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A study on the modular structures of Z_2S_3 and Z_5S_3

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A STUDY ON THE MODULAR STRUCTURES OF $\mathbb{Z}_2 S_3$ AND $\mathbb{Z}_5 S_3$

A Thesis

Presented to the

Faculty of

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San Bernardino

In Partial Fulfillment

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ABSTRACT

This project is a study of the properties of the modules $\mathbb{Z}_2S_3$ and $\mathbb{Z}_5S_3$, which are examined both as modules over themselves and as modules over their respective integer fields. We will examine each of these modules separately, since they each hold distinct properties. We explore the properties that these modules hold and how those properties differ when we change the module it is examined over. The overall goal is to determine the simplicity and semisimplicity of each module. In order to achieve this goal, we will study the structure of their modules, their radical, and their submodules.

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Chapter 1

Introduction

The following work first began as an undergraduate-level independent study and honors project with Dr. Llosent. Initially I began to study the properties of quivers in Representation Theory. As I progressed through the Masters Program, the study grew into a study of modules and their properties, particularly those properties that relate to the simplicity and semisimplicity of modules.

There are a handful of concepts from Ring Theory and Module Theory that are necessary to fully understand the concepts that are presented here. In the beginning chapters we will submit definitions and prove some of the necessary theorems from Ring Theory and Module Theory that are required to move forward, including the notions of the radical of a module, comaximal submodules, and the Chinese Remainder Theorem for modules. There are many concepts and theorems that originate in Ring Theory, but have a likeness in Module Theory. For example, the notion of a submodule of a module carries properties similar to those of an ideal within a ring. These definitions and properties will be referenced and used frequently throughout this paper.

The next chapter explores the properties of Representation Theory. First we look at the definition of a linear representation and examples of quivers and their representations. Within Representation Theory we find the vital definition of a RG-module and an algebra. This chapter lays the final ground work so that we may begin to study the structure of our modules.

Next we will examine the modules $\mathbb{Z}_2 S_3$ and $\mathbb{Z}5_S 3$. Although their structures are similar, they have properties that make them different from each other. We do not directly compare these two modules to each other, rather we consider each as its own module with its own qualities. Because of its importance to the structure of these modules, we begin with a review of the group S_3. This group is necessary to form the modules so it is important that the reader has an understanding of S_3, particularly with the way that the permutations act upon each other.

As we look at the module $\mathbb{Z}_5S_3$, we will show that the module $\mathbb{Z}_2S_3$ is only semisimple as a module over $\mathbb{Z}_2$, but not as a module over itself. This will be accomplished both by the use of Machke's Theorem as well as by direct examination of the module. We will describe much of the structure of the module $\mathbb{Z}_2S_3$, both as a module over itself and as a module over the field $\mathbb{Z}_2$.

Finally we will examine the module $\mathbb{Z}_5S_3$. This module differs from $\mathbb{Z}_2S_3$ in structure, because we are using the field $\mathbb{Z}_5$ instead of the field $\mathbb{Z}_2$. At minimum, this changes the structure of the module because it gives us a module with more elements to consider. More importantly, it changes our methods for determining semisimplicity because the order of $\mathbb{Z}_5$ does not divide the order of S_3. We look again at the simplicity and semisimplicity of the module $\mathbb{Z}_5S_3$.

This paper concludes with a chapter on future research. The overall goal of this project is to explore the structures of the modules $\mathbb{Z}_2S_3$ and $\mathbb{Z}_5S_3$. There is still much to discover about these two modules, so we have included a chapter to discuss intentions of continued research into the structure of these modules.

Chapter 2

Ring Theory

This chapter serves as a basis for understanding the terminology that will be used throughout this project. It will provide definitions, theorems, and set the foundations for the connections that will form between Ring Theory and Module Theory.

We begin with Ring Theory. This field of study examines sets with two binary operations, namely addition and multiplication and relates the two operations by the distributive laws [DF91].When most begin to study Ring Theory, they begin with the concept of a group and expand their understanding to what are known as rings. Connections are formed between subgroups and subrings, normal subgroups and ideals, and group homomorphisms and ring homorphisms.

There are some terms and definitions from ring theory that are used in module theory. Their relationship is similar to the relationship between groups and rings. Let us first examine these properties as they pertain to rings. In Chapter 3 they will be re-examined in relationship to modules.

2.1 Definitions.

We will begin with the definition of a ring.

Definition 2.1. *A ring R is a set together with two binary operations " $+$ " and " $\times$ ", called addition and multiplication respectively, satisfying the following axioms:*

1. *$(R, +)$ is an Abelian group,*
2. *" $\times$ " is closed in R:*
   $a \times b \in R$ for all $a, b \in R$, and
3. *" $\times$ " is associative: $(a \times b) \times c = a \times (b \times c)$ for all $a, b, cinR$,*
4. *the distributive laws hold in R: for all $a, b, c \in R$*
   $(a + b) \times c = (a \times c) + (b \times c)$ and $a \times (b + c) = (a \times b) + (a \times c)$.

The ring R is said to be commutative if multiplication is commutative. It is said to have identity if there is an element $1 \in R$ with

$$1 \times a = a \times 1 = a$$

for all $a \in R$. Neither of these properties are assumed to exist within a ring R. If R is either commutative or has an identity element then we will say so [DF91].

Example 2.2. *The ring of integers $\mathbb{Z}$, the rational numbers $\mathbb{Q}$, the real numbers, $\mathbb{R}$, and the complex numbers $\mathbb{C}$ are all examples of rings, more specifically, they are all commutative rings with identity.*

Now we will give the definition of a subring. The relationship between a ring and one of its subrings is similar to that of a group and one of its subgroups.

Definition 2.3. *Let R be a ring. A subset S of R is a subring if the operations of addition and multiplication in R when restricted to S give S the structure of a ring. That is, the subset S has the same operations of addition and multiplication as the ring R so that S itself is a ring [DF91].*

The Subring Criterion gives the test that may be used to see if a subset of a ring is indeed a subring. It requires that a subset S of a ring R be nonempty and closed under subtraction and under multiplication. Then, to test whether or not a subset is a subring, one must first show that the subset is nonempty. Second, one must show that the operations of subtraction and multiplication are preserved within S. Then S is a subring of R.

Now examine a subring I of R which has special properties.

Definition 2.4. *Let R be a ring. A subset I of R is a left ideal of R if*

1. *I is a subring of R, and*
2. *I is closed under left multiplication by elements from R, that is to say that $rI \subseteq I$ for all $r \in R$.*

This ideal I functions in the ring R similar to the way that a normal subgroup N functions within its group S. In order to show that a subset is in fact an ideal, it must be nonempty, closed under addition of its elements, and closed under multiplication with elements of its ring R [DF91].

Example 2.5. *In the ring $\mathbb{Z}$, the subring $n\mathbb{Z}$ is an ideal for any $n \in \mathbb{Z}$ [DF91].*

Example 2.6. *More specifically, the subring $2\mathbb{Z}$, representing the even integers, is an ideal of the ring of integers $\mathbb{Z}$.*

There are a few terms which relate to the study of ideals. They bring attention to the properties that ideals hold.

Definition 2.7. *An ideal I in an arbitrary ring R is called a maximal ideal if $I \neq R$ and the only ideals containing I are I and R. Not all rings will have maximal ideals [DF91].*

Example 2.8. *The ideal $n\mathbb{Z}$ is maximal in the ring $\mathbb{Z}$ if and only if n is a prime number.*

Example 2.9. *The ideal $2\mathbb{Z}$ from Example 2.6 is a maximal ideal in $\mathbb{Z}$ because 2 is a prime number. The ideal $2\mathbb{Z}$ is not equal to the ring $\mathbb{Z}$. We can also see that the only other ideal that will contain $2\mathbb{Z}$ is the entire ring $\mathbb{Z}$.*

Definition 2.10. *Let A be any subset of a ring R. (A) denote the smallest ideal of a ring R which contains A. Then (A) is called the ideal generated by A. An ideal generated by a finite set is called a finitely generated ideal [DF91].*

Another type of ring is the quotient ring, which we define below:

Definition 2.11. *Let R be a ring and let I be an ideal of R. Then the additive quotient group R/I is also a ring under the binary operations:*

$$(r+I)+(s+I)=(r+s)+I \text{ and } (r+I)\times(s+I)=(rs)+I$$

for all $r, s \in R$.

Conversely, if I is any subring of R which satisfies the above operations such that the operations are well-defined, then the subring I is an ideal of R. We then call the ring R/I the quotient ring of R by I [DF91].

Elements of a ring R may also have other useful properties. Some of their many properties are listed here:

Definition 2.12. *Let R be a ring. An element $x \in R$ is called nilpotent if there exists some positive integer n such that $x^n = 0$ [DF91].*

Example 2.13. *In the ring $\mathbb{Z}_4$, the elements $\{0\}$ and $\{2\}$ are nilpotent. The element $\{0\}$ is trivially nilpotent in every ring since it is already equal to 0. Additionally,*

$$\begin{aligned} 2^2 &= 4 \\ &= 0 \textit{ in } \mathbb{Z}_4. \end{aligned}$$

The remaining elements $\{1, 3\}$ are not nilpotent in $\mathbb{Z}_4$ since there does not exist an $n \in \mathbb{Z}^+$ such that

$$\begin{aligned} 1^n &= 0 \textit{ and} \\ 3^n &= 0. \end{aligned}$$

Example 2.14. *The ring $\mathbb{Z}_5$ has no nontrivial nilpotent elements, since there does not exist an $n \in \mathbb{Z}^+$ such that:*

$$\begin{aligned} 1^n &= 0, \\ 2^n &= 0, \\ 3^n &= 0, \\ 4^n &= 0. \end{aligned}$$

Again, the element $\{0\}$ is trivially nilpotent in every ring.

