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The Banach-Tarski paradox

Matthew Jacob Norman

The Banach-Tarski Paradox

A Thesis

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Matthew Jacob Norman

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Abstract

When mathematician Georg Cantor began to question what a number actually represents he began to set the groundwork for a whole new field of mathematics, known as set theory. This thesis discusses the early history of set theory as it is documented as well as the necessary basics of set theory in order to further understand the contents within. Set theory not only proved to be for the mathematical at heart but also struck interest into the mind of philosophers, theologians, and logicians. The interest of the "non-believers" produced a world of set-theoretical paradoxes which is a large portion of this paper. This thesis discusses the earliest of paradoxes from the time before Georg Cantor to the first of set-theoretical paradoxes such as Russell's paradox, leading up towards the Hausdorff Paradox. Ultimately, the main purpose of this thesis is to establish the history and motivation leading up to the Banach-Tarski Paradox, as well as it's proof.

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Chapter 1

Introduction

As I was nearing the end of my coursework to obtain a masters degree in mathematics from Cal State San Bernardino I began to turn my focus towards the content of my thesis. The options seemed limitless (yes, pun intended) but I wanted to focus on a topic which I found fascinating enough to keep my interest throughout the process of preparing, writing, researching, editing, formulating, and presenting my thesis. Number theory has always been high on my list of interests but as I began to travel down that road I was introduced to another field of mathematics, set theory, of which of course I have taken courses and exams on, but never knew could be so amusing.

I had the privilege of having Dr. Freiling for my last elective course to satisfy my graduation requirements and found out that he was a set theorist. This is a topic I haven't given much thought to but after a brief period of researching my interest began to rise when reading the ideas and theories related to this field. What really caught my attention was the amount of philosophical beliefs or disbeliefs that live in the topics of Set Theory. I approached Dr. Freiling to discuss some ideas I had found to focus my thesis on including the axiom of choice, concept of infinity, and other findings of set theory. He told me of some of the interesting things he has heard of throughout his time in the field which leads to the focus of my thesis.

He had mentioned to me that it can be mathematically shown that you could take a solid ball and cut it up into several pieces and put it back together in a way to create two solid balls of the same shape and size. Now I remember upon first hearing this I was found speechless as I tried to understand what he had just said. That seems completely absurd, but mostly impossible. I also thought though that this concept was just fascinating enough that I would be willing to spend countless hours of my life preparing, writing, researching, editing, formulating, and presenting this topic as my thesis. This paradox as it is called is known as the Banach-Tarski Paradox. The proof in itself is amazing but there are a countable number of steps we must first take in order to fully grasp the ideas behind the proof. I start at the beginning............

Chapter 2

Early History

Something that scientists, philosophers, theologians, and mathematicians have deliberated for eternity is the concept of infinity. The first written records of the idea of infinity are dated back to Zeno of Elea around fourth century BC. I predict that these ideas had circulated since the time of man but it wasn't until the fourth century that evidence was able to be journalized in some way. Zeno was thought to be a philosopher of his time and it seemed as if his goal was to stir controversy and thought into the minds of the people. He had proposed several famous paradoxes, or logical inconsistencies, which deal with the notion of infinity.

The first to mention is the "Achilles and the tortoise". The story says that the Greek god Achilles is in a race with a tortoise, in which the tortoise gets a lead, and even though Achilles is much quicker than the tortoise, he will never overtake the tortoise. This is due to the fact that Achilles must first reach the starting point, then he must reach the point of where the tortoise was, and by this time the tortoise would have moved a greater amount, forcing Achilles to reach the next point, and this continues. He was able to put this story into a mathematical explanation known as the dichotomy paradox.

The dichotomy paradox says that in order to reach a destination, one must travel half of the distance, but before we travel half of the distance we must travel a quarter of the distance. Before traveling a quarter of a distance we must travel an eight of a distance, and before an eighth, we must travel a sixteenth, and a thirty-secondth, and so on. So to begin our travels we must start with an infinite number of tasks which would seem to be impossible. Numerically this can be represented as an infinite sum

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots.$$

His ideas of the infinite were answered by Greek philosophers Democritus and Leucippus saying that it is obvious that Achilles would catch the tortoise. Their idea was that you can not subdivide something forever. After a certain amount of time you reach the "indivisibles" which have no distance and Achilles would surpass the tortoise. These ideas of "indivisibles" became known as "infinitesimals" and opened up a whole new area of mathematics much later around the 17th century.

Since the time of Zeno of Elea, Democritus, and Lecippus, questions of these "infinitesimals" went unanswered. Questions of continuity begin to arise and people had no answer to them. "What is an 'instant' of time, or an "atom" of matter, if it is nothing? Is the universe continuous or not? Is a line continuous?...If parts have no size, how can even an infinity of them make a whole?"[Eve97, p.34]. Questions of these sort relate to the Banach-Tarski Paradox in parallel fashion. We will see that indeed an infinite amount of points can make a whole.

Continuity ideas remained unrequited until mathematicians Newton and Leibniz created the calculus in the seventeenth century. The main focus of the calculus was to answer the question of how to find a speed when it kept changing. Every object in motion would have to have a certain speed at a particular time and finding this would require division by zero since it was known that $d = rt$ (distance equals rate times time). In order to find rate r you would have to divide through by time t which would give $\frac{d}{t} = r$ and at an instantaneous time $t = 0$. Calculus was able to answer some questions that were circulating around that time but it still left an uncertainty about the concept of continuity and infinity.

I mention the creation of calculus before set theory because it is what brought the ideas of philosophers and mathematicians way before their time back to the forefront. The calculus that Newton and Leibniz created became known as "infinitesimal calculus" because you had to accept the ideas of the infinities. "Accept continuity and you get infinities; reject infinities and you are left with discontinuity"[Eve97, p.35]. It was then around this era that many mathematicians were getting clever and almost careless in a way. Many theorems were put forth about infinities and continuity and at times many of them came without proofs. Mathematics almost no longer became a science but a philosophical debate, and even though many advances were made towards correcting that, it's philosophical underlaying still exist today.

So where does set theory lie in all of this? Let me introduce Georg Cantor, to whom the origins of set theory almost solely belong. Georg Ferdinand Ludwig Philipp Cantor was born in 1845 in Saint Petersburg, Russia. Cantor spent much of his schooling years focusing on number theory and trigonometry. He completed his dissertation on number theory in 1867 and became a professor of mathematics. Through his study of trigonometric functions Cantor began to become consumed with the ideas of "infinitesimals" and then devoted his work to finding answers concerning continuity and the infinities. "While mathematicians and philosophers before Georg Cantor only look to infinity with mathematical telescopes, treating it as a potential, Cantor consummated, or actualized the infinite, dropping it in our laps to be manipulated and explored" [Wap05, p.7]. I will now introduce the necessities of set theory that will be required for proof of the Banach-Tarski Paradox, most all of which is credited to Georg Cantor.

Chapter 3

Set Theory Basics

A *set* is a collection of objects and each object in the set is called an *element*, or *member* of the set. The basic principal of set theory is whether an element belongs to a set. If an element belongs to a set, we may say that it is contained in that set. Using notation, let A be a set and x be an object. If x is contained in the set we write

$$x \in A,$$

and if x is not an element of A we write

$$x \notin A.$$

If we want to say that A contains precisely the elements x, y, z then we write

$$A = \{x, y, z\}.$$

We can also use *set builder notation* to classify a set and its elements. For example if we wanted A to represent the points along the line $x = y$ than we would write

$$A = \{(x, y) : x = y\}$$

and we could say that $(2, 2) \in A$ and $(1, 2) \notin A$.
A set which contains no elements is known as the *null*, or *empty set* and is written as $\emptyset$.
If two sets have the same elements then they are considered *equal* and equality of sets is denoted by

$$A = B,$$

where A, B are both sets and contain the same elements. And, of course, if A, B are not equal sets than we write

$$A \neq B.$$

We are also allowed to have a subset of a set. A set A is a *subset* of set B if and only if each element of A is also an element of B. We would write that

$$A \subseteq B.$$

If we have two sets then we can take their *intersection* by creating a new set with all the elements that are in both A and B, denoted

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The *union* of sets is when we create a new set with all elements belonging to A or B, denoted

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

As an example let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then

$$A \cap B = \{2, 3\}$$

$$A \cup B = \{1, 2, 3, 4\}.$$

Let's take a set U which A and B are both subsets. U is known as a *universal set*. For example, we could let $U = \{0, 1, 2, 3, 4, 5, 6\}$.

Using the notion of unions we are also allowed to make a partition. A *partition* of a set X is a disjoint collection, C, of non-empty subsets of X whose union is X. Two sets, A and B are *disjoint* if $A \cap B = \emptyset$. As an example of a partition:

$$\text{let } X = \{1, 2, 3, 4, 5\} \text{ then, as an example } C = \{\{1, 2\}, \{3, 4\}, \{5\}\}$$

where we would say that C is a partition of X.

The *complement* of set A is taken to be all elements in U not in A, denoted A^C,

$$A^C = \{0, 4, 5, 6\}.$$

Using A, B as defined above, the *difference of sets*, $A - B$ is the set of all elements in A that are not in B, here

$$A - B = \{1\}.$$

3.1 Cantor's Theorem and Diagonalization Proof

Of the many things that philosophers and mathematicians had pondered for years, as mentioned before, was the concept of continuity. Since the sixth century B.C., the time of Pythagoras, the world had known of the different classes (sets) of numbers. Using todays standard notion:

$\mathbb{N} = \{1, 2, 3, ...\}$ are known as the natural numbers. Zero is only sometimes included.

$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ are known as the integers.

$\mathbb{Q}$ represents all numbers of the form $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$, known as the rational numbers. It is worth noting that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$.

$\mathbb{I}$ represents the irrational numbers which are any numbers that cannot be expressed as $\frac{a}{b}$. Decimal representations never end or repeat. Among the most famous is $\sqrt{2}$.

$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$. The set $\{\mathbb{Q}, \mathbb{I}\}$ is a partition of the real numbers, $\mathbb{R}$.

