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Leonhard Euler's contribution to infinite polynomials

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Leonhard Euler's Contribution to Infinite Polynomials

A Thesis

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In Partial Fulfillment

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in

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by

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Abstract

This thesis will show and compare two methods for solving infinite polynomials: Leonhard Euler's and Issac Newton's. It will show the path Euler took, which made him famous. Also, it will show Newton's method; both methods will be applied to several polynomials. Furthermore, it will compare the ease of Newton's method to the challenging path of Euler's success.

Leonhard Euler (1707-83), during his formative years as a mathematician, became fixated with alternating series. He worked on several different series to determine if they diverged or converged, and what they converged to. Some of his most famous work was in the area of infinite polynomials. He claimed "what holds for a finite polynomial holds for an infinite polynomial.: (infinite series of surprises). Using this claim, Euler was able to achieve solving for an infinite polynomial, which his predecessors, including Issac Newton, had not been able to do. This finding by Euler projected him into mathematical fame and prominence.

This thesis will focus on Euler's famous method for solving the infinite polynomial. It will show how he manipulated the sine function to find all possible points along the sine function such that the sine A would equal to y; these would be the roots of the polynomial. It also shows how Euler set the infinite polynomial equal to the infinite product allowing him to determine which coefficients were equal to which reciprocals of the roots, roots squared, roots cubed, etc. This method was repeated to find the third coefficient, thus showing the difficulty in the mathematical computation using Euler's method.

Issac Newton (1642-1727), Euler's predecessor, discovered a method for solving infinite polynomials. His method involved using two of his famous identities. Newton's path was much simpler when solving any polynomial than Euler's.

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Chapter 1

Introduction

Leonhard Euler is considered one of the most productive mathematicians of the 18th century and certainly one of the most renowned of all time. "He pioneered much of the mathematical notation that continues to be in use today. His ideas are responsible for many great advances in the field" [AFDnd]. Euler was born in Basel, Switzerland on April 15, 1707 and died in St. Petersburg, Russia on September 18, 1783 at age 76. His primary fields of expertise were mathematics and physics. Euler was the eldest of six children. His father had studied mathematics under Jacob Bernoulli, a prominent Swiss mathematician of that time, and wished Leonhard to do likewise. However, Leonhard's father, Paul, a Protestant priest, wanted Leonhard to follow in his profession. Therefore, Euler followed his father's wishes and studied theology along with taking lessons under the tutelage of Johann Bernoulli, the younger brother. During this period, Leonhard demonstrated a brilliant mind, receiving a master's degree from Basel University at age sixteen. The Bernoulli's suggested that Leonhard study mathematics instead of the ministry.

In 1727, the Bernoulli's were influential in obtaining Leonhard a position at the St Petersburg Academy of Sciences where he eventually became professor of mathematics. It was here, and in 1733 that he met and married Katharina Gsell with whom he had thirteen children. He was so focused and single minded regarding his work that the oftentimes bedlam of his household, with small children playing under his feet, did not interrupt his train of thought. Only five of his children lived to adulthood, and their descendants became influential, themselves, in 19th century Russia. In 1735, after repeatedly staring directly into the sun, while studying problems in astronomy, Euler went blind in his right eye [Gek07].

In 1741, he left Russia, mainly because of political strife, and went to Germany becoming director of the mathematical class at the Prussian Academy of Sciences in Berlin. He worked long and hard during his tenure, but his health suffered as a result. He worked on his ideas on the calculus of variations. "In 1748 Euler published the first of three volumes of calculus analysis, entitled **Introduction to Analysis of the Infinite**. All modern treatments of exponential, logarithmic, and trigonometric functions are derived from Euler's publication" [Kat98]. He was not favored by the king, which only added to his problems. In 1766, he returned to St. Petersburg where he spent the remainder of his life. By this time, Euler was almost fully blind; a cataract had formed on his left eye. Surgery was able to restore part of his vision, but complications caused a setback. Aside from his blindness, his general health was always good. However, because of his extraordinary memory, he was able to continue his work. "Nearly half of his 886 books and manuscripts were created during this period" [Eul99]. The following quote is attributed to Euler: "For since that fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear " [Eul12], [Gek07].

Euler was a devout Christian who often opposed the intellectuals of his period who were atheists. He was a man who led a full and productive existence. During his life, Euler won numerous prizes which brought him substantial monetary profits especially during his latter years. He was not envious of other mathematicians; he delighted in complimenting others for their mathematical achievements; he was secure within himself of his own accomplishment in the field of mathematics.

Leonhard Euler worked on anything that had mathematical influence. It has been estimated that the entirety of his works would fill sixty to eighty volumes. He was married twice; his first wife, Katharina, died after forty years of marriage, and his second wife was the half-sister to his first wife. He died of a heart attack while playing with his grandson. Having come from a relatively common background, Leonhard Euler left his mark of mathematical genius that will last throughout the ages [Gek07], [Mwnd].

Euler's predecessor, Issac Newton, found a common method to express sums of powers of roots of finite polynomials as an expression of its coefficients. Although the process Euler used worked, it became extremely difficult with higher powers, when summing of the roots occurred. During Euler's productive years, he claimed: "What holds for a finite polynomial holds for an infinite polynomial" [San01]. This was the time Euler's work began involving finite polynomials, consequently overlapping Newton's.

Chapter 2

Alternating Series

In this chaper we will dicuss Euler's work with alternating series. We will show his methods used to manipulate a series to determine if they would converge or diverge. Around 1734/35 Euler started with the alternating series [Kli 83], [Dun83]:

$$y = \sin x = x - x^3/3! + x^5/5! - .... \tag{E1}$$

In an attempt to try to evaluate one of the major sticking points of Sir Issac Newton's era. At this time, Euler was fixated on determining convergent and divergent series. One of the well known series which brought him fame and eluded some famous mathematicians before him, including Sir Issac Newton, was the illustrious and incomprehensible alternating series below:

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + ...$$

In a successful endeavor to determine if the series actually converges or diverges, Euler attemptedto use the already known series in E1 to find the answer. He divided this series through by y, then rearranged it to get the following [kli83], [Dun83].

2.1 The Finite Polynomial

$$1 - x/y + x^3/3!y - x^5/5!y + ... = 0. \tag{E2}$$

Next, Euler decided to treat the left-hand side of this infinite series in E2 as a finite polynomial. He then knew that the roots of a finite polynomial could be represented as $A, B, C, ...$. Since the constant term of this polynomial was set equal to one, using algebra, Euler was able to set the infinite series equal to the infinite product as shown below.

2.2 The Infinite Product

$$1 - x/y + x^3/3!y - x^5/5!y + ... = (1 - \tfrac{x}{A})(1 - \tfrac{x}{B})(1 - \tfrac{x}{C})... \qquad \text{(E3)}$$

where $A, B, C, ...$, represent the roots of the infinite polynomial. In order to find these roots, Euler needed to find a value for x, such that $\sin x$ would equal to y as is in equation (E1). Looking at series (E2), Euler could find multiple values for x such that $\sin x = 1$. These would become the roots of the infinite polynomial. Letting $A = \frac{\pi}{2}$ would be the first or smallest positive solution/root of the polynomial, and all the other different roots $B, C, ...$ would be $n\pi - A$ for all odd integers n and $n\pi + A$ for all even integers n. In reality, the value A is a representation of all possible points along the sine function such that the $\sin A$ would equal to y. To help us visualize this pattern, we should look at the roots for (E2) knowing that when $A = \frac{\pi}{2}$ then $\sin \frac{\pi}{2} = 1$, when $B = \frac{5\pi}{2}$ then $\sin \frac{5\pi}{2} = 1$, and when $C = \frac{9\pi}{2}$ then $\sin \frac{9\pi}{2} = 1$. Continuing the pattern for $\sin x = 1$, should be comprehensible as to say these roots are, A, $\pi - A, 2\pi + A, 3\pi - A, ..., -\pi - A, -2\pi + A, -3\pi - A, ...$ and so on, with our initial value $A = \frac{\pi}{2}$.

Euler decided to view the infinite polynomial in (E3) as being a special case of the following.

2.3 The General Case

$$a_0 + a_1x + a_2x^2 + a_3x^3 + ... = a_0(1 - \tfrac{x}{A})(1 - \tfrac{x}{B})(1 - \tfrac{x}{C}).... \qquad \text{(E4)}$$

He assumed that all infinite polynomials were equal to their infinite product. Therefore the coefficients $a_1, a_2, a_3, ...$ of this polynomial on the left hand side could be set equal to the comparable powers of the roots of the infinite product on the right-hand side. Expanding this infinite product out a few terms would help us see the progression of terms to construct and build an equation from. Multiplying the first three products of the right-hand side of (E4) above gives the following [kli83], [Dun83]:

$$(1-\tfrac{x}{A})(1-\tfrac{x}{B})(1-\tfrac{x}{C}) = 1 - \tfrac{x}{A} - \tfrac{x}{B} + \tfrac{x^2}{AB} - \tfrac{x}{C} + \tfrac{x^2}{AC} + \tfrac{x^2}{AC} + \tfrac{x^2}{BC} - \tfrac{x^3}{ABC} \,. \qquad \text{(E5)}$$

Setting the term on the left-hand of (E4) side equal to the comparable term on the right-hand side, Euler noticed he could state that $a_1 = a_0(-\frac{1}{A} - \frac{1}{B} - \frac{1}{C} - ...)$. He had surmised the equation for finding the a_1 coefficient of the polynomial [Dun83], that being:

$$a_1 = -a_0(\tfrac{1}{A} + \tfrac{1}{B} + \tfrac{1}{C} + ...). \qquad \text{(E6)}$$

Chapter 3

Leonhard Euler's Solutions

In equation (E6), a_1 represented the coefficient of the x term on the left-hand side of the finite polynomial. Euler knew he could set $a_1$equal to the same given degree terms of the infinite product multiplied out on the right-hand side of (E4) which is shown expanded below in equation (E5). Doing this, he obtained the roots of the x-terms on the right-hand side, allowing Euler to construct equation (E6). He knew the sum of the terms in the parentheses of (E6) were reciprocals of the roots of the infinite polynomial. This allowed Euler to use the calculated roots from above, which were, $A, \pi-A, 2\pi+A, 3\pi-A, ..., -\pi-A, -2\pi+A, -3\pi-A, ..$ to figure out the needed coefficient a_1 in equation (E6). Substituting these roots into (E6) manipulated the equation into what we'll call equation (E7) below [kli83], [Dun83].

