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Whitney's 2-isomorphism theorem for hypergraphs

Eric Anthony Taylor

WHITNEY'S 2-ISOMORPHISM THEOREM FOR HYPERGRAPHS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Eric Anthony Taylor

September 2013

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ABSTRACT

An important invariant of a graph is its cycle matroid. Whitney's 2-isomorphism theorem characterizes when two graphs have isomorphic cycle matroids. This expository paper will outline a proof of Whitney's theorem using blocks of cycle matroids. Then we will generalize Whitney's 2-isomorphism Theorem to hypergraphs and polymatroids by characterizing when two hypergraphs have isomorphic associated polymatroids. An associated polymatroid is an invariant of a hypergraph and carries the same information as a cycle matroid. Polymatroids generalize matroids by lifting the restriction that singletons have rank at most one. An associated polymatroid of a 2-uniform hypergraph (that is a graph) will be the usual cycle matroid.

ACKNOWLEDGEMENTS

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Dedicated

To my loving wife Stephanie: your support and dedication has been paramount to me achieving success. Also, to my family: I appreciate your patience. To my mentors in mathematics and in life: I thank you for your wisdom.

Sincerely,

Eric Taylor

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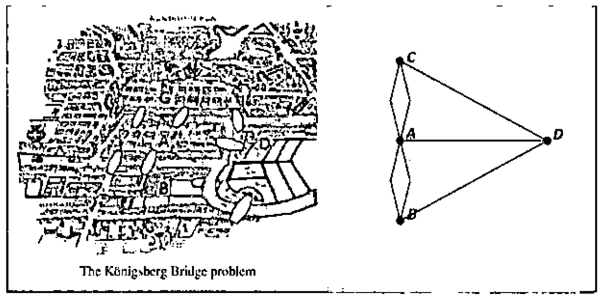
1 Graphs



Euler (1735) showed that no path exists that crosses over each bridge exactly once and returns to the starting point.

Figure 1.1: A graph representing the bridges of Konigsberg.

This expository paper will examine a fundamental theorem from graph theory: Whitney's 2-Isomorphism Theorem. Whitney's 2-isomorphism theorem characterizes when two graphs have isomorphic cycle matroids. We will outline a proof of Whitney's theorem using blocks of cycle matroids. Then we will generalize Whitney's 2-isomorphism Theorem to hypergraphs and polymatroids by characterizing when two hypergraphs have isomorphic associated polymatroids. We will expound upon Dirk Vertigan and Geoff Whittle's seminal work "*A 2-Isomorphism Theorem for Hypergraphs*" [VW97]. We are interested in investigating the connections between graphs, matroids, polymatroids and hypergraphs.

We begin by stating basic graph definitions and results. We will follow the standard graph notation of Diestel's book [D10].

1.1 Preliminaries

Definition 1.1 [1]*Let V be a finite set and E be a family of subsets of the Cartesian product $V \times V$, where each element of E is an unordered pair of elements of V. A graph is a pair $G = (V, E)$ of sets where the elements of V are called vertices, denoted $V(G)$, and the elements of E are called edges, denoted $E(G)$. For an arbitrary graph G, the elements of E are of the form $\{v_1, v_2\}$ where v_1, $v_2 \in V$, and $V \cap E = \emptyset$.*

Definition 1.2 *The order of a graph G, denoted $|G|$, is its number of vertices, also denoted $|V(G)|$ (or simply $|V|$). Similarly, $|E(G)|$ (or simply $|E|$) denotes the number of edges of a graph G.*

Definition 1.3 *Let $G = (V, E)$. We say that two vertices v_1 and v_2 of G are neighbors if $\{v_1, v_2\}$ is an edge of G. A vertex $v \in V$ is incident with an edge $e \in E$ if $v \in e$.*

Example 1.1 *The vertices $v_2 \in V$ and $v_7 \in V$ are neighbors since there exists an edge $\{v_2, v_7\}$ in $E(G)$ (also $v_2v_7 \in E(G)$) that contains both. See Figure* 1.2.

[1] In this section we will be utilizing the basic graph notation of Diestel's book [D10].

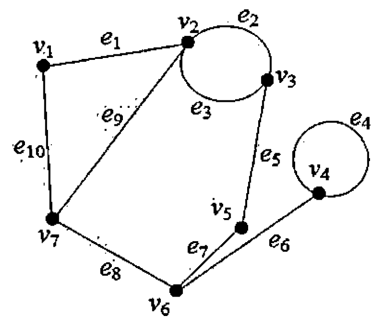


Figure 1.2: A graph G = (6,7).

Definition 1.4 *A loop is an edge whose endpoints are the same vertex. An edge is a multiple if there exists another edge with the same endpoints; if they have the same direction (say from x to y), they are parallel.*

Definition 1.5 *A graph is a simple graph if it does not contain multiple edges or loops. [See Figure 1.1. Edges e_2 and e_3 are multiple and parallel. Edge e_4 is a loop.]*

Definition 1.6 *If all vertices of a graph G are pairwise adjacent, then G is complete.*

Definition 1.7 *Let $G = (V, E)$ and $G' = \left(V', E'\right)$ be two graphs. If $V' \subseteq V$ and $E' \subseteq E$, then G' is a subgraph of G, written as $G' \subseteq G$. Less formally, we say G contains G'. (Note: G is a supergraph of G'.)*

Definition 1.8 *If $G' \subseteq G$ and G' contains all the vertices of G, that is $V' = V$, we say that G' is a spanning subgraph of G.*

Definition 1.9 *For $G' \subseteq G$, if E' consists of all edges of G spanned by V', G' is an induced subgraph, denoted by $G[V']$, so that $G' = G[V']$.*

Definition 1.10 *A graph G is minimal when it possesses a property R and does not contain a proper induced subgraph of G that also contains this property. Similarly, a graph G is maximal when it possesses a property R and G is not contained in a proper supergraph H that also possesses this property.*

Definition 1.11 *For any graph $G = (V, E)$, if $U \subseteq V$, we denote $G - U$ for $G[V \setminus U]$, that is $G - U$ is obtained from G by deleting all vertices in U and their incident edges. Likewise, if $F \subseteq E$, we say $G - F$ for $G = (V, E \setminus F)$.*

