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A learning progression for identifying and placing fractions on a number line

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A LEARNING PROGRESSION FOR IDENTIFYING AND PLACING FRACTIONS ON A NUMBER LINE

A Project

Presented to the

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ABSTRACT

Many researchers and teachers have concluded that students have particular difficulties representing fractions on a number line. However, the number line is increasing in importance because of its prevalence within the soon-to-be adopted Common Core Content Standards for Mathematics in which number line fluency is expected from middle grades through upper division mathematics. Since students have some ease in identifying and understanding integer placement on a number line but struggle greatly with fraction placement, it is critical that teachers of both lower and upper grades embrace teaching fractions incorporating the number line. This project provides a learning progression for identifying and placing fractions on a number line that teachers can consider when introducing fractions on number lines. This progression was developed from observing and analyzing high school students' performance on an assessment consisting of various number line tasks involving fractions of various forms and from individual interviews with students regarding their assessment responses.

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CHAPTER ONE

INTRODUCTION

Purpose of Project

At the end of the 2010-2011 school year, I noticed my pre-algebra students, who had studied fractions and decimals the majority of the school year, could not do a problem on the final exam review that asked them to place a series of positive and negative fractions, decimals, and whole numbers on a number line. I was rather surprised and taken aback that even the highest performing students struggled greatly with this task. Consequently, I decided that I was going to incorporate the number line into my teaching the following year and see if the teaching would produce different responses from my students. Incorporating number line tasks with which students are not familiar, however, requires research and experimentation in order to present it to students in a way that is effective and can build upon their prior knowledge.

Thus, this project came about in order to determine and understand how I, and other teachers teaching rational numbers, should go about teaching fraction placement on the number line. Fractions – specifically proper, improper, and mixed fractions – were chosen as a more specific subset of rational numbers because it is evident from previous research and from colleagues in all education levels that students can fairly easily identify and place integers on a number line but have great difficulty placing rational numbers that are in fraction form.

Description of Project

This project was conducted by giving a group of pre-algebra students a written assessment with various number line and fraction tasks after the unit instruction on fractions was taught. The instruction of fractions was based off of a predicted learning progression and incorporated the various number line tasks that were presented to students on the written assessment. The written assessment was also given to AP calculus students, however, they did not receive the number line instruction that was given to the pre-algebra students. Following the assessments, interviews with pre-algebra students were conducted to determine a more clear understanding of their reasoning in their responses on the written assessment. Data were analyzed using Rasch's simple logistic model and compared with student interview responses to develop a learning progression for students placing and identifying fractions on a number line.

Significance of the Project

The number line is one of the most useful ways to represent all real numbers in relation to each other (Widjaja, Stacey, & Steinle, 2011). It allows students to experience the density of the number system since all real numbers – natural numbers, whole numbers, integers, rational numbers, and irrational numbers – have an exact and unique location on the number line (C. A. Pearn, 2007). Discrete models and hands-on manipulatives are effective and beneficial for conceptual knowledge and understanding, however they cannot capture the vast array of numbers that students need to learn and use throughout their education in mathematics and in real life situations (Widjaja et al., 2011). Therefore, having students familiarize themselves with the number line will ultimately improve their number sense in the present and long term.

In addition, "the understandings students construct about rational number[s] as well as their familiarity with number line conventions will provide them important resources for algebra in the middle and high school grades," (Saxe et al., 2007, p. 2). Throughout the middle and high school grades and also through the most advanced levels of mathematics, students graph whole number, rational number, and irrational numbers on both a single number line as well as in a Cartesian coordinate system where the coordinates of a point are the distances from a set of perpendicular lines (axes) that intersect at a point called the origin. Without familiarity in number line tasks and fraction placement on number lines, students will not be able to depict nor understand the entirety of graphical representations in algebra and throughout higher mathematics.

The number line is also important in measurement – a key concept throughout all grade levels as well as a subject used in daily living. Starting in elementary grades, the number line is used most prominently in finding length when discussing various forms of measurement. The popular ruler or tape measure is simply a number line that has partitions of fractions with powers of two (if measured in inches) or powers of ten (if measured in centimeters) as the denominators (i.e. for inches the partitions are halves, fourths, eighths, and sixteenths; for centimeters the partitions are tenths representing millimeters). In real life, lengths do not simply consist of whole number measurements but also of every form of rational number. For a rational number to be considered as a measurement, the distance from zero, it has a place on the number line (Lesh, Bradbard, ERIC Clearinghouse for Science, & Georgia Univ., 1976).

Due to its clear number representation and application within graphing and measurement, the soon-to-be-implemented Common Core Content Standards for Mathematics have also put a great emphasis on students being familiar with the number line. At the elementary levels, these standards recommend that instruction on fractional concepts incorporate fractions as points on the number line to emphasize relationships between fractions and whole numbers, fraction equivalence, and proficiency in adding, subtracting, multiplying, and dividing of proper, improper, and mixed fractions. Therefore, whether or not the number line is currently emphasized in mathematics instruction, these standards will soon require students to be familiar with number line and be able to use it to depict fraction relationships and concepts. Thus, identifying and placing fractions on a number line is of great significance for all levels of students. Students' knowledge of fractions and their position in relation to whole numbers and other fractions is necessary for success in algebra and higher levels of mathematics, as well as in demonstrating a basic understanding of measurement.

Research Questions

The main question that this project answers is, "What is a learning progression for placing and identifying fractions on a number line?" Although there are various ways a learning progression is defined, a *learning progression* (LP), according to the National Research Council, is specifically a description of "the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic" (as cited in Battista, 2011, p. 508). In more general terms, a LP describes students' different ways of reasoning about a certain subject, regardless of curriculum. Moreover, "it focuses on understanding and reacting to students' current cognitive structures," and then provides some sequence in which aspects of a subject should be taught to most benefit learners, very similar to a learning trajectory (Battista, 2011, p. 513). Therefore, this project presents a learning progression for placing and identifying fractions on a number line.

Another question that will be answered in the analysis of this project is, "What is the match between the predicted progression and the item difficulties along the Rasch logit scale?" A progression was predicted and used to teach the students before the assessment was given in the pre-algebra classes. The Rasch logit scale will help locate the items in terms of their difficulties and either support or suggest modifications to the predicted progression.

CHAPTER TWO

REVIEW OF LITERATURE

Current State of Affairs on Fractions and Number Lines

Much of the studies done on students' learning throughout primary and secondary grades have uncovered that fractions are one of the main subject areas in which students have difficulty understanding and becoming proficient (Behr & And Others, 1984; Misquitta, 2011). In a study done by Van Steenbrugge, Valcke, and Desoete, fractions, division, and problem solving were the three topics out of about 60 that were identified as most difficult for students to learn among all the mathematical topics the primary grades covered (Van Steenbrugge, Valcke, & Desoete, 2010, p. 65). Consequently, by the time most students reach high school, most knowledge and comprehension of fractions is forgotten, not practiced, or non-existent. In an error analysis by Brown, it reports that "students of age seventeen recurrently demonstrated a lack in proficiency with fractional concepts" (Brown & Quinn, 2006, p. 1). The study also noted that in an analysis by the 1990 NAEP mathematics achievement, "46 percent of all high school seniors demonstrated success with a grasp of decimals, percentages, fractions, and simple algebra." Furthermore, the National Mathematics Advisory Panel (NMAP) concluded that in 2008 nearly 50 percent of middle and high school students struggle with elementary fraction concepts and content. Likewise, in the past nine years I've been interacting with students, either through tutoring, substitute teaching, student teaching, or in a contracted teaching position, I've rarely come across students who feel confident and display proficiency in understanding and performing fractional computations. If students do possess some knowledge of fractions, their understanding is unfortunately "characterized by a knowledge of rote procedures, which are often incorrect, rather than by the concepts underlying the procedures," (Mack, 1990, p. 17) This is a major concern, especially, when such knowledge is expected to be common knowledge by the time students reach high school and are given algebraic problems involving numerous kinds of rational numbers and asked to plot rational numbers when graphing and representing solutions.

There have been numerous studies trying to answer the question "Why are fractions so difficult for students to learn?" (Behr & And Others, 1984; Brown & Quinn, 2006; Misquitta, 2011; Mitchell, 1990; Niemi, 1996; Van Steenbrugge et al., 2010; Widjaja et al., 2011). Gallistel and Gelman (as cited in Misquitta, 2011) reasoned that "because of students' previous experiences with counting in discrete wholes, students fail to see how fractions fit on the number line" and therefore do not understand them as numbers themselves but as two whole numbers; students do not consider numerators and denominators in relation to each other but as separate quantities that can be operated on (Misquitta, 2011, p.109; Moss & Case, 1999).

In addition, Kieren (1980) suggested that there are five fractional or rational number thinking patterns that students need to comprehend before they can comprehend rational numbers: "part-whole relationships, ratios, quotients, measures, and operators" (p. 134). From studies mentioned already, personal experience, and the experiences of other teachers, it is clear that most students do not have one or more of these foundational concepts, and consequently do not have fractional comprehension or computation skills.

Besides the various concepts needed to understand fractions, there are common traits that appear in many current teaching methods that may explain why students have difficulties learning fractional concepts. Moss and Case (1999) offer four specific reasons that may aid in understanding the confusion of students: syntactic versus semantic emphasis in training (too much teaching of procedural knowledge rather than conceptual knowledge), adult versus child-centered instruction (teachers presenting rote procedures rather than having students discover and make sense of these numbers themselves), use of representation in which rational and whole numbers are easily confused (such as the use of pie charts insufficiently differentiating the difference between whole and rational numbers), and problems with notation. With a revision of these four core instructional methods, Moss and Case (1999) stated that "children can be led to a deeper understanding of some aspects of the rational number system," (p. 124).

Surrounded by messages of teaching reform such as the above mentioned, teachers are often seeking new methods and curricula to help students grasp this difficult number category. Some of the specific methods teachers have used to present fractions to students have been through fraction bars, centimeter rods, pie charts, partitioning, paper folding, and the number line. The most popular and conventional among these methods have been the ones that have "focused on part-whole models of fractions (e.g. a pie). Part-whole illustrations make it easy to generate language about fractions because already acquired whole number language can be used, and part-whole models can be easily assimilated to children's counting schemas" (Niemi, 1996, p. 352; Psycharis, Latsi, & Kynigos, 2009). However, Gelman, Cohen, and Hartnett's 1989 study (as cited in Niemi, 1996) as well as Amato's study (2005) concluded that such representations detour one's understanding of fractions as numbers. Consequently, this leads "many educators to propose that quantitative models [such as the number line] be used to introduce the fraction concept," (Niemi, 1996, p. 352). Amato adds, however, that the use of a variety of experiences, not just the number line, with fractions equal to, greater than, and less than the unit will help students understand fractions as an extension to the number system (2005). Yet, the most widely adopted publishers for public elementary schools in California, i.e. Holt, Reinhart, and Winston; Hougton Mifflin Harcourt; Glencoe/McGraw Hill; Pearson Prentice Hall; and Scott Foresman, primarily use part-whole relationships and area models for teaching students in grades 4 through 8 learning fractional concepts. Little or no references are made for the use of number lines.