3.1 Solving for the Coefficient a_1

$$\frac{1}{A} + \frac{1}{\pi-A} + \frac{1}{2\pi+A} + \dots - \frac{1}{\pi+A} - \frac{1}{2\pi-A} - \frac{1}{3\pi+A} - \dots = \frac{1}{y} \tag{E7}$$

In equation (E7), actually $a_1 = -\frac{1}{y}$. This is because if we divide both sides of equation (E6) by $-a_0$ we get $\frac{a_1}{-a_0}$ and this represents the right-hand side of (E7) above. This would be the x-coefficient of equation (E3) as it should be. Now that Euler had finalized an equation, which found the a_1 term of an finite polynomial, he could finally test it by substituting in the root. To do this we need to look back at the series (E1) and what it was manipulated to be, that being, equation (E2). Euler let $y = 1$, which meant $\sin x$ had to also equal one. The first root which allowed this to happen was when $A = \frac{\pi}{2}$.

Plugging $\frac{\pi}{2}$ into equation (E6) then gave the following [kli83], [Dun83]:

$$(\frac{1}{\frac{\pi}{2}} + \frac{1}{\frac{\pi}{2}} + \frac{1}{\frac{5\pi}{2}} + \frac{1}{\frac{5\pi}{2}} + \frac{1}{\frac{9\pi}{2}} + \frac{1}{\frac{9\pi}{2}} ... - \frac{1}{\frac{3\pi}{2}} - \frac{1}{\frac{3\pi}{2}} - \frac{1}{\frac{7\pi}{2}} - \frac{1}{\frac{7\pi}{2}} - \frac{1}{\frac{11\pi}{2}} - \frac{1}{\frac{11\pi}{2}} ...) = 1$$

Next, Euler simplified the equation to:

$$(\frac{4}{\pi} + \frac{4}{5\pi} + \frac{4}{9\pi} + ... - \frac{4}{3\pi} - \frac{4}{7\pi} - \frac{4}{11\pi} - ...) = 1.$$

Then he factored out what's in common and put the remaining terms in a converging sequence:

$$\frac{4}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + ...) = 1.$$

Multiplying both sides by $\frac{\pi}{4}$ showed:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}.$$

3.2 Euler's Challenges

A good question to ask is, Why didn't any of Euler's predecessors working on infinite series before him, figure this out? Why didn't Issac Newton, Gottfried Leibniz, Brook Taylor, John Wallis or Colin Maclaurin or others come up with this method for finding the roots of infinite polynomials? The reason behind Euler's success was his ascertained claim that, "What holds for a finite polynomial holds for an infinite polynomial" [San01]. Claiming this meant he could write the infinite series in E2 as a infinite polynomial, and it could be set equal to an infinite product. This claim allowed him to come to the solution above for a_1. At this time in history, not many prominent mathematicians accepted Euler's claim; no one anticipated a value of $\frac{\pi}{4}$. This was an irrational number; π was associated with the circumference of a circle. For this to materialize in the formula for the sum was, at the time, reasonably contestable.

Euler continued with the same technique to determine the outcome for a_2 also. Could he come up with a comparable manner to solve for the coefficient a_2? Euler looked back at the infinite product in equation (E5). He wanted to set the right-hand side of it equal to the second-degree term of the finite polynomial on the left-hand side. Euler knew he could match up the second-degree terms with the a_2 coefficient. That is when

Euler realized that $a_2 = \frac{1}{AB} + \frac{1}{AC} + \frac{1}{BC} + ... = \sum_{i<j} \frac{1}{r_i r_j}$. He could manipulate the a_2 term here to have a pattern similar to (E7). Euler examined his discovery, and the discovery of the a_1 coefficient in (E6) and squared both sides forming the new equation below:

$$a_1^2 = [-a_0(\tfrac{1}{A} + \tfrac{1}{B} + \tfrac{1}{C} + ...)]^2. \qquad \text{(E8)}$$

Squaring the right-hand side of (E8) and rearranging the terms would give: $(a_0)^2[\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} + ... + \frac{1}{AB} + \frac{1}{BA} + \frac{1}{AC} + \frac{1}{CA} + \frac{1}{BC} + \frac{1}{CB} + ...] = (a_0)^2[\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} + ... + 2\sum_{i<j} \frac{1}{r_i r_j}]$. So, (E8) could be rewritten as: $(a_1)^2 = a_0^2(\sum_{i=1}^{n} \frac{1}{(r_i)^2} + 2\sum_{i<j}^{n} \frac{1}{r_i r_j})$ which implies $(a_1)^2 = \sum_{i=1}^{n} \frac{1}{(r_i)^2} + 2a_2$, since $2\sum_{i<j}^{n} \frac{1}{r_i r_j} = 2a_2$. Euler then rearranged these final terms to get: $(a_1)^2 - 2a_2 = \sum_{i=1}^{n} \frac{1}{(r_i)^2}$. He then applied this work to equation (E2) and the following equation below [kli83], [Dun83].

3.3 Solving for the Coefficient a_2

$$(a_1)^2 - 2a_2 = \frac{1}{A^2} + \frac{1}{(\pi-A)^2} + ... + \frac{1}{(\pi+A)^2} + \frac{1}{(2\pi+A)^2} + \frac{1}{(3\pi+A)^2} + ... = \frac{1}{y^2}. \qquad \text{(E9)}$$

Why did Euler set the equation in (E9) equal to $\frac{1}{y^2}$? Well, $(a_1)^2 - 2a_2 = \frac{1}{y^2}$. If we review equation (E2), we see that $a_1 = -\frac{1}{y}$, and $a_2 = 0$. Therefore $(-\frac{1}{y})^2 - 2(0) = \frac{1}{y^2}$. Euler did it again. He finalized an equation which found the a_2 term of a finite polynomial and it looked simular to (E7). He could again find the coefficient by plugging in the roots. Looking back at the series (E1) and what it was manipulated to be, equation (E2), Euler let $y = 1$, which meant $\sin x$ had to also equal one. The first root which allowed this to occur again would be the same, $A = \frac{\pi}{2}$. Substituting $\frac{\pi}{2}$ into equation (E9) gave:

$$\frac{1}{(\frac{\pi}{2})^2} + \frac{1}{(\pi-\frac{\pi}{2})^2} + \frac{1}{(2\pi+\frac{\pi}{2})^2} + \frac{1}{(3\pi-\frac{\pi}{2})} ... + \frac{1}{(-\pi-\frac{\pi}{2})^2} + \frac{1}{(-2\pi+\frac{\pi}{2})^2} + \frac{1}{(-3\pi-\frac{\pi}{2})^2} + \frac{1}{(-4\pi+\frac{\pi}{2})^2} + ... = 1$$

Next was squaring the denominator which gave:

$$\frac{1}{\frac{\pi^2}{4}} + \frac{1}{\frac{\pi^2}{4}} + \frac{1}{\frac{25\pi^2}{4}} + \frac{1}{\frac{25\pi^2}{4}} + ... \frac{1}{\frac{9\pi^2}{4}} + \frac{1}{\frac{9\pi^2}{4}} + \frac{1}{\frac{49\pi^2}{4}} + \frac{1}{\frac{49\pi^2}{4}} + ... = 1$$

Euler then simplified the equation as well as added some continuing terms to get:

$$\frac{8}{\pi^2} + \frac{8}{25\pi^2} + \frac{8}{81\pi^2} + ...\frac{8}{9\pi^2} + \frac{8}{49\pi^2} + \frac{8}{121\pi^2} + ... = 1$$

Then he factored out what is in common and put the remaining terms in a converging sequence giving [kli83], [Dun83]:

$$\frac{8}{\pi^2}(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + ...) = 1$$

Multiplying both sides by $\frac{\pi^2}{8}$ showed that:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... = \frac{\pi^2}{8}.$$

3.4 Euler's Problem Resolution

Here was another solution Euler had found for the sums of the reciprocals squared that his famous mathematical predicessors were stuck on. Now that he had found the convergence of the sum of the squares of the reciprocals of the roots of the polynomial, he noticed a pattern. First, finding the sum of the reciprocals of this polynomial had the pattern of summing one over the roots to determine convergence. Next, finding the sum of the squares of the reciprocals of the roots had the pattern of summing one over the squares of the roots to determine convergence. Was Euler foreseeing the same pattern also for summing one over the cubes of the roots? If so, this pattern would likely indefinitely continue for all possible roots of this given polynomial. This was the next step in the succession of determining a convergence or divergence of this series.

