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Solutions to a generalized Pell equation

Kyle Christopher Castro

Solutions to a Generalized Pell Equation

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

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Master of Arts

in

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Abstract

The classic study of Pell's Equation consists of solutions to the diophantine equation $x^2 - dy^2 = n$ where d and n are fixed integers. This thesis aims to extend the notion of continued fractions to a new field, $\mathbb{Q}(x)^*$, in order to find solutions to generalized Pell's Equations in $\mathbb{Q}[x]$. Historically, mathematicians were interested in the specific Pell Equation $x^2 - dy^2 = 1$ which has influenced the primary goal of this project to become the solvability over $\mathbb{Q}[x]$ where $n = 1$. The difficulty in generalizing solutions over $\mathbb{Q}[x]$ stems from the complexity of the continued fraction expansions of the elements belonging to the field extension $\mathbb{Q}(x)^*$. The investigation of these new solutions to Pell's Equation will begin with the necessary extensions of theorems as they apply to polynomials with rational coefficients and fractions of such polynomials in order to describe each "family" of solutions.

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Chapter 1

Introduction

A common topic of Number Theory courses is the study of Continued Fractions which was first introduced in 1202 by Leonardo Fibonacci in his work *Liberation*. Today, the continued fraction expansions of most real numbers can be completely characterized thus creating the opportunity for many applications. The study of continued fractions is a method mathematicians have used to explore the notion of the extension of $\mathbb{Z}$ to $\mathbb{Q}$ and its completion to $\mathbb{R}$ with respect to absolute value. The idea behind this extension is one that can be paralleled to a new ring, $\mathbb{Q}[x]$. We would like to consider the extension of $\mathbb{Q}[x]$ to $\mathbb{Q}(x)$ and its completion to $\mathbb{Q}(x)^*$ with respect to a Non-Archimedian Valuation.

The traditional study of Continued Fractions can be segmented into two main categories: those with finite continued fraction expansions and those with infinite continued fraction expansions. All rational numbers are expressible as finite continued fractions while irrational numbers are expressible as infinite continued fractions. Infinite continued fraction expansions can be subdivided further into two categories: periodic (repeating) or non-periodic expansions. In order to characterize real numbers by their continued fraction expansions, the idea of convergents is used; convergents are approximations found through partial continued fraction expansions whose limits converge to the real number they represent.

Although all irrational numbers are expressible as infinite continued fractions, not all irrational numbers have periodic or eventually periodic continued fraction expansions. It can be shown that if $\alpha \in \mathbb{R} - \mathbb{Q}$ is a root of a quadratic polynomial of the form $Ax^2 + Bx + C$ where $A, B, C \in \mathbb{Z}$ and $A \neq 0$, then α has either a periodic or eventually periodic infinite simple continued fraction expansion. The expansion is periodic from the its start if $\alpha > 1$ and $-1 < \alpha' < 0$ where α' is the conjugate of α. This result plays a vital role in the characterization of the solutions to the Pell Equation $x^2 - dy^2 = 1$, because if d is a positive integer that is not a perfect square, then $\sqrt{d}$ is a root of the polynomial $x^2 - d$. Therefore, $\sqrt{d}$ has an eventually periodic simple continued fraction expansion. The investigation which ensues from this property eventually leads to Theorems 8.6 and 8.7 from Strayer's text:

> "Theorem 8.6: Let d be a positive integer that is not a perfect square. Let $\frac{p_i}{q_i}$ denote the i^{th} convergent of the eventually periodic simple continued fraction expansion of $\sqrt{d}$ and let p be the period length of this expansion. If p is even, the positive solutions of the Pell Equation $x^2 - dy^2 = 1$ are given precisely by $x = p_{np-1}$ and $y = q_{np-1}$ where n is a positive integer; if p is odd, the positive solutions of the Pell Equation are given precisely by $x = p_{2np-1}$ and $y = q_{2np-1}$ where n is a positive integer."

> "Theorem 8.7: Let d be a positive integer that is not a perfect square and let x_1, y_1 be the [least positive] solution of the Pell Equation $x^2 - dy^2 = 1$. Then all positive solutions of this equation are given precisely by x_n, y_n where $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$ and n is a positive integer." [Str94]

In summary, the solutions to the Pell Equation $x^2 - dy^2 = 1$ can be found by substituting the numerator and denominator of the convergents of $\sqrt{d}$ for x & y. Once the least positive solution, also known as the fundamental solution, is found, all other positive solutions can be generated through the final equation given in Theorem 8.7. This is truly a fantastic result. For a review of the traditional Continued Fraction theory see [NZM91], [Old77], or [Str94].

Although this thesis is similar to the works of Crawford and Vicknair, we have extended the theory as done in [Str94]. The works of Crawford and Vicknair are generally inaccessible, yielding the need to duplicate some proofs in detail; however, the proofs will be different because they are an adaptation of [Str94], whereas the others used [NZM91].

It is the types of results stated above which we will try to duplicate in the new field $\mathbb{Q}(x)$. $\mathbb{Q}(x)$ is the field of fractions of polynomials with rational coefficients and is also an extension of $\mathbb{Q}[x]$ which is the domain consisting of polynomials with rational coefficients. Ultimately, we will be creating an extension mapping from $\mathbb{Q}[x]$ to $\mathbb{Q}(x)$ and completing it to $\mathbb{Q}(x)^*$ with respect to a Non-Archimedean valuation similar to the extension mapping from $\mathbb{Z}$ to $\mathbb{Q}$ and its completion to $\mathbb{R}$ as mentioned above with respect to absolute value.

Elements belonging to $\mathbb{Q}[x]$ are polynomials with rational coefficients, that is, elements have the form $A_nx^n + A_{n-1}x^{n-1} + \ldots + A_0$ where $A_i \in \mathbb{Q}\ \forall\ i \geq 0$. Elements belonging to $\mathbb{Q}(x)$ are fractions of polynomials with rational coefficients or fractions of the elements belonging to $\mathbb{Q}[x]$; therefore, elements may have the form $\frac{f(x)}{g(x)}$ where $f(x), g(x) \in \mathbb{Q}[x]$ and $g(x) \neq 0$. It is the elements in $\mathbb{Q}(x)$ which we complete to $\mathbb{Q}(x)^*$ by taking the roots these elements; elements in $\mathbb{Q}(x)^*$ may have the form $\frac{\sqrt{f(x)}}{g(x)}$ where all terms are defined as above, but n must be an even integer. Continuing on, the $\sqrt{A_nx^n + A_{n-1}x^{n-1} + \ldots + A_0} = C_kx^k + C_{k-1}x^{k-1} + \ldots + C_0 + C_{-1}x^{-1} + C_{-2}x^{-2} + \ldots = \sum_{i=k}^{-\infty} c_ix^i$ where $k = n/2$ and $C_k \in \mathbb{Q}$ for some $k \in \mathbb{Z}$ such that $(C_lx^l + C_{l-1}x^{l-1} + \ldots + C_0 + C_{-1}x^{-1} + C_{-2}x^{-2} + \ldots)^2 = A_nx^n + A_{n-1}x^{n-1} + \ldots + A_0$. It is the continued fraction expansions of these elements which we will be interested in.

The continued fraction expansions of the elements in $\mathbb{Q}[x]$, $\mathbb{Q}(x)$, and $\mathbb{Q}(x)^*$ have many features in common with $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ when it comes to their expansions, but there are important differences as well. Continued fraction expansions in our new rings and fields have the form $[a_0, a_1, \ldots]$, where $deg(a_i) > 0$ for $i > 0$. Similar to the elements of $\mathbb{Q}$, elements of $\mathbb{Q}(x)$ have finite continued fraction expansions. The elements $\mathbb{Q}(x)^* - \mathbb{Q}(x)$ have infinite continued fraction expansions similar to elements of $\mathbb{R} - \mathbb{Q}$. For our purposes, the infinite continued fraction expansions can be categorized as *Eventually Periodic* if $\alpha = [a_0, a_1, \ldots \overline{a_i, a_{i+1}, \ldots, a_n}]$ or *Purely Periodic*, or *Periodic* for short, if $\alpha = [\overline{a_0, a_1, \ldots, a_n}]$. A special case called *Almost Periodic* also occurs, but is best discussed at a subsequent time. A myriad of theorems and definitions from the traditional study of continued fractions hold true for our new rings and fields, with only minor changes or variations. For example, the idea of the greatest integer function no longer exists now that we have moved away from $\mathbb{Z}$. Instead, we will be using the "integral part" of an element.

Definition 1.1. *Define α to be $\sum_{i=k}^{-\infty} c_ix^i$. Then the* <u>*integral part*</u> *of α, denoted $[[\alpha]]$, is defined as $\sum_{i=k}^{0} c_ix^i$.*

The reader can observe that the "integral part" is merely a generalization of the greatest integer function where $[[\sqrt{\alpha}]] = \sum_{i=k}^{0} c_i(10)^i$ for $c_i \in \{0, 1, 2 \ldots, 9\}$.

Many of the differences that arise in our setting are due to the fact that we are no longer dealing with measures of distance. Since we are not considering the extensions to these new rings and fields with respect to absolute value (because it ceases to hold in $\mathbb{Q}[x]$), we need a new way of comparing two elements. In fact, we are not even considering an Archimedean valuation anymore (i.e. absolute value), but we will be considering the extension from $\mathbb{Q}[x]$ to $\mathbb{Q}(x)$ and completing it to $\mathbb{Q}(x)^*$ with respect to the Non-Archimedean valuation which will be defined as follows:

Definition 1.2. *Let P be an ordered field and K be a field with $a, b \in K$. Then a mapping $v : K \longrightarrow P$ is called a Multiplicative Non-Archimedean valuation if*
(1) $v(a) > 0$ for $a \neq 0$; $v(0) = 0$
(2) $v(ab) = v(a)v(b)$
(3) $v(a+b) \leq max\{v(a), v(b)\}$.

Using this valuation, we define $v(\frac{f}{g}) = e^{deg(f)-deg(g)}$ where e is the transcendental number 2.718.... Traditionally, older texts tend to use multiplicative valuations while modern texts tend to use additive valuations, but translating from one to the other is possible. The Non-Archimedean valuation differs from the Archimedean valuation (i.e. absolute value) in that the Archimedean case, $\phi(m \cdot 1) = \phi(1+1+1+1+\ldots+1) > 1$ compared to the Non-Archimedean case $\phi(m \cdot 1) = \phi(1+1+1+1+\ldots+1) \leq 1$. Also in the multiplicative Archimedean valuation, the triangle inequality holds, $\phi(a+b) \leq \phi(a) + \phi(b)$; whereas, this is not true in a Multiplicative Non-Archimedean valuation by (3). For an in-depth study on valuation theory, see Zariski and Samuel (1991) and van der Waerdan(1970).

Since we have picked $\mathbb{Q}[x]$ as the fixed ring similar to $\mathbb{Z}$, we expect vital proprieties to hold; specifically, the division algorithm.

Theorem 1.3. (The Division Algorithm)
Given $f(x), g(x) \in \mathbb{Q}[x]$ with $g(x) \neq 0$, then there exists unique $q(x), r(x) \in \mathbb{Q}[x]$ such that $f(x) = q(x)g(x) + r(x)$ where $deg(r(x)) < deg(g(x))$ or $r(x) = 0$

Proof. Notice, if $deg(f) < deg(g)$, then $f = 0 \cdot g + f$ where $q = 0$ and $r = f$ and the argument is done. Now, consider $deg(f) \geq deg(g)$ and let $f = a_m x^m + a_{m-1}x^{m-1} + \ldots + a_0$ and $g = b_k x^k + b_{k-1}x^{k-1} + \ldots + b_0$ with $a_m, b_k \neq 0$. Letting $q_0 = \frac{a_m}{b_k} x^{m-k}$ and $f_1 = f - g(q_0)$, we have $deg(f_1) < deg(f)$.

Notice that if $deg(f_1) < deg(g)$ then $q = q_0, r = f_1$ finishes the proof, so assume not. Then letting $q_1 = \frac{a'_{m-1}}{b_k}x^{m-k-1}$ and $f_2 = f_1 - g(q_1)$ where a'_{m-1} is the leading coefficient of f_1 yields $deg(f_2) < deg(g)$. Then letting $q = q_0 + q_1, r = f_2$ finishes the proof, therefore assume not. Continuing on in this manner, we notice $deg(f) > deg(f_1) > deg(f_2)\ldots$ creates a decreasing sequence. Therefore, there exists some f_n such that $deg(f_n) < deg(g)$ or $deg(f_n) = 0$. Letting $q = q_0 + q_1 + \ldots q_{n-1}$, $r = f_n \Rightarrow f(x) = g(x)q(x) + r(x)$ as desired.

Now, to prove uniqueness, suppose there exists q_1, q_2, r_1, r_2 such that $f = q_1(g) + r_1$ and $f = q_2g + r_2$ with $deg(r_1) < deg(g)$ and $deg(r_2) < deg(g)$. Then we have,

$$q_1(g) + r_1 = q_2(g) + r_2$$
$$\Rightarrow r_1 - r_2 = (q_2 - q_1)g$$
$$\Rightarrow g|(r_1 - r_2)$$

but $deg(r_1 - r_2) < deg(g) \Rightarrow r_1 - r_2 = 0$

$$\Rightarrow r_1 = r_2$$

Also, $(q_2 - q_1)g = 0 \Rightarrow g = 0$ or $q_2 - q_1 = 0$

but $g \neq 0$ so $q_2 - q_1 = 0 \Rightarrow q_2 = q_1$ □

Having the division algorithm for elements in $\mathbb{Q}[x]$, we can define other properties such as the greatest common divisor.

Definition 1.4. *Let $f(x)$ & $g(x) \in \mathbb{Q}[x]$. The greatest common divisor of $f(x)$ and $g(x)$ is a polynomial $d(x)$ of highest degree such that $d(x)$ divides $f(x)$ & $g(x)$. We denote the greatest common divisor of $f(x)$ & $g(x)$ by $gcd(f(x), g(x))$.*

With the division algorithm and the greatest common divisor, we are now ready to discuss continued fractions.

Chapter 2

Continued Fractions

Before we can generalize solutions to Pell's Equation, we must generalize the notion of continued fractions. Through this generalization we can create infinitely many more solutions by comparing and contrasting the continued fraction expansions in the new rings and fields with those from the real numbers.

2.1 Finite Continued Fractions

As previously stated, having the division algorithm for elements in $\mathbb{Q}[x]$ allows for a formal definition of a finite continued fraction:

Definition 2.1. *The expression*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}$$

where $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Q}(x)$*,* $deg(a_i) > 0$ *for* $0 < i \leq n$*, and* $a_1, a_2, \ldots, a_n \neq 0$ *is said be a* *finite continued expansion* *and is denoted by* $[a_0, a_1, a_2, \ldots, a_n]$*. A* *finite simple continued fraction* *is a continued fraction expansion in which* $a_0, a_1, a_2, \ldots, a_n \in \mathbb{Q}[x]$*.*

It is important to note that if the numerator has smaller degree than the denominator, continued fraction expansions may begin with 0. Moreover, $deg(a_i)$ must be greater than 0 for all i to ensure that each finite continued fraction expansion is unique. For our purposes, we are only interested in the finite *simple* continued fractions and their expansions. Here are a few examples of finite simple continued fractions.

Example 2.2. *Find the finite simple continued fraction expansion of*

$$\frac{6x^9+21x^8-41x^7-217x^6-428x^5+6x^3+42x^2+102x+128}{6x^7+3x^6-80x^5+6x+24}.$$

By the division algorithm,

$$\begin{aligned} 6x^9 + 21x^8 - 41x^7 - 217x^6 - 428x^5 + 6x^3 + 42x^2 + 102x + 128 \\ = (6x^7 + 3x^6 - 80x^5 + 6x + 24) \cdot (x^2 + 3x + 5) + (8x^6 - 28x^5 + 8) \\ 6x^7 + 3x^6 - 80x^5 + 6x + 24 = (8x^6 - 28x^5 + 8) \cdot (\tfrac{3}{4}x + 3) + 4x^5 \\ 8x^6 - 28x^5 + 8 = (4x^5)(2x - 7) + 8 \\ 4x^5 = (8)(1/2x^5) + 0. \end{aligned}$$

Dividing each equation by the first factor on the right hand side yields,

$$\begin{aligned} \frac{6x^9+21x^8-41x^7-217x^6-428x^5+6x^3+42x^2+102x+128}{6x^7+3x^6-80x^5+6x+24} &= x^2 + 3x + 5 + \frac{8x^6-28x^5+8}{6x^7+3x^6-80x^5+6x+24} \\ \frac{6x^7+3x^6-80x^5+6x+24}{8x^6-28x^5+8} &= \frac{3}{4}x + 3 + \frac{4x^5}{8x^6-28x^5+8} \\ \frac{8x^6-28x^5+8}{4x^5} &= 2x - 7 + \frac{8}{4x^5} \\ \frac{4x^5}{8} &= \frac{1}{2}x^5. \end{aligned}$$

So combining these ratios we have,

$$\frac{6x^9+21x^8-41x^7-217x^6-428x^5+6x^3+42x^2+102x+128}{6x^7+3x^6-80x^5+6x+24} = x^2 + 3x + 5 + \frac{1}{\frac{6x^7+3x^6-80x^5+6x+24}{8x^6-28x^5+8}}$$

$$= x^2 + 3x + 5 + \frac{1}{(\frac{3}{4}x+3)+\frac{1}{\frac{8x^6-28x^5+8}{4x^5}}} = x^2 + 3x + 5 + \frac{1}{(\frac{3}{4}x+3)+\frac{1}{(2x-7)+\frac{1}{\frac{4x^5}{8}}}}$$

$$= x^2 + 3x + 5 + \frac{1}{(\frac{3}{4}x+3)+\frac{1}{(2x-7)+\frac{1}{\frac{1}{2}x^5}}}.$$

Thus, $\frac{6x^9+21x^8-41x^7-217x^6-428x^5+6x^3+42x^2+102x+128}{6x^7+3x^6-80x^5+6x+24} = [x^2 + 3x + 5, \frac{3}{4}x + 3, 2x - 7, \frac{1}{2}x^5]$.

Example 2.3. *Find the finite simple continued fraction expansion of*

$$\frac{6x^3+15x+4}{30x^5+18x^4+81x^3+65x^2+30x+4}.$$

Again by the division algorithm,

$$\begin{aligned} 30x^5+18x^4+81x^3+65x^2+30x+4 &= (6x^3+15x+4)\cdot(5x^2+3x+1)+(3x) \\ 6x^3+15x+4 &= (3x)\cdot(2x^2+5)+4) \\ 3x &= (4)(\tfrac{3}{4}x)+0. \end{aligned}$$

Dividing each equation by the first factor on the right hand side yields,

$$\begin{aligned} \tfrac{30x^5+18x^4+81x^3+65x^2+30x+4}{6x^3+15x+4} &= 5x^2+3x+1+\tfrac{3x}{6x^3+15x+4} \\ \tfrac{6x^3+15x+4}{3x} &= 2x^2+5+\tfrac{4}{3x} \\ \tfrac{3x}{4} &= \tfrac{3}{4}x. \end{aligned}$$

So combining these ratios and inverting we have,

$$\frac{6x^3+15x+4}{30x^5+18x^4+81x^3+65x^2+30x+4} = 0+\frac{1}{5x^2+3x+1+\frac{3x}{6x^3+15x+4}}$$

$$= 0+\frac{1}{(5x^2+3x+1)+\frac{1}{\frac{6x^3+15x+4}{3x}}} = 0+\frac{1}{(5x^2+3x+1)+\frac{1}{(2x^2+5)+\frac{4}{3x}}}$$

$$= 0+\frac{1}{(5x^2+3x+1)+\frac{1}{(2x^2+5)+\frac{1}{(\frac{3}{4}x)}}}.$$

Thus, $\frac{6x^3+15x+4}{30x^5+18x^4+81x^3+65x^2+30x+4} = [0, 5x^2+3x+1, 2x^2+5, \frac{3}{4}x]$.