Furthermore, an ideal I may be nilpotent if there exists an $n \in \mathbb{Z}^+$ such that I^n is the zero ideal. We define I^n to be the set $\{x_1 \cdot x_2 \cdot \ldots \cdot x_n \mid x_i \in I \text{ for all } i = 1, ..., n\}$. Since the subring R of a ring R is also considered to be an ideal, it follows that a ring R may be nilpotent if there exists an $n \in \mathbb{Z}^+$ such that R^n is the zero ideal.

Rings may also have properties which show how their ideals interact with each other.

Definition 2.15. *Let A and B be subrings of a ring R. Define*

$$A + B = \{a + b \mid a \in A, b \in B\}$$

to be the sum of two subrings [DF04].

Definition 2.16. *Let R be a ring and let A and B be ideals of R. Then A and B are said to be comaximal if $A + B = R$ [DF91].*

Definition 2.17. *Let R and S be rings. A ring homomorphism is a map $\varphi : R \to S$ satisfying*

1. *$\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in R$ and*

2. *$\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$.*

[DF91]

Theorem 2.18. *The Chinese Remainder Theorem.*

Let $A_1, A_2, ..., A_k$ be ideals in a ring R. The map

$$R \to R/A_1 \times R/A_2 \times \cdots \times R/A_k$$

defined by $r \mapsto (r + A_1, r + A_2, ..., r + A_k)$ is a ring homomorphism with kernel $A_1 \cap A_2 \cap \cdots \cap A_k$. If the ideals A_i and A_j are comaximal for each $i, j \in \{1, 2, ..., k\}$ with $i \neq j$, then this map is surjective and $A_1 \cap A_2 \cap \cdots \cap A_k = A_1 \cdot A_2 \dots A_k$. Then

$$R/(A_1 A_2 \cdots A_k) = R/(A_1 \cap A_2 \cap \cdots \cap A_k) \cong R/A_1 \times R/A_2 \times \cdots \times R/A_k.$$

[DF91]

2.2 The Ascending and Descending Chain Conditions.

The Ascending and Descending Chain Conditions are directly related to what are known as Noetherian and Artinian rings. These are named after the mathematicians Emmy Noether and Emil Artin respectively.

We begin with the definition of a Noetherian ring. Noetherian rings are a particularly important part of commutative algebra. A large portion of the material studied in commutative algebra is aimed at discovering properties of Noetherian rings.

Definition 2.19. *Let R be a ring and let $I_1, I_2, ...$ be ideals of R for all $i = 1, 2, ...$. We say that R is a Noetherian ring if*

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots \subseteq I_k \subseteq I_{k+1} \subseteq \ldots$$

is any ascending chain of ideals of R, then there is a positive integer N such that $I_k = I_N$ for all $k \geq N$ where $k, N \in \mathbb{N}$. That is to say that the chain becomes stationary at I_N for some $N \in \mathbb{N}$, meaning

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_N = I_{N+1} = \ldots$$

Then R satisfies the Ascending Chain Condition, or the A.C.C. [DF91]

We also say that a ring R is a Noetherian ring if every ideal of R is finitely generated, that is $I = (a_1, a_2, ..., a_n)$, for some $n \in \mathbb{N}$, for every ideal I of R.

The definition of a Noetherian Ring is used in the following proposition:

Proposition 2.20. *Let R be a ring. The following conditions are equivalent:*

1. *R is a Noetherian ring, that is, R satisfies the A.C.C.*

2. *Every ideal of R is finitely generated, $I = (a_1, a_2, ..., a_n)$ for every ideal I of R.*

3. *Every nonempty collection of ideals of R contains a maximal element under inclusion.*

[DF91]

Now we look at Artinian rings, these rings satisfy the ascending chain condition with respect to reverse inclusion. This is also known as the Descending Chain Condition.

Definition 2.21. *A ring R is said to satisfy the minimum condition or Descending Chain Condition, abbreviated as the D.C.C., on left ideals if:*

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_\ell \supseteq I_{\ell+1} \supseteq \ldots$$

is any descending chain of left ideals of R, then there is a positive integer N such that $I_\ell = I_L$ for all $\ell \geq L$, where $\ell, L \in \mathbb{N}$. Then the chain becomes stationary at I_L for some $L \in \mathbb{N}$, meaning

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_L = I_{L+1} = \ldots$$

Then R satisfies the Descending Chain Condition, or the D.C.C. Rings that satisfy the D.C.C. are also known as Artinian rings [DF91].

We may also say that a ring R is an Artinian ring if every ideal of R is finitely generated, that is $I = (a_1, a_2, ..., a_n)$ for some $n \in \mathbb{N}$, for every ideal of R.

2.3 The Characteristic of a Ring.

Another property of a ring R is its characterisitc.

Definition 2.22. *The characteristic of a ring R is the smallest positive integer n such that $1_R + 1_R + \cdots + 1_R = 0_R$ is in R, where 1_R is added n times. If no such integer exists the characteristic of R is said to be 0 [DF91].*

The characteristic of a ring will prove useful when we are examining modules in Chapters 5 and 6. Consider the following examples:

Example 2.23. *In $\mathbb{Z}_5$, 5 is the smallest possible positive integer such that*

$$\begin{aligned} 1+1+1+1+1 &= 5 \\ &= 0 \in \mathbb{Z}_5. \end{aligned}$$

Therefore $\mathbb{Z}_5$ is a ring of characteristic 5.

Example 2.24. *We may generalize Example 2.24 to say that $\mathbb{Z}_n$ is a ring of characteristic n.*

Chapter 3

Module Theory

The theory of modules and their structure will prove to be essential to understanding the proofs in this thesis. Although Module Theory can also be used to understand vector spaces, we will be focusing on its relationship to Ring Theory. Module Theory also lays a base for some of the concepts from Representation Theory, which will be explored in the next chapter. Modules themselves can be viewed as a representation of groups; in fact, rings act upon modules as algebraic objects [DF91]. The modules are a representation for rings- the action of a ring on an Abelian group preserves the structure of the group.

Many of the properties that are examined in this chapter bear a likeness to those in Chapter 2. This is because Ring Theory serves as a foundation for Module Theory. There is a correlation between the properties of rings and the properties of modules. These connections will be explored throughout the chapter in order to help the reader see the relationship between rings and modules.

3.1 Definitions.

We begin with the definition of a module.

Definition 3.1. *Let R be a ring, not necessarily commutative nor with 1. A left R-module, or a left module over R, denoted R-mod, is a set M which satisfies the following requirements:*

1. *A binary operation " + " on M, under which M is an Abelian group.*

2. *A ring action of R on M, that is a map $R \times M \to M$, denoted by rm, for all $m \in M$ and for all $r \in R$ which satisfies the following properties:*

   (a) $(r+s)m = rm + sm$, *for all* m $M, r, s \in R$,

   (b) $(rs)m = r(sm)$, *for all* $m \in M, r, s \in R$, *and*

   (c) $r(m+n) = rm + rn$, *for all* $m, n \in M, r \in R$.

   If the ring R has an identity element 1_R we impose the additional condition:

   (d) $1_R m = m$, *for all* $m \in M$.

The term "left" module is used because the ring elements appear on the left-hand side [DF91].

Modules over a field F instead of a ring R are the same as vector spaces over a field F. Additionally, the modules that we deal with will all be left modules, denoted as R-modules. Finally, we will only be considering finite-dimensional modules.

Example 3.2. *The most basic example of a module is to let the module $M = R$, where R is any ring. We may then say that R is a left R-module, where the action of a ring element on a module element is the usual multiplication of the ring R [DF91].*

Example 3.3. *In Chapter 5 we will be examining the module $\mathbb{Z}_2 S_3$. Here we have the group S_3, which is Abelian under the operation " + ". We then impose the action of the ring $\mathbb{Z}_2$. Together, under the ring action, the two give us the $\mathbb{Z}_2$-module $\mathbb{Z}_2 S_3$.*

The elements of this module look like sums of the elements of S_3, where each element of S_3 has a coefficient from $\mathbb{Z}_2$. For example, the elements

$$0e + 1(12) + 0(13) + 0(23) + 1(123) + 0(132) \text{ and}$$

$$1e + 1(12) + 0(13) + 1(23) + 0(123) + 0(132)$$

belong to the module $\mathbb{Z}_2 S_3$. The example below shows both action of $\mathbb{Z}_2 S_3$ as well as the addition of elements in $\mathbb{Z}_2S_3$.

$$\begin{aligned}
& [0e+1(12)+0(13)+0(23)+1(123)+0(132)]\times[1e+1(12)+0(13)+1(23) \\
& +0(123)+0(132)] \\
= \ & 0e[1e+1(12)+0(13)+1(23)+0(123)+0(132)] \\
& +1(12)[1e+1(12)+0(13)+1(23)+0(123)+0(132)] \\
& +0(13)[1e+1(12)+0(13)+1(23)+0(123)+0(132)] \\
& +0(23)[1e+1(12)+0(13)+1(23)+0(123)+0(132)] \\
& +1(123)[1e+1(12)+0(13)+1(23)+0(123)+0(132)] \\
& +0(132)[1e+1(12)+0(13)+1(23)+0(123)+0(132)] \\
= \ & 1(12)[1e+1(12)+0(13)+1(23)+0(123)+0(132)] \\
& +1(123)[1e+1(12)+0(13)+1(23)+0(123)+0(132)] \\
= \ & 1(12)+1e+0(132)+1(123)+0(23)+0(13) \\
& +1(123)+1(13)+0(23)+1(12)+0(132)+0e \\
= \ & 1e+0(12)+1(13)+0(23)+0(123)+0(132) \text{ in } \mathbb{Z}_2S_3.
\end{aligned}$$

Since this is one of the two modules which we will be focusing on, we will go into greater detail regarding the interaction among the elements of this module in Chapters 5.