Richard Dedekind, a mathematician and good friend of Georg Cantor, created the Dedekind cut which filled some holes in the concept of continuity. The Dedekind cut showed that the number line was continuous by splitting it into two separate subsets at a certain point, most likely irrational, and then the union of those subsets would again create the number line. It seems elementary but was a huge advancement at the time.

Richard Dedekind used the idea of a *well-ordered set* when discussing the Dedekind cut. A *well-ordered set* is an ordered set in which every nonempty subset has a first element. Cantor described a well-ordered set in 1883 as a set satisfying the three following conditions:

1. it contains a first element
2. any element with a successor has an immediate successor
3. any finite or infinite set of elements which has a successor has an immediate successor

$\mathbb{N}$ has a well-ordering since each subset of $\mathbb{N}$ would have a least element. Take for example $N = \{2n : n \in \mathbb{N}\}$ which has a least element. The set $\mathbb{R}$ is not a well-ordering set under the usual ordering since each subset would not have a least element. Consider the set $R = \{x : x > 0, x \in \mathbb{R}\}$ which does not have a least element.

Now that questions of continuity were beginning to be answered, Georg Cantor began to explore ideas which opened up the mathematical world into a deeper understanding of the concept of infinity. This began with his notion of the cardinality of a set.

The *cardinality of a set* refers to the size of the respected set. If $A = \{1, 2, 3, 4, 5\}$ then the cardinality of A is 5, denoted $|A| = 5$, since there are 5 elements in A.

The *power set* of any set A is the set of all subsets of A. The power set is denoted as $\mathcal{P}(A)$. Let $A = \{1, 2, 3\}$ then

$$\mathcal{P}(A) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \emptyset\},$$

notice that $|A| = 3$ and $|\mathcal{P}(A)| = 8$. It can be shown that the cardinality of the power set of a finite set A is $2^{|A|}$, or that $|\mathcal{P}(A)| = 2^{|A|}$.

The proof is shown here using mathematical induction. Mathematical induction has been seen used in history since around the time of Plato and Zeno of Elea. Related techniques were used by Francesco Maurolico in 1575 to show that the sum of the first n odd integers is n^2 and by other earlier mathematicians such as Blaise Pascal, Pierre de Fermat, and Jakob Bernoulli who all hold their respective spots in the history of mathematics. The rigorous step-by-step system of mathematical induction wasn't widely used until the early nineteenth century, by mathematicians such as Richard Dedekind. Mathematical induction consists of three steps:

1. Basis step- Show what you are trying to prove is true for a simple case, usually $n = 0$ or $n = 1$

2. Assumption step- Assume what you are trying to prove is true for a general case $n = k$

3. Induction step- Show the statement holds true for $n = k + 1$

Theorem 3.1.1. $|\mathcal{P}(A)| = 2^{|A|}$*, where A is a finite set.*

Proof. Step 1: Let $A = \emptyset$, then $|A| = 0$ and $\mathcal{P}(A) = \{\emptyset\}$ which implies $|\mathcal{P}(A)| = 1$ and so $|\mathcal{P}(A)| = 2^{|A|} = 2^0 = 1$.
Step 2: let $|B| = k$ so $|\mathcal{P}(B)| = 2^k$.
Step 3: Show that if $|A| = k + 1$ then $|\mathcal{P}(A)| = 2^{k+1}$. Let $|(A)| = k + 1$ and $x \in A$.

Take $B = A - x$ so that $|B| = k$. Now if we take $\mathcal{P}(B)$ and adjoin x with every subset of B to create new subsets we really have just doubled $|\mathcal{P}(B)|$. That is to say $|\mathcal{P}(A)| = |\mathcal{P}(B \cup \{x\})| = 2(|\mathcal{P}(B)|) = 2(2^k) = 2^{k+1}$. □

Cantor was able to create a another relationship between the cardinality of a set and it's power set. The relationship is known as Cantor's theorem which states the following.

Theorem 3.1.2 (Cantor's Theorem). *For any set A, $|A| < |\mathcal{P}(A)|$.*

This states that the cardinality of any set is strictly less than the cardinality of its power set.

Proof. It is easy to see that $\mathcal{P}(A)$ has at least as many elements as A since for every $a \in A$ there exists an $\{a\} \in \mathcal{P}(A)$, so this gives us that $|A| \leq |\mathcal{P}(A)|$. Now we must show that $|A| \neq |\mathcal{P}(A)|$.

Build a one-to-one and onto mapping, f, from A to $\mathcal{P}(A)$. For any $a \in A$, $a \in f(A)$ or $a \notin f(A)$. Let $B = \{x : x \in A, x \notin f(x)\}$ so $B \subseteq A$ and $B \in \mathcal{P}(A)$. B is the set of all $x \in A$ that get mapped to subsets of $\mathcal{P}(A)$ which do not contain x. Since f is onto there exists a $b \in A$ such that $f(b) = B$, $B \in \mathcal{P}(A)$.

If $b \in f(b)$ then $b \in B$ and $b \notin f(b)$ which is a contradiction. If $b \notin f(b)$ then $b \in B$ and $b \in f(b)$ which is also a contradiction. This implies that f can not exist and that $|A| \neq |\mathcal{P}(A)|$. Therefore $|A| < |\mathcal{P}(A)|$. □

Cantor was able to easily understand the concept and arithmetic of cardinality pertaining to finite sets but he began to wonder about the cardinality of sets such as $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{R}$. To understand his thinking let's first define what it means for a function to be one-to-one or onto. For a function to be *one-to-one* means that every distinct element of the first set(*range*) gets mapped to a distinct element in the second set(*domain*). As an example the mapping $x \mapsto x$ is one-to-one whereas the mapping $x \mapsto x^2$ is not, since, for example, -1 and 1 both get mapped to 1. A function which is one-to-one is said to be injective.

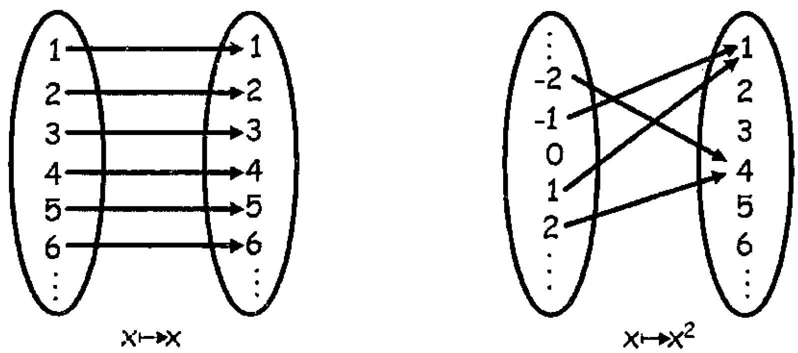

Figure 3.1: Example of injective and non-injective functions

A function is said to be *onto*, or surjective, if every element of the range is mapped to by at least one element of the domain. If we consider the mapping $x \mapsto x^2$ into $\mathbb{N}$ when $x \in \mathbb{Z}$ we do not have an onto mapping, but if $x \in \mathbb{R}$ then we create an onto mapping into $\mathbb{N}$. Notice when $x \in \mathbb{Z}$ nothing will get mapped to 3, but if we take $x \in \mathbb{R}$ then we may consider radicals.

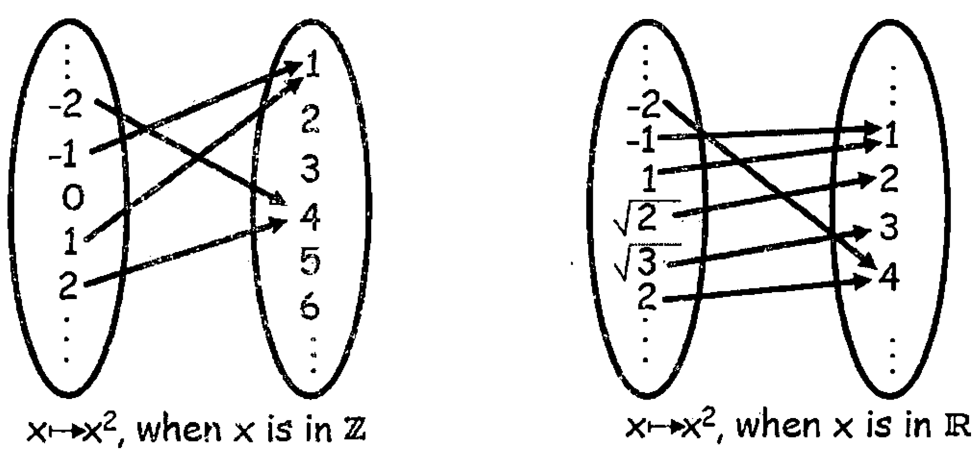

Figure 3.2: Example of surjective and non-surjective functions

If a function is both injective and surjective then it is said to be *bijective.* If a function is bijective then it is equivalent to having a *one-to-one correspondence.* In a bijection every element of the range is paired with exactly one element of the domain, and every element of the domain is paired with exactly one element of the range.

Cantor used these explanation to define his idea of what an infinite set actually was. He defined an infinite set as one that could be put in a one-to-one correspondence with a proper subset of itself. As an example of an infinite set we look at $\mathbb{N}$ and take its proper subset of odd numbers. We may form a bijection as shown:

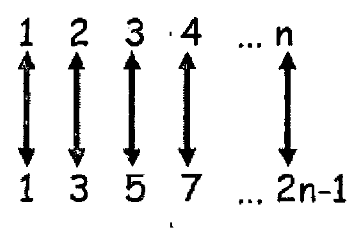

Figure 3.3: Bijection from the natural numbers to the proper subset of odd numbers

Using similar graphical representation techniques Cantor was surprisingly able to show that the natural numbers could be placed in a one-to-one correspondence with the rational numbers, implying they have the same *transfinite* cardinality, that is $|\mathbb{N}| = |\mathbb{Q}|$. This also implies that $\mathbb{Q}$ is a *countable* set. A set, S, is considered *countable* if there exists an injective function from $S \to \mathbb{N}$.