The preceding examples seem to suggest that elements of $\mathbb{Q}(x)$ have finite simple continued fraction expansions; in fact, this is the topic of our next theorem.

Theorem 2.4. *Let $\alpha \in \mathbb{Q}(x)^*$. Then $\alpha \in \mathbb{Q}(x)$ if and only if α is expressible as a finite simple continued fraction.*

Proof. ($\Rightarrow$) Assume $\alpha \in \mathbb{Q}(x)$. Then $\alpha = \frac{f(x)}{g(x)}$ with $f(x), g(x) \in \mathbb{Q}[x]$ and $g(x) \neq 0$. Then by the division algorithm of polynomials,

$$\begin{aligned} f(x) &= g(x)q_0 + r_0 \text{ where } \deg r_0 < \deg g(x) \\ g(x) &= r_0 q_1 + r_1 \text{ where } \deg r_1 < \deg r_0 \\ r_0 &= r_1 q_2 + r_2 \text{ where } \deg r_2 < \deg r_1 \end{aligned}$$

$$\vdots$$

$$r_{n-3} = r_{n-2}q_{n-1} + r_{n-1}$$

$$r_{n-2} = r_{n-1}q_n$$

with $q_0, q_1, \cdots, q_n \in \mathbb{Q}[x]$. Thus $\frac{f(x)}{g(x)} = \frac{g(x)q_0+r_0}{g(x)} = q_0 + \frac{r_0}{g(x)} = q_0 + \frac{1}{\frac{g(x)}{r_0}}$.
Similarly, $\frac{g(x)}{r_0} = q_1 + \frac{r_1}{r_0} = q_1 + \frac{1}{\frac{r_0}{r_1}}$. Continuing on in this manner,

$$\frac{r_0}{r_1} = q_2 + \frac{1}{\frac{r_1}{r_2}}$$

$$\frac{r_1}{r_2} = q_3 + \frac{1}{\frac{r_2}{r_3}}$$

$$\vdots$$

$$\frac{r_{n-3}}{r_{n-2}} = q_{n-1} + \frac{1}{\frac{r_{n-2}}{r_{n-1}}}$$

$$\frac{r_{n-2}}{r_{n-1}} = q_n.$$

Substitution yields, $\frac{f(x)}{g(x)} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{\ddots + \cfrac{1}{q_{n-1} + \cfrac{1}{q_n}}}}}$

$= [q_0, q_1, \ldots, q_n]$ as desired.

($\Leftarrow$) Let α be expressible as a finite simple continued fraction, say $\alpha = [a_0, a_1, \ldots, a_n]$ where $a_0, a_1, \ldots, a_n \in \mathbb{Q}[x]$. Notice if $n = 0$, $\alpha = a_0 \in \mathbb{Q}(x)$ and if $n = 1$, $\alpha = a_0 + \frac{1}{a_1} = \frac{a_0a_1+1}{a_1} \in \mathbb{Q}(x)$. Assume $\alpha \in \mathbb{Q}(x)$ for $n = k$ where $k \geq 1$. Consider $\alpha = [a_0, a_1, \ldots, a_{k+1}] = a_0 + \frac{1}{[a_1,a_2,\ldots,a_{k+1}]}$. By the induction hypothesis, $[a_1, a_2, \ldots, a_{k+1}] \in \mathbb{Q}(x)$. Let $\frac{f(x)}{g(x)} = [a_1, a_2, \ldots, a_{k+1}]$ for $f(x), g(x) \in \mathbb{Q}[x]$.
Then, $\alpha = a_0 + \frac{1}{\frac{f(x)}{g(x)}} = a_0 + \frac{g(x)}{f(x)} = \frac{a_0f(x)+g(x)}{f(x)} \in \mathbb{Q}(x)$. So by Mathematical Induction, $\alpha \in \mathbb{Q}(x)$. □

2.2 Convergents

In order to develop an infinite continued fraction expansion, we are going to use the idea of convergents to assess how adding terms to the expansion affects it. For now we will consider the convergents of finite continued fraction expansions.

Definition 2.5. *Let $\alpha = [a_0, a_1, \ldots, a_n]$ be expressible as a finite continued fraction. The finite continued fraction $C_i = [a_0, a_1, \ldots, a_i]$, $0 \le i \le n$ is said to be the $\underline{i^{th}\text{ convergent}}$ of α. If $i = n$, then $\alpha = C_n = [a_0, a_1, \ldots, a_n]$.*

Consider the following example:

Example 2.6. *Find the convergents of $\alpha = [x^4, x^3, x^2, x]$.*

The convergents of α are:

$$C_0 = [x^4] = x^4$$
$$C_1 = [x^4, x^3] = x^4 + \frac{1}{x^3} = \frac{x^7+1}{x^3}$$
$$C_2 = [x^4, x^3, x^2] = x^4 + \frac{1}{x^3+\frac{1}{x^2}} = \frac{x^9+x^4+x^2}{x^5+1}$$
$$C_3 = [x^4, x^3, x^2, x] = x^4 + \frac{1}{x^3+\frac{1}{x^2+\frac{1}{x}}} = \frac{x^{10}+x^7+x^5+x^3+1}{x^6+x^3+x}$$

and $\alpha = C_3 = \frac{x^{10}+x^7+x^5+x^3+1}{x^6+x^3+x}$.

Notice in our example, each of the convergents is becoming a "better" approximation of α; it converges as the name suggests. This method of computing each of the convergents is not practical; in fact, we can define each convergent by the following recurrence relation so that we may compute them efficiently.

Proposition 2.7. *Let $\alpha = [a_0, a_1, \ldots, a_n]$ be expressible as a finite simple continued fraction. Define $p_0, p_1, \ldots, p_n$ and $q_0, q_1, \ldots, q_n$ by the following recurrence relations:*

$$p_0 = a_0$$
$$p_1 = a_1 a_0 + 1$$
$$p_i = a_i p_{i-1} + p_{i-2}$$

$$q_0 = 1$$
$$q_1 = a_1$$
$$q_i = a_i q_{i-1} + q_{i-2}$$

for $2 \le i \le n$. Then,

$$C_i = \frac{p_i}{q_i}, \text{ for } 0 \le i \le n.$$

Proof. Notice, if $i = 0$,

$$C_0 = [a_0] = a_0 = \frac{a_0}{1} = \frac{p_0}{q_0}.$$

If $i = 1$,

$$C_1 = [a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0a_1+1}{a_1} = \frac{p_1}{q_1}.$$

If $i = 2$,

$$C_2 = [a_0, a_1, a_2] = a_0 + \frac{1}{a_1+\frac{1}{a_2}} = \frac{a_2(a_0a_1+1)}{a_2a_1+1} = \frac{a_2p_1+p_0}{a_2q_1+q_0} = \frac{p_2}{q_2}.$$

Assume, for the sake of induction, that $C_k = [a_0, a_1, \dots, a_k] = \frac{p_k}{q_k}$. Consider,

$$\begin{aligned} C_{k+1} &= [a_0, a_1, \dots, a_k, a_{k+1}] = [a_0, a_1, \dots, a_k + \tfrac{1}{a_{k+1}}] \\ &= \frac{(a_k+\frac{1}{a_{k+1}})p_{k-1}+p_{k-2}}{(a_k+\frac{1}{a_{k+1}})q_{k-1}+q_{k-2}} \quad (\text{Since } \frac{p_k}{q_k} = \frac{a_kp_{k-1}+p_{k-2}}{a_kq_{k-1}+q_{k-2}}) \\ &= \frac{a_{k+1}(a_kp_{k-1}+p_{k-2})+p_{k-1}}{a_{k+1}(a_kq_{k-1}+q_{k-2})+q_{k-1}} \\ &= \frac{a_{k+1}p_k+p_{k-1}}{a_{k+1}q_k+q_{k-1}} \quad (\text{By the induction hypothesis}) \\ &= \frac{p_{k+1}}{q_{k+1}}. \end{aligned}$$

Thus by Mathematical Induction, $C_i = \frac{p_i}{q_i}$, for $0 \le i \le n$. □

We can compute the convergents of Examples 2.2 & 2.6 using this alternate technique.

Example 2.8. *Compute the convergents of*

$$\frac{6x^9+21x^8-41x^7-217x^6-428x^5+6x^3+42x^2+102x+128}{6x^7+3x^6-80x^5+6x+24} = [x^2+3x+5, \tfrac{3}{4}x+3, 2x-7, \tfrac{1}{2}x^5]$$

using Proposition 2.7.

Then $a_0 = x^2+3x+5, a_1 = \frac{3}{4}x+3, a_2 = 2x-7, a_3 = \frac{1}{2}x^5$. So we have,

$$\begin{aligned} p_0 &= x^2+3x+5 \\ p_1 &= (\tfrac{3}{4}x+3)\cdot(x^2+3x+5)+1 = \tfrac{3}{4}x^3+\tfrac{21}{4}x^2+\tfrac{51}{4}x+16 \\ p_2 &= (2x-7)\cdot(\tfrac{3}{4}x^3+\tfrac{21}{4}x^2+\tfrac{51}{4}x+16)+(x^2+3x+5) = \tfrac{3}{2}x^4+\tfrac{21}{4}x^3-\tfrac{41}{4}x^2-\tfrac{217}{4}x-107 \\ p_3 &= (\tfrac{1}{2}x^5)\cdot(\tfrac{3}{2}x^4+\tfrac{21}{4}x^3-\tfrac{41}{4}x^2-\tfrac{217}{4}x-107)+(\tfrac{3}{4}x^3+\tfrac{21}{4}x^2+\tfrac{51}{4}x+16) \\ &= \tfrac{3}{4}x^9+\tfrac{21}{8}x^8-\tfrac{41}{8}x^7-\tfrac{217}{8}x^6-\tfrac{107}{2}x^5+\tfrac{3}{4}x^3+\tfrac{21}{4}x^2+\tfrac{51}{4}x+16 \end{aligned}$$

$$q_0 = 1$$
$$q_1 = \frac{3}{4}x + 3$$
$$q_2 = (2x-7)\cdot(\frac{3}{4}x+3) + 1 = 3x^2 + \frac{9}{2}x - 20$$
$$q_3 = (\frac{1}{2}x^5)\cdot(3x^2 + \frac{9}{2}x - 20) + (\frac{3}{4}x + 3) = \frac{3}{2}x^7 - \frac{9}{4}x^6 - 10x^5 + \frac{3}{4}x + 3.$$

So,

$$C_0 = \frac{p_0}{q_0} = \frac{x^2+3x+5}{1} = x^2 + 3x + 5$$
$$C_1 = \frac{p_1}{q_1} = \frac{\frac{3}{4}x^3 + \frac{21}{4}x^2 + \frac{51}{4}x + 16}{\frac{3}{4}x+3} = \frac{3x^3+21x^2+51x+64}{3x+12}$$
$$C_2 = \frac{p_2}{q_2} = \frac{\frac{3}{4}x^4 + \frac{21}{4}x^3 - \frac{41}{4}x^2 - \frac{217}{4}x - 107}{3x^2 + \frac{9}{2}x - 20} = \frac{6x^4+21x^3-41x^2-217x-428}{12x^2-18x-80}$$
$$C_3 = \frac{p_3}{q_3} = \frac{\frac{3}{4}x^9 + \frac{21}{8}x^8 - \frac{41}{8}x^7 - \frac{217}{8}x^6 - \frac{107}{2}x^5 + \frac{3}{4}x^3 + \frac{21}{4}x^2 + \frac{51}{4}x + 16}{\frac{3}{2}x^7 - \frac{9}{4}x^6 - 10x^5 + \frac{3}{4}x + 3}$$
$$= \frac{6x^9+21x^8-41x^7-217x^6-428x^5+6x^3+42x^2+102x+128}{12x^7-18x^6-80x^5+6x+24}.$$

We can compare these new results of the convergents of this recurrence relationship with those of our first convergent computations.

Example 2.9. *Compute the convergents* $\frac{x^{10}+x^7+x^5+x^3+1}{x^6+x^3+x} = [x^4, x^3, x^2, x]$ *using Proposition 2.7 and compare the results with Example 2.6.*

We have $a_0 = x^4, a_1 = x^3, a_2 = x^2, a_3 = x$. So,

$$p_0 = x^4$$
$$p_1 = (x^3)\cdot(x^4) + 1 = x^7 + 1$$
$$p_2 = (x^2)\cdot(x^7+1) + (x^4) = x^9 + x^4 + x^2$$
$$p_3 = (x)\cdot(x^9+x^4+x^2) + (x^7+1) = x^{10} + x^7 + x^5 + x^3 + 1$$
$$q_0 = 1$$
$$q_1 = x^3$$
$$q_2 = (x^2)\cdot(x^3) + 1 = x^5 + 1$$
$$q_3 = (x)\cdot(x^5+1) + (x^3) = x^6 + x^3 + x.$$

Thus,

$$C_0 = \frac{p_0}{q_0} = \frac{x^4}{1} = x^4$$
$$C_1 = \frac{p_1}{q_1} = \frac{x^7+1}{x^3}$$
$$C_2 = \frac{p_2}{q_2} = \frac{x^9+x^4+x^2}{x^5+1}$$
$$C_3 = \frac{p_3}{q_3} = \frac{x^{10}+x^7+x^5+x^3+1}{x^6+x^3+x}.$$

Notice that in Examples 2.8 & 2.9 the convergents of $\alpha = \frac{f(x)}{g(x)} \in \mathbb{Q}(x)$ with $gcd(f(x), g(x)) = 1$ which can be proven to always be true by the following proposition and its corollary.

Proposition 2.10. *Let $\alpha = [a_1, a_2, \ldots, a_n]$ be expressible as a finite simple continued fraction and let all notation be as in Proposition 2.7. Then, $p_i q_{i-1} - p_{i-1} q_i = (-1)^{i-1}$ for $1 \leq i \leq n$.*

Proof. Notice, if $i = 1$, we have $p_1 q_0 - p_0 q_1 = (a_0 a_1 + 1)1 - a_0 a_1 = 1 = (-1)^{1-1}$.
Assume, for the sake of induction, $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$.
Consider $p_{k+1} q_k - p_{k-1} q_k$

$$\begin{aligned}
&= (a_{k+1} p_k + p_{k-1}) q_k - p_k (a_{k+1} q_k + q_{k-1}) \\
&= p_{k-1} q_k - p_k q_{k-1} \\
&= -(p_k q_{k-1} - p_{k-1} q_k) \\
&= -(-1)^{k-1} \text{ by the induction hypothesis} \\
&= (-1)^k.
\end{aligned}$$

Thus, by Mathematical Induction, $p_i q_{i-1} - p_{i-1} q_i = (-1)^{i-1}$ for $1 \leq i \leq n$. □

One can see that this proposition is illustrated in Examples 2.8 and 2.9. The corollary will show $gcd(f(x), g(x)) = 1$.

Corollary 2.11. *Let $\alpha = [a_1, a_2, \ldots, a_n]$ be a finite simple continued fraction and let all notation be as in Proposition 2.7. Then, $gcd(p_i, q_i) = 1$ for $0 \leq i \leq n$.*

Proof. Clearly, $gcd(p_0, q_0) = 1$. Now, let $d_i = gcd(p_i, q_i)$ for $1 \leq i \leq n$.
Then we have $d_i | p_i q_{i-1}$ and $d_i | p_{i-1} q_i$ for $1 \leq i \leq n$.
So, $d_i | (p_i q_{i-1} - p_{i-1} q_i) \Rightarrow$ by Proposition 2.10, $d_i | (-1)^{i-1}$ for $1 \leq i \leq n$.
Thus, $d_i = 1$ for $1 \leq i \leq n$. □

As we begin to extend our theory to infinite simple continued fractions we need another important corollary.

Corollary 2.12. *Let $\alpha = [a_0, a_1, \ldots, a_n]$ be a finite simple continued fraction with all notation as in Proposition 2.7.*
Then,

$$C_i - C_{i-1} = \frac{(-1)^{i-1}}{q_i q_{i-1}} \quad 1 \leq i \leq n$$
$$\text{and } C_i - C_{i-2} = \frac{(-1)^i a_i}{q_i q_{i-2}} \quad 2 \leq i \leq n$$

Proof. First, $C_i - C_{i-1} = \frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}}$

$$= \frac{p_i q_{i-1} - p_{i-1} q_i}{q_i q_{i-1}}$$

$$= \frac{(-1)^{i-1}}{q_i q_{i-1}} \qquad \text{for } 1 \leq i \leq n \text{ (by Proposition 2.10).}$$

Also, $C_i - C_{i-2} = \frac{p_i}{q_i} - \frac{p_{i-2}}{q_{i-2}}$

$$= \frac{p_i q_{i-2} - p_{i-2} q_i}{q_i q_{i-2}}$$

$$= \frac{(a_i p_{i-1} + p_{i-2}) q_{i-2} - p_{i-2}(a_i q_{i-1} + q_{i-2})}{q_i q_{i-2}}$$

$$= \frac{a_i (p_{i-1} q_{i-2} - p_{i-2} q_{i-1})}{q_i q_{i-2}}$$

$$= \frac{a_i (-1)^{i-2}}{q_i q_{i-2}} \qquad \text{(by Proposition 2.10)}$$

$$= \frac{(-1)^i a_i}{q_i q_{i-2}} \qquad \text{for } 2 \leq i \leq n.$$

□

We are now prepared for infinite continued fractions.

2.3 Infinite Continued Fractions

As previously stated, elements belonging $\mathbb{Q}(x)^* - \mathbb{Q}(x)$ have infinite continued fraction expansions. These elements include the square roots of polynomials with even degrees and rational coefficients; moreover, we will only consider the square roots where the leading coefficients are positive squares. It is in $\mathbb{Q}(x)^*$ that many of the theorems and proofs from traditional Number Theory do not hold, or they require some extra assumptions. This occurs most notably because we lack the notion of absolute value, which is necessary for these proofs. Instead, using the idea of valuations, defined in the introduction, changes the way which we compare two elements creating new issues in these proofs.

Finite continued fraction expansions may be unpredictable, but they will eventually come to an end. In the infinite case, many of these continued fraction expansions will be unpredictable and they never terminate. After discussing the general infinite simple continued fraction expansions we will shift our focus to those with more desirable expansions, i.e periodic and eventually periodic expansions. We will start by proving that the expression as an infinite simple continued fraction is unique before precisely defining when such expansions occur. The following proposition has our first use of valuations.