In Chapter 5 we will see that $\mathbb{Z}_2S_3$ may be examined as both a $\mathbb{Z}_2S_3$-module, that is, a module over itself, and a $\mathbb{Z}_2$-module. In this instance we are examining elements of $\mathbb{Z}_2S_3$. Alternatively, we could multiply one of the elements above by one of the elements from $\mathbb{Z}_2$. Then we would be working with a $\mathbb{Z}_2$-module. For example, within the $\mathbb{Z}_2$-module we may have

$$\begin{aligned}
& 1\times[1e+1(12)+0(13)+1(23)+0(123)+0(132)] \\
= \ & [1\times 1]e+[1\times 1](12)+[1\times 0](13)+[1\times 1](23)+[1\times 0](123)+[1\times 0](132) \\
= \ & 1e+1(12)+0(13)+1(23)+0(123)+0(132)
\end{aligned}$$

Now we will look at the definition of a submodule. A submodule's function is similar to that of a subgroup or a subring within its respective group or ring.

Definition 3.4. *Let R be a ring and let M be a left R-module. Then a R-submodule of M is a subgroup N of M which is closed under the action of ring elements. That is, $rn \in N$, for all $n \in N$ and for all $r \in R$.*

In other words, a subset N of M is a submodule of M if and only if

1. *$N \neq \emptyset$, and*
2. *$x + \alpha y \in N$ for all $\alpha \in R$ and for all $x, y \in N$.*

[DF91]

The two conditions listed above are known as the Submodule Criterion. They are used to test whether or not a subset is a submodule of a given module. This criterion will be referenced frequently throughout the following chapters.

Example 3.5. *If we let R act as a module in the same way we did in Example 3.2, that is, if we treat the ring R as a module, then the left submodules of the module R are exactly the left ideals of the ring R.*

Now let us look at some of the properties and terminology associated with submodules.

Definition 3.6. *For any subset A of M let*

$$RA = \{\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_m a_m \mid \alpha_1, ..., \alpha_m \in R, a_1, ..., a_m \in A, m \in \mathbb{Z}^+\}.$$

[DF04]

We refer to RA as the submodule of M generated by the subset A. If $A = 0$ then $RA = \{0\}$. If A is a finite set $(a_1, a_2, ..., a_n)$, then we write RA as $Ra_1 + Ra_2 + \ldots + Ra_n$. In this case we call RA a finitely generated subset of A.

If N is a submodule of M, not necessarily proper, and $N = RA$, for some subset A of M, we call A a set of generators, or generating set, for N. We also say the submodule N is generated by the set A. If the set A is finite, then we call N a finitely generated submodule of M [DF04].

We may extend this definition to say that, when we consider M as a submodule of itself such that M meets the above criteria, the module M is finitely generated. We may state this definition more formally:

Definition 3.7. *The left R-module M is finitely generated if and only if there exist $a_1, a_2, ..., a_n$ in M such that for all x in M, there exist $r_1, r_2, ..., r_n$ in R with $x = r_1a_1 + r_2a_2 + \cdots + r_na_n$. We refer to the set $\{a_1, a_2, ..., a_n\}$ as a generating set for M [DF04].*

Definition 3.8. *Let M be a R-module and let N be a submodule of M. Then the submodule N, not necessarily a proper submodule of M, is called cyclic if there exists an element $a \in M$ such that $N = Ra$, that is, if N is generated by one element:*

$$N = Ra = \{ra \mid r \in R\}.$$

[DF04]

Again, in the case where the submodule N is equal to the module M, we may say that the module itself is cyclic.

Now we should be beginning to see some of the similarities between Ring Theory and Module Theory.

Definition 3.9. *A submodule N in an arbitrary module M is called a maximal submodule if $N \neq M$ and the only submodules containing N are the submodule N itself and the module M [DF91].*

Definition 3.10. *Let M be a module. An element $x \in M$ is called nilpotent if there exists some positive integer n such that $x^n = 0_M$ [DF91].*

The next definition will prove to be very important to the modules we study in this paper.

Definition 3.11. *The radical of a module M to be the intersection of all of the maximal submodules of M. It is also the largest nilpotent ideal of a ring. It is denoted as $Rad(M)$ [Alp86].*

Definition 3.12. *Let R be a ring and let M and N be R-modules. Then a map $\varphi : M \to N$ is an R-module homomorphism if it respects the R-module structures of M and N. That is to say*

1. *$\varphi(x + y) = \varphi(x) + \varphi(y)$, for all $x, y \in M$, and*

2. $\varphi(\alpha x) = \alpha\varphi(x)$, *for all* $\alpha \in R, x \in M$.

An R-module homomorphism is an isomorphism of R-modules if it is both injective and surjective. The modules M and N are said to be isomorphic, denoted $M \cong N$, if there is some R-module isomorphism $\varphi : M \to N$.
If $\varphi : M \to N$ is an R-module homomorphism, let $\ker\varphi = \{m \in M \mid \varphi(m) = 0\}$, the kernel of φ, and let $\varphi(M) = \{n \in N \mid n = \varphi(m) \text{ for some } m \in M\}$, the image of of φ [DF91].

3.2 The Ascending and Descending Chain Conditions.

Modules also have Ascending and Descending Chain Conditions. These conditions are similar to the Ascending and Descending Chain Conditions of rings and ring ideals from Chapter 2.

Definition 3.13. *Let R be a ring. The left R-module M is said to be a Noetherian R-module, or to satisfy the Ascending Chain Condition on submodules, often denoted as the A.C.C. on submodules, if there are no infinite increasing chains of submodules.*

That is whenever

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots \subseteq M_k \subseteq M_{k+1} \cdots$$

is an increasing chain of submodules of M and there is a positive integer N such that for all $k \geq N, M_k = M_N$. The chain will then become stationary at stage N and

$$M_1 \subseteq M_2 \subseteq \ldots \subseteq M_N = M_{N+1} = M_{N+2} = \cdots.$$

This type of left module is also said to be a Noetherian R-module [DF91]. A ring R is said to be a Noetherian ring if it is Noetherian as a left module over itself.

Theorem 3.14. *Let R be a ring and let M be a left R-module. Then the following are equivalent:*

1. *M is a Noetherian R-module, that is, it satisfies the A.C.C. on modules.*

2. *Every nonempty set of submodules of M contains a maximal element under inclusion.*

3. Every submodule of M is finitely generated.

[DF91]

Definition 3.15. *Similarly, we say that the left R-module satisfies the Descending Chain Condition on submodules, often written as the D.C.C. on submodules, if there are no infinite decreasing chains of submodules. Whenever*

$$M_1 \supseteq M_2 \supseteq \ldots \supseteq M_k \supseteq M_{k+1} \ldots$$

is a decreasing chain of submodules of M, then there is a positive integer L such that for all $\ell \geq L, M_\ell = M_L$. This chain will also become stationary at stage L and

$$M_1 \supseteq M_2 \supseteq \ldots M_L = M_{L+1} = \ldots.$$

Often a module that satisfies the Descending Chain Condition on submodules is referred to as an Artinian module. A ring R is said to be an Artinian ring if it is Artinian as a left module over itself.

3.3 Simple and Semisimple Modules.

The first definition will prove to be very important. It will be referenced frequently throughout this paper.

Definition 3.16. *The module M is said to be irreducible, or simple, if its only submodules are 0 and M. Otherwise M is called reducible [DF91].*

If a module is not simple, it may have a different property:

Definition 3.17. *The module M is said to be completely reducible if it is a direct sum of irreducible submodules. In other words, a module is completely reducible, or semisimple, if it can be written as a direct sum:*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n,$$

for some $n \in \mathbb{N}$, and where each M_i is a simple submodule of M [DF91].

We will also be focusing on the semisimplicity of modules frequently throughout Chapters 5 and 6.

Theorem 3.18. *Let R be a nonzero ring with 1, not necessarily commutative. Then every left R-module is completely reducible if and only if the ring R, considered as a left R-module over itself, is a direct sum:*

$$R = L_1 \oplus L_2 \oplus \cdots \oplus L_n$$

where each L_i is a simple submodule of the module R.

That is to say that every R-module is completely reducible if and only if the ring R is semisimple.

Rings R satisfying the conditions of this theorem also satisfy the D.C.C. on submodules[DF91].

Definition 3.19. *The module M is said to be indecomposable if M cannot be written as $M_1 \oplus M_2$ for any nonzero submodules M_1 and M_2. Otherwise M is called decomposable [DF91].*

Definition 3.20. *Let R be a ring, let M be a left R-module, and let N be a submodule of M. The quotient group M/N can be made into a left module, called a quotient module, with the action on the elements of R defined as follows:*

$$\alpha(x + N) = (\alpha x) + N,$$

for all $\alpha \in R$ and $x + N \in M/N$ [DF91].