Cantor's visual representation of the proof that $\mathbb{Q}$ is countable is shown below. This represents a injective function from $\mathbb{Q} \to \mathbb{N}$.

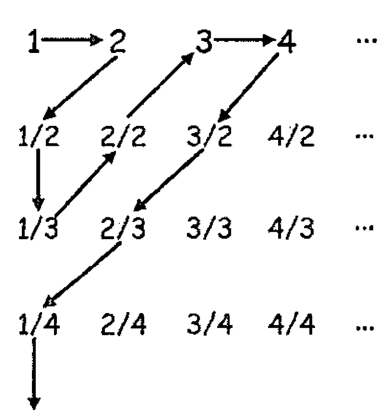

Figure 3.4: Cantor's mapping from the rational numbers to the natural numbers

An example of such an injective function mapping $\mathbb{Q} \to \mathbb{N}$ is

$$f(x) = \begin{cases} 2^m \cdot 3^n, & \text{if } x > 0 \text{ and } \frac{m}{n} \text{ is in simplified form} \\ 1, & \text{if } x = 0 \\ 2^m \cdot 3^n \cdot 5, & \text{if } x < 0 \text{ and } -\frac{m}{n} \text{ is in simplified form} \end{cases}$$

where $x = \frac{m}{n} \in \mathbb{Q}$ and $m, n \in \mathbb{N}$.

If we use Cantor's own definition of infinite sets it is clear to see that both $\mathbb{N}$ and $\mathbb{Q}$ are infinite, but to see they have the same cardinality was surprising. Cantor was amazed himself as he found these results and began to wonder if all infinite sets of numbers would have the same cardinality as $\mathbb{N}$.

What shook the mathematical world was that Cantor was able to show that the cardinality of $\mathbb{R}$ was greater than the cardinality of $\mathbb{N}$. He showed that the infinite set of real numbers could not be put into a one-to-one correspondence with the infinite set of integers implying that $|\mathbb{R}| > |\mathbb{N}|$. This has become one of the more famous proofs of mathematical history, now known as the Cantor's diagonalization proof. Although not his first proof that the real numbers are uncountable, it presented some nice techniques that became useful for many later proofs.

Cantor knew he could show, by definition, that the set of natural numbers, $\mathbb{N}$, is a countable set and he knew that if he could establish a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{R}$ then he could show that the real numbers were also countable. Cantor's diagonalization proof shows that such a injection does not exist, proving that $\mathbb{R}$ is an uncountable set.

Proof. Cantor's Diagonalization proof- First note that we can establish a bijection from $\mathbb{R}$ to the closed unit interval $[0, 1]$. As an example of this bijection consider the function $f(x) = \frac{1}{2} + \frac{1}{\pi} arctan(x)$ which is bijective function from $\mathbb{R}$ to $[0, 1]$. If we eliminate surjectivity we may stretch $(0, 1)$ to the closed interval $[0, 1]$ using the shifting from infinity method discussed in section 6.2. Let $g(x) = x$ which is an injective function from $[0, 1]$ to $\mathbb{R}$. We may now use the Schröder-Bernstein Theorem (a version of this theorem is proved in section 8.1) which says that if an injective function exists from each set into the other then there exists a bijection from one set to the other. This guarantees a bijection between $\mathbb{R}$ and $[0, 1]$. This bijection allows us to show that if a one-to-one correspondence does not exist between $\mathbb{N}$ and $[0, 1]$ then a one-to-one correspondence also does not exist between $\mathbb{N}$ and $\mathbb{R}$. The idea behind the proof is to show that given an infinite sequence of real numbers we can create another real number that is not contained in the sequence, showing that we can not enumerate the real numbers. Considering a sequence contained in $[0,1]$ we may write out the decimal expansion of the numbers in this sequence using non-negative integers d_{ij} as follows:

$$\begin{array}{ccccc} 0. & d_{11} & d_{12} & d_{13} & \cdots \\ 0. & d_{21} & d_{22} & d_{23} & \cdots \\ 0. & d_{31} & d_{32} & d_{33} & \cdots \\ 0. & d_{41} & d_{42} & d_{43} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

If $\mathbb{R}$ was countable this sequence would represent all possible decimal expansions contained in $[0,1]$. Now consider a new decimal expansion, let's say e, such that $e = 0.d_1d_2d_3\ldots$ and for each $n = 1, 2, \ldots$, $d_n \neq d_{nn}$ which would imply that e is not represented in the sequence of expansions above, showing that it is impossible to list all decimal expansions between $[0,1]$. This proves that $\mathbb{R}$ can not be placed in a one-to-one correspondence with $\mathbb{N}$, showing that the real numbers are uncountable. □

What this proof had accomplished was showing that in actuality there are different cardinalities of infinities. Cantor used the notation $\aleph_0$ to express the size of the first cardinality of infinity, basically $|\mathbb{N}| = \aleph_0$. Cantor expressed that the cardinality of the continuum is $|\mathbb{R}| = \mathfrak{c}$. From the proof above we know that $\aleph_0 < \mathfrak{c}$. Cantor proposed that there exist an infinite amount of transfinite cardinalities. His continuum hypothesis however states that perhaps there is no infinite cardinality between $\aleph_0$ and $\mathfrak{c}$, such as to say that $\aleph_1 = \mathfrak{c}$. Cantor was able to show that we could arrange an inequality such as $\aleph_0 < \aleph_1 < \aleph_2 < \ldots < \aleph_n < \aleph_{n+1} < \ldots$ with an infinite sequence of transfinite cardinal numbers. He simply stated that "given a finite or infinite set, it was always possible to show the existence of a greater set, in particular the set of all its subsets" [Dan92, p.8].

3.1.1 Transfinite Arithmetic

Ideas of infinite sets and infinite cardinal numbers were shunned because of their paradoxical results which will be discussed later in this paper. A popular argument against these theories was that infinite sets simply do not exist in nature. Even if you would consider the set of all the atoms that make up nature, you still would not have a infinite set. Cantor agreed with this notion, but his argument was that they do not appear in nature because they do not act like they belong in nature. With finite sets and finite cardinal numbers, basic arithmetic rules can be applied. Cantor defined his concept of a finite cardinal number and the arithmetic associated upon it.

In his published papers Cantor describes a single element of a set as e_0 and names the set that contains e_0 as E_0 so that $E_0 = \{e_0\}$. Then by previous definition of cardinality we get that

$$|E_0| = 1.$$

If we are able to obtain a single element, then we may create another element, e_1 and take the union of E_0 and $\{e_1\}$ we get

$$E_1 = E_0 \cup \{e_1\} = \{e_0, e_1\},$$

so that

$$|E_1| = 2.$$

Continuing on in this process we would arrive at

$$|E_2| = 3, |E_3| = 4, |E_4| = 5, \ldots.$$

Cantor then claimed that we may begin to understand finite cardinal arithmetic if we see that

$$|E_{v-1}| = v,$$

where

$$E_v = E_{v-1} \cup \{e_v\} = \{e_0, e_1, e_2, \ldots, e_v\}.$$

Through this Cantor was able to show that cardinal numbers follow the arithmetic rules of the natural numbers, that is to say that "every cardinal number, except 1, is the sum of the immediately preceding cardinal number and 1"[Dan92, p.5]. Cantor showed that

$$|E_v| = |E_{v-1}| + 1.$$

Cantor then proceeded to define his ideas on transfinite arithmetic. Recall from earlier that $\aleph_0$ represents the first of the transfinite cardinal numbers. He continued as he did with the finite cardinal numbers. If we begin with a set of cardinality $\aleph_0$, such as $\mathbb{N}$ and add a single element e_0 we arrive at

$$\mathbb{N} \cup \{e_0\} \text{ with cardinality } \aleph_0 + 1.$$

After some thought you may realize that we find a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{N} \cup \{e_0\}$ implying that in terms of cardinality

$$\aleph_0 = \aleph_0 + 1.$$

You may recall that Cantor defined an infinite set as one that could be put in a one-to-one correspondence with a proper subset of itself. Transfinite arithmetic confirmed in Cantor's mind that $\aleph_0$ was truly infinite since

$$\aleph_0 = \aleph_0 + \aleph_0.$$

Since this was true we may notice that $\aleph_0 + \aleph_0 = 2(\aleph_0)$ and so we may say that if we take any arbitrary natural number, v, as Cantor had used we get

$$\aleph_0 = v\,(\aleph_0)\,.$$

"How do we get from $\aleph_0$ to $\aleph_1$ by arithmetic operations? We know by now that the most elementary steps, involving sums and products, just lead from $\aleph_0$ back to $\aleph_0$ again. The simplest thing we know to do starts with $\aleph_0$ and ends up with something larger is to form $2^{\aleph_0}$."[Hal74] The continuum hypothesis, through using exponents of transfinite arithmetic, states that $\aleph_1 = 2^{\aleph_0}$.

Chapter 4

Early Set Theoretic Paradoxes

4.1 Burali-Forti and Cantor's Paradox

As I am building the background and necessary skills that will be applied in proof of the Banach-Tarski paradox, I would like to discuss the earliest of paradoxes discovered which were centered around Cantorian set theory. As Cantor began to publish his findings of set theory and transfinite arithmetic, all of the critics began to voice their opinions. Critics felt that Cantor was just trying to disrupt everything they had known about traditional mathematics. The first published paradox was in 1897 by Cesare Burali-Forti who was an Italian mathematician that lived from 1861-1931. First let me give the definition of paradox as it is written in Merriam-Webster.

Definition 4.1.1. *Paradox-*

1. *a statement that is seemingly contradictory or opposed to common sense and yet is perhaps true*
2. *a self-contradictory statement that at first seems true*
3. *an argument that apparently derives self-contradictory conclusions by valid deduction from acceptable premises*

In order to understand the paradox that Burali-Forti had found in Georg Cantor's published set theory we must understand what Cantor described to be ordinal numbers.