Euler obviously knew what the a_0 term was for the polynomial $p(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + ... + a_nx^n$ and now he had found a way to calculate the a_1 and a_2 terms also. Next he needed to calculate another term, the a_3 term, to see if a pattern would emerge similar to that of equation (E7). Since he was working with a polynomial in (E3) that dealt with only the odd terms, Euler needed to look at the complete infinite polynomial containing both even and odd terms to see what would occur with a_2. This infinite polynomial is shown as:

$$a_0 + a_1x + a_2x^2 + a_3x^3 + ...a_nx^n = a_0(1 - \frac{x}{\alpha_1})(1 - \frac{x}{\alpha_2})(1 - \frac{x^3}{\alpha_3})...(1 - \frac{x}{\alpha_n}). \qquad \text{(E10)}$$

Now looking at equation (E10), Euler needed to use some previous mathematical calculations from equations (E6) and (E8) to verify this theory for the higher powers of reciprocals, in particular, one over the cubes of the roots. He knew $a_0 = 1$, $a_1 = -a_0[\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + ... + \frac{1}{\alpha_n}]$ and that $a_1^2 = (a_1)^2 = \sum_{i=1}^{n} \frac{1}{(r_i)^2} + 2a_2$.

Knowing the following about the coefficients, Euler then cubed the a_1 term getting:

$$(a_1)^3 = a_1.(a_1)^2 = -a_0[\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + ... + \frac{1}{\alpha_n}].(a_0)^2[\frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + ... + \frac{1}{\alpha_n^2} + 2\sum_{i<j} \frac{1}{\alpha_i\alpha_j}].$$

Now multiplying it out gave:

$$(a_1)^3 = (-a_0)^3[\frac{1}{\alpha_1^3} + \frac{1}{\alpha_2^3} + ... + \frac{1}{\alpha_n^3} + 3\sum_{i<j} \frac{1}{\alpha_i\alpha_j^2} + 6\sum_{i<j<k} \frac{1}{\alpha_i\alpha_j\alpha_k}].$$

3.5 Solving for the Complexity of the Coefficient a_3

Here is where Euler saw the sum of the cubes. What was needed now was to solve for them and find the given summations. He isolated the sum of the cubes getting:

$$(-a_1)^3 - 3\sum_{i<j} \frac{1}{\alpha_i\alpha_j^2} - 6\sum_{i<j<k} \frac{1}{\alpha_i\alpha_j\alpha_k} = \frac{1}{\alpha_1^3} + \frac{1}{\alpha_2^3} + ... + \frac{1}{\alpha_n^3}, \text{ since } a_0 = 0.$$

The next step was to find the summations of this equation and this had to be completed by looking at the infinite polynomial. The first summation was $\sum \frac{1}{\alpha_i\alpha_j^2}$. This needed to be expanded and manipulated. Euler knew the expanded sum represented specific powers of terms of the infinite product on the right hand side. These specific powers were related to the coefficients of the same powers of the infinite polynomial, the left hand side. Euler's task now was to manipulate the summation so he could set it equal to the given coefficients of the infinite polynomial. The manipulation of the first summation $\sum_{i<j} \frac{1}{\alpha_i\alpha_j^2}$ was tricky. The summation expanded was:

$$\sum_{i<j} \frac{1}{\alpha_i\alpha_j^2} = \frac{1}{AB^2} + \frac{1}{AC^2} + \frac{1}{AD^2} + ... + \frac{1}{BA^2} + \frac{1}{BC^2} + \frac{1}{BD^2} + ... + \frac{1}{CA^2} + \frac{1}{CB^2} + \frac{1}{CD^2} + ... + \frac{1}{DA^2} + \frac{1}{DB^2} + \frac{1}{DC^2} + ...$$

Seeing the first expanded summation written out, Euler did something unusual [Dun83]. He added the one missing term to each different changing sum within the summation itself, and then subtracted each one out, being careful not to alter the original summation itself. This gave Euler no missing terms within the summation, and by subtracting the one term that was added in throughout, gave the difference of cubes. This was a cumbersome process that wasn't guaranteed to lead anywhere, but it had to be tried. So, the summation above was manipulated to look like the following:

$$\sum_{i<j} \frac{1}{\alpha_i\alpha_j^2} = (\frac{1}{AA^2} + \frac{1}{AB^2} + \frac{1}{AC^2} + ... - \frac{1}{AA^2}) + (\frac{1}{BA^2} + \frac{1}{BB^2} + \frac{1}{BC^2} + ... - \frac{1}{BB^2}) + (\frac{1}{CA^2} + \frac{1}{CB^2} + \frac{1}{CC^2} + ... - \frac{1}{CC^2}) + ... + (\frac{1}{RA^2} + \frac{1}{RB^2} + ... + \frac{1}{RR^2} - \frac{1}{RR^2}).$$

Now, when factoring out the common terms and the terms that Euler added in, the summation was manipulated into the following:

$$\sum_{i<j} \frac{1}{\alpha_i\alpha_j^2} = \frac{1}{A}(\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} + ...) + \frac{1}{B}(\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} + ...) + \frac{1}{C}(\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} + ...) + ... - (\frac{1}{A^3} + \frac{1}{B^3} + \frac{1}{C^3} + ...).$$

Factoring out the same common sum above, Euler was able to still manipulate the summation into:

$$\sum_{i<j} \frac{1}{\alpha_i\alpha_j^2} = (\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + ...)(\frac{1}{A^2} + \frac{1}{B^2} + \frac{1}{C^2} + ...) - (\frac{1}{A^3} + \frac{1}{B^3} + \frac{1}{C^3} + ...).$$

Euler had manipulated the first sum into something he could work with. The product of the first two here were found earlier. The equation for finding the a_1 coefficient was $a_1 = -a_0(\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + ...)$, and the equation for finding the a_2 coefficient was $(a_1)^2 - 2a_2 = \sum_{i=1}^{n} \frac{1}{(l_i)^2}$. Figuring this out, Euler now was able to state the first summation:

$$\sum_{i<j} \frac{1}{\alpha_i\alpha_j^2} = (-a_1)(a_1^2 - a_2) - (\frac{1}{A^3} + \frac{1}{B^3} + \frac{1}{C^3} + ...).$$

Euler had one more summation that needed to be manipulated. This was $6\sum_{i<j<k} \frac{1}{\alpha_i\alpha_j\alpha_k}$. It had to be expanded also, while setting the specific powers related to the coefficients of the same powers of the infinite polynomial on the left-hand side. Thus, Euler needed to look at the expanded sum to see the relationship. He expanded the sum and arrived at:

$$\sum_{i<j<k} \frac{1}{\alpha_i\alpha_j\alpha_k} = \frac{1}{ABC} + \frac{1}{ABD} + \frac{1}{ABE} + \frac{1}{ABF} + ... + \frac{1}{ACD} + \frac{1}{ACE} + \frac{1}{ACF} + \frac{1}{ACG} + ... + \frac{1}{ADE} + \frac{1}{ADF} + \frac{1}{ADG} + \frac{1}{ADH} + ...$$

Looking back at the infinite polynomial $a_0 + a_1 + a_2x^2 + a_3x^3 + ...a_nx^n = (1 - \frac{x}{A})(1 - \frac{x}{B})(1 - \frac{x}{C})...(1 - \frac{x}{\alpha_n})$, Euler knew the infinite product on the right-hand side had what his last sum was equal to. This sum dealt with the x^3 terms, but with a small manipulation. If the infinite product on the right-hand side was expanded, the x^3 terms would look like:

$$-\frac{1}{ABC} - \frac{1}{ABD} - \frac{1}{ABE} - \frac{1}{ABF} - ... - \frac{1}{ACD} - \frac{1}{ACE} - \frac{1}{ACF} - \frac{1}{ACG} - ... - \frac{1}{ADE} - \frac{1}{ADF} - \frac{1}{ADG} - \frac{1}{ADH} - ...$$

Now, factoring out the negative, matched the summation he needed. The negative coefficient of the x^3 term$(-a_3)$ of the infinite polynomial was what this summation would be equal to:

$$\sum_{i<j<k} \frac{1}{\alpha_i\alpha_j\alpha_k} = \frac{1}{ABC} + \frac{1}{ABD} + \frac{1}{ABE} + \frac{1}{ABF} + ... + \frac{1}{ACD} + \frac{1}{ACE} + \frac{1}{ACF} + \frac{1}{ACG} + ... + \frac{1}{ADE} + \frac{1}{ADF} + \frac{1}{ADG} + \frac{1}{ADH} + ... = -a_3.$$

With both summations found, Euler was able to rewrite the sum the cubes:

$$(-a_1)^3 - 3\sum_{i<j} \frac{1}{\alpha_i\alpha_j^2} - 6\sum_{i<j<k} \frac{1}{\alpha_i\alpha_j\alpha_k} = \frac{1}{\alpha_1^3} + \frac{1}{\alpha_2^3} + ... + \frac{1}{\alpha_n^3} \text{ as,}$$

$$(-a_1)^3 - 3(-a_1)(a_1^2 - 2a_2) - (\frac{1}{A^3} + \frac{1}{B^3} + \frac{1}{C^3} + ...) - 6(-a_3) = \frac{1}{A^3} + \frac{1}{B^3} + \frac{1}{C^3} + ...$$

He then rearranged the equation to see what the sum of the cubes were equal to:

$$a_1^3 - 3a_1a_2 - 3(-a_3) = \frac{1}{A^3} + \frac{1}{B^3} + \frac{1}{C^3} + ...$$

Going back to $y = \sin x$, the task was to replace the first root of the sum of the cubes by the smallest root that would allow $A = \sin\frac{\pi}{2} = 1$ and all proceeding roots $B, C, ...$by $\pm\pi$ multiples of the sine function which would also equal one. Examples of these multiples are: $A, \pi - A, 2\pi + A, 3\pi - A, ..., -\pi - A, -2\pi + A, -3\pi - A, ...$