Proposition 2.13. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$. Then the expression of α as an infinite simple continued fraction is unique.*

Proof. First, by definition, $a_0 = [[\alpha]]$. Also, $\alpha = \lim_{i\to\infty}[a_0, a_1, \ldots, a_i]$ where $v(a_i) > 0$ for $i \geq 0$.

$$\begin{aligned}\Rightarrow \alpha &= \lim_{i\to\infty}(a_0 + \tfrac{1}{[a_1,a_2,\ldots,a_i]}) \\ &= a_0 + \tfrac{1}{\lim_{i\to\infty}[a_1,a_2\ldots,a_i]} \\ &= a_0 + \tfrac{1}{[a_1,a_2,\ldots]}.\end{aligned}$$

To prove uniqueness, let $\alpha = [a_0, a_1, a_2, \ldots] = [b_0, b_1, b_2, \ldots]$ be two expressions of α as infinite continued fractions. By definition, $[[\alpha]] = a_0 = b_0$. By the previous result,

$$\begin{aligned}&a_0 + \tfrac{1}{[a_1,a_2,\ldots]} = b_0 + \tfrac{1}{[b_1,b_2,\ldots]} \\ &\Rightarrow [a_1, a_2, \ldots] = [b_1, b_2, \ldots].\end{aligned}$$

Assume, for the sake of induction, $a_k = b_k$ for $k \geq 0$ and that $[a_{k+1}, a_{k+2}, \ldots] = [b_{k+1}, b_{k+2}, \ldots]$. By the same reasoning as above, $a_{k+1} = b_{k+1}$ and $[a_{k+2}, a_{k+3}, \ldots] = [b_{k+2}, b_{k+3}, \ldots]$. Thus, by Mathematical Induction, $a_i = b_i$ for all $i \geq 0$. □

The fact that an infinite simple continued fraction expansion is unique yields the "only if" direction of the next proposition with ease.

Proposition 2.14. $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$ *if and only if* α *is expressible as an infinite simple continued fraction.*

Proof. ($\Rightarrow$) Let $\alpha = \alpha_0 \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$ and define $a_0, a_1, \ldots$ and $\alpha_1, \alpha_2, \ldots$ by the following recurrence relations:

$$\begin{aligned}a_i &= [[\alpha_i]] \; i \geq 0 \\ \alpha_{i+1} &= \tfrac{1}{\alpha_i - a_i} \; i \geq 0.\end{aligned}$$

Clearly, $a_0, a_1, \ldots \in \mathbb{Q}[x]$. Also, $v(\alpha_i - a_i) < 1 \Rightarrow v(\alpha_{i+1}) = \frac{1}{v(\alpha_i - a_i)} > 1$.
So, $v(\alpha_{i+1}) > 1$ and $v(a_1), v(a_2), \ldots > 0$.
Now, to show $\alpha = [a_0, a_1, \ldots]$, rewrite $\alpha_{i+1} = \frac{1}{\alpha_i - a_i}$ as $\alpha_i = a_i + \frac{1}{\alpha_{i+1}}$

$$\begin{aligned}\Rightarrow \alpha &= \alpha_0 = a_0 + \tfrac{1}{\alpha_1} \\ &= a_0 + \tfrac{1}{a_1 + \frac{1}{\alpha_2}} \\ &\vdots\end{aligned}$$

$$= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_i + \cfrac{1}{\alpha_{i+1}}}}}}$$

$$= [a_0, a_1, \ldots, a_i, \alpha_{i+1}] = \tfrac{\alpha_{i+1}p_i + p_{i-1}}{\alpha_{i+1}q_i + q_{i-1}}\ i > 0 \text{ (by Proposition 2.7).}$$

Then, $v(\alpha - C_i) < 1$ since $v(\frac{1}{x}) = e^{-1}$. In fact, $\lim_{i\to\infty} v(\alpha - C_i) = \lim_{x\to\infty} e^{-x} = 0$. Therefore, $\lim_{i\to\infty} v(\alpha - C_i) = 0 \Rightarrow \lim_{i\to\infty} \alpha - \lim_{i\to\infty} C_i = 0$
$\Rightarrow \alpha = \lim_{i\to\infty} \alpha = \lim_{i\to\infty} C_i = [a_0, a_1, a_2, \ldots]$ as desired.

($\Leftarrow$) Let α be expressible as an infinite simple continued fraction, say $\alpha = [a_0, a_1, \ldots]$. Assume for the sake of contradiction, $\alpha \in \mathbb{Q}(x)$. Then by Theorem 2.4, α is expressible as a finite simple continued fraction, say $\alpha = [b_0, b_1, \ldots, b_n]$. Moreover, by Proposition 2.13, $a_i = b_i$ for $0 \leq i \leq n \Rightarrow \lim_{n\to\infty}[a_{n+1}, a_{n+2}, \ldots] = 0$; a contradiction. □

We can illustrate the above proposition by creating an infinite simple continued fraction expansion which we know is unique by Proposition 2.13.

Example 2.15. *Find the infinite simple continued fraction expansion of $\sqrt{x^2+1} \in \mathbb{Q}(x)^*$ using Proposition 2.13.*

$$\alpha = \alpha_0 = \sqrt{x^2+1} = x + \tfrac{1}{2}x^{-1} + \ldots$$

$$\begin{aligned} a_0 &= [[\alpha_0]] = x & \alpha_1 &= \tfrac{1}{\alpha_0 - a_0} = 2x + \tfrac{1}{2}x^{-1} + \ldots \\ a_1 &= [[\alpha_1]] = 2x & \alpha_2 &= \tfrac{1}{\alpha_1 - a_1} = 2x + \tfrac{1}{2}x^{-1} + \ldots \\ a_2 &= [[\alpha_2]] = 2x & \alpha_3 &= \tfrac{1}{\alpha_2 - a_2} = 2x + \tfrac{1}{2}x^{-1} + \ldots \end{aligned}$$

$$\vdots$$

Thus $a_i = 2x$ for $i > 0$ making the infinite simple continued fraction expansion of $\sqrt{x^2+1} = [x, 2x, 2x, \ldots]$.

We can change the polynomial slightly to observe how the infinite simple continued fraction expansion is affected:

Example 2.16. *Find the infinite simple continued fraction expansion of* $\sqrt{x^2+4x+1} \in \mathbb{Q}(x)^*$ *using Proposition 2.13.*

$$\alpha = \alpha_0 = \sqrt{x^2+4x+1} = x + 2 - \tfrac{3}{2}x^{-1} + \dots$$

$$a_0 = [[\alpha_0]] = x+2 \qquad \alpha_1 = \tfrac{1}{\alpha_0 - a_0} = -\tfrac{2}{3}x - \tfrac{4}{3} + \tfrac{1}{2}x^{-1} - \dots$$

$$a_1 = [[\alpha_1]] = -\tfrac{2}{3}x - \tfrac{4}{3} \qquad \alpha_2 = \tfrac{1}{\alpha_1 - a_1} = 2x + 4 - \tfrac{3}{2}x^{-1} + \dots$$

$$a_2 = [[\alpha_2]] = 2x+4 \qquad \alpha_3 = \tfrac{1}{\alpha_2 - a_2} = -\tfrac{2}{3}x - \tfrac{4}{3} + \tfrac{1}{2}x^{-1} - \dots$$

$$a_3 = [[\alpha_3]] = -\tfrac{2}{3}x - \tfrac{4}{3} \qquad \alpha_4 = \tfrac{1}{\alpha_3 - a_3} = 2x + 4 - \tfrac{3}{2}x^{-1} + \dots$$

$$a_4 = [[\alpha_4]] = 2x+4 \qquad \alpha_5 = \tfrac{1}{\alpha_4 - a_4} = -\tfrac{2}{3}x - \tfrac{4}{3} + \tfrac{1}{2}x^{-1} - \dots$$

$$\vdots$$

where the pattern continues on infinitely many times. Thus, the infinite simple continued fraction expansion of $\sqrt{x^2+4x+1} = [x+2, -\frac{2}{3}x - \frac{4}{3}, 2x+4, -\frac{2}{3}x - \frac{4}{3}, 2x+4, \dots]$.

Not all infinite simple continued fractions expansions have repeating patterns as the last two examples seem to suggest.

Example 2.17. *Find the infinite simple continued fraction expansion of* $\sqrt{x^4+8x^3+16x^2+x+1}$ *using Proposition 2.13.*

$$\alpha = \alpha_0 = \sqrt{x^4+8x^3+16x^2+x+1} = x^2 + 4x + \tfrac{1}{2}x^{-1} - \tfrac{3}{2}x^{-2} + \dots$$

$$a_0 = [[\alpha_0]] = x^2 + 4x \qquad \alpha_1 = \tfrac{1}{\alpha_0 - a_0} = \frac{2x^2+8x+\frac{1}{2}x^{-1}-\frac{3}{2}x^{-2}+\dots}{x+1}$$

$$a_1 = [[\alpha_1]] = 2x+6 \qquad \alpha_2 = \tfrac{1}{\alpha_1 - a_1} = \frac{-2x^2-8x-6-\frac{1}{2}x^{-1}-\frac{3}{2}x^{-2}+\dots}{12x+35}$$

$$a_2 = [[\alpha_2]] = -\tfrac{1}{6}x - \tfrac{13}{72} \qquad \alpha_3 = \tfrac{1}{\alpha_2 - a_2} = \frac{10368x^2+41472x+1656+\frac{1}{2}x^{-1}-\frac{3}{2}x^{-2}+\dots}{276x-133}$$

$$a_3 = [[\alpha_3]] = \tfrac{864}{23}x + \tfrac{89064}{529}$$

$$\alpha_4 = \tfrac{1}{\alpha_3 - a_3} = \frac{559682x^2+2238728x-1298166+\frac{1}{2}x^{-1}-\frac{3}{2}x^{-2}+\dots}{x+1}$$

$$\vdots$$

where no pattern seems readily visible. Although we have only reviewed three iterations of the recurrence relationship, the coefficients are growing rapidly without any sign of possibly repeating. Finding out whether or not the continued fraction expansion will ever repeat is difficult to determine.

As we continue to create more examples of infinite simple continued fractions, we will also be considering the convergents of these expansions. In the finite case, we have found that $\alpha = C_n$ when $i = n$ if $\alpha = [a_0, a_1, \ldots, a_n]$; however, in the infinite case C_i is simply an approximation of α for all i. Therefore, it is worthwhile to discover how accurate of an approximation of α the i^{th} convergent, $C_i = \frac{p_i}{q_i}$, really is. First, we need a proposition as a preliminary result. For this, we will use the valuation as defined in the introduction where $v(\frac{f}{g}) = e^{deg(f)-deg(g)}$.

Proposition 2.18. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$ and let $\frac{p_i}{q_i}, i = 0, 1, 2, \ldots$ be the convergents of the infinite simple continued fraction expansion of α. If $a, b \in \mathbb{Q}[x]$ and $0 < v(b) < v(q_{i+1})$, then $v(q_i\alpha - p_i) \leq v(b\alpha - a)$.*

Proof. Consider the system of equations given by

$$p_i x + p_{i+1} y = a$$
$$q_i x + q_{i+1} y = b.$$

Notice, if $x = 0$ we have $bp_{i+1} = aq_{i+1}$; since $(p_{i+1}, q_{i+1}) = 1$ by Corollary 2.11, we have $q_{i+1}|b$ which is a contradiction to $0 < v(b) < v(q_{i+1})$. Also, if $y = 0$ we have $a = p_i x$ and $b = q_i x$; since $v(x) \geq 1$, we have $v(b\alpha - a) = v(x)v(q_i\alpha - p_i) \geq v(q_i\alpha - p_i)$. Therefore, we may assume $x \neq 0$ and $y \neq 0$. Now, we will show $v(x(q_i\alpha - p_i)) > v(y(q_{i+1}\alpha - p_{i+1}))$.
First, since $b = q_i x + q_{i+1} y$, $q_i x = b - q_{i+1} y \Rightarrow v(q_i x) = v(b - q_{i+1} y)$.
However, $v(b) < v(q_{i+1}) \Rightarrow v(q_i x) = v(q_{i+1} y) \Rightarrow v(q_i)v(x) = v(q_{i+1})v(y)$.
We know that $v(q_i) < v(q_{i+1}) \Rightarrow v(x) > v(y)$.
Also, Consider

$$v(\alpha - \tfrac{p_i}{q_i}) = v(\tfrac{\alpha_{i+1}p_i + p_{i-1}}{\alpha_{i+1}q_i + q_{i-1}} - \tfrac{p_i}{q_i}) = v(\tfrac{-(p_i q_{i-1} - p_{i-1} q_i)}{(\alpha_{i+1}q_i + q_{i-1})q_i})$$
$$= v(\tfrac{-(p_i q_{i-1} - p_{i-1} q_i)}{(\alpha_{i+1}q_i + q_{i-1})q_i}) = v(\tfrac{(-1)^i}{q_i(\alpha_{i+1}q_i + q_{i-1}})$$
$$= \tfrac{1}{v(q_i)v(\alpha_{i+1}q_i + q_{i-1})}.$$

But $v(\alpha_{i+1}q_i) > v(q_{i-1})$

$$\Rightarrow = \tfrac{1}{v(q_i)v(\alpha_{i+1}q_i)} = \tfrac{1}{v(q_i)v(a_{i+1}q_i)}$$
$$= \tfrac{1}{v(q_i)v(a_{i+1}q_i - q_{i-1})} = \tfrac{1}{v(q_i)v(q_{i+1})}.$$

So, $v(q_i\alpha - p_i) = \frac{1}{v(q_{i+1})}$ and $v(q_{i+1}\alpha - p_{i+1}) = \frac{1}{v(q_{i+2})}$ similarly. Since $v(q_{i+1}) < v(q_{i+2})$, $\frac{1}{v(q_{i+1})} > \frac{1}{v(q_{i+2})}$. Thus, $v(x(q_i\alpha - p_i)) > v(y(q_{i+1}\alpha - p_{i+1}))$. Finally, consider

$$v(b\alpha - a) = v((q_i x + q_{i+1}y)\alpha - (p_i x + p_{i+1}y))$$
$$= v(x(q_i\alpha - p_i) + y(q_{i+1}\alpha - p_{i+1})).$$

But $v(x(q_i\alpha - p_i)) > v(y(q_{i+1}\alpha - p_{i+1}))$

$$\Rightarrow = v(x(q_i\alpha - p_i))$$
$$= v(x)v(q_i\alpha - p_i)$$
$$> v(q_i\alpha - p_i) \text{ as desired.}$$

□

We are now ready to prove that the i^{th} convergent of α is the "best" approximation for α.

Corollary 2.19. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$ and let $\frac{p_i}{q_i}, i = 0, 1, 2, \ldots$ be the convergents of the infinite simple continued fraction expansion of α. If $a, b \in \mathbb{Q}[x]$ and $0 \leq v(b) \leq v(q_i)$, then $v(\alpha - \frac{p_i}{q_i}) \leq v(\alpha - \frac{a}{b})$.*

Proof. Assume, by contradiction, $v(\alpha - \frac{p_i}{q_i}) > v(\alpha - \frac{a}{b})$. Then,

$$v(q_i\alpha - p_i) = v(q_i)v(\alpha - \frac{p_i}{q_i})1 > v(b)v(\alpha - \frac{a}{b}) = v(b\alpha - a)$$

which contradicts Proposition 2.18. □

This corollary shows that given any infinite simple continued fraction expansion, $C_i = \frac{p_i}{q_i}$ is the closest approximation of α for a denominator with value less than or equal to q_i. If an approximation closer to α is desired, one must consider $\frac{a}{b}$ with the denominator, b, having greater value than q_i. The next proposition will show that $\frac{a}{b}$ is a "close" approximation to α if and only if it is a convergent of the infinite simple continued fraction expansion.

Proposition 2.20. *Let $a, b \in \mathbb{Q}[x]$ with $gcd(a, b) = 1$ and $v(b) > 0$. Let $\alpha \in \mathbb{Q}(x)^*$, then $v(\alpha - \frac{a}{b}) \leq \frac{1}{v(xb^2)}$ if and only if $\frac{a}{b}$ is a convergent of the infinite simple continued fraction expansion of α.*

Proof. ($\Rightarrow$) Let $\frac{p_i}{q_i}$ be a convergent of the infinite simple continued fraction expansion of α and assume, by contradiction, that $\frac{a}{b}$ is not such a convergent. Since the values of the denominators form a non-decreasing sequence, there exists a unique k such that $v(q_k) \leq v(b) < v(q_{k+1})$. Then by proposition 2.18,

$$v(q_k\alpha - p_k) \leq v(b\alpha - a) = v(b)v(\alpha - \frac{a}{b}) \leq \frac{1}{v(xb)} \text{ by hypothesis.}$$

So $v(\alpha - \frac{p_k}{q_k}) \leq \frac{1}{v(xbq_k)}$. Now, $\frac{a}{b} \neq \frac{p_k}{q_k} \Rightarrow bp_k - aq_k \neq 0 \Rightarrow v(bp_k - aq_k) \geq 1$.

Thus,

$$\frac{1}{v(bq_k)} \leq \frac{v(bp_k - aq_k)}{v(bq_k)}$$
$$= v(\tfrac{p_k}{q_k} - \tfrac{a}{b}) = v((\tfrac{p_k}{q_k} - \alpha) + (\alpha - \tfrac{a}{b}))$$
$$\leq max\{v(\tfrac{p_k}{q_k} - \alpha), v(\alpha - \tfrac{a}{b})\}.$$

We know from Corollary 2.19, $v(\frac{p_k}{q_k} - \alpha) = v(\alpha - \frac{p_k}{q_k}) \leq v(\alpha - \frac{a}{b}) \Rightarrow max\{v(\frac{p_k}{q_k} - \alpha), v(\alpha - \frac{a}{b})\}$

$$\leq max\{v(\alpha - \tfrac{a}{b}), v(\alpha - \tfrac{a}{b})\}$$
$$= v(\alpha - \tfrac{a}{b}) \leq \frac{1}{v(xb^2)} < \frac{1}{v(xb^2)} + \frac{1}{v(xbq_k)}.$$

So we have, $\frac{1}{v(bq_k)} < \frac{1}{v(xb^2)} + \frac{1}{v(xbq_k)}$

$$\Rightarrow v(xb) < v(q_k) + v(b)$$
$$\Rightarrow v(x-1)v(b) < v(q_k)$$

but $v(x-1) > 1 \Rightarrow v(b) < v(q_k)$. Contradiction.

($\Leftarrow$) Let $\frac{a}{b}$ be a convergent of the infinite simple continued fraction expansion of α. Consider $v(\alpha - \frac{a}{b}) = v(\alpha - \frac{p_i}{q_i})$ for some $i \geq 0$. Then,

$$v(\alpha - \tfrac{p_i}{q_i}) = v(\tfrac{\alpha_{i+1}p_i + p_{i-1}}{\alpha_{i+1}q_i + q_{i-1}} - \tfrac{p_i}{q_i}) = v(\tfrac{-(p_i q_{i-1} - p_{i-1}q_i)}{(\alpha_{i+1}q_i + q_{i-1})q_i})$$
$$= v(\tfrac{-(-1)^{i-1}}{(\alpha_{i+1}q_i + q_{i-1})q_i}) \text{ by Proposition 2.10}$$
$$= \frac{1}{v((\alpha_{i+1}q_i + q_{i-1})q_i)}$$
$$= \frac{1}{v((a_{i+1}q_i + q_{i-1})q_i)} \text{ since } v(\alpha_i) = v(a_i) \text{ for all } i$$
$$= \frac{1}{v(q_{i+1}q_i)} \leq \frac{1}{v(xq_iq_i)} = \frac{1}{v(xq_i^2)}$$

since $v(q_i) < v(q_{i+1})$ for all i. □

Example 2.21. *To illustrate Proposition 2.20, consider* $\alpha = \sqrt{x^2+1} = [x, 2x, 2x, \ldots]$.