An *ordinal number* references the order of a well-ordered set, it is a measure of rank or position. In set theory, the ordinal number 0 is represented as $\emptyset$, the ordinal number 1 is represented as $\{\emptyset\}$, the ordinal number 2 is represented as $\{\emptyset, \{\emptyset\}\}$, etc. In general the ordinal number n can be represented as the set of ordinal numbers $\{0, 1, 2, 3, \ldots, n-1\}$.

The *Burali-Forti Paradox* says that if we take the set of all ordinal numbers would be an ordinal number greater than any number in the set. This implies that the set of all ordinal numbers does not form a set, which is contradictory.

Although the critics were trying hard to find any inconsistencies in the Cantorian set theory which Burali-Forti had done, it is speculated in history that Burali-Forti most likely was not a critic of Cantor, in fact he is noted "as an admirer and follower of Cantor"[Dan92, p.21].

Around the time that Burali-Forti had discovered an error in Cantor's set theory, or simply an error in defining the true definition of ordinal numbers, Cantor had discovered his own. *Cantor's Paradox* considers the sets of all sets, S. Every subset would also be a member of S so $\mathcal{P}(S)$ would be a subset of S such that $\mathcal{P}(S) \subset S$. If this was actually true then $|\mathcal{P}(A)| \leq |S|$ which contradicts Cantor's Theorem stated previously in section 3.1.

4.2 Russell's Paradox

Although there were a few before him, Bertrand Russell almost singlehandedly is credited with introducing set-theoretic paradoxes into the world of mathematics. Bertrand Russell (1872-1970) was credited as not only being a mathematician but also a philosopher, logician, and historian. Bertrand Russell was an agnostic and atheist, whereas Georg Cantor thought of himself at times to be a messenger of God. They obviously did not share religious views. Russell also did not generally accept some of the views of set theory that Cantor had proposed. Russell, frustrated with inconceivable notions and unclear philosophical validity, set out to write a book to settle the debate of mathematical foundations. The title of his book was "The Principles of Mathematics", which in itself would deserve the attention of a thesis paper. The paper in itself defines and shapes mathematical logic. Earlier versions were published, however the complete version was not published until 1903. In an autobiography Bertrand Russell had written the following on the New Year's Eve of 1900 talking about his origination of mathematical logic.

> Thank goodness a new age will begin....In October I invented a new subject, which turned out to be all mathematics for the first time treated in its essence. Since then I have written 200,000 words, and I think they are all better than any I had written before.[Eve97]

Although Bertrand Russell had proposed completion of the book in 1901, he stumbled upon an inconsistency that forced him to reinvent his own foundations.

As previously mentioned, Russell did not accept many of Cantor's views. It was almost as if he spent much of his time trying to disprove ideas that Cantor had suggested. As an example, Cantor had shown that aside from infinity there is no greatest number. Bertrand Russell took the view that the number of all the things in the world ought to be the greatest number possible, that being the set of all atoms in nature.

Cantor had also considered that if you take the sequence of transfinite cardinal numbers

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_v, \ldots$$

in its totality, you may then consider another sequence of even larger transfinite cardinal numbers. Russell thought of this to be absurd. "There was no doubt that Cantor's methods were ingenious, but Russell questioned their philosophical validity. If the succession of natural numbers

$$1, 2, 3, \ldots, n, \ldots$$

was unlimited, how then is it possible for someone to talk of the first number following all these numbers?. . .if it was said that the natural numbers have no limit, then it could not be possible to demand, in the next step, that one had a number larger than any natural number."[Dan92, p.62-63] As Russell continued to critique and analyze Cantor's work he discovered that infinity will forever remain contradictory. Russell began to consider sets of sets and more specifically sets being members of themselves. "But at some point in May (1901), Russell asked himself not about sets that include other sets as members but about sets that include themselves as members-and sets that specifically exclude themselves as members"[Eve97, p.180]. This has become famously known as Russell's paradox.

Russell's Paradox. *Let S be the set of all sets which do not contain themselves as members, that is $S = \{X : X \notin X\}$*

We have to ask ourselves, is S a member of itself? If it is, then by construction, S is not a member of itself. If S is not a member of itself, then it is a member of itself. Which creates a clear contradiction.

Russell had discovered this paradox in 1901 but had chosen to keep it quiet for almost another year as he continued work on his book, hoping that any errors were a result of his own faults. Out of what seemed like frustration, Russell proposed a solution to these type of contradicting sets calling it the *theory of types*. The theory of types was based on a logical hierarchy. He stated that his paradox was a proposition about a proposition which could neither be true nor false but meaningless He denied that these types of sets simply cannot exist.

Russell's paradox is most commonly explained in layman's terms using the Barber Paradox which Russell introduced in 1918 as it was suggested to him by someone else.

> If the town barber shaves the men, and only those men, who do not shave themselves, then who shaves the barber?

You must note that the contradiction arises since if the barber shaves himself, then he must not shave himself since the barber shaves only those men that do not shave themselves. If the barber does not shave himself then he must go to the barber which implies he shaves himself.

Chapter 5

Zermelo-Fränkel Set Theory

As paradoxes began to surface at the beginning of the 19th century, Ernst Zermelo(1871-1953), a German mathematician, began to formalize set theory to save set theory from its critics. Although Zermelo published his first axioms in 1908, it wasn't until 1922 that logician Abraham Fränkel(1891-1965) contributed to his axioms and together created the Zermelo-Fränkel set theory that is used today. Zermelo-Fränkel set theory most commonly consists of the following nine axioms.

1. The Axiom of Existence.
2. The Axiom of Extension.
3. The Axiom Schema of Specification.
4. The Axiom of Pairing.
5. The Axiom of Unions.
6. The Axiom of Powers.
7. The Axiom of Infinity.
8. The Axiom Schema of Replacement.
9. The Axiom of Choice.

Let's briefly state each axiom.

The Axiom of Existence simply says that there exists a set which has no elements, namely $\emptyset$.

The Axiom of Extension says that two sets are equal if and only if they have the same elements.

The Axiom Schema of Specification allows us to choose members of a set so as to create a subset of that set.

The Axiom of Pairing says that for any elements a and b there exists a set {a,b} that contains exactly a and b.

The Axiom of Unions states that for every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.

The Axiom of Powers says that for each set there exists a set that contains among its elements all the subsets of the given set. This was discussed earlier as the power set, $\mathcal{P}$.

The Axiom of Infinity as written by Paul Halmos says that there exists a set containing 0 and containing the successor of each of its elements. Basically, simply accepting the fact that infinite sets exist.

The Axiom Schema of Replacement says that if we have a set A along with a mapping, f, then $f(A)$ is also a set. I'm inclined to compare this to a previous mapping used earlier. Define $A = \{1, 2, 3\}$ which is clearly a set and the mapping $f : x \mapsto x^2$ we obtain our new set $A' = \{1, 4, 9\}$ which would also be a set according to the axiom schema of replacement.

The Axiom of Choice says that for any collection C of nonempty sets, we are allowed to choose an element from each set in that collection.

What Zermelo-Fränkel had essentially accomplished through these axioms was an answer to Russell's theory of types. Zermelo-Fränkel set theory does not allow a set such as Russell's Paradox to be created, avoiding any discrepancies. Although Russell had suggested a solution, the axioms provided a logical axiomatic road map.

5.1 Axiom of Choice

Although the Axiom of Choice may most deservingly be awarded it's own section I have it listed here as a sub-section so as to not stray from my overall purpose in building up to the Banach-Tarski Paradox.

Zermelo-Fränkel set theory was mostly widely accepted and used by mathematicians and logicians. I say mostly because there continues to be debates on the true meanings of the axiom of specification, and the axiom of replacement, but that as it may be they are still generally accepted. The axiom of choice, however, is not so readily accepted for use. Some consider it such an unacceptable axiom to mathematics and they solely use only the first eight axioms. When mathematicians are using only the first eight axioms it is referred to as the ZF axiomatic system. For the believers of the axiom of choice they will use all nine axioms referred to as the ZFC axiomatic system. The C, hopefully, obviously representing use of the axiom of choice.

The axiom of choice allows us to do exactly that, choose. Let me again define the axiom of choice.

Definition 5.1.1. *Axiom of Choice- For any set C of nonempty sets, we can choose a member from each set in that collection. In other words, there exists a choice function f defined on C such that for each set $S \in C$, $f(S) \in S$.*

For finite sets, this does not seem unrealistic. As an example, if we consider several subsets of $\mathbb{N}$ we may create a choice function, f, to pick the smallest member of each subset. If we take finitely many closed intervals of the real number line we may consider a choice function, f, to be the midpoint of each interval. The controversy arises when we consider arbitrary sets. We no longer may take the smallest member or the midpoint of each set as these points might not even exist. I give an example of such a set in section 6.3. When we choose to use the axiom of choice we are simply believing that some choice function exists.

Bertrand Russell would of course not let his opinion about the axiom of choice go without being heard. He thought of the axiom of choice to be slightly absurd. He compared it to having infinite sets of pairs of shoes and socks and said this, " To choose one sock from each of infinetly many pairs of socks requires the Axiom of Choice, but for shoes the Axiom is not needed". What this meant was that we can construct a clear choice function for the shoes, such as let f be the set of all right shoes. However for the infinetly many pairs of socks we just have to believe that a choice function would exist to allow us to get a sock from each pair since there really is no right or left.

Mathematician Kurt Gödel (1906-1978) showed that the Axiom of Choice could not be disproved since it is consistent with the axioms of the ZF system. This was published in 1940 in the paper titled *The consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis with the Axioms of Set Theory.* Six years before this paper was published Paul Cohen(1934-2007) was born and would eventually show in 1963 that the Axiom of Choice could not be derived from the ZF system of set theory claiming its independence. What Gödel and Cohen had collaboratively shown was that the Axiom of Choice could neither be proved nor disproved, in turn, proving that it's use still remains a choice.

Certain mathematicians choose to believe the Axiom of Choice simply to make their lives simpler. The Axiom of Choice can lead to some very interesting results like the Vitali set which will be shown in the next chapter and the Banach-Tarski paradox which is proven in chapter 8.

