounded for all $t \geq 0$.

3. Let $a(t) \leq a < 0$ and $b(t) \geq b > 0$. Let's construct a function $\bar{y}(t) \to -\infty$ such that $\bar{y}(t) \geq y(t)$ to prove $y(t) \to -\infty$ for some finite $t_b > 0$.

$$\begin{aligned}
y(t) &\leq \frac{e^{\int_0^t b(\hat{t}) d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t a e^{(n-1)\int_0^{\tilde{t}} b d\hat{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\
&= \frac{e^{\int_0^t b(\hat{t}) d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - a(n-1)\int_0^t e^{b(n-1)\tilde{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\
&= \frac{e^{\int_0^t b(\hat{t}) d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - \frac{a}{b}\left(e^{b(n-1)t} - 1\right)\right]^{\frac{1}{n-1}}} \\
&= \bar{y}(t).
\end{aligned}$$

Let

$$\bar{I}(t) = \frac{a}{b}\left(e^{b(n-1)t} - 1\right),$$

and then

$$\bar{I}'(t) = a(n-1)e^{b(n-1)t}.$$

From above, $\bar{I} \leq 0$ and $\bar{I}' < 0$ for all $t \geq 0$. $\bar{I}$ is decreasing for all $t \geq 0$. If $d < 0$, then

$$\bar{I}(t) > \left(\frac{1}{d^{n-1}}\right)$$

on $[0, t_{\bar{b}})$, where

$$t_{\bar{b}} = \frac{1}{b(n-1)} ln\left(1 + \frac{b}{ad^{n-1}}\right).$$

Therefore, $\bar{y}(t) \to -\infty$ as $t \to t_{\bar{b}}$. Since $\bar{y}(t) \geq y(t)$, $y(t) \to -\infty$ as $t \to t_b \leq t_{\bar{b}}$.

If $d > 0$, then $\left(\frac{1}{d^{n-1}}\right) > 0$, and the denominator of solution $y(t)$ is positive for all $t \geq 0$. Therefore, $y(t)$ will remain bounded for all $t \geq 0$.

4. Let $a_1 \leq a(t) \leq a < 0$ and $b_1 \leq b(t) \leq b < 0$. Let's construct a function $\bar{y}(t) \to -\infty$ such that $y(t) \leq \bar{y}(t)$.

$$\begin{aligned} y(t) &\leq \frac{e^{\int_0^t b d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t a e^{(n-1)\int_0^{\tilde{t}} b_1 d\hat{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= \frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - a(n-1)\int_0^t e^{b_1(n-1)\tilde{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= \frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - \frac{a}{b_1}\left(e^{b_1(n-1)t} - 1\right)\right]^{\frac{1}{n-1}}} \\ &= \bar{y}(t). \end{aligned}$$

Let

$$\bar{I}(t) = \frac{a}{b_1}\left(e^{b_1(n-1)t} - 1\right),$$

and then

$$\bar{I}'(t) = a(n-1)e^{b_1(n-1)t}.$$

From above, $\frac{-a}{b_1} < \bar{I} \leq 0$ and $\bar{I}' < 0$ for all $t \geq 0$. $\bar{I}$ is decreasing for all $t \geq 0$. If $d < \left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}}$, then

$$\bar{I}(t) > \left(\frac{1}{d^{n-1}}\right)$$

on $[0, t_{\bar{b}})$, where

$$t_{\bar{b}} = \frac{1}{b_1(n-1)} ln \left(1 + \frac{b_1}{ad^{n-1}}\right).$$

Therefore, $\bar{y}(t) \to -\infty$ as $t \to t_{\bar{b}}$. Since $y(t) \leq \bar{y}(t)$, $y(t) \to -\infty$ as $t \to t_b \leq t_{\bar{b}}$.