Then,

$$\alpha = x + \tfrac{1}{2}x^{-1} - \tfrac{1}{8}x^{-3} + \tfrac{1}{16}x^{-5} - \tfrac{5}{128}x^{-7} + \tfrac{7}{256}x^{-9} - \tfrac{21}{1024}x^{-11} + \tfrac{33}{2048}x^{-13} - \tfrac{143}{65536}x^{-19} + \ldots.$$

Consider

$$\frac{a}{b} = \frac{64x^7 + 112x^5 + 56x^3 + 7x}{64x^6 + 80x^4 + 24x^2 + 1}$$

$$= x + \tfrac{1}{2}x^{-1} - \tfrac{1}{8}x^{-3} + \tfrac{1}{16}x^{-5} - \tfrac{5}{128}x^{-7} + \tfrac{7}{256}x^{-9} - \tfrac{21}{1024}x^{-11} + \tfrac{1048x^4 + 476x^2 + 21}{65536x^{17} + 81920x^{15} + 24576x^{13} + 1024x^{11}}.$$

Since $v(\alpha - \frac{a}{b}) = e^{-13} \leq \frac{1}{v(xb^2)} = e^{-13}$, $\frac{64x^7+112x^5+56x^3+7x}{64x^6+80x^4+24x^2+1}$ is a convergent of the infinite simple continued fraction expansion of $\sqrt{x^2+1}$. In fact,

$$C_6 = [x, 2x, 2x, 2x, 2x, 2x, 2x] = \frac{64x^7+112x^5+56x^3+7x}{64x^6+80x^4+24x^2+1} = \frac{a}{b}.$$

On the other hand, consider each of the convergents of the infinite simple continued fraction expansion of $\sqrt{x^2+1}$.

$$\begin{aligned}
C_0 &= \tfrac{p_0}{q_0} = x \\
C_1 &= \tfrac{p_1}{q_1} = x + \tfrac{1}{2}x^{-1} = \tfrac{2x^2+1}{2x} \\
C_2 &= \tfrac{p_2}{q_2} = x + \tfrac{1}{2}x^{-1} - \tfrac{1}{8}x^{-3} + \tfrac{1}{32x^5+8x^3} = \tfrac{4x^3+3x}{4x^2+1} \\
C_3 &= \tfrac{p_3}{q_3} = x + \tfrac{1}{2}x^{-1} - \tfrac{1}{8}x^{-3} + \tfrac{1}{16}x^{-5} - \tfrac{1}{32x^7++16x^5} = \tfrac{8x^4+8x^2+1}{8x^3+4x} \\
&\vdots
\end{aligned}$$

Then,

$$\begin{aligned}
v(\alpha - C_0) &= v(\alpha - \tfrac{x}{1}) = e^{-1} \leq \tfrac{1}{v(x(1)^2)} = e^{-1} \\
v(\alpha - C_1) &= v(\alpha - \tfrac{2x^2+1}{2x}) = e^{-3} \leq \tfrac{1}{v(x(2x)^2)} = e^{-3} \\
v(\alpha - C_2) &= v(\alpha - \tfrac{4x^3+3x}{4x^2+1}) = e^{-5} \leq \tfrac{1}{v(x(4x^2+1)^2)} = e^{-5} \\
v(\alpha - C_3) &= v(\alpha - \tfrac{8x^4+8x^2+1}{8x^3+4x}) = e^{-7} \leq \tfrac{1}{v(x(8x^3+4x)^2)} = e^{-7} \\
&\vdots
\end{aligned}$$

In this example, $v(\alpha - \frac{p_i}{q_i}) = \frac{1}{v(xq_i^2)}$ for all i because $v(a_0) = v(a_1) = v(a_2) = \ldots$. Choosing a new example where $v(a_i)$ varies will cause $v(\alpha - \frac{p_i}{q_i}) \leq \frac{1}{v(xq_i^2)}$.

2.4 Eventually Periodic Continued Fractions

In the introduction we discussed what it meant to be periodic and eventually periodic and Examples 2.15 and 2.16 are those where the infinite simple continued fraction expansions repeat. Here is the definition:

Definition 2.22. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$ and let $\alpha = [a_0, a_1, \ldots]$ be the infinite simple continued fraction expansion of α. α is said to be eventually periodic if there exists non-negative integers p and N such that $a_n = a_{n+p}$ for all $n \geq N$. We call the sequence $a_N, a_{N+1}, \ldots, a_{N+(p-1)}$ the period of α where p is minimal. The eventually periodic continued fraction expansion is denoted,*

$$\alpha = [a_0, a_1, \ldots a_{N-1}, \overline{a_N, a_{N+1}, \ldots a_{N+(p-1)}}].$$

If N=0, the infinite simple continued fraction expansion is called purely periodic, or periodic for short.

Example 2.23. *Use Example 2.15 to demonstrate purely periodic and eventually periodic infinite simple continued fraction expansions.*

By Example 2.15 we have that $\sqrt{x^2+1} = [x, \overline{2x}]$ is an eventually periodic continued fraction expansion. So,

$$\sqrt{x^2+1} = [x, \overline{2x}] = x + \cfrac{1}{2x + \cfrac{1}{2x + \cfrac{1}{\ddots}}}$$

then adding x to both sides yields

$$x + \sqrt{x^2+1} = 2x + \cfrac{1}{2x + \cfrac{1}{2x + \cfrac{1}{\ddots}}} = [\overline{2x}]$$

is a purely periodic infinite simple continued fraction expansion.

We would now like to categorize some particular elements of $\mathbb{Q}(x)^* - \mathbb{Q}(x)$ beginning with a definition.

Definition 2.24. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$. Then α is a quadratic surd if α is a root of a quadratic polynomial $Ax^2 + Bx + C$ with $A, B, C \in \mathbb{Q}[x]$ and $A \neq 0$.*

Then the following proposition categorizes the above elements elegantly.

Proposition 2.25. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$, then α is a quadratic surd if and only if $\alpha = \frac{a+\sqrt{b}}{c}$ where $a, b, c \in \mathbb{Q}[x]$, $v(b) > 0$, b is not a perfect square, and $c \neq 0$.*

Proof. ($\Rightarrow$) Assume that α is a quadratic surd. Then there exists $A, B, C \in \mathbb{Q}[x]$ with $A \neq 0$ such that $A(\alpha)^2 + B(\alpha) + C = 0$. By the quadratic formula, we have

$$\alpha = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Note, $B^2 - 4AC$ is not a perfect square and $v(B^2 - 4AC) > 0$ since α is a quadratic surd. Letting $a = -B$, $b = B^2 - 4AC$, and $c = 2A$ and taking the plus sign yields $\alpha = \frac{a+\sqrt{b}}{c}$. Otherwise, letting $a = B$, $b = B^2 - 4AC$, and $c = -2A$ and taking the minus sign also yields the desired result.

($\Leftarrow$) Assume that $\alpha = \frac{a+\sqrt{b}}{c}$ where $a, b, c \in \mathbb{Q}[x]$, $v(b) > 0$, b is not a perfect square, and $c \neq 0$. Then $c^2 \neq 0 \Rightarrow \alpha$ is a root of the quadratic polynomial, $c^2x^2 - 2acx + (a^2 - b)$. Let $A = c^2$, $B = -2ac$, $C = a^2 - b$. Then α is a root of the polynomial $Ax^2 + Bx + C$ with $A, B, C \in \mathbb{Q}[x]$ and $A \neq 0$ as desired. □

We will explore the notion of a quadratic surd by examining its continued fraction expansion. The following lemma will begin our study of the quadratic surd by yielding the characteristics which it possesses.

Lemma 2.26. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$, then α is a quadratic surd if and only if $\alpha = \frac{P+\sqrt{d}}{Q}$ where $P, Q, d \in \mathbb{Q}[x]$, $v(d) > 0$, d is not a perfect square, $Q \neq 0$, and $Q|d - P^2$.*

Proof. ($\Rightarrow$) Assume that α is a quadratic surd. Then by Proposition 2.25, $\alpha = \frac{a+\sqrt{b}}{c}$ where $a, b, c \in \mathbb{Q}[x]$, $v(b) > 0$, b is not a perfect square, and $c \neq 0$. Multiplying the numerator and denominator by c^2 (to ensure that the leading coefficient remains positive) yields,

$$\alpha = \frac{ac^2+c^2\sqrt{b}}{c^3} = \frac{ac^2+\sqrt{bc^4}}{c^3}.$$

Letting $P = ac^2$, $d = bc^4$, and $Q = c^3$ produces the desired result.

($\Leftarrow$) Let $\alpha = \frac{P+\sqrt{d}}{Q}$ where $P, Q, d \in \mathbb{Q}[x]$, $v(d) > 0$, d is not a perfect square, $Q \neq 0$, and $Q|d - P^2$; then, by Proposition 2.25, α is a quadratic surd. □

Computing the infinite simple continued fraction expansions of a quadratic surd can seem somewhat mysterious at times when using the recurrence relationship defined in the first half of Proposition 2.14. We now use the last lemma to define an alternate method for finding the continued fraction expansion of a quadratic surd that not only offers more information, but the new information we obtain will have many unique properties.

Proposition 2.27. *By Lemma 2.26, let*

$$\alpha = \frac{P_0+\sqrt{d}}{Q_0}$$

be a quadratic surd where $P_0, d, Q_0 \in \mathbb{Q}[x]$, $d \neq 0$, d is not a perfect square, $Q_0 \neq 0$, and $Q_0|(d - P_0^2)$. Define $\alpha_0, \alpha_1, \alpha_2, \ldots, a_0, a_1, a_2, \ldots, P_1, P_2, \ldots, Q_1, Q_2, \ldots$ by the following recurrence relations:

$$\alpha_i = \frac{P_i+\sqrt{d}}{Q_i}$$
$$a_i = [[\alpha_i]]$$
$$P_{i+1} = a_iQ_i - P_i$$
$$Q_{i+1} = \frac{d-P_{i+1}^2}{Q_i}$$

for all $i \geq 0$. *Then* $\alpha = [a_0, a_1, a_2, \ldots]$.

Proof. First, we can show $P_i, Q_i \in \mathbb{Q}[x], Q_i \neq 0$ and $Q_i|(d-P_i^2)$ for all $i \geq 0$ by induction. Notice, if i = 0, then $\alpha = \frac{P_0+\sqrt{d}}{Q_0}$ for which the conditions are true by assumption. Now, assume that $k \geq 0$ and $P_k, Q_k \in \mathbb{Q}[x]$, $Q_k \neq 0$, and $Q_k|(d-P_k^2)$. Since $P_{k+1} = a_kQ_k - P_k$, clearly $P_{k+1} \in \mathbb{Q}[x]$ since $P_k, Q_k \in \mathbb{Q}[x]$ by the induction hypothesis. Now, consider

$$Q_{k+1} = \frac{d-P_{k+1}^2}{Q_k} = \frac{d-(a_kQ_k-P_k)^2}{Q_k} = \frac{d-P_k^2}{Q_k} + 2a_kP_k - a_k^2Q_k.$$

Then $Q_{k+1} \in \mathbb{Q}[x]$. Furthermore, since d is not a perfect square, $d - P_{k+1}^2 \neq 0$ so $Q_{k+1} \neq 0$. Finally, $Q_{k+1} = \frac{d-P_{k+1}^2}{Q_k} \Rightarrow Q_k = \frac{d-P_{k+1}^2}{Q_{k+1}}$. Since $Q_k|(d - P_{k+1}^2), Q_{k+1} \in \mathbb{Q}[x] \Rightarrow Q_{k+1}|(d - P_{k+1}^2)$. Thus, by Mathematical Induction, $P_i, Q_i \in \mathbb{Q}[x]$, $Q_i \neq 0$, and $Q_i|(d - P_i^2)$ for all $i \geq 0$.

Now, we want to show $\alpha = [a_0, a_1, a_2, \ldots]$, but by the first half of Proposition 2.14 it suffices to show $\alpha_{i+1} = \frac{1}{\alpha_i - a_i}$. Consider

$$\begin{aligned}
\alpha_i - a_i &= \frac{P_i+\sqrt{d}}{Q_i} - a_i = \frac{\sqrt{d}-(a_iQ_i-P_i)}{Q_i} \\
&= \frac{\sqrt{d}-P_{i+1}}{Q_i} = \frac{(\sqrt{d}-P_{i+1})(\sqrt{d}+P_{i+1})}{Q_i(\sqrt{d}+P_{i+1})} \\
&= \frac{\sqrt{d}-P_{i+1}^2}{Q_i(\sqrt{d}+P_{i+1})} = \frac{Q_iQ_{i+1}}{Q_i(\sqrt{d}+P_{i+1})} \text{ (since } Q_{i+1} = \frac{d-P_{i+1}^2}{Q_i}) \\
&= \frac{Q_{i+1}}{\sqrt{d}+P_{i+1}} = \frac{1}{\alpha_{i+1}}.
\end{aligned}$$

So $\alpha_{i+1} = \frac{1}{\alpha_i - a_i} \Rightarrow \alpha = [a_0, a_1, a_2 \ldots]$. ☐

A few more results:

Lemma 2.28. *Let* α *be a quadratic surd and let* $a, b, c, d \in \mathbb{Q}[x]$. *Then,*

$$\alpha = \frac{a\alpha+b}{c\alpha+d}$$

belongs to either $\mathbb{Q}(x)$ *or* $\mathbb{Q}(x)^* - \mathbb{Q}(x)$.

Proof. By Lemma 2.26, $\alpha = \frac{m+\sqrt{n}}{l}$ with $m, n, l \in \mathbb{Q}[x]$ and n not a perfect square. Thus,

$$\begin{aligned}\frac{a\alpha+b}{c\alpha+d} &= \frac{a\frac{m+\sqrt{n}}{l}+b}{c\frac{m+\sqrt{n}}{l}+d}\\ &= \frac{(am+lb)+a\sqrt{n}}{(cm+ld)+c\sqrt{n}}\\ &= \frac{(am+lb+a\sqrt{n})(cm+ld-c\sqrt{n})}{(cm+ld+c\sqrt{n})(cm+ld-c\sqrt{n})}\\ &= \frac{(am+lb)(cm+ld)+(ald-clb)\sqrt{n}-acn}{(cm+ld)^2-c^2n}.\end{aligned}$$

Letting $P = (am+lb)(cm+ld) - acn$, $Q = (cm+ld)^2 - c^2n$, and $r = ald - clb$ we have, $\frac{a\alpha+b}{c\alpha+d} = \frac{p+r\sqrt{n}}{Q}$ where $P, Q, r \in \mathbb{Q}[x]$. Notice, if $b = \frac{a}{c}d$, then $r = 0 \Rightarrow \frac{a\alpha+b}{c\alpha+d} = \frac{P}{Q} \in \mathbb{Q}(x)$. Otherwise $\frac{a\alpha+b}{c\alpha+d}$ is a quadratic surd by Proposition 2.25. So $\frac{a\alpha+b}{c\alpha+d} \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$. □

As in the traditional study of Number Theory, we would like to consider the conjugate of a quadratic surd. The conjugate will be referred to multiple times throughout this section.

Definition 2.29. *Let $\alpha = \frac{a+\sqrt{b}}{c}$ be a quadratic surd where $a, b, c \in \mathbb{Q}[x]$ and $c \neq 0$. The conjugate of α, denoted α', is*

$$\alpha' = \frac{a-\sqrt{b}}{c}.$$

We now present a lemma which summarizes the elementary properties of the conjugate.

Lemma 2.30. *Let $\alpha_1 = \frac{a_1+\sqrt{b}}{c_1}$ and $\alpha_2 = \frac{a_2+\sqrt{b}}{c_2}$ where $a_1, a_2, b, c_1, c_2 \in \mathbb{Q}[x]$ and $c_1, c_2 \neq 0$. Then*

a) $(\alpha_1 + \alpha_2)' = \alpha_1' + \alpha_2'$
b) $(\alpha_1 - \alpha_2)' = \alpha_1' - \alpha_2'$
c) $(\alpha_1\alpha_2)' = \alpha_1'\alpha_2'$
d) $(\frac{\alpha_1}{\alpha_2})' = \frac{\alpha_1'}{\alpha_2'}$.

Proof. The following proofs are straightforward computations.

a) $$\begin{aligned}(\alpha_1+\alpha_2)' &= \left(\frac{a_1+\sqrt{b}}{c_1} + \frac{a_2+\sqrt{b}}{c_2}\right)'\\ &= \left(\frac{c_2a_1+c_1a_2+(c_2+c_1)\sqrt{b}}{c_1c_2}\right)' = \frac{c_2a_1+c_1a_2-(c_2+c_1)\sqrt{b}}{c_1c_2}\\ &= \frac{c_2a_1-c_2\sqrt{b}}{c_1c_2} + \frac{c_1a_2-c_1\sqrt{b}}{c_1c_2} = \frac{a_1-\sqrt{b}}{c_1} + \frac{a_2-\sqrt{b}}{c_2} = \alpha_1' + \alpha_2'.\end{aligned}$$

b) $(\alpha_1 - \alpha_2)' = (\frac{a_1+\sqrt{b}}{c_1} - \frac{a_2+\sqrt{b}}{c_2})'$

$= \frac{a_1+\sqrt{b}}{c_1} + \frac{-a_2-\sqrt{b}}{c_2})' = \frac{a_1-\sqrt{b}}{c_1} + \frac{-a_2+\sqrt{b}}{c_2}$ by a)

$= \frac{a_1-\sqrt{b}}{c_1} - \frac{a_2-\sqrt{b}}{c_2} = \alpha_1' - \alpha_2'.$

c) $(\alpha_1\alpha_2)' = ((\frac{a_1+\sqrt{b}}{c_1})(\frac{a_2+\sqrt{b}}{c_2}))'$

$= (\frac{a_1a_2+b+(a_1+a_2)\sqrt{b}}{c_1c_2})' = \frac{a_1a_2+b-(a_1+a_2)\sqrt{b}}{c_1c_2}$

$= \frac{(a_1-\sqrt{b})(a_2-\sqrt{b})}{c_1c_2} = \frac{a_1-\sqrt{b}}{c_1}\frac{a_2-\sqrt{b}}{c_2} = \alpha_1'\alpha_2'.$

d) $(\frac{\alpha_1}{\alpha_2})' = (\frac{\frac{a_1+\sqrt{b}}{c_1}}{\frac{a_2+\sqrt{b}}{c_2}})' = (\frac{a_1c_2+c_2\sqrt{b}}{a_2c_1+c_1\sqrt{b}})'$

$= (\frac{(a_1c_2+c_2\sqrt{b})(a_2c_1-c_1\sqrt{b})}{(a_2c_1+c_1\sqrt{b})(a_2c_1-c_1\sqrt{b}}))'$

$= \frac{(a_1c_2-c_2\sqrt{b})(a_2c_1+c_1\sqrt{b})}{(a_2c_1)^2-c_1^2)}$ by c)

$= \frac{(a_1c_2-c_2\sqrt{b})(a_2c_1+c_1\sqrt{b})}{(a_2c_1+c_1\sqrt{b})(a_2c_1-c_1\sqrt{b})}$

$= \frac{a_1c_2-c_2\sqrt{b}}{a_2c_1-c_1\sqrt{b}} = (\frac{\frac{a_1-\sqrt{b}}{c_1}}{\frac{a_2-\sqrt{b}}{c_2}}) = \frac{\alpha_1'}{\alpha_2'}.$ □

The previous results are exactly what we need to create the best possible characterization of a quadratic surd. The following proposition and its modified converse are vital to finding the solutions to Pell's Equation.