Chapter 6

Preliminaries

6.1 Isometries and Matrix Rotations

Now that we have a good background of set theory needed to help understand the Banach-Tarski paradox I would like to write about some needed information to further understand it's meaning all while raising interest towards the Banach-Tarski Paradox.

An *isometry* is a distance preserving transformation. This means that if we take a shape, or line segment and transform it using a reflection, glide reflection, rotation, or translation, the distance between each set of points is unchanged. This is true since reflections, glide reflections, rotations, and translations are isometries. The proof of the Banach-Tarski Paradox uses rotations and translations.

A *translation* is simply shifting an image to a new location. If we are working in the Cartesian coordinate system, a horizontal shift of a units can be represented as $(x, y) \mapsto (x + a, y)$ and a vertical shift of b units can be represented as $(x, y) \mapsto (x, y + b)$.

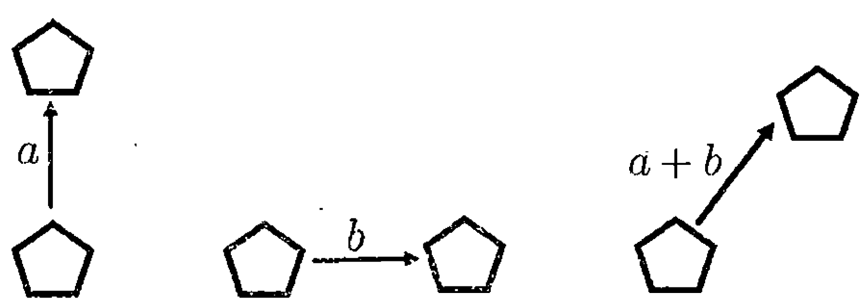

Figure 6.1: Translations

A *rotation* is a transformation in which a figure is rotated θ degrees around a fixed point, P, or line, l, in three dimensional space.

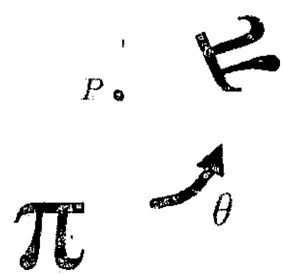

Figure 6.2: An example of a rotation

Rotations are represented using matrices. Two dimensional rotations about the origin are represented by

$$R(\theta) = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}.$$

A three dimensional rotation around the z-axis is represented by

$$R_z = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The proof of the Banach-Tarski Paradox requires two rotations in three-dimensional space.

The first rotation, τ, is a 120° rotation about the z axis represented by

$$\tau = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and σ which represents a 180° rotation about the line $z = x$ in the xz plane. To obtain this rotation a few steps are required.

Using a transformation rotation matrix, M^{-1}, I will be able to take the line $z = x$ into the z axis, perform my needed 180° rotation, R, then use M to take it back to the line $z = x$. Let

$$M = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{-\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \text{ and } R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so that

$$\begin{aligned} \sigma &= MRM^{-1} \\ &= M \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{-\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \\ &= M \begin{bmatrix} \frac{-\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{-\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{-\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{aligned}$$

6.2 Equidecomposable, Lebesgue Measure, and "Hilbert's Hotel"

Is it possible to take a square, of side lengths 1 unit, and reassemble it to create an isosceles right triangle?

Originally upon first hearing this, I was quick to say yes. I could clearly see that you could just simply cut the square along its diagonal and rearrange the two pieces to create the wanted isosceles right triangle. It would be isosceles since the length of the diagonal would be used to create two sides of the triangle and it would be a right triangle since the diagonals of a square also bisect each angle creating two 45° angles which would give us our 90° angle. Here is an illustration of the proposed solution.

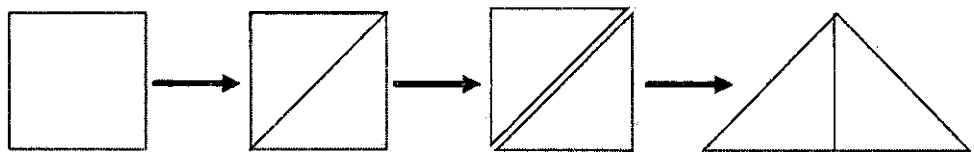

Figure 6.3: Creating an isosceles right triangle from a square

Seems simple enough. The square and the isosceles right triangle are congruent by what can be called scissors congruence, or scissors dissection, meaning the construction can be done using paper and scissors. Actually a theorem exists for this type of problem.

Theorem 6.2.1 (Wallace-Bolyai-Gerwien Theorem). *Two polygons are congruent by dissection if and only if they have the same area.*

Essentially the Wallace-Bolyai-Gerwien Theorem would allow us to perform this type of rearranging pieces process to any two shapes of equal area. So our question is solved and we may move on to the next. Well, not quite so fast. If we begin to think of these figures as a set of points something interesting happens.

If we start to think about the mathematical point-set model of what is being asked, some obstacles arise. What happens is when we cut along the diagonal the points associated with that line can only go to one of the hypotenuses of our two right triangles being used to create the larger isosceles right triangle.

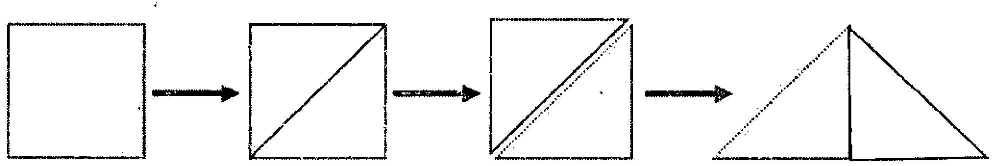

Figure 6.4: Square to incomplete isosceles right triangle

In actuality we have not achieved our goal, you can see we are still missing the line segment representing on of our legs of our isosceles right triangle. To advance further we need to state the definition of equidecomposable.

Definition 6.2.1 (Equidecomposable). *Sets A, B are* equidecomposable *if A and B can be partitioned into the same finite number of congruent pieces. Denoted as $A \sim B$.*

In 1925 Alfred Tarski (co-founder of the Banach-Tarski Paradox) asked if it was possible to rearrange the interior of a circle to form a square of equal area, known as Tarski's circle-squaring problem. This has yet to be done through construction with paper and scissors but has been proven possible through the use of the axiom of choice and equidecomposability. Mathematician Miklós Laczkovich proved this 65 years later around 1990. It was shown that the amount of finite congruent pieces would be around 10^{50}, which is a very large amount of pieces, although still finite. This solution requires the use of non-Lebesgue measure, which I will now define.

Lebesgue measurement can be thought of as our basic notion of length, area, or volume. As an example the interval $[0, 1]$ would have a Lebesgue measure of 1.

Definition 6.2.2. *Lebesgue measure, which I will denote as μ, satisfies the following conditions:*

1. *If $A = (a, b)$ or $[a, b]$ then $\mu(A) = |b - a|$.*
2. *If $A \subset \mathbb{R}$ is a bounded subset of $\mathbb{R}$, $c \in \mathbb{R}$ and $\mu(A)$ exists, then $\mu(A + c) = \mu(A)$ where $(A + c)$ represents a translation. Translations preserve Lebesgue measure.*
3. *If A, B are disjoint subsets of $\mathbb{R}$ and both $\mu(A)$,$\mu(B)$ exist, then $\mu(A \cup B) = \mu(A) + \mu(B)$ and under similar hypotheses $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.*

Tarski's circle-squaring problem is unsolvable using Lebesgue measure, as would our square to isosceles right triangle problem. Their solutions require the use of point sets with non-Lebesgue measure, as will the proof of the Banach-Tarski Paradox.

Let's go back to the square to isosceles right triangle problem using this idea of non-Lebesgue measure. We are close to having it complete minus the left leg of our isosceles right triangle. Our goal in completing this segment is to keep track of every single point we have available. Remember that when we cut down the diagonal we created two segments of length of $\sqrt{2}$ which our goal now is to fill one of those. When we place the two triangles together to form the one we get an overlapping of line segments of length 1 down the middle which we could take one of them and rotate it to fill our missing segment so we would still only be missing a length of $\sqrt{2} - 1$.

In order to complete this we are going to use a process which Wapner [Wap05] refers to as *shifting to/from infinity.* The basic idea is summed up in the story of "Hilbert's

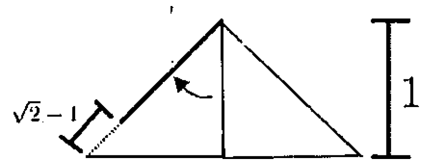

Figure 6.5: Nearly complete isosceles right triangle

Hotel" proposed by mathematician David Hilbert. The story goes as is, there is a hotel with infinetly many rooms which is fully occupied and a guest arrives looking for a place to stay. To make room for this guest what we could do is shift each occupant up one room, so $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 4$, ... freeing up room # 1 for the guest.

We also could create a way to accommodate an infinite amount of new guests by shifting each current guest to a new room using the mapping $n \mapsto 2n$. This would open up all odd numbered rooms which would create an infinite number of vacant rooms for the infinite amount of new guests. Using Cantor's transfinite arithmetic notion, the guests are being absorbed in the cardinality of $\aleph_0$. Applying this idea to a geometrical figure we could fill the circumference of a circle with one point missing. Picking any arbitrary point on the circumference, denote as 0, we could create an infinite set, A, of equally spaced points on the circumference. $A = \{0, 1, 2, 3, 4, ...\}$ where no two points would be the same. This is possible since the circumference of the circle is 2π which is irrational.

Figure 6.6: Shifting from infinity

Now by shifting from infinity we could fill the point at 0 by sending ..., $3 \mapsto 2$, $2 \mapsto 1$, $1 \mapsto 0$. This could also be done to fill or create segments from the circle which have a distance equal to or less than the radius as shown above on the right hand side.