Definition 3.21. *If N, K are submodules of M, then define $N + L = \{n + \ell \mid n \in N, \ell \in L\}$.*

This definition is similar to the one in Chapter 2. The addition of submodules is similar to the addition of ideals.

In Chapter 2 we gave the definition for comaximal ideals. The same property exists in Module Theory.

Definition 3.22. *We may extend the definition of comaximal ideals A and B in a ring R to modules. Then, given a module M and two submodules N, K of M, we say that N and K are comaximal if $N + K = M$.*

Definition 3.23. *The socle of a module M, denoted $socM$, is the largest semisimple submodule of M.*

Similarly we define the top of a module M, denoted $topM$, to be the largest semisimple factor module of M, that is $topM = M/MJ$, where J is the radical of the algebra we are working with [Erd90].

Definition 3.24. *We may define the socle series of M to be the sequence of submodules*

$$0 \subseteq soc_1(M) \subseteq soc_2(M) \subseteq \cdots \subseteq soc_k(M) \subseteq \cdots \subseteq M$$

with $soc_1(M) = socM$ and $soc_k(M)/soc_{k-1}(M) = soc(M/soc_{k-1}(M))$. We define the socle length of M to be the first integer r such that $soc_r(M) = soc_{r+1}(M)$.

The radical series of M is defined to be the sequence of submodules

$$0 \subseteq \cdots \subseteq rad^k(M) \subseteq \cdots \subseteq rad^2(M) \subseteq rad(M) \subseteq M$$

with $rad^k(M) = rad(rad^{k-1}(M))[\cong MJ^k]$.

It is important to note that $M \neq 0$ implies $socM \neq 0$ and $topM \neq 0$ [Erd90].

3.4 Theorems.

Now examine some theorems which apply to modules. These theorems will allow us to use properties of modules.

First we will look at a theorem called the Krull-Schmidt Theorem. This theorem requires that modules satisfy the A.C.C. and the D.C.C., according to their definitions in Section 3.2. We may apply this theorem to the modules we will be looking at in Chapters 5 and 6.

Theorem 3.25. *The Krull-Schmidt Theorem.*

Let M be a module that satisfies the Ascending Chain Condition and the Descending Chain Condition. That is to say that M has finite length. Then M is a direct sum of indecomposable modules and such a decomposition is unique up to isomorphism[Erd90].

In general, the Krull-Schmidt Theorem fails if M is only Artinian and not Noetherian, that is, if it only satisfies the Descending Chain Condition.

The next two theorems will help to prove the Chinese Remainder Theorem.

Theorem 3.26. *The First Isomorphism Theorem for Modules.*

Let R be a ring and let M, N be R-modules and let $\varphi : M \to N$ be an R-module homomorphism. Then $ker\varphi$ is a submodule of M and $M/ker\varphi \cong \varphi(M)$ [DF91].

Theorem 3.27. *The Fourth Isomorphism Theorem for Modules, which is also known as the Lattice Isomorphism Theorem.*

Let N be a submodule of the R-module M. Then there exists a bijection between the submodules of M which contain N and the submodules of M/N. The correspondence is given by $A \leftrightarrow (A+N)/N$, for all $A \supseteq N$. This correspondence commutes with the processes of taking sums and intersections, meaning there is a lattice isomorphism between the lattice of submodules of M/N and the lattice of submodules of M which contain N [DF91].

Now we may apply the Chinese Remainder Theorem to modules by considering submodules of a module M instead of ideals of a ring R as we did in Chapter 2.

Theorem 3.28. *The Chinese Remainder Theorem for modules.*

We then let $N_1, N_2, ..., N_k$ be submodules of a module M. The map

$$M \to M/N_1 \times M/N_2 \times \cdots \times M/N_k$$

defined by $m \mapsto (m + N_1, m + N_2, ..., m + N_k)$ is a module homomorphism with kernel $N_1 \cap N_2 \cap \cdots \cap N_k$. If for each $i, j \in \{1, 2, ..., k\}$ with $i \neq j$ the submodules N_i and N_j are comaximal, then this map is surjective and $N_1 \cap N_2 \cap \cdots \cap N_k = N_1N_2 \cdots N_k$, so

$$M/(N_1N_2 \cdots N_k) = M/(N_1 \cap N_2 \cap \cdots \cap N_k) \cong M/N_1 \times M/N_2 \times \cdots \times M/N_k.$$

[DF91]

Proof. (by induction)
Let M be a module and let $N_1, N_2, ..., N_k$ be submodules of M.

Consider the basis for induction: the case where $k = 2$. That is, the case where there are two submodules N_1 and N_2 of the module M. Then consider the map $\varphi : M \to M/N_1 \times M/N_2$ which is defined by $\varphi(m) = (m + N_1, m + N_2)$. The submodules N_1 and N_2 are comaximal.

We claim that this map φ is a ring homomorphism. Recall from Definition 3.12 that φ must satisfy the following conditions:

1. $\phi(x + y) = \phi(x) + \phi(y)$, for all $x, y \in M$ and
2. $\phi(\alpha x) = \alpha\phi(x)$, for all $\alpha \in R, x \in M$.

We check these conditions. Let x, y be elements of the module M and let α be an element of the ring R. Then

$$\begin{aligned} 1.\ \varphi(x+y) &= ((x+y)+N_1, (x+y)+N_2) \\ &= ((x+N_1)+(y+N_1), (x+N_2)+(y+N_2)) \\ &= ((x+N_1), (x+N_2)) + ((y+N_1), (y+N_2)) \\ &= \varphi(x) + \varphi(y). \end{aligned}$$

$$\begin{aligned} 2.\ \varphi(\alpha x) &= (\alpha x + N_1, \alpha x + N_2) \\ &= \alpha(x + N_1, x + N_2) \\ &= \alpha\varphi(x). \end{aligned}$$

Then φ is in fact a R-module homomorphism. Now we want to show that the kernel of φ, denoted $\ker\varphi = N_1 \cap N_2$.

Let $x \in M$ such that x is also an element of $\ker\varphi$. Then

$$\begin{aligned} \varphi(x) &= (0_M + N_1, 0_M + N_2) \\ &= (N_1, N_2). \end{aligned}$$

However, according to the definition of φ,

$$\begin{aligned}
\varphi(x) &= (x+N_1, x+N_2). \\
\text{So } \varphi(x) &= (x+N_1, x+N_2) \\
&= (N_1+N_2).
\end{aligned}$$

$$\text{This implies that } x+N_1 = N_2 \text{ and } x+N_2 = N_2.$$

$$\text{Then } x \in N_1 \text{ and } x \in N_2.$$

$$\text{Thus } x \in N_1 \cap N_2 \text{ and } \ker\varphi = N_1 \cap N_2.$$

It remains to show that when N_1 and N_2 are comaximal submodules of M, then φ is a surjective mapping and $N_1 \cap N_2 = N_1N_2$.

To show that φ is a surjective mapping, let $(x+N_1, y+N_2) \in M/N_1 \times M/N_2$, where $x, y \in M$. We want to find a $m \in M$ such that m is the preimage of $(x+N_1, y+N_2)$ such that $\varphi(m) = (x+N_1, y+N_2)$.

Now we know that $1_M \in M$. Since N_1 and N_2 are comaximal, $N_1 + N_2 = M$. This implies that there exists an element $n_1 \in N_1$ and a $n_2 \in N_2$ such that $n_1 + n_2 = 1_M$. Then $n_1 = 1 - n_2 \in 1_M + N_2$ and $n_2 = 1 - n_1 \in 1_M + N_1$. Furthermore, $n_1 + N_2 = 1_M + N_2$ and $n_2 + N_1 = 1_M + N_1$.

Consider an element $m \in M$ such that $m = yn_1 + xn_2$. Then

$$\begin{aligned}
\varphi(m) &= \varphi(yn_1 + xn_2) \\
&= \varphi(yn_1) + \varphi(xn_2) \\
&= \varphi(y)\varphi(n_1) + \varphi(x)\varphi(n_2) \\
&= (y + N_1, y + N_2)(n_1 + N_1, n_1 + N_2) + (x + N_1, x + N_2)(n_2 + N_1, n_2 + N_2) \\
&= (y + N_1, y + N_2)(N_1, 1M + N_2) + (x + N_1, x + N_2)(1_M + N_1, N_2), \\
&\quad \text{since } N_1, N_2 \text{ are ideals of } M \text{and by the work above.} \\
&= (y + N_1, y + N_2)(0 + N_1, 1M + N_2) + (x + N_1, x + N_2)(1_M + N_1, 0 + N_2), \\
&= (y \cdot 0 + N_1, y \cdot 1_M + N_2) + (x \cdot 1_M + N_1, x \cdot 0 + N_2) \\
&= (0 + N_1, y + N_2) + (x + N_1, 0 + N_2) \\
&= ((0 + x) + N_1, (y + 0) + N_2) \\
&= (x + N_1, y + N_2)
\end{aligned}$$

So φ is a surjective R-module homomorphism when N_1 and N_2 are comaximal submodules.

Finally, we know that the submodule $N_1N_2 \subseteq N_1 \cap N_2$ according to the definition of a submodule from Definition 3.9, which says that $rn \in N$ for all n in a submodule N and all r in a ring R. Since N_1 and N_2 are comaximal submodules, we may define n_1 and n_2 as we did above so that $n_1 + n_2 = 1_M$. Then for any $c \in N_1 \cap N_2$,

$$c = c1_M = c(n_1 + n_2) = cn_1 + cn_2$$

$$\in N_1N_2.$$

By double inclusion, $N_1 \cap N_2 = N_1N_2$.