To see boundedness for solution $y(t)$, let's construct a bounded function, $\tilde{y}(t)$, such that $\tilde{y}(t) \leq y(t)$.

$$\begin{aligned} y(t) &\geq \frac{e^{\int_0^t b_1 d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1) \int_0^t a_1 e^{(n-1)\int_0^{\tilde{t}} b d\hat{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= \frac{e^{b_1 t}}{\left[\left(\frac{1}{d^{n-1}}\right) - a_1(n-1) \int_0^t e^{b(n-1)\tilde{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= \frac{e^{b_1 t}}{\left[\left(\frac{1}{d^{n-1}}\right) - \frac{a_1}{b}\left(e^{b(n-1)t} - 1\right)\right]^{\frac{1}{n-1}}} \\ &= \tilde{y}(t). \end{aligned}$$

Let

$$\tilde{I}(t) = \frac{a_1}{b}\left(e^{b(n-1)t} - 1\right),$$

and then

$$\tilde{I}'(t) = a_1(n-1)e^{b(n-1)t}.$$

From above, $\frac{-a_1}{b} < \tilde{I} \leq 0$ and $\tilde{I}' < 0$ for all $t \geq 0$. $\tilde{I}$ is decreasing for all $t \geq 0$.

If $d \geq \left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$, then

$$\tilde{I}(t) > \left(\frac{1}{d^{n-1}}\right)$$

for all $t \geq 0$. Therefore, If $d \geq \left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$, then $\tilde{y}(t)$ is bounded for all $t \geq 0$. Since $\tilde{y}(t) \leq y(t)$, $y(t)$ is bounded for all $t \geq 0$. □

Example Let's consider the following initial-value problem,

$$\begin{cases} y'(t) = \frac{e^{4t}}{(t+1)^3} y^4 + \left(\frac{2t+3}{t+1}\right) y \\ y(0) = d, \end{cases} \tag{6}$$

with two different values of d as investigated below.

Notice that

$$a(t) = \frac{e^{4t}}{(t+1)^3} \geq 1 > 0$$

and

$$b(t) = \frac{2t+3}{t+1} \geq 2 > 0$$

are continuous for all $t \geq 0$. According to Theorem 3.1.1, the solution to problem (6) should blow up in finite time for any $d > 0$. The solution to problem (6) is

$$y(t) = \frac{e^{2t}(t+1)}{\left[\frac{1}{d^3} - \frac{3}{10}e^{10t} + \frac{3}{10}\right]^{\frac{1}{3}}}.$$

If $d = \sqrt[3]{10/7}$, then

$$y(t) = \frac{\sqrt[3]{10}e^{2t}(t+1)}{[10 - 3e^{10t}]^{\frac{1}{3}}},$$

where $y(t) \to +\infty$ as $t \to t_b = \frac{1}{10}ln\left(\frac{10}{3}\right)$.

If $d = -1/2$, then

$$y(t) = \frac{-\sqrt[3]{10}e^{2t}(t+1)}{[77 + 3e^{10t}]^{\frac{1}{3}}},$$

which is bounded for all $t \geq 0$.

3.2 Case $n = \frac{p}{q}$, Where p is an Odd Constant and q is an Odd Constant

Let us continue our investigation of the solution behavior of (1) when $n = \frac{p}{q} > 1$, where p is an odd number and q is an odd number. With this n, we obtain the solution

$$y(t) = \frac{e^{\int_0^t b(\hat{t})d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t a(\bar{t})e^{(n-1)\int_0^{\bar{t}} b(\hat{t})d\hat{t}}d\bar{t}\right]^{\frac{1}{n-1}}}.$$

The sign of solution $y(t)$ will depend on the sign of d. If d is positive or negative, then $y(t)$ is positive or negative, respectively. Another important note is that the expression $(n-1)$ will be a rational number with an even numerator. Therefore, for any non-zero d, $d^{n-1} > 0$.

Theorem 3.2.1. *The following is true for solution $y(t)$ of (1) when $n = \frac{p}{q} > 1$, where p is an odd number and q is an odd number.*

1. *Let $a(t) \geq a > 0$ and $b(t) \geq b > 0$. If $d > 0$, then $y(t) \to +\infty$ as $t \to t_b$. If $d < 0$, then $y(t) \to -\infty$ as $t \to t_b$.*

2. *Let $a_1 \geq a(t) \geq a > 0$ and $b_1 \leq b(t) \leq b < 0$. If $d > \left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}}$, then solution $y(t) \to +\infty$ as $t \to t_b$. If $0 < d \leq \left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$, then solution $y(t)$ is bounded for all $t \geq 0$.*

   If $d < -\left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}}$, then $y(t) \to -\infty$ as $t \to t_b$. If $0 > d \geq -\left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$, then solution $y(t)$ is bounded for all $t \geq 0$.