Definition 1.12 *Let $G = (V, E)$ and $G' = \left(V', E'\right)$ be two graphs. We say G and G' are isomorphic, denoted by $G \cong G'$, if there exists a bijection $\varphi : V \to V'$ with $xy \in E \iff \varphi(x)\varphi(y) \in E'$ for all $x, y \in V$. A map φ is called an isomorphism.*

Definition 1.13 *A map that takes graphs as arguments is called a graph invariant if it assigns equal values to isomorphic graphs.*

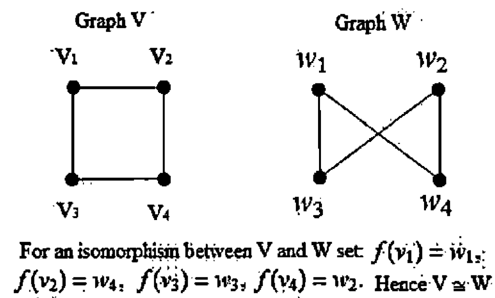


Figure 1.3: Graph Isomorphism between V and W.

1.2 Connectivity

Definition 1.14 *A path is a non-empty graph $P = (V, E)$, where $V = \{x_0, x_1, \ldots, x_k\}$, $E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$ and each x_i is distinct. The length of a path P is denoted by P^k. A path between two vertices $x \in V$ and $y \in V$ is denoted by xPy.*

Definition 1.15 *A cycle C is a path where $E = \{x_0x_1, \ldots, x_{k-1}x_0\}$ and $k \geq 3$. The length of a cycle is denoted by C^k.*

Remark 1.1 *A graph may contain multiple paths and cycles.*

Definition 1.16 *A non-empty graph G is called connected if there exists a path between every pair of distinct vertices in G.*

Definition 1.17 *A component is a maximally connected subgraph.*

Definition 1.18 *A graph $G = (V, E)$ that is not connected is said to be disconnected.*

Definition 1.19 *A cut vertex is a vertex whose removal disconnects the remaining subgraph, that is $G - v$ has more components than G has.*

Definition 1.20 *A subset U of V is said to be disconnecting vertex set or a separator if $G - U$ is disconnected.*

1.3 Trees

Definition 1.21 *A tree is a connected graph without cycles.*

Remark 1.2 *A spanning tree is a spanning subgraph that is a tree.*

Definition 1.22 *A forest is a disconnected graph whose components are trees.*

Remark 1.3 *It follows that a forest is a graph without cycles.*

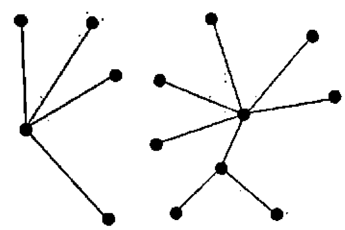


Figure 1.4: A Forest with two components.

Proposition 1.1 *A graph $G = (V, E)$ has a spanning tree if and only if G is connected.*

Proof. Suppose G has a spanning tree. This implies that there exists a path between any two vertices of G. Thus G is connected.

Suppose G is connected. If G is not a tree, then it must contain a cycle. Select a cycle and remove an edge from it. The resulting subgraph G' will still contain the same number of vertices as G and remain connected. Repeat process until G' contains no cycles. The resulting subgraph G' will be a spanning tree. ■

Corollary 1.1 *Every graph has a spanning forest.*

Corollary 1.2 *For any forest, $|E| = |V| - k$ where k is the number of components of (V, F). For a tree, $|E| = |V| - 1$.*

Definition 1.23 *Let $G = (V, E)$ be a connected graph. We say that G is $k-$connected (for $k \in \mathbb{N}$) if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. In other words, for a $k-$connected graph it is possible to form a path from any vertex to every other vertex in a graph after removing $k - 1$ vertices.*

Definition 1.24 *Let $G = (V, E)$ and $F \subseteq E$. If $G \setminus F$ has more components than G, we say that F is an edge cut (or simply a cut).*

Definition 1.25 *A bond is a non-empty minimal edge cut.*

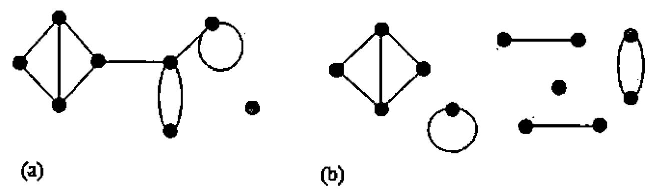
Remark 1.4 *If G is connected, its bonds are its minimal cuts. If G is disconnected, then its bonds are the minimal cuts of its components.*

Remark 1.5 *An edge cut in a connected graph is minimal if and only if both sides of the resulting vertex partition induce connected subgraphs.*

An arbitrary connected graph G can be decomposed into subgraphs that cover G and capture precisely the structure of G.

Definition 1.26 *A block is a maximally connected subgraph without a cut vertex.*

Remark 1.6 *Every block is a maximal 2-connected subgraph, or a bridge, or an isolated vertex.*



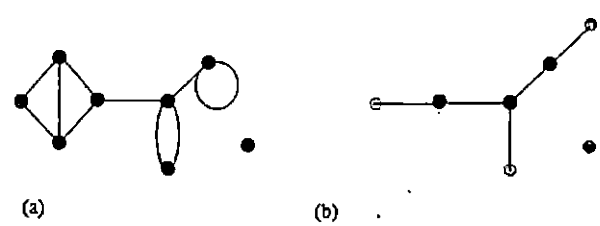
(a) A graph G. (b) The six blocks of G.

Figure 1.5: A graph G and its six block subgraphs.

Proposition 1.2 *Let $G = (V, E)$ be a graph. Then the cycles of G are exactly the cycles of its blocks.*

Proof. Each cycle in G is a connected subgraph without a cut vertex. Therefore, by definition, it lies in some maximal subgraph which is a block of G. ∎

Definition 1.27 *The block graph of a graph G is a tree graph that depicts all of the blocks of G as emphasized vertices. See Figure 1.6.*



(a) A graph G. (b) The block graph of G.

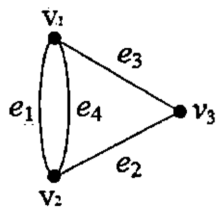
Figure 1.6: A graph G and its block graph representation.