Theorem 2.31. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$. If the expression of α as an infinite simple continued fraction is eventually periodic, then α is a quadratic surd.*

Proof. Assume the expression of α as an infinite simple continued fraction is eventually periodic. Then there exists integers p,N such that $p, N \geq 0$ with,

$$\alpha = [a_0, a_1, \ldots, a_{N-1}, \overline{a_N, a_{N+1}, + \ldots, +a_{N+(p-1)}}].$$

Letting $\beta = [\overline{a_N, a_{N+1}, + \ldots, +a_{N+(p-1)}}]$ we have

$$\beta = [a_N, a_{N+1}, + \ldots, +a_{N+(p-1)}, \beta]$$
$$\Rightarrow \beta = \frac{\beta P_{p-1}+P_{p-2}}{\beta Q_{p-1}+Q_{p-2}} \text{ by Proposition 2.7}$$

where $\frac{P_{p-2}}{Q_{p-2}}$ and $\frac{P_{p-1}}{Q_{p-1}}$ are convergents of the purely periodic continued fraction expansion $[\overline{a_N, a_{N+1}, +\dots, +a_{N+(p-1)}}]$. Since the expression of β as a simple continued fraction is infinite, $\beta \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$. Also, β is a root of the quadratic polynomial $Q_{p-1}x^2 + (Q_{p-2} - P_{p-1})x - P_{p-2}$, so β is a quadratic surd. Now,

$$\alpha = [a_0, a_1, \dots, a_{N-1}, \beta] = \frac{\beta P_{N-1} + P_{N-2}}{\beta Q_{N-1} + Q_{N-2}} \text{ by Proposition 2.7}$$

where $\frac{P_{N-2}}{Q_{N-2}}$ and $\frac{P_{N-1}}{Q_{N-1}}$ are convergents of $[a_0, a_1, \dots, a_{N-1}]$. Since the expression of α as a simple continued fraction is infinite, $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$; however, since $\beta \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$, Lemma 2.28 shows that α is a quadratic surd. □

We can compute the quadratic surd that the eventually periodic simple continued fraction represents by first finding the β as described above and adding the a_i not belonging to the period.

Example 2.32. *Find the quadratic surd represented by* $\alpha = [x^2, \overline{2x, 3x^2}]$.

First, we will find the quadratic surd represented by the periodic infinite simple continued fraction $\beta = [\overline{2x, 3x^2}]$. Notice $\beta = [2x, 3x^2, \overline{2x, 3x^2}] = [2x, 3x^2, \beta]$ so we have,

$$\beta = 2x + \frac{1}{3x^2 + \frac{1}{\beta}}$$

$$\Rightarrow \beta = \frac{6x^3\beta + 2x + \beta}{3x^2\beta + 1}$$

$$\Rightarrow 3x^2\beta^2 - 6x^3\beta - 2x = 0$$

which yields, by the quadratic formula, $\beta = \frac{6x^3 + \sqrt{36x^6 + 24x^3}}{6x^2}$. Thus

$$\alpha = [x^2, \overline{2x, 3x^2}] = x^2 + \frac{1}{\frac{6x^3 + \sqrt{36x^6 + 24x^3}}{6x^2}}$$

$$= x^2 + \frac{6x^2}{6x^3 + \sqrt{36x^6 + 24x^3}}$$

$$= \frac{6x^5 + x^2\sqrt{36x^6 + 24x^3} + 6x^2}{6x^3 + \sqrt{36x^6 + 24x^3}}$$

$$= \frac{-2x^3 + \sqrt{36x^6 + 24x^3}}{4x}.$$

Therefore $[x^2, \overline{2x, 3x^2}] = \frac{-2x^3 + \sqrt{36x^6 + 24x^3}}{4x}$.

Although it was not proven in Example 2.17, it is suggested that $\sqrt{x^4 + 8x^3 + 16x^2 + x + 1}$ does not have an eventually periodic continued fraction expansion leading us to believe that the converse of Theorem 2.31 does not hold; that is, given a quadratic surd, its infinite simple continued fraction expansion may not necessarily be eventually periodic. The following example is a counterexample to the converse of Theorem 2.31.

Example 2.33. *Find the infinite simple continued fraction expansion of* $\alpha = \sqrt{x^6 + 2x^4}$ *using Proposition 2.27.*

$$P_0 = 0 \quad Q_0 = 1 \quad \alpha = \alpha_0 = \sqrt{x^6 + 2x^4} = x^3 + x - \tfrac{1}{2}x^{-1} + \tfrac{1}{2}x^{-3} + \dots$$

$$a_0 = [[\alpha_0]] = x^3 + x$$

$P_1 = x^3 + x$	$Q_1 = -x^2$	$\alpha_1 = -\frac{x^3+x+\sqrt{x^6+2x^4}}{x^2}$	$a_1 = [[\alpha_1]] = -2x$
$P_2 = x^3 - x$	$Q_2 = -4x^2 + 1$	$\alpha_2 = -\frac{x^3-x+\sqrt{x^6+2x^4}}{4x^2-1}$	$a_2 = [[\alpha_2]] = -\frac{1}{2}x$
$P_3 = x^3 + \frac{1}{2}x$	$Q_3 = -\frac{1}{4}x^2$	$\alpha_3 = -\frac{4x^3+2x+4\sqrt{x^6+2x^4}}{x^2}$	$a_3 = [[\alpha_3]] = -8x$
$P_4 = x^3 - \frac{1}{2}x$	$Q_4 = -12x^2 + 1$	$\alpha_4 = -\frac{2x^3-x+2\sqrt{x^6+2x^4}}{24x^2-2}$	$a_4 = [[\alpha_4]] = -\frac{1}{6}x$
$P_5 = x^3 + \frac{1}{3}x$	$Q_5 = -\frac{1}{9}x^2$	$\alpha_5 = -\frac{9x^3+3x+9\sqrt{x^6+2x^4}}{x^2}$	$a_5 = [[\alpha_5]] = -18x$
$P_6 = x^3 - \frac{1}{3}x$	$Q_6 = -24x^2 + 1$	$\alpha_6 = -\frac{3x^3-x+3\sqrt{x^6+2x^4}}{72x^2-3}$	$a_6 = [[\alpha_6]] = -\frac{1}{12}x$
$P_7 = x^3 + \frac{1}{4}x$	$Q_7 = -\frac{1}{16}x^2$	$\alpha_7 = -\frac{16x^3+4x+16\sqrt{x^6+2x^4}}{x^2}$	$a_7 = [[\alpha_7]] = -32x$
$P_8 = x^3 - \frac{1}{4}x$	$Q_8 = -40x^2 + 1$	$\alpha_8 = -\frac{4x^3-x+4\sqrt{x^6+2x^4}}{160x^2-4}$	$a_8 = [[\alpha_8]] = -\frac{1}{20}x$

$$\vdots$$

So $\alpha = [x^3+x, -2x, -\frac{1}{2}x, -8x, -\frac{1}{6}x, -18x, -\frac{1}{12}x, -32, -\frac{1}{20}x, \dots]$ where the infinite simple continued fraction expansion is not periodic, but a pattern does present itself. It turns out that $a_{2i-1} = -2(i)^2x$ and $a_{2i} = -(\frac{1}{(i)(i+1)})x$ for $i \geq 1$. Following this pattern, the infinite simple continued fraction expansion of $\sqrt{x^6 + 2x^4}$ is guaranteed to never be periodic. More on this example is covered in [Vic78].

The best result that we could possibly hope for is to characterize exactly which quadratic surds have eventually periodic infinite simple continued fraction expansions. However, while there are many which are characterizable, equally as many of them are not. In the following sections we will assume that the quadratic surds which we are considering have eventually periodic infinite simple continued fraction expansions.

2.5 Periodic Continued Fractions

Although we were unable to characterize the infinite simple continued fraction expansions of quadratic surds as precisely as in the traditional study, we will be able to examine those with eventually periodic expansions and classify them precisely by their expansions in this section. We will start with a theorem.

Theorem 2.34. *If α is a quadratic surd with a periodic continued fraction expansion, then the expansion is purely periodic if and only if $v(\alpha) > 1$ and $v(\alpha') < 1$, where α' denotes the conjugate of α. For ease of notation, we will call such quadratic surds 'reduced'.*

Proof. Let α be a quadratic surd with a periodic continued fraction expansion.
($\Rightarrow$) Assume that $\alpha = \alpha_0$ with $v(\alpha) > 1$ and $v(\alpha') < 1$. Then, by definition, $\alpha = [a_0, a_1, \ldots]$ where

$$a_i = [[\alpha_i]],\ i \geq 0$$
$$\alpha_{i+1} = \frac{1}{\alpha_i - a_i},\ i \geq 0.$$

By the second equation, $\alpha_i = a_i + \frac{1}{\alpha_{i+1}}$. Then $\alpha_i' = a_i + \frac{1}{\alpha_{i+1}'}$ for all i. Now, we have $\alpha = \alpha_0 \Rightarrow v(\alpha_0') < 1 \Rightarrow v(\alpha_0') = v(a_0 + \frac{1}{\alpha_1'})$, but $v(\alpha_0') < 1$ and $v(\alpha_0) > 1 \Rightarrow v(\frac{1}{\alpha_1'}) > 1$. Thus, $v(\alpha_1') < 1$. For the sake of induction, assume $v(\alpha_i') < 1$ for $i = k$. Then, $v(\alpha_k') = v(a_k + \frac{1}{v(\alpha_{k+1}')})$. However, $v(\alpha_k') < 1 \Rightarrow v(a_k + \frac{1}{\alpha_{k+1}'}) < 1$ and $v(a_k) > 1 \Rightarrow v(\frac{1}{\alpha_{k+1}'}) > 1$. So, $v(\alpha_{k+1}') < 1$. Therefore, by Mathematical Induction, $v(\alpha_i') < 1$ for all i. Furthermore, $v(\alpha_i') < 1 \Rightarrow a_i + \frac{1}{\alpha_{i+1}'} = m_1 x^{-1} + m_2 x^{-2} + \ldots$ where $m_i \in \mathbb{Q}$ for $i = 1, 2, \ldots$. Thus, $a_i = -\frac{1}{\alpha_{i+1}'} + m_1 x^{-1} + m_2 x^{-2} + \ldots \Rightarrow a_i = [[\frac{-1}{\alpha_{i+1}'}]]$ for $i \geq 0$.
Now, since α is a quadratic surd with a periodic continued fraction expansion, $\alpha_j = \alpha_k$ for some $j, k \in \mathbb{N}$ with $j < k$. Therefore, $\alpha_j' = \alpha_k'$ and $a_{j-1} = [[\frac{-1}{\alpha_j'}]] = [[\frac{-1}{\alpha_k}]] = a_{k-1}$. So, $\alpha_{j-1} = a_{j-1} + \frac{1}{\alpha_j} = a_{k-1} + \frac{1}{\alpha_k} = \alpha_{k-1}$. Iterating this process j times yields $\alpha_0 = \alpha_{k-j}$.

$$\begin{aligned}\Rightarrow \alpha = \alpha_0 &= [a_0, a_1, \ldots, a_{k-j-1}, \alpha_{k-j}] \\ &= [a_0, a_1, \ldots, a_{k-j-1}, \alpha_0] \\ &= [\overline{a_0, a_1, \ldots, a_{k-j-1}}].\end{aligned}$$

Thus, the infinite simple continued fraction expansion of α is purely periodic.

($\Leftarrow$) Assume that the expression of α as an infinite simple continued fraction is purely periodic with period length $p+1$ such that $\alpha = [\overline{a_0, a_1, \ldots, a_p}]$ where $a_0, a_1, \ldots, a_{p-1} \in \mathbb{Q}[x]$. Since α is purely periodic, $a_0 \in \mathbb{Q}[x]$, but $a_0 \notin \mathbb{Q}$. Thus $v(\alpha) > 1$. Now, $\alpha = [a_0, a_1, \ldots, a_p, \alpha] = \frac{\alpha p_p + p_{p-1}}{\alpha q_p + q_{p-1}}$ (from proposition 7.6) where $\frac{p_{p-1}}{q_{p-1}}$ and $\frac{p_p}{q_p}$ are the $(p-1)^{th}$ and p^{th} convergents of $[\overline{a_0, a_1, \ldots, a_p}]$, respectively. So, $\alpha' = \frac{\alpha' p_p + p_{p-1}}{\alpha' q_p + q_{p-1}}$.

Algebraic manipulations of these two equations yield that α and α' are roots of the quadratic polynomial

$$q_p X^2 + (q_{p-1} - p_p)X - p_{p-1}.$$

Moreover, $\alpha' = \frac{-1}{\beta}$ where $\beta = [\overline{a_p, a_{p-1}, \ldots, a_0}]$.
But, $v(\alpha) = v([\overline{a_0, a_1, \ldots, a_p}]) > 1 \Rightarrow v(\beta) > 1$. So, $v(\frac{-1}{\beta}) = v(\alpha') < 1$ as desired. $\square$

To illustrate this theorem, consider the following example.

Example 2.35. *Find the infinite simple continued fraction expansion of*

$$\alpha = x^2 + \tfrac{1}{2}x + 1 + \sqrt{x^4 + x^3 + \tfrac{9}{4}x^2 + 3x + 2}.$$

First, notice $v(\alpha) > 1$ and since $\sqrt{x^4 + x^3 + \frac{9}{4}x^2 + 3x + 2} = x^2 + \frac{1}{2}x + 1 + x^{-1} + \ldots$, $v(\alpha') = v(x^2 + \frac{1}{2}x + 1 - \sqrt{x^4 + x^3 + \frac{9}{4}x^2 + 3x + 2}) = v(x^2 + \frac{1}{2}x + 1 - (x^2 + \frac{1}{2}x + 1 + x^{-1} + \ldots)) = v(-x^{-1} - \ldots) < 1$ so α is reduced. Moreover, $\sqrt{x^4 + x^3 + \frac{9}{4}x^2 + 3x + 2}$ has been chosen due to its eventually periodic continued fraction expansion. By Proposition 2.27,

$$P_0 = x^2 + \tfrac{1}{2}x + 1 \qquad Q_0 = 1 \qquad \alpha = \alpha_0 = x^2 + \tfrac{1}{2}x + 1 + \sqrt{x^4 + x^3 + \tfrac{9}{4}x^2 + 3x + 2}$$
$$a_0 = [[\alpha_0]] = 2x^2 + x + 2$$

$$P_1 = x^2 + \tfrac{1}{2}x + 1 \qquad Q_1 = 2x + 1 \qquad \alpha_1 = \frac{x^2 + \frac{1}{2}x + 1 + \sqrt{x^4 + x^3 + \frac{9}{4}x^2 + 3x + 2}}{2x+1} \qquad a_1 = [[\alpha_1]] = x$$

$$P_2 = x^2 + \tfrac{1}{2}x - 1 \qquad Q_2 = 2x + 1 \qquad \alpha_2 = \frac{x^2 + \frac{1}{2}x - 1 + \sqrt{x^4 + x^3 + \frac{9}{4}x^2 + 3x + 2}}{2x+1} \qquad a_2 = [[\alpha_2]] = x$$

$$P_3 = x^2 + \tfrac{1}{2}x + 1 \qquad Q_3 = 1 \qquad \alpha_3 = x^2 + \tfrac{1}{2}x + 1 + \sqrt{x^4 + x^3 + \tfrac{9}{4}x^2 + 3x + 2}$$
$$a_3 = [[\alpha_3]] = 2x^2 + x + 2.$$

Since $\alpha_3 = \alpha_1$, the infinite simple continued fraction expansion of

$$x^2 + \tfrac{1}{2}x + 1 + \sqrt{x^4 + x^3 + \tfrac{9}{4}x^2 + 3x + 2} = [\overline{2x^2 + x + 2, x, x}]$$

is purely periodic.

It seems surprising that if α has an infinite simple continued fraction expansion that is periodic and α is reduced, the expansion is purely periodic. As it turns out, Theorem 2.34 holds because of the specific pattern that the eventually periodic infinite simple continued fraction expansion of $\alpha = \sqrt{d}$ has.

Proposition 2.36. *Let $d \in \mathbb{Q}[x]$, where d is not a perfect square, with the infinite simple continued fraction expansion of $\sqrt{d}$ being eventually periodic. Then, the infinite simple continued fraction expansion of $\sqrt{d}$ takes the form*

$$[a_0, \overline{a_1, a_2 \ldots, a_{p-1}, 2a_0}]$$

where p is the period length and $a_0 = [[\sqrt{d}]]$.

Proof. Consider $\alpha = [[\sqrt{d}]] + \sqrt{d}$. Clearly, $v(\alpha) > 1$ and $v(\overline{\alpha}) < 1$. So by Theorem 2.34, the infinite simple continued fraction expansion of α is purely periodic, say

$$[\overline{a_0, a_1, a_2 \ldots, a_{p-1}}]$$

where p is the period length. Noting that $a_0 = [[\ [[\sqrt{d}]] + \sqrt{d}\ ]] = 2[[\sqrt{d}]]$, we have that

$$\begin{aligned}
\sqrt{d} &= ([[\sqrt{d}]] + \sqrt{d}) - [[\sqrt{d}]] = \alpha - [[\sqrt{d}]] \\
&= [\overline{a_0, a_1, a_2 \ldots, a_{p-1}}] - [[\sqrt{d}]] \\
&= [a_0, \overline{a_1, a_2 \ldots, a_{p-1}, a_0}] - [[\sqrt{d}]] \\
&= [2[[\sqrt{d}]], \overline{a_1, a_2 \ldots, a_{p-1}, 2[[\sqrt{d}]]}] - [[\sqrt{d}]] \\
&= [\ [[\sqrt{d}]], \overline{a_1, a_2 \ldots, a_{p-1}, 2[[\sqrt{d}]]}\ ] \text{ as desired.}
\end{aligned}$$

□

The following corollary sums up the two previous results by comparing the similarities of the continued fraction expansions of $\sqrt{d}$ and $[[\sqrt{d}]] + \sqrt{d}$.

Corollary 2.37. *Let $d \in \mathbb{Q}[x]$, where d is not a perfect square, with the infinite simple continued fraction expansion of $\sqrt{d}$ being periodic. Then, the infinite simple continued fraction expansion of $\sqrt{d}$ and $[[\sqrt{d}]] + \sqrt{d}$ differ only in the first component (with the first component of the latter being twice the first component of the former), and the period lengths are equal. Furthermore, the values generated by $\alpha = \alpha_0 = \sqrt{d}$, $P_0 = 0$, $Q_0 = 1$ differ from those generated by $\alpha = \alpha_0 = [[\sqrt{d}]] + \sqrt{d}$, $P_0 = [[\sqrt{d}]]$, $Q_0 = 1$ only at P_0 and a_0.*

Proof. The first claim follows directly from the proof of Proposition 2.36. To prove the values generated by $\alpha = \alpha_0 = \sqrt{d}$, $P_0 = 0$, $Q_0 = 1$ differ from those generated by $\alpha = \alpha_0 = [[\sqrt{d}]] + \sqrt{d}$, $P_0 = [[\sqrt{d}]]$, $Q_0 = 1$ only in P_0 and a_0, we use a straightforward computation. If $\alpha = \sqrt{d}$, then we have

$$P_0 = 0 \qquad Q_0 = 1 \qquad \alpha = \alpha_0 = \sqrt{d} \qquad a_0 = [[\alpha_0]] = [[\sqrt{d}]]$$

$$P_1 = a_0 Q_0 - P_0 = [[\sqrt{d}]] \qquad Q_1 = \frac{d - P_1^2}{Q_0} = d - [[\sqrt{d}]] \qquad \alpha_1 = \frac{[[\sqrt{d}]] + \sqrt{d}}{d - [[\sqrt{d}]]} \qquad a_1 = [[\alpha_1]]$$

whereas if $\alpha = [[\sqrt{d}]] + \sqrt{d}$, we have

$$P_0 = [[\sqrt{d}]] \qquad Q_0 = 1 \qquad \alpha = \alpha_0 = [[\sqrt{d}]] + \sqrt{d} \qquad a_0 = [[\alpha_0]] = 2[[\sqrt{d}]]$$

$$P_1 = a_0Q_0 - P_0 = [[\sqrt{d}]] \quad Q_1 = \frac{d-P_1^2}{Q_0} = d - [[\sqrt{d}]] \quad \alpha_1 = \frac{[[\sqrt{d}]]+\sqrt{d}}{d-[[\sqrt{d}]]} \quad a_1 = [[\alpha_1]].$$

Notice that the values differ only in P_0 and a_0. Since the values of P_i, Q_i, α_i, a_i are generated recursively, the remaining values will be identical for $i > 1$. □

2.6 Examples

While the previous sections had examples throughout, we did not investigate what makes each example unique. In the traditional study of Number Theory, $\sqrt{d}$ is always eventually periodic for $d \in \mathbb{Z}$, but we are not as fortunate. While we have not been able to classify when $\sqrt{d}$ is eventually periodic for $d \in \mathbb{Q}[x]$ thus far, there are unique properties that do allow us to classify certain expansions based on the period length. Example 2.15 had a period length of one so we will classify it as period 1, but Examples 2.16 and 2.32 used period 2 expansions. Here are a few more examples of period 2, period 3, and period 4 expansions using notation as defined in Proposition 2.27 .