3. *Let $a(t) \leq a < 0$ and $b(t) \geq b > 0$. For any d, solution $y(t)$ is bounded for all $t \geq 0$.*

4. *Let $a(t) \leq a < 0$ and $b_1 \leq b(t) \leq b < 0$. For any d, solution $y(t)$ is bounded for all $t \geq 0$.*

Proof. For each case below, if $d = 0$, then solution $y(t) = 0$ for all $t \geq 0$.

1. Let $a(t) \geq a > 0$ and $b(t) \geq b > 0$. If $d > 0$, then solution $y(t) > 0$ for all $t \geq 0$. We need to show that there exists a function $\bar{y}(t) \to +\infty$ in finite time, such that $y(t) \geq \bar{y}(t)$. Proof is the same as in Theorem 3.1.1 part 1. Therefore, solution $y(t) \to +\infty$ as $t \to t_b \leq t_{\bar{b}}$.

If $d < 0$, then solution $y(t) < 0$ for all $t \geq 0$. We need to show that there exists a function $\bar{y}(t) \to -\infty$ in finite time, such that $y(t) \leq \bar{y}(t)$.

$$
\begin{aligned}
y(t) &\leq -\frac{e^{\int_0^t b d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t a e^{(n-1)\int_0^{\tilde{t}} b d\hat{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\
&= -\frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - a(n-1)\int_0^t e^{b(n-1)\tilde{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\
&= -\frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - \frac{a}{b}\left(e^{b(n-1)t} - 1\right)\right]^{\frac{1}{n-1}}} \\
&= \bar{y}(t).
\end{aligned}
$$

Let

$$
\bar{I}(t) = \frac{a}{b}\left(e^{b(n-1)t} - 1\right),
$$

and then

$$
\bar{I}'(t) = a(n-1)e^{b(n-1)t}.
$$

From above, $\bar{I}(t) \geq 0$ and $\bar{I}'(t) > 0$ for all $t \geq 0$. $\bar{I}$ is increasing for all $t \geq 0$. If $d < 0$, then $\left(\frac{1}{d^{n-1}}\right) > 0$ and

$$\bar{I}(t) < \left(\frac{1}{d^{n-1}}\right)$$

on $[0, t_{\bar{b}})$, where

$$t_{\bar{b}} = \frac{1}{b(n-1)} ln\left(1 + \frac{b}{ad^{n-1}}\right).$$

Therefore, $\bar{y}(t) \to -\infty$ as $t \to t_{\bar{b}}$. Since $y(t) \leq \bar{y}(t)$, $y(t) \to -\infty$ as $t \to t_b \leq t_{\bar{b}}$.

2. Let $a(t) \geq a > 0$ and $b_1 \leq b(t) \leq b < 0$. If we let $d > \left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}}$, then solution $y(t) > 0$ for all $t \geq 0$. The proof for blow-up and boundedness on solution $y(t)$ is the same as in Theorem 3.1.1 part 2. Therefore, if $d > \left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}}$, then $y(t) \to +\infty$ as $t \to t_b \leq t_{\bar{b}}$. If $d \leq \left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$, then $y(t)$ is bounded for all $t \geq 0$.