By Theorem 3.26, the First Isomorphism Theorem, we know that given a ring R, a module M, and its submodule N, we may define a R-module homomorphism $\psi : M \to N$. Then kerψ is a submodule of M and $M/\text{ker}\psi \cong \psi(M)$.

Since we know that φ is a R-module homomorphism and we have shown that ker$\varphi =$

$N_1 \cap N_2$, we may now say:

$$\begin{aligned} M/ker\varphi &\cong \varphi(M) \\ M/N_1 \cap N_2 &\cong M/N_1 \times M/N_2. \end{aligned}$$

Thus, for the case when $k = 2$, the Chinese Remainder Theorem for modules is true.

The general case follows by induction when we define the first submodule to be N_1 as before and allow $A = N_2 \ldots N_k$ and show that N_1 and A are comaximal. By hypothesis, for each $i \in \{2, 3, ..., k\}$, there are elements $n_i \in N_1$ and $a_i \in N_i$ such that $n_i + a_i = 1_M$. Since $n_i + a_i \equiv a_i + N_i$, it follows that

$$1_M = (n_2 + a_2) + \ldots (n_k + a_k) \in N_1 + (A_2 \ldots A_k).$$

Thus, by induction, the Chinese Remainder Theorem holds. □

Theorem 3.29. *Let R be a ring. A R-module M is semisimple if and only if $Rad(R)M = 0$ [Erd90].*

Lemma 3.30. *Let R be a ring, let M be a R-module, and let $Rad(R)$ be the radical of the ring R. If $a \in Rad(R)$ then $1 - a$ has a left inverse in R [Ben98].*

Lemma 3.31. *Nakayama's Lemma.*

Let R be a ring, let M be a finitely generated R-module, and let $Rad(R)$ be the radical of R. If the product $Rad(R)M = M$ then $M = 0$[Ben98].

Proof. Suppose that $M \neq 0$. Choose elements $m_1, ..., m_n \in M$ such that the m_i generate M, where n is minimal. Since $\mathrm{Rad}(R)M = M$, we can write elements m_n of $\mathrm{Rad}(R)M$ as

$$m_n = \sum_{i=1}^{n} a_i m_i,$$

with $a_i \in \mathrm{Rad}(R)$. By Lemma 3.30, we know that $1 - a_n$ has a left inverse in R. Let b be the inverse of $1 - a_n$. Note that $a_n m_n$ is the last term of the sum $m_n = \sum_{i=1}^{n} a_i m_i$. Then the product

$$\begin{aligned} (1 - a_n)m_n &= 1m_n - a_n m_n \\ &= m_n - a_n m_n \\ &= \sum_{i=1}^{n-1} a_i m_i \end{aligned}$$

Since b is the left inverse of $1-a_n$, we may multiply the last line of the equation above on the left side by b. Then

$$\begin{aligned}
(1-a_n)m_n &= \sum_{i=1}^{n-1} a_i m_i \\
b(1-a_n)m_n &= b(\sum_{i=1}^{n-1} a_i m_i) \\
1m_n &= b(\sum_{i=1}^{n-1} a_i m_i) \\
m_n &= b(\sum_{i=1}^{n-1} a_i m_i)
\end{aligned}$$

This contradicts the minimality of n. Then our original assumption, $M \neq 0$ is not true, and $M = 0$. □

Theorem 3.32. *Rad(R)M ≅ Rad(M) [Erd90].*

Proof. Let M' be a maximal submodule of M. Then $\text{Rad}(R)(M/M') = M/M'$, so by Nakayama's lemma, Lemma 3.31, $M/M' = 0$. We have $\text{Rad}(R)M \subsetneq M'$, and $\text{Rad}(R)M \subseteq \text{Rad}(M)$.

Conversely $M/\text{Rad}(R)M$ is completely reducible by Lemma 2.2 and so $\text{Rad}(M/\text{Rad}(R)M) = 0$, which implies that $\text{Rad}(M) \subseteq \text{Rad}(R)M$. □

Lemma 3.33. *Let M be a finitely generated module over an arbitrary ring R. Then a submodule N of M is small in M if and only if $N \subset RadM$*

Chapter 4

Representation Theory

So far we have examined rings and modules. In Chapter 3 we examined modules as the "representation objects" of a ring R [DF91]. In this case, the ring R imposed an action upon an Abelian group. This is an example of Representation Theory. This chapter will serve as the final foundation for understanding the modules we will examine in Chapters 5 and 6.

In the next two chapters, we will be examining the structure of two types of modules- modules over a field and modules over an RG-ring, which will be defined in this chapter. Changing the type of ring that we represent a module over will change the structure of the module itself.

4.1 Definitions.

We begin with the definition of a representation.

Definition 4.1. *A linear representation of a group G is any homomorphism from G into $GL(V)$, where $GL(V)$ is the group of nonsingular linear transformations from the vector space V to itself [DF91].*

Before we examine some examples of representations, we will need the following definition:

Definition 4.2. *A quiver Q is a directed graph $Q = (Q_0, Q_1)$ where Q_0 is the set of verticies and Q_1 is the set of arrows; together Q_0 and Q_1 form quivers. We define a map by $s : Q_1 \to Q_0$ where $s(\alpha) = i$ and the map $e : Q_1 \to Q_0$ where $e(\alpha) = j$. The maps s and e form the vertices which yield $\alpha : i \to j$, an arrow of Q [ARO97].*

One may think of these maps as the respective starting and ending points of an arrow α of a quiver Q. That is, given an arrow $\alpha \in Q_1$, we say it starts at vertex $s(\alpha)$ and terminates at $e(\alpha)$.

The quiver is finite if both Q_0 and Q_1 are finite.

Example 4.3. *Let Q be a quiver and let K be a fixed field. A representation $\underline{V}$ of the quiver Q over the field K is given by (V_i, φ_α). For any vertex $i \in Q_0$ there is a vector space V_i, and for any arrow $\alpha : i \to j$, there is a linear transformation $\varphi_\alpha : V_i \to V_j$.*

Denote the category of representations over Q over K by $\mathcal{R}(Q)$.

Now we give some basic examples of quivers. The graphs are representations of the quivers.

Example 4.4. *The finite quiver $Q = (Q_0, Q_1)$ where $Q_0 = \{1, 2, 3\}$ and $Q_1 = \{\alpha, \beta\}$ with $e(\alpha)$=1, $s(\alpha)$=$e(\beta)$=2 and $s(\beta)$=3.*

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

Example 4.5. *The finite quiver $Q = (Q_0, Q_1)$ with $Q_0 = \{1, 2, 3\}$ and $Q_1 = \{\alpha, \beta, \gamma, \phi\}$ with $e(\alpha)$=$s(\beta)$=1, $s(\alpha)$=$e(\beta)$=$e(\gamma)$=$s(\phi)$=2, $e(\phi)$=$s(\gamma)$=3.*

$$\begin{tikzcd}
1 \arrow[dr, shift left, "\alpha"] & & \\
& 2 \arrow[ul, shift left, "\beta"] \arrow[r, shift left, "\gamma"] & 3 \arrow[l, shift left, "\phi"]
\end{tikzcd}$$

Definition 4.6. *Let G be an Abelian group, let R be a ring, and let M be a module such that M is formed using G. A representation may be given by $\varphi : G \to M$ on a module over R. Under this correspondence we say that the module M affords the representation φ of G [DF91].*

Definition 4.7. *Define the group ring RG of a group G over a ring R to be the set of all formal sums of the form*

$$\sum_{g \in G} \alpha_g g, \ \alpha_g \in R.$$

This group ring is also referred to as a RG-ring [DF91].

Two formal sums are equal if and only if all corresponding coefficients of group elements are equal. Addition in RG is defined as

$$\sum_{i=1}^{n} \alpha_i g_i + \sum_{i=1}^{n} \beta_i g_i = \sum_{i=1}^{n} (\alpha_i + \beta_i) g_i.$$

Multiplication is defined as

$$\left(\sum_{i=1}^{n} \alpha_i g_i\right) \left(\sum_{i=1}^{n} \beta_i g_i\right) = \sum_{k=1}^{n} \left(\sum_{i,j,g_i g_j = g_k}^{n} \alpha_i \beta_j\right) g_k.$$

Definition 4.8. *Let G be a group and let F be a field and let V be a vector space. The degree of a representation of G is the dimension of V [DF91].*

Now we will look at some properties of representations. Many of the following properties focus on how one representation may relate to another representation.

Definition 4.9. *Let M be a module over a ring R. We may then form the trivial representation by setting $gm = m$ for all $g \in G$ and all $m \in M$ [DF91].*

Definition 4.10. *Two representations of G are equivalent if the RG-modules affording them are isomorphic modules. Representations which are not equivalent are called inequivalent [DF91].*

Definition 4.11. *Let R be a ring and let M be a nonzero R-module. A representation is said to be irreducible, reducible, or completely reducible according to whether the RG-module affording it has the corresponding property [DF91].*

4.2 Algebras.

Now we will examine algebras and their properties within Representation Theory.

Before we give the definition for an algebra, we need to define the center of a ring.