Euler now replaced the sum of the cubes by these roots, knowing what the sum of cubes was equal to [kli83], [Dun83]:

$$\frac{1}{A^3} + \frac{1}{(\pi-A)^3} + \frac{1}{(2\pi+A)^3} + ... + \frac{1}{(\pi+A)} + \frac{1}{(2\pi+A)} + \frac{1}{(3\pi+A)} + ... = a_1^3 - 3a_1a_2 - 3(-a_3)$$

The summation now needed to be simplified. Euler did just that and obtained several steps in the simplification process. Also, since $a_2 = 0$ in the Tayor series for $\sin x$, Euler's infinite polynomial, the second term $-3a_1a_2$ on the R.H.S. went to zero leaving:

$$\frac{1}{(\frac{\pi}{2})^3} + \frac{1}{(\pi-\frac{\pi}{2})^3} + \frac{1}{(2\pi+\frac{\pi}{2})^3} + \frac{1}{(3\pi-\frac{\pi}{2})^3} + \frac{1}{(4\pi+\frac{\pi}{2})^3} + ... - \frac{1}{(\pi+\frac{\pi}{2})^3} - \frac{1}{(2\pi-\frac{\pi}{2})^3} - \frac{1}{(3\pi+\frac{\pi}{2})^3} - \frac{1}{(4\pi-\frac{\pi}{2})^3} - ... = a_1^3 - 3(-a_3).$$

Next, Euler simplified and cubed the denominator, then knowing $a_1 = \frac{1}{y}$, $a_3 = \frac{1}{3!y}$ and $y = 1$ he continued to simplify getting:

$$\frac{8}{\pi^3} + \frac{8}{\pi^3} + \frac{8}{125\pi^3} + \frac{8}{125\pi^3} + \frac{8}{729\pi^3} + \frac{8}{729\pi^3} + ... - \frac{8}{27\pi^3} - \frac{8}{27\pi^3} - \frac{8}{343\pi^3} - \frac{8}{343\pi^3} - \frac{8}{1331\pi^3} - \frac{8}{1331\pi^3} - ... = \frac{1}{2}.$$

He now combined the like terms getting:

$$\frac{16}{\pi^3} + \frac{16}{125\pi^3} + \frac{16}{729\pi^3} + ... - \frac{16}{27\pi^3} - \frac{16}{343\pi^3} - \frac{16}{1331\pi^3} - ... = \frac{1}{2}.$$

Euler then factored out what was common within the summation getting:

$$\frac{16}{\pi^3}(1 + \frac{1}{125} + \frac{1}{729} + ... - \frac{1}{27} - \frac{1}{343} - \frac{1}{1331} - ...) = \frac{1}{2}.$$

Next, he multiplied both sides by $\frac{\pi^3}{16}$ and put the terms of the summation in a converging order getting:

$1 - \frac{1}{27} + \frac{1}{125} - \frac{1}{343} + \frac{1}{729} - \frac{1}{1331} + ... = \frac{\pi^3}{16} \cdot \frac{1}{2}$

Euler finally realized the summation of the cubes converged and that it converged to the following:

$1 - \frac{1}{27} + \frac{1}{125} - \frac{1}{343} + \frac{1}{729} - \frac{1}{1331} + ... = \frac{\pi^3}{32}$.

Furthermore, if we manipulated the above equation, we could also say the cubes converged to the following [Dun83]:

$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + ... = \frac{\pi^3}{32}$.

Again! Euler found another solution for the alternating sums of the reciprocals of the odd positive integers cubed, yet another solution that his famous mathematical predecessors were not able to find a solution to. Pretty clearly, this technique could continue, but is increasingly much more difficult to compute as we try finding coefficients beyond a_3.

Chapter 4

Issac Newton's Influence

This is where Newton's material on infinte polynomials comes into play. In fact, finding the coefficient for a_4 and beyond, would be extremely problematical. Euler's predecessor and a distinguished mathematician, Sir Isaac Newton, revealed his own method for solving infinite polynomials. He is thought to be one of the greatest scientists ever to have lived.

Issac Newton was born December 25, 1642 in Lincolnshire, England, and he died March 20, 1727 in Kensington, England at 84 years of age. He was a physicist, mathematician, astronomer, philosopher, alchemist, and a theologian.

His most famous work, the Principia, is "considered to be one of the most important scientific books ever written laying the foundations in which he describes universal gravitations and the three laws of motion" [Bur01]. Legend, it seems, is that Newton saw an apple fall while in his Lincolnshire garden and then contended that gravity causes an object to fall constantly to the earth's center. Eventually he "guessed" that this force was responsible for other "orbital motions" and subsequently named it "universal gravitation" [Ham11], [Fow12].

Around 1661, Issac attended Trinity College. Here he worked while studying. He was considered a "sizar" or servant to other students. In 1664 the university closed down as the Great Plague was breaking out across Europe. Newton returned home where he focused on physics and mathematics. It is claimed that "in less than two years, he began revolutionary advances in mathematics, optic, physics, and astronomy" [OR00]. In 1667 the university reopened, and Newton "put himself forward for a candidate for a fellowship." Throughout his academic years, his scientific achievements continued. His later began writing religious tracts "dealing with the literal interpretation of the Bible." He became a member of the Parliament of England; he moved to London to become warden of the Royal Mint which provided additional wealth to his estate. In 1705, Queen Anne knighted Newton–Sir Issac Newton. By this time, Newton had become a rich man. He is buried in Westminster Abbey. It is claimed that Newton's work "has distinctly advanced every branch of mathematics then studied" [Rou08].

4.1 Newton's First Identity

Newton established a "general scheme for expressing sums of powers of roots for a polynomial in terms of its coefficients" [Kli83], [kal09]. This material is a precursor to Euler's work. Newton discovered a universal method to allow a mathematician to relate the sums of powers of roots of any given polynomial in terms of the coefficients. This would apply to any polynomial no matter what the power was and eventually bacame known as Newton's Identities. Looking at a polynomial, say $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$, we could use Newton's first identity to show that $\sum_{j=0}^{k} a_{n-j}s_{k-j} = 0$ when $k \geq n$. This is where s_k represents the sums of the roots to the k^{th} power and where n is represents the degree of the polynomial. Using the restrictions on this identity, we will show an example of a degree three polynomial $p(x) = x^3 - 2x^2 - 5x + 6$, verifying that indeed $\sum_{j=0}^{n} a_{n-j}s_{k-j} = 0$ for $k \geq n$. If we look at the polynomial $p(x)$, we know that it is a cubic polynomial, therefore $n = 3$ and $k \geq n$ represent the powers of the roots. If we let $k = 3$, then knowing $n = 3$, gives the summation: $\sum_{j=0}^{n} a_{n-j}s_{k-j} = a_3s_3 + a_2s_2 + a_1s_1 + a_0s_0 = 0$. Setting the leading coefficient $a_n = 1$ gives the summation: $\sum_{j=0}^{n} a_{n-j}s_{k-j} = s_3 + a_2s_2 + a_1s_1 + a_0s_0 = 0$. To prove Newton's summation actually equals zero, we will expand the $s_{k's}$ as a symmetric function of $r's$. This means that the summation $s_3 + a_2s_2 + a_1s_1 + a_0s_0$ can be rewritten as the following summation: $(r_1^3 + r_2^3 + r_3^3) + a_2(r_1^2 + r_2^2 + r_3^2) + a_1(r_1^1 + r_2^1 + r_3^1) + a_0(r_1^0 + r_2^0 + r_3^0)$. Since $a_2 = -2, a_1 = -5, a_0 = 6$ and the roots of $p(x)$ are $r_1 = 1, r_2 = -2, r_3 = 3$, we can now verify the summation actualy equals to zero. Replacing all known variables gives [kal09]:

$(1^3 + (-2)^3 + (3)^3) - 2(1^2 + (-2)^2 + 3^2) - 5(1 - 2 + 3) + 6(1^0 + (-2)^0 + 3^0)$
$= (1 - 8 + 27) - 2(1 - 4 + 9) - 5(1 - 2 + 3) + 6(1 + 1 + 1)$
$= 20 - 28 - 10 + 18$
$= 0.$

Thus, this verifies that $\sum_{j=0}^{n} a_{n-j}s_{k-j} = 0$ for $k \geq n$ when $k = 3$ and $n = 3$ for this polynomial.

Newton's identity will work given any value k such that $k \geq n$. A another example is using the same $p(x)$; let us verify this is true when $k = 5$. Knowing $n = 3$, then the summation would be: $\sum_{j=0}^{n} a_{n-j}s_{k-j} = a_3s_5 + a_2s_4 + a_1s_3 + a_0s_2 = 0$, rewriting the summation so that the leading coefficient $a_3 = 1$ gives $s_5 + a_2s_4 + a_1s_3 + a_0s_2$. Expanding the $s_{k's}$ as a symmetric function of $r's$ allows us to rewrite it as $(r_1^5 + r_2^5 + r_3^5) + a_2(r_1^4 + r_2^4 + r_3^4) + a_1(r_1^3 + r_2^3 + r_3^3) + a_0(r_1^2 + r_2^2 + r_3^2)$. All that is left is to plug in all known roots and coefficients to verify the summation equals zero [kal09]:

$(1^5 + (-2)^5 + 3^5) - 2(1^4 + (-2)^4 + 3^4) - 5(1^3 + (-2)^3 + 3^3) + 6(1^2 + (-2)^2 + 3^2)$
$= (1 - 32 + 243) - 2(1 + 16 + 81) - 5(1 - 8 + 27) + 6(1 + 4 + 9)$
$= (212) - 2(98) - 5(20) + 6(14)$
$= 212 - 196 - 100 + 84$
$= 0.$

Once more, this verifies that $\sum_{j=0}^{n} a_{n-j}s_{k-j} = 0$ for $k \geq n$;this time $k = 5$ and $n = 3$. Continuing for other values of $k \geq 3$ for $p(x)$ would prove true as well [Kal09].