Now let's construct a function $\bar{y}(t) \to -\infty$ such that solution $y(t) \leq \bar{y}(t)$. Since we're letting $d < -\left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}}$, solution $y(t) < 0$ for all $t \geq 0$.

$$\begin{aligned} y(t) &\leq -\frac{e^{\int_0^t b_1 d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t a e^{(n-1)\int_0^{\tilde{t}} b_1 d\hat{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= -\frac{e^{b_1 t}}{\left[\left(\frac{1}{d^{n-1}}\right) - a(n-1)\int_0^t e^{b_1(n-1)\tilde{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= -\frac{e^{b_1 t}}{\left[\left(\frac{1}{d^{n-1}}\right) - \frac{a}{b_1}\left(e^{b_1(n-1)t} - 1\right)\right]^{\frac{1}{n-1}}} \\ &= \bar{y}(t). \end{aligned}$$

Let

$$\bar{I}(t) = \frac{a}{b_1}\left(e^{b_1(n-1)t} - 1\right),$$

and then

$$\bar{I}'(t) = a(n-1)e^{b_1(n-1)t}.$$

From above, $0 \leq \bar{I} < \frac{-a}{b_1}$ and $\bar{I}' > 0$ for all $t \geq 0$. $\bar{I}$ is increasing for all $t \geq 0$. If $d < -\left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}}$, then

$$\bar{I}(t) < \left(\frac{1}{d^{n-1}}\right)$$

on $[0, t_{\bar{b}})$, where

$$t_{\bar{b}} = \frac{1}{b_1(n-1)} ln\left(1 + \frac{b_1}{ad^{n-1}}\right).$$

Therefore, $\bar{y}(t) \to -\infty$ as $t \to t_{\bar{b}}$. Since $y(t) \leq \bar{y}(t)$, $y(t) \to -\infty$ as $t \to t_b \leq t_{\bar{b}}$.

To see boundedness for solution $y(t)$, let's construct a bounded function, $\tilde{y}(t)$, such that $\tilde{y}(t) \leq y(t)$.

$$\begin{aligned} y(t) &\geq -\frac{e^{\int_0^t bd\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t a_1 e^{(n-1)\int_0^{\tilde{t}} bd\hat{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= -\frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - a_1(n-1)\int_0^t e^{b(n-1)\tilde{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= -\frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - \frac{a_1}{b}\left(e^{b(n-1)t} - 1\right)\right]^{\frac{1}{n-1}}} \\ &= \tilde{y}(t). \end{aligned}$$

Let

$$\tilde{I}(t) = \frac{a_1}{b}\left(e^{b(n-1)t} - 1\right),$$

and then

$$\tilde{I}'(t) = a_1(n-1)e^{b(n-1)t}.$$

From above, $0 \leq \tilde{I} < \frac{-a_1}{b}$ and $\tilde{I}' > 0$ for all $t \geq 0$. $\tilde{I}$ is increasing for all $t \geq 0$.

If $0 < d \leq \left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$, then

$$\tilde{I}(t) < \left(\frac{1}{d^{n-1}}\right)$$

for all $t \geq 0$. This also implies that if $0 > d \geq -\left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$, then

$$\tilde{I}(t) < \left(\frac{1}{d^{n-1}}\right),$$

since $d^{n-1} > 0$ for any negative d. Therefore, if $d \geq -\left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$, then $\tilde{y}(t)$ is bounded for all $t \geq 0$. Since $\tilde{y}(t) \leq y(t)$, $y(t)$ is bounded for all $t \geq 0$.

3. Let $a(t) \leq a < 0$ and $b(t) \geq b > 0$. Let's consider any non-zero d. To show boundedness for $y(t)$, let's construct bounded functions $\tilde{y}(t)$ and $\tilde{y_o}(t)$ such that $y(t) \leq \tilde{y}(t)$ for when $d > 0$, and $\tilde{y_o}(t) \leq y(t)$ for when $d < 0$.