The unpopularity of the number line may be due to the fact that it is one of the more difficult methods for students to grasp (Bright, Behr, Post, & Wachsmuth, 1988; Kurt & Cakiroglu, 2009; Psycharis et al., 2009). This may be because it requires "an integration of two forms of information, visual and symbolic; this integration does not seem essential with other models" (Bright et al., 1988, p. 215). In Kurt and Cakiroglu's study (2009) analyzing how well students could translate between four representations of fractions – a numeric symbol, discrete objects, a region model, and the number line – the five most difficult translations of representations out of the twelve all included the number line. The easiest, however, were translating between numeric symbols and discrete models. This corresponds to the observations made among the pre-algebra students in this project mentioned earlier.

Another reason for the unpopularity of the number line could be that students struggle with the fundamental notion that "the unit for the number line, once chosen, can be divided into any number of congruent parts," (Lesh et al., 1976, p. 124). The misunderstanding of the "unit" on the number line can also be attributed to language being used at the elementary levels; "the word 'unit' is substituted by the word 'whole'" when learning fractions as part-whole relationships (Amato, 2005, p. 52). Therefore, the difference in language used within a number line system versus part-whole relationships needs to be reconciled for students to begin understanding fractions on a number line.

Although the number line is not popular among the methods for teaching fractions, the soon-to-be-implemented Common Core Content Standards for Mathematics features the number line's prominent use starting in second grade (standard 2.MD.6). In the third grade, students represent unit fractions (fractions with a numerator of 1) on a number line by defining the interval from 0 to 1 as the whole and then partitioning it into as many parts as the denominator (standard 3.NF.2). Continuing through sixth grade, the number line is incorporated in representing equivalence among fractions and between fractions and decimals as well as a model to understand operations on fractions. Therefore, it is important for teachers to begin embracing the use of the number line more heavily in their mathematics teaching and modeling of fractions. Also, it is important for teachers to begin embracing the reality that students need to be given tasks involving fractions on a number line so that the number line can become something students can use on a regular basis in their understanding of numbers. This brings me to my specific topic of placing fractions on the number line.

Previous Studies with Fractions and Number Lines

Specifically regarding number lines and fractions, four articles were uncovered that asked students to locate and identify fractions on a number line. One article titled *Identifying Fractions on Number Lines* (Bright et al., 1988) conducted three experiments where each was a series of a pre-test followed by instruction followed by a post-test and interviews. Experiments 1 and 2 included five to eight fourth and fifth grade students while experiment 3 included a whole class of 30 students. The experiments included multiple choice tests and interviews. In Experiment 1, instruction focused on point representation on a number line and making equivalent fractions on the number line. The test items included a combination of several variations. One variation gave a numerical fraction and asked students to identify the model (pie chart or number line marking) that corresponded to the fraction. The reverse was given as well – the model was given and asked the students to identify the numerical fraction that corresponded to the model. Another variation included a number line marked with consecutive units from 0 (i.e. 0, 1, 2, ...) versus a number line marked with only every 2 units from 0 (i.e. 0, 2, 4, ...). Finally, representation on a number line that showed a reduced fraction versus an unreduced fraction was another variation of test items. Results showed that students could not choose a reduced fraction name when an unreduced representation on a number line was shown and finding equivalent fractions from reduced to unreduced was done without any usage of the number line.

In Experiment 2, instruction focused again on fraction equivalence and point representation on a number line but built upon translation between discrete area models and number lines. The test items included area models and more number line placement and equivalence questions with different partitioning and unit labeling. The results once again showed that students had most difficulty with unreduced fraction representation on a number line. The final experiment was a large group experiment where the instruction and test given were identical to that of Experiment 2 to show the difference between small group and whole class instruction. The results of this test showed several things: items of number lines with consecutive units marked were easier than items of number lines with every 2 units marked; items where the numerical fraction was given were easier than items where representation was given; items where students had to create or add in partitions were harder than when students did not. Common errors that appeared among students included using the wrong unit, counting marks instead of intervals, and representing the reciprocal of the fraction.

The second article, titled *Identify Fractions and Decimals on a Number Line* (Shaughnessy, 2011), outlines a series of concepts that students must master in order to be able to identify decimals and fractions on the number line. This article summarizes results from a study by Saxe et al., (2007) titled *Learning about fractions as points on a number line*. The main concepts that teachers should focus on are as follows. First, students need to have an understanding of the number line itself. Students should understand that larger values are to the right and smaller values are to the left. Second, students need to understand that the number line is a measurement of distance, specifically defined by the unit distance between zero and one. After the unit distance is defined, and only after, can other values be found or placed on the number line. Third, students need to understand that the numerator and denominator in fraction have specific meaning: the denominator shows the number of equal parts there are in the unit and the numerator is how many parts it took to get to the target. A side note of this concept is that the decimal is just a special kind of fraction where the denominator is simply a power of ten. Fourth, students need to understand that the partitioning between zero and one needs to be equal for fractions to be identified. This can be assessed by teachers giving number lines with equal and unequal partitions to see if students are considering distance or if they are simply counting parts.

In this article, Shaughnessy outlines common errors that stem from misunderstanding the core concepts mentioned above such as counting marks rather than distances to name the denominator. Shaughnessy emphasized that teachers therefore need to be aware of these misconceptions and offer students a variety of number line tasks such as number lines that are both partitioned and not partitioned, tasks that give a fraction or a decimal and ask to locate it on the number line and also tasks that ask students to label a marked point.

In the third article, titled *Meanings for Fraction as Number-Measure by Exploring the Number Line* (Psycharis et al., 2009), a study is described that used digital media software to see if number line tasks could help students understand that fractions are static numbers as well as dynamic measurements from zero. The tasks included ordering, comparing, and performing operations on fractions. The results indicated that the use of the technology was helpful in student comprehension of fractions as both numbers and measurements on the number line since the technology was able to provide immediate feedback to the students and was able to be flexible in unit length representation.

Finally, the fourth article, titled *Whole Number Knowledge and Number Lines Help to Develop Fraction Concepts* (C. Pearn & Stephens, 2007), described a study that presented 5th and 6th grade students of three different schools with several number line tasks including locating fractions on number lines given whole numbers as well as identifying where the unit 1 is when given another fraction. Teachers of one of the three schools were given extensive professional development while the other two were not, and student performances were compared. The results showed that students of all three schools had more difficulty locating a fraction given consecutive units than locating the position of the unit 1 when given a fraction. Follow up interviews were also conducted with three students that included number line tasks mentioned above and questions that asked students to place whole numbers in relation to other whole numbers in order to see how well students could connect their whole number knowledge in a fraction context. Results of the interviews concluded that when students first identified whole numbers it benefited their understanding of fraction placement between 0 and 1. Also, students corrected some of their incorrect responses when they reviewed their written test they had taken previously at their own school.

What's Missing in the Current Research/Literature

In summary, there have been an abundant number of studies regarding rational number concepts and why students struggle with them. What is missing in the current research literature are studies done in which understanding of fractions on a number line are conducted at the high school level containing both high and low performing students. Also, there have not been any studies done where a learning progression is given for various types of fractions being placed or identified on various number line structures. One study used high school students as subjects but the purpose of that study was to discover students' understanding of fractions and operations with them (Brown & Quinn, 2006). Another study used college-aged adults but it tested whether the subjects could represent fractional magnitude in various forms (Schneider & Siegler, 2010). However, the majority of studies found regarding fractions were conducted at the elementary levels from grades two through sixth. This is not surprising because students are taught fractions at the elementary levels and aren't taught them extensively again in secondary courses. For many students, however, fractional computation is not learned, and therefore is having to be re-taught and incorporated throughout high school math lessons. I have yet to uncover a study pertaining to high school students placing fractions on a number line or understanding them in relation to a number line.

My Project Within the Existing Research/Literature

This project provided a new perspective and brought a fresh understanding of students' thinking and of their struggles in regard to fractional values on a number line. Since the students assessed in this project were in high school ranging from ninth to twelfth grade in a remedial pre-algebra class and an advanced placement calculus class, this study documented what low and high achieving students misunderstand about fractional values, specifically relating to their placement on a number line.

After reviewing the current research literature, I took the differing number line tasks with their reported levels of difficulty and variation and included them in this project's assessment. I also included some questions that were posed to students in the studies within interview questions. Student responses to both the written test and interviews were considered when a learning progression was developed.

CHAPTER THREE

METHODOLOGY

Setting and Participants

The project was conducted in a public high school in San Bernardino County in Southern California within three sections of pre-algebra courses and two sections of an advanced placement (AP) calculus course. There were 58 pre-algebra students and 17 AP calculus students who took the written assessment and three pre-algebra students were interviewed. Participants included 14 to 17 year old male and female students who were in grades 9 through 12. Students were predominantly of Hispanic or Latino ethnicity with a few Caucasian, African American, and Asian ethnicity. In addition, there was a wide range of abilities for mathematics among the students. A few had specific learning disabilities and a few were classified as gifted and talented students (GATE students).

The instruction on fractions was given over a time period of 3 months to the pre-algebra students and prior to the written assessment. The assessment was given to both the pre-algebra and AP calculus students individually during one class period with forty-five minutes allotted for students to take the test. The three interviews were conducted individually after school on three separate days and took between 30 and 60 minutes each.

The Design and Materials

To achieve the goal of identifying a learning progression for placing fractions on a number line, a 20-item assessment was developed that covered a variety of fractions and structures to determine what students could do.

Assessment Questions

Table 1 shown below is a chart of the various types of fractions with various structures in the assessment.

Table 1. Assessment Question Analysis

Assessment Question Analysis			**Various Kinds of Fractions**				
Format of questions			Proper fractions: Even Denominators	Proper fractions: Odd Denominators	Unreduced fractions	Mixed numbers	Improper fractions
Identify point given on number line	Equal Partitioning	0 – 1 labeling	1	1			
		0 – 2 labeling	1		1	1	1
	Unequal Partitioning	0 – 1 labeling	1	1		1	1
Place given fraction on number line	0 to 1 labeling		1	1	1		1
	0 to 2 labeling		1			1	1
	Placing fractions on same number line with 0 to 1 labeling		1		1	1	1
	Given fraction locate where 1 whole is.		1	1		1	

Note: The number in each box indicates the number of questions on the assessment that have the corresponding kind of fraction and number line format.

Six out of the 20 questions asked students to explain their reasoning for placing the given fraction or unit where they chose to do so. The other 14 items simply asked students to identify or place a fraction on a number line. A copy of the assessment as given to students is included in Appendix A.

Learning Progression Prediction

The following was my prediction of the order in which fractions on number lines should be taught to students:

1. Fractions between 0 and 1 with denominators of powers of 2: $\frac{1}{2}, \frac{3}{4}$, and $\frac{5}{8}$.
2. Fractions between 0 and 1 with denominators that are not powers of 2 including odd numbers and even multiples of 3: $\frac{1}{3}, \frac{3}{5}$, and $\frac{1}{6}$.
3. Mixed numbers combined with the same progression as 1 and 2 above for their fractional part.
4. Improper fractions combined with the same progression as 1 and 2 above for denominators.
5. Identify where 1 is in relation to a given fraction in the same progression as 1 and 2 above for types of fractions.

Regarding 1 and 2, fractions that have denominators with powers of 2 were predicted to be easier than those that do not because, for these fractions, students can partition the whole by creating half segments progressively. For example, to make halves, the whole is divided in half once; to make fourths, the whole is divided in half, and then in half again; to make eighths, the whole is divided in half, then half again, and then in half a third time; etc. Regarding the format of number lines, the following beliefs were an integral part of the study: (1) teaching students to name fractions that are given on a number line already partitioned for them should precede teaching students to create partitions themselves, (2) teaching students to identify and place a fraction on a number line where consecutive integers are marked should be taught before non-consecutive integers are marked, and finally (3) teaching students to identify fractions with equal partitioning given should precede ones with unequal partitions, leaving it up to the student to fill in the gaps with tick marks of their own.