Definition 4.12. *The center of a ring R is $\{z \in R \mid zr = rz$ for all $r \in R\}$. That is, it is the set of all elements which commute with every element of R [DF91].*

Definition 4.13. *Let R be a commutative ring with identity. An R-algebra is a ring A with identity together with a ring homomorphism $f : R \to A$ mapping 1_R to 1_A such that the subring $f(R)$ of the ring A is contained in the center of A [DF04].*

Definition 4.14. *A path in the quiver Q is an ordered sequence of arrows $p = \alpha_n \dots \alpha_1$ with $e(\alpha_t) = s(\alpha_{t+1})$ for $1 \leq t < n$ [ARO97].*

Definition 4.15. *A path algebra KQ of the quiver Q over the field K is the K-vector space with paths of Q as a basis. We may define a linear map $f : KQ \to End_K(KQ)$, where $End_K(KQ)$ is the set of endomorphisms of KQ [ARO97].*

Definition 4.16. *Let K be a field. A ring R is a K-algebra if K is contained in the center of R and the identity of K is the identity of R [DF04].*

Definition 4.17. *Let R be a ring. The ring S is a finitely generated R-algebra if S is generated as a ring by R together with some finite set $r_1, r_2, ..., r_n$ of elements of S, for some $n \in \mathbb{N}$ [DF04].*

Definition 4.18. *Let R be a ring and let S and T be R-algebras. A map $\psi : S \to T$ is a R-algebra homomorphism if ψ is a ring homomorphism that is the identity on R [DF04].*

Corollary 4.19. *The ring R is a finitely generated K-algebra if and only if there is some surjective K-algebra homomorphism*

$$\varphi : K[x_1, x_2, ..., x_n] \to R$$

from the polynomial ring in a finite number of variables onto R that is the identity map on K. Any finitely generated K-algebra is therefore Noetherian[DF04].

The Noetherian algebra above satisfies similar conditions to Noetherian rings and Noetherian modules. Then a Noetherian algebra satisfies the Ascending Chain Condition on its left ideals. Algebras may also be Artinian, satisfying the Descending Chain Condition on its left ideals in the same way that rings and modules satisfy the Descending Chain Condition.

The notion of a radical also applies to algebras.

Definition 4.20. *The radical of an algebra A consists of the elements of A which annihilate every simple A-module, in other words, the radical of A consists of the elements which annihilate every semisimple A-module. Note that the radical of A is an ideal [DF04].*

We will see that the radical has many interesting properties. In addition to the properties listed in the theorem below, we may say that the algebra A is semisimple if rad $A = 0$. It follows that the algebra A is semisimple if and only if A is semisimple.

Theorem 4.21. *The radical of A is equal to each of the following:*

1. *the smallest submodule of A whose corresponding quotient is semisimple;*
2. *the intersection of all the maximal submodules of A;*
3. *the largest nilpotent ideal.*

[Alp86]

4.3 Theorems.

The following are very important theorems in Representation Theory.

The first theorem we will be examining is best-known as Machke's Theorem. It will be very useful in discovering properties of the modules which will be examined in Chapters 5 and 6.

Theorem 4.22. *Machke's Theorem.*

Let G be a finite group and let F be a field whose characteristic does not divide $\mid G \mid$. If V is any FG-module, and U is any submodule of V, then V has a submodule W such that $V = U \oplus W$. That is, every submodule is a direct summand of its submodules [DF91].

Lemma 4.23. *Let R be an arbitrary nonzero ring. If M and N are simple R-modules and $\varphi : M \to N$ is a nonzero R-module homomorphism, then φ is an isomorphism [DF91].*

Lemma 4.24. *Schur's Lemma.*

Let R be a nonzero ring. If M is a simple R-module, then $Hom_R(M, M)$ is a division ring, where $Hom_R(M, M)$ is the set of all R-module homomorphisms from M into M [DF91].

Theorem 4.25. *The categories $\mathcal{R}(Q)$ and KQ are equivalent, where $\mathcal{R}(C)$ is the category of representations and KQ is a path algebra according to Definition 4.15 [Erd90].*

Theorem 4.26. *Gabriel's Theorem.*

Any basic finite dimensional K-agebra is of the form KQ/I for a unique quiver Q and some ideal I with $(KQ^+)^n \subseteq I \subseteq (KQ^+)^2$, for some $n \geq 2$ [Erd90].

Theorem 4.27. *Let A be a KG-algebra where K is a field, G is a group, and A has an identity 1_A. Let Rad(A) denote the radical of A. Then $A/Rad(A)$ is semisimple [Erd90].*

Proof. Let A be an algebra and let $\text{Rad}(A)$ be the radical of A. To show that $A/\text{Rad}(A)$ is semisimple, we need to show that

$$A/\text{Rad}(A) \cong < a_1 > \oplus < a_2 > \oplus \cdots \oplus < a_n >,$$

where $< a_i >= \{aa_i \mid a \in A\}$ is an ideal of A. This is true if and only if $A/\text{Rad}(A) \cong k_1 \oplus k_2 \oplus \cdots \oplus k_n$ where k_i are fields, $i = 0, ..., n$ and $n \in \mathbb{N}$, such that $k_i = k$ for all i.

□

Chapter 5

A Look at the Module $\mathbb{Z}_2S_3$

5.1 The Group S_3.

Before we examine the module $\mathbb{Z}_2S_3$, let us first describe the elements of the permutation group in three letters, which is better-known as S_3. We will reference these permutations throughout this chapter as well as Chapter 6.

The even permutations of S_3:

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = e$$

$$\rho = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132)$$

$$\rho\prime = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$$

The odd permutations of S_3:

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (23)$$

$$\mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (13)$$

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)$$

So the elements of S_3 are $\{e, (12), (13), (23), (123), (132\}$.

Define the action between two elements of S_3 to be the normal multiplication between permutations. For reference, the products of the elements of S_3 are given below in Table 5.1:

Table 5.1: The Multiplication Table of S_3

	e	(12)	(13)	(23)	(123)	(132)
e	e	(12)	(13)	(23)	(123)	(132)
(12)	(12)	e	(132)	(123)	(23)	(13)
(13)	(13)	(123)	e	(132)	(12)	(23)
(23)	(23)	(132)	(123)	e	(13)	(12)
(123)	(123)	(13)	(23)	(12)	(132)	e
(132)	(132)	(23)	(12)	(13)	e	(123)

We will use these products throughout Chapters 5 and 6.

It will also be useful to define the subgroups of S_3 since they will also be referenced throughout Chapters 5 and 6.
The subgroups of S_3 are:

$$\begin{aligned} H_1 &= \{e, (12)\} \\ H_2 &= \{e, (13)\} \\ H_3 &= \{e, (23)\} \\ H_4 &= \{e, (123), (132)\} \end{aligned}$$

It is easy to prove that these are indeed subgroups of S_3, since each subset is closed under multiplication among permutations and each element has an inverse within the subset. For example, in the subset H_4:

$$e \text{ is its own inverse,}$$
$$(123)(132) = e, \text{ and}$$
$$(132)(123) = e,$$

making (132) the inverse element of (123) and (123) the inverse element of (132). Thus H_4 is a subgroup of S_3.

5.2 The Left Module $\mathbb{Z}_2S_3$.

Now let us take the ring of integers mod 2, $\mathbb{Z}_2 = \{[0], [1]\}$, and the group S_3 that was described above to form the left module $\mathbb{Z}_2S_3$.

First we will examine $\mathbb{Z}_2S_3$ as a left $\mathbb{Z}_2S_3$-module. The elements of the set $\mathbb{Z}_2S_3$ are formed using formal sums of the elements of $\mathbb{Z}_2$, which are $\{0, 1\}$, and the permutations of S_3 described in Section 5.1. This is to say that we are taking formal sums of the elements

$$\begin{array}{cccccc} 0e, & 0(12), & 0(13), & 0(23), & 0(123), & 0(132), \\ 1e, & 1(12), & 0(13), & 0(23), & 0(123), & 1(132). \end{array}$$

Then the elements of $\mathbb{Z}_2S_3$ are of the form:

$$a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132),$$

where $a_k \in \mathbb{Z}_2$. Since there are only two elements in $\mathbb{Z}_2$, the a_i are equal to either 0 or 1. For example, an element of the form $0e + 0(12) + 1(13) + 1(23) + 1(123) + 0(132)$ would belong to $\mathbb{Z}_2S_3$.

We make $\mathbb{Z}_2S_3$ into an Abelian group by defining an operation:

Addition " $+$ " is defined as:

$$\begin{aligned} a_ig_1 + a_jg_1 &= (a_i + a_j)g_1 \\ a_ig_1 + a_jg_2 &= a_ig_1 + a_jg_2, \end{aligned}$$

where the $a_i, a_j = 0$ or 1 since they are elements of $\mathbb{Z}_2$, and the g_k are elements of S_3.

We make the Abelian group $\mathbb{Z}_2S_3$ into a left $\mathbb{Z}_2S_3$-module by defining an action " $\cdot$ ": $\mathbb{Z}_2S_3 \times \mathbb{Z}_2S_3 \to \mathbb{Z}_2S_3$:

$$\begin{aligned} (a_ig_1) \cdot (a_jg_2) &= (a_i \cdot a_j)(g_1 \cdot g_2) \\ &= a_kg_3 \text{ where } a_k = a_i \cdot a_j \text{ and } g_3 = g_1 \cdot g_2, \end{aligned}$$

where $a_i, a_j, a_k \in \mathbb{Z}_2$ and $g_1, g_2, g_3 \in S_3$.

The additive identity in $\mathbb{Z}_2S_3$ is:

$$0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132).$$

The multiplicative identity in $\mathbb{Z}_2S_3$ is:

$$1e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132).$$

5.3 The Simplicity and Semisimplicity of the Module $\mathbb{Z}_2S_3$.

First let us examine $\mathbb{Z}_2S_3$ according to the structure of an RG-group ring. Recall from Chapter 4 that an RG-ring is a set of formal sums of the form

$$\sum_{g\in G} \alpha_g g, \alpha_g \in R.$$

Then we will be examining $\mathbb{Z}_2S_3$ as a $\mathbb{Z}_2S_3$-module.