Definition 1.28 *The incidence matrix $B = V \times E$ of a graph $G = (V, E)$ is the matrix B with*

$$B_{v,e} = \begin{Bmatrix} 1 \text{ if } v \in e \\ 0 \text{ if } v \notin e \end{Bmatrix}$$

for $v \in V$ and $e \in E$.

Example 1.2 *The graph on $V = \{v_1, v_2, v_3\}$ with the edge set $E = \{e_1, e_2, e_3, e_4\}$. It has the incidence matrix $B = V \times E$*



$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

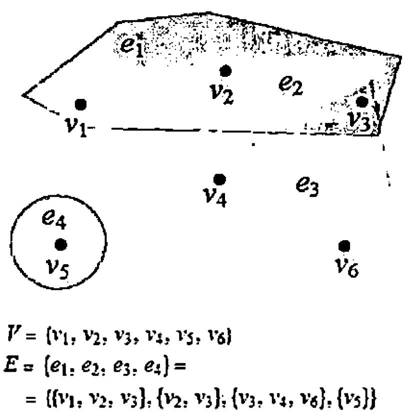
Figure 1.7: A graph G and its incidence matrix $B = V \times E$

2 Hypergraphs

In this section we will introduce hypergraphs which generalize graphs. We will utilize the notations and expound on the related theorems found in Vertigan and Whittle's "*A 2-Isomorphism Theorem for Hypergraphs*" [VW97].

2.1 Hypergraph Introduction

Definition 2.1 [2]*Let V be a finite set and E a family of subsets of V. A hypergraph H is a triple (V, E, I) where V and E are called vertices and edges respectively and $I \subseteq V \times E$ is the incidence relation of H.*



$$I = \begin{array}{c c} & \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Figure 2.1: A hypergraph H=(6,4,(6,4)).

Remark 2.1 *In this expository paper, every edge will be incident with at least one vertex.*

Definition 2.2 *A subset W of vertices is a separator of H if, for any edge e, either $\bar{e}$ is contained in W or $\bar{e}$ is disjoint from W.*

Remark 2.2 *A component of H is a minimal non-empty separator.*

Definition 2.3 *A (non-trivial) hypergraph H is connected if it has exactly one component.*

A hypergraph H is simple if it has no loops or parallel edges. A hypergraph is 2-connected if it is connected and contains no cut vertices.

Notation 2.1 *For a subset A of edges, $\bar{A}$ denotes the set of vertices incident with at least one edge in A.*

$$H \mid A = (\bar{A},\ A,\ I \cap (\bar{A} \times A)).$$

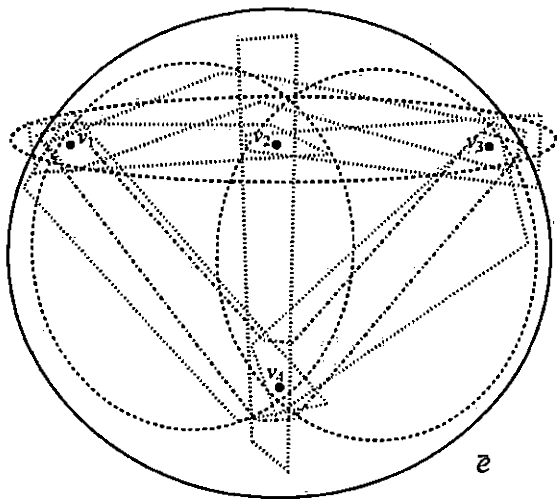
Notation 2.2 *For a subset W of vertices, $\bar{W}$ denotes the maximal set F of edges such that $\bar{F}$ is contained in W.*

$$H \mid W = (W,\ \bar{W},\ I \cap (W \times \bar{W})).$$

[2]In this section we will be utilizing the hypergraph notation of Vertigan and Whittle [VW97]. For futher details see also [B73].

2.2 Complete Hypergraph

Definition 2.4 *A complete hypergraph is a hypergraph (V, E, I), for $|V| \geq 2$ having the property that for every subset W of V having at least two elements, there is a unique edge $\bar{e}$ such that $\bar{e} = W$.*



Complete Hypergraph on Four Vertices

Figure 2.2: A complete hypergraph on four vertices.

Remark 2.3 *The maximum size of a simple complete hypergraph on n vertices is $2^n - n - 1$(See Remark 2.1).*

3 Matroids

In this section we will introduce matroids, which are an abstraction of linear independence. Matroids can be characterized by their bases and circuits, with the latter being related to cycles in graphs. We will utilize the notations of and expound upon the related theorems found in Oxley's book "*Matroid Theory*" [O92].

3.1 Independent Sets and Rank

Definition 3.1 [3] *A pair $(E,\mathcal{I})$ is called a matroid if E is a finite set and $\mathcal{I}$ a non-empty collection of subsets of E satisfying:*

1. *If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.(hereditary)*
2. *If $I,\ J \in \mathcal{I}$ and $|I| \leq |J|$, then $I \cup z \in \mathcal{I}$ for some $z \in J - I$.(independence augmentation)*

Note: These axioms were given by Whitney in 1935[4].

Definition 3.2 *For a given a matroid $M = (E,\mathcal{I})$, a subset I of E is called independent, denoted $\mathcal{I}(M)$, if it belongs to $\mathcal{I}$, and dependent otherwise.*

Remark 3.1 *The members of $\mathcal{I}$ are the independent sets of M, and E is the ground set of M, denoted $E(M)$.*

Remark 3.2 *A matroid on E is a hypergraph. The independent sets correspond to the edges of hypergraph $(E,\mathcal{I})$*[5].

Definition 3.3 *For $U \subseteq E$, a subset B of U is called a base if it is an inclusionwise maximal subset of U. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$.*

Corollary 3.1 *For any subset U of E, any two bases of U have the same size.*

Definition 3.4 *The common size of the bases of a subset U of S is called the rank of U, denoted $r_M(U)$ or simply $r(U)$ when the matroid M is clear from the context.*

Definition 3.5 *Let E be a finite set. The function $r : 2^E \to \mathbb{Z}$ is the rank function of a matroid M on E if it has the following properties:*

1. If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.
2. If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.

[3] In this section we will be utilizing the matroid notation of Oxley's book [O92].
[4] See Whitney's historic papers [W32] and [W33].
[5] See Berge's historic book [B73] for further details.