Combining these beliefs and previously mentioned predictions together, I taught my students about fractions incorporating number line questions within the 'warm up' activities, notes, and homework questions. A log was kept of what number line and fraction structures I taught and my observation of student responses to verify that every variation of questions presented to students in the assessment was presented during instruction in order to give validity to the assessment results.

<u>Observations Prior to Assessment</u>

Prior to conducting this project's assessment, I made several observations as instruction with the number line began in the three pre-algebra classes. When the number line was first introduced in these pre-algebra classes with fractions, it was immediately apparent that students lacked familiarity with a number line as a form for representing fractions. For example, when the fraction $\frac{1}{2}$ was placed on the number line, several students asked, "Why is it between 0 and 1 if it has a 1 on top?" Yet when given a circle and told to represent $\frac{1}{2}$, many did so with ease. Furthermore, during a lesson on mixed numbers, students were asked to identify the fraction $2\frac{3}{4}$ when given pie graphs and then were asked to identify the same fraction when given it's placement on the number line (see Figure 1).

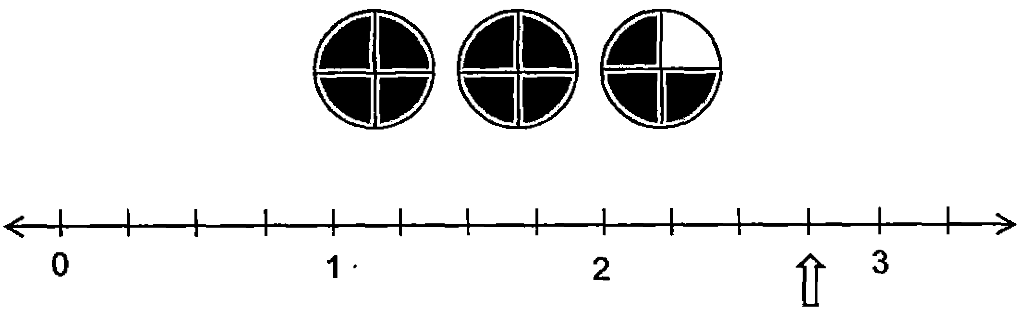


Figure 1. Pie Graphs Versus Number Line Representing $2\frac{3}{4}$

About 70% of students could identify the pie graphs value but only about 5% of students could state the number line value. It was apparent from the examples that not only were students extremely unfamiliar with the number line as a form of fraction representation but that they were also unable to state the value of a fraction in relation to whole numbers.

Another interesting observation was made regarding students' process for partitioning the number line. When partitioning the unit segment on a number line, many students tended to divide the segment into one additional segment than necessary. For example, when asked to plot the fraction $\frac{3}{4}$, students would draw four tick marks between 0 and 1 instead of three and plot the fraction on the third tick mark from 0 (see Figure 2):

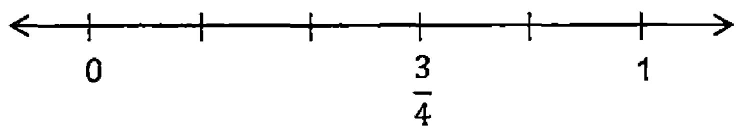


Figure 2. Inserting More Partitions than Necessary

This, however, creates five equal spaces instead of four. At first, this was a very common mistake among many students. It wasn't until the students were asked to label the tick marks to show that $\frac{4}{4}$ needed to be 1 that students realized their error and began making a conscious effort to draw the correct number of partitions needed to plot a fraction. This misconception was addressed several times throughout the rest of the instruction since it revealed that students were not making equal distances but rather simply counting tick marks to plot fractions.

Data Analysis and Procedures

Scoring the Assessment

After one month following the end of the instruction on fractions, students were given the assessment and their responses were scored using a rubric and point system outlined as follows. For the items that required students to identify the fraction to which the arrow pointed (questions 1 – 7) a score of 0 or 1 was given since these questions did not have varying levels of correctness. For the items that required students to locate the fraction (questions 8 – 14) a score of 0, 1, or 2 was given since these questions included a range of correctness (i.e. within a 25% error margin of placement). For the items that required explanation for fraction or unit placement (items 15 – 20), a score of 0, 1, or 2 was given for placing the fraction on the number line and a score of 0, 1, or 2 was given for the degree of correctness a written response gave. Consequently, items 15 – 20 were re-numbered as items 15 – 26 for the Rasch analysis where the odd items asked students to place the fraction or unit and the even items asked for the students' explanation for their placement. The rubric created to score these free response items was determined by viewing all varying responses that students gave on these questions. The assessment key and rubric used to score the items outlined in more detail is included in Appendix A.

Following the scoring of the assessments, the following descriptive statistics were obtained: item difficulties, item discriminations, reliability coefficient, and standard error of measurement. Student scores were further analyzed using Rasch's simple logistic model. Winsteps software program version 3.74.0 (Linacre, 2012) was used for the Rasch analysis. The following outputs were obtained: descriptive statistics, a variable map that shows the location of item difficulties and student abilities along a common interval scale, and item and student fit statistics.

Interviews

A month and a half after the assessment was given to the pre-algebra classes, semi-structured individual interviews were conducted with three students, A, B, and C, from these three classes. Students A, B, and C had varying levels of understanding in their respective order from below basic to proficient. In the interviews students were asked to explain their responses in as much depth and as clearly as possible. I often probed with questions to determine if they had transferred their understanding to other number line tasks with similar parameters. Students responded to the interview questions on the written assessment or on an individual whiteboard. The interviews were audio recorded so responses and answers could be documented accurately. The following are the types of questions that guided the interview:

1. What fraction would you say this point represents on the number line?
    a. Why do you think so? How do you know for sure?

   This process was repeated for equivalent and non-equivalent fractions between 0 and 1, fractions greater than 1 and 2, with equal partitions given, with non-equal partitions, and with no partitions to see if they could estimate what the point is by making their own partitions.

2. Show where [fraction] is on the number line.
    a. Please explain why and how you put it there.

   This process was repeated with varying fractions as listed in question 1 above but with fractions that were different from the ones given in question 1. As the student located the fraction, I closely observed how they partitioned the number line and how they counted any segments to locate the given fraction. I also carefully watched for common errors such as counting/marking partitions rather than distances or inversing the fraction.

3. If this fraction is given, where is 1 on the number line?
    a. Can you explain how you determined that?

This process was repeated for various fractions including unit fractions, mixed numbers and improper fractions.

4. What was easier for you?
    a. identifying this fraction (pointing out a specific fraction) or this one (pointing out another)?
    b. placing this fraction or this one?
    c. identifying this fraction or placing it without partitions?
    d. identifying this fraction or being given its location and finding 1?

These interview questions were asked to gain insights into what the students personally preferred as a possible learning progression, i.e. what they thought was the easiest and what was the hardest. These responses were considered and compared with the more formal findings from the data analysis of the assessment scores.

CHAPTER FOUR

RESULTS

The Rasch Analysis

The assessment scores were analyzed using the software program Winsteps, version 3.74.0 (Linacre, 2012) to obtain descriptive statistics as well as output tables showing item and student measures.

Descriptive Statistics

Examining the results from the Rasch analysis, the assessment had a person reliability of 0.91 indicating good separation among persons. Also, the mean measure for students was 0.30 logits higher than the mean calibration of the items (see Tables 2 and 3) indicating the average ability of the students was higher than the average difficulty of the items. The standardized fit statistics for items were within the acceptable range of -2.0 and +2.0. For additional descriptive statistics, refer to Appendix B for Tables 4, 5, and 6 that include the calibration of each item, the standard error of measurement, the fit statistics, and the point of measure correlation for both items and students.

The mean raw score of pre-algebra and calculus students was 24.3 out of a total of 45 possible points (54% of the total); the median score was 26 (i.e. 57.8%). The maximum and minimum scores were 44 (i.e. 97.8%) and 1 (i.e. 2.2%) respectively. The minimum score, however, was that of a student who only answered the first 7 out of the 26 items. The standard deviation of the raw scores was 13.5.

TABLE 2. Summary of Measured Person

	TOTAL SCORE	COUNT	MEASURE	MODEL ERROR	INFIT MNSQ	INFIT ZSTD	OUTFIT MNSQ	OUTFIT ZSTD
MEAN	24.3	26.0	.30	.42	.97	.0	1.04	.1
S.D.	13.5	.0	1.64	.17	.28	.8	.48	.8
MAX.	44.0	26.0	3.56	1.02	1.93	2.1	2.74	2.0
MIN.	1.0	26.0	-3.59	.29	.50	-2.0	.18	-2.1

REAL RMSE .48 TRUE SD 1.56 SEPARATION 3.27 PERSON RELIABILITY .91
MODEL RMSE .46 TRUE SD 1.57 SEPARATION 3.45 PERSON RELIABILITY .92
S.E. OF PERSON MEAN = .19

DELETED: 1 PERSON
PERSON RAW SCORE-TO-MEASURE CORRELATION = .98
CRONBACH ALPHA (KR-20) PERSON RAW SCORE "TEST" RELIABILITY = .95

TABLE 3. Summary of Measured Item

	TOTAL SCORE	COUNT	MEASURE	MODEL ERROR	INFIT MNSQ	INFIT ZSTD	OUTFIT MNSQ	OUTFIT ZSTD
MEAN	69.2	74.0	.00	.23	.97	-.2	1.04	.2
S.D.	22.1	.0	.81	.05	.22	1.2	.38	1.0
MAX.	107.0	74.0	1.39	.38	1.57	2.5	1.90	1.9
MIN.	33.0	74.0	-2.22	.18	.68	-1.9	.48	-1.6

REAL RMSE .24 TRUE SD .77 SEPARATION 3.20 ITEM RELIABILITY .91
MODEL RMSE .23 TRUE SD .77 SEPARATION 3.34 ITEM RELIABILITY .92
S.E. OF ITEM MEAN = .16

UMEAN=.0000 USCALE=1.0000
ITEM RAW SCORE-TO-MEASURE CORRELATION = -.45
1924 DATA POINTS. LOG-LIKELIHOOD CHI-SQUARE: 2283.76 with 1806 d.f. p=.0000
Global Root-Mean-Square Residual (excluding extreme scores): .5329
Capped Binomial Deviance = .2059 for 518.0 dichotomous observations

When comparing the performance of the pre-algebra students separately from the AP calculus students, the pre-algebra students, i.e. students 1 – 58 (with student 9 not counted in the analysis due to being an outlier), averaged a raw score of 21.0 (i.e. 46.7%). The AP calculus students, i.e. students 59 – 75, averaged a raw score of 36.2 (i.e. 80.4%).

Item Difficulty Analysis

The overall difficulty of the items did not appear to increase as the assessment questions progressed, as was expected, but seemed to jump around the different categories of questions. The most difficult select response item, not considering the even numbered items 16 – 26 that asked students to explain their choice of placement, was item 3. This item gave students a number line labeled with integers 0 and 2 with partitions of halves that asked students to identify the fraction $1\frac{1}{2}$ or $\frac{3}{2}$. The easiest item, by far, was item 1 that gave students a number line labeled with consecutive integers and partitions of halves that asked students to identify $\frac{1}{2}$. The item that appeared in the middle of the logit scale was item 4. This item gave students a number line labeled with integers 0 and 2 and partitions of fourths and asked students to identify $\frac{1}{2}$.