Example 2.38. *Find the infinite simple continued fraction expansion for $\alpha = \sqrt{4x^6 + x}$.*

Let $\alpha = \sqrt{4x^6 + x}$. Then,

$$P_0 = 0 \quad Q_0 = 1 \quad \alpha = \alpha_0 = \sqrt{4x^6 + x} = 2x^3 + \tfrac{1}{4}x^{-2} - \dots \quad a_0 = [[\alpha_0]] = 2x^3$$

$$P_1 = 2x^3 \qquad Q_1 = x \qquad \alpha_1 = \frac{2x^3+\sqrt{4x^6+x}}{x} \qquad a_1 = [[\alpha_1]] = 4x^2$$

$$P_2 = 2x^3 \qquad Q_2 = 1 \qquad \alpha_2 = 2x^3 + \sqrt{4x^6 + x} \qquad a_2 = [[\alpha_2]] = 4x^3$$

$$P_3 = 2x^3 \qquad Q_3 = x \qquad \alpha_3 = \frac{2x^3+\sqrt{4x^6+x}}{x} \qquad a_3 = [[\alpha_3]] = 4x^2$$

$$\vdots$$

Again, since $\alpha_1 = \alpha_3$, the infinite simple continued fraction will be periodic; therefore, $\alpha = \sqrt{4x^6 + x} = [2x^3, \overline{4x^2, 4x^3}]$ is period 2.

Example 2.39. *Find the infinite simple continued fraction expansion for*

$$\alpha = \sqrt{\tfrac{1}{16}x^4 + \tfrac{1}{2}x + \tfrac{1}{2}}.$$

Let $\alpha = \sqrt{\frac{1}{16}x^4 + \frac{1}{2}x + \frac{1}{2}}$, then

$$P_0 = 0 \quad Q_0 = 1 \quad \alpha = \alpha_0 = \sqrt{\tfrac{1}{16}x^4 + \tfrac{1}{2}x + \tfrac{1}{2}} = \tfrac{1}{4}x^2 + x^{-1} + x^{-2} - \dots$$
$$a_0 = [[\alpha_0]] = \tfrac{1}{4}x^2$$

$$P_1 = \tfrac{1}{4}x^2 \qquad Q_1 = \tfrac{1}{2}x + \tfrac{1}{2} \qquad \alpha_1 = \frac{\frac{1}{4}x^2 + \sqrt{\frac{1}{16}x^4 + \frac{1}{2}x + \frac{1}{2}}}{\frac{1}{2}x + \frac{1}{2}} \qquad a_1 = [[\alpha_1]] = x - 1$$

$$P_2 = \tfrac{1}{4}x^2 - \tfrac{1}{2} \qquad Q_2 = \tfrac{1}{2}x + \tfrac{1}{2} \qquad \alpha_2 = \frac{\frac{1}{4}x^2 - \frac{1}{2} + \sqrt{\frac{1}{16}x^4 + \frac{1}{2}x + \frac{1}{2}}}{\frac{1}{2}x + \frac{1}{2}} \qquad a_2 = [[\alpha_2]] = x - 1$$

$$P_3 = \tfrac{1}{4}x^2 \qquad Q_3 = 1 \qquad \alpha_3 = \tfrac{1}{4}x^2 + \sqrt{\tfrac{1}{16}x^4 + \tfrac{1}{2}x + \tfrac{1}{2}} \qquad a_3 = [[\alpha_3]] = \tfrac{1}{2}x^2$$

$$P_4 = \tfrac{1}{4}x^2 \qquad Q_4 = \tfrac{1}{2}x + \tfrac{1}{2} \qquad \alpha_4 = \frac{\frac{1}{4}x^2 + \sqrt{\frac{1}{16}x^4 + \frac{1}{2}x + \frac{1}{2}}}{\frac{1}{2}x + \frac{1}{2}} \qquad a_4 = [[\alpha_4]] = x - 1$$

$$\vdots$$

But $\alpha_1 = \alpha_4$, implies that $\alpha = \sqrt{\frac{1}{16}x^4 + \frac{1}{2}x + \frac{1}{2}} = [\frac{1}{4}x^2, \overline{x - 1, x - 1, \frac{1}{2}x^2}]$ is period 3.

Example 2.40. *Find the infinite simple continued fraction expansion for*

$$\alpha = \sqrt{x^{14} - x^9 - 2x^8 - x^6 + \tfrac{1}{4}x^4 + \tfrac{3}{2}x^3 + x^2 + \tfrac{1}{2}x + \tfrac{1}{2}}.$$

Then

$$P_0 = 0 \qquad Q_0 = 1$$

$$\alpha = \alpha_0 = \sqrt{x^{14} - x^9 - 2x^8 - x^6 + \tfrac{1}{4}x^4 + \tfrac{3}{2}x^3 + x^2 + \tfrac{1}{2}x + \tfrac{1}{2}}$$

$$= x^7 - \tfrac{1}{2}x^2 - x - \tfrac{1}{2}x^{-1} + \tfrac{1}{4}x^{-4} - \tfrac{1}{4}x^{-7} + \dots \qquad a_0 = [[\alpha_0]] = x^7 - \tfrac{1}{2}x^2 - x$$

$$P_1 = x^7 - \tfrac{1}{2}x^2 - x \qquad Q_1 = -x^6 + \tfrac{1}{2}x^3 + \tfrac{1}{2}x + \tfrac{1}{2}$$

$$\alpha_1 = \frac{x^7 - \frac{1}{2}x^2 - x + \sqrt{x^{14} - x^9 - 2x^8 - x^6 + \frac{1}{4}x^4 + \frac{3}{2}x^3 + x^2 + \frac{1}{2}x + \frac{1}{2}}}{-x^6 + \frac{1}{2}x^3 + \frac{1}{2}x + \frac{1}{2}} \qquad a_1 = [[\alpha_1]] = -2x$$

$$P_2 = x^7 - x^4 - \tfrac{1}{2}x^2 \qquad Q_2 = -2x^5 + 2x^2 + 1$$

$$\alpha_2 = \frac{x^7 - x^4 - \frac{1}{2}x^2 + \sqrt{x^{14} - x^9 - 2x^8 - x^6 + \frac{1}{4}x^4 + \frac{3}{2}x^3 + x^2 + \frac{1}{2}x + \frac{1}{2}}}{-2x^5 + 2x^2 + 1} \qquad a_2 = [[\alpha_2]] = -x^2$$

$$P_3 = x^7 - x^4 - \tfrac{1}{2}x^2 \qquad Q_3 = -x^6 + \tfrac{1}{2}x^3 + \tfrac{1}{2}x + \tfrac{1}{2}$$

$$\alpha_3 = \frac{x^7 - x^4 - \frac{1}{2}x^2 + \sqrt{x^{14} - x^9 - 2x^8 - x^6 + \frac{1}{4}x^4 + \frac{3}{2}x^3 + x^2 + \frac{1}{2}x + \frac{1}{2}}}{-x^6 + \frac{1}{2}x^3 + \frac{1}{2}x + \frac{1}{2}} \qquad a_3 = [[\alpha_3]] = -2x$$

$$P_4 = x^7 - \tfrac{1}{2}x^2 - x \qquad\qquad Q_4 = 1$$

$$\alpha_4 = x^7 - \tfrac{1}{2}x^2 - x + \sqrt{x^{14} - x^9 - 2x^8 - x^6 + \tfrac{1}{4}x^4 + \tfrac{3}{2}x^3 + x^2 + \tfrac{1}{2}x + \tfrac{1}{2}}$$

$$a_4 = [[\alpha_4]] = 2x^7 - x^2 - 2x$$

$$P_5 = x^7 - \tfrac{1}{2}x^2 - x \qquad\qquad Q_5 = -x^6 + \tfrac{1}{2}x^3 + \tfrac{1}{2}x + \tfrac{1}{2}$$

$$\alpha_5 = \frac{x^7 - \frac{1}{2}x^2 - x + \sqrt{x^{14} - x^9 - 2x^8 - x^6 + \frac{1}{4}x^4 + \frac{3}{2}x^3 + x^2 + \frac{1}{2}x + \frac{1}{2}}}{-x^6 + \frac{1}{2}x^3 + \frac{1}{2}x + \frac{1}{2}} \qquad a_5 = [[\alpha_5]] = -2x$$

$$\vdots$$

But $\alpha_1 = \alpha_5$, we have that $\alpha = \sqrt{x^{14} - x^9 - 2x^8 - x^6 + \frac{1}{4}x^4 + \frac{3}{2}x^3 + x^2 + \frac{1}{2}x + \frac{1}{2}} = [x^7 - \frac{1}{2}x^2 - x, \overline{-2x, -x^2, -2x, 2x^7 - x^2 - 2x}]$ is period 4.

We can discover the properties that make each of the above infinite simple continued fraction expansions eventually periodic by considering the properties the elements have. For example, if $\alpha = \sqrt{d}$ is period 2 ($[a_0, \overline{a_1, 2a_0}]$), it turns out that $a_1 | (2a_0)$ and $d = a_0^2 + \frac{2a_0}{a_1}$. If $\alpha = \sqrt{d}$ is period 3 ($[a_0, \overline{a_1, a_2, 2a_0}]$), then $a_1^2 + 1 | (2a_1 a_0 + 1)$ and $d = a_0^2 + \frac{2a_1 a_0 + 1}{a_1^2 + 1}$. Lastly, if $\alpha = \sqrt{d}$ is period 4 ($[a_0, \overline{a_1, a_2, a_3, 2a_0}]$), then $a_2 a_1^2 + 2a_1 | (2a_0 a_1 a_2 + 2a_0 + a_2)$ and $d = a_0^2 + \frac{2a_0 a_1 a_2 + 2a_0 + a_2}{a_2 a_1^2 + 2a_1}$. The correlation between each of the expansions and the properties they satisfy is not obvious; to find them we must consider the following expansions:

Example 2.41. *Consider the general cases for periods 2, 3, and 4 infinite simple continued fraction expansions.*

If $\alpha = \sqrt{d}$ is period 2, say $\alpha = [a_0, \overline{a_1, 2a_0}]$, then by definition

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{\alpha + a_0}} \Rightarrow \alpha = \frac{a_0 a_1 \alpha + a_1 a_0^2 + 2a_0 + \alpha}{a_1 \alpha + a_1 a_0 + 1}.$$

Multiplying both sides by $a_1 \alpha + a_1 a_0 + 1$ and subtracting yields

$$a_1 \alpha^2 - a_1 a_0^2 - 2a_0 = 0$$

$$\Rightarrow \alpha = \sqrt{\frac{a_1 a_0^2 + 2a_0}{a_1}} = \sqrt{a_0^2 + \frac{2a_0}{a_1}}.$$

If we are considering $d \in \mathbb{Q}[x]$, the desired conditions appear.

If $\alpha = \sqrt{d}$ is period 3, say $\alpha = [a_0, \overline{a_1, a_2, 2a_0}]$, then by definition

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\alpha + a_0}}} \Rightarrow \alpha = \frac{a_0 a_1 a_2 \alpha + a_1 a_2 a_0^2 + a_0 a_1 + a_0 \alpha + a_0^2 + a_2 \alpha + a_2 a_0 + 1}{a_1 a_2 \alpha + a_0 a_1 a_2 + a_1 + \alpha + a_0}.$$

Multiplying both sides by $a_1a_2\alpha + a_0a_1a_2 + a_1 + \alpha + a_0$ and subtracting yields

$$a_1a_2\alpha^2 + a_1\alpha + \alpha^2 - a_0^2a_1a_2 - a_0a_1 - a_0^2 - a_2\alpha - a_2a_0 - 1 = 0$$

since $\alpha \in \mathbb{Q}(x)^*$, $a_1 = a_2$ so that the $a_1\alpha$ and $a_2\alpha$ terms disappear. Then

$$a_1a_2\alpha^2 + \alpha^2 - a_0^2a_1a_2 - a_0a_1 - a_0^2 - a_2a_0 - 1 = 0$$
$$\Rightarrow \alpha = \sqrt{\tfrac{a_0^2a_1a_2+a_0a_1+a_0^2+a_2a_0+1}{a_1a_2+1}} = \sqrt{a_0^2 + \tfrac{2a_0a_1+1}{a_1^2+1}}.$$

Again, if we are considering $d \in \mathbb{Q}[x]$, the desired conditions appear.

If $\alpha = \sqrt{d}$ is period 4, say $\alpha = [a_0, \overline{a_1, a_2, a_3, 2a_0}]$, then by definition

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\alpha + a_0}}}}$$

$$\Rightarrow \alpha = \frac{a_0a_1a_2a_3\alpha + a_0^2a_1a_2a_3 + a_0a_1a_2 + a_0a_1\alpha + a_0^2a_1 + a_0a_3\alpha + a_3a_0^2 + 2a_0 + a_2a_3\alpha + a_0a_2a_3 + a_2 + \alpha}{a_1a_2a_3\alpha + a_0a_1a_2a_3 + a_1a_2 + a_1\alpha + a_0a_1 + a_3\alpha + a_3a_0 + 1}.$$

Multiplying both sides by $a_1a_2a_3\alpha + a_0a_1a_2a_3 + a_1a_2 + a_1\alpha + a_0a_1 + a_3\alpha + a_3a_0 + 1$ and subtracting yields

$$a_1a_2a_3\alpha^2 + a_1a_2\alpha + a_1\alpha^2 + a_3\alpha^2 - a_0^2a_1a_2a_3 - a_0a_1a_2 - a_0^2a_1 - a_0^2a_3 - 2a_0 - a_2a_3\alpha - a_0a_2a_3 - a_2 = 0$$

since $\alpha \in \mathbb{Q}(x)^*$, $a_1a_2 = a_2a_3 \Rightarrow a_1 = a_3$ so that the $a_1a_2\alpha$ and $a_2a_3\alpha$ terms disappear. Then

$$a_1^2a_2\alpha^2 + 2a_1\alpha^2 - a_0^2a_1^2a_2 - 2a_0a_1a_2 - 2a_0^2a_1 - 2a_0 - a_2 = 0$$
$$\Rightarrow \alpha = \sqrt{\tfrac{a_0^2a_1^2a_2+2a_0a_1a_2+2a_0^2a_1+2a_0+a_2}{a_1^2a_2+2a_1}} = \sqrt{a_0^2 + \tfrac{2a_0a_1a_2+2a_0+a_2}{a_1^2a_2+2a_1}}.$$

where if we are considering $d \in \mathbb{Q}[x]$, the desired conditions appear.

In theory, we could repeat this process multiple times and build the conditions for each infinite simple continued fraction expansion with a given period length; however, as is shown in the previous examples, the algebra involved in solving for α is increasing in complexity. In fact, if $\alpha = \sqrt{d}$ is period 5, then there seems to be several cases for which different conditions apply. To approach the problem from the other direction we could consider the infinite simple continued fraction expansions of various polynomials and explore how the expansions vary based on minor changes to their coefficients.

Example 2.42. *Consider $\alpha = \sqrt{d}$ when $d \in \mathbb{Q}[x]$ is of degree 2.*

Let $\alpha = \sqrt{a^2x^2 + bx + c} = ax + \frac{b}{2a} + \frac{c}{2a}x^{-1} + \ldots$ where $a, b, c \in \mathbb{Q}$ with $a \neq 0$ and $a^2x^2 + bx + c$ is not a perfect square. For ease of notation, we will find the infinite simple continued fraction as defined in Proposition 2.14. Then,

$$\begin{aligned} a_0 &= [[\alpha_0]] = ax + \tfrac{b}{2a} & \alpha_1 &= \tfrac{1}{\alpha_0 - a_0} = \tfrac{4a^3x + 2ab + 4a^2\sqrt{a^2x^2+bx+c}}{-b^2+4ac} \\ a_1 &= [[\alpha_1]] = \tfrac{8a^3x+4ab}{-b^2+4ac} & \alpha_2 &= \tfrac{1}{\alpha_1 - a_1} = \tfrac{2ax^2 + b + 2a\sqrt{a^2x^2+bx+c}}{2a} \\ a_2 &= [[\alpha_2]] = 2ax + \tfrac{b}{a} & \alpha_3 &= \tfrac{1}{\alpha_2 - a_2} = \tfrac{4a^3x + 2ab + 4a^2\sqrt{a^2x^2+bx+c}}{-b^2+4ac} \end{aligned}$$

since $\alpha_1 = \alpha_3$ we have $\alpha = \sqrt{a^2x^2 + bx + c} = [ax + \frac{b}{2a}, \overline{\frac{8a^3x+4ab}{-b^2+4ac}, 2ax + \frac{b}{a}}]$.

Every quadratic polynomial with rational coefficients and a perfect square as a leading coefficient has an eventually periodic infinite simple continued fraction expansion. We can now create an infinite family of quadratic surds that are period 2 and compute their continued fraction expansions with very little effort! We can also raise the degree of these polynomials by replacing x with x^n where $n \geq 2$. Since we are taking elements in $\mathbb{Q}(x)^*$, the leading degree of d should be divisible by 2 and all leading coefficients should should be perfect squares. We will now examine $\alpha = \sqrt{a^2x^4 + bx^3 + cx^2 + dx + e}$ by starting with the case where $b = c = d = 0$ (note: the case when $b = d = 0$ is covered by Example 2.42; $\alpha = \sqrt{a^2x^4 + cx^2 + e} = [ax + \frac{c}{2a}, \overline{\frac{8a^3x^2+4ac}{-c^2+4ae}, 2ax^2 + \frac{c}{a}}]$).

Example 2.43. *Consider $\alpha = \sqrt{d}$ when $d = a^2x^4 + e$.*

Let $\alpha = \sqrt{a^2x^4 + e} = ax^2 + \frac{e}{2a}x^{-2} + \ldots$ where $a, e \in \mathbb{Q}$ with $a \neq 0$ and $a^2x^4 + e$ is not a perfect square. Using Proposition 2.14, we have

$$\begin{aligned} a_0 &= [[\alpha_0]] = ax^2 & \alpha_1 &= \tfrac{ax^2 + \sqrt{a^2x^4+e}}{e} \\ a_1 &= [[\alpha_1]] = \tfrac{2ax^2}{e} & \alpha_2 &= ax^2 + \sqrt{a^2x^4 + e} \\ a_2 &= [[\alpha_2]] = 2ax^2 & \alpha_3 &= \tfrac{ax^2 + \sqrt{a^2x^4+e}}{e} \end{aligned}$$

since $\alpha_1 = \alpha_3$, $\alpha = \sqrt{a^2x^4 + e} = [ax^2, \overline{\frac{2ax^2}{e}, 2ax^2}]$.

