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# Prouhet-Tarry-Escott problem

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# Prouhet-Tarry-Escott Problem

A Thesis

Presented to the

Faculty of

California State University,

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In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Juan Manuel Gutierrez III

June 2012

PROUHET-TARRY-ESCOTT PROBLEM

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## Abstract

The Prouhet-Tarry-Escott Problem is a complex problem that still has much to be discovered. My goal for this masters thesis is to organize the known results in a systematic way and to provide further insight using original ideas. I also intend to show the proofs of my findings in order to provide the most rigorous and complete outline possible for the problem. Some of the proofs will use original ideas that I have developed with Professor Fejzić over the course of the year.

Many results and problems in Number Theory are often easy to comprehend but difficult to prove. The Prouhet-Tarry-Escott Problem is no different. This problem is still unsolved in that there are no known methods for finding ideal solutions of size twelve or higher. The solutions to the problem are so difficult to find manually that many are obtained by extensive computer searches. This fascinating problem shows up in many areas of mathematics such as the study of polynomials, graph theory, and the theory of integral quadratic forms. In fact, its solution would not only put to rest an old problem in Number theory but would also make breakthroughs in these other areas of mathematical research.

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# Chapter 1

# Introduction

The purpose of this research paper is to gain a deeper understanding of a famous unsolved mathematical problem known as the Prouhet-Tarry-Escott Problem. The problem consists of finding two disjoint multi-sets of integers $\{a_1, a_2, \cdots, a_n\}$ and $\{b_1, b_2, \cdots, b_n\}$ such that

$$a_1^s + a_2^s + \cdots + a_n^s = b_1^s + b_2^s + \cdots + b_n^s \quad \text{for } s = 1, 2, \ldots, k \tag{1.1}$$

When two sets of integers $\{a_1, a_2, \cdots, a_n\}$ and $\{b_1, b_2, \cdots, b_n\}$ satisfy (1.1) we use the notation $[a_1, a_2, \cdots, a_n] =_k [b_1, b_2, \cdots, b_n]$. The number $n$ is called the size of the solution, while $k$ is the degree of the solution. A simple example is $[1, 8, 8] =_2 [2, 5, 10]$, which has size three and degree 2. Note that $1+8+8 = 2+5+10$ and $1^2+8^2+8^2 = 2^2+5^2+10^2$. This example will be used to illustrate some of the definitions that will be used throughout the paper. Note that in the example the size of the solution and degree differ by one. In general, when the degree of the solution and the size differ by one it is referred to as an ideal solution. The example mentioned above is an example of an ideal solution of size three and degree two.

Ideal solutions are not the only solutions that have unique characteristics. Solutions can also be symmetric. Symmetric solutions are of the form

$$[T + a_1, \ldots, T + a_m, T - b_1, \ldots, T - b_m] =_k [T + b_1, \ldots, T + b_m, T - a_1, \ldots, T - a_m]$$
$$\text{when } k = 1, 2, \ldots, 2n$$

or in the form

$$[T+a_1,\ldots,T+a_m,T-a_1,\ldots,T-a_m] =_k [T+b_1,\ldots,T+b_m,T-b_1,\ldots,T-b_m]$$

$$\text{when } k=1,2,\ldots,2n+1$$

Any solution not in either of these forms will be considered non-symmetric. An example of a symmetric solution is $[0,4,7,11] =_3 [1,2,9,10]$. This is a symmetric solutions because it verifies the definition above. Let $T=3$ we see that we get $[3,7,10,14,2,1,-6,-7] =_6 [4,5,12,13,3,-1,-4,-8]$. Note any repeats can be cancelled out and we are left with an ideal solution $[7,10,14,2,1,-6,-7] =_6 [4,5,12,13,-1,-4,-8]$.

In addition to ideal symmetric solutions there are even and odd ideal symmetric solutions. An even ideal symmetric solution of size $n$ is of the form $\left[\pm a_1,\ldots,\pm a_{\frac{n}{2}}\right] =_{n-1} \left[\pm b_1,\ldots,\pm b_{\frac{n}{2}}\right]$, while an odd ideal symmetric solution of size $n$ and with even degree $n-1$ is of the form $[a_1,\ldots,a_n] =_{n-1} [-a_1,\ldots,-a_n]$. These definitions will have implications for sine and cosine polynomials on the unit disk.

The Prouhet-Tarry-Escott Problem has seemingly easy criteria to satisfy, however ideal solutions are difficult to find. There are currently only ideal solutions for degree eleven and smaller, excluding ten for which there is no current known solution. The following is a list of the smallest ideal solutions currently known [Bor02]. The solutions of degree eight and nine have two different but inequivalent solutions, i.e. solutions not dependent on another. In addition, all of the following solutions are symmetric as well.

$$[\pm 2] =_1 [\pm 1]$$

$$[-2,-1,3] =_2 [2,1,-3]$$

$$[-5,-1,2,6] =_3 [-4,-2,4,5]$$

$$[-8,-7,1,5,9] =_4 [8,7,-1,-5,-9]$$

$$[\pm 1, \pm 11, \pm 12] =_5 [\pm 4, \pm 9, \pm 13]$$

$$[-50,-38,-13,-7,24,33,51] =_6 [50,38,13,7,-24,-33,-51]$$

$$[\pm 5, \pm 14, \pm 23, \pm 24] =_7 [\pm 2, \pm 16, \pm 21, \pm 25]$$

$$[-98,-82,-58,-34,13,16,69,75,99] =_8 [98,82,58,34,-13,-16,-69,-75,-99]$$

$$[174,148,132,50,8,-63,-119,-161,-169] =_8 [-174,-148,-132,-50,-8,63,119,161,169]$$

$$[\pm 99, \pm 100, \pm 188, \pm 301, \pm 313] =_9 [\pm 71, \pm 131, \pm 180, \pm 307, \pm 308]$$

$$[\pm 103, \pm 189, \pm 366, \pm 452, \pm 515] =_9 [\pm 18, \pm 245, \pm 331, \pm 471, \pm 508]$$

$$[\pm 151, \pm 140, \pm 127, \pm 86, \pm 61, \pm 22] =_{11} [\pm 148, \pm 146, \pm 121, \pm 94, \pm 47, \pm 35]$$

These solutions show the extent of the currrent known ideal solutions. The solutions are the most concise to this date, however they are a result of a collaboration of many mathematicans. The next chapter is dedicated to the history of the Prouhet-Tarry-Escott Problem, more specifically the work before the 1920's. The majority of the material for this section will be taken from Leonard Eugene Dickson who published a book History of The Theory of Numbers, which includes a thorough account of the problem and its origins [Dic66].

The list of ideal solutions above is relatively short considering the amount of time spent on the problem. This paper will include a list of these ideal solutions and the corresponding general formulas used to find them. These formulas will be referred to as parametric equations. In order to find these parametric equations we will reference Peter Borwein's Computational Excursions in Analysis and Number Theory and Jack Chernick Ideal Solutions of the Tarry-Escott Problem [Bor02, Che37]. Examples will also be used in order to clarify the method used to find these solutions.

Finally, the last chapter will include original ideas on the problem. The work will include a combinatorics proof that will reduce the upper limit necessary to find solutions. This proof will be given in its entirety and will include examples.

# Chapter 2

# The History

The solutions and interest for Prouhet-Tarry-Escott Problem is not credited to any one person, rather a collection of mathematicians. Many mathematicians devoted their time to this problem beginning with finding a general solution for degree two. One of first more notable mathematicians to work on this problem was Goldbach. In 1750, Goldbach wrote a letter to Euler about his findings of the first general solutions of degree two,

$$\begin{aligned}[a+b+d, a+c+d, b+c+d, d] =_2 [a+d, b+d, c+d, a+b+c+d]\\ \text{for any } a, b, c \text{ and } d \in \mathbb{Z}\end{aligned} \tag{2.1}$$

Clearly there are infinitely many solutions in this form but the size of these solutions are four which is not ideal. One year later Euler wrote Goldbach back with a slight alteration to (2.1); he found by setting $d = 0$ the resulting equation will still form a parametric solution, i.e. $[a+b, a+c, b+c] =_2 [a, b, c, a+b+c]$. For example, if $a = 3, b = 5, c = 7$ we get $[8, 10, 12] =_2 [3, 5, 7, 15]$. This shows the beginning of the problem we know today, however it differs in that the size on the left is not equal to size of the right. The size of the solutions will later become more important as the research continued.

In 1851, Prohout was the first to prove how the size of the set related to the degree of the solution. He noted that the first $n^m$ numbers, $\{1, 2, 3, \ldots, n^m\}$ can be separated into $n$ sets each with $n^{m-1}$ terms such that the sum of the $k^{th}$ powers of the terms is the same for all sets $k < m$. For example, when $n = 3$ and $m = 4$ means the numbers $1, \ldots, 81$ are separable into 3 sets each of size 27. The sum of these sets are equal for all degree less than or equal to 4. This result gives an upper bound to the size of the set in relation to the degree. That is given a degree k then a solution can be found with size $2^{m-1}$, when $n = 2$.

Later in 1861, Pollock found an parametric equation of an ideal solution of size three and degree two.

$$[a, a+b, a+2b+3c] =_2 [a-c, a+b+2c, a+2b+2c] \quad \text{for any } a, b, c \in \mathbb{Z}. \tag{2.2}$$

We can illustrate by letting $a = 7, b = 4, c = 3$ then the solution $[7, 11, 24] =_2 [4, 17, 21]$ is produced. It is now clear that for degree two there are infinitely many ideal solutions.

Other mathematicians followed Pollock and continued working toward finding properties of second degree equations. In 1906, A. Gérardin found that $x^3 + y^3 + z^3 = (x+1)^3 + (y-2)^3 + (z+1)^3$ is equivalent to $\Delta_x + \Delta_z = (y-1)^2$, where $\Delta_x = x(x+1)/2$. He found picking $\Delta_x = 1, 3, 6, 10, 15, \ldots$ produced values of $z's \leq 100$. One particular solution was produced when $x = 1, y = 12, z = 15$. Using these numbers we get

$$\begin{aligned} 1^3 + 12^3 + 15^3 &= (2)^3 + (10)^3 + (16)^3 \\ \text{but} \quad 1^2 + 12^2 + 15^2 &\neq (2)^2 + (10)^2 + (16)^2 \\ 1 + 12 + 15 &= (2) + (10) + (16) \end{aligned}$$

This is not a solution to the Prouhet-Tarry-Escott Problem because it does not work for all the degrees less than or equal to three but the sums for degree two differ by ten. Gérardin noticed that if you pick two solutions like the one shown above, where the second degree differs by some square $m^2$ then they can be manipulated in such a way to get a complete solution to the Prouhet-Tarry-Escott Problem. For instance, if you multiply the numbers of the first solution by $m$ and add it to the second solution you can get new solutions. Using the method described you can produce the following solutions:

$$\begin{aligned} [2, 4, 20, 22, 33] &=_3 [1, 6, 16, 26, 32] \\ [1, 4, 12, 13, 20] &=_3 [2, 3, 10, 16, 19] \\ [3, 4, 15, 20, 23, 26] &=_3 [2, 5, 17, 18, 22, 27] \\ [2, 6, 30, 46, 53, 73] &=_3 [3, 4, 34, 44, 51, 74] \\ [2, 6, 44, 58, 63, 91] &=_3 [1, 8, 40, 60, 65, 90] \end{aligned}$$

Solutions of this type gave Gérardin the information he needed to find parametric equations for degree three and five.

$$[1, m+3, 2m-2, 4m+2, 5m-3, 6m-1]$$
$$=_3 [2, m-1, 2m+3, 4m-3, 5m+1, 6m-2]$$
$$[x, x+3, x+5, x+6, x+9, x+10, x+12, x+15]$$
$$=_5 [x+1, x+2, x+4, x+7, x+8, x+11, x+13, x+14]$$

Gérardin continued his work for another two years and his worked inspired Tarry and Escott to pursue parametric solutions of degrees greater than three.