Now to finish our square to isosceles right triangle problem we know that we are going to be missing a distance of $\sqrt{2} - 1$. To fix this, we can remove a line segment of our needed distance from our original square, but then we will need to fill the gap that we create. To fill the gap we use the "Hilbert's Hotel" concept. and shift from infintiy. We inscribe a circle of radius .5, notice $.5 > \sqrt{2} - 1$, which will allow us to fill the gap as discussed in the previous paragraph, and shown on the right hand side of figure 6.6. Here I show the segment being removed from our original square and being translated to complete our isosceles right triangle.

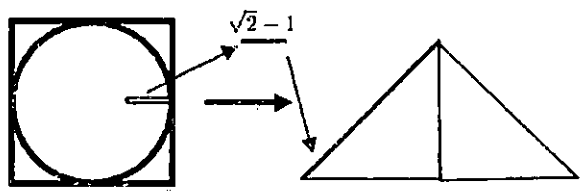

Figure 6.7: Complete isosceles right triangle

6.3 The Vitali Set

The first requirement of Lebesgue measure says that for a bounded interval $[a, b]$ its Lebesgue measure would be $|b - a|$. Seems reasonable enough, but let's ask ourselves, must every bounded set be Lebesgue measurable? In 1905, Giuseppe Vitali (1875-1932) created a subset of the real numbers which is not Lebesgue measurable, creatively known as the Vitali set. The Vitali set gears us up for the Banach-Tarski proof by letting us see that we may have sets of points which have no Lebesgue measure, or point sets which have no volume as in the case of the Banach-Tarski paradox.

Let's walk through the construction of a Vitali set and then show why this set has no measure. For $a, b \in [0, 1]$ call a, b equivalent if $a - b$ is a rational number. Partition $[0, 1]$ into uncountably many equivalence classes where each class contains a countable number of elements. These classes are constructed so that if you choose two elements from the same class they will differ by a rational number and if you choose two elements from different classes they will differ by an irrational number. The Vitali set, V, is the set constructed from the Axiom of Choice which allows us to choose exactly one element from each of the equivalence classes.

To show V has no measure let $V_q = V + q = \{x + q : x \in V\}$ where $q \in \mathbb{Q}$. By construction $\bigcup V_q = \mathbb{R}$. This shows that the set of real numbers partitions into a countable collection of disjoint sets.

Theorem 6.3.1. *The Vitali Set is non-measurable*

Proof. Assume V is measurable. If V is measurable then $\mu(V_q) = \mu(V)$ where μ represents Lebesgue measure as previously stated. Now, if $\mu(V) = 0$ then $\mu(\mathbb{R}) = 0$ since $\bigcup V_q = \mathbb{R}$ which is impossible. But if $\mu(V) > 0$ then

$$\mu([0,2]) \geq \mu(\bigcup\{V_q : q \in \mathbb{Q} \text{ and } 0 \leq q \leq 1\}) = \infty$$

which is also impossible since $\mu([0,2]) = 2 \ngeq \infty$. This contradicts our assumption that V is measurable, showing that the Vitali set, V, is non-measurable. □

Using the Vitali set we can produce a result that closely resembles the Banach-Tarski paradox. Imagine if we take the interval $[a, b]$ and wrap it around so that point a is joined to point b. We have then constructed a circle.

Constructing a circle in this fashion from $[0, 1]$ we can form the Vitali set on this circle and reassemble it to form two complete circles from the one. Using the argument to construct the classes of the Vitali set we may construct classes of points saying two points are equivalent if one can be obtained through the other by a rational multiple of a rotation. We create uncountably many classes, each with a countable number of elements. Since the rationals between zero and one form a countable set we can denote the rotations as $\vartheta_1, \vartheta_2, \vartheta_3, \ldots$.

We then may use the axiom of choice to select one point from each of these equivalence classes to form a set, C, where $C_i = \vartheta_i$ for $i = 1, 2, 3, \ldots$. The complete circle is represented as $\cup\{C_i : i = 1, 2, 3, \ldots\}$. Since all C_i are congruent, we may use a mapping to form the complete circle from a subset of $\cup\{C_i : i = 1, 2, 3, \ldots\}$, such as $C_{2n} \mapsto C_n$.

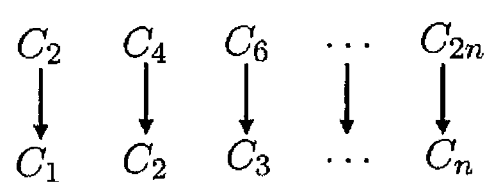

Figure 6.8: Example of a mapping

This shows that we are able to partition a circle into two sets using rotations, which are isometric and equidecomposable to our original circle.

I have shown that we are able to create the two circles from the one but we could actually create any amount of circles from the one using appropriate mappings. For example, if we wanted to create four complete circles we could use the mappings: $C_{4n-1} \mapsto C_n$, $C_{4n-2} \mapsto C_n$, $C_{4n-3} \mapsto C_n$, and $C_{4n} \mapsto C_n$. We will use a similar argument in the proof of the Banach-Tarski paradox.

6.4 Hausdorff Paradox

As I have mentioned before the proof of the Banach-Tarski Paradox is fascinating, and although not mathematically difficult to comprehend, it does need some additional theorems and definitions to arrive at our conclusion. I will list those here.

In 1914 a mathematician named Felix Hausdorff published what is called the Hausdorff Paradox. The Hausdorff Paradox is the main ingredient of the Banach-Tarski Paradox and in it's proof (which will be embedded into the proof of the Banach-Tarski Paradox) the Axiom of Choice is required. I will state it formally here, and an explanation of the terms will follow.

Theorem 6.4.1. *Hausdorff Paradox- There is a countable subset D of S^2 such that S^2/D is SO_3-paradoxical.*

D represents the collection of points which remained fixed on the sphere, S^2, throughout each rotation. This will be more clear in presentation of the Banach-Tarski proof. S^2/D is equivalent to $S^2 - D$. SO_3 refers to the group of rotations in $\mathbb{R}^3$. By saying that S^2/D is SO_3-paradoxical we mean that we can partition S^2/D into finitely many subsets and using SO_3, map each subset back onto the entire S^2/D. That is to say we can take a disjoint decomposition of $S^2 \in \mathbb{R}^3$ into four subsets A, B, C, D such that $A \cong B \cong C \cong B \cup C$ ($\cong$ represents congruent) and D is countable. D may be infinite, but must be countable. If you look back at the relation $A \cong B \cong C \cong B \cup C$ it should seem odd that any of of the sets A, B, C is congruent to the set $B \cup C$. This implies that each A, B, or C is $\frac{1}{3}$ of S^2 and since and A, B, and C are congruent to $B \cup C$ then each A, B, or C is also $\frac{1}{2}$ of S^2. Hence, the paradox. Since $A \cong B \cup C$, $B \cong B \cup C$, and $C \cong B \cup C$ we may map each subset back onto the entire S^2/D as will be explained in the Banch-Tarski proof in the next chapter.

6.5 Elementary Group Theory

When you apply a binary operation to a set you can create a group. A *group* is any set, H, along with a group operation $(*)$ that satisfies four requirements. A group must have:

1. closure- For any two elements, $a, b \in H$, $a * b \in H$.

2. associativity- For elements $a, b, c \in H$, $(a * b) * c = a * (b * c)$..

3. an identity element, $e \in H$, such that for any element $a \in H$, $a * e = e * a = a$ holds true.

4. for any element $a \in H$ there is an inverse element, $n \in H$ such that , $n * a = a * n = e$.

As an example, $\mathbb{Z}$ is a group under the operation of addition. It is easy to show that it satisfies the requirements of a group.

1. Closure: Addition of any two integers is also an integer.

2. Associativity: $(a + b) + c = a + (b + c)$.

3. Identity: The identity is 0. For any element $a \in \mathbb{Z}$, $a + 0 = 0 + a = a$.

4. Inverse: The inverse element of $a \in \mathbb{Z}$ is $-a$. $a + -a = -a + a = 0$.

In the proof of Banach-Tarski paradox we establish a group of rotations created by τ and σ of section 6.1.

Chapter 7

The Banach-Tarski Paradox

Theorem 7.0.1. *A solid ball may be separated into a finite number of pieces and re-assembled in such a way as to create two solid balls, each identical in shape and volume to the original.*

The proof follows in the next few pages.

Proof. Without loss of generality we may focus on a ball of radius one centered at the origin of the rectangular coordinate system, $(0,0,0)$. We may rewrite the statement of the theorem as *the unit ball* $B = \{(x,y,z) : x^2 + y^2 + z^2 \leq 1\}$ *can be partitioned into two sets* B_1 *and* B_2 *such that* $B \sim B_1$ *and* $B \sim B_2$*.(here* $\sim$ *implies equidecomposability as defined in section 6.2.)*

Let τ and σ denote rotations of B. Define τ to be a clockwise rotation of $120°$ about the z axis and σ be a clockwise rotation of $180°$ about the line $z = x$.

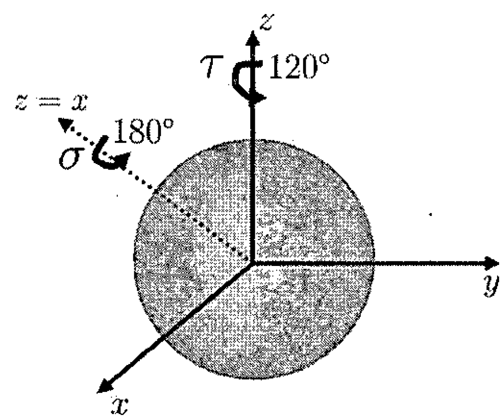

Figure 7.1: τ and σ rotations

The rotations can be represented in matrix form by

$$\tau = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \sigma = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The identity matrix, I, is represented as

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that $\tau^3 = \sigma^2 = I$

$$\begin{aligned}
\tau^3 &= \tau\tau^2 \\
&= \tau \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \tau \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= I
\end{aligned}$$

$$\begin{aligned}
\sigma^2 &= \sigma\sigma \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= I
\end{aligned}$$

Using τ and σ we can establish a group, G, which consists of all the countably infinite number of rotations created by τ and σ. Since $\tau^3 = \sigma^2 = I$, we may be able to reduce some combinations. Let's make up an example. σ^5 can be reduced since

$$\sigma^5 = \sigma^2\sigma^2\sigma = II\sigma = \sigma$$

or $\tau^5\sigma^2\tau$ can similarly be reduced

$$\tau^5\sigma^2\tau = \tau^3\tau^2\sigma^2\tau = I\tau^2 I\tau = \tau^2\tau = \tau^3 = I.$$

As each rotation combination is being made they are representing rotations of different lengths. Length is the number of symbols needed in reduced form. For reference, the identity rotation, I, would have a length of 0. τ, σ, and τ^2 have a length of 1. $\tau\sigma, \tau^2\sigma, \sigma\tau$, and $\sigma\tau^2$ all have length 2. Lengths must be given in reduced form. These specific rotations are chosen in a way such that every possible combination of τ and σ has a unique, reduced form and represent different rotations as shown in section 7.2.1 below.