We know from Example 2.24 that the characterisitc of $\mathbb{Z}_2$ is 2. The order of the group S_3 is found by calculating $3! = 6$. In this case, 2 does divide 6, so when we apply Theorem 4.22, which is better known as Machke's Theorem, we find that $\mathbb{Z}_2S_3$ is not semisimple as a $\mathbb{Z}_2S_3$-module.

Let us momentarily examine $\mathbb{Z}_2S_3$ as a $\mathbb{Z}_2$-module instead. Then we need to find simple submodules G_i of $\mathbb{Z}_2S_3$ as a $\mathbb{Z}_2$-module so that $\mathbb{Z}_2S_3$ may be written in the following way:

$$\mathbb{Z}_2S_3 \cong G_1 \oplus G_2 \oplus \cdots \oplus G_k$$

for some $k \in \mathbb{N}$.

As a $\mathbb{Z}_2$-module, we can represent $\mathbb{Z}_2S_3$ in the following way:

$$\mathbb{Z}_2S_3 \cong \mathbb{Z}_2e \oplus \mathbb{Z}_2(12) \oplus \mathbb{Z}_2(13) \oplus \mathbb{Z}_2(23) \oplus \mathbb{Z}_2(123) \oplus \mathbb{Z}_2(132).$$

Since each of the submodules $\mathbb{Z}_2e, \mathbb{Z}_2(12), \mathbb{Z}_2(13), \mathbb{Z}_2(23), \mathbb{Z}_2(123)$, and $\mathbb{Z}_2(132)$ are generated by a single element, they are all simple submodules. For example, the entire submodule $\mathbb{Z}_2(12)$ is generated by the element (12).

So $\mathbb{Z}_2S_3$ is semisimple as a $\mathbb{Z}_2$-module, since it can be written as the direct sum of simple submodules.

What then can we learn about $\mathbb{Z}_2S_3$ as a $\mathbb{Z}_2S_3$-module?

The number of elements in $\mathbb{Z}_2S_3$, written $\mid \mathbb{Z}_2S_3 \mid$, is found by calculating $2^6 = 64$. At the beginning of Section 4.3 we used Machke's Theorem, Theorem 4.22, to show that $\mathbb{Z}_2S_3$ is not semisimple as a $\mathbb{Z}_2S_3$-module. Therefore, it cannot be simple as a $\mathbb{Z}_2S_3$-module. What about the simplicity of $\mathbb{Z}_2S_3$ as a $\mathbb{Z}_2$-module? Earlier in Section 5.3 we showed that $\mathbb{Z}_2S_3$ is semisimple as a $\mathbb{Z}_2$-module.

Since $\mathbb{Z}_2S_3$ is finite, we know that it is Artinian. This means that it satisfies the Descending Chain Condition. If $\mathbb{Z}_2S_3$ is both cyclic and generated by every one of its elements then it is simple. If $\mathbb{Z}_2S_3$ is simple then its only submodules are 0 and M, as defined in Chapter 3. So we will examine the submodules of $\mathbb{Z}_2S_3$. In doing so, we will be able to determine whether or not $\mathbb{Z}_2S_3$ is simple as a $\mathbb{Z}_2S_3$-module.

5.4 Submodules of $\mathbb{Z}_2S_3$.

We will examine the submodules of $\mathbb{Z}_2S_3$. Recall the Submodule Criterion from Chapter 3: Let R be a ring and let M be a module. Then a subset N of M is a submodule of M if and only if

1. $N \neq \emptyset$, and
2. $x + \alpha y \in N$ for all $\alpha \in R$ and for all $x, y \in N$.

We will be using the field $\mathbb{Z}_2$ as R and the module $\mathbb{Z}_2S_3$ as M. Define a subset A of $\mathbb{Z}_2S_3$ where each component of an element shares the same coefficient from $\mathbb{Z}_2$. Then an arbitrary element α of A is of the form:

$$\alpha = ke + k(12) + k(13) + k(23) + k(123) + k(132),$$

where k is an element of $\mathbb{Z}_2$. Since $k \in \mathbb{Z}_2$, α of A is either of the form

$$\alpha = \begin{cases} 0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132), \text{ or} \\ 1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132) \end{cases}$$

For simplicity, define these two elements of A as:

$$\begin{aligned} \alpha_0 &= 0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132), \text{ and} \\ \alpha_1 &= 1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132). \end{aligned}$$

If A is in fact a $\mathbb{Z}_2$-submodule of $\mathbb{Z}_2S_3$, then it will satisfy the Submodule Criterion. First, we know that A is nonempty, since, by definition of A, α_0 and α_1 belong to A. Next, consider an element of the form $x + \beta y$, where $x = k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)$ and $y = ke + k(12) + k(13) + k(23) + k(123) + k(132)$ belong to A, and β is an element of $\mathbb{Z}_2$. Then

$$\begin{aligned} x + \beta y &= [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\ &\quad + \beta[ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\ &= [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\ &\quad + [(\beta k)e + (\beta k)(12) + (\beta k)(13) + (\beta k)(23) + (\beta k)(123) + (\beta k)(132)] \\ &= [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\ &\quad + [ce + c(12) + c(13) + c(23) + c(123) + c(132)], \text{ where } c = \beta k \in \mathbb{Z}_2 \\ &= (k' + c)e + (k' + c)(12) + (k' + c)(13) + (k' + c)(23) + (k' + c)(123) + (k' + c)(132) \\ &= de + d(12) + d(13) + d(23) + d(123) + d(132), \text{ where } d = k' + c \in \mathbb{Z}_2 \\ &\quad \in A, \text{ since each element of } S_3 \text{ shares the same coefficient } d \text{ from } \mathbb{Z}_2. \end{aligned}$$

Note that c and d are indeed elements of $\mathbb{Z}_2$ since $\mathbb{Z}_2$ is a field and is closed under both addition and multiplication. Additionally, $\mathbb{Z}_2$ is closed, so the product of β and y remains in $\mathbb{Z}_2$.

Therefore, by the Submodule Criterion, $A = \{\alpha_0, \alpha_1\}$ is a $\mathbb{Z}_2$-submodule of $\mathbb{Z}_2S_3$. By definition of a submodule, A is closed under addition and multiplication.

Since A is a proper submodule of $\mathbb{Z}_2S_3$, we know that $\mathbb{Z}_2S_3$ is not simple as a $\mathbb{Z}_2$-module. According to its definition in Chapter 3 the only submodules of a simple submodule are 0 and the module itself.

5.5 Generating $\mathbb{Z}_2S_3$.

Consider further evidence against the simplicity of $\mathbb{Z}_2S_3$ as a $\mathbb{Z}_2S_3$-module. If $\mathbb{Z}_2S_3$ is simple, then it will be generated by each of its elements. Then is it possible for $\mathbb{Z}_2S_3$ to be generated by each of its elements?
First let us consider $\mathbb{Z}_2S_3 \cdot [1e+0(12)+0(13)+0(23)+0(123)+0(132)]$. In this case, it is possible for $\mathbb{Z}_2S_3$ to be generated by the element $1e+0(12)+0(13)+0(23)+0(123)+0(132)$.

We will refer to the sum $1e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)$ as $1e$.

Then

$$\begin{aligned}
[1e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)] \cdot 1e &= 1e, \\
[0e + 1(12) + 0(13) + 0(23) + 0(123) + 0(132)] \cdot 1e &= 1(12), \\
[0e + 0(12) + 1(13) + 0(23) + 0(123) + 0(132)] \cdot 1e &= 1(13), \\
[0e + 0(12) + 0(13) + 1(23) + 0(123) + 0(132)] \cdot 1e &= 1(2e), \\
[0e + 0(12) + 0(13) + 0(23) + 1(123) + 0(132)] \cdot 1e &= 1(123), \\
[0e + 0(12) + 0(13) + 0(23) + 0(123) + 1(132)] \cdot 1e &= 1(132).
\end{aligned}$$

The remaining elements of $\mathbb{Z}_2S_3$ can be formed by sums of these products. For example,

$$[1e + 1(12) + 0(13) + 0(23) + 0(123) + 0(132)]1e = 1e + 1(12).$$

Other elements of $\mathbb{Z}_2S_3$, such as the element $1(13) + 1(123)$ can be formed in a similar fashion. So we can say that $\mathbb{Z}_2S_3 = \mathbb{Z}_2S_3 \cdot 1e$.

Now consider $\mathbb{Z}_2S_3 \cdot [0e + 1(12) + 0(13) + 0(23) + 0(123) + 0(132)]$. Is it possible to generate $\mathbb{Z}_2S_3$ by an element other than the multiplicative identity element?

We will refer to the sum $0e + 1(12) + 0(13) + 0(23) + 0(123) + 0(132)$ as $1(12)$.