3. If X and Y are subsets of E, then

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y).$$

Definition 3.6 *Let M_1 and M_2 be matroids. We say M_1 and M_2 are isomorphic, denoted by $M_1 \cong M_2$, if there is a bijection φ from $E(M_1)$ to $E(M_2)$ such that for all $X \subseteq E(M_1)$, the set $\varphi(X)$ is independent in M_2 if and only if X is independent in M_1. A bijection φ is an isomorphism from M_1 to M_2.*

Definition 3.7 *Let M_1 and M_2 be matroids. A matroid invariant is a function f defined on matroids such that $f(M_1) = f(M_2)$ whenever $M_1 \cong M_2$.*

Remark 3.3 *The rank and cardinality of bases are matroid invariants.*

3.2 Circuits

Definition 3.8 *A circuit C of a matroid M is an inclusionwise minimum dependent set, whose proper subsets are independent.*

A circuit of M that has n elements is called an n-circuit. We denote the set of circuits of a matroid M by $\mathcal{C}(M)$ or simply $\mathcal{C}$.

Definition 3.9 *The set $\mathcal{C}$ of circuits of a matroid M has the following properties:*

1. $\varnothing \notin \mathcal{C}$.
2. If C_1 and C_2 are members of $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.
3. If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

Theorem 3.1 *Let E be a set and $\mathcal{C}$ be a collection of subsets of E satisfying properties of Definition 3.9. Let $\mathcal{I}$ be the collection of subsets of E that contain no member of $\mathcal{C}$. Then (E, I) is a matroid that has $\mathcal{C}$ as its collection of circuits.*

Corollary 3.2 *It follows that if C is a circuit of M, then $C \notin \mathcal{I}(M)$ and $C - x \in \mathcal{I}(M)$ for all x in C.*

A matroid can be characterized in three specific ways: by the collection of its independent sets, by the collection of its circuits, and by the collection of its bases.

Theorem 3.2 *Let E be the set of column labels of an $m \times n$ matrix A over a field $\mathbb{F}$, and let $\mathcal{I}$ be the set of subsets X of E for which the multiset of columns labelled by X is a set and is linearly independent in the vector space $V(m, \mathbb{F})$. Then (E, I) is a matroid.*

Example 3.1 *Let A be the following matrix over the field $\mathbb{R}$ of real numbers:*

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Then $E = \{1,2,3,4,5\}$ and $\mathcal{I} = \left\{ \begin{array}{c} \varnothing, \{1\}, \{2\}, \{4\}, \{5\}, \{1,2\}, \\ \{1,5\}, \{2,4\}, \{2,5\}, \{4,5\} \end{array} \right\}$.

Thus the set of circuits of this matroid is $\mathcal{C} = \{\{3\}, \{1,4\}, \{1,2,5\}, \{2,4,5\}\}$ whose proper subsets are independent.

Lemma 3.1 *Suppose that I is an independent set in a matroid M and e is an element of M such that $I \cup e$ is dependent. Then M has a unique circuit contained in $I \cup e$, and this circuit contains e.*

Proof. By Corollary 3.2, $I \cup e$ contains a circuit, and all such circuits must contain e. Suppose C and C' are distinct circuits that contain e. Then Definition 3.9 implies that $(C \cup C') - e$ contains a circuit. So $(C \cup C') - e \subseteq I$, which is a contradiction. Thus, C is unique. ■

Theorem 3.3 *Let E be the set of edges of a graph G and $\mathcal{C}$ be the set of edge sets of cycles of G. Then $\mathcal{C}$ is the set of circuits of a matroid on E.*

Proof. The set $\mathcal{C}$ satisfies Properties 1 and 2 of Definition 3.9.

To prove that it satisfies Property 3, we want to examine two distinct cycles of G and construct a cycle with an edge set contained in $(C_1 \cup C_2) - e$.

Let C_1 and C_2 be the edge sets of two distinct cycles of G that have e as a common edge. Let u and v be endpoints of e. Let P_1 be the path from u to v in G whose edge set is $C_1 - e$. Likewise, let P_2 be the path from u to v in G whose edge set is $C_2 - e$. Starting at u, we traverse P_1 towards v, and let w be the first vertex at which the next edge of P_1 is not in P_2. Continue traversing P_1 from w towards v until the first time we reach a vertex x that is distinct from w but is also in P_2. Since P_1 and P_2 both end at v, such a vertex must exist. So now we adjoin the section of P_1 from w to x to the section of P_2 from x to v, the result is a cycle (see Figure 3.1), with the edge set contained in $(C_1 \cup C_2) - e$. Hence $\mathcal{C}$ satisfies property 3.

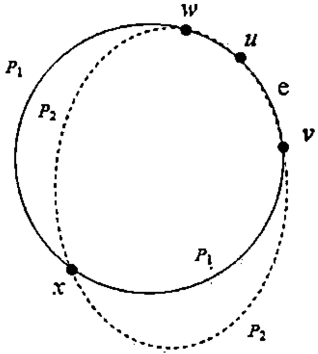


Figure 3.1: A unique circuit of a matroid M.

■

The matroid derived from the graph G (Figure 3.1) is called a *cycle matroid* of G. It follows that a set X of edges is independent in the matroid of G if and only if X does not contain the edge set of a cycle, that is the subgraph induced by X is a forest.

3.3 . Graphic Matroid

Definition 3.10 *Let $G = (V, E)$ be a graph and let $\mathcal{I}$ be a collection of all subsets of E that form a forest. Then $M = (E, \mathcal{I})$ is a matroid and called a cycle matroid, denoted $M(G)$.*

Remark 3.4 *Any matroid obtained by this construction or isomorphic to such a matroid is a graphic matroid.*

Remark 3.5 *The circuits of $M(G)$ are exactly the cycles of G.*

Remark 3.6 *The bases of $M(G)$ are exactly the inclusionwise maximal forests F of G. So if G is connected, the bases are the spanning trees.*

Let G be the graph in Figure 3.2 and let $M = M(G)$.