Using the lengths of rotations generated by τ and σ we partition G into three subsets, G_1, G_2, G_3. Every rotation is assigned to a subset by a specific rule. The identity matrix, I, is an element of G_1. The possible rotations are appended to I resulting in σ,τ, and τ^2. σ and τ are sent to G_2 and τ^2 is sent to G_3. The possible rotations are again appended to these rotations creating new rotations which will be assigned to a subset. The rotations must be in reduced form to be appropriately assigned. For example, when we attach σ to σ, a new rotation is not created since $\sigma^2 = I$ and I has already been assigned to G_1. The assigning process is shown in Table 8.1 below. A rotation is represented by υ.

	If $\upsilon \in G_1$	If $\upsilon \in G_2$	If $\upsilon \in G_3$
If leftmost character of υ is τ or τ^2 assign ...	$\sigma\upsilon$ to G_2	$\sigma\upsilon$ to G_1	$\sigma\upsilon$ to G_1
If leftmost character of υ is σ assign ...	$\tau\upsilon$ to G_2	$\tau\upsilon$ to G_3	$\tau\upsilon$ to G_1
	$\tau^2\upsilon$ to G_3	$\tau^2\upsilon$ to G_1	$\tau^2\upsilon$ to G_2

Table 7.1: Creating Subsets of G

G_2 is the set of all rotations in G_1 which get a rotation of τ added to the front end and rotations in G_3 which get a τ^2 attached. G_3 is the set of all rotations in G_1 which get a τ^2 rotation added and rotations in G_2 which get a τ attached to the front end. G_1 is the set of all rotations in G_2 which get a τ^2 attached and G_3 which get a τ attached as well as those rotations in G_2 and G_3 which get a σ attached to the front end. As an example, let's look at the reduced rotation of $\sigma\tau^2\sigma$ which is an element of G_2. Attaching

σ, τ, and τ^2 to the left end we create the rotations:

$$\sigma\sigma\tau^2\sigma = \tau^2\sigma \in G_1$$
$$\tau\sigma\tau^2\sigma \in G_3$$
$$\tau^2\sigma\tau^2\sigma \in G_1$$

These basic relations exist amongst the subsets:

$$\tau G_1 = G_2 \quad (7.1)$$
$$\tau^2 G_1 = G_3 \quad (7.2)$$
$$\sigma G_1 = G_2 \cup G_3 \quad (7.3)$$

The proof of equation (8.1) is shown in section 7.2 below.

Now we may focus on the surface of our ball, S, upon which we will be able to construct two copies of S from the original.

Keeping our focus on just the surface we can see that through every rotation we have two points, which remained fixed at the end of each corresponding axis (pole). Each rotation in G will have two more points such as these that remained fixed. I am going to put all of these points into a set denoted as P. P is countable since G is countable. This creates two set of points, P and $S - P$. Taking any point in $S - P$ we may connect it to another point in $S - P$ through a rotation established in G. These connected points are thought of as being in the same *orbit*. This process creates an uncountable infinity of *orbits* and through the Axiom of Choice we are allowed to select exactly one point from each of these *orbits* creating a new set, C. The set C is chosen such that any point in C can not be rotated to another point in C by rotations in G. Also if every point in C was to be rotated by every rotation in G, we would eventually get every point in $S - P$ similarly to the Vitali set in section 6.3.

Taking the points in C through each rotation of the subsets of G we create a partition of $S - P$. Let G_1C denote the rotations of G_1 applied to C, G_2C denote the rotations of G_2 applied to C, and G_3C denote the rotations of G_3 applied to C. Now we have that

$$S = G_1C \cup G_2C \cup G_3C \cup P$$

$$S - P = G_1C \cup G_2C \cup G_3C$$

From (8.1), (8.2), (8.3) above we see that when G_1C is rotated by τ we get G_2C. Similarly, τ^2 applied to G_1C creates G_3C, and lastly, σ applied to G_1C creates $G_2C \cup G_3C$ so we get

$$G_1C \cong G_2C \tag{7.4}$$
$$G_1C \cong G_3C \tag{7.5}$$
$$G_1C \cong G_2C \cup G_3C \tag{7.6}$$

Thus implying

$$G_1C \cong G_2C \cong G_3C \cong G_2C \cup G_3C$$

This was a proof of the Hausdorff Paradox as discussed in Theorem 6.4.1.

Since each of G_1, G_2, G_3 are congruent to $G_2C \cup G_3C$ we may split each one so that we create three sets of G_2C and three sets of G_3C.

$$\begin{aligned} G_1C &= G_2C \cup G_3C \\ G_2C &= G_2C \cup G_3C \\ G_3C &= G_2C \cup G_3C \end{aligned}$$

There are now six disjoint subsets that are all congruent to each other, this allows us to construct two copies of $S - P$. I will denote these copies as S_1 and S_2, both of which are only missing a countable set of points, P.

To complete the surface of the spheres we are going to have to fill the holes created by the set P. This is essentially a quick fix. To fill the holes created by the poles in S_1 we may simply use the set P of the original ball. In order to fill the holes created by the poles in S_2 we may use the concept of shifting from infinity/ Hilbert's Hotel as discussed in section 6.2.

Now that we have two complete surfaces we must fill in the interior in order to create two solid balls. This can be obtained through a bijection from S_1 and S_2 through the ball up to, but not including $(0, 0, 0)$. We may complete the proof by filling in the center similarly to the way we filled the holes created by P on S_1 and S_2. That is, we may complete the first ball by plugging the center point with the center point from our original ball, and the second ball may be completed by using the shifting from infinity method.

We now have proved the Banach-Tarski paradox by creating two solid balls of identical shape and volume to a single solid ball. □

7.1 Banach-Schröder-Bernstein Theorem and the Strong Form of Banach-Tarski

What has been proven in the previous section is considered the *duplication* version of the Banach-Tarski Paradox. A stronger version is stated as followed.

Theorem 7.1.1 (Strong Version of Banach-Tarski). *If A and B are two bounded subsets of $\mathbb{R}^3$ with nonempty interiors, then A and B are equidecomposable (see section 6.2), that is $A \sim B$.*

Proof. We will show that A is piecewise congruent to a subset of B, $A \preceq B$, and that B is piecewise congruent to a subset of A, $B \preceq A$ so that $A \sim B$ by the Banach-Schröder-Bernstein Theorem (discussed and proven next). Choose two solid balls, M, N so that $M \subseteq A$ and $N \subseteq B$. We may produce copies of N by the Banach-Tarski duplication theorem so that the copies completely cover M. Assume n copies are required. Then,

$$A \subseteq M \subseteq n \text{ overlapping copies of } N \preceq n \text{ disjoint copies of } N \sim N \subseteq B$$

which shows that $A \preceq B$. By a similar argument $B \preceq A$, and then by applying the Banach-Schröder-Bernstein Theorem, $A \sim B$. □

"$A \preceq B$ if and only if A is equivalent to a subset of B. Then $\preceq$ is a relation on the equivalence classes and, in fact, is reflexive and transitive[Wag93]." The classical Schröder-Bernstein Theorem says that A and B are equivalent if there is a bijection from A to B. In terms of cardinality this says that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. "Banach realized that the proof of the Schröder-Bernstein Theorem could be applied to any equivalence relation (such as equidecomposability) satisfying two abstract properties [Wag93]."

Theorem 7.1.2 (Banach-Schröder-Bernstein Theorem). *Suppose G acts on X and let $A, B \subseteq X$. If $A \preceq B$ and $B \preceq A$, then A is piecewise congruent to B, $A \sim B$. Thus $\preceq$ is a partial ordering of the equidecomposability classes in $\mathcal{P}(X)$.*

Proof. The relation $\sim$ is an equivalence relation on $\mathcal{P}(X)$ satisfying the following two conditions:

(i) if $A \sim B$ then there is a bijection $g : A \to B$ such that $C \sim g(C)$ whenever $C \subseteq A$

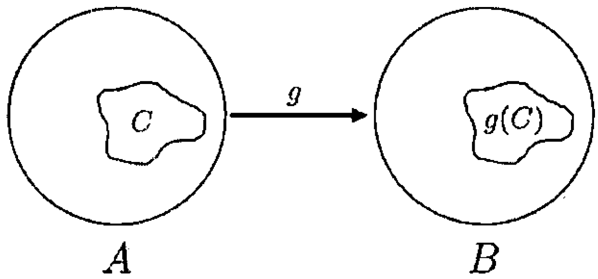

Figure 7.2: Banach-Schröder-Bernstein condition (i)

(ii) if $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$, and if $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1 \cup A_2 \sim B_1 \cup B_2$.

Let $f : A \to B_1$ and $g : A_1 \to B$, where $B_1 \subseteq B$ and $A_1 \subseteq A$ be bijections by (i). Let $C_0 = A \backslash A_1$ and, by induction, let C_{n+1} be $g^{-1}f(C_n)$ where $C = \bigcup_{n=0}^{\infty} C_n$. We see that $g(A \backslash C) = B \backslash f(C)$ and by choice of g we get that $A \backslash C \sim B \backslash f(C)$. Function f gives us that $C \sim f(C)$ and using (ii) from above we get $(A \backslash C) \cup C \sim (B \backslash f(C)) \cup f(C)$ which means that $A \sim B$. □

7.2 Clean Up

In this section I will prove the claim that every rotation in G has a unique, reduced form representation and also show the construction of G_1, G_2, G_3 as well as the proof of equation (8.1) in the proof of the Banach-Tarski paradox.