Looking at another case that requires a great deal more calculation, possibly leading to a proof of the general case could be where $n = 6, k = 8$. For notational simplicity, we will let the leading coefficient $a_n = 1$, when using Newton's identities. Therefore, all polynomials need to be prearranged such that they are monic. When $n = 6, k = 8$, then the summation gives $\sum_{j=0}^{n} a_{n-j}s_{k-j} = s_8 + a_5s_7 + a_4s_6 + a_3s_5 + a_2s_4 + a_1s_3 + a_0s_2 = 0$. To prove the summation equals zero we need to expand the $s_{k's}$ as a symmetric function of $r's$ which represent the roots. Then the summation $s_8 + a_5s_7 + a_4s_6 + a_3s_5 + a_2s_4 + a_1s_3 + a_0s_2$ can be written as: $(r_1^8 + r_2^8 + r_3^8 + ... + r_n^8) + a_5(r_1^7 + r_2^7 + r_3^7 + ... + r_n^7) + a_4(r_1^6 + r_2^6 + r_3^6 + ...r_n^6) + a_3(r_1^5 + r_2^5 + r_3^5 + ...r_n^5) + a_2(r_1^4 + r_2^4 + r_3^4 + ... + r_n^4) + a_1(r_1^3 + r_2^3 + r_3^3 + ... + r_n^3) + a_0(r_1^2 + r_2^2 + r_3^2 + ... + r_n^2)$.

Distributing the coefficients, we can rearrange the finite summations into descending order to generate the following pattern below [kal09]:

$$\begin{array}{l}
r_1^8 + a_5 r_1^7 + a_4 r_1^6 + a_3 r_1^5 + a_2 r_1^4 + a_1 r_1^3 + a_0 r_1^2 \\
r_2^8 + a_5 r_2^7 + a_4 r_2^6 + a_3 r_2^5 + a_2 r_2^4 + a_1 r_2^3 + a_0 r_2^2 \\
r_3^8 + a_5 r_3^7 + a_4 r_3^6 + a_3 r_3^5 + a_2 r_3^4 + a_1 r_3^3 + a_0 r_3^2 \\
+ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \\
. \quad . \qquad\qquad\qquad\qquad\qquad\qquad\qquad . \\
. \qquad\quad . \qquad\qquad\qquad\qquad\qquad\quad . \\
. \qquad\qquad\qquad\quad . \qquad\qquad\qquad\quad . \\
+ \qquad\qquad\qquad\qquad\qquad\qquad + \qquad + \\
r_n^8 + a_5 r_n^7 + a_4 r_n^6 + a_3 r_n^5 + a_2 r_n^4 + a_1 r_n^3 + a_0 r_n^2.
\end{array} \qquad (**)$$

When adding the n-summations above, each column will generate the following desired sums represented by summing all like roots. This desired sum is written below as:

$$s_8 + a_5 s_7 + a_4 s_6 + a_3 s_5 + a_2 s_4 + a_1 s_3 + a_0 s_2. \qquad (1)$$

Now, if s_k is replaced by x^k, then the polynomial would look like $x^8 + a_5 x^7 + a_4 x^6 + a_3 x^5 + a_2 x^4 + a_1 x^3 + a_0 x^2 = 0$ and since each term has a common factor of x^2, then we can factor out x^2 getting $x^2 P(x)$ if we let $P(x) = x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0 = 0$ implying $x^2 P(x) = x^2(x^6 + a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x^1 + a_0) = 0$ Thus, the left hand side is $x^2 P(x)$ and this verifies $r_j^2 P(r_j) = 0$ for each r_j s.t. $j = 1, ..., n$ understanding that $r_j^2 P(r_j) = r_j^6 + a_5 r_j^5 + a_4 r_j^4 + a_3 r_j^3 + a_2 r_j^2 + a_1 r_j^1 + a_0 = 0$. This says that each row of (**) is zero. Finally, this allows us to conclude that $s_8 + a_5 s_7 + a_4 s_6 + a_3 s_5 + a_2 s_4 + a_1 s_3 + a_0 s_2 = 0$ [kal09].

Now, showing that $\sum_{j=0}^{n} a_{n-j} s_{k-j} = 0$ for any case when $k \geq n$ where k and n are arbitrary positive integers, gives the following summation:

$\sum_{j=0}^{n} a_{n-j} s_{k-j} = a_n s_k + a_{n-1} s_{k-1} + a_{n-2} s_{k-2} + ... + a_0 s_{k-n}$. This can be proven also equal to zero by expanding the $s_{k's}$ as a symmetric function of $r's$ giving:

$a_n s_k + a_{n-1} s_{k-1} + a_{n-2} s_{k-2} + ... + a_0 s_{k-n} = a_n(r_1^k + r_2^k + r_3^k + ... + r_n^k) + a_{n-1}(r_1^{k-1} + r_2^{k-1} + r_3^{k-1} + ... + r_n^{k-1}) + a_{n-2}(r_1^{k-2} + r_2^{k-2} + r_3^{k-2} + ... + r_n^{k-2}) + ... + a_0(r_1^{k-n} + r_2^{k-n} + r_3^{k-n} + ... + r_n^{k-n})$.

Again distributing the coefficients(the a-terms) and breaking up the summation by the same roots, we can further expand this summation for $k \geq n$ where n and k are arbitrary, to look like [kal09]:

$$
\begin{array}{l}
a_n r_1^k + a_{n-1} r_1^{k-1} + a_{n-2} r_1^{k-2} + \dots + a_0 r_1^{k-n} \\
a_n r_2^k + a_{n-1} r_2^{k-1} + a_{n-2} r_2^{k-2} + \dots + a_0 r_1^{k-n} \\
a_n r_2^k + a_{n-1} r_3^{k-1} + a_{n-2} r_3^{k-2} + \dots + a_0 r_3^{k-n} \\
\quad + \qquad\qquad\qquad\qquad + \\
\quad \cdot \quad \cdot \qquad\qquad\qquad \cdot \qquad\qquad\qquad (*) \\
\quad \cdot \qquad \cdot \qquad\qquad \cdot \\
\quad \cdot \qquad\quad \cdot \qquad\quad \cdot \\
\quad + \qquad\qquad\quad + \qquad + \\
a_n r_n^k + a_{n-1} r_n^{k-1} + a_{n-2} r_n^{k-2} + \dots + a_0 r_n^{k-n}.
\end{array}
$$

Again, adding the sums of the similar roots of the summation above obtains [Kal09]:

$a_n s_k + a_{n-1} s_{k-1} + a_{n-2} s_{k-2} + \dots + a_0 s_{k-n}$. If we replace the $s_{k's}$ above by $x^{k's}$ then we get the following equation:

$a_n x^k + a_{n-1} x^{k-1} + a_{n-2} x^{k-2} + \dots + a_0 x^{k-n}$.

(Case #1)

If $k = n$ then the summation simplifies to:

$a_n x^k + a_{n-1} x^{k-1} + a_{n-2} x^{k-2} + \dots + a_0 = 0$ and thus is non-factorable when $k = n$.

(Case #2)

If $k > n$ the summation simplifies to:

$a_n x^k + a_{n-1} x^{k-1} + a_{n-2} x^{k-2} + \dots + a_0 x^{k-n} = 0$ here, each term has a common factor of x^{k-n}. Therefore, in this case, we can factor out x^{k-n} getting $x^{k-n}P(x)$ if we let $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_0 = 0$ implying $x^{k-n}P(x) = x^{k-n}(a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_0) = 0$. This is similar to the summation above when where we had $n = 6$ and $k = 8$. Now that $r_j^{k-n}P(r_j) = 0$ for each r_j such that, $j = 1, \dots, n$, I can finally say $a_n r_j^k + a_{n-1} r_j^{k-1} + a_{n-2} r_j^{k-2} + \dots + a_0 r_j^{k-n} = 0$. Again, this says that each row of (*) is zero and allows use to implys that $a_n s_k + a_{n-1} s_{k-1} + a_{n-2} s_{k-2} + \dots + a_0 s_{k-n} = 0$.

This method was Newton's universal method, to be able to relate the sums of the powers to the roots of any given polynomial $P(x)$. Eventually this bacame known as one of Newton's two identies which is remembered as his first Identity, that being, given that $k \geq n$, we could show that $\sum_{j=0}^{k} a_{n-j} s_{k-j} = 0$ [kal09].