$$
\begin{aligned}
y(t) &\leq \frac{e^{\int_0^t b(\hat{t})d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t ae^{(n-1)\int_0^{\tilde{t}} bd\hat{t}}d\tilde{t}\right]^{\frac{1}{n-1}}} \\
&= \frac{e^{\int_0^t b(\hat{t})d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - a(n-1)\int_0^t e^{b(n-1)\tilde{t}}d\tilde{t}\right]^{\frac{1}{n-1}}} \\
&= \frac{e^{\int_0^t b(\hat{t})d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - \frac{a}{b}\left(e^{b(n-1)t} - 1\right)\right]^{\frac{1}{n-1}}} \\
&= \tilde{y}(t).
\end{aligned}
$$

and

$$
\begin{aligned}
y(t) &\geq -\frac{e^{\int_0^t b(\hat{t})d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t ae^{(n-1)\int_0^{\tilde{t}} bd\hat{t}}d\tilde{t}\right]^{\frac{1}{n-1}}} \\
&= -\frac{e^{\int_0^t b(\hat{t})d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - a(n-1)\int_0^t e^{b(n-1)\tilde{t}}d\tilde{t}\right]^{\frac{1}{n-1}}} \\
&= -\frac{e^{\int_0^t b(\hat{t})d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - \frac{a}{b}\left(e^{b(n-1)t} - 1\right)\right]^{\frac{1}{n-1}}} \\
&= \tilde{y}_o(t).
\end{aligned}
$$

Let

$$\tilde{I}(t) = \frac{a}{b}\left(e^{b(n-1)t} - 1\right)$$

for both inequalities above. Then,

$$\tilde{I}'(t) = a(n-1)e^{b(n-1)t}.$$

From above, $\tilde{I} \leq 0$ and $\tilde{I}' < 0$ for all $t \geq 0$. $\tilde{I}$ is decreasing for all t. For any non-zero d,

$$\left(\frac{1}{(+d)^{n-1}}\right) = \left(\frac{1}{(-d)^{n-1}}\right) > 0$$

and

$$\tilde{I}(t) \leq 0 < \left(\frac{1}{d^{n-1}}\right)$$

for all $t \geq 0$. Therefore, $\tilde{y}(t)$ and $\tilde{y_o}(t)$ are bounded for all $t \geq 0$. Therefore $y(t)$ is bounded for all $t \geq 0$.

4. Let $a(t) \leq a < 0$ and $b_1 \leq b(t) \leq b < 0$. Let's consider any non-zero d. To show boundedness for $y(t)$, let's construct bounded functions $\tilde{y}(t)$ and $\tilde{y_o}(t)$ such that $y(t) \leq \tilde{y}(t)$ for when $d > 0$, and $\tilde{y_o}(t) \leq y(t)$ for when $d < 0$.

$$\begin{aligned} y(t) &\leq \frac{e^{\int_0^t b d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t a e^{(n-1)\int_0^{\tilde{t}} b_1 d\hat{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= \frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - a(n-1)\int_0^t e^{b_1(n-1)\tilde{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= \frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - \frac{a}{b_1}\left(e^{b_1(n-1)t} - 1\right)\right]^{\frac{1}{n-1}}} \\ &= \tilde{y}(t) \end{aligned}$$

and

$$\begin{aligned} y(t) &\geq -\frac{e^{\int_0^t b d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t a e^{(n-1)\int_0^{\tilde{t}} b_1 d\hat{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= -\frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - a(n-1)\int_0^t e^{b_1(n-1)\tilde{t}} d\tilde{t}\right]^{\frac{1}{n-1}}} \\ &= -\frac{e^{bt}}{\left[\left(\frac{1}{d^{n-1}}\right) - \frac{a}{b_1}\left(e^{b_1(n-1)t} - 1\right)\right]^{\frac{1}{n-1}}} \\ &= \tilde{y}(t) \end{aligned}$$

Let

$$\tilde{I}(t) = \frac{a}{b_1}\left(e^{b_1(n-1)t} - 1\right)$$

for both inequalities above. Then,

$$\tilde{I}'(t) = a(n-1)e^{b_1(n-1)t}.$$

From above, $\tilde{I} \leq 0$ and $\tilde{I}' < 0$ for all $t \geq 0$. $\tilde{I}$ is decreasing for all t. For any non-zero d,

$$\left(\frac{1}{(+d)^{n-1}}\right) = \left(\frac{1}{(-d)^{n-1}}\right) > 0$$

and

$$\tilde{I}(t) \leq 0 < \left(\frac{1}{d^{n-1}}\right)$$

for all $t \geq 0$. Therefore, $\tilde{y}(t)$ and $\tilde{y_o}(t)$ are bounded for all $t \geq 0$. Therefore $y(t)$ is bounded for all $t \geq 0$. □

Example Let's consider the following initial-value problem,

$$\begin{cases} y'(t) = \frac{e^t}{(t+1)^{2/5}} y^{\frac{7}{5}} + \left(\frac{4t+5}{t+1}\right) y \\ y(0) = d, \end{cases} \tag{7}$$

with two different values of d as investigated below.