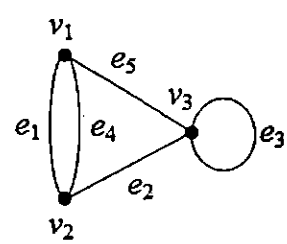


Figure 3.2: A graph G and its graphic matroid $M(G)$.

Then $E(M) = \{e_1, e_2, e_3, e_4, e_5\}$ and $C(M) = \{\{e_3\}, \{e_1, e_4\}, \{e_1, e_2, e_5\}, \{e_2, e_4, e_5\}\}$.
If we compare M with the matroid $M[A]$ in Example 3.1, we can see that there exists a

bijection φ from $\{1,2,3,4,5\}$ to $\{e_1,e_2,e_3,e_4,e_5\}$ defined by $\varphi(i)=e_i$. Thus, a set X is a circuit in $M[X]$ if and only if $\varphi(X)$ is a circuit in M.

From Definition 3.10, we know that each base U of F is an inclusionwise maximal forest contained in F. So U has $|V|-k$ elements, where k is the number of components of (V,F). So each base of F has $|V|-k$ elements.

Remark 3.7 *So U forms a spanning tree in each component of the graph (V,F).*

Definition 3.11 *For each subset F of E, let $k(V,F)$ denote the number of components of the graph (V,F). Then for each $F\subseteq E$:*

$$r_{M(G)}(F)=|V|-k(V,F)$$

which is called the rank function of $M(G)$.

3.3.1 Matroid Connectivity

Proposition 3.4 *Let M be a graphic matroid. Then $M\cong M(G)$ for some connected graph G.*

Proof. Suppose M is graphic, that is $M\cong M(H)$ for some graph H. If H is connected, we are done. If not, suppose the connected components are $H_1,H_2,\ldots,H_n$ of H. We can form a new graph G by identifying with each block H_i a vertex v_i (for $i\in[1,2,\ldots,n]$) and joining $v_1,v_2,\ldots,v_n$ into a single vertex. So $E(H)=E(G)$ and G is connected. Observe, if $X\subseteq E(H)$, then X is the set of edges of a cycle in H if and only if X is the set of edges of a cycle in G. Hence $M(H)\cong M(G)$. ■

Corollary 3.3 *If H is a disconnected graph , then there is a connected graph G, such that $M(H)\cong M(G)$.*

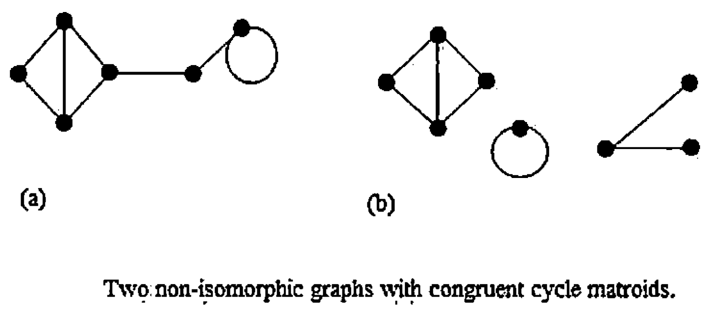


Figure 3.3: Two non-isomorphic graphs with congruent cycle matroids.

Definition 3.12 *A matroid M is connected if and only if for every pair of distinct elements of $E(M)$, there is a circuit containing both.*

Definition 3.13 *Let G be a connected loopless graph and suppose that $|V(G)|\geq 3$. Then $M(G)$ is a connected matroid if and only if G is a 2-connected graph.*

3.4 Whitney's 2-Isomorphism Theory for Graphs

Definition 3.14 *A graph is 2-isomorphic to a graph H if H can be transformed into a graph isomorphic to G by a sequence of the following operations:*

1. *Vertex joining.* Let v_1 and v_2 be vertices of distinct components of G. We modify G by identifying v_1 and v_2 as new vertex $\bar{v}$.

2. *Vertex splitting.* We modify G by the reverse operation of vertex joining, namely we split G at a *cut vertex.*

3. *Whitney twist (or twisting).* Suppose that the graph G is obtained from disjoint graphs G_1 and G_2 by joining the vertices u_1 of G_1 with v_1 of G_2 as the vertex u of G, and joining the vertices u_2 of G_1 with v_2 of G_2 as the vertex v of G. In a *Whitney twist of G about $\{u, v\}$*, we instead join u_1 with v_2 as the vertex u of G', and join u_2 with v_1 as the vertex v of G'. We say G_1 and G_2 are the *pieces* of the twisting.

Remark 3.8 *A 2-isomorphism transformation does not alter the edge sets of cycles of graph.*

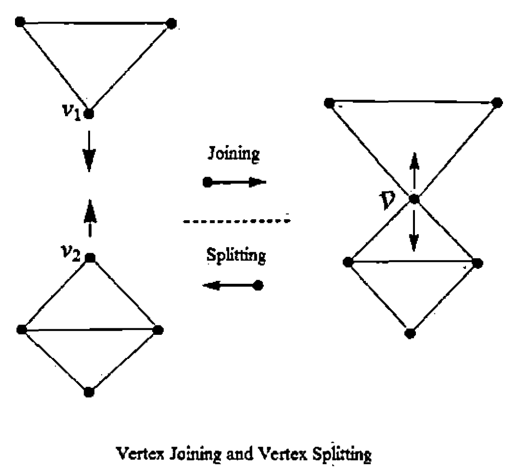


Figure 3.4: A vertex joining and splitting.

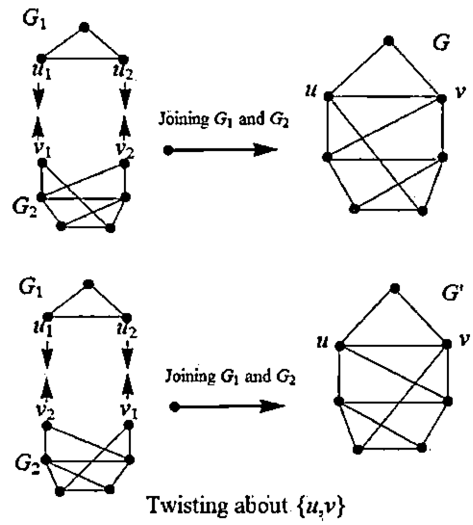


Figure 3.5: A twisting about $\{u,v\}$.