4.2 Newton's Second Identity

The other case Newton produced became known as his second identy. Which says that given a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1 x^1 + a_0$, then $\sum_{j=0}^{k} a_{n-j}s_{k-j} = (n-k)a_{n-k}$ for $k < n$. This is where s_k represents the sums of the roots to the k^{th} power and n represents by the degree of the polynomial. Within the summation, we obviously know that a_j is the coefficient of the x_j term, but more over, s_n is the sum of the n^{th} powers of the roots, however many roots there might be. Using Newton's identity, we will show an example of a degree three polynomial $p(x) = x^3 - 2x^2 - 5x + 6$, verifying that indeed $\sum_{j=0}^{k} a_{n-j}s_{k-j} = (n-k)a_{n-k}$ works for $k < n$. If we take a look at the polynomial in question, we can see that the coefficients are $a_3 = 1, a_2 = -2, a_1 = -5, a_0 = 6$ and then with some work, realize the roots of the polynomial are $r_1 = 1, r_2 = -2, r_3 = 3$. Knowing that the degree of $p(x)$ is three, verifies that $n = 3$ for the boundaries of Newton's identity and thus allows us to show in detail, one of the two expansions for k, when k is a positive integer less than n. This is the case here, when $k = 2$, the summation would be: $\sum_{j=0}^{2} a_{3-j}s_{2-j} = a_3 s_2 + a_2 s_1 + a_1 s_0 = a_1$. Subtracting a_1 from both sides of the expression gives $a_3 s_2 + a_2 s_1 + a_1 s_0 - a_1 = 0$. Since we know that the $s_{k's}$ represent the sums of the roots to the k^{th} power of the polynomial $p(x)$, we can expand this summation into something more comprehensible for our purpose in verifying Newton's identity. Writing s_2, s_1, s_0 as the sums of the roots to the power they represent, we get the following expansion: $a_3(r_1^2+r_2^2+r_3^2)+a_2(r_1^1+r_2^1+r_3^1)+a_1(r_1^0+r_2^0+r_3^0)-a_1 = 0$. Now all that is left is to substitute in all the known roots and given coefficients to this expression to verify Newton's identity checks [Kal09]. This is shown below in a few steps:

$$\begin{aligned}
&a_3(r_1^2 + r_2^2 + r_3^2) + a_2(r_1^1 + r_2^1 + r_3^1) + a_1(r_1^0 + r_2^0 + r_3^0) - a_1 \\
&= 1(1^2 + (-2)^2 + 3^2) - 2(1 + (-2) + 3) - 5(1 + 1 + 1) + 5 \\
&= 1(14) - 2(2) - 5(3) + 5 \\
&= 14 - 4 - 15 + 5 \\
&= 0.
\end{aligned}$$

Thus, Newton's identity holds for this polynomial.

Now, working with the same polynomial, $p(x)$, where n still equals 3, for the boundaries of Newton's identity, we can show the following expansion for the other case, the case when $k = 1$. The summation, this time would be:

$\sum_{j=0}^{1} a_{3-j}s_{1-j} = a_3s_1 + a_2s_0 = 2a_2$. Subtracting $2a_2$ from both sides of the expression gives $a_3s_1 + a_2s_0 - 2a_2 = 0$. Since we know that the $s_{k's}$ represent the sums of the roots to the k^{th} power, this allows us to represent the polynomial $p(x)$ as something more explicable for our purpose in verifying Newton's identity. This time we will also be working with s_1, s_0 also as the sums of the roots they represent. Manipulating the summation above to equal zero gives: $a_3(r_1^1 + r_2^1 + r_3^1) + a_2(r_1^0 + r_2^0 + r_3^0) - 2a_2 = 0$. All that is left is to substitute all the known roots and given coefficients to verify Newton's identity checks and this is shown below:

$$\begin{aligned}
&a_3(r_1^1 + r_2^1 + r_3^1) + a_2(r_1^0 + r_2^0 + r_3^0) - 2a_2 \\
&= 1(1 + (-2) + 3) - 2(1 + 1 + 1) - 2(2) \\
&= 1(2) - 2(3) - 2(-2) \\
&= 0
\end{aligned}$$

Again, Newton's identity holds for this polynomial [Kal09]. We have just shown that the polynomial $p(x) = x^3 - 2x^2 - 5x + 6$ verifies a particular case using Newton's summation $\sum_{j=0}^{k} a_{n-j}s_{k-j} = (n-k)a_{n-k}$ where $k < n$. This is the case where $n = 3$, which is a degree three polynomial and $k = 1$ or 2, the only positive integers less than n.

We will now show another case that requires much more computation and might point us to the proof of the general case. Let us look at the case when $n = 6$ and k decreases from 5 to 1.

When $k = 5$ then $\sum_{j=0}^{k} a_{n-j}s_{k-j} =$ $\quad s_5 + a_5s_4 + a_4s_3 + a_3s_2 + a_2s_1 + a_1s_0 = a_1$.

When $k = 4$ then we get: $\quad s_4 + a_5s_3 + a_4s_2 + a_3s_1 + a_2s_0 = 2a_2$.

When $k = 3$ then we get: $\quad s_3 + a_5s_2 + a_4s_1 + a_3s_0 = 3a_3$.

When $k = 2$ then we get: $\quad s_2 + a_5s_1 + a_4s_0 = 4a_4$.

When $k = 1$ then we get: $\quad s_1 + a_5s_0 = 5a_5$.

In the cases for $n = 6$ and $k = 5, 4, 3, ...1$, a noticeable pattern is created. If we take the case for $k = 5$ and replace all the $s_{k's}$ with $x^{k's}$ this will produce the polynomial $p(x) = x^5 + a_5x^4 + a_4x^3 + a_3x^2 + a_2x^1 + a_1x^0 - a_1 = 0$. Dividing $p(x)$ by one of its known roots, say r_j, produces exactly the coefficients of this polynomial as shown below:

$\frac{p(x)}{x-r_j} = x^5 + (r_j + a_5)x^4 + (r_j^2 + a_5r_j + a_4)x^3 + (r_j^3 + a_5r_j^2 + a_4r_j + a_3)x^2 + (r_j^4 + a_5r_j^3 + a_4r_j^2 + a_3r_j + a_2)x + (r_j^5 + a_5r_j^4 + a_4r_j^3 + a_3r_j^2 + a_2r_j + a_1)$.

Looking at the polynomial above, and using our knowledge of calculus, we should remember that $\sum_{j=1}^{6} \frac{p(x)}{x-r_j} = p'(x)$. What this is saying in words is that, if you take the sum of the polynomial divided by any of it roots, one at a time as you sum over all the roots, this will give you the derivative of the polynomial [Kal09]. This is explicitly the case above, and if we let $j = 1, 2, 3, ...5$, the roots of $p'(x)$, then expanding $p'(x)$ out, gives the summation over all the roots to look like the following:

$p'(x) = \frac{p(x)}{x-r_1} + \frac{p(x)}{x-r_2} + \frac{p(x)}{x-r_3} + ... \frac{p(x)}{x-r_6} = 6x^5 + ((r_1^1 + r_2^1 + r_3^1 + ...r_6^1) + a_5(r_1^0 + r_2^0 + r_3^0 + ...r_6^0))x^4 + ((r_1^2 + r_2^2 + r_3^2 + ...r_6^2) + a_5(r_1^1 + r_2^1 + r_3^1 + ...r_6^1) + a_4(r_1^0 + r_2^0 + r_3^0 + ...r_6^0))x^3 + ((r_1^3 + r_2^3 + r3 + ...r_6^3) + a_5(r_1^2 + r_2^2 + r_3^2 + ...r_6^2) + a_4(r_1^1 + r_2^1 + r_3^1 + ...r_6^1) + a_3(r_1^0 + r_2^0 + r_3^0 + ...r_6^0)x^2 + ((r_1^4 + r_2^4 + r_3^4 + ...r_6^4) + a_5(r_1^3 + r_2^3 + r_3^3 + ...r_n^3) + a_4(r_1^0 + r_2^0 + r_3^0 + ..r_6^0) + a_3(r_1^1 + r_2^1 + r_3^1 + ...r_6^1) + a_2(r_1^0 + r_2^0 + r_3^0 + ...r_6^0)x + ((r_1^5 + r_2^5 + r_3^5 + ...r_6^5) + a_5(r_1^4 + r_1^4 + r_1^4 + ...r_6^4) + a_4(r_1^3 + r_2^3 + r_3^3 + ...r_6^3) + a_3(r_1^2 + r_2^2 + r_3^2 + ...r_6^2) + a_2(r_1^1 + r_2^1 + r_3^1 + ...r_6^1) + a_1(r_1^0 + r_2^0 + r_3^0 + ...r_6^0)).$

If we reintroduce the $s_{k's}$ again as the sum of the roots, we can simplify what above to look like the following [kal09]:

$\frac{p(x)}{x-r_1} + \frac{p(x)}{x-r_2} + \frac{p(x)}{x-r_3} + ... \frac{p(x)}{x-r_6} = 6x^5 + (s_1 + a_5 s_0)x^4 + (s_2 + a_5 s_1 + a_4 s_0)x^3 + (s_3 + a_5 s_2 + a_4 s_1 + a_3 s_0)x^2 + (s_4 + a_5 s_3 + a_4 s_2 + a_3 s_1 + a_2 s_0)x + (s_5 + a_5 s_4 + a_4 s_3 + a_3 s_2 + a_2 s_1 + a_1 s_0).$

Observing Newton's identity here, we should now realize that the right-hand side is repeated by synthetic division on all roots r_j such that $j = 1, 2, 3, ...5$. When the computation is complete, the end result is $p'(x)$, the derivative of the polynomial $p(x)$. The left-hand side of $\frac{p(x)}{x-r_1} + \frac{p(x)}{x-r_2} + \frac{p(x)}{x-r_3} + ... \frac{p(x)}{x-r_6}$ is the factored form of the polynomial $p(x)$ written as $(x - r_1)(x - r_2)(x - r_3)...(x - r_6)$. Since we know $p'(x) = 6x^5 + 5a_5x^4 + 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$ we can set the coefficients of the terms accurately equal to Newton's identity $\sum_{j=0}^{k} a_{n-j}s_{k-j} = (n-k)a_{n-k}$ for $k < n$ when $n = 6$ and $k = 5, 4, 3, ...1$, for which these k^{th} values are shown above, once again verifying Newton's identities [Kal09].

If we review the two unique cases above, these patterns might potentially help construct a proof of Newton's identity: $\sum_{j=0}^{k} a_{n-j}s_{k-j} = (n-k)a_{n-k}$ when $k < n$, n is finite and $k = n, n-1, n-2, ...3, 2, 1$. Knowing these values for k and n, we are able to expand the summation to get:

$$\sum_{j=0}^{k} a_{n-j}s_{k-j} = a_n s_k + a_{n-1}s_{k-1} + a_{n-2}s_{k-2} + ... + a_{n-k}s_0 = (n-k)a_{n-k}.$$

Replacing all the $s_{k's}$ with $x^{k's}$ produces the following polynomial:

$p(x) = a_n x^{k-0} + a_{n-1}x^{k-1} + a_{n-2}x^{k-2} + ... + a_{n-k}x^0$.