Furthermore, an analysis of the item difficulty and student ability interval scale was done in three ways: 1) comparing items between the different format of number line tasks, 2) comparing the items with even versus odd denominators as appeared on the assessment, and 3) comparing items with consecutive integers versus non-consecutive integers (see Figures 3, 4, and 5).

```
74 PERSON   26 ITEM
--------------------------------------------------------------------------------

                         PERSON - MAP - ITEM
                             <more>|<rare>
 4                                 +
                           |
                           |
              60   64 T|
                           |
                           |
                           |
 3                         +
   03  62  63  65  68  |
                           |
                           |
       14  43  48  66  |
                   11  |
                           |
 2                 26 S+
                   19  |
                   72  |
                   02  |T
               20  57  |                                      E20
   34  46  55  61  75  |
                           |                                  E18,E24,E26
 1             59  70  +
           05  41  73  |S                                     E22
   16  42  50  67  69  |   A03                                E16
           04  49  74  |           B10,B11
           23  24  27  |           B09
           06  44  56 M|
                   51  |                            D19
 0             33  71  +M  A04
               35  36  |   A06
                   39  |   A05              C14,C17       D23,D25
               08  25  |
           28  29  45  |   A02     B08      C13,C15
               40  53  |   A07
                   31  |S
-1                 52  +                    C12           D21
                           |
       07  32  37  54 S|
                           |
               10  21  |T
                   58  |
       01  13  17  22  |
-2                         +
           18  38  47  |
                           |  A01
                           |
                           |
                           |
               12  15  |
-3                        T+
                           |
                           |
                           |
                   30  |
                           |
-4                         +
                    <less>|<frequ>
```

KEY

A = identifying fraction on a given number line with partitions.

B = Placing a fraction on 1 number line without partitions in relation to other fractions

C = Placing a given fraction on a number line without partitions.

D = Identifying 1 given another fraction.

E = Written response explaining placement of fraction or 1.

Figure 3. Variable Map of Student Measures and Item Calibration: Formatting Comparison

```
74 PERSON   26 ITEM
------------------------------------------------------------------------------

                          PERSON - MAP - ITEM
                              <more>|<rare>
 4                                  +
                         |
                         |
               60  64 T|
                         |
                         |
                         |
 3                       +
   03  62  63  65  68   |
                         |
                         |
       14  43  48  66   |
                   11   |
                         |
 2                 26 S+
                   19   |
                   72   |
                   02   |T
               20  57   |                                          20
   34  46  55  61  75   |
                         |                                          18,24,26
 1             59  70   +
           05  41  73   |S                                          22
   16  42  50  67  69   |   F03                                     16
           04  49  74   |                     G10,G11
           23  24  27   |   F09
           06  44  56 M|
                   51   |                     G19
 0             33  71   +M  F04
               35  36   |                     G06
                   39   |   F14,F17,F23,      G25
               08  25   |
           28  29  45   |   F08,F13,F15
               40  53   |   F07
                   31   |S
-1                 52   +                     G12,G21
                         |
       07  32  37  54 S|
                         |
               10  21   |T
                   58   |
       01  13  17  22   |
-2                       +
           18  38  47   |
                         |   F01
                         |
                         |
                         |
               12  15   |
-3                      T+
                         |
                         |
                         |
                   30   |
                         |
                         |
-4                       +
                  <less>|<frequ>
```

KEY

F = Fractions with even denominators

G = Fractions with odd denominators

Figure 4. Variable Map of Student Measures and Item Calibration: Even vs. Odd Denominators

```
74 PERSON 26 ITEM
--------------------------------------------------------------------------------

                         PERSON - MAP - ITEM
                             <more>|<rare>
 4                                 +
                         |
                         |
           60   64 T|
                         |
                         |
                         |
 3                       +
   03   62   63   65   68  |
                         |
                         |
        14   43   48   66  |
                       11  |
                         |
 2                     26 S+
                       19  |
                       72  |
                       02  |T
                  20   57  |                                                 20
   34   46   55   61   75  |
                         |                                                   18,24,26
 1                59   70  +
             05   41   73  |S                                                22
   16   42   50   67   69  |        I03                                      16
             04   49   74  |             J10,J11
             23   24   27  |             J09
             06   44   56 M|
                       51  |                                 K19
 0                33   71  +M       I04
                  35   36  |   H06
                       39  |   H05           J14,J17                          23,25
                  08   25  |
             28   29   45  |   H02           J08,J15         K13,
                  40   53  |   H07
                       31  |S
-1                     52  +                 J12                              21
                         |
        07   32   37   54 S|
                         |
                  10   21  |T
                       58  |
        01   13   17   22  |
-2                         +
             18   38   47  |
                         |   H01
                         |
                         |
                         |
                  12   15  |
-3                        T+
                         |
                         |
                         |
                       30  |
                         |
                         |
-4                         +
                    <less>|<frequ>
```

KEY

H = Number lines with consecutive integers with partitions

I = Number lines with non-consecutive integers with partitions

J = Number lines with consecutive integers without partitions

K = Number lines with non-consecutive integers without partitions

Figure 5. Variable Map of Student Measures and Item Calibration: Varying Number Line Scales

After analyzing these comparisons, several observations were made:

a) Identifying fractions with partitions given did not appear substantially more difficult or easier for students than placing fractions on a number line without partitions with only integers given.

b) The denominators being even or odd did not appear to make any difference in the item difficulty.

c) With the exception of the fraction $\frac{7}{8}$, unit fractions showed a strong correlation of being easier than any other types of fraction to identify, place, and be used to locate 1 on a number line.

d) After unit fractions, proper fractions were easier for students to place and identify than mixed numbers or improper fractions.

e) Improper fractions were generally easier for students to place or identify than mixed numbers.

f) Identifying and placing fractions on a number line where consecutive integers were marked was easier for students than identifying or placing fractions on a number line where non-consecutive integers were marked.

These observations were combined with the results from the student interviews in order to develop the progression that is outlined in a later section.

The Student Interviews

Interviewing students revealed more interesting observations in addition to some misconceptions. Many times, however, when the students were probed on their reasoning and were referred back to previous answers and reasons, they corrected themselves. It is interesting to note, however, some of their initial responses. For example, all three students initially failed to recognize the location of 1 in question 3 of the assessment to identify the fraction $1\frac{1}{2}$. This question included a number line showing partitions of halves with only 0 and 2 labeled (see Figure 6).

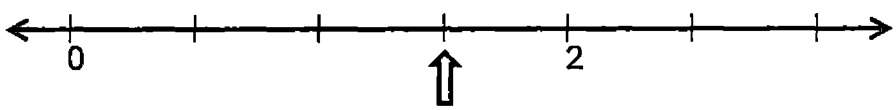


Figure 6. Assessment Question 3

Students B and C identified it as $\frac{3}{4}$ and student A identified it as $1\frac{3}{4}$. When student C was asked to explain his answer of $\frac{3}{4}$, he said "I broke it into fourths... one fourth, two fourths, three fourths, and four fourths, and so three-fourths is right there" (student C, personal communication, March 2012).

I replied with, "Ok, and even though this is a two, does that matter, versus a one?"

Student C answered, "no..., I mean like, you basically just double the denominator".

"Ok, and so you say, this is one fourth, two fourths, three fourths, and four fourths, so..."

"Yeah, one whole."

"One whole? So the two is the same thing as one whole?"

"Yeah."

Later in referencing assessment question 13, student C realizes his error and locates 1 before counting the partitions and concludes the fraction is $1\frac{1}{2}$.

Student B also confirmed that the fraction was $\frac{3}{4}$ because "it is broken up into four pieces" (student B, personal communication, March 2012). When I asked, "Would it matter, then, if [the 2] was a 1 instead of a 2?" student B simply responded, "uhh...no." Unlike student C, student B did not self-correct herself.

When student A was asked to explain his answer of $1\frac{3}{4}$, he responded that he saw it was "before the 2" so he knew it had to be "after 1" (student A, personal communication, March 2012). When I questioned him about his answer of $\frac{3}{4}$, he noticed his error and corrected himself identifying 1 in the correct location and concluded the fraction was $1\frac{1}{2}$.

A misconception that appeared in two of the three students was the idea that reducing a fraction or changing the form of a fraction, i.e. improper to mixed, may change its location on the number line. For example, in question 11, which asked students to place $\frac{4}{3}$ on the same number line as the three previous questions, student B concluded the position correctly between 1 and 2. When asked if she could write this number differently, she divided the two numbers by doing the division algorithm and concluded $\frac{4}{3} = 1\frac{1}{3}$. When asked if she could show how it was both forms on the number line, she could not and soon concluded that they would be different places on the number line but didn't know how to draw it differently.

Similarly, for questions 8 and 9, student C partitioned the whole into six equal pieces and knew that to make twelve pieces, he simply needed to split each sixth into two pieces because "six times two is twelve....you just double the denominator," (student C, personal communication, March 2012). When I asked him where $\frac{4}{12}$ was he said, "one twelfth, two twelfths, three twelfths, four twelfths... which would be the same as two sixths."

I replied, "Same as two-sixths? Ok, could it be written as even something else?" He proceeds to reduce the fraction numerically and says, "Divide it by two...you get one-third."

"Ok, could you maybe picture this [the reducing] on the number line, too?"

"Yeah."

"How would you do that?"

"You break it down into thirds." He begins to break the whole into three parts making different partitions than the ones he already made for twelfths. As he's drawing, he proceeds to say, "one third, two thirds...."

I then question, "Drawing different lines?" He didn't respond but kept going. He continues again, "one third, two thirds, three thirds or one whole."

"Ok, this is one-third?" pointing to his most recent drawn tick mark he made for thirds as he affirms (see Figure 7). "And four twelfths, here, you're saying is two sixths, which is the same thing as one third, and you would say this is one third?" pointing to the recently drawn "one third" tick mark.

He pauses then hesitantly replies, "yeah, but...I tried to space them out,...but...one-third, yeah."

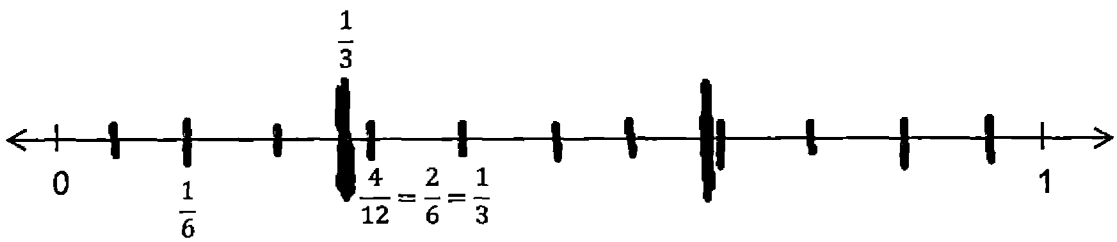


Figure 7. Reducing Fractions Creating Different Placement on a Number Line

I continued, "Ok, so both of them would be one third, then? Or would one of them be one third and the other would have to be something else?"

"They both would be one-third."

"But different spots on the number line?"

"Yeah, ...well because,....they have to be spaced out evenly."