In 1908, Escott showed he could find not one, but all the solutions of the sets in the form:

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \quad \text{and} \quad \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2 \tag{2.3}$$

for $n = 3$. He did this by substitution and reducing the problem to solving two unknowns rather than $n$ of them. The equation reduced the number of terms necessary to solve for and resulted in the following, $X_1^2 + X_1X_2 + X_2^2 = Y_1^2 + Y_1Y_2 + Y_2^2$. Using this simplified equation, Escott let $N$ be any number whose prime factors are of the form $6n+1$ or 3 and any square factors in common to $X_1, X_2, Y_1, Y_2$. The solutions would then be $N$ in the form $x^2 + xy + y^2$. Later in 1912, H.B. Mathieu gave a general solution to (2.3) for $n = 3$, $l \pm (ab + ac), l(1 - bd) + qab \mp ac, l(cd + 1) \mp ab - qac$.

In 1912, G. Tarry built upon the work of another mathematician, M. Frolov, who found that if $\sum_{k=1}^{n} a^k = \sum_{k=1}^{n} b^k$ and $\sum_{k=1}^{n} c^k = \sum_{k=1}^{n} d^k$, then $\sum_{k=1}^{n} (a+c)^k = \sum_{k=1}^{n} (b+d)^k$. Frolov correctly found that there exists ideal solutions for degree two, but thought more than four terms were necessary for an ideal solution of degree three. Tarry generalized this idea of size and degree and stated that the first $2^n(2a+1)$ integers can be separated into two sets of integers with size $2^{n-1}(2a+1)$ and degree $t$ for $t = 1, 2, \ldots, n$. For example, let $a = 1$ and $n = 3$, then the first twenty-four integers can be separated into two sets of integers each of size 12 with degree 3. Using these conditions the following solution is produced, $[1, 3, 7, 8, 9, 11, 14, 16, 17, 18, 22, 24] =_3 [2, 4, 5, 6, 10, 12, 13, 15, 19, 20, 21, 23]$. E. Miot, in 1913, found a similar result to Tarry. He found that any $2^n(2a+1)$ numbers in arithmetical progression can be separated into two equal sets having the same sum for the $t$th powers for $t = 1, 2, \ldots, n$ if $a > 0$ and $n > 1$. However if $a = 0$, then it is only true for degree $1, \ldots, n-1$. Using the same example above we get

$[a, a+2r, a+6r, \ldots, a+23r] =_3 [a+r, a+3r, \ldots, a+22r]$. Tarry noted that the number of terms in each set of the equation is $2k-d$. This occurs if $k$ is the number of terms in each member of the given equation and $x$ is expressible in $d$ ways as a difference of two numbers that belong to the same set. For example, given the solution $[1,5,10,16,27,28,38,39] =_6 [2,3,13,14,25,31,36,40]$ we see that if $x = 11$ then we can manipulate the set so that each member in the set will have a difference of 11, i.e. $11 = 16-5 = 27-16 = 38-27 = 39-28 = 13-2 = 14-3 = 25-14 = 36-25$. By taking $x$ to be 11, we get that $d = 8$ and the following solution is produced $[1,5,10,24,28,42,47,51] =_7 [2,3,12,21,31,40,49,50]$.

Later Tarry also found that if $t = \frac{2}{3}(a+b+c)$ the following solution can be found $[a,b,c] =_2 [t-a, t-b, t-c]$. Tarry referred to this as the double property. He also found that for some $x$, $[a,b,\ldots,h] =_n [p,q,\ldots,t]$ implies $[a,\ldots,h,p+x,\ldots,t+x] =_{n+1} [p,\ldots,t,a+x,\ldots,h+x]$. This is a property that will later be proved in the properties section. Applying this lemma he found a solution of degree five

$$[6a-3b-8c, 5a-9c, 4a-4b-3c, 2a+2b-5c, a-2b-5c, a-2b+c, b]$$
$$=_5 [6a-2b-9c, 5a-4b-5c, 4a+b-8c, 2a-3b, a+2b-3c, c]$$

Later Tarry republished this result and found that given $A_1, \ldots, A_k =_{2n} B_1, \ldots, B_k$ and $A_i + A_{k-i} = 2h = B_i + B_{k-i}$ implies $A_1, \ldots, A_k =_{2n+1} B_1 + \ldots, B_k$. These equations are subtracting an $h$ from every term of the given equation as opposed to adding like the earlier proposition.

Mathematicians L. Bastien and A. Aubry each contributed work of a different kind. In 1913, Bastien proved that it is impossible for a solution to be in the form $[x_1, \ldots, x_n] =_n [y_1, \ldots, y_n]$ unless the $x_i's$ do not form a permutation of the $y_i's$. Alternatively, Aubry proved new as well as known solutions for the first and second degree. He also proved the impossibility of a solution of the form $[x, y,] =_3 [t, u, v,]$. The work of these mathematicians began the work of ideal solutions and when such solutions exist.

# Chapter 3

# Ideal Solutions

Recall that an ideal solution is when the size of the solution and the degree differ by one. Ideal solutions are the most difficult to find; over the past 100 years there are still only 13 known ideal parametric solutions. We now present the first 10 parametric ideal solutions. [Che37]

The general solution for the second degree was already mentioned in the previous section. The solution was of the form

$$[AD + k, AG + BD + k, BG + k] =_2 [AD + BG + K, AG + k, BD + k]. \tag{3.1}$$

To illustrate an example of this solution let $A = 3, B = 7, D = 8, G = 2$, and $k = 0$. Using these numbers we get the solution

$$[24, 62, 14] =_2 [38, 6, 56]$$

$$\text{i.e.} \quad 24 + 62 + 14 = 38 + 6 + 56 = 100$$

$$24^2 + 62^2 + 14^2 = 38^2 + 6^2 + 56^2 = 4776$$

The parametric solution of the third degree can be found rather easily under certain conditions. The general formula for the third degree is

$$[a_1, a_2, -a_1, -a_2] =_3 [b_1, b_2, -b_1, -b_2] \tag{3.2}$$

or

$$[Ma_1 + k, Ma_2 + k, -Ma_1 + k, -Ma_2 + k] =_3 [Mb_1 + k, Mb_2 + k, -Mb_1 + k, -Mb_2 + k] \tag{3.3}$$

This equation is only true when $a_1, a_2, b_1, b_2$ satisfy $a_1^2 + a_2^2 = b_1^2 + b_2^2$. This occurs when

$$a_1 = p_1p_2 + p_3p_4 \qquad b_1 = p_1p_2 - p_3p_4$$

$$a_2 = p_1p_3 - p_2p_4 \qquad b_2 = p_1p_3 + p_2p_4$$

This parametric equation will produce all the integers solutions. One possible integers solution can be produced by letting $p_1 = 13$, $p_2 = 10$, $p_3 = 3$, and $p_4 = 5$. Picking these as our $p's$ then:

$$a_1 = 130 + 15 = 145 \qquad b_1 = 130 - 15 = 115$$

$$a_2 = 39 - 50 = -11 \qquad b_2 = 39 + 50 = 89$$

So the solution $[145, -11, -145, 11] =_3 [115, 89, -115, -89]$ satisfies all conditions. [Che37]

Parametric solutions for the fourth degree begin by assuming that

$$[a, a+2k, -a, -a-2k] =_3 [b, k, -b, -k]$$

is a solution for degree three. Applying a proposition that will be discussed in the next section we get the general formula for degree four.

$$[a_1, a_2, a_3, a_4, a_5] =_4 [-a_1, -a_2, -a_3, -a_4, -a_5] \tag{3.4}$$

where $a_i = \left(m^2, -2n^2, -m^2 - mn + 2n^2, -m^2 + mn + 2n^2, m^2 - 2n^2\right)$. This parametric solution was based upon a solution for degree three, if that solution was changed then so would the solutions for the fourth degree. For instance if the third degree solution was changed to $[-3k, -k, k, 3k] =_3 [u, v, -u, -v]$, then 3.4 holds but under the conditions that $a_i = \left(-2m^2 - 2n^2, -m^2 + mn + 2n^2, -3mn + n^2, m^2 + 3mn, 2m^2 - mn - n^2\right)$. Letting $m = 2$ and $n = 8$ we see that the first condition produces $[4, -128, 108, 140, -124] =_4 [-4, 128, -108, -140, 124]$. However using the second solution for degree three we get the solution $[-136, 140, 16, 52, -72] =_4 [136, -140, -16, -52, 72]$. Note that using the same parameters we get different solutions, in general this will hold for all values of $m$ and $n$. [Che37]

Borwein also has a parametric solution for fourth degree equations. His solution is only one-parameter which is an advantage over Chernick. His solution is presented below.

$$\left[2m^2, -1, 2m^2 - 1, -2m^2 + 1 - m, -2m^2 + m + 1\right]$$
$$=_4 \left[-2m^2, 1, -2m^2 + 1, 2m^2 - 1 + m, 2m^2 - m - 1\right]$$

To illustrate this solution let $m = 3$. Then $[-18, 1, -17, 20, 14] =_5 [18, -1, 17, -20, -14]$.

In "Ideal Solutions of Tarry-Escott Problem" [Che37], Chernick has a parametric solution for the fifth degree. Since then another parametric solution was published in "Computational Excursions in Analysis and Number Theory" [Bor02]. For the remaining degrees we will continue showing both solutions to see the contrast over the years. We will begin with the fifth degree.

$$[a_1, a_2, a_3, -a_1, -a_2, -a_3] =_5 [b_1, b_2, b_3, -b_1, -b_2, -b_3] \tag{3.5}$$

where

$$\begin{aligned} a_1 &= -5m^2 + 4mn - 3n^2 & b_1 &= -5m^2 + 6mn + 3n^2 \\ a_2 &= -3m^2 + 6mn + 5n^2 & b_2 &= -3m^2 - 4mn - 5n^2 \\ a_3 &= -m^2 - 10mn - n^2 & b_3 &= -m^2 + 10mn - n^2 \end{aligned}$$

Applying this equation we see that we can get a solution if $m = 2$ and $n = 3$. We get the ideal solution $[-23, 69, -73, 23, -69, 73] =_5 [43, -81, 47, -43, 81, -47]$. The corresponding solution by Borwein is as follows,

$$\begin{aligned} &[\pm(2n + 2m), \pm(nm + n + m - 3), \pm(nm - n - m - 3)] \\ &=_5 [\pm(2n - 2m), \pm(n - nm - m - 3), \pm(m - nm - n - 3)] \end{aligned} \tag{3.6}$$

Let $n = 4$ and $m = 5$ and we get the symmetric solution $[\pm 18, \pm 26, \pm 8] =_5 [\pm 2, \pm 24, \pm 22]$.

As in the previous solutions Chernick begins by assuming a solution to the previous degree (3.5). This specific solution is not parametric but infinitely many solutions can be found using the recursion formula. Chernick begins this solution by letting

$$\begin{aligned} a_1 &= a & b_1 &= k \\ a_2 &= a + 2k & b_2 &= 3k \\ a_3 &= b & b_3 &= b + 2r \end{aligned}$$

We can now assume these parameters hold for the sixth degree therefore, then the following holds.

$$\begin{aligned} a^2 + (a + 2k)^2 + b^2 &= k^2 + (3k)^2 + (b + 2r)^2 \\ a^4 + (a + 2k)^4 + b^4 &= k^4 + (3k)^4 + (b + 2r)^4 \end{aligned}$$

These two equations can be rewritten in terms of $M, r$, and $Q$. More specifically the second degree can be rewritten as $a^2+2ak-3k^2=2Mr$, $b=M-r$, $a+k=2Q$ while the fourth degree can be rewritten as $M^2-Mr+r^2=7k^2$. This equation has a parametric solution of

$$k=m^2+mn+n^2 \qquad r=m^2+6mn+2n^2 \qquad M=-2m^2+2mn+3n^2$$

Using these variables we can substitute them into the second degree equation to get $2Q^2=n(n+2m)(2n+3m)(4n-m)$. Substituting once more where

$$Q=2u_1u_2u_3u_4 \qquad n=2u_1^2 \qquad n+2m=4u_2^2$$
$$2n+3m=u_3^2 \qquad 4n-m=u_4^2$$

This final substitution yields $9u_1^2-2u_2^2=u_4^2$ and $u_1^2+6u_2^2=u_3^2$. Letting $u_1=1$ and $u_2=2$ we obtain an non-trivial solution for the sixth degree.

$$[x_1,x_2,\ldots,x_7]=_6[x_1,x_2,\ldots,x_7] \tag{3.7}$$

where

$$x_1=-134 \qquad x_2=-75 \qquad x_3=-66 \qquad x_4=-8$$
$$x_5=47 \qquad x_6=87 \qquad x_7=133$$

Assuming a different solution of the fifth degree we see that we can get infinitely many solutions using this method.