Dividing this finite polynomial by one of its known roots, say r_j, produces the coefficients of the polynomial shown below where we know the leading coefficient is monic, by letting $a_n = 1$:

$\frac{p(x)}{x-r_j} = a_n x^{k-1} + (r_j + a_{n-1})x^{k-2} + (r_j^2 + a_{n-1}r_j + a_{n-2})x^{k-3} + (r_j^3 + a_{n-1}r_j^2 + a_{n-2}r_j + a_{n-3})x^{k-4} + ... + (r_j^{k-0} + a_{n-1}r_j^{k-1} + a_{n-2}r_j^{k-2} + ...a_1)$.

Now, letting $j = 1, 2, 3, ...n$, all the roots of the finite polynomial $p(x)$ and then expanding $p(x)$ out, allows the summation over all the roots to become:

$\frac{p(x)}{x-r_1} + \frac{p(x)}{x-r_2} + \frac{p(x)}{x-r_3} + ...\frac{p(x)}{x-r_n} = nx^{k-1} + ((r_1^1 + r_2^1 + r_3^1 + ...r_n^1) + a_{n-1}(r_1^0 + r_2^0 + r_3^0 + ...r_n^0))x^{k-2} + ((r_1^2 + r_2^2 + r_3^2 + ...r_n^2) + a_{n-1}(r_1^1 + r_2^1 + r_3^1 + ...r_n^1) + a_{n-2}(r_1^0 + r_2^0 + r_3^0 + ...r_n^0))x^{k-3} + ((r_1^3 + r_2^3 + r_3^3 + ...r_n^3) + a_{n-1}(r_1^2 + r_2^2 + r_3^2 + ...r_n^2) + a_{n-2}(r_1^1 + r_2^1 + r_3^1 + ...r_n^1) + a_{n-3}(r_1^0 + r_2^0 + r_3^0 + ...r_n^3))x^{k-4} + ... + ((r_1^{k-0} + r_2^{k-0} + r_3^{k-0} + ...r_n^{k-0}) + a_{n-1}(r_1^{k-1} + r_2^{k-1} + r_3^{k-1} + ...r_n^{k-1}) + a_{n-2}(r_1^{k-2} + r_2^{k-2} + r_3^{k-2} + ...r_n^{k-2}) + ...a_1)$.

If we reintroduce the s's as the sum of the roots, we can simplify what is above to look like:

$\frac{p(x)}{x-r_1} + \frac{p(x)}{x-r_2} + \frac{p(x)}{x-r_3} + ...\frac{p(x)}{x-r_n} = nx^{k-1} + (s_1 + a_{n-1}s_0)x^{k-2} + (s_2 + a_{n-1}s_1 + a_{n-2}s_0)x^{k-3} + (s_3 + a_{n-1}s_2 + a_{n-2}s_1 + a_{n-3}s_0)x^{k-4} + ... + (s_{k-0} + a_{n-1}s_{k-1} + a_{n-2}s_{k-2} + ...a_1 s_0)$.

Observing the right-hand side of the finite sum above and the examples above, we can say the sum is also a repeated synthetic division on the finite sum over all the roots r_j such that $j = 1, 2, 3, ...n$. When the computation is complete, the end result is again $p'(x)$ the derivative of the original polynomial $p(x)$. The left-hand side of $\frac{p(x)}{x-r_1} + \frac{p(x)}{x-r_2} + \frac{p(x)}{x-r_3} + ...\frac{p(x)}{x-r_n}$ is the factored form of the polynomial $p(x)$ written as $(x - r_1)(x - r_2)(x - r_3)...(x - r_n)$.

Since we know $p'(x) = nx^{k-1} + (n-1)a_{n-1}x^{k-1} + (n-2)a_{n-2}x^{k-2} + (n-3)a_{n-3}x^{k-3} + ... + a_1$ we can set the coefficients of the terms accurately, equal to Newton's identity $\sum_{j=0}^{k} a_{n-j}s_{k-j} = (n-k)a_{n-k}$ for $k < n$ when $n = n$ and $k = k, ...1$, for which these k^{th} values are shown above, once again verifying Newton's second identity [Kal09].

Chapter 5

Issac Newton's Solutions

What became known as the infinite polynomial above in (E2) was one of the most famous, yet devisive polynomials of the 17th century. We have shown above the methods used by both Leonhard Euler and Sir Isaac Newton in their accomplishments to a solution to this polynomial. Euler's success to this polynomial brought him fame as a mathematician, but Newton's method was less demanding when it came to evalution. Using Newton's second identity: $\sum_{j=0}^{k} a_{n-j}s_{k-j} = (n-k)a_{n-k}$ when $k < n$, we will show the steps that can be used to solve this infinite polynomial (E2) above. Plugging $k = 1, 2, 3, 4$ into our summation will help us see how less involved Newton's method was. Applying the second identity for k gives us the following four results below.

When $k = 0$ then $\sum_{j=0}^{k} a_{n-j}s_{k-j} = a_n s_0 = na_n$ this implies $s_0 = \frac{na_n}{a_n}$ which implies $s_0 = n$.

When $k = 1$ then $\sum_{j=0}^{1} a_{n-j}s_{k-j} = a_n s_1 + a_{n-1}s_0 = (n-1)a_{n-1}$ now, we need to simplify this and solve for s_1. So, $a_n s_1 = (n-1)a_{n-1} - a_{n-1}s_0$ using the distriutive property and substituting in for s_0, we get, $a_n s_1 = na_{n-1} - a_{n-1} - a_{n-1}n$ which implies $s_1 = \frac{-a_{n-1}}{a_n}$. What this means is that: $s_1 = r_1 + r_2 + r_3 + ... + r_n = \frac{-a_{n-1}}{a_n}$. By knowing the sums of the roots, we can calculate the reciprocals of the sums of the roots to be: $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + ... + \frac{1}{r_n} = \frac{-a_1}{a_0}$. This is because given any polynomial whenever it's coefficients are reversed then it's roots are replaced by reciprocals of roots of that polynomial [kal09]. For example is we look at the polynomial $p(x) = x^2 + x - 6$, it factors to $(x-2)(x+3)$. That means the roots of this polynomail $p(x)$ is $r_1 = 2$ and $r_2 = -3$. If we reverse the coefficients of the polynomail $p(x)$, then $revp(x) = -6x^2 + x + 1$. Now, $revp(x)$ factors to $(-2x+1)(3x+1)$ so, the roots of $revp(x)$ are $r_1 = \frac{1}{2}$ and $r_2 = -\frac{1}{3}$ which shows that by reversing the coefficients of a polynomial we get the roots replace by the reciprocals of the roots. This allows us to create what we are calling, the reciprocal of the sums of the roots equation above.

When $k = 2$ then $\sum_{j=0}^{2} a_{n-j}s_{k-j} = a_n s_2 + a_{n-1}s_1 + a_{n-2}s_0 = (n-2)a_{n-2}$. Now, we need to simplify this and solve for s_2. so, $a_n s_2 = (n-2)a_{n-2} - a_{n-1}s_1 - a_{n-2}s_0$ using the distributive property and substituting in for s_0 and s_1, we get, $a_n s_2 = na_{n-2} - 2a_{n-2} - a_{n-1}(\frac{-a_{n-1}}{a_n}) - a_{n-2}n$. Simplifying further, we get, $a_n s_2 = \frac{(a_{n-1})^2}{a_n} - 2a_{n-2}$ which implies $s_2 = \frac{(a_{n-1})^2}{a_n^2} - \frac{2a_{n-1}}{a_n}$. What this means is that: $s_2 = r_1^2 + r_2^2 + r_3^2 + ... + r_n^2 = \frac{(a_{n-1})^2}{a_n^2} - \frac{2a_{n-2}}{a_n}$. By knowing the sums of the roots squared, we can calculate the reciprocals of the sums of the roots square to be: $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + ... + \frac{1}{r_n^2} = \frac{a_1^2}{a_0^2} - \frac{2a_2}{a_0}$.

When $k = 3$ then $\sum_{j=0}^{3} a_{n-j}s_{k-j} = a_n s_3 + a_{n-1}s_2 + a_{n-2}s_1 + a_{n-3}s_0 = (n-3)a_{n-3}$. Now, we need to simplify this and solve for s_3. So, $a_n s_3 = (n-3)a_{n-3} - a_{n-1}s_2 - a_{n-2}s_1 - a_{n-3}s_0$ using the distributive property and substituting in for s_0, s_1, and s_2 we get, $a_n s_3 = na_{n-3} - 3a_{n-3} - a_{n-1}(\frac{a_{n-1}}{a_n^2} - \frac{2a_{n-2}}{a_n}) - (\frac{-a_{n-1}}{a_n}) - a_{n-3}n$. Simplifying further, we get, $a_n s_3 = na_{n-3} - 3a_{n-3} - \frac{a_{n-1}^3}{a_n^2} + \frac{2a_{n-2}a_{n-1}}{a_n} + \frac{a_{n-2}a_{n-1}}{a_n} - na_{n-3}$. Continuing to simplify, gives: $s_3 = -\frac{3a_{n-3}}{a_n} - \frac{a_{n-1}^3}{a_n^3} + \frac{3a_{n-2}a_{n-1}}{a_n 2}$. What this means is that: $s_3 = r_1^3 + r_2^3 + r_3^3 + ... + r_n^3 = -\frac{3a_{n-3}}{a_n} - \frac{a_{n-1}^3}{a_n^3} + \frac{3a_{n-2}a_{n-1}}{a_n^2}$. By knowing the sums of the roots cubed, we can calculate the reciprocals of the sums of the roots cubed to be: $\frac{1}{r_1^3} + \frac{1}{r_2^3} + \frac{1}{r_3^3} + ... + \frac{1}{r_n^3} = -\frac{a_1^3}{a_0^3} - \frac{3a_3}{a_0} + \frac{3a_2 a_1}{a_0^2}$.