Now we consider the cases where $b = c = e = 0$ and $b = c = 0$.

Example 2.44. *Consider $\alpha = \sqrt{d}$ when $d = a^2x^4 + dx$.*

Let $\alpha = \sqrt{a^2x^4 + dx} = ax^2 + \frac{d}{2a}x^{-1} + \ldots$ where $a, d \in \mathbb{Q}$ with $a \neq 0$ and $a^2x^4 + dx$ not a perfect square. By Proposition 2.14,

$$a_0 = [[\alpha_0]] = ax^2 \qquad \alpha_1 = \frac{ax^2+\sqrt{a^2x^4+dx}}{dx}$$
$$a_1 = [[\alpha_1]] = \frac{2ax}{d} \qquad \alpha_2 = ax^2 + \sqrt{a^2x^4+dx}$$
$$a_2 = [[\alpha_2]] = 2ax^2 \qquad \alpha_3 = \frac{ax^2+\sqrt{a^2x^4+dx}}{dx}$$

since $\alpha_1 = \alpha_3$, $\alpha = \sqrt{a^2x^4+dx} = [ax^2, \overline{\frac{2ax}{d}, 2ax^2}]$.

Example 2.45. *Consider $\alpha = \sqrt{d}$ when $d = a^2x^4 + dx + e$.*

Let $\alpha = \sqrt{a^2x^4 + dx + e} = ax^2 + \frac{d}{2a}x^{-1} + \frac{e}{2a}x^{-2} + \ldots$ where $a, d, e \in \mathbb{Q}$ with $a \neq 0$ and $a^2x^4 + dx + e$ not a perfect square. By Proposition 2.14,

$$a_0 = [[\alpha_0]] = ax^2 \qquad \alpha_1 = \frac{ax^2+\sqrt{a^2x^4+dx+e}}{dx+e}$$
$$a_1 = [[\alpha_1]] = \frac{2ax}{d} - \frac{2ae}{d^2} \qquad \alpha_2 = \frac{(d^2ax^2-2e^2a+d^2\sqrt{a^2x^4+dx+e})d^2}{d^4+4a^2e^2dx-4a^2e^3}$$
$$a_2 = [[\alpha_2]] = \frac{d^3x}{2ae^2} + \frac{d^2(-d^4+4a^2e^3)}{8a^3e^4} \qquad \alpha_3 = -\frac{8a^3e^4(8a^4e^4x^2-d^6+8d^2a^2e^3+8a^3e^4\sqrt{a^2x^4+dx+e})}{32d^5a^4e^5x+16d^4a^4e^6-4d^9a^2e^2x-12d^8a^2e^3+d^{12}}$$
$$a_3 = [[\alpha_3]] = -\frac{32e^6a^5x}{d^5(-d^4+8a^2e^3)} + \frac{8a^3e^4(16a^4e^6-12d^4a^2e^3+b^8)}{d^6(-d^4+8a^2e^3)^2}$$
$$\vdots$$

where the infinite simple continued fraction expansion does not appear as though it is going to have an eventually periodic form as in Examples 2.43 & 2.44 (though it is difficult to say with certainty).

Similar problems arise when the other general cases of $a^2x^4+bx^3+cx^2+dx+e$ are considered. The expansions grow rapidly, lessening the chances of becoming eventually periodic. It is important to note that although the general case may not always appear to be eventually periodic, adjusting the coefficients can create such expansions as in Example 2.35. In Example 2.45, letting $a^2 = \frac{d^4}{8e^3}$ forces $\alpha_7 = \alpha_1$ so the infinite simple continued fraction expansion is period 6. The period 6 that is created is not the common form of a period 6; instead the expansions are "almost period 3", but $a_3 = r(2a_0)$ (instead of $2a_0$) and $a_6 = 2a_0$. We will define such expansions as follows:

Definition 2.46. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$. α is said to be* <u>*Almost Period n*</u> *for $n \in \mathbb{N}$ if*

$$\alpha = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n, r(\alpha + a_0)] \text{ where } r \in \mathbb{Q}.$$

Notice if n is odd, then the expression of α as a continued fraction expansion has period length $2n$; such an expansion has the form,

$$[a_0, \overline{a_1, a_2, \ldots, a_{n-1}, r(2a_0), \tfrac{a_1}{r}, r(a_2), \tfrac{a_3}{r}, \ldots, r(a_{n-1}), r(\tfrac{2a_0}{r}) = 2a_0}] \text{ where } r \in \mathbb{Q}.$$

Example 2.47. *Find the infinite simple continued fraction expansions of a) $\alpha = \sqrt{x^2-1}$ and b) $\alpha = \sqrt{4x^4+4x+2}$ (a specific case of Example 2.45) using Proposition 2.27.*

a) Let $\alpha = \sqrt{x^2-1}$. Then by Proposition 2.27,

$P_0 = 0$	$Q_0 = 1$	$\alpha = \alpha_0 = \sqrt{x^2-1} = x - \frac{1}{2}x^{-1} + \ldots$	$a_0 = [[\alpha_0]] = x$
$P_1 = x$	$Q_1 = -1$	$\alpha_1 = -x - \sqrt{x^2-1}$	$a_1 = [[\alpha_1]] = -2x$
$P_2 = x$	$Q_2 = 1$	$\alpha_2 = x + \sqrt{x^2-1}$	$a_2 = [[\alpha_2]] = 2x$
$P_3 = x$	$Q_3 = -1$	$\alpha_3 = -x - \sqrt{x^2-1}$	$a_3 = [[\alpha_3]] = -2x$
		$\vdots$	

where the infinite simple continued fraction expansion is "almost period 1", but $\alpha = [x, \overline{-2x, 2x}] = [x, \overline{-1(2x), \frac{-2(x)}{-1}}]$ where $r = -1$. Thus, as noted above, the infinite simple continued fraction of α has period $2(1) = 2$.

b) If $\alpha = \sqrt{4x^4+4x+2}$. Then by Proposition 2.27,

$P_0 = 0 \quad Q_0 = 1 \quad \alpha = \alpha_0 = \sqrt{4x^4+4x+2} = 2x^2 + x^{-1} + \frac{1}{2}x^{-2} + \ldots \quad a_0 = [[\alpha_0]] = 2x^2$

$P_1 = 2x^2$	$Q_1 = 4x+2$	$\alpha_1 = \frac{2x^2+\sqrt{4x^4+4x+2}}{4x+2}$	$a_1 = [[\alpha_1]] = x - \frac{1}{2}$
$P_2 = 2x^2 - 1$	$Q_2 = x + \frac{1}{2}$	$\alpha_2 = \frac{2x^2-1+\sqrt{4x^4+4x+2}}{x+\frac{1}{2}}$	$a_2 = [[\alpha_2]] = 4x - 2$
$P_3 = 2x^2$	$Q_3 = 4$	$\alpha_3 = \frac{2x^2+\sqrt{4x^4+4x+2}}{4}$	$a_3 = [[\alpha_3]] = x^2$
$P_4 = 2x^2$	$Q_4 = x + \frac{1}{2}$	$\alpha_4 = \frac{2x^2+\sqrt{4x^4+4x+2}}{x+\frac{1}{2}}$	$a_4 = [[\alpha_4]] = 4x - 2$
$P_5 = 2x^2 - 1$	$Q_5 = 4x+2$	$\alpha_5 = \frac{2x^2-1+\sqrt{4x^4+4x+2}}{4x+2}$	$a_5 = [[\alpha_5]] = x - \frac{1}{2}$
$P_6 = 2x^2$	$Q_6 = 1$	$\alpha_6 = 2x^2 + \sqrt{4x^4+4x+2}$	$a_6 = [[\alpha_6]] = 4x^2$
$P_7 = 2x^2$	$Q_7 = 4x+2$	$\alpha_7 = \frac{2x^2+\sqrt{4x^4+4x+2}}{4x+2}$	$a_7 = [[\alpha_7]] = x - \frac{1}{2}$
		$\vdots$	

where the $\alpha_1 = \alpha_7$ so infinite simple continued fraction expansion is period 6. Notice the expansion is "almost period 3", but $\alpha = [2x^2, \overline{x - \frac{1}{2}, 4x - 2, \frac{4x^2}{4}, 4(x - \frac{1}{2}), \frac{4x-2}{4}, \frac{4(4x^2)}{4}}]$ where $r = \frac{1}{4}$; thus, the infinite simple continued fraction of α is period 6.

It is worth mentioning that it can be proven that there are no "almost period 2" or "almost period 4" continued fraction expansions. That is, if $\sqrt{d} = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n, r(\sqrt{d}+a_0)]$ for $n = 2$ or $n = 4$, then $r = 1$ causing α to be an ordinary eventually periodic simple continued fraction expansion. For the complete details of these proofs see [Vic78].

If this pattern persists, it seems that there does not exist an α such that the infinite simple continued fraction expansion is "almost period n" for n even. When considering $n = 6$, even with the assistance of Maple, the equations became unwieldy. We formalize this idea with a conjecture:

Conjecture 2.48. *Let $\alpha \in \mathbb{Q}(x)^* - \mathbb{Q}(x)$ have an infinite simple continued fraction expansion. Then α may have an "almost period n" continued fraction expansion if and only if n is odd.*

To sum up the previous examples as well as other examples not previously done, consider the tables on the following pages:

Table 2.1: Generalized Expansions (with examples)

	Period
$\sqrt{a_0^2 + \frac{2a_0}{a_1}} = [a_0, \overline{a_1, 2a_0}]$	2
$\sqrt{4x^6 + x} = [2x^3, \overline{4x^2, 4x^3}]$	2
$\sqrt{\frac{1}{4}x^2 + \frac{1}{3}x} = [\frac{1}{2}x, \overline{3x, x}]$	2
$\sqrt{a_0^2 + \frac{2a_0a_1+1}{a_1^2+1}} = [a_0, \overline{a_1, a_1, 2a_0}]$	3
$\sqrt{x^4 + x^3 + \frac{9}{4}x^2 + 3x + 2} = [x^2 + \frac{1}{2}x + 1, \overline{x, x, 2x^2 + x + 2}]$	3
$\sqrt{\frac{1}{4}x^{20} + x^8 - x^4 + 1} = [\frac{1}{2}x^{10}, \overline{x^2, x^2, x^{10}}]$	3
$\sqrt{a_0^2 + \frac{2a_0a_1a_2+2a_0+a_2}{a_1^2a_2+2a_1}} = [a_0, \overline{a_1, a_2, a_1, 2a_0}]$	4
$\sqrt{x^{14} - x^9 - 2x^8 - x^6 + \frac{1}{4}x^4 + \frac{3}{2}x^3 + x^2 + \frac{1}{2}x + \frac{1}{2}}$ $= [x^7 - \frac{1}{2}x^2 - x, \overline{-2x, -x^2, -2x, 2x^7 - x^2 - 2x}]$	4
$\sqrt{\frac{1}{4}x^{18} + \frac{1}{2}x^{14} + x^{11} + \frac{1}{4}x^{10} + 2x^7 + x^4 + x^3 + 1}$ $= [\frac{1}{2}x^9 + \frac{1}{2}x^5 + x^2, \overline{x^2, x^5, x^2, x^9 + x^5 + 2x^2}]$	4

Table 2.2: Generalized Polynomials (with examples)

	Period
$\sqrt{a^2x^2+bx+c} = [ax+\frac{b}{2a}, \overline{\frac{8a^3x+4ab}{-b^2+4ac}, 2ax+\frac{b}{a}}]$	1, 2
$\sqrt{x^2+1} = [x, \overline{2x}]$	1*
$\sqrt{x^2-1} = [x, \overline{-2x, 2x}]$	2**
$\sqrt{\frac{1}{4}x^2+x+\frac{1}{2}} = [\frac{1}{2}x+1, \overline{-2x-4, x+2}]$	2
$\sqrt{a^2x^4+dx} = [ax^2, \overline{\frac{2ax}{d}, 2ax^2}]$	2
$\sqrt{4x^4-x} = [2x^2, \overline{-4x, 4x^2}]$	2
$\sqrt{\frac{1}{16}x^4+3x} = [\frac{1}{4}x^2, \overline{\frac{1}{6}x, \frac{1}{2}x^2}]$	2
$\sqrt{a^2x^4+e} = [ax^2, \overline{\frac{2ax^2}{e}, 2ax^2}]$	1, 2
$\sqrt{x^4+1} = [x^2, \overline{2x^2}]$	1*
$\sqrt{16x^4+4} = [4x^2, \overline{2x^2, 8x^2}]$	2**
$\sqrt{a^2x^4+cx^2+e} = [ax^2+\frac{c}{2a}, \overline{\frac{8a^3x^2+4ac}{-c^2+4ae}, 2ax^2+\frac{c}{a}}]$	1, 2
$\sqrt{\frac{1}{4}x^4+x^2+\frac{1}{2}} = [\frac{1}{2}x^2+1, \overline{-2x^2-4, x^2+2}]$	2
$\sqrt{x^4+8x^2+8} = [x^2+4, \overline{-\frac{1}{4}x^2-1, 2x^2+8}]$	2
$\sqrt{a^2x^4+dx+e} = [ax^2, \frac{2ax}{d}-\frac{2ae}{d}, 2ax^2, \ldots]$	3, 6
$\sqrt{\frac{1}{16}x^4+\frac{1}{2}x+\frac{1}{2}} = [\frac{1}{4}x^2, \overline{x-1, x-1, \frac{1}{2}x^2}]$	3
$\sqrt{4x^4+4x+2} = [2x^2, \overline{x-\frac{1}{2}, 4x-2, x^2, 4x-2, x-\frac{1}{2}, 4x^2}]$	6**

** - Almost Period   * - Special Case

Chapter 3

Pell's Equation

The entire study of Continued Fractions over $\mathbb{Q}(x)^*$ is the core study of this project due to its relevance to the solvability of Pell's Equation, $X^2 - dY^2 = n$ where $d \in \mathbb{Q}[x]$ and $X, Y \in \mathbb{Q}[x]$. As in the previous chapter, we will only be interested in the cases where the leading coefficient of d is a positive square. When comparing solutions to Pell's Equation over $\mathbb{Z}$, there are many notable differences from the beginning. Traditionally, if $d < 0$ or d is a perfect square, there are only finitely many solutions; however, transitioning to $\mathbb{Q}(x)^*$, there are an infinite number of solutions for any given d. We choose to restrict d to not be a perfect square in hopes of being able to classify all solutions precisely. This restriction is what creates the relationship between continued fractions and solutions to Pell's Equation.

Theorem 3.1. *Let $d \in \mathbb{Q}[x]$ and $n \in \mathbb{Q}$, with $v(d) > 0$, where d is not a perfect square, $0 < v(n) < v(\sqrt{d})$, and the infinite simple continued fraction expansion of $\sqrt{d}$ being eventually periodic. If $f^2 - dg^2 = n$ where the leading coefficients of f and g have the same sign, then $\frac{f}{g}$ is a convergent of the infinite simple continued fraction expansion of $\sqrt{d}$.*

Proof. Let $f^2 - dg^2 = n$. Then, $(f + g\sqrt{d})(f - g\sqrt{d}) = n \Rightarrow (f - g\sqrt{d}) = \frac{n}{(f+g\sqrt{d})}$

$$\Rightarrow (\tfrac{f}{g} - \sqrt{d}) = \frac{n}{g^2(\frac{f}{g}+\sqrt{d})}$$

$$\Rightarrow v(\tfrac{f}{g} - \sqrt{d}) = v(\frac{n}{g^2(\frac{f}{g}+\sqrt{d})})$$

but $v(\frac{f}{g} + \sqrt{d}) \geq v(x)$ since 'f' and 'g' have the same sign. So $v(\frac{f}{g} - \sqrt{d}) \leq v(\frac{1}{xg^2})$. Therefore, $\frac{f}{g}$ is a convergent of the infinite simple continued fraction expansion of $\sqrt{d}$ by Proposition 2.20. $\square$

As stated in the introductory chapter, we will be interested in characterizing solutions to the specific Pell Equation $X^2 - dY^2 = 1$ rather than a general n; it is at this point that we will begin to look only at this specific case for future results. We will pursue the solutions by examining their relationships to the convergents of the infinite simple continued fraction expansions of $\sqrt{d}$.

Lemma 3.2. *Let $d \in \mathbb{Q}[x]$ where d is not a perfect square and let $\frac{p_i}{q_i}$ be the i^{th} convergent of the periodic infinite simple continued fraction expansion of $\sqrt{d}$. Then, $p_i^2 - dq_i^2 = (-1)^{i-1}Q_{i+1}$ for $i \geq 0$ where $Q_1, Q_2, \ldots$ are defined as in Proposition 2.27.*

Proof. Let all notation be as in Proposition 2.27 with $\alpha_0 = \sqrt{d}$. Since $\sqrt{d} = \alpha_0 = [a_0, a_1, \ldots, a_i, \alpha_{i+1}]$ for $i > 0$, Proposition 2.7 yields $\sqrt{d} = \frac{\alpha_{i+1}p_i + p_{i-1}}{\alpha_{i+1}q_i + q_{i-1}}$. But

$$\alpha_{i+1} = \frac{P_{i+1}+\sqrt{d}}{Q_{i+1}}$$
$$\Rightarrow \sqrt{d} = \frac{(P_{i+1}+\sqrt{d})p_i + Q_{i+1}p_{i-1}}{(P_{i+1}+\sqrt{d})q_i + Q_{i+1}q_{i-1}}$$

and so,

$$dq_i + (P_{i+1}q_i + Q_{i+1}q_{i-1})\sqrt{d} = (P_{i+1}p_i + Q_{i+1}p_{i-1}) + p_i\sqrt{d} \text{ for } i > 0$$
$$\Rightarrow dq_i = P_{i+1}p_i + Q_{i+1}p_{i-1} \text{ and } P_{i+1}q_i + Q_{i+1}q_{i-1} = p_i$$
$$\Rightarrow dq_i^2 = P_{i+1}p_iq_i + Q_{i+1}p_{i-1}q_i \text{ and } P_{i+1}p_iq_i + Q_{i+1}p_iq_{i-1} = p_i^2$$
$$\Rightarrow p_i^2 - dq_i^2 = (p_iq_{i-1} - p_{i-1}q_i)Q_{i+1} \text{ for } i > 0$$

but by Proposition 2.10, $(p_iq_{i-1} - p_{i-1}q_i) = (-1)^{i-1} \Rightarrow p_i^2 - dq_i^2 = (-1)^{i-1}Q_{i+1}$ for $i > 0$. If $i = 0 \Rightarrow p_0^2 - dq_0^2 = a_0^2 - d = (-1)(d - a_0^2) = (-1)(\frac{d-P_1^2}{1}) = (-1)Q_1$. □

Lemma 3.3. *Let $d \in \mathbb{Q}[x]$ where d is not a perfect square with the infinite simple continued fraction expansion of $\sqrt{d}$ being periodic. Let p be the period length of the infinite simple continued fraction expansion of $\sqrt{d}$ and let all notation be defined as in Proposition 2.27. If $\alpha = \alpha_0 = \sqrt{d}$, $P_0 = 0$, and $Q_0 = 1$, then $Q_i = 1$ if and only if $p|i$.*

Proof. By Corollary 2.37, it suffices to show the desired result given $\alpha = \alpha_0 = [[\sqrt{d}]] + \sqrt{d}$, $P_0 = [[\sqrt{d}]]$, and $Q_0 = 1$ with $\alpha_0 = [\overline{a_0, a_1, \ldots, a_{p-1}}]$ so that α is reduced. If $i \in \mathbb{Z}$ with $i \geq 0$ and $\alpha_i = [a_i, a_{i+1}, a_{i+2}, \ldots]$, we have $\alpha_0 = \alpha_p = \alpha_{2p} = \ldots$. Furthermore, $\alpha_i = \alpha_0 \Rightarrow p|i$.
Now if $p|i$, we have $\frac{P_i+\sqrt{d}}{Q_i} = \alpha_i = \alpha_0 = [[\sqrt{d}]] + \sqrt{d} \Rightarrow P_i - Q_i[[\sqrt{d}]] = (Q_i - 1)\sqrt{d}$. Notice, if $Q_i \neq 1$, the left hand side of the previous equation belongs to $\mathbb{Q}(x)$, but the right hand side does not. Conversely, if $Q_i = 1$, we have that $\alpha_i = P_i + \sqrt{d}$; since α_i is

periodic, we have $v(\overline{\alpha_i}) < 1 \Rightarrow v(P_i - \sqrt{d}) < 1 \Rightarrow P_i = [[\sqrt{d}]]$. So $\alpha_i = \alpha_0$ and $p|i$ as desired. □

In the traditional study of continued fractions, it holds that Q_i is never equal to -1; however, Example 2.47 shows that we are not able to say the same in $\mathbb{Q}(x)^*$.