Borwein has a purely parametric solution for the sixth degree. His solution is also of the form (3.7), however his $x_i's$ are found by two parameters.

$$x_1=-(-3j^2k+k^3+j^3)(j^2-kj+k^2) \qquad x_2=(j+k)(j-k)(j^2-3kj+k^2)j$$
$$x_3=(j-2k)(j^2+kj-k^2)kj \qquad x_4=-(j-k)(j^2-kj-k^2)(-k+2j)k$$
$$x_5=-(j-k)(-2kj^3+j^4-j^2k^2+k^4 \qquad x_6=(j^4-4kj^3+j^2k^2+2k^3j-k^4)k$$
$$x_7=(j^4-4kj^3+5j^2k^2-k^4)j$$

One example of a sixth degree solution where $j=3$ and $k=4$ yields,

$$[221,231,-300,-152,-23,-316,339]=_6[-221,-231,300,152,23,316,339]$$

The most basic solutions for degree 7 are of the symmetric form. The generic solution is of the form

$$[a_1, \ldots, a_4, -a_1, \ldots, -a_4] =_7 [b_1, \ldots, b_4, -b_1, \ldots, -b_4] \tag{3.8}$$

It is clear that this equation is equivalent to finding $a_i$'s and $b_i$'s of the form, $[a_1, \ldots, a_4] =_k [b_1, \ldots, b_4]$ for $k = 2, 4, 6$. We can find that one possible solution for the previous equation is

$$[u - 7w, u - 2v + w, 3u + w, 3u + 2v + w] =_k [u + 7w, u - 2v - w, 3u - w, 3u + 2v - w]$$

We see that this equation can be reduced to $u^2 + uv + v^2 = 7w^2$, which a parametric solution was given earlier for the sixth degree. One example leads to the following solution $[7, 24, 25, 34] =_k [14, 15, 31, 32]$. Now this yields the actually solution for degree seven, $[7, 24, 25, 34, -7, -24, -25, -34] =_k [14, 15, 31, 32, -14, -15, -31, -32]$. Borwein states this solution more directly by immediately giving the parameters for (3.8).

$$\begin{aligned} a_1 &= 5m^2 + 9mn + 10n^2 & b_1 &= 9m^2 + 5mn + 4n^2 \\ a_2 &= m^2 - 13mn - 6n^2 & b_2 &= m^2 + 15mn + 8n^2 \\ a_3 &= 7m^2 - 5mn - 8n^2 & b_3 &= 5m^2 - 7mn - 10n^2 \\ a_4 &= 9m^2 + 7mn - 4n^2 & b_4 &= 7m^2 + 5mn - 6n^2 \end{aligned}$$

The rest of the solutions will be those given by Borwein. Chernick ended his parametric solutions with degree seven and there are still no known parametric solution of degree 8. There are two non-equivalent symmetric solutions, they are,

$$\begin{aligned} &[-98, -82, -58, -34, 13, 16, 69, 75, 99] \\ &=_8 [98, 82, 58, 34, -13, -16, -69, -75, -99] \\ &[174, 148, 132, 50, 8, -63, -119, -161, -169] \\ &=_8 [-174, -148, -132, -50, -8, 63, 119, 161, 169] \end{aligned}$$

The parametric solutions for degree nine are also symmetric. Borwein attributes the solutions to Letac, who found them using rational solutions on an elliptic curve.

$$[a_1, \ldots, a_5, -a_1, \ldots, -a_5] =_9 [b_1, \ldots, b_5, -b_1, \ldots, -b_5] \tag{3.9}$$

where

$$a_1 = 4n + 4m \qquad b_1 = 4n - 4m$$

$$a_2 = mn + n + m - 11 \qquad b_2 = -mn + n - m - 11$$

$$a_3 = mn - n - m - 11 \qquad b_3 = -mn - n + m - 11$$

$$a_4 = mn + 3n - 3m + 11 \qquad b_4 = -mn + 3n + 3m + 11$$

$$a_5 = mn - 3n + 3m + 11 \qquad b_5 = -mn - 3n - 3m + 11$$

The final two ideal solutions that will be discussed currently do not have parametric solutions. More specifically, not only are there no known ideal solutions for the tenth degree, there are no solutions altogether. As for degree 11 there is only one ideal solution where all the entries are less than 1000. That particular solution is $[\pm151, \pm140, \pm127, \pm86, \pm61, \pm22] =_{11} [\pm148, \pm146, \pm121, \pm94, \pm47, \pm35]$.

# Chapter 4

# Basic Properties

Ideal solutions are the most sought after solutions and as the previous section established there are only a small number of parametric ideal solutions. The exact method by which these solutions were produced was overlooked but Chernick often found them using previous solutions and applying simple properties. This section will list those simple properties he used to find those ideal solutions as well as a corresponding proof for each. However we will begin this section with a congruence statement for the Prouhet-Tarry-Escott Problem. The Prouhet-Tarry-Escott Problem can be expressed many ways, one of which is in terms of polynomials. In a definition above, it was noted that certain solutions can be expressed as sine and cosine polynomials, i.e. even and odd ideal symmetric solutions. The following theorem will establish the relationships between the Prouhet-Tarry-Escott Problem and the polynomials it represents.

**Theorem 1.** *The following are equivalent:*

*a)* $\sum_{i=1}^{n} \alpha_i^j = \sum_{i=1}^{n} \beta_i^j$ *for* $j = 1, \ldots, k-1$

*b)* $\deg \left( \prod_{i=1}^{n}(z-\alpha_i) - \prod_{i=1}^{n}(z-\beta_i) \right) \leq n-k$

*c)* $(z-1)^k | \sum_{i=1}^{n} z^{\alpha_i} - \sum_{i=1}^{n} z^{\beta_i}$

*Proof.* $(a) \Rightarrow (b)$

Assume $\sum_{i=1}^{n} \alpha_i^i = \sum_{i=1}^{n} \beta_i^i$ for $j = 1, \ldots, k-1$. By definition,

$$\begin{array}{ccccccccccc} \alpha_1 & + & \alpha_2 & + \ldots + & \alpha_n & = & \beta_1 & + & \beta_2 & + \ldots + & \beta_n \\ \alpha_1^2 & + & \alpha_2^2 & + \ldots + & \alpha_n^2 & = & \beta_1^2 & + & \beta_2^2 & + \ldots + & \beta_n^2 \\ \vdots & + & \vdots & + \ldots + & \vdots & = & \vdots & + & \vdots & + \ldots + & \vdots \\ \alpha_1^{k-1} & + & \alpha_2^{k-1} & + \ldots + & \alpha_n^{k-1} & = & \beta_1^{k-1} & + & \beta_2^{k-1} & + \ldots + & \beta_n^{k-1} \end{array}$$

Let $p(x)$ and $q(x)$ be polynomials of the form:

$$\begin{aligned} p(x) &= (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) \\ &= [x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_n] \\ q(x) &= (x-\beta_1)(x-\beta_2)\cdots(x-\beta_n) \\ &= [x^n + d_1 x^{n-1} + d_2 x^{n-2} + \ldots + d_n] \\ \text{then} \quad p(x) - q(x) &= \left[x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_n\right] \\ &\quad - \left[x^n + d_1 x^{n-1} + d_2 x^{n-2} + \ldots + d_n\right] \end{aligned}$$

Let

$$\begin{aligned} c_1 &= -\sum \alpha_k & \qquad s_1 &= \alpha_1 + \alpha_2 + \ldots + \alpha_n \\ c_2 &= \sum_{i \neq j} \alpha_i \alpha_j & s_2 &= \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2 \\ \vdots &= \vdots & \vdots &= \vdots \\ c_n &= \sum \alpha_k & s_{k-1} &= \alpha_1^{k-1} + \alpha_2^{k-1} + \ldots + \alpha_n^{k-1} \end{aligned}$$

Similarly for $d_i$. Let

$$\begin{aligned} d_1 &= -\sum \beta_k & \qquad t_1 &= \beta_1 + \beta_2 + \ldots + \beta_n \\ d_2 &= \sum_{i \neq j} \beta_i \beta_j & t_2 &= \beta_1^2 + \beta_2^2 + \ldots + \beta_n^2 \\ \vdots &= \vdots & \vdots &= \vdots \\ d_n &= \sum \beta_k & t_{k-1} &= \beta_1^{k-1} + \beta_2^{k-1} + \ldots + \beta_n^{k-1} \end{aligned}$$

By Newton's Identities, we know that we can express each coefficient in terms of $\alpha$ and $\beta$. For the first degree we get:

$$s_1 + c_1 = 0 \qquad\qquad t_1 + d_1 = 0$$
$$c_1 = -s_1 \qquad\qquad d_1 = -t_1$$

But by assumption, $s_1 = t_1$ and so must $-s_1 = -t_1$. Therefore $c_1 = d_1$.

Applying Newtons Identities again we can the following relationship between $s$ and $c$.

$$s_2 + c_1 s_1 + 2c_2 = 0 \qquad\qquad t_2 + d_1 t_1 + 2d_2 = 0$$
$$\Rightarrow c_2 = \frac{-(s_2 + c_1 s_1)}{2} \qquad\qquad \Rightarrow d_2 = \frac{-(t_2 + d_1 t_1)}{2}$$

By the previous identity we know that $c_1 s_1 = d_1 t_1$ and $s_2 = t_2$ by assumption. Therefore $c_2 = d_2$.

We can continue using Newton's Identities to show that $c_i = d_i$ for each $i \leq k-1$. Referring back to

$$p(x) - q(x) = \left[x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_n\right] - \left[x^n + d_1 x^{n-1} + d_2 x^{n-2} + \ldots + d_n\right]$$

we see that every term will cancel till the $k-1$ term. The term after the last cancelation is of the form $c_k x^{n-k}$ and $d_k x^{n-k}$ respectively for $p(x)$ and $q(x)$. Therefore we get the fact that $\deg[(c_k x^{n-k} + \ldots + c_n) - (d_k x^{n-k} + \ldots + d_n)] \leq n - k$, which is precisely part b of the theorem. □

*Proof.* (a) $\Rightarrow$ (c)
Assume $\sum_{i=1}^{n} \alpha_i^i = \sum_{i=1}^{n} \beta_i^i$ for $j = 1, \ldots, k-1$. i.e.,

$$\begin{array}{ccccccccccccc}
\alpha_1 & + & \alpha_2 & + \ldots + & \alpha_n & = & \beta_1 & + & \beta_2 & + \ldots + & \beta_n \\
\alpha_1^2 & + & \alpha_2^2 & + \ldots + & \alpha_n^2 & = & \beta_1^2 & + & \beta_2^2 & + \ldots + & \beta_n^2 \\
\vdots & + & \vdots & + \ldots + & \vdots & = & \vdots & + & \vdots & + \ldots + & \vdots
\end{array}$$
$$\alpha_1^{k-1} + \alpha_2^{k-1} + \ldots + \alpha_n^{k-1} = \beta_1^{k-1} + \beta_2^{k-1} + \ldots + \beta_n^{k-1}$$

Let $p(x)$ be a polynomial defined as

$$\begin{aligned}
p(x) &= \sum_{i=1}^{n} x^{\alpha_i} - \sum_{i=1}^{n} x^{\beta_i} \\
&= (x^{\alpha_1} + x^{\alpha_2} + \cdots + x^{\alpha_n}) - (x^{\beta_1} + x^{\beta_2} + \cdots + x^{\beta_n})
\end{aligned}$$

Taking the derivative of both sides of the equation yields

$$p'(x) = (\alpha_1 x^{\alpha_1 - 1} + \alpha_2 x^{\alpha_2 - 1} + \cdots + \alpha_n x^{\alpha_n - 1}) - (\beta_1 x^{\beta_1 - 1} + \beta_2 x^{\beta_2 - 1} + \cdots + \beta_n x^{\beta_n - 1})$$

Note when $x = 1$,

$$p'(x) = (\alpha_1 + \alpha_2 + \cdots + \alpha_n) - (\beta_1 + \beta_2 + \cdots + \beta_n)$$

By assumption, $\sum_{i=1}^{n} \alpha_i^i = \sum_{i=1}^{n} \beta_i^i$ for $j = 1, \ldots, k-1$. Therefore $p'(1) = 0$.