Proposition 3.5 *If a graph G is 2-isomorphic to a graph H, then $M(G) \cong M(H)$.*

Theorem 3.6 *Whitney's 2-isomorphism Theorem. Let G and H be graphs having no isolated vertices. Then $M(G)$ and $M(H)$ are isomorphic if and only if G and H are 2-isomorphic.*

Remark 3.9 *Unfortunately, the operations of 2-isomorphism cannot be applied to the class of graphs that are 3-connected loopless graphs*[6].

Remark 3.10 *Fortunately, 3-connected graphs are uniquely determined by their cycle matroids.*

Proposition 3.7 *Let G and H be loopless graphs without isolated vertices. Suppose that $\Phi : E(G) \to E(H)$ is an isomorphism from $M(G)$ to $M(H)$. If G is 3-connected, then Φ induces an isomorphism between graphs G and H.*

Proof. (Sketch of proof) Since G is 3-connected, $G - v$ is 2-connected for every vertex v of G. By Definition 3.13, we know the $M(G-v)$ is connected, so $M(G)$ has precisely $|V(G)|$ connected hyperplanes[7]. But $M(G) \cong M(H)$, and $M(G)$ is connected and loopless, so H is a loopless block. It follows that $|V(G)| = r(M(G))+1 = r(M(H))+1 = |V(H)|$. Thus $M(H)$ has precisely $|V(H)|$ connected hyperplanes. Hence $G \cong H$. ∎

Proposition 3.8 *Let G be a block having at least four vertices and let G not be 3-connected. Then G has a representation as a generalize cycle, where each part is a block.*

[6]See Oxley's book [O92] for further details of Tutte's results.
[7]Refer to [O92] for further details and proofs of this result.

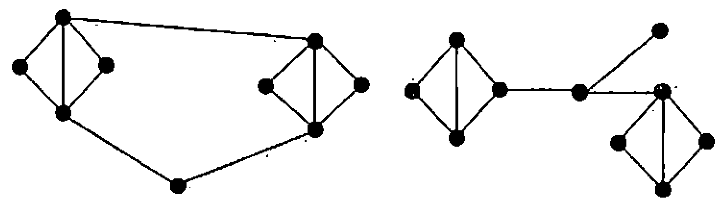


Figure 3.6: A generalized cycle of a graph G.

Proposition 3.9 *Let G have a block representation as a generalized cycle with the block parts of $G_1, G_2, \ldots, G_n$. Let H be a graph for which there is an isomorphism Ψ from $M(G)$ to $M(H)$ and, for each i, let H_i be the subgraph of H induced by $\Psi(E(G_i))$. Then H is a generalized cycle with parts $H_1, H_2, \ldots, H_n$.*

Proof of Theorem 3.6

Proof. (Sketch of proof)[8]

1. By Proposition 3.7, if G is 3-connected, then $G \cong H$.
2. Suppose G is not 3-connected. Then by Proposition 3.8, G has a representation as a generalized cycle in which each part is block, say $G_1, G_2, \ldots, G_n$. It follows from Definition 3.16, H has a representation as a generalized cycle, the parts of which are $H_1, H_2, \ldots, H_n$, where $H[\Psi(E(G_i))]$.
3. We transform H into a generalized cycle of H'.
4. We transform H', by a well-defined sequence of twistings of the individual blocks of the generalized cycle of H', into a graph isomorphic to G .
5. Hence we transformed H into a graph isomorphic to G.

■

[8]Refer to [O92] for further details and proofs of the results in this section.

4 Polymatroids and Rank Equivalence

In this section we will introduce polymatroids which are generalization of matroids. An associated polymatroid is an invariant of a hypergraph and carries the same information as a cycle matroid. Polymatroids generalize matroids by lifting the restriction that singletons have rank at most one. An associated polymatroid of a 2-uniform hypergraph (that is a graph) will be the usual cycle matroid. We will utilize the notations of and expound upon the related theorems found in Vertigan and Whittle's "*A 2-Isomorphism Theorem for Hypergraphs*" [VW97] and in Schrijver's book "*Combinatorial Optimization: Polyhedra and Efficiency*" [S03].

4.1 Polymatroids

Definition 4.1 [9]*Let f be a set function on a finite set E, that is, a function defined on the $P(E)$ of all subsets of E. Then f is called submodular if*

$$f(T \cap U) + f(T \cup U) \leq f(T) + f(U)$$

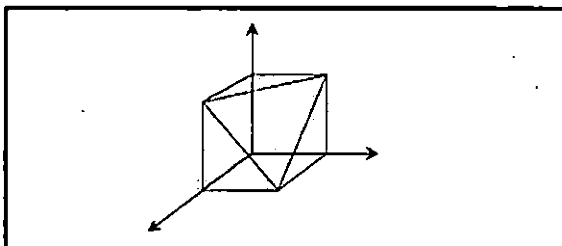
for all subsets T, U of E.

Definition 4.2 *Let E be a finite set and consider an integer valued set function $\rho : 2^E \to \mathbb{Z}$. Then ρ is called a polymatroid if it has the following properties:*

1. *ρ is normalized, that is $\rho(\varnothing) = 0$,*
2. *ρ is increasing, that is $\rho(A) \leq \rho(B)$ whenever $A \subseteq B \subseteq E$,*
3. *ρ is submodular, that is $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$ for all subsets A and B of E.*

Remark 4.1 *Each submodular function gives a polymatroid, which is a generalization of the independent set polytope of a matroid.*

Remark 4.2 *In a sense, submodularity is the discrete analogue of convexity.*



The independent set polytope of $U_{2,3}$

Figure 4.1: An independent set polytope $U_{2,3}$.

[9]In this section we will be utilizing the polymatroid notation of Schrijver's book [S03].

Definition 4.3 *A separator of a polymatroid ρ on E is a subset A of E with the property that $\rho(A) + \rho(E - A) = \rho(E)$.*

Remark 4.3 *A component of a polymatroid ρ is a minimal non-empty separator.*

A polymatroid ρ is connected if its only separators are E and $\varnothing$. A subset A of a polymatroid ρ on E is a spanning if $\rho(A) = \rho(E)$.