Another observation that was made in the interviews was that all students did not have an efficient strategy for equally partitioning a number line. As noted in the data analysis, it didn't matter the size or parity of the denominators for the difficulty of the fraction; making fifths didn't appear to be any harder than making fourths. It was apparent that the students had not learned the strategy of partitioning fractions based on factors. For example, to make perfect fourths, one could find half first, and then take half of each of those to create four equal pieces. Also, to find sixths, one could find half and then break each of those halves into thirds. When student B was asked, "Do you have a strategy that you could use if you wanted to make these pieces exactly even? Meaning, instead of starting from the left, like you did, and working your way to the right, could you have done it a different way to try to get six equal pieces? In other words, could you start from the middle or from the right or anywhere else besides the left?" she replied, "Well, starting from the left is actually easier, so I can kinda, like, measure each tick mark," (student A, personal communication, March 2012).

When the same question was posed to student C, he answered, "You can put fingers in between the spaces to make them exactly even," (student C, personal communication, March 2012). He also stated that starting from the left, and specifically "from the whole", is the only way he would make partitions.

These students had no notion of partitioning by smaller factors and then partitioning the remaining spaces equally. Rather, they always began from the left counting as they went in hopes the right spacing appeared somewhat equal. And although they stated the tick marks had to be equally spaced, students A and C didn't seem to give much attention to their inaccurate spacing. Sometimes they erased and started again from the left to obtain better spacing, but they often eye-balled their ticks and left the spacing how it was if it "was close enough." From this, it is also evident that the idea of exact placement wasn't present in their method of partitioning.

Student Opinion on a Learning Progression

When asked what format they thought teachers should teach first when learning about fractions on a number line (i.e. partitions given, only integers given, finding 1 given a fraction, or placing various fractions on the same number line), students B and C said "the ones with the ticks," or partitions given, should probably be given first so students can simply count instead of having to draw tick marks. Student A said placing them where there were no partitions given was easier than when given ticks because it's easy to "mess up counting like [he] did at first". All students were in agreement that placing several different fractions on the same number line was the hardest thing for them to do on the assessment.

When asked about what they thought about placing 1 when given a fraction, they all also agreed that "it's pretty easy," but going backwards (being given any fraction besides a unit fraction) instead of going forwards was the only "confusing part." This supports the data analysis that concluded identifying 1 when given a unit fraction was easier than identifying 1 when given a proper fraction (other than a unit fraction) or an improper fraction.

When asked if they thought there was an order teachers should teach the various types of fractions (i.e. proper fractions, improper fractions, and mixed numbers), student B said she thought the "popular fractions" such as $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}$, and $\frac{1}{4}$ should be taught first. She also said that improper fractions were easier for her than mixed numbers because you can just "keep on counting" on the number line. Students C said it he thought proper fractions were the "same difference" as improper fractions and mixed numbers; they were all "pretty easy" except that "non-reduced fractions were easier than reduced fractions". From observing student C's work during the interview, what might have been meant by this last conclusion could be that the reducing process on the number line was harder than just plotting the non-reduced fraction.

CHAPTER FIVE

CONCLUSION

Developing a Learning Progression

The first and most critical concept that students must understand is the idea of the unit. As similarly concluded in Shaughnessy's article (2011), identifying the unit distance such as from zero to one, or its equivalent, must be done before identifying or placing any other fraction on a number line. With this concept embedded in every placement or identification of any fraction on the number line, the order in which the different types of fractions should be taught based on the assessment and interview analysis are1) unit fractions, 2) proper fractions, 3) improper fractions, and 4) mixed numbers. As noted in the assessment observation, whether the denominator is even or odd did not appear to show any correlation of difficulty. However, this is probably due to the fact that students did not have an efficient strategy to partition. Otherwise, even denominators would be far simpler to partition than odd because the half-distance can be located with ease. Therefore, when students begin to partition unit distance, it is important for teachers to give students a strategy to partition based on factors for students to more accurately understand fraction placement.

The progression of the various formats for number line tasks, as taken from the analyses, is as follows:

1) Identify fractions on a number line where partitions are given with consecutive units labeled.
2) Identify fractions on a number line where partitions are given with consecutive units labeled but with some partitions missing.
3) Identify the placement of 1 given a fraction.
4) Place a fraction on a number line without partitions but consecutive integers are labeled.
5) Place a fraction on a number line without partitions but given non-consecutive integers (i.e. 0, 2, 4...).
6) Place a variety of fractions on one number line labeled with consecutive integers.
7) Identify fractions on a number line where partitions are given with non-consecutive integers labeled.

Although the fraction type progression is listed separately from this format progression, the two progressions should be imbedded within each other. As various fractions are introduced to students, the various formats should be introduced in the order of the format progression. For example, unit fractions and then proper fractions should be placed and identified first with partitions and consecutive integers given, followed by partitions missing, then locating 1 given a fraction, then placing the fractions on a number line with consecutive integers but without partitions,...etc. As various types of fractions are introduced, the various formats are imbedded as well.

It is interesting to note that identifying the placement of 1 came more naturally to students when partitions were not given as oppose to when they were. For example, assessment question 13, where partitions were not given, was 0.5 logits lower than question 4, where partitions were given, and both of these questions asked for the same fraction, $\frac{1}{2}$, to be identified (see Table 5 in Appendix B). A hypothesis can only be given as a reason for this difference. Perhaps students are less inclined to locate 1 if the partitioning is already given; they may simply assume the whole is the given integer and then begin to count the equal spaces not observing that the whole number is 2 instead of 1. Rather, if they are given only 0 and 2 without partitions, the need for partitioning is natural and this may spur them to identify 1 first. This difference in formatting is something that can be pursued with further research.

There are a several differences between the data based learning progression and the predicted learning progression. First, unit fractions, rather than simply proper fractions with denominators of powers of 2, should be taught first. The data displayed that the parity of the denominators did not seem to make a difference in the difficulty of the questions. It is important to note that this idea is supported within the Common Core Content Standards for Mathematics. When fractions are heavily introduced in third grade, the first fractions that students must identify and begin placing on a number line after identifying the unit distance are unit fractions (standard 3.NF.2). Continuing to fourth grade, students begin building fractions from unit fractions by applying their previous understanding of operations on whole numbers (standard 4.NF.3). For example, students will decompose fractions by representing it as a sum of unit fractions (i.e. $\frac{3}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$).

Second, improper fractions appeared to be easier than mixed numbers, rather than the reverse. This conclusion, however, was not definitive and should be researched further with more assessment questions of varying formats to determine if this is true. Third, identifying 1 when given a fraction precedes placing fractions on number lines. This was expected to be more difficult than identifying or placing a fraction on a number line, but instead it was easier than most varying formats. Fourth, all number lines already partitioned did not necessarily precede number lines without partitions. The order appeared in the number line scaling of consecutive versus non-consecutive integers. In other words, having partitions or not did not make much of a difference in the difficulty but the labeling of consecutive integers versus non-consecutive integers did.

Finally, having equal partitions did not necessarily precede non-equal partitions. There was not a strong difference in difficulty between questions that provided all the tick marks between integers and those that did not. Students seemed to automatically account for the unequal spacing in their identification of a fraction.

Limitations of Project

It is important to note that the results of the assessment and interviews were heavily contingent upon the instruction students received. Although the various formats and types of fractions were given during the instruction, some may have been emphasized more than others unintentionally. Also, the strategy of using factors for partitioning fractions was demonstrated briefly during instruction, however, it was not taught specifically. If the strategy had been taught specifically, results from pre-algebra students may have differed. Furthermore, there are additional number line tasks that this learning progression does not include. Other formats of number lines and types of fractions such as the absence of zero and negative fractions were not explored within this project.

After analyzing the item calibrations in comparison with student ability measures, it was evident that there were high performing students whose abilities measured higher than the calibration of the most difficult item; there were no items targeted at the highest student ability levels. Similarly, for the low performing students, the fraction $\frac{1}{2}$ was the only fraction that appeared at their ability level. The items were not well targeted to the students, particularly at the upper and lower ends of the student ability distribution. Additional items that are easier and more difficult would have produced measures with lower measurement errors.

Limited time and human factors were also limitations to this project. Since this project had subjects that attended a public high school, there was limited access to the participants. The allotted 59 minutes of each period was all the time available with students to teach the number line, conduct the assessment, and schedule possible times to interview students. The time allotted to teach students number line tasks was limited because it was combined with teaching the regular course curriculum that did not incorporate the number line. Also, there were time limitations and scheduling difficulties for conducting student interviews because they were held on the students' own time after school and only on specified days available.

In addition, there were a large number of students who lacked motivation and interest for this study and submitted incomplete assessments; many students did not give explanations for their responses to questions 15 through 20 (items 16, 18, 20, 22, 24, and 26). Many students also rushed through the questions resulting in careless mistakes and unclear written explanations.

Summary

Although placing and identifying fractions on a number line proves unpopular and difficult, it is critical that students learn to do so in order to increase their understanding of fractions in relation to other numbers and also fraction equivalence. In order for students to learn how to identify and place fractions on a number line, instruction with the number line needs to begin as soon as fractions are introduced in the elementary levels emphasizing the foundational concept of the unit distance and its implications of measurement.

The number line's underlying application of measurement is crucial in a variety of work and everyday life activities. Since number line representation of fractions is often being overlooked at the lower levels this may be a key component attributing to students' overall lack of number sense. Hopefully, with the Common Core Standards quickly approaching school curriculum, which will require the use of number lines in representing all numbers, the unfamiliarity of the number line among students will begin to fade. Undoubtedly, students' comprehension of fractional values and their relationship to other numbers will increase as well.

Furthermore, the notion of equivalence between various representations of fractions remains to be a stumbling block in students' development of number sense. Within this project, the assessment questions that asked students to place different fractions on one number line was one task that highlighted this difficulty. Students struggled with understanding that equivalent numeric representation also meant equivalent graphical representation. Referring to Figure 7, it was evident that Student C realized there had to be some error for both these tick marks to be one-third, but he didn't seem to know how to fix it. He didn't seem to grasp the concept that partitions can be taken away just like they can be added to show equivalent fractions. The fact that if two numbers are numerically equal then its representation is equal was not understood by student C and student B.

This inflexibility of representation is not only seen with fractions but in higher levels of math as well. For example, when students are asked to find the output value of a function given an input value, they can compute the output value algebraically. However, when asked to show what this means graphically, they often cannot. I've heard it said by some former high-achieving students that they went through all four years of high school mathematics not knowing that one can tell what the graph of a function will look like based on its equation. This is most likely due to a lack of understanding of the relationship between the numeric or algebraic representation and the graphical representation. Continually embedding multiple representations and different interpretations of numbers and mathematical concepts, especially fractions, can aid in student comprehension of equivalence (Kurt & Cakiroglu, 2009). The number line is only one, but perhaps the most beneficial, of representations.

APPENDIX A

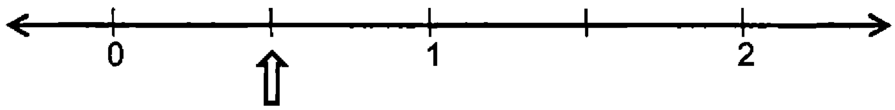
THE NUMBER LINE ASSESSMENT, ANSWER KEY, AND SCORING RUBRIC

Name: ____________________ Per.____

Identifying and Placing Fractions on a Number Line Assessment (by Corinne Marshall)

Name the fraction that is indicated by the arrow on the number line.

Answers:

1. 