Taking the derivative again yields

$$p''(x) = \sum_{i=1}^{n} \alpha_i(\alpha_i - 1)x^{\alpha_i - 2} - \sum_{i=1}^{n} \beta_i(\beta_i - 1)x^{\beta_i - 2}$$

When $x = 1$ we get

$$\begin{aligned} p''(1) &= \sum_{i=1}^{n} \alpha_i(\alpha_i - 1) - \sum_{i=1}^{n} \beta_i(\beta_i - 1) \\ &= \sum_{i=1}^{n} \alpha_i^2 - \sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \beta_i^2 + \sum_{i=1}^{n} \beta_i \end{aligned}$$

But by assumption, $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i$ and $\sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} \beta_i^2$. So we get that $p''(1) = 0$ as well.

Taking the derivative a third time yields

$$\begin{aligned} p'''(x) &= \sum_{i=1}^{n} \alpha_i(\alpha_i - 1)(\alpha_i - 2)x^{\alpha_i - 3} - \sum_{i=1}^{n} \beta_i(\beta_i - 1)(\beta_i - 2)x^{\beta_i - 3} \\ \Rightarrow p'''(1) &= \sum_{i=1}^{n} \alpha_i(\alpha_i - 1)(\alpha_i - 2) - \sum_{i=1}^{n} \beta_i(\beta_i - 1)(\beta_i - 2) \\ &= \sum_{i=1}^{n} \alpha_i^3 - 3\sum_{i=1}^{n} \alpha_i^2 + 2\sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \beta_i^3 + 3\sum_{i=1}^{n} \beta_i^2 - 2\sum_{i=1}^{n} \beta_i \end{aligned}$$

But by assumption, $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i, \sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} \beta_i^2$ and $\sum_{i=1}^{n} \alpha_i^3 = \sum_{i=1}^{n} \beta_i^3$. These equations hold true even with a constant. So we get that $p'''(1) = 0$ as well.

This is true for all derivatives up to $k-1$. When we take that derivative we get

$$
\begin{aligned}
p^{k-1}(x) &= \sum_{i=1}^{n} \alpha_i(\alpha_i - 1)\dots(\alpha_i - (k-2))x^{\alpha_i-(k-1)} \\
&\quad - \sum_{i=1}^{n} \beta_i(\beta_i - 1)\dots(\beta_i - (k-2))x^{\beta_i-(k-1)} \\
\Rightarrow p^{k-1}(1) &= \sum_{i=1}^{n} \alpha_i(\alpha_i - 1)\dots(\alpha_i - (k-2)) \\
&\quad - \sum_{i=1}^{n} \beta_i(\beta_i - 1)\dots(\beta_i - (k-2))
\end{aligned}
$$

As seen in the previous examples multiplying out the terms will result in a constant in front of each term but the assumption will hold and will result in $p^{k-1}(1) = 0$.

By the Taylor Series we can write $p(x)$ in the form

$$p(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + \dots + a_{k-1}(x-1)^{k-1} + a_k(x-1)^k + \dots + a_n(x-1)^n$$

where

$$
\begin{aligned}
a_0 &= p(1) = 0 \\
a_1 &= p'(1) = 0 \\
a_2 &= \frac{p''(1)}{2!} = 0 \\
\vdots &= \vdots \\
a_{k-1} &= \frac{p^{k-1}(1)}{(k-1)!} = 0
\end{aligned}
$$

Thus we can rewrite $p(x)$ as

$$
\begin{aligned}
p(x) &= a_k(x-1)^k + a_{k+1}(x-1)^{k+1} + \dots + a_n(x-1)^n \\
&= (x-1)^k \left[a_k + a_{k+1}(x-1) + \dots + a_n(k-1)^{n-k}\right]
\end{aligned}
$$

Now let $m = a_k + a_{k+1}(x-1) + \dots + a_n(k-1)^{n-k}$ so that $p(x) = (x-1)^k m \Rightarrow (x-1)^k | p(x) \Rightarrow (x-1)^k | \sum_{i=1}^{n} x^{\alpha_i} - \sum_{i=1}^{n} x^{\beta_i}$

□

*Proof.* (c) $\Rightarrow$ (a)

Assume $(x-1)^k | \sum_{i=1}^{n} x^{\alpha_i} - \sum_{i=1}^{n} x^{\beta_i}$. Then $\sum_{i=1}^{n} x^{\alpha_i} - \sum_{i=1}^{n} x^{\beta_i} = (x-1)^k m$. Let $m = a_k + a_{k+1}(x-1) + \ldots + a_n(k-1)^{n-k}$. Then

$$\begin{aligned}(x-1)^k m &= (x-1)^k \left[a_k + a_{k+1}(x-1) + \ldots + a_n(k-1)^{n-k}\right] \\ &= a_k(x-1)^k + a_{k+1}(x-1)^{k+1} + \ldots + a_n(x-1)^n\end{aligned}$$

Define $p(x)$ as $p(x) = a_k(x-1)^k + a_{k+1}(x-1)^{k+1} + \ldots + a_n(x-1)^n$. Every polynomial can be rewritten using the Taylor Formula with $c = 1$.

$$\begin{aligned}a_0 &= p(1) = 0 && \Rightarrow p(1) = 0 \\ a_1 &= p'(1) = 0 && \Rightarrow p'(1) = 0 \\ a_2 &= \frac{p''(1)}{2!} = 0 && \Rightarrow p''(1) = 0 \\ \vdots &= \vdots \\ a_{k-1} &= \frac{p^{k-1}(1)}{(k-1)!} = 0 && \Rightarrow p^{k-1}(1) = 0\end{aligned}$$

By assumption $p(x) = \sum_{i=1}^{n} x^{\alpha_i} - \sum_{i=1}^{n} x^{\beta_i}$. So,

$$\begin{aligned}
p(1) &= \sum_{i=1}^{n}(1)^{\alpha_i} - \sum_{i=1}^{n}(1)^{\beta_i} = 0 \\
&\Rightarrow \sum_{i=1}^{n}(1)^{\alpha_i} = \sum_{i=1}^{n}(1)^{\beta_i} \\
p'(1) &= \sum_{i=1}^{n} \alpha_i(1)^{\alpha_i - 1} - \sum_{i=1}^{n} \beta_i(1)^{\beta_i - 1} = 0 \\
&\Rightarrow \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i \\
p''(1) &= \sum_{i=1}^{n} \alpha_i(\alpha_i - 1)(1)^{\alpha_i - 2} - \sum_{i=1}^{n} \beta_i(\beta_i - 1)(1)^{\beta_i - 2} = 0 \\
&= \sum_{i=1}^{n} \alpha_i(\alpha_i - 1) - \sum_{i=1}^{n} \beta_i(\beta_i - 1) = 0 \\
&= \sum_{i=1}^{n} \alpha_i^2 - \sum_{i=1}^{n} \alpha_i - \sum_{i=1}^{n} \beta_i^2 + \sum_{i=1}^{n} \beta_i = 0 \\
&\Rightarrow \sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} \beta_i^2
\end{aligned}$$

Similarly, for derivatives up to $k-1$ will yield,

$$\begin{aligned}
p^{k-1}(1) = &\sum_{i=1}^{n} \alpha_i(\alpha_i - 1)\ldots(\alpha_i - k + 2)(1)^{\alpha_i - k + 1} \\
&- \sum_{i=1}^{n} \beta_i(\beta_i - 1)\ldots(\beta_i - k + 2)(1)^{\beta_i - k + 1} = 0 \\
&= \sum_{i=1}^{n} \alpha_i(\alpha_i - 1)\ldots(\alpha_i - k + 2) - \sum_{i=1}^{n} \beta_i(\beta_i - 1)\ldots(\beta_i - k + 2) = 0 \\
&= \sum_{i=1}^{n} \alpha_i^{k-1} - c_1 \sum_{i=1}^{n} \alpha_i^{k-2} - \ldots - c_{k-1} \sum_{i=1}^{n} \alpha_i \\
&+ \sum_{i=1}^{n} \beta_i^{k-1} + c_1 \sum_{i=1}^{n} \beta_i^{k-2} + \ldots + c_{k-1} \sum_{i=1}^{n} \beta_i = 0 \\
&\Rightarrow \sum_{i=1}^{n} \alpha_i^{k-1} = \sum_{i=1}^{n} \beta_i^{k-1}
\end{aligned}$$

Putting together the results of these derivatives we get precisely (1), $\sum_{i=1}^{n} \alpha_i^j = \sum_{i=1}^{n} \beta_i^j$ for $j = 1, \ldots, k-1$. □

*Proof.* (b) ⇒ (a)
Assume $\deg \left[ \prod_{i=1}^{n} (z - \alpha_i) - \prod_{i=1}^{n} (z - \beta_i) \right] \leq n - k$. Let $p(x)$ and $q(x)$ be polynomials of the form:

$$p(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n) \qquad q(x) = (x - \beta_1)(x - \beta_2) \cdots (x - \beta_n)$$
$$= [x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_n] \qquad = [x^n + d_1 x^{n-1} + d_2 x^{n-2} + \ldots + d_n]$$

Then our assumption can be rewritten as
$\deg \left[ (x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_n) - (x^n + d_1 x^{n-1} + d_2 x^{n-2} + \ldots + d_n) \right] \leq n - k$
Since the degree must be less than or equal to $n - k$ the terms up to $n - k$ must cancel. Therefore

$$x^n = x^n$$
$$c_1 x^{n-1} = d_1 x^{n-1} \qquad \Rightarrow c_1 = d_1$$
$$c_2 x^{n-2} = d_2 x^{n-2} \qquad \Rightarrow c_2 = d_2$$
$$\vdots = \vdots$$
$$c_{k-1} x^{n-(k-1)} = d_{k-1} x^{n-(k-1)} \qquad \Rightarrow c_{k-1} = d_{k-1}$$

We can use Newtons Identities in order to rewrite each term.
For the first degree, where $s, c$ are coefficients for $p(x)$ and $t, d$ are for $q(x)$.

$$s_1 + c_1 = 0 \qquad t_1 + d_1 = 0$$
$$\Rightarrow s_1 = -c_1 \qquad \Rightarrow t_1 = -d_1$$

However, we saw that $c_1 = d_1 \Rightarrow s_1 = t_1$. So we get $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i$.

$$s_2 + c_1 s_1 + 2c_2 = 0 \qquad t_2 + d_1 t_1 + 2d_2 = 0$$
$$\Rightarrow s_2 = -c_1 s_1 - 2c_2 \qquad \Rightarrow t_2 = -d_1 t_1 - 2d_2$$

By the previous identity we know $-c_1s_1 = -d_1t1$ and by assumption $c_2 = d_2$. So we get $-2c_2 = -2d_2$ which means that $s_2 = t_2$. Which mens that $\sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} \beta_i^2$.

This will continue for $s_{k-1} = t_{k-1}$. So $\sum_{i=1}^{n} \alpha_i^i = \sum_{i=1}^{n} \beta_i^i$ for $j = 1, \ldots, k-1$ which is precisely (1). □

The previous theorem showed how the original Prouhet-Tarry-Escott Problem can be reduced into equivalent statements regarding polynomials. The first form is the most used version and because of this many properties have been discovered in order to help distinguish equivalent solutions from one another.