Notice that

$$a(t) = \frac{e^t}{(t+1)^{2/5}} \geq 1 > 0$$

and

$$b(t) = \left(\frac{4t+5}{t+1}\right) \geq 4 > 0,$$

which are continuous for all $t \geq 0$. According to Theorem 3.2.1, the solution to problem (7) should blow up in finite time for any non-zero d. The solution to problem (7) is

$$y(t) = \pm \frac{e^{4t}(t+1)}{\left[\frac{1}{d^{2/5}} + \frac{2}{13} - \frac{2}{13} e^{13t/5}\right]^{\frac{5}{2}}}.$$

If $d = 1/32$, then

$$y(t) = \frac{e^{4t}(t+1)}{\left[\frac{54}{13} - \frac{2}{13} e^{13t/5}\right]^{\frac{5}{2}}},$$

where $y(t) \to +\infty$ as $t \to t_b = \frac{5}{13} ln(27)$.

If $d = -1/243$, then

$$y(t) = -\frac{e^{4t}(t+1)}{\left[\frac{119}{13} - \frac{2}{13} e^{13t/5}\right]^{\frac{5}{2}}},$$

where $y(t) \to -\infty$ as $t \to t_b = \frac{5}{13} ln(119/2)$.

3.3 Case $n = \frac{p}{q}$, Where p is an Odd Constant and q is an Even Constant

In this final section on Bernoulli IVPs with variable coefficients, we'll consider $n = \frac{p}{q} > 1$, where p is an odd number and q is an even number. Clearly, the expression $(n-1)$ will be a rational number with an even denominator. This implies d^{n-1} is defined only on $d > 0$. We can expect that when $d < 0$, solution of (1) will not exist. Unbounded growth will only occur when $d > 0$. In this section, the expression $\frac{1}{n-1}$ will be a rational number with an odd denominator. Therefore, we obtain the solution to (1) as

$$y(t) = \frac{e^{\int_0^t b(\hat{t})d\hat{t}}}{\left[\left(\frac{1}{d^{n-1}}\right) - (n-1)\int_0^t a(\tilde{t})e^{(n-1)\int_0^{\tilde{t}} b(\hat{t})d\hat{t}}d\tilde{t}\right]^{\frac{1}{n-1}}}.$$

Theorem 3.3.1. *The following is true for solution $y(t)$ of (1) when $n = \frac{p}{q} > 1$, where p is an odd number and q is an even number.*

1. *Let $a(t) \geq a > 0$ and $b(t) \geq b > 0$. If $d > 0$, then solution $y(t) \to +\infty$ as $t \to t_b$. If $d = 0$, then solution $y(t)$ is bounded for all $t \geq 0$.*

2. *Let $a_1 \geq a(t) \geq a > 0$ and $b_1 \leq b(t) \leq b < 0$. If $d > \left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}}$, then solution $y(t) \to +\infty$ as $t \to t_b$.*
   If $0 \leq d \leq \left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$, then solution $y(t)$ is bounded for all $t \geq 0$.

3. *Let $a(t) \leq a < 0$ and $b(t) \geq b > 0$. For any positive d, $y(t)$ does not blow up in finite time.*

4. *Let $a(t) \leq a < 0$ and $b_1 \leq b(t) \leq b < 0$. For any positive d, $y(t)$ does not blow up in finite time.*

Proof. For each case below, if $d = 0$, then solution $y(t) = 0$ for all $t \geq 0$.