Theorem 3.4. *Let $d \in \mathbb{Q}[x]$ where d is not a perfect square with the infinite simple continued fraction expansion of $\sqrt{d}$ being periodic. Let $\frac{p_i}{q_i}$ denote the i^{th} convergent of the infinite simple continued fraction expansion of $\sqrt{d}$ and let p be the period length of this expansion. If $Q_i \neq -1$ for all i and p is even, then the solutions with positive leading coefficients to the Pell Equation $X^2 - dY^2 = 1$ are given by $X = p_{np-1}$ and $Y = q_{np-1}$ where n is a positive integer; if p is odd, the solutions are given by $X = p_{2np-1}$ and $Y = q_{2np-1}$.*

Proof. By Lemma 3.2, we have $p_i^2 - dq_i^2 = (-1)^{i-1}Q_{i+1}$ for $i \geq 0$. Note if $Q_i \neq -1$ for all i, $p_i^2 - dq_i^2 = 1$ only if $p|i+1$ by Lemma 3.3. Moreover, if $p|i+1$, then, $i = np-1$ for some positive integer n. So we have,

$$p_{np-1}^2 - dq_{np-1}^2 = (-1)^{np-2}.$$

If p is even, then $X = p_{np-1}$, $Y = q_{np-1}$ solve $X^2 - dY^2 = 1$; otherwise, if p is odd, then $X = p_{2np-1}$, $Y = q_{2np-1}$ solve $X^2 - dY^2 = 1$. These solutions are the only solutions with positive leading coefficients since any such solution must be a convergent of the infinite simple continued fraction expansion of $\sqrt{d}$ by Theorem 3.1. □

Theorem 3.4 allows us to find all solutions with positive leading coefficients by not only considering the convergents of the infinite simple continued fraction expansions, but by looking in a precise location.

Example 3.5. *Using Theorem 3.4, find all solutions with positive leading coefficients to the Pell Equation a) $X^2 - (\frac{1}{4}x^{18} + x^8 - x^6 + x^4 - x^2 + 1)Y^2 = 1$ and b) $X^2 - (\frac{1}{4}x^{18} + \frac{1}{2}x^{14} + x^{11} + \frac{1}{4}x^{10} + 2x^7 + x^4 + x^3 + 1)Y^2 = 1$.*

a) Consider $X^2 - (\frac{1}{4}x^{18} + x^8 - x^6 + x^4 - x^2 + 1)Y^2 = 1$. Since $Q_i \neq -1$ for all i and $\sqrt{\frac{1}{4}x^{18} + x^8 - x^6 + x^4 - x^2 + 1} = [\frac{1}{2}x^9, \overline{x, x, x^9}]$, we have that $p = 3$; thus, the solutions are given precisely by $X = p_{6n-1}$ and $Y = q_{6n-1}$ where n is a positive integer. The solution with the least positive value, given when $n = 1$, is

$$X = p_5 = \tfrac{1}{2}x^{22} + x^{20} + \tfrac{1}{2}x^{18} + 2x^{12} + 2x^{10} + 2x^2 + 1$$

and

$$Y = q_5 = x^{13} + 2x^{11} + x^9 + 2x^3 + 2x.$$

That is, $(\frac{1}{2}x^{22} + x^{20} + \frac{1}{2}x^{18} + 2x^{12} + 2x^{10} + 2x^2 + 1)^2 - (\frac{1}{4}x^{18} + x^8 - x^6 + x^4 - x^2 + 1)(x^{13} + 2x^{11} + x^9 + 2x^3 + 2x)^2 = 1$.

b) Consider $X^2 - (\frac{1}{4}x^{18} + \frac{1}{2}x^{14} + x^{11} + \frac{1}{4}x^{10} + 2x^7 + x^4 + x^3 + 1)Y^2 = 1$. Since $Q_i \neq -1$ for all i and

$$\sqrt{\tfrac{1}{4}x^{18} + \tfrac{1}{2}x^{14} + x^{11} + \tfrac{1}{4}x^{10} + 2x^7 + x^4 + x^3 + 1}$$
$$= [\tfrac{1}{2}x^9 + \tfrac{1}{2}x^5 + x^2, \overline{x^2, x^5, x^2, x^9 + x^5 + 2x^2}],$$

we have that $p = 4$; thus, the solutions are given precisely by $X = p_{4n-1}$ and $Y = q_{4n-1}$ where n is a positive integer. The solution with the least positive value, given when $n = 1$, is

$$X = p_3 = \tfrac{1}{2}x^{18} + \tfrac{1}{2}x^{14} + 2x^{11} + 2x^7 + 2x^4 + 1$$

and

$$Y = q_3 = x^9 + 2x^2.$$

That is, $(\frac{1}{2}x^{18} + \frac{1}{2}x^{14} + 2x^{11} + 2x^7 + 2x^4 + 1)^2 - (\frac{1}{4}x^{18} + \frac{1}{2}x^{14} + x^{11} + \frac{1}{4}x^{10} + 2x^7 + x^4 + x^3 + 1)(x^9 + 2x^2)^2 = 1$.

Although we have only found a single solution, substituting new values for n will provide us with infinitely many solutions we would like. The solutions found in the previous examples are those of least degree and are known as the *fundamental solutions.* Rather than compute the n^{th} convergents for every desired n using our previous methods, we can use the fundamental solution to generate all others.

Theorem 3.6. *Let $d \in \mathbb{Q}[x]$ where d is not a perfect square with the infinite simple continued fraction expansion of $\sqrt{d}$ being periodic. Let x_1, y_1 be the fundamental solution of the Pell Equation $X^2 - dY^2 = 1$. Then all solutions with positive leading coefficients are given by x_n, y_n where $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$ and n is a positive integer.*

Proof. First, consider $x_n^2 - dy_n^2 = (x_n + y_n\sqrt{d})(x_n - y_n\sqrt{d})$

$$= (x_1 + y_1\sqrt{d})^n(x_1 - y_1\sqrt{d})^n \text{ from the definition}$$
$$= (x_1^2 - dy_1^2)^n$$
$$= (1)^n \text{ since } x_1, y_1 \text{ is the fundamental solution}$$
$$= 1.$$

Now, assume by contradiction that X, Y is a solution not of the form x_n, y_n for any n. Then $\exists\ k \in \mathbb{Z}^+$ such that

$$v((x_1 + y_1\sqrt{d})^k) \leq v(X + Y\sqrt{d}) < v((x_1 + y_1\sqrt{d})^{k+1}).$$

Multiplying both sides by $v((x_1 + y_1\sqrt{d})^{-k})$ yields,

$$v(1) \leq v(X + Y\sqrt{d})v((x_1 + y_1\sqrt{d})^{-k}) < v(x_1 + y_1\sqrt{d})$$
$$\Rightarrow 1 \leq v(X + Y\sqrt{d})v((x_1 + y_1\sqrt{d})^k) < v(x_1 + y_1\sqrt{d})$$

Now, let $r, s \in \mathbb{Q}[x]$ such that $r + s\sqrt{d} = (X + Y\sqrt{d})(x_1 + y_1\sqrt{d})^k$. Then

$$1 \leq v(r + s\sqrt{d}) < v(x_1 + y_1\sqrt{d}).$$

Moreover, $v(r^2 - ds^2) = v(r + s\sqrt{d})v(r - s\sqrt{d})$

$$= v(X + Y\sqrt{d})v((x_1 + y_1\sqrt{d})^k)v(X - Y\sqrt{d})v((x_1 - y_1\sqrt{d})^k)$$
$$= v(X^2 - dY^2)v((x_1^2 - dy_1^2)^k)$$
$$= v(1)$$
$$= 1.$$

So since x_1, y_1 is the fundamental solution, $v(x_1) \leq v(r)$ and $v(y_1) \leq v(s) \Rightarrow v(x_1 + y_1\sqrt{d}) \leq v(r + s\sqrt{d})$ a contradiction. □

Example 3.7. *Using Example 3.5 and Theorem 3.6, find all solutions with positive leading coefficients to the Pell Equation a)* $X^2 - (\frac{1}{4}x^{18} + x^8 - x^6 + x^4 - x^2 + 1)Y^2 = 1$ *and b)* $X^2 - (\frac{1}{4}x^{18} + \frac{1}{2}x^{14} + x^{11} + \frac{1}{4}x^{10} + 2x^7 + x^4 + x^3 + 1)Y^2 = 1$.

a) The fundamental solution to the Pell Equation

$$X^2 - dY^2 = 1$$

where $d = \frac{1}{4}x^{18} + x^8 - x^6 + x^4 - x^2 + 1$ is

$$X = p_5 = \frac{1}{2}x^{22} + x^{20} + \frac{1}{2}x^{18} + 2x^{12} + 2x^{10} + 2x^2 + 1$$

and

$$Y = q_5 = x^{13} + 2x^{11} + x^9 + 2x^3 + 2x.$$

But we have established that all solutions with positive leading coefficients are given by x_n, y_n where

$$x_n+y_n\sqrt{d} = ((\tfrac{1}{2}x^{22}+x^{20}+\tfrac{1}{2}x^{18}+2x^{12}+2x^{10}+2x^2+1)+(x^{13}+2x^{11}+x^9+2x^3+2x)\sqrt{d})^n.$$

We can compute the next solution with positive leading coefficients and second smallest value by expanding

$$((\tfrac{1}{2}x^{22}+x^{20}+\tfrac{1}{2}x^{18}+2x^{12}+2x^{10}+2x^2+1)+(x^{13}+2x^{11}+x^9+2x^3+2x)\sqrt{d})^2$$
$$= U + V\sqrt{d}$$

where,

$$U = \tfrac{1}{2}x^{44}+2x^{42}+3x^{40}+2x^{38}+\tfrac{1}{2}x^{36}+4x^{34}+12x^{32}+12x^{30}+4x^{28}+12x^{24}+26x^{22}+ 16x^{20}+2x^{18}+16x^{14}+24x^{12}+8x^{10}+8x^4+8x^2+1$$

and

$$V = \tfrac{1}{2}x^{35}+2x^{33}+3x^{31}+2x^{29}+\tfrac{1}{2}x^{27}+3x^{25}+9x^{23}+9x^{21}+3x^{19}+6x^{15}+13x^{13}+ 8x^{11}+x^9+4x^5+6x^3+2x$$

b) The fundamental solution to the Pell Equation

$$X^2 - dY^2 = 1$$

where $d = \frac{1}{4}x^{18}+\frac{1}{2}x^{14}+x^{11}+\frac{1}{4}x^{10}+2x^7+x^4+x^3+1$ is

$$X = p_3 = \tfrac{1}{2}x^{18}+\tfrac{1}{2}x^{14}+2x^{11}+2x^7+2x^4+1$$

and

$$Y = q_3 = x^9+2x^2.$$

But all solutions with positive leading coefficients are given by x_n, y_n where

$$x_n+y_n\sqrt{d} = ((\tfrac{1}{2}x^{18}+\tfrac{1}{2}x^{14}+2x^{11}+2x^7+2x^4+1)+(x^9+2x^2)\sqrt{d})^n.$$

We can compute the next solution with positive leading coefficients and second smallest value by expanding

$$((\tfrac{1}{2}x^{18}+\tfrac{1}{2}x^{14}+2x^{11}+2x^7+2x^4+1)+(x^9+2x^2)\sqrt{d})^2 = U + V\sqrt{d}$$

where,

$$U = \tfrac{3}{4}x^{36} + x^{32} + 5x^{29} + \tfrac{1}{2}x^{28} + 8x^{25} + 13x^{22} + 4x^{21} + 22x^{18} + \\ 16x^{15} + 10x^{14} + 24x^{11} + 8x^{8} + 8x^{7} + 8x^{4} + 1$$

and

$$V = \tfrac{1}{2}x^{27} + \tfrac{1}{2}x^{23} + 3x^{20} + 3x^{16} + 6x^{13} + 5x^{9} + 4x^{6} + 2x^{2}$$

Chapter 4

Conclusion

The process of finding solutions to $X^2 - dY^2 = 1$ in $\mathbb{Q}[x]$ can be laborious, and nearly impossible, if one does not know about the theory of infinite simple continued fraction expansions that has been developed in Chapter 2. While analysis of solutions to the traditional Pell's Equation, $X^2 - dY^2 = n$, is not as complete as when $n = 1$ in $\mathbb{Z}$, the specific case has properties which can be generalized nicely.

Ultimately, the vital piece of information we lack is the characterization of when $\alpha = \sqrt{d}$ has an eventually periodic continued fraction expansion. The transition from $\mathbb{R}$ to $\mathbb{Q}(x)^*$ has created changes to the traditional theory which have impacted the final results in a major way, primarily the converse of Theorem 2.31. Knowing that a quadratic surd has an eventually periodic infinite simple continued fraction expansion enables us to determine where Pell's Equation has solutions given an arbitrary $d \in \mathbb{Q}[x]$. However at this stage, except for the specific cases outlined in Section 2.6, one must compute the infinite simple continued fraction expansion of $\sqrt{d}$ in order to determine if it is periodic. It is important to note that the expansion could be of any given period length so the process could be time consuming. This problem could be due to a few things. The new field and its properties could be the reason that the expansions are not as nicely categorized. Also, changing to the Non-Archimedean valuation from the absolute value seems to have played the largest role in the converse failing. Lastly, it is possible that many more of the expansions we have considered, at some point, eventually become periodic, but as of now the proof simply cannot be done.

A characteristic that is new to elements of $\mathbb{Q}(x)^*$ is the fact that there exists an

$i \in \mathbb{N}$ such that $Q_i = -1$, which causes the fundamental solutions to their Pell's Equation to show up earlier on. In the traditional study of Pell's Equation, the version of Lemma 3.3 over $\mathbb{R}$ proves that there does not exist an i with $Q_i = -1$ and so the solutions that are outlined in Theorem 3.4 are the only possible solutions in $\mathbb{Z}$. However, having $Q_{i+1} = -1$ over $\mathbb{Q}(x)^*$ for some i creates solutions earlier on since $(-1)^{i-1}Q_{i+1} = 1$ when $i-1$ is odd.

One property that was surprising was the examples where $Q_i = r$ for $r \in \mathbb{Q}$ $r \neq 1; -1$. This is noteworthy because it appears in every "almost periodic" case. Our research into a few of the general cases of polynomials has led to the observation that it is the coefficients which affect exactly what r can be; for example, in the case of $\sqrt{a^2x^4 + dx + e}$ where $a = \frac{d^4}{8e^3}$ so that the infinite simple continued fraction is "almost period 3", $Q_3 = r = 2e$. A problem is that our collection of "almost periodic" examples are limited in number so our ability to explore any patterns they might have is hindered.

At first, the process of converting the traditional continued fraction theory into the new field was difficult having just learned of valuations. It turns out that the properties we lose (due to no longer having absolute value), have an almost direct translation with valuations. Since we are no longer dealing with $\mathbb{R}$, we do not have decimals, but clearly the new corollary to that is the infinite expansions where $b_0x^{-1} + b_1x^{-2} + \ldots$, which look like Lagrange expansions. Also, instead of considering the distance between two elements, we measure the elements by the remaining values, or degrees, when they are subtracted; that is, we are no longer taking $|\alpha - \frac{a}{b}|$, but we have $v(\alpha - \frac{a}{b})$. Lastly, the idea of $\alpha < 0$ translates to be $v(\alpha) < 1$ which is notable in the definition of "reduced" elements in Theorem 2.34.

Since the Non-Archimedean Valuation is stronger than the traditional absolute value, we are able to arrive at stronger conclusions in certain cases. Proposition 2.20 has two noteworthy changes when considered over $\mathbb{Q}(x)^*$. Traditionally, $|\alpha - \frac{a}{b}| < \frac{1}{2b^2}$; whereas, now we have the possibility of equality so that $v(\alpha - \frac{a}{b}) \leq \frac{1}{v(xb^2)}$. Also, more surprisingly, the "only if" direction of the proposition holds. In the traditional study, $\sqrt{3}$ can be used as a counter-example since $|\sqrt{3} - C_0| = |\sqrt{3} - 1| > \frac{1}{2}$. When extended to $\mathbb{Q}(x)^*$, it is sometimes the case $v(\alpha - \frac{a}{b}) = \frac{1}{v(xb^2)}$, but never greater, allowing the converse to hold.

The use of Maple was essential to computing many examples at an expedited rate, but it had its advantages and limitations. The main advantage of Maple was its ease of computing due to its ability to handle recursive definitions and its simplifying abilities. The limitations were that in using it, the outputs generated were less familiar and seemed detached. It is highly suggested that when studying continued fraction expansions, one does many examples by hand as well as using Maple. Maple proves to be a great source for checking and computing those expressions for which one already has an understanding, similar to a calculator. Even though Maple is an amazing computation tool, the numbers involved in some of the non-periodic continued fractions grow too rapidly to continue the expansions.

In terms of material, Chapter 3 summarizes the theorems and lemmas by combining them to find solutions to Pell's Equation. Initially, beginning the study of solutions to Pell's Equation with a subject that seems as disconnected as the study of continued fractions seems counter-intuitive, but once the reader understands how the solutions are discovered, each of the chapters on continued fraction expansions form the stepping stones to finalizing the results. The solutions come from the convergents of the infinite simple continued fraction expansions which were defined after finite simple continued fractions. Each of the examples, lemmas, theorems, and corollaries were precisely those necessary to further the progress in each chapter.

Despite the number of theorems and properties of continued fraction expansions which we are able to generalize over our new rings and fields, there are many problems left to be considered. The idea that there may not exist any almost periodic continued fractions of even length has been discussed and although the result seems likely, we have not proven it. Moreover, only expansions with "almost periods 1 and 3" have been found so finding other "almost periods" is another problem left to consider. As stated earlier, the most important of these open problems is to discover the general conditions an element of $\mathbb{Q}(x)^* - \mathbb{Q}(x)$ must meet for its infinite continued fraction expansion to be periodic. With such a result, we would be able to examine an element and immediately determine whether or not the expansion is periodic. Knowing whether expansions are periodic or not will allow for the complete characterization of the solutions to Pell's Equation.

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