Last, when $k = 4$ then $\sum_{j=0}^{4} a_{n-j}s_{k-j} = a_n s_4 + a_{n-1}s_3 + a_{n-2}s_2 + a_{n-3}s_1 + a_{n-4}s_0 = (n-4)a_{n-4}$. Now, we need to simplify this and solve for s_4. So, $a_n s_4 = (n-4)a_{n-4} - a_{n-1}s_3 - a_{n-2}s_2 - a_{n-3}s_1 - a_{n-4}s_0$. Using the distributive property and substituting in for s_0, s_1, s_2 and s_3,we get: $a_n s_4 = na_{n-4} - 4a_{n-4} - a_{n-1}(-\frac{a_{n-1}^3}{a_n^3} - \frac{3a_{n-3}}{a_n} + \frac{3a_{n-2}a_{n-1}}{a_n^2}) - a_{n-2}(\frac{a_{n-1}^2}{a_n^2} - \frac{2a_{n-2}}{a_n}) - a_{n-3}(-\frac{a_{n-1}}{a_n}) - a_{n-4}n$. Again using the distributive property and simplifying further, we get, $a_n s_4 = na_{n-4} - 4a_{n-4} + \frac{a_{n-1}^4}{a_n^3} + \frac{3a_{n-3}a_{n-1}}{a_n} - \frac{3a_{n-2}a_{n-1}^2}{a_n^2} - \frac{a_{n-1}^2 a_{n-2}}{a_n^2} + \frac{2a_{n-2}^2}{a_n} + \frac{a_{n-3}a_{n-1}}{a_n} - na_{n-4}$. Dividing both sides by a_n and adding like terms gives: $s_4 = -\frac{4a_{n-4}}{a_n} + \frac{a_{n-1}^4}{a_n^4} + \frac{2a_{n-2}^2}{a_n^2} - \frac{4a_{n-2}a_{n-1}^2}{a_n^3} + \frac{4a_{n-3}a_{n-1}}{a_n^2}$. This means: that: $s_4 = r_1^4 + r_2^4 + r_3^4 + ... + r_n^4 = -\frac{4a_{n-4}}{a_n} + \frac{a_{n-1}^4}{a_n^4} + \frac{2a_{n-2}^2}{a_n^2} - \frac{4a_{n-2}a_{n-1}^2}{a_n^3} + \frac{4a_{n-3}a_{n-1}}{a_n^2}$.

By knowing the sums of the roots to the fourth power, we can calculate the reciprocals of the sums of the roots to the fourth power to be: $\frac{1}{r_1^4}+\frac{1}{r_2^4}+\frac{1}{r_3^4}+...+\frac{1}{r_n^4}=-\frac{4a_4}{a_0}+\frac{a_1^4}{a_0^4}+\frac{2a_2^2}{a_0^2}-\frac{4a_2a_1^2}{a_0^3}+\frac{4a_3a_1}{a_0^2}$.

Finally, for this final summation, we will show the actual solution for the sums of the reciprocals of the roots to the 4^{th} power. First, it is necessary to remember a few things stated earlier. The equation (E1) $y=1$ meant that the smallest positive root for $\sin x=1$ was$\frac{\pi}{2}$; So, we let $\frac{\pi}{2}$ equal to A because $\sin\frac{\pi}{2}=1$. Also, all other proceeding roots that allowed $\sin x=1$, would be $\pm\pi$ multiples of the sine function, that being $\pi-A, 2\pi+A, 3\pi-A, ...-\pi-A, -2\pi+A, -3\pi-A, ...$. Knowing this, we are able to replace the reciprocals of the roots to the 4^{th} power on the left-hand side of the summation above by all possible roots that allow $\sin x=1$ starting with A which equals $\frac{\pi}{2}$. On the right-hand side of this equation, we need to substitute what the following coefficients are equal to, by matching them with the coefficients in equation (E2). Doing all this gives us:

$\frac{1}{(\frac{\pi}{2})^4}+\frac{1}{(\pi-\frac{\pi}{2})^4}+\frac{1}{(2\pi+\frac{\pi}{2})^4}+\frac{1}{(3\pi-\frac{\pi}{2})^4}+\frac{1}{(4\pi+\frac{\pi}{2})^4}+...+\frac{1}{(-\pi-\frac{\pi}{2})^4}+\frac{1}{(-2\pi+\frac{\pi}{2})^4}+\frac{1}{(-3\pi-\frac{\pi}{2})^4}+\frac{1}{(-4\pi+\frac{\pi}{2})^4}-...=-\frac{4(0)}{1}+\frac{(-\frac{1}{y})^4}{1^4}+\frac{2(0)^2}{1^2}-\frac{4(0)(1)^2}{1^3}+\frac{4(\frac{1}{3y})(-\frac{1}{y})}{1^2}$.

Now taking the 4^{th} root of the denominator and simplifying the double fractions as well as the right hand side of the equation above gives:

$\frac{16}{\pi^4}+\frac{16}{\pi^4}+\frac{16}{625\pi^4}+\frac{16}{625\pi^4}+\frac{16}{6561\pi^4}+...+\frac{16}{81\pi^4}+\frac{16}{81\pi^4}+\frac{16}{2104\pi^4}+\frac{16}{2104\pi^4}=\frac{1}{y^4}-\frac{4}{6y^2}$.

Combining the like terms on the left hand side of the equation and plugging one into y on the right hand side gives:

$\frac{32}{\pi^4}+\frac{32}{625\pi^4}+\frac{32}{6561\pi^4}+...+\frac{32}{81\pi^4}+\frac{32}{2104\pi^4}+\frac{32}{14641\pi^4}+...=\frac{1}{3}$.

We know need to factor out the like terms on the left hand side equation getting:

$\frac{32}{\pi^4}(1+\frac{1}{625}+\frac{1}{6561}+...+\frac{1}{81}+\frac{1}{2104}+\frac{1}{14641}+...)=\frac{1}{3}$.

Next, multiplying both sides by $\frac{\pi^4}{32}$ and putting the terms of the summation into descending order we get:

$1+\frac{1}{81}+\frac{1}{625}+\frac{1}{2104}+\frac{1}{6561}+\frac{1}{14641}+...=\frac{\pi^4}{32}\cdot(\frac{1}{3})$.

If we take this one more step, we can say the reciprocals of the fourth root converge to the following:

$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\frac{1}{11^4}+...=\frac{\pi^4}{96}$.

Chapter 6

Conclusion

Leonhard Euler, one of the greatest mathematicians of his time, in the 18th century, found a unique way to solve infinite polynomials. Solving infinite polynomials, unknown converging and diverging series as well as other unsolved mathematical uncertainties, was the focus of some brilliant mathematical minds during this period in history. Euler's method for solving the infinite polynomial was often criticized by other mathematicians of his time, but it was to be the turning point in terms of recognizing him as a distinguished mathematician. His claim: "what holds for a finite polynomial holds for an infinite polynomial" [San01] is what allowed him to succeed and proved him to be one of the greatest mathematicians in history [San01], [AFDnd].

Euler's well known predecessor and distinguished mathematician of the 17th century, Sir Isaac Newton, spent much time on the same topic, taking an identical path as Euler, but without success. Although Euler was able to prove his method worked for finding the coefficients of polynomials as well as the infinite polynomial given the known roots, it was extremely cumbersome and laborious. In fact, finding the coefficient of a_3, using Euler's method, has filled more than four pages in this thesis. Each time the coefficient increased, the difficulty amplified exponentially becoming more complex than previously.

Isaac Newton found another way to work with the infinite polynomials that was actually simpler to solve. He established a "general scheme for expressing sums of powers of roots for a polynomial in terms of its coefficients" [Kli83]. This was a universal method that related the sums of the powers of roots of any polynomial in terms of the coefficients.

Finding the a_3 coefficient using Newton's method took less than half a page. Newton's method would be the best way to solve for the coefficients of a_3 and beyond for an infinite polynomial. The sheer work involved in Euler's method would be overwhelming.

Using the method of reverse polynomials, enabled this thesis to be finalized. Using this, we were able to combine both Euler's and Newton's techniques. Newton's method for expressing sums of powers of roots of the polynomial was able to be manipulated, using reverse polynomials into the sums of the reciprocals of the roots, we were able to compare his results to those of Euler's. This occurred because Euler's method dealt specifically with the sums of the reciprocals of the roots [Kal09].

Using Newton's method and the technique of reverse polynomials to find the coefficients a_1, a_2, a_3 took two pages of work; whereas, using Euler's method took eleven pages of work. By using Newton's method and the technique of reverse polynomials to find the coefficients a_1, a_2, a_3, it took eleven pages using Euler's method. The final step in this thesis was the calculation of a_4 using Newton's method due to the immense amount of work that was completed at the beginning of the thesis to find a_3 using Euler's method.

Two extremely important mathematicians, born almost a century apart, have left their mark on history to this day, and will continue to do so for centuries to come. Mathematics would not be what it is without their profound insight and contribution. As Euler stated, "To those who ask what the infinitely small quantity in mathematics is, we answer that it is actually zero. Hence there are not so many mysteries hidden in this concept as they are usually believed to be [Eul12].

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