**Proposition 1.** *If* $[a_1, a_2, \ldots, a_m] =_k [b_1, b_2, \ldots, b_m]$ *for* $k = 1, \ldots, n$ *then*

$$[Sa_1 + T, Sa_2 + T, \ldots, Sa_m + T] =_k [Sb_1 + T, Sb_2 + T, \ldots, Sb_m + T]$$

*for* $k = 1, \ldots, n$

*Proof.* Assume $[a_1, a_2, \ldots, a_m] =_k [b_1, b_2, \ldots, b_m]$ for $k = 1, \ldots, n$.
When $k = 1$, by the addition and multiplication property of equality we get the two equivalent equations:

$$a_1, a_2, \ldots, a_m = b_1, b_2, \ldots, b_m$$
$$\Rightarrow (Sa_1 + T) + \ldots + (Sa_m + T) = (Sb_1 + T) + \ldots + (Sb_m + T)$$

Need to check when $k = n$ the equation $(Sa_1 + T)^n + \ldots + (Sa_m + T)^n = (Sb_1 + T)^n + \ldots + (Sb_m + T)^n$ will hold. The left hand side of the equation yields:

$$\begin{aligned} LHS &= \left[\binom{n}{0}(Sa_1)^n + \binom{n}{1}(Sa_1)^{n-1}T + \ldots + \binom{n}{n}T^n\right] + \\ &\ldots + \left[\binom{n}{0}(Sa_m)^n + \binom{n}{1}(Sa_m)^{n-1}T + \ldots + \binom{n}{n}T^n\right] \\ &= \left[\binom{n}{0}(S)^n(a_1)^n + \binom{n}{1}(S)^{n-1}(a_1)^{n-1}T + \ldots + \binom{n}{n}T^n\right] + \\ &\ldots + \left[\binom{n}{0}(S)^n(a_m)^n + \binom{n}{1}(S)^{n-1}(a_m)^{n-1}T + \ldots + \binom{n}{n}T^n\right] \\ &= \binom{n}{0}S^n\left[a_1^n + \ldots + a_m^n\right] + \binom{n}{1}S^{n-1}T\left[a_1^{n-1} + \ldots + a_m^{n-1}\right] + \ldots + \binom{n}{n}mT^n \end{aligned}$$

Similarly on the right hand side of the equation we get

$$\binom{n}{0} S^n \left[b_1^n + \ldots + b_m^n\right] + \binom{n}{1} S^{n-1}T \left[b_1^{n-1} + \ldots + b_m^{n-1}\right] + \ldots + \binom{n}{n} mT^n$$

. Since $S$ and $T$ are integers and by assumption we get the desired equality. □

A consequence of this proposition is the standard form of a solution. Standard form of a solution is when the sum of the $a$ terms and the sum of the $b$ terms is zero. This is considered an equivalent solution to the original solution. For example, take the solution $[0, 4, 7, 11] =_3 [1, 2, 9, 10]$. To the find the correct $S, T \in \mathbb{Z}$ we set one side equal to zero.

$$\begin{aligned} LHS = (M \cdot 0 + K) + (M \cdot 4 + K) + (M \cdot 7 + K) + (M \cdot 11 + K) &= 0 \\ 22M + 4K &= 0 \\ 4K &= -22M \\ K &= \frac{-11}{2}M \end{aligned}$$

Referring back to the original solution $[0, 4, 7, 11] =_3 [1, 2, 9, 10]$, we need to multiply each side by 2 and then subtract each number by 11. Multiplying each term by two yields $[0, 8, 14, 22] =_3 [2, 4, 18, 20]$. Finally subtracting each term by 11 gives the standard form which is $[-11, -3, 3, 11] =_3 [-9, -7, 7, 9]$.

**Proposition 2.** *If* $[a_1, a_2, \ldots, a_m] =_k [b_1, b_2, \ldots, b_m]$ *for* $k = 1, \ldots, n$ *then*

$$[a_1, \ldots, a_m, b_1 + T, \ldots, b_m + T] =_k [b_1, \ldots, b_m, a_1 + T, \ldots, a_m + T]$$

*for* $k = 1, \ldots, n + 1$.

*Proof.* Assume $[a_1, a_2, \ldots, a_m] =_k [b_1, b_2, \ldots, b_m]$ for $k = 1, \ldots, n$. By proposition 1, if we set $S = 1$ we get

$$[a_1 + T, \ldots, a_m + T] =_k [b_1 + T, \ldots, b_m + T]$$

for $k = 1, \ldots, n$.

When k=1, we see that using property 1 as well as the addition property we get:

$$a_1 + \ldots + a_m = b_1 + \ldots + b_m$$

$$(a_1 + T) + \ldots + (a_m + T) = (b_1 + T) + \ldots + (b_m + T)$$

$$\Rightarrow a_1 + \ldots + a_m + (b_1 + T) + \ldots + (b_m + T) = b_1 + \ldots + b_m + (a_1 + T), \ldots, (a_m + T)$$

Similarly, when $k = n$,

$$(a_1)^n + \ldots + (a_m)^n = (b_1)^n + \ldots + (b_m)^n$$
$$(a_1+T)^n + \ldots + (a_m+T)^n = (b_1+T)^n + \ldots + (b_m+T)^n$$
$$\Rightarrow (a_1)^n + \ldots + (a_m)^n + (b_1+T)^n + \ldots + (b_m+T)^n$$
$$= (b_1)^n + \ldots + (b_m)^n + (a_1+T)^n + \ldots + (a_m+T)^n$$

Need to check for $k = n+1$

$$\begin{aligned}
&(a_1)^{n+1} + \ldots + (a_m)^{n+1} + (b_1+T)^{n+1} + \ldots + (b_m+T)^{n+1} \\
&= (b_1)^{n+1} + \ldots + (b_m)^{n+1} + (a_1+T)^{n+1} + \ldots + (a_m+T)^{n+1} \\
(a_1)^{n+1} + \ldots + (a_m)^{n+1} &+ \left[\binom{n+1}{0} b_1^{n+1} + \binom{n+1}{1} b_1^n T + \ldots + \binom{n+1}{n+1} T^{n+1}\right] \\
&+ \ldots + \left[\binom{n+1}{0} b_m^{n+1} + \binom{n+1}{1} b_m^n T + \ldots + \binom{n+1}{n+1} T^{n+1}\right] \\
= (b_1)^{n+1} + \ldots + (b_m)^{n+1} &+ \left[\binom{n+1}{0} a_1^{n+1} + \binom{n+1}{1} a_1^n T + \ldots + \binom{n+1}{n+1} T^{n+1}\right] \\
&+ \ldots + \left[\binom{n+1}{0} a_m^{n+1} + \binom{n+1}{1} a_m^n T + \ldots + \binom{n+1}{n+1} T^{n+1}\right] \\
\Rightarrow \left[\binom{n+1}{1} b_1^n T + \ldots + \binom{n+1}{n+1} T^{n+1}\right] &+ \ldots + \left[\binom{n+1}{1} b_m^n T + \ldots + \binom{n+1}{n+1} T^{n+1}\right] \\
= \left[\binom{n+1}{1} a_1^n T + \ldots + \binom{n+1}{n+1} T^{n+1}\right] &+ \ldots + \left[\binom{n+1}{1} a_m^n T + \ldots + \binom{n+1}{n+1} T^{n+1}\right] \\
\Rightarrow \binom{n+1}{1} T\left[b_1^n + \ldots + b_m^n\right] + \ldots &+ \binom{n+1}{n} T^n \left[b_1 + \ldots + b_m\right] + m\left[\binom{n+1}{n+1} T^{n+1}\right] \\
= \binom{n+1}{1} T\left[a_1^n + \ldots + a_m^n\right] + \ldots &+ \binom{n+1}{n} T^n \left[a_1 + \ldots + a_m\right] + m\left[\binom{n+1}{n+1} T^{n+1}\right]
\end{aligned}$$

Setting $S = \binom{n+1}{i} T^i$ we can use property 1 together with the assumption to get the desired result. $\square$

This proposition shows that given a solution we can manipulate the data so that the solution is for that of one degree higher. However, in this process it is clear that if the given $T$ is not chosen wisely then the size of the solution can be double in the process. More specifically a solution of degree 3 with size 4 can be manipulated to a solution of degree 4 with size 8. In order to keep the sizes of these solutions down it is clear that choosing a $T$ such that some $a_i + T = a_j$ for some $i, j, \in \mathbb{Z}$ would decrease the size. To illustrate this begin with a solution $[1,5,10,16,27,28,38,39] =_6 [2,3,13,14,25,31,36,40]$ using the previous theorem let $T = 11$. The result is

$$[1,5,10,16,27,28,38,39,13,14,24,25,36,42,47,51]$$
$$=_7 [2,3,13,14,25,31,40,12,16,21,27,38,39,49,50]$$

Note the repeats can be cancelled out to get the final ideal solution

$$[1,5,10,24,28,42,47,51] =_7 [2,3,12,21,31,40,49,50]$$

It stands to reason that given a solution in the proper form we can get create a solution for a smaller degree. Escott noted that if you reversed the process stated in proposition 2 you could create another solution in a method he called integration. This method is stated more formally below.

**Proposition 3.** *If* $[a_1, a_2, \ldots, a_m] =_k [b_1, b_2, \ldots, b_m]$ *for some* $k, m \in \mathbb{Z}$ *and if* $a_i \equiv b_i \mod h$ *for* $i \in [1,2,\ldots,m]$ *with* $a_i < b_i$ *for* $i \in [1,2,\ldots,r]$ *and* $a_i > b_i$ *for* $i \in [r+1, r+2, \ldots, m]$, *then*

$$[a_1, a_1+T, a_1+2T, \ldots, b_1-T, \ldots, a_r, a_r+T, a_r+2T, \ldots, b_r-T]$$
$$=_{k-1} [b_{r+1}, b_{r+1}+T, \ldots, a_{r+1}-T, \ldots, b_m, b_m+T, \ldots, a_m-T]$$

The implications of this proposition are easy to see with an example. Consider the ideal solution $[1,5,9,17,18] =_4 [2,3,11,15,19]$ where $h = 7$. By the first condition $a_i \equiv b_i (\mod 7)$, so the solution has to be reordered in order to properly identify the $a_i's$ and $b_i's$, i.e. $[1,5,9,17,18] =_4 [15,19,2,3,11]$. Note that $a_i < b_i$ for $i \in [1,2]$, so in this example $r = 2$. Note that

$$a_1, a_1+T, a_1+2T, \ldots, b_1-T = 1, 8, 15, \ldots, 8$$
$$a_2, a_2+T, a_2+2T, \ldots, b_2-T = 5, 12, 19, \ldots, 12$$

Similarly,

$$b_{2+1}, b_{2+1}+T, \ldots, a_{2+1}-T = 2, 9, 16, \ldots, 2$$
$$b_{2+2}, b_{2+2}+T, \ldots, a_{2+2}-T = 3, 10, 17, \ldots, 10$$
$$b_{2+3}, b_{2+3}+T, \ldots, a_{2+3}-T = 11, 18, 25, \ldots, 11$$

Taking the first four smallest numbers from the first set and finding the corresponding solution in the next set we get the ideal solution $[1,5,8,12] =_3 [2,3,10,11]$.

**Proposition 4.** *If* $[a_1, a_2, \ldots, a_m] =_k [b_1, b_2, \ldots, b_m]$ *for* $k = 1, \ldots, 2n-1$ *then*

$$[T + a_1, \ldots, T + a_m, T - b_1, \ldots, T - b_m] =_k [T + b_1, \ldots, T + b_m, T - a_1, \ldots, T - a_m]$$

*for* $k = 1, \ldots, 2n$.