1. Let $a(t) \geq a > 0$ and $b(t) \geq b > 0$. Proof is same as in Theorem 3.1.1 part 1 for when $d > 0$. Therefore, $y(t) \to +\infty$ as $t \to t_b$. If $d < 0$, then $\left(\frac{1}{d^{n-1}}\right)$ is undefined. Therefore, solution $y(t)$ does not exist. Therefore, $y(t)$ does not blow up in finite time.

2. Let $a_1 \geq a(t) \geq a > 0$ and $b_1 \leq b(t) \leq b < 0$. Proof is same as in Theorem 3.1.1 part 2 for when $d > \left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}}$ and $d \leq \left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}$. Therefore, $y(t) \to +\infty$ as $t \to t_b$

and $y(t)$ is bounded for all $t \geq 0$, respectively. However, if $d < 0$, then $\left(\frac{1}{d^{n-1}}\right)$ is undefined. Therefore, solution $y(t)$ does not exist. Therefore, $y(t)$ does not blow up in finite time.

3. Let $a(t) \leq a < 0$ and $b(t) \geq b > 0$. Proof is same as in Theorem 3.2.1 part 3 for when $d > 0$. Therefore, $y(t)$ is bounded for all $t \geq 0$. If $d < 0$, then $\left(\frac{1}{d^{n-1}}\right)$ is undefined. Therefore, solution $y(t)$ does not exist. Therefore, $y(t)$ does not blow up in finite time.

4. Let $a(t) \leq a < 0$ and $b_1 \leq b(t) \leq b < 0$. Proof is same as in Theorem 3.2.1 part 4 for when $d > 0$. Therefore, $y(t)$ is bounded for all $t \geq 0$. If $d < 0$, then $\left(\frac{1}{d^{n-1}}\right)$ is undefined. Therefore, solution $y(t)$ does not exist. Therefore, $y(t)$ does not blow up in finite time.

□

Example Let's consider the following initial-value problem,

$$\begin{cases} y'(t) = \left(\frac{t+2}{t+1}\right) y^{\frac{3}{2}} + \left(\frac{-t-2}{t+1}\right) y \\ y(0) = d, \end{cases} \tag{8}$$

with two different values of d as investigated below.

Notice that

$$2 \geq a(t) = \frac{t+2}{t+1} \geq 1 > 0$$

and

$$-2 \leq b(t) = \frac{-t-2}{t+1} \leq -1 < 0$$

for all $t \geq 0$, and

$$\left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}} = (2/1)^{\frac{1}{3/2-1}} = 4$$

and

$$\left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}} = (1/2)^2 = \frac{1}{4}.$$

According to Theorem 3.3.1, the solution to problem (8) will blow up in finite time for any $d > \left(\frac{-b_1}{a}\right)^{\frac{1}{n-1}}$. The solution to problem (8) is

$$y(t) = \frac{(t+1)^{-1}e^{-t}}{\left[\left(\frac{1}{d^{1/2}}\right) + \frac{e^{-t/2}}{\sqrt{t+1}} - 1\right]^2}.$$

If $d = 9$, then

$$y(t) = \frac{(t+1)^{-1}e^{-t}}{\left[\frac{e^{-t/2}}{\sqrt{t+1}} - \frac{2}{3}\right]^2},$$

where $y(t) \to +\infty$ as $t \to t_b \approx 0.443712$.

Now for $y(t)$ to be bounded, Theorem 3.3.1 says to consider any d such that

$$0 \leq d \leq \left(\frac{-b}{a_1}\right)^{\frac{1}{n-1}}.$$

If $d = 1/9$, then

$$y(t) = \frac{(t+1)^{-1}e^{-t}}{\left[\frac{e^{-t/2}}{\sqrt{t+1}} + 2\right]^2},$$

which is bounded for all $t \geq 0$.

Chapter 4

Conclusion

Long term behavior of Bernoulli IVP with constant coefficients has been completely investigated. We now know the conditions on the coefficients and initial values that lead to the unbounded growth of solutions in finite time. This information can be used to investigate behavior of more complicated problems. For example, we investigated behavior of solutions for some Bernoulli problems with variable coefficients. Comparison theorem can be applied to extend the results to other nonlinear ordinary and partial differential equations.

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