Definition 4.4 *A hypergraphic polymatroid χ_H of H is defined, for all subsets A of E to be*

$$\chi_H(A) = |\bar{A}| - k(H \mid A).$$

The hypergraphic χ_H is an invariant of a hypergraph, which essentially contains the same information as a cycle matroid contains for a graph (See Definition 3.5 and Definition 3.11). In fact, if H is a graph, that is a 2-uniform hypergraph, then χ_H is the rank function of the cycle matroid of the graph.

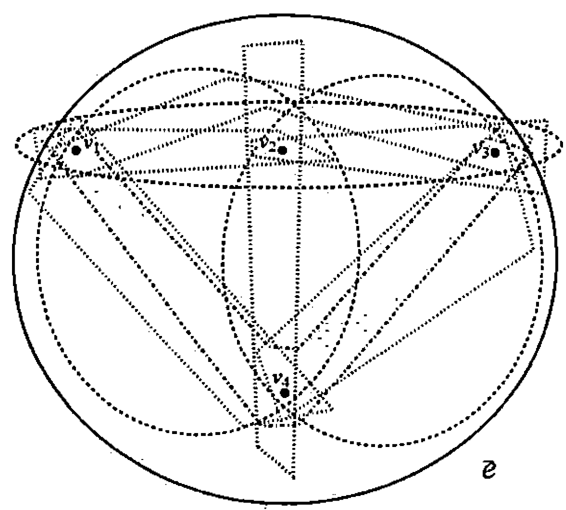
4.2 Rank-equivalent Hypergraphs

Definition 4.5 *Let H and I be hypergraphs. If H and I have the same edge set E and $\chi_H(A) = \chi_I(A)$ for every $A \subseteq E$, then H and I are rank-equivalent, that is, $\chi_H = \chi_I$.*

Corollary 4.1 *It follows that two hypergraphs have isomorphic hypergraphic polymatroids if and only if, up to relabelling of edges, they are rank equivalent.*

Theorem 4.1 *The hypergraph H is rank-unique if the only hypergraph that is rank-equivalent to H is H itself (up to vertex labelling and isolated vertices).*

Definition 4.6 *Complete hypergraphs are rank-unique.*



Complete Hypergraph on Four Vertices

Figure 4.2: A complete hypergraph on four vertices.

4.3 Hypergraph Transformations

Definition 4.7 *A twisting partition of $H = (V, E, I)$ is a partition $\{U, W, u, w\}$ of V such that for every edge e of H, either $\bar{e} \subseteq U \cup \{u, w\}$ or $\bar{e} \subseteq W \cup \{u, w\}$ or $\{u, w\} \subseteq \bar{e}$.*

Definition 4.8 *A twisting of H is defined to be when each edge e of E with $\bar{e} \subseteq W \cup \{u, w\}$ and $|\bar{e} \cap \{u, w\}| = 1$, change $\bar{e}$ to $\bar{e} \triangle \{u, w\}$ where $\triangle$ denotes the symmetric difference.*

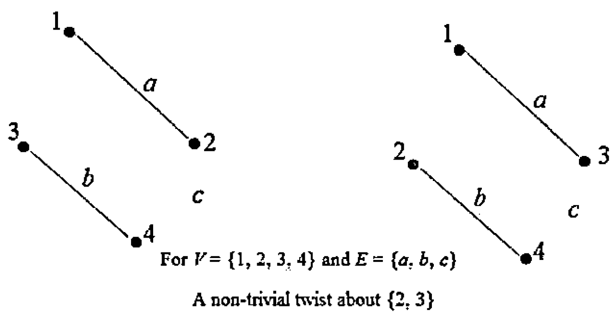


Figure 4.3: A non-trivial twist of a hypergraph H.

Remark 4.4 *If H is a 2-uniform hypergraph (i.e. a graph), then a twisting is as defined in Definition 3.14.*

Definition 4.9 *A splitting of H is to remove a cut vertex v of H such that its edges are partitioned into two subsets A and B.*

Definition 4.10 *A joining of H is a reverse splitting operation where vertices v_1 and v_2 in components A and B are joined into a single cut vertex v such that $\bar{A} \cap \bar{B} = \{v\}$.*

Theorem 4.2 *Hypergraphs H and I are 2-isomorphic if H can be transformed into I by a sequence of twistings, splittings and joinings. (Note: This includes the addition and removal of isolated vertices and the relabelling of vertices.)*

A hypergraph H is rank-unique if the only hypergraph rank-equivalent to H is H itself. It follows then that a hypergraph is rank-unique if and only if it has non-trivial twisting partitions.

5 2-Isomorphism Theorem for Hypergraphs

In this section we will generalize Whitney's 2-isomorphism Theorem to hypergraphs and polymatroids by characterizing when two hypergraphs have isomorphic associated polymatroids. We will utilize the notations of and expound upon the related theorems found in Vertigan and Whittle's "*A 2-Isomorphism Theorem for Hypergraphs*" [VW97].

5.1 Whitney's 2-Isomorphism Theory for Hypergraphs

Theorem 5.1 *Let H and I be hypergraphs. Then H and I are rank-equivalent if and only if they are 2-isomorphic.*

Proof. ($\Longleftarrow$) If H and I are 2-isomorphic, then they are rank-equivalent. Without loss of generality, assume I is obtained by H via a single twist. So I is obtained from H by twisting on the twisting partition $\{U, W, u, w\}$. Let A be a subset of edges. Observe, if $\bar{A} \cap \{u, w\}$ has the same cardinality in H as in I, then clearly $\chi_H(A) = \chi_I(A)$.

If not then, without loss of generality, assume $\bar{A}$ contains more elements of $\{u, w\}$ in I than in H. Observe, if $\bar{A}$ contains both elements of $\{u, w\}$ in I and only one element of $\{u, w\}$ in H, then $k(I \mid A) = k(H \mid A) + 1$.

Set $|\bar{A}|_H = n$. So, $|\bar{A}|_I = n + 1$.

$$
\begin{aligned}
\text{So, } \chi_I(A) &= |\bar{A}|_I - k(I \mid A) \\
&= (n+1) - (k(H \mid A) + 1) \\
&= n - k(H \mid A) \\
&= \chi_H(A).
\end{aligned}
$$

Hence, $\chi_H(A) = \chi_I(A)$.