*Proof.* Assume $[a_1, a_2, \ldots, a_m] =_k [b_1, b_2, \ldots, b_m]$ for $k = 1, \ldots, 2n-1$. We need to verify that is it is for every $k = 1, 2, \ldots, 2n.\Rightarrow$

When $k = 1$,

$$\begin{aligned}
&(T + a_1) + \ldots + (T + a_m) + (T - b_1) + \ldots + (T - b_m) \\
&\qquad = (T + b_1) + \ldots + (T + b_m) + (T - a_1) + \ldots + (T - a_m) \\
&\Rightarrow 2mT + (a_1 + \ldots + a_m) - (b_1 + \ldots + b_m) = 2mT + (b_1 + \ldots + b_m) - (a_1 + \ldots + a_m) \\
&\qquad \Rightarrow 2(a_1 + \ldots + a_m) = 2(b_1 + \ldots + b_m) \quad \text{which is true by the assumption.}
\end{aligned}$$

We need to check when $k = 2n$,

$$\begin{aligned}
&(T + a_1)^{2n} + \ldots + (T + a_m)^{2n} + (T - b_1)^{2n} + \ldots + (T - b_m)^{2n} \\
&\qquad = (T + b_1)^{2n} + \ldots + (T + b_m)^{2n} + (T - a_1)^{2n} + \ldots + (T - a_m)^{2n}
\end{aligned}$$

The left hand side of the equation will yield:

$$\begin{aligned}
&\left[\binom{2n}{0}T^{2n} + \binom{2n}{1}T^{2n-1}a_1 + \ldots + \binom{2n}{2n}(a_1)^{2n}\right] + \\
&\ldots + \left[\binom{2n}{0}T^{2n} + \binom{2n}{1}T^{2n-1}a_m + \ldots + \binom{2n}{2n}(a_m)^{2n}\right] \\
&+ \left[\binom{2n}{0}T^{2n} - \binom{2n}{1}T^{2n-1}b_1 + \ldots + \binom{2n}{2n}(b_1)^{2n}\right] + \\
&\ldots + \left[\binom{2n}{0}T^{2n} - \binom{2n}{1}T^{2n-1}b_m + \ldots + \binom{2n}{2n}(b_m)^{2n}\right] \\
&\Rightarrow 2mT^{2n} + \binom{2n}{1}T^{2n-1}(a_1 + \ldots + a_m) + \binom{2n}{2}T^{2n-2}(a_1^2 + \ldots + a_m^2) + \\
&\ldots + \binom{2n}{2n}(a_1^{2n} + \ldots + a_m^{2n}) - \binom{2n}{1}T^{2n-1}(b_1 + \ldots + b_m) \\
&+ \binom{2n}{2}T^{2n-2}(b_1^2 + \ldots + b_m^2) - \ldots + \binom{2n}{2n}(b_1^{2n} + \ldots + b_m^{2n})
\end{aligned}$$

Similarly the right hand side of the equation will yield:

$$2mT^{2n} + \binom{2n}{1}T^{2n-1}(b_1 + \ldots + b_m) + \binom{2n}{2}T^{2n-2}(b_1^2 + \ldots + b_m^2) +$$
$$\ldots + \binom{2n}{2n}(b_1^{2n} + \ldots + b_m^{2n}) - \binom{2n}{1}T^{2n-1}(a_1 + \ldots + a_m)$$
$$+\binom{2n}{2}T^{2n-2}(a_1^2 + \ldots + a_m^2) - \ldots + \binom{2n}{2n}(a_1^{2n} + \ldots + a_m^{2n})$$

So every even term will cancel, to get

$$2\binom{2n}{1}T^{2n-1}(a_1 + \ldots + a_m) + \ldots + 2\binom{2n}{2n-1}T(a_1^{2n-1} + \ldots + a_m^{2n-1})$$
$$= 2\binom{2n}{1}T^{2n-1}(b_1 + \ldots + b_m) + \ldots + 2\binom{2n}{2n-1}T(b_1^{2n-1} + \ldots + b_m^{2n-1})$$

but by assumption $[a_1, a_2, \ldots, a_m] =_k [b_1, b_2, \ldots, b_m]$ for $k = 1, \ldots, 2n-1$
$\Rightarrow 2\binom{2n}{1}T^{2n-1}(a_1, a_2, \ldots, a_m) =_k 2\binom{2n}{1}T^{2n-1}(b_1, b_2, \ldots, b_m)$, so we get the desired conclusion.

□

This proposition states how starting with an odd degree solution can be manipulated to yield a complete solution to the Prouhet-Tarry-Escott Problem. These solutions can also result in an ideal solution if $m = n+1$ we can yield a solution of degree$2n+1$. For our example, $[0, 24, 33, 51] =_k [7, 13, 38, 50]$ for $k = 1, 3, 5$ where $n = 3$[Pie10]. Using the proposition we see that this odd leads to infinitely many complete solutions

$$[T+0, T+24, T+33, T+51, T-0, T-24, T-33, T-51]$$
$$=_5 [T+7, T+13, T+38, T+50, T-7, T-13, T-38, T-50]$$

If we let $T = 5$, then we get one of the many ideal solutions

$$[5, 29, 38, 56, 5, -19, -28, -46] =_7 [12, 18, 43, 55, -2, -8, -33, -45]$$

**Proposition 5.** *If* $[a_1, a_2, \ldots, a_m] =_k [b_1, b_2, \ldots, b_m]$ *for* $k = 2, 4, \ldots, 2n$ *then*

$$[T+a_1, \ldots, T+a_m, T-a_1, \ldots, T-a_m] =_k [T+b_1, \ldots, T+b_m, T-b_1, \ldots, T-b_m]$$

*for* $k = 1, \ldots, 2n-1$.

*Proof.* Assume $[a_1, a_2, \ldots, a_m] =_k [b_1, b_2, \ldots, b_m]$ for $k = 2, 4, \ldots, 2n$.

When $k = 2$,

$$(T+a_1)^2 + \ldots + (T+a_m)^2 + (T-a_1)^2 + \ldots + (T-a_m)^2$$
$$= (T+b_1)^2 + \ldots + (T+b_m)^2 + (T-b_1)^2 + \ldots + (T-b_m)^2$$
$$(T^2 + 2a_1 + a_1^2) + \ldots + (T^2 + 2a_m + a_m^2) + (T^2 - 2a_1 + a_1^2) + \ldots + (T^2 - 2a_m + a_m^2)$$
$$= (T^2 + 2b_1 + b_1^2) + \ldots + (T^2 + 2b_m + b_m^2) + (T^2 - 2b_1 + b_1^2) + \ldots + (T^2 - 2b_m + b_m^2)$$
$$2mT^2 + 2(a_1^2 + \ldots + a_m^2) = 2mT^2 + 2(b_1^2 + \ldots + b_m^2)$$

Check when $k = 2n-1$,

$$(T+a_1)^{2n-1} + \ldots + (T+a_m)^{2n-1} + (T-a_1)^{2n-1} + \ldots + (T-a_m)^{2n-1}$$
$$= (T+b_1)^{2n-1} + \ldots + (T+b_m)^{2n-1} + (T-b_1)^{2n-1} + \ldots + (T-b_m)^{2n-1}$$

So the left hand side of the equation yields,

$$\left[\binom{2n-1}{0}T^{2n-1} + \binom{2n-1}{1}T^{2n-2}a_1 + \ldots + \binom{2n-1}{2n-1}a_1^{2n-1}\right] + \ldots$$
$$+ \left[\binom{2n-1}{0}T^{2n-1} + \binom{2n-1}{1}T^{2n-2}a_m + \ldots + \binom{2n-1}{2n-1}a_m^{2n-1}\right]$$
$$+ \left[\binom{2n-1}{0}T^{2n-1} - \binom{2n-1}{1}T^{2n-2}a_1 + \ldots - \binom{2n-1}{2n-1}a_1^{2n-1}\right] +$$
$$\ldots + \left[\binom{2n-1}{0}T^{2n-1} - \binom{2n-1}{1}T^{2n-2}a_m + \ldots - \binom{2n-1}{2n-1}a_m^{2n-1}\right]$$

So all odd powers of $a_i$ will cancel out for all $1 \leq a_i \leq 2n-1$. So we get,

$$2m\binom{2n-1}{0}T^{2n-1} + 2\binom{2n-1}{2}T^{2n-3}(a_1^2 + a_2^2 + \ldots + a_m^2) +$$
$$\ldots + 2\binom{2n-1}{2n-2}T(a_1^{2n} + a_2^{2n} + \ldots + a_m^{2n})$$

Similarly the right hand side would be,

$$2m\binom{2n-1}{0}T^{2n-1} + 2\binom{2n-1}{2}T^{2n-3}(b_1^2 + b_2^2 + \ldots + b_m^2) +$$
$$\ldots + 2\binom{2n-1}{2n-2}T(b_1^{2n} + b_2^{2n} + \ldots + b_m^{2n})$$

The $2m\binom{2n-1}{0}T^{2n-1}$ would cancel. By assumption, $[a_1, a_2, \ldots, a_m] =_k [b_1, b_2, \ldots, b_m]$ for $k = 2, 4, \ldots, 2n$.
$\Rightarrow 2\binom{2k-1}{k}T^{2n-(k+1)}(a_1 + a_2 + \ldots + a_m) = 2\binom{2k-1}{k}T^{2n-(k+1)}(b_1 + b_2 + \ldots + b_m)$
To get the desired conclusion. □

This proposition is similar to the previous one except we are starting with an even solution. Similarly, starting with an even solution we can get a complete solution if $m = n + 1$ we will get a solution of degree $2n + 1$. For our example, $[2, 16, 21, 25] =_k [5, 14, 23, 24]$ for $k = 2, 4, 6$where $n = 3$[Pie10]. Using the proposition we see that this odd leads to infinitely many complete solutions

$$[T+2, T+16, T+21, T+25, T-2, T-16, T-21, T-25]$$
$$=_5 [T+5, T+14, T+23, T+24, T-5, T-14, T-23, T-24]$$

If we let $T = 3$, then we get one of the many ideal solutions

$$[5, 19, 24, 28, 1, -13, -18, -22] =_7 [8, 17, 26, 27, -2, -11, -20, -21]$$

# Chapter 5

# Prouhet-Tarry-Escott Problem Revisited

The following result is well known and it follows easily from Newton's identities.

**Proposition 6.** *Let* $p(x) = \prod_{i=1}^{s}(x - c_i)$ *and* $q(x) = \prod_{i=1}^{s}(x - d_i)$. *Then* $[c_1, c_2, \ldots, c_s] =_k [d_1, d_2, \ldots, d_s]$ *if and only if* $p(x) - q(x)$ *is a polynomial of degree* $s - k - 1$. *In particular if* $[c_1, c_2, \ldots, c_s] =_k [d_1, d_2, \ldots, d_s]$ *then* $s \geq k + 1$.

This proposition is a special case of Theorem 1 from the Basic Properties Chapter in which the proof was given.

The theorem below is attributed to Golden.

**Theorem 2.** *If* $[c_1, c_2, \cdots, c_{k+1}] =_k [d_1, d_2, \cdots, d_{k+1}]$ *is an ideal solution and* $\sum_{i=1}^{k+1} c_i = 0$, *then* $\sum_{i=1}^{k+1} c_i^{k+2} - \sum_{i=1}^{k+1} d_i^{k+2} = 0$.

*Proof.* The condition $\sum_{i=1}^{k+1} c_i = 0$ implies that $\sum_{i=1}^{k+1} d_i = 0$ and thus the corresponding polynomials, $p(x)$ and $q(x)$ from Proposition (6) are of the form $p(x) = x^{k+1} + a_{k-1}x^{k-1} +$

$\cdots + a_1x + a$ and $q(x) = x^{k+1} + a_{k-1}x^{k-1} + \cdots + a_1x + b$. Since $p(c_i) = 0$, we have

$$0 = \sum_{i=1}^{k+1} c_i p(c_i) = \sum_{i=1}^{k+1} c_i^{k+2} + a_{k-1}\sum_{i=1}^{k+1} c_i^k + \cdots + a_1 \sum_{i=1}^{k+1} c_i^2 + a\sum_{i=1}^{k+1} c_i$$
$$= \sum_{i=1}^{k+1} c_i^{k+2} + a_{k-1}\sum_{i=1}^{k+1} c_i^k + \cdots + a_1 \sum_{i=1}^{k+1} c_i^2$$

and similarly

$$0 = \sum_{i=1}^{k+1} d_i p(d_i) = \sum_{i=1}^{k+1} d_i^{k+2} + a_{k-1}\sum_{i=1}^{k+1} d_i^k + \cdots + a_1 \sum_{i=1}^{k+1} d_i^2$$

By subtracting the last two equalities we obtain $\sum_{i=1}^{k+1} c_i^{k+2} - \sum_{i=1}^{k+1} d_i^{k+2} = 0$. □

A simple observation that $[c_1, c_2, \ldots, c_s] =_k [d_1, d_2, \ldots, d_s]$ if and only if the function $F(x) = \sum_{i=1}^{s} e^{c_i x} - \sum_{i=1}^{s} e^{d_i x}$ satisfies $F(0) = F'(0) = \cdots = F^k(0) = 0$ prompts us to study the functions of the form $F(x) = \sum_{i=1}^{m} a_i e^{b_i x}$ where $a_i$'s and $b_i$'s are integers. In this chapter we will refer to $a_i$'s are the coefficients and to $b_i$'s as the nodes of $F(x) = \sum_{i=1}^{m} a_i e^{b_i x}$. Let $\Psi = \left\{ \sum_{i=1}^{m} a_i e^{b_i x} : a_i, b_i \in \mathbb{Z} \text{ and } \mathbb{N} \right\}$ denote the set of all such functions. If $F, G \in \Psi$, then clearly $F - G \in \Psi$ and $FG \in \Psi$, thus $\Psi$ is a commutative subring of $\mathbb{C}^\infty(\mathbb{R})$. A nonnegative integer $k$ will be called an order of $F \in \Psi$ if $F(0) = F'(0) = \cdots = F^k(0) = 0$. It will be convenient to assign order of $-1$ to $F \in \Psi$ for which $F(0) \neq 0$. The reader should keep in mind the equivalent definition of order that is, $F$ has order $k \geq 0$ if $\sum a_i = 0, \sum a_i b_i = 0, \ldots, \sum a_i b_i^k = 0$. Notice that according to our definition if $F$ is order $k$ then it is also of every order that is smaller than $k$. The number of distinct nodes, $b_i$'s, will be called the size of $F$ and by $\|F\| = \sum_{i=1}^{m} |a_i|$ we denote the norm of $F$. From Proposition (6) it follows that $\|F\| \geq 2k+2$ and thus an ideal solution of Prouhet-Tarry-Escott problem of order $k$ equivalent to finding a function $F \in \Psi$ of order $k$ with $\|F\| = 2k+2$.