Theorem 7.2.1. *There are two independent rotations, τ and σ, about axes through the origin in $\mathbb{R}^3$.*

This is the same as saying that every rotation in G (except I,σ,τ,and τ^2) has a unique, reduced form representation.

Proof. Let's notice that every reduced rotation in G can be expressed in one of the four forms:

$$\begin{aligned}
\text{form } \alpha &= \tau^{P_1}\sigma\tau^{P_2}\sigma\tau^{P_3}\ldots\tau^{P_n}\sigma \\
\text{form } \beta &= \sigma\tau^{P_1}\sigma\tau^{P_2}\sigma\tau^{P_3}\ldots\sigma\tau^{P_n} \\
\text{form } \gamma &= \tau^{P_1}\sigma\tau^{P_2}\sigma\tau^{P_3}\ldots\sigma\tau^{P_n} \\
\text{form } \delta &= \sigma\tau^{P_1}\sigma\tau^{P_2}\sigma\tau^{P_3}\ldots\sigma\tau^{P_n}\sigma
\end{aligned}$$

where $n \geq 1$ and each $P_i = 1$ or 2. For form $\gamma, n > 1$ since if $n = 1$ then form γ would be a form of α. We first want to show that any reduced rotation of form $\alpha, \beta, \gamma, \delta$ can not equal I, then we can show that any reduced form in G is unique.

Using the rotations from our Banach-Tarski proof we have that

$$\tau = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \sigma = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Notice in every form α, β, γ,and δ, the rotation $\tau^{P_i}\sigma$ repeats itself. Calculating the product $\tau\sigma$ and $\tau^2\sigma$ we arrive at

$$\tau^{P_i}\sigma = \frac{1}{2}\begin{bmatrix} 0 & \pm\sqrt{3} & -1 \\ 0 & 1 & \pm\sqrt{3} \\ 2 & 0 & 0 \end{bmatrix}$$

using $+\sqrt{3}$ if $P_i = 1$ and $-\sqrt{3}$ if $P_i = 2$.

Since form α begins with $\tau^{P_1}\sigma$ then repeats itself n times we can arrive at

$$\text{form } \alpha = \frac{1}{2^n}\begin{bmatrix} m_{1\,1} & m_{1\,2}\sqrt{3} & m_{1\,3} \\ m_{2\,1}\sqrt{3} & m_{2\,2} & m_{2\,3}\sqrt{3} \\ m_{3\,1} & m_{3\,2}\sqrt{3} & m_{3\,3} \end{bmatrix}$$

where each of $m_{1\,1}, m_{2\,1}, m_{3\,1}, m_{3\,2},$ and $m_{3\,3}$ are even integers and the rest are odd integers. To see that form α could never equal I we notice that $m_{1\,2}\sqrt{3}$ can never equal 0, implying that form α could never equal I.

To show that form β could never equal the Identity matrix, assume that form $\beta = I$. If this was true then $\sigma\beta\sigma = \sigma I\sigma = \sigma^2 = I$, but since $\sigma\beta\sigma$ is of the form α which we have shown cannot equal I, form β also could never equal I.

To show form $\gamma \neq I$ assume form γ has the smallest possible n. If $P_1 = P_n$ then either $P_1 = P_n = 1$ or $P_1 = P_n = 2$. Assume first that $P_1 = P_n = 1$ so form $\gamma = \tau\sigma\tau^{P_2}\sigma\tau^{P_3}\ldots\sigma\tau$ and $\tau^2\gamma\tau$ is of the form β. If form $\gamma = I$ then $\tau^2\gamma\tau$ should remain of the form β which is clearly not the case since $\tau^2\gamma\tau = \tau^2 I\tau = \tau^3 = I$. If $P_1 = P_n \neq 1$ then let's assume that $P_1 = P_n = 2$ so form $\gamma = \tau^2\sigma\tau^{P_2}\sigma\tau^{P_3}\ldots\sigma\tau^2$. Now notice that $\tau\gamma\tau^2$ is of the form β and if form $\gamma = I$ then $\tau\gamma\tau^2$ should remain of the form β which is clearly not the case since $\tau\gamma\tau^2 = \tau I\tau^2 = \tau^3 = I$.

We must also consider the case that P_1 and P_n of form γ are not equal each other. Continue assuming that form $\gamma = I$. If $n > 3$ then $I = \sigma\tau^{P_n}\gamma\tau^{P_1}\sigma = \tau^{P_2}\sigma\ldots\sigma\tau^{P_{n-1}}$ which is of γ form but this would imply that $P_{n-1} = 1$ which is not allowed by definition of form γ. Now we have that $n = 2$ or $n = 3$. If $n = 2$ then we have that $I = \tau^{P_2}\gamma\tau^{P_1} = \sigma$ which is not true. If $n = 3$ then then $I = \sigma\tau^{P_2}\gamma\tau^{P_1}\sigma = \tau^{P_2}$ which is also not true. Therefore form $\gamma \neq I$.

To show that form $\delta \neq I$ we can say that $\sigma\delta\sigma$ is of γ form and if form $\delta = I$ then $\sigma\delta\sigma = \sigma I\sigma = \sigma^2 = I$ which is impossible since form $\gamma \neq I$.

This has shown that each rotation, other than the identity itself, can not reduce down to the identity rotation. Now I will show that all reduced forms are unique. Assume there does exist two distinct reduced rotations that represent the same rotation, $\lambda_1\lambda_2\ldots\lambda_m = \rho_1\rho_2\ldots\rho_n$ where each λ and ρ represent τ, τ^2 or σ. Taking the inverse of $\rho_1\rho_2\ldots\rho_n$ we should get that $(\lambda_1\lambda_2\ldots\lambda_m)(\rho_1\rho_2\ldots\rho_n)^{-1} = I$ which would require each $\lambda_m\rho_n$ to reduce which would imply that each $\lambda_i = \rho_i$ which contradicts the statement that λ and ρ are distinct. □

I will now prove equation (8.1) from above.

Claim. $\tau G_1 = G_2$

Proof. Let $r \in G_1$ and show that $\tau r \in G_2$. There are three possibilities for r. Either $r = \sigma$, $r = \tau$ or $r = \tau^2$. Notice that if $r = \tau^2$ then the leftmost character of τr would be σ since $\tau\tau^2 = I$.

Case 1. $r = \sigma$

Let $r \in G_1$ then by table 7.1, $\tau r \in G_2$.

Now let $\tau r \in G_2$. Since $r = \sigma$ then the only way that $\tau r \in G_2$ was if $r \in G_1$. Therefore if $r = \sigma$ then $\tau G_1 = G_2$.

Case 2. $r = \tau$

In this case we must notice that $r = \tau\nu$ where the leftmost character of ν cannot be τ nor τ^2 so it must be σ. Let $r \in G_1$ so that $\nu \in G_3$ and our rotation would be of the form $\tau^2\sigma\ldots$ which belongs to G_2.

Now let $\tau r = \tau^2\nu = \tau^2\sigma\ldots \in G_2$. Since $\nu = \sigma$ then $\nu \in G_3$. Then by table 7.1, $r \in G_1$. Therefore if $r = \tau$ then $\tau G_1 = G_2$.

Case 3. $r = \tau^2$

r must equal $\tau^2\nu$ where ν must equal σ again. If $r \in G_1$ then $\nu \in G_2$ and we get that $\tau r = \tau^3\nu = \nu \in G_2$.

Now let $\tau r \in G_2$. since $\tau r = \nu$ as above, and $\nu = \sigma$ then $r = \tau^2\sigma\ldots \in G_1$. Therefore if $r = \tau^2$ then $\tau G_1 = G_2$. □

The proofs of equation (8.2) and (8.3) follow similarly as above.

Chapter 8

Conclusion

Although restricted to the mathematical world, the Banach-Tarski Paradox gives some quite fascinating results. The proof is mathematically sound, there are no errors in the results, it is what it is. You are faced with several options. There are essentially two possible mathematical viewpoints. A "platonic" viewpoint allows you to either accept the results, viewing it almost a child accepting the wonders of the world, or you may choose to disregard the results, shutting down at first mention of the use of the Axiom of Choice. Understanding the results are simply too absurd to believe in and we do not yet hold enough mathematical foundation to decide on an answer. From the viewpoint of a "formalist" you would choose to not choose. A "formalist" believes the axioms of mathematics are like the rules of chess. You can change the rules any way you would like, but then you are playing a different game. Similarly, you can say that there is more than one possible concept of a "set" and you don't have to choose between them.

I would consider myself to be "platonic" and accept the results in a childish light. I consider the results to be entertaining and amusing, making for an interesting and educational thesis topic. It has been fascinating reading and researching about the years that set theory came into existence, bringing with it philosophical debate and mathematical discoveries. In the end, this thesis paper has been an enriched learning and growing experience for me as I continually seek to expand my knowledge and foundation of mathematics.

Bibliography

[Dan92] A.R.G. Dantan. *Bertrand Russell and the origins of the set-theoretic 'paradoxes'*. Birkhäuser Verlag, 1992.

[Eve97] W.R. Everdell. *The First Moderns: Profiles in the Origins of Twentieth-Century Thought.* University of Chicago Press, 1997.

[Hal74] Paul R. Halmos. *Naive set theory.* Springer-Verlag, New York, 1974. Reprint of the 1960 edition, Undergraduate Texts in Mathematics.

[Wag93] Stan Wagon. *The Banach-Tarski paradox.* Cambridge University Press, Cambridge, 1993. With a foreword by Jan Mycielski, Corrected reprint of the 1985 original.

[Wap05] Leonard M. Wapner. *The Pea & the Sun.* A K Peters Ltd., Wellesley, MA, 2005. A mathematical paradox.