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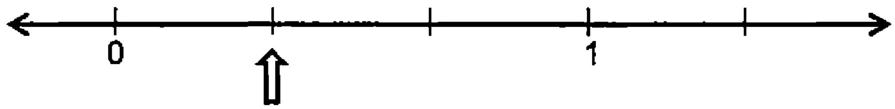
2. 

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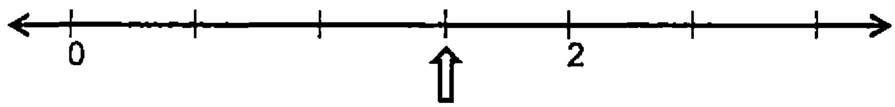
3. 

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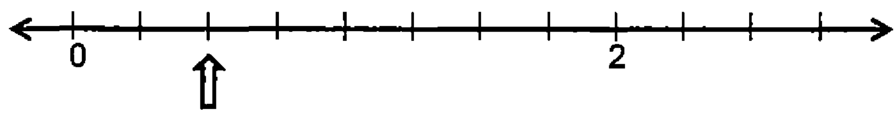
4. 

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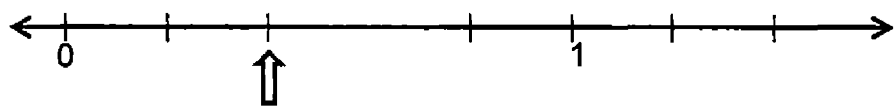
5. 

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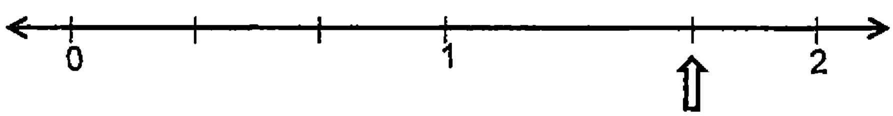
6. 

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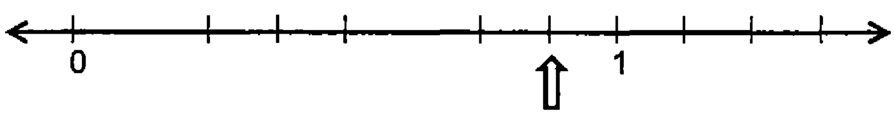
7. 

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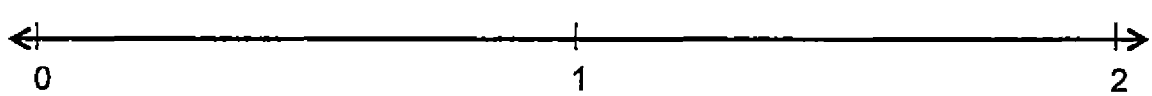
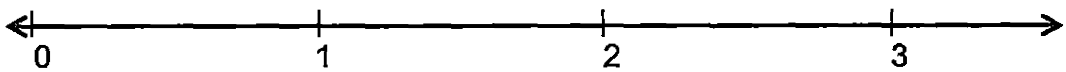
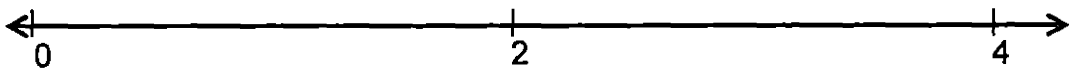
For problems 8 – 11, place and <u>label</u> the given fraction on the one number line with a bold dot as accurately as you can.

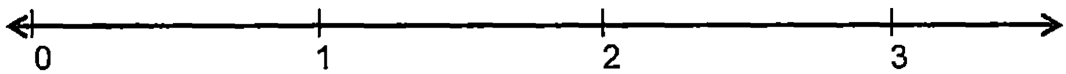
8. $\frac{1}{6}$ 9. $\frac{4}{12}$ 10. $1\frac{2}{3}$ 11. $\frac{4}{3}$



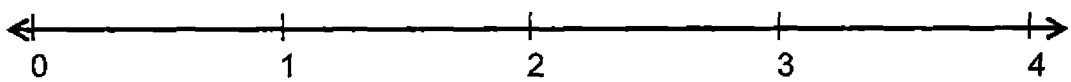
Show the <u>location of the given fraction</u> by placing a dot as accurately as you can on the indicated number line.

12. $\frac{1}{3}$ 

13. $\frac{1}{2}$ 

14. $\frac{8}{6}$ 

Show the <u>location of the given fraction</u> by placing a dot as accurately as you can on the indicated number line. Include a brief explanation **how** you placed it there.

15. $\frac{1}{4}$ 

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16. $\frac{3}{2}$ 

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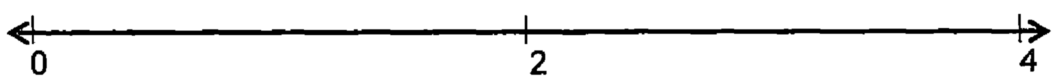
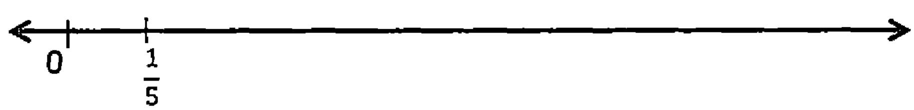
17. $1\frac{2}{3}$ 

Given the following fraction, locate as accurately as possible the <u>location of 1</u>. Include a brief explanation of **how** you chose its location.

18. 

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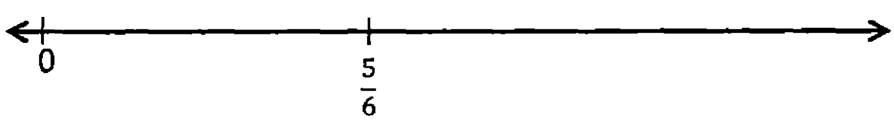
19. 

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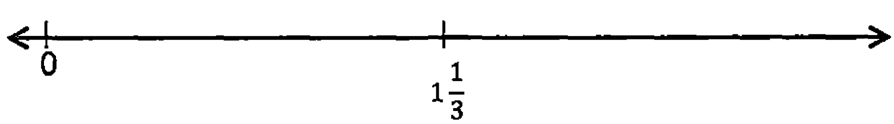
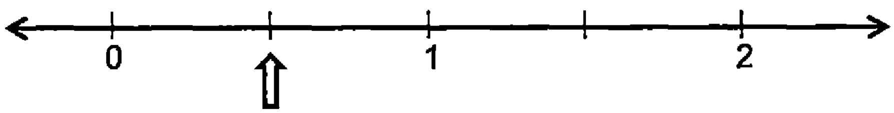
20. 

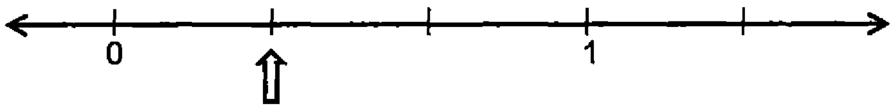
ANSWER KEY

Identifying and Placing Fractions on a Number Line Assessment (by Corinne Marshall)
Name the fraction that is indicated by the arrow on the number line.

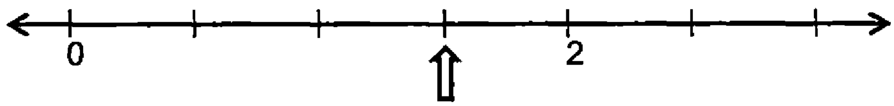
Answers:

8.


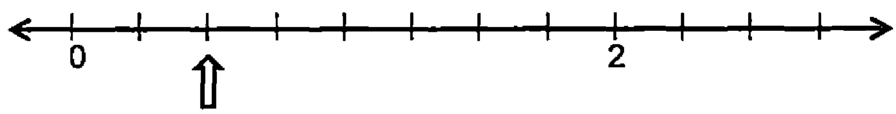
Answer: $\frac{1}{2}$

9.


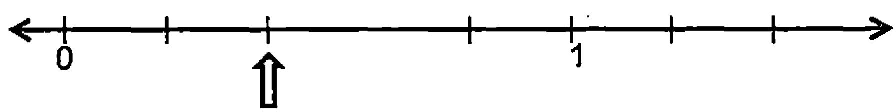
Answer: $\frac{1}{3}$

10.


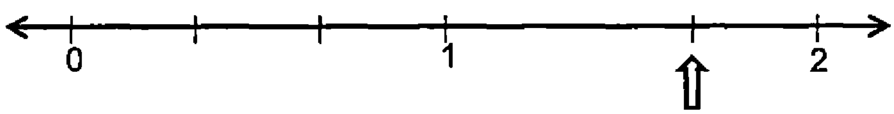
Answer: $1\frac{1}{2} \text{ or } \frac{3}{2}$

11.


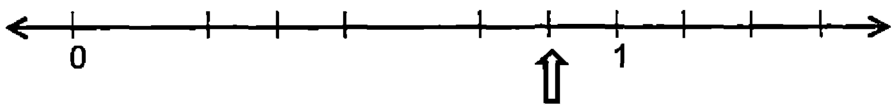
Answer: $\frac{2}{4} \text{ or } \frac{3}{2}$

12.


Answer: $\frac{2}{5}$

13.


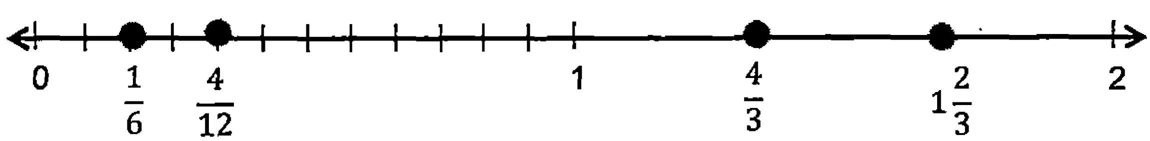
Answer: $1\frac{2}{3} \text{ or } \frac{5}{3}$

14.


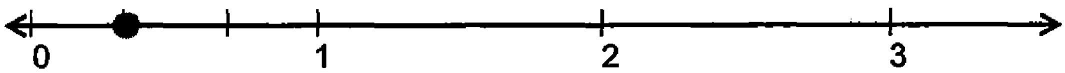
Answer: $\frac{7}{8}$

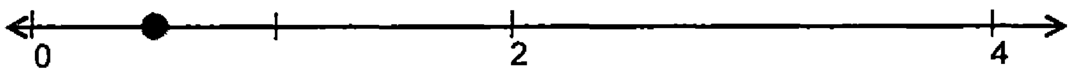
For problems 8 – 11, place and <u>label</u> the given fraction on the one number line with a bold dot as accurately as you can.

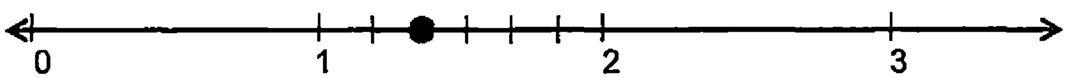
8. $\frac{1}{6}$ 9. $\frac{4}{12}$ 10. $1\frac{2}{3}$ 11. $\frac{4}{3}$



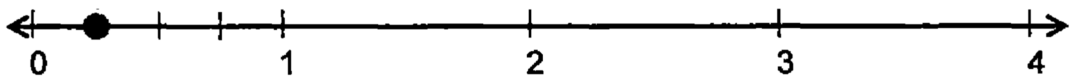
Show the <u>location of the given fraction</u> by placing a dot as accurately as you can on the indicated number line.