($\Longrightarrow$) (Sketch of proof). We want to show that if H and I are rank-equivalent hypergraphs and H is simple and 2-connected, then H and I are 2-isomorphic. We do this by showing that two rank-equivalent hypergraphs H and I can be transformed to rank-equivalent complete hypergraphs via a well-defined sequence of operations, which consists of twist partitions and edge extensions, all of which preserve rank-equivalence. We must first develop this sequence of operations and then apply it to finish the other direction of our proof. ■

5.2 Rank-equivalent Complete Hypergraphs

Definition 5.1 *Let H be a hypergraph. Then χ_H is connected if and only if H is 2-connected.*

Definition 5.2 *If H and I are rank-equivalent hypergraphs and A is a set of edges for which $H \mid A$ is 2-connected, then $I \mid A$ is 2-connected.*

Proposition 5.2 *Let H and I be 2-isomorphic hypergraphs and let F be a subset of edges. If F is spanning in χ_H and $H \mid F$ is 2-connected, then $H \mid F$ and $I \mid F$ are 2-isomorphic.*

Proof. We assume H and I have no isolated vertices. Since F is a spanning in χ_H, so $\bar{F} = V(H)$. So a twisting partition of H is also a twisting partition of $H \mid F$. Observe, if we choose the twists corresponding to the sequence of twisting partitions that transforms H to I, then we can transform $H \mid F$ to $I \mid F$. ∎

5.2.1 Edge Extensions

Now we wish to consider adding edges to rank-equivalent hypergraphs in particular way that preserves rank-equivalence[10].

Proposition 5.3 *If H and I are rank-equivalent and e is an edge, then $H \mid \bar{e}$ and $I \mid \bar{e}$ are rank-equivalent.*

Remark 5.1 *By Definition 4.5, H and I have the same edge set, so rank-equivalence for hypergraphs restricted to particular edge sets also follows.*

Definition 5.3 *Let H and H' be a simple hypergraphs. We say the hypergraph H' is an extension of H if $H = H' \mid E(H)$.*

Definition 5.4 *Let H and I be rank-equivalent hypergraphs. We say that (H', I') is a coherent extension of (H, I) if H' and I' are extensions of H and I respectively, and H' and I' are rank equivalent.*

Definition 5.5 *A hypergraph H is closed under coherent extensions if it has no non-trivial coherent extensions.*

Proposition 5.4 *Let H and I be rank-equivalent hypergraphs and assume that $H \mid V_H(f)$ and $I \mid V_I(f)$, then H and I can be extended to rank-equivalent hypergraphs H' and I' where $H' \mid \bar{f}$ and $I' \mid \bar{f}$ are equal complete hypergraphs.*

Remark 5.2 *Recall Definition 2.4. By Definition 4.6, complete hypergraphs are rank-unique. It follows that a pair of rank-equivalent complete hypergraphs are equal.*

5.2.2 Twisting Partitions

Definition 5.6 *Let e be an edge of a simple hypergraph H. We say that e is minimal incomplete if $H \mid \bar{e}$ is not complete but for every edge f with $\bar{f} \subset \bar{e}$, the hypergraph $H \mid \bar{f}$ is complete.*

Definition 5.7 *A hypergraph H is near-complete if it has a minimal incomplete edge e with $\bar{e} = V(H)$.*

Proposition 5.5 *Let H be a simple 2-connected hypegraph that is closed under coherent extensions. Suppose I is rank-equivalent to H. Let e be a minimal incomplete edge of H. Then $H \mid \bar{e}$ and $I \mid \bar{e}$ are 2-isomorphic.*

[10]Refer to [VW97] for further details and proofs of the results in this section.

Proposition 5.6 *Let H be simple 2-connected hypegraph that is closed under coherent extensions. Suppose I is rank-equivalent to H. Let e be a minimal incomplete edge of H. Then H is 2-isomorphic to a hypergraph H_1 such that $H_1 \mid \bar{e} = I \mid \bar{e}$.*

Proof. Let $H^* = \{h \in E \mid \bar{h} \subseteq \bar{e}$ or $h = f$ or $h = g\}$. It follows that $H \mid H^*$ is rank unique and that no twisting partition of $H \mid \bar{e}$ extends to a twisting partition of H. By Proposition 5.5, there is a sequence of twistings (each associated with a twisting partition) from $H \mid \bar{e} = I \mid \bar{e}$. If we can extend every twisting partition to the whole hypergraph then we are done. If not, H has a restriction, such as hypergraph H^*. But then $H \mid H^* = I \mid H^*$, so that $H \mid \bar{e} = I \mid \bar{e}$. ∎

5.2.3 Proof of Theorem 5.1

Recall Theorem 5.1: Let H and I be hypergraphs. Then H and I are rank-equivalent if and only if they are 2-isomorphic.

Proof. ($\Longrightarrow$) Let H be a 2-connected simple hypergraph. Let I be a hypergraph that is rank-equivalent to H, (that is H and I have the same edge set E and $\chi_H(A) = \chi_I(A)$ for every $A \subseteq E$, so that $\chi_H = \chi_I$). Let $E(H) = E(I) = E$. Assume H and I are not 2-isomorphic. Suppose $|V(H)| = |V(I)| = n$, where n is of a minimum order, that is both H and I are near-complete and bound above by $|E| = m$, where $m = 2^n - n - 1$ (see Remark 2.3). So H is closed under coherent extensions and is not complete. So there exist an incomplete edge e, and by Proposition 5.6, H is 2-isomorphic to a hypergraph H_1 such that $H_1 \mid \bar{e} = I \mid \bar{e}$. By Proposition 5.4, there exists a non-trivial coherent extension of (H_1, I) to (H', I') such that $H' \mid \bar{e} = I' \mid \bar{e}$ is complete. Since $E(H') > m$, then by our maximality assumption, H' is 2-isomorphic to I'. However, $H_1 = H' \mid E$ is 2-isomorphic to $I = I' \mid E$, by Proposition 5.2. It follows that H is 2-isomorphic to I. ∎

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