**Proposition 7.** *If $F$ is of order $k$, then the size, $m$, of $F$ satisfies $m \geq k+2$.*

*Proof.* If $c_i$'s are distinct integers then the vectors $\left\{c_i := \left[1, c_i, c_i^2, \ldots, c_i^k\right]\right\}_{i=1}^{k+1}$ are linearly independent. This follows easily from the fact that the determinant of a matrix whose column vectors are $\{c_i : i = 1, \ldots, k+1\}$ is Vandermonde determinant and this it is not equal to 0. Now the system $\sum a_i = 0, \sum a_i b_i = 0, \ldots, \sum a_i b_i^k = 0$ is equivalent to $\sum a_i \mathbf{b}_\mathrm{i} = \mathbf{0}$; thus if there are fewer than $(k+2)$ distinct $b_i$'s then each $a_i = 0$ making $F \equiv 0$. □

The following Leibniz rule for higher order differentiation will be frequently used.

**Proposition 8.** *If $f$ and $g$ are two functions that are $n$ times differentiable at $x$, then $fg$ is also $n$ times differentiable at $x$ and*

$$(fg)^n(x) = \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(x) g^{(n-i)}(x)$$

**Proposition 9.** *If $F$ is of order $k$ and $G$ is of order $r$, then $FG$ is order $k+r+1$.*

*Proof.* First assume that $k \geq 0$ and $r \geq 0$. By Leibniz rule,

$$(FG)^{(k+r+1)}(0) = \sum_{i=0}^{k+r+1} \binom{k+r+1}{i} F^{(i)}(0) G^{(k+r+1-i)}(0).$$

If $i \leq k$, then $F^{(i)}(0)$, while if $i > k$ then $k+r+1-i \leq r$ and hence in this case $G^{(k+r+1-i)}(0) = 0$. Similarly one shows that $FG^{(l)}(0)$ for all $0 \leq l \leq k+r$. If $k \geq 0$ and $r = -1$, then $FG$ is of order $k$ follows from Leibniz rule again. If $k = -1, r = -1$ then $FG$ is at least of order $-1$ by definition. □

**Proposition 10.** *If $F(x) = \sum_{i=1}^{m} a_i e^{b_i x}$ is of order $k$ then for any integers $c$ and $d$ the function $G(x) = \sum_{i=1}^{m} a_i e^{(db_i + c)x}$ is also of order $k$.*

*Proof.* Write $G(x) = \sum_{i=1}^{m} a_i e^{db_i x} e^{cx}$. Then by Leibniz rule,

$$G^{(l)}(0) = \sum_{i=0}^{l} \binom{l}{i} d^l \sum_{i=1}^{m} a_i b_i^l c^{l-i} = \sum_{i=0}^{l} \binom{l}{i} d^l c^{l-i} F^{(l)}(0)$$

which is 0 for $0 \leq l \leq k$. □

It follows from Proposition (9) that if $F$ has order $k$ then $FG$ has order at least $k$. The next result is also attributed to Gloden.

**Corollary 1.** *If* $[c_1, c_2, \cdots, c_{k+1}] =_k [d_1, d_2, \cdots, d_{k+1}]$ *is an ideal solution,* $c = \sum_{i=1}^{k+1} c_i$ *then* $F(x) = \sum_{i=1}^{k+1} e^{((k+1)c_i - c)x} - \sum_{i=1}^{k+1} e^{((k+1)d_i - c)x}$ *is also an ideal solution and moreover* $F^{(k+2)}(0) = 0$.

*Proof.* By Proposition (10) $F(x)$ is also an ideal solution. The nodes, $\{(k+1)c_i - c\}_{i=1}^{k+1}$ of $F(x)$ satisfy $\sum_{i=1}^{k+1}((k+1)c_i - c) = (k+1)\sum_{i=1}^{k+1} c_i - \sum_{i=1}^{k+1} c = 0$ so by Theorem (2) $F^{(k+2)} = 0$. □

An immediate consequence is the following corollary.

**Theorem 3.** *Let* $F(x) = \sum_{i=1}^{m} a_i e^{b_i x}$ *be of order* $k$ *but not of order* $k+1$*. Then there is a unique number* $c$ *such that* $G(x) = \sum_{i=1}^{m} a_i e^{(k+1)b_i x - cx}$ *is of order* $k$ *and* $G^{(k+2)}(0) = 0$*. Moreover* $c = \frac{(k+1)F^{(k+2)}(0)}{(k+2)F^{(k+1)}(0)}$.

*Proof.* It follows from Proposition (10) that $G(x)$ is of order $k$ for any $c$. By Leibniz rule,

$$\begin{aligned} G^{(k+2)}(0) &= \sum_{i=0}^{k+2} \binom{k+2}{i} (k+1)^i F^{(i)}(0)(-c)^{k+2-i} \\ &= (k+1)^{k+2} F^{(k+2)}(0) - c(k+2)(k+1)^{k+1} F^{(k+1)}(0) \end{aligned}$$

which is 0 if and only if $c = \frac{(k+1)F^{(k+2)}(0)}{(k+2)F^{(k+1)}(0)}$. □

The next Corollary is an immediate consequence of Corollary (1) and Theorem (3).

**Corollary 2.** *If* $F(x) = \sum_{i=1}^{m} a_i e^{b_i x}$ *is an ideal solution of order* $k$*, then*

$$(k+1)\sum_{i=1}^{m} a_i b_i^{k+2} = \frac{1}{2}\sum_{i=1}^{m} |a_i|\, b_i (k+2) \sum_{i=1}^{m} a_i b_i^{k+1}$$

*Proof.* By Corollary (1) $G(x) = \sum_{i=1}^{m} a_i e^{b_i x - cx}$ where $c = \sum_{i=1}^{k+1} c_i = \frac{1}{2}\sum_{i=1}^{m} |a_i|\, b_i$ satisfies $G^{(k+2)}(0) = 0$. By Theorem (3) $c = \frac{(k+1)F^{(k+2)}(0)}{(k+2)F^{(k+1)}(0)}$. □

The following result might be interesting in its own.

**Lemma 1.** *If* $\sum_{i=1}^{m} a_i b_i^j = 0$ *for* $j = 0, 1, \ldots, k$ *and* $\sum_{l=1}^{n} c_l d_l^j = 0$ *for* $j = 0, 1, \ldots, r$ *then* $\sum_{i=1}^{m}\sum_{l=1}^{n} a_i c_l (b_i + d_l)^{k+r+2} = \binom{k+r+2}{k+1} \sum_{i=1}^{m} a_i b_i^{k+1} \sum_{l=1}^{n} c_l d_l^{r+1}$.

*Proof.* Let $F(x) = \sum_{i=1}^{m} a_i e^{b_i x}$ and $G(x) = \sum_{l=1}^{n} c_l e^{d_l x}$. By Leibniz rule,

$$FG^{(k+r+2)}(0) = \sum_{i=0}^{k+r+2} \binom{k+r+2}{i} F^{(i)}(0) G^{(k+r+2-i)}(0).$$

Since for $i \leq k$, $F^{(i)}(0) = 0$, and for $i > k+2$ then $(k+r+2-i \leq r), G^{(k+r+2-i)}(0) = 0$ the last equality simplifies to $FG^{(k+r+2)}(0) = \binom{k+r+2}{k+1} F^{(k+1)}(0) G^{(r+1)}(0)$. □

**Proposition 11.** *The function* $(e^x - 1)^{n+1}$ *is of order* $n$.

*Proof.* Let $F(x) = e^x - 1 = e^x - e^{0x}$. Then $F$ is of order 0. By Proposition (9) $F^2(x)$ is of order 1, $F^3(x) = F^2(x)F(x)$ is of order 2, and so on $F^{n+1}(x)$ is of order $n$. □

Since the nodes of $(e^x - 1)^{n+1}$ are $0, 1, \ldots, (n+1)$ by Proposition (7) it follows, that the maximum order of $(e^x - 1)^{n+1}$ is $n$. In fact the following is true.

**Proposition 12.** *Let* $F(x) = (e^x - 1)^{n+1}$. *Then* $F^{(n+1)}(0) = n+1!$.

*Proof.* Let $G(x) = (e^x - 1)^n$ and $g(x) = e^x - 1$. By Leibniz rule,

$$F^{(n+1)}(0) = \sum_{i=0}^{n+1} \binom{n+1}{i} G^{(i)}(0) g^{(n+1-i)}(0).$$

Since $G$ is of order $n-1$, the last equality reduces to

$$\begin{aligned} F^{(n+1)}(0) &= G^{(n+1)}(0)g(0) + (n+1)G^{(n)}(0)g'(0) \\ &= (n+1)G^{(n)}(0) \end{aligned}$$

Now the proof can be completed by induction on $n$. □

**Proposition 13.** *Let* $F(x) = (e^x - 1)^{n+1}$. *Then* $\|F\| = 2^{n+1}$.

*Proof.* From $F(x) = \sum_{i=0}^{n+1} \binom{n+1}{i} (-1)^{n+1-i} e^{ix}$ we get

$$\|F\| = \sum_{i=0}^{n+1} \binom{n+1}{i} = (1+1)^{n+1} = 2^{n+1}$$

□

**Proposition 14.** *A product of $r$ consecutive integers is divisible by $r!$.*

*Proof.* We have to show that if $r$ is a positive integer, then $r! \,|\, (t+1)(t+2)\cdots(t+r)$ for any integer $t$. The proof is by induction on $r$. Clearly it is true for $r = 1$. Now assume that there is an $r \geq 1$ for which the statement is true. Let $p(t) = (t+1)(t+2)\cdots(t+r)(t+r+1)$. From the identity $p(t+1) - p(t) = (r+1)(t+2)\cdots(t+r+1)$ and the induction hypothesis $r! \,|\, (r+1)(t+2)\cdots(t+r+1)$, since the right hand side is a product of $r$ consecutive integers it follows that $(r+1)! \,|\, p(t+1) - p(t)$ for every $t$. If $t \geq 1$ we write $p(t) = (p(t) - p(t-1)) + (p(t-1) - p(t-2)) + \cdots + (p(1) - p(0)) + (r+1)!$ and because $(r+1)!$ divides every term on the left hand side we have $(r+1)! \,|\, p(t)$. If $t < 0$ we write $p(t) = (p(t) - p(t+1)) + (p(t+1) - p(t+2)) + \cdots + (p(-1) - p(0)) + (r+1)!$ and again we conclude that $(r+1)! \,|\, p(t)$. □

**Theorem 4.** *If $F$ is order $k$ then $(k+1)!$ divides $F^{(r)}(0) \left(= \sum a_i b_i^{(r)}\right)$ for every non-negative integer $r$.*