12. $\frac{1}{3}$ 

13. $\frac{1}{2}$ 

14. $\frac{8}{6}$ 

Show the <u>location of the given fraction</u> by placing a dot as accurately as you can on the indicated number line. Include a brief explanation **how** you placed it there.

15. $\frac{1}{4}$ 

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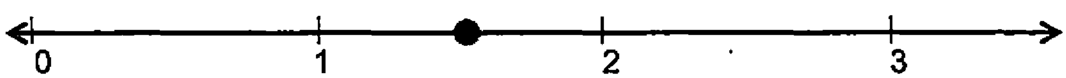
16. $\frac{3}{2}$ 

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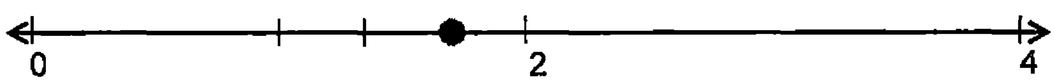
17. $1\frac{2}{3}$ 

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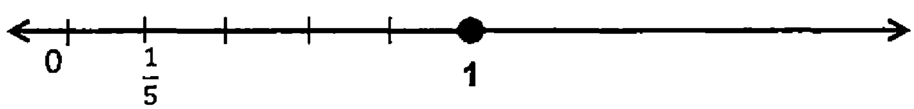
Given the following fraction, locate as accurately as possible the <u>location of 1</u>. Include a brief explanation of **how** you chose its location.

18. 

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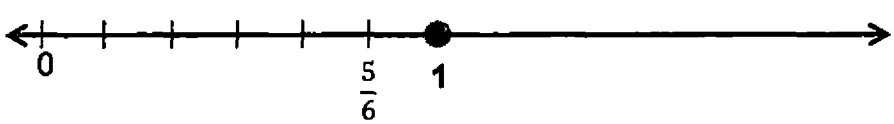
19. 

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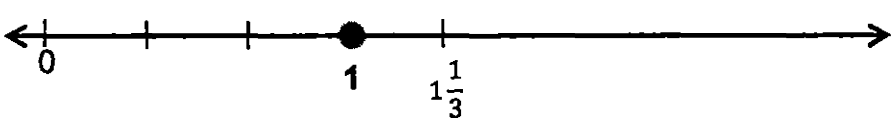
20. 

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Rubric for Scoring Number Line Assessment Items
(by Corinne Marshall)

Items 1 – 7 are given a score of 0 or 1, wrong or right. There is no partial correctness on these items.

Items 8 – 14 are given a score of 0, 1, or 2 depending on correctness:

0	1	2
If the fraction is placed outside of the set range of correctness (25%)	The fraction is placed within the range but there is no evidence of exact placement with equal portions assigned within the two whole numbers it lies between.	The fraction is placed within the range and there is evidence of exact placement with equal portions assigned within the two whole numbers it lies between.

Items 15 – 17 are given a score of 0, 1, or 2 for placing the fraction on the number line correctly and then a score of 0, 1, or 2 for a written explanation.

Placement of fraction on number line			Explanation of placement		
0	1	2	0	1	2
If the fraction is placed outside of the set range of correctness (25%)	The fraction is placed within the range but there is no evidence of exact placement with equal portions (or an incorrect number of portions).	The fraction is placed within the range and there is evidence of exact placement with equal portions assigned within the two whole numbers it lies between.	No explanation written; Reasoning has nothing to do with portions based on denominator or equal parts.	Written explanation including what the fraction is closest to; may include reference to fraction parts to create portions but still incorrect.	Written explanation with reference to equal portions or pieces created by the denominator.

Items 18 – 20 are given a score of 0, 1, or 2 for placing 1 on the number line correctly and then a score of 0, 1, or 2 for a written explanation.

Placement of 1 on number line			Explanation of placement		
0	1	2	0	1	2
The placement of 1 is placed outside of the set range of correctness (25%)	1 is placed within the range but there is no evidence of exact placement with equal portions (or an incorrect number of portions).	1 is placed within the range with evidence of exact placement based on equal portions assigned before or after the given fraction to locate 1.	No explanation written; No reference to creating portions based on the given fraction's denominator or 1's position in relation to the given fraction.	Explanation including where 1 is in relation to the given fraction but without reference to creating equal portions from the given fraction's denominator; or incorrect portions.	Written explanation with reference to equal portions from the given fraction's denominator.

APPENDIX B

ADDITIONAL OUTPUT TABLES

TABLE 4. Item Category/Option/Distractor Frequencies: Misfit Order

ENTRY NUMBER	DATA CODE	SCORE VALUE	DATA COUNT	%	AVERAGE ABILITY	S.E. MEAN	OUTF MNSQ	PTMEA CORR.	ITEM
13 A	0	0	18	24	-.83	.33	3.8	-.46	13
	1	1	9	12	-.40	.46	.8	-.20	
	2	2	44	59	1.09	.20	1.1	.55	
	MISSING	0	3	4	-2.23	.71	.2	-.32	
2 B	0	0	25	34	-.57	.33	2.0	-.39	02
	1	1	48	65	.79	.21	1.3	.39	
	MISSING	0	1	1	-.86		.4	-.08	
9 C	0	0	21	28	-.69	.32	3.8	-.52	09
	1	1	13	18	.17	.35	1.0	-.10	
	2	2	30	41	1.47	.21	1.1	.57	
	MISSING	0	10	14	-.95	.45	.8	-.30	
16 D	0	0	6	8	-1.09	.36	.3	-.52	16
	1	1	10	14	-.24	.46	3.8	-.42	
	2	2	27	36	1.89	.18	.5	.74	
	MISSING	0	31	42	-.64	.21	.8	-.49	
3 E	0	0	41	55	-.34	.26	1.8	-.44	03
	1	1	33	45	1.11	.22	1.1	.44	
19 F	0	0	20	27	-.76	.23	1.1	-.49	19
	1	1	15	20	-.28	.39	2.0	-.24	
	2	2	33	45	1.52	.21	1.1	.65	
	MISSING	0	6	8	-1.38	.50	.4	-.31	
10 G	0	0	32	43	-.59	.21	1.5	-.66	10
	1	1	2	3	-.72*	.78	1.9	-.14	
	2	2	32	43	1.63	.19	1.3	.71	
	MISSING	0	8	11	-1.16	.53	.3	-.31	
12 H	0	0	12	16	-1.87	.19	.4	-.66	12
	1	1	12	16	.19	.41	2.5	-.06	
	2	2	47	64	1.05	.17	.7	.57	
	MISSING	0	3	4	-2.23	.71	.3	-.32	
5 I	0	0	29	39	-.74	.30	1.6	-.53	05
	1	1	44	59	1.02	.19	.9	.53	
	MISSING	0	1	1	-.86		.4	-.08	
4 J	0	0	31	42	-.54	.26	1.3	-.49	04
	1	1	41	55	1.05	.22	1.2	.49	
	MISSING	0	2	3	-1.86	1.00	.2	-.22	
8 K	0	0	11	15	-1.10	.32	1.3	-.43	08
	1	1	19	26	-.39	.32	1.1	-.33	
	2	2	39	53	1.27	.19	.8	.61	
	MISSING	0	5	7	-1.53	.82	1.6	-.30	
23 L	0	0	8	11	-1.44	.30	.6	-.52	23
	1	1	22	30	.22	.31	1.8	-.19	
	2	2	33	45	1.38	.19	.8	.53	
	MISSING	0	11	15	-1.48	.35	.8	-.46	
11 M	0	0	23	31	-.72	.23	.8	-.58	11
	1	1	11	15	-.32	.45	2.2	-.23	
	2	2	29	39	1.78	.18	.5	.73	
	MISSING	0	11	15	-.82	.42	.7	-.29	
1 m	0	0	11	15	-1.44	.33	.7	-.45	01
	1	1	63	85	.61	.19	1.2	.45	

TABLE 4. Item Category/Option/Distractor Frequencies: Misfit Order (continued)

25 l	0	0	7	9	-1.58	.32	.7	-.50	25
	1	1	30	41	.30	.22	.9	-.17	
	2	2	28	38	1.41	.25	1.2	.48	
	MISSING	0	9	12	-1.66	.40	.8	-.45	
14 k	0	0	14	19	-1.53	.27	.7	-.66	14
	1	1	11	15	.31	.37	1.0	-.05	
	2	2	40	54	1.26	.19	.8	.60	
	MISSING	0	9	12	-1.11	.38	.7	-.32	
21 j	0	0	6	8	-1.45	.27	.6	-.44	21
	1	1	11	15	-.59	.42	.9	-.35	
	2	2	48	65	1.12	.18	.7	.59	
	MISSING	0	9	12	-1.77	.34	.5	-.47	
20 i	0	0	8	11	-.46	.57	1.2	-.49	20
	1	1	17	23	.63	.26	1.2	-.25	
	2	2	15	20	2.41	.23	.6	.66	
	MISSING	0	34	46	-.61	.21	.8	-.52	
26 h	0	0	10	14	-1.15	.51	1.0	-.66	26
	1	1	15	20	.71	.23	.6	-.07	
	2	2	19	26	2.09	.24	.8	.63	
	MISSING	0	30	41	-.55	.21	.8	-.43	
22 g	0	0	8	11	-1.53	.47	.5	-.67	22
	1	1	15	20	.69	.32	1.1	-.09	
	2	2	23	31	1.88	.20	.6	.59	
	MISSING	0	28	38	-.67	.20	.7	-.46	
18 f	0	0	10	14	-.93	.53	3.9	-.71	18
	1	1	10	14	1.04	.24	.2	.00	
	2	2	20	27	2.05	.20	.5	.61	
	MISSING	0	34	46	-.57	.20	.6	-.49	
6 e	0	0	30	41	-1.03	.21	.6	-.68	06
	1	1	43	58	1.26	.19	.7	.68	
	MISSING	0	1	1	-.86		.4	-.08	
24 d	0	0	10	14	-1.32	.44	.6	-.71	24
	1	1	17	23	.98	.25	.6	.03	
	2	2	18	24	2.08	.23	.7	.57	
	MISSING	0	29	39	-.63	.20	.6	-.46	
17 c	0	0	23	31	-1.31	.21	.6	-.75	17
	1	1	2	3	.10	.65	.2	-.03	
	2	2	45	61	1.29	.16	.4	.75	
	MISSING	0	4	5	-1.46	.73	.3	-.26	
7 b	0	0	23	31	-1.21	.25	.6	-.67	07
	1	1	49	66	1.10	.17	.6	.67	
	MISSING	0	2	3	-1.86	1.00	.2	-.22	
15 a	0	0	15	20	-1.80	.16	.3	-.72	15
	1	1	14	19	.17	.28	.8	-.07	
	2	2	41	55	1.29	.18	.7	.66	
	MISSING	0	4	5	-1.46	.73	.7	-.26	

TABLE 5. Item Statistics: Misfit Order

ENTRY NUMBER	TOTAL SCORE	TOTAL COUNT	MEASURE	MODEL S.E.	INFIT MNSQ	INFIT ZSTD	OUTFIT MNSQ	OUTFIT ZSTD	PT-MEASURE CORR.	PT-MEASURE EXP.	EXACT MATCH OBS%	EXACT MATCH EXP%	ITEM
13	97	74	-.51	.20	1.57	2.5	1.90	1.7	A .58	.71	60.8	71.5	13
2	48	74	-.60	.30	1.30	1.9	1.69	1.9	B .40	.56	71.6	78.1	02
9	73	74	.37	.19	1.36	2.0	1.66	1.7	C .62	.71	52.7	63.5	09
16	64	74	.68	.19	.81	-1.2	1.45	1.1	D .72	.71	63.5	61.7	16
3	33	74	.65	.29	1.24	1.7	1.39	1.4	E .44	.57	70.3	74.9	03
19	81	74	.07	.19	1.14	.9	1.34	1.1	F .66	.71	56.8	61.8	19
10	66	74	.60	.18	.98	.0	1.32	.6	G .70	.71	66.2	65.8	10
12	106	74	-.93	.21	.74	-1.4	1.28	.7	H .70	.68	71.6	70.5	12
5	44	74	-.26	.29	1.04	.4	1.27	1.0	I .53	.57	74.3	76.8	05
4	41	74	-.01	.29	1.11	.8	1.22	.9	J .51	.57	73.0	76.1	04
8	97	74	-.61	.21	1.19	1.1	1.10	.4	K .64	.69	60.8	65.9	08
23	88	74	-.25	.20	.94	-.3	1.13	.6	L .70	.70	56.8	62.9	23
11	69	74	.51	.19	.92	-.4	1.05	.3	M .71	.71	58.1	64.6	11
1	63	74	-2.22	.38	1.03	.2	.77	-.1	m .45	.44	86.5	86.8	01
25	86	74	-.22	.21	1.03	.2	.94	-.3	l .68	.68	59.5	62.5	25
14	91	74	-.29	.20	1.02	.2	.81	-.4	k .71	.71	64.9	67.9	14
21	107	74	-.96	.21	.91	-.4	.72	-.5	j .70	.68	67.6	70.8	21
20	47	74	1.39	.21	.80	-1.2	.91	-.2	i .71	.67	71.6	65.9	20
26	53	74	1.11	.20	.89	-.6	.74	-.7	h .72	.68	71.6	64.8	26
22	61	74	.81	.19	.82	-1.1	.77	-.6	g .74	.70	66.2	62.7	22
18	50	74	1.19	.19	.80	-1.1	.66	-.6	f .73	.68	68.9	66.1	18
6	43	74	-.17	.29	.77	-1.7	.62	-1.6	e .69	.57	83.8	76.6	06
24	53	74	1.13	.20	.77	-1.4	.65	-1.1	d .75	.68	68.9	64.6	24
17	92	74	-.27	.19	.74	-1.3	.48	-.5	c .77	.72	77.0	73.6	17
7	49	74	-.69	.30	.74	-1.9	.58	-1.4	b .68	.56	86.5	78.6	07
15	96	74	-.51	.20	.68	-1.9	.60	-1.2	a .77	.70	71.6	68.3	15
MEAN	69.2	74.0	.00	.23	.97	-.2	1.04	.2			68.5	69.4	
S.D.	22.1	.0	.81	.05	.22	1.2	.38	1.0			8.6	6.4	

TABLE 6. Person Statistics: Measure Order

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ENTRY NUMBER	TOTAL SCORE	TOTAL COUNT	MEASURE	MODEL S.E.	INFIT MNSQ	INFIT ZSTD	OUTFIT MNSQ	OUTFIT ZSTD	PT-MEASURE CORR.	PT-MEASURE EXP.	EXACT MATCH OBS%	EXACT MATCH EXP%	PERSON
60	44	26	3.56	.99	.97	.3	.96	.4	-.06	.11	96.2	96.4	60
64	44	26	3.56	.99	.94	.3	.74	.3	.16	.11	96.2	96.4	64
3	43	26	2.89	.70	1.93	1.2	1.88	1.0	.09	.16	96.2	93.0	03
62	43	26	2.89	.70	1.04	.3	2.74	1.4	-.16	.16	92.3	93.0	62
63	43	26	2.89	.70	.98	.2	1.70	.9	.01	.16	92.3	93.0	63
65	43	26	2.89	.70	.76	-.1	.37	-.3	.35	.16	92.3	93.0	65
68	43	26	2.89	.70	1.77	1.1	.75	.2	.29	.16	96.2	93.0	68
14	42	26	2.50	.57	.92	.1	.99	.3	.09	.20	88.5	89.9	14
43	42	26	2.50	.57	.96	.2	1.06	.4	.14	.20	88.5	89.9	43
48	42	26	2.50	.57	.60	-.5	.29	-.7	.50	.20	88.5	89.9	48
66	42	26	2.50	.57	.90	.1	1.61	.9	-.05	.20	88.5	89.9	66
11	41	26	2.22	.50	1.53	1.0	1.52	.8	.12	.23	84.6	86.0	11
26	40	26	1.99	.45	.58	-.9	.44	-.7	.36	.26	80.8	82.4	26
19	39	26	1.81	.41	1.50	1.2	2.14	1.6	.30	.28	80.8	77.6	19
72	38	26	1.65	.39	.97	.1	.91	.0	.19	.30	73.1	76.0	72
2	37	26	1.51	.37	.85	-.3	1.44	.9	.26	.32	84.6	73.2	02
20	36	26	1.38	.35	.82	-.5	.75	-.4	.29	.34	65.4	69.9	20
57	36	26	1.38	.35	.50	-1.8	.94	.0	.27	.34	80.8	69.9	57
34	35	26	1.26	.34	1.04	.2	.87	-.1	.38	.35	69.2	68.4	34
46	35	26	1.26	.34	.67	-1.2	.74	-.5	.47	.35	69.2	68.4	46
55	35	26	1.26	.34	.62	-1.4	.67	-.7	.43	.35	69.2	68.4	55
61	35	26	1.26	.34	1.00	.1	1.22	.6	-.04	.35	53.8	68.4	61
75	35	26	1.26	.34	1.00	.1	1.13	.4	.09	.35	53.8	68.4	75
59	33	26	1.04	.32	1.16	.7	.94	.0	.42	.38	53.8	59.8	59
70	33	26	1.04	.32	.79	-.8	.64	-.9	.61	.38	69.2	59.8	70
5	31	26	.84	.31	1.15	.7	1.06	.3	.55	.41	53.8	56.5	05
41	31	26	.84	.31	1.58	2.1	1.50	1.4	.21	.41	46.2	56.5	41
73	31	26	.84	.31	.93	-.2	.90	-.2	.50	.41	50.0	56.5	73
16	30	26	.75	.30	1.07	.4	.92	-.1	.38	.42	69.2	54.7	16
42	30	26	.75	.30	.86	-.6	1.60	1.7	.03	.42	57.7	54.7	42
50	30	26	.75	.30	.77	-1.0	.61	-1.3	.75	.42	57.7	54.7	50
67	29	26	.66	.30	.93	-.2	1.06	.3	.04	.42	53.8	53.8	67
69	29	26	.66	.30	1.44	1.8	1.37	1.2	.08	.42	42.3	53.8	69
4	28	26	.57	.30	.93	-.2	.88	-.4	.53	.43	65.4	53.0	04
49	28	26	.57	.30	1.05	.3	.99	.1	.56	.43	46.2	53.0	49
74	28	26	.57	.30	.81	-.8	.79	-.7	.68	.43	46.2	53.0	74
23	27	26	.49	.29	1.49	2.0	1.49	1.7	.26	.44	38.5	52.0	23
24	26	26	.40	.29	.91	-.3	.77	-.9	.69	.45	57.7	52.5	24
27	26	26	.40	.29	.92	-.3	.98	.0	.35	.45	50.0	52.5	27
56	25	26	.31	.29	.92	-.3	.93	-.2	.20	.45	61.5	49.7	56
6	24	26	.23	.29	.73	-1.2	.67	-1.4	.66	.45	57.7	49.2	06
44	24	26	.23	.29	.81	-.8	.81	-.8	.49	.45	46.2	49.2	44
51	23	26	.15	.29	.59	-2.0	.56	-2.1	.81	.46	53.8	47.3	51
33	22	26	.06	.29	.77	-1.0	.79	-.9	.63	.46	65.4	47.3	33
71	22	26	.06	.29	1.08	.4	1.08	.4	.50	.46	38.5	47.3	71
35	19	26	-.20	.29	.81	-.8	.73	-1.1	.74	.46	50.0	50.7	35
36	19	26	-.20	.29	1.04	.3	1.17	.7	.45	.46	42.3	50.7	36
39	18	26	-.28	.30	.92	-.3	.88	-.4	.59	.46	46.2	53.2	39
25	17	26	-.37	.30	1.26	1.1	1.15	.6	.50	.46	42.3	54.0	25
8	16	26	-.46	.30	1.34	1.4	1.16	.6	.40	.45	53.8	56.3	08
28	15	26	-.56	.31	.85	-.6	.84	-.5	.46	.45	61.5	57.0	28
29	15	26	-.56	.31	.91	-.3	.99	.1	.46	.45	53.8	57.0	29
45	15	26	-.56	.31	.81	-.7	.70	-1.0	.61	.45	73.1	57.0	45
40	14	26	-.65	.31	.94	-.2	.75	-.7	.60	.45	61.5	57.7	40
53	14	26	-.65	.31	1.05	.3	.95	.0	.51	.45	46.2	57.7	53
31	12	26	-.86	.33	1.02	.2	1.05	.2	.27	.43	57.7	63.6	31
52	11	26	-.97	.34	.79	-.7	.74	-.6	.65	.43	61.5	64.6	52
32	9	26	-1.21	.36	1.32	1.0	2.11	2.0	.30	.41	53.8	68.3	32
7	8	26	-1.35	.38	.76	-.6	1.27	.7	-.04	.40	73.1	73.5	07
37	8	26	-1.35	.38	1.48	1.3	1.09	.3	.45	.40	73.1	73.5	37

TABLE 6. Person Statistics: Measure Order (continued)

54	8	26	-1.35	.38	.80	-.5	.59	-.8	.38	.40	76.9	73.5	54
10	7	26	-1.51	.40	1.04	.2	1.58	1.1	-.10	.38	69.2	75.0	10
21	7	26	-1.51	.40	.78	-.5	1.24	.6	.40	.38	69.2	75.0	21
58	6	26	-1.68	.43	.74	-.5	.83	-.1	.49	.37	80.8	78.2	58
1	5	26	-1.88	.47	.74	-.4	.77	-.1	.50	.35	84.6	83.6	01
13	5	26	-1.88	.47	1.18	.5	1.98	1.3	-.17	.35	76.9	83.6	13
17	5	26	-1.88	.47	.81	-.3	.80	.0	.22	.35	76.9	83.6	17
22	5	26	-1.88	.47	.89	-.1	.97	.2	.39	.35	84.6	83.6	22
18	4	26	-2.13	.52	.75	-.3	.71	-.1	.46	.33	88.5	86.2	18
38	4	26	-2.13	.52	.80	-.2	.58	-.3	.18	.33	80.8	86.2	38
47	4	26	-2.13	.52	.63	-.6	.44	-.6	.54	.33	88.5	86.2	47
12	2	26	-2.87	.73	1.14	.4	2.60	1.3	-.13	.26	92.3	92.7	12
15	2	26	-2.87	.73	1.00	.3	.85	.3	.16	.26	92.3	92.7	15
30	1	26	-3.59	1.02	.71	.0	.18	-.4	.55	.20	96.2	96.3	30
MEAN	24.3	26.0	.30	.42	.97	.0	1.04	.1			68.5	69.4	
S.D.	13.5	.0	1.64	.17	.28	.8	.48	.8			17.2	15.8	

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