*Proof.* If $r \leq k$ this is obvious since in the case $F^{(r)}(0) = 0$. Assume that $r > k$. Let $p_r(t) = (t+1)(t+2)\cdots(t+r) = \sum_{j=0}^{r} c_j t^j$. If $b$ is an integer then $p_r(b)$ is a product of $r$ consecutive integers and thus $r! \,|\, p_r(b)$. Thus $r! \,|\sum a_i p_r(b_i)$. But

$$\begin{aligned}
\sum a_i p_r(b_i) &= \sum a_i \sum_{j=0}^{r} c_j b_i^j \\
&= \sum_{j=0}^{r} c_j \sum a_i b_i^j \\
&= \sum_{j=0}^{k} c_j F^{(j)}(0) + \sum_{j=k+1}^{r-1} c_j F^{(j)}(0) + F^{(r)}(0) \\
&= \sum_{j=k+1}^{r-1} c_j F^{(j)}(0) + F^{(r)}(0)
\end{aligned}$$

We complete the proof by induction on $r > k$. For $r = k+1$ the previous equality simplifies to $\sum a_i p_r(b_i) = F^{(r)}(0)$ and hence $(k+1)! \,|\, F^{(r)}(0)$. If the statement is true for some $r \geq k+1$ then from $(r+1)! \,|\sum a_i p_{r+1}(b_i)$ the identity $\sum a_i p_{r+1}(b_i) = \sum_{j=k+1}^{r} c_j F^{(j)}(0) + F^{(r+1)}(0)$ and induction hypothesis it follows that $(k+1)! \,|\, F^{(r+1)}(0)$. □

**Corollary 3.** *If* $F(x) = \sum_{i=1}^{m} a_i e^{b_i x}$ *is an ideal solution of order* $k$*, then* $(k+2)!$ *divides* $F^{(k+2)}(0)$.

*Proof.* By Corollary (2) $(k+2)$ divides $F^{(k+2)}(0)$. By Theorem (4) $(k+1)!$ divides $F^{(k+2)}(0)$. □

**Theorem 5.** *A function* $F(x) = \sum_{i=1}^{m} a_i e^{b_i x}$ *where* $b_1 < b_2 < \cdots < b_m$ *is order* $k$ *if and only if* $F(x) = e^{b_1 x} \sum_{i=1}^{d} c_i (e^x - 1)^{k+i}$ *where* $c_i$ *are integers and* $d = b_m - b_1 - k$.

*Proof.* If $F(x) = e^{b_1 x} \sum_{i=1}^{d} c_i e^{d_i x}$ then by Proposition (11) and the comment after Proposition (9) the order of $F$ is at least $k$. Next assume that $F = \sum_{i=1}^{m} a_i e^{b_i x}$ has order $k$. First we will assume that $0 = b_1$. By Proposition (7), $b_m \geq k+1$ and by Proposition (12) $\frac{F^{k+1}(0)}{(k+1)!}$ is an integer $c_1$. Let $F_1(x) = F(x) - c_1(e^x - 1)^{k+1}$. By Proposition (12) $F_1^{(k+1)}(0) = F^{(k+1)}(0) - \frac{F^{k+1}(0)}{(k+1)!}(k+1)! = 0$ so $F_1(x)$ is of order $k+1$. Since the nodes of $c_1(e^x - 1)^{k+1}$ are $0, 1, 2, \ldots, k+1$, we have $F_1(x)$ has order $k+1$ and nodes between 0 and $b_m$. If $b_m \geq k+2$, the same argument applied to $F_1(x)$ shows that there is an integer $c_2$ such that $F_2(x) = F_1(x) - c_2(e^x - 1)^{k+2} = F(x) - c_1(e^x - 1)^{k+1} - c_2(e^x - 1)^{k+2}$ has order $k+2$ and nodes between 0 and $b_m$. Notice that continuing this argument the order is increasing by 1 while the nodes remain between 0 and $b_m$. We continue this argument until the order, $k+d$ of $F_d$ satisfies $k+d = b_m$. By Proposition (7) $F_d(x) \equiv 0$. Hence in this case $F(x) = \sum_{i=1}^{d} c_i (e^x - 1)^{k+i}$. If $b_1 \neq 0$, then we can apply the argument to $F(x)e^{-b_1 x}$, for the in this case $F(x)e^{-b_1 x}$ has order $k$ and its nodes that start at 0.

□

**Corollary 4.** *The coefficients* $a_i$*'s and the nodes* $b_1 < b_2 < \ldots < b_m$ *satisfy* $\sum_{i=1}^{m} a_i b_i^r = 0$ *for* $r = 0, 1, \ldots, k$ *if and only if* $\sum_{i=1}^{m} a_i x^{b_i} = x^{b_1}(x-1)^{k+1} p(x)$ *where* $p(x)$ *is a polynomial with integer coefficients.*

*Proof.* The proof follows immediately from Theorem (5) with the substitution $x = e^x$. □

If $F(x)$ is of order $k$, then by Proposition (9) $F(x)(e^x-1)$ is order $k+1$. The next corollary states that the converse also holds.

**Corollary 5.** *If* $F(x) = \sum_{i=1}^{m} a_i e^{b_i x}$ *where* $0 \le b_1 < b_2 < \cdots < b_m$ *is of order* $k$ *then there is* $G(x)$ *with nodes between* $b_1$ *and* $b_m + b_1 - 1$ *such that* $G$ *is of order* $k-1$ *and* $F(x) = G(x)(e^x - 1)$.

*Proof.* We can write $\sum_{i=1}^{m} a_i e^{b_i x} = \sum_{i=1}^{m} a_i (e^{b_i x} - 1) = (e^x - 1) \sum_{i=1}^{m} a_i \sum_{j=0}^{b_i - 1} e^{jx}$. Let $G(x) = \sum_{i=1}^{m} a_i \sum_{j=0}^{b_i-1} e^{jx}$. Then by Theorem (5) $(e^x-1)G(x) = e^{b_1 x} \sum_{i=1}^{d} c_i (e^x - 1)^{k+i}$. Thus $G(x) = e^{b_1 x} \sum_{i=0}^{d-1} c_i (e^x-1)^{k+i}$ and hence it is of order $k-1$. □

**Proposition 15.** *Let* $F(x) = (e^x - 1)^{n+1}$. *Then* $F^{(n+2)}(0) = \frac{n+1}{2}(n+2)!$.

*Proof.* The proof is by induction on $n$. For $n=1$, direct calculation shows that $F^{(3)}(0) = 2^3 - 2 = 6 = \frac{2}{3}3!$. We write $F(x) = G(x)(e^x - 1)$, then by Leibniz rule $F^{(n+2)}(0) = (n+2)G^{(n+1)}(0) + \binom{n+2}{2} G^n(0)$ and by Proposition (12) and the induction hypothesis $F^{(n+2)}(0) = (n+2)\frac{n}{2}(n+1)! + \frac{(n+2)(n+1)}{2} n! = \frac{n+1}{2}(n+2)!$ □

The following corollary should be compared to Corollary (3).

**Corollary 6.** *Let* $F(x)$ *be of order* $k$, *then for* $k$ *odd* $(k+2)!$ *divides* $F^{(k+2)}(0)$ *while for* $k$ *even* $\frac{1}{2}(k+2)!$ *divides* $F^{(k+2)}(0)$.

*Proof.* By Theorem (5) $F(x) = e^{b_1 x} \sum_{i=1}^{d} c_i (e^x - 1)^{k+i}$ and so

$$\begin{aligned} F^{(k+2)}(0) &= (e^{b_1 x}(c_1(e^x-1)^{k+1} + c_2(e^x-1)^{k+2}))^{(k+2)}_{|x=0} \\ &= c_1 \frac{k+1}{2}(k+2)! + c_2(k+2)! + (k+2)(k+1)!\, b_1 \end{aligned}$$

Now if $k$ is odd $(k+2)!$ divides the left hand side of the last equality, while if $k$ is even then $\frac{k+1}{2}$ is not an integer so the best that can be said in this case is that $\frac{1}{2}(k+2)!$ divides $F^{(k+2)}(0)$. □

The last two theorems use ideas from Wright [Wri59], to give a combinatorial solution to Prouhet-Tarry-Escott problem. Our results are slightly better than that of

Wright since unlike his solutions, we will show that there are solutions with distinct integers.

**Theorem 6.** *For every $k$ there is an $s \leq \frac{k(k+1)}{2}+1$ such that the system $[a_1, a_2, \ldots, a_s] =_k [b_1, b_2, \ldots, b_s]$ has an integer solution.*

*Proof.* Let $s$ be an integer to be determined later. To each collection of $s$ distinct numbers $\{a_1, a_2, \ldots, a_s\} \subset \{1, 2, \ldots, n\}$ we will assign an "address" $[\alpha_1, \alpha_2, \ldots, \alpha_k]$ where for $1 \leq i \leq k$ the $i$-th term is defined to be $\alpha_i = a_1^i + a_2^i + \ldots + a_s^i$. Since $\alpha_1 < sn, \alpha_2 < sn^2, \ldots, \alpha_k < sn^k$ there are fewer than $sn \times sn^2 \times \cdots \times sn^k = s^k n^{\frac{k(k+1)}{2}}$ different addresses. On the other hand there are $\binom{n}{s}$ different collections of $s$ distinct numbers $\{a_1, a_2, \ldots, a_s\}$. Thus if $\binom{n}{s} > s^k n^{\frac{k(k+1)}{2}}$ there will be at least two different collections sharing the same address. But two collections $\{a_1, a_2, \ldots, a_s\}$ and $\{b_1, b_2, \ldots, b_s\}$ share the same address if and only if $[a_1, a_2, \ldots, a_s] = [b_1, b_2, \ldots, b_s]$. So it remains to show that for sufficiently large $n$, and for $s < \frac{k(k+1)}{2} + 1$ the inequality $\binom{n}{s} > s^k n^{\frac{k(k+1)}{2}}$ has a solution. For fixed $s$ we can think of $\binom{n}{s} = \frac{(n)\times(n-1)\cdots\times(n-(s+1))}{s!}$ as a polynomial in $n$ and of degree $s$. Hence if this polynomial has a larger degree than the degree of the polynomial $s^k n^{\frac{k(k+1)}{2}}$ then for all large $n$ we will have $\binom{n}{s} > s^k n^{\frac{k(k+1)}{2}}$. But this is the case if $s = \frac{k(k+1)}{2} + 1$. Note that it is possible to have $s < \frac{k(k+1)}{2} + 1$ in the case that some of the terms in the two collections are the same. □

The last proof can be easily modified to give us the following stronger result.

**Theorem 7.** *For every $k$, and every $p$ there is $s \leq \frac{k(k+1)}{2} + 1$ such that the system $[a_{11}, a_{12}, \ldots, a_{1s}] =_k [a_{21}, a_{22}, \ldots, a_{2s}] =_k \cdots =_k [a_{p1}, a_{p2}, \ldots, a_{ps}] =_k$ has integer solutions.*

*Proof.* It is enought to prove that for $s \leq \frac{k(k+1)}{2} + 1$ the inequality $\binom{n}{s} > ps^k n^{\frac{k(k+1)}{2}}$ has a solution, for in this case there will be at least $p$ distinct collections, $\{a_{11}, a_{12}, \cdots, a_{1s}\}, \ldots, \{a_{p1}, a_{p2}, \cdots, a_{ps}\}$ to share the same address. But as in the proof of Theorem (6) this inequality will certainly have a solution if the degree, $s$ of the polynomial is greater than the degree, $\frac{k(k+1)}{2}$ of the polynomial $ps^k n^{\frac{k(k+1)}{2}}$. □

# Chapter 6

# Conclusion

The Prouhet-Tarry-Escott Problem is still unsolved and requires more research. This thesis has included the necessary background and history necessary to understand the origin of the problem. In addition it has shown the known properties the problem possess in order to show how new solutions can be acquired. The previous chapter showed the problem in a new perspective in order to find and verify new properties. The final theorem in the last chapter is a small contribution in the effort toward solving this problem. Though this result is only slightly stronger than Wright's, it shows how progress is being made toward the ultimate goal of solving this problem.

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