Then

$$
\begin{aligned}
[1e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)]\, 1(12) &= 1(12) \\
[0e + 1(12) + 0(13) + 0(23) + 0(123) + 0(132)]\, 1(12) &= 1e \\
[0e + 0(12) + 1(13) + 0(23) + 0(123) + 0(132)]\, 1(12) &= 1(123) \\
[0e + 0(12) + 0(13) + 1(23) + 0(123) + 0(132)]\, 1(12) &= 1(132) \\
[0e + 0(12) + 0(13) + 0(23) + 1(123) + 0(132)]\, 1(12) &= 1(13) \\
[0e + 0(12) + 0(13) + 0(23) + 0(123) + 1(132)]\, 1(12) &= 1(23)
\end{aligned}
$$

Then the element $1(12)$ can also generate $\mathbb{Z}_2S_3$, meaning each element of $\mathbb{Z}_2S_3$ can be formed using combinations of the products above, similar to the way that they were formed in the previous example. So $\mathbb{Z}_2S_3 = \mathbb{Z}_2S_3 \cdot 1(12)$ as well.

Similarly, the elements $1(13), 1(23), 1(123)$, and $1(132)$ generate $\mathbb{Z}_2S_3$:

$$
\begin{aligned}
\mathbb{Z}_2S_3 = \mathbb{Z}_2S_3 \cdot 1(13) : [1e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)]\, 1(13) &= 1(13) \\
[0e + 1(12) + 0(13) + 0(23) + 0(123) + 0(132)]\, 1(13) &= 1(132) \\
[0e + 0(12) + 1(13) + 0(23) + 0(123) + 0(132)]\, 1(13) &= 1e \\
[0e + 0(12) + 0(13) + 1(23) + 0(123) + 0(132)]\, 1(13) &= 1(123) \\
[0e + 0(12) + 0(13) + 0(23) + 1(123) + 0(132)]\, 1(13) &= 1(23) \\
[0e + 0(12) + 0(13) + 0(23) + 0(123) + 1(132)]\, 1(13) &= 1(12)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{Z}_2S_3 = \mathbb{Z}_2S_3 \cdot 1(23) : [1e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)]\,1(23) &= 1(23) \\
[0e + 1(12) + 0(13) + 0(23) + 0(123) + 0(132)]\,1(23) &= 1(123) \\
[0e + 0(12) + 1(13) + 0(23) + 0(123) + 0(132)]\,1(23) &= 1(132) \\
[0e + 0(12) + 0(13) + 1(23) + 0(123) + 0(132)]\,1(23) &= 1e \\
[0e + 0(12) + 0(13) + 0(23) + 1(123) + 0(132)]\,1(23) &= 1(12) \\
[0e + 0(12) + 0(13) + 0(23) + 0(123) + 1(132)]\,1(23) &= 1(13)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{Z}_2S_3 = \mathbb{Z}_2S_3 \cdot 1(123) : [1e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)]\,1(123) &= 1(123) \\
[0e + 1(12) + 0(13) + 0(23) + 0(123) + 0(132)]\,1(123) &= 1(23) \\
[0e + 0(12) + 1(13) + 0(23) + 0(123) + 0(132)]\,1(123) &= 1(12) \\
[0e + 0(12) + 0(13) + 1(23) + 0(123) + 0(132)]\,1(123) &= 1(13) \\
[0e + 0(12) + 0(13) + 0(23) + 1(123) + 0(132)]\,1(123) &= 1(132) \\
[0e + 0(12) + 0(13) + 0(23) + 0(123) + 1(132)]\,1(123) &= 1e
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{Z}_2S_3 = \mathbb{Z}_2S_3 \cdot 1(132) : [1e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)]\,1(132) &= 1(132) \\
[0e + 1(12) + 0(13) + 0(23) + 0(123) + 0(132)]\,1(132) &= 1(13) \\
[0e + 0(12) + 1(13) + 0(23) + 0(123) + 0(132)]\,1(132) &= 1(23) \\
[0e + 0(12) + 0(13) + 1(23) + 0(123) + 0(132)]\,1(132) &= 1(12) \\
[0e + 0(12) + 0(13) + 0(23) + 1(123) + 0(132)]\,1(132) &= 1e \\
[0e + 0(12) + 0(13) + 0(23) + 0(123) + 1(132)]\,1(132) &= 1(123)
\end{aligned}
$$

It is the case that $\mathbb{Z}_2S_3$ can be generated by any of its single elements; $\mathbb{Z}_2S_3$ is cyclic. However, this is not enough to determine whether or not $\mathbb{Z}_2S_3$ is simple. In addition to being cyclic, it must also be generated by every one of its elements, not only the single elements $\{1e, 1(12), 1(13), 1(23), 1(123), 1(132)\}$.

5.6 Generating $\mathbb{Z}_2S_3$ from $1e+1(12)$.

Can we also generate $\mathbb{Z}_2S_3$ from the other, non-single elements? Let us try $1e+1(12)$:

$$\begin{aligned}
[1e+0(12)+0(13)+0(23)+0(123)+0(132)]\,[1e+1(12)] &= 1e+1(12)\\
[0e+1(12)+0(13)+0(23)+0(123)+0(132)]\,[1e+1(12)] &= 1(12)+1e\\
[0e+0(12)+1(13)+0(23)+0(123)+0(132)]\,[1e+1(12)] &= 1(13)+1(123)\\
[0e+0(12)+0(13)+1(23)+0(123)+0(132)]\,[1e+1(12)] &= 1(23)+1(132)\\
[0e+0(12)+0(13)+0(23)+1(123)+0(132)]\,[1e+1(12)] &= 1(123)+1(23)\\
[0e+0(12)+0(13)+0(23)+0(123)+1(132)]\,[1e+1(12)] &= 1(132)+1(13)
\end{aligned}$$

This is not injective, as the first two products equal the same sum, $1e+1(12)$. So $\mathbb{Z}_2S_3$ is not generated by the element $1e+1(12)$.

We may see further evidence that $\mathbb{Z}_2S_3$ is not generated by $1e+1(12)$ by examining the product of an arbitrary element from $\mathbb{Z}_2S_3$ and the element $1e+1(12)$.
Is it possible to obtain the element $1e$ as the product of $1e+1(12)$ and some element a of $\mathbb{Z}_2S_3$?
Suppose that this element a does exist in $\mathbb{Z}_2S_3$. Then a is of the form $a_1e+a_2(12)+a_3(13)+a_4(23)+a_5(123)+a_6(132)$ such that

$$a[1e+1(12)] = 1e+0(12)+0(13)+0(23)+0(123)+0(132).$$

Then,

$$\begin{aligned}
a[1e+1(12)] &= [a_1e+a_2(12)+a_3(13)+a_4(23)+a_5(123)+a_6(132)][1e+1(12)]\\
&= 1e+0(12)+0(13)+0(23)+0(123)+0(132)\\
&= a_1e+a_2(12)+a_3(13)+a_4(23)+a_5(123)+a_6(132)\\
&\quad +a_2e+a_1(12)+a_5(13)+a_6(23)+a_3(123)+a_4(132)\\
&= (a_1+a_2)e+(a_1+a_2)(12)+(a_3+a_5)(13)+(a_4+a_6)(23)\\
&+ (a_3+a_5)(123)+(a_6+a_4)(132)
\end{aligned}$$

According to the structure of the element a, we know that $a_i = 0$ for $i = 3, 4, 5$, and 6, or $a_i = 1$ for $i = 3, 4, 5$, and 6, since $(a_3+a_5) = 0, (a_4+a_6) = 0, (a_3+a_5) = 0$, and $(a_6+a_4) = 0$. More importantly, since this sum is equal to $1e+0(12)+0(13)+0(23)+0(123)+0(132)$, we may equate the a_i and conclude that

$$(a_1 + a_2) = 1 \text{ and } (a_2 + a_1) = 0.$$

However, this is a contradiction, since $\mathbb{Z}_2S_3$ is closed under addition and the sum of the elements a_1 and a_2 cannot be equal to both 1 and 0.
We may then conclude that there does not exist an element $a \in \mathbb{Z}_2S_3$ such that $a[1e + 1(12)] = 1e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)$. This means that $1e + 1(12)$ does not generate $\mathbb{Z}_2S_3$. We know that $\mathbb{Z}_2S_3$ is generated by six of its sixty-four elements. However $\mathbb{Z}_2S_3$ is not generated by all of its elements. In order to be a simple module, $\mathbb{Z}_2S_3$ must be generated by each of its elements.

Finally we may conclude that $\mathbb{Z}_2S_3$ is not simple.
Consider the following proposition, which follows as a consequence of the properties we examined above.

Proposition 5.1. *If a module is generated by a one-element it does not follow that the module is simple. A module is simple if and only if it is generated by each of its elements.*

5.7 Other Submodules of $\mathbb{Z}_2S_3$.

Let us return to examining the submodules of $\mathbb{Z}_2S_3$ which we began to examine in Section 5.4. We proved in Section 5.4 that the set $A = \{0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132), 1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)\}$ was a $\mathbb{Z}_2$-submodule of $\mathbb{Z}_2S_3$.

Now consider that same subset A of $\mathbb{Z}_2S_3$. Is A a $\mathbb{Z}_2S_3$-submodule of $\mathbb{Z}_2S_3$? Again, we will apply the Submodule Criterion from Section 5.4:
First, we know that A is nonempty, since $A = \{0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132), 1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)\}$.

Now let $x, y \in A$ and let $\beta = b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) \in \mathbb{Z}_2S_3$.

According to the Submodule Criterion, let us show that $x + \beta y$ belongs to A.

$$
\begin{aligned}
& x + \beta y \\
= \ & [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\
& +[b_1 e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132)] \cdot [ke + k(12) + k(13) \\
& +k(23) + k(123) + k(132)] \\
= \ & [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\
& +b_1 e[ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\
& +b_2(12)[ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\
& +b_3(13)[ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\
& +b_4(23)[ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\
& +b_5(123)[ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\
& +b_6(132)[ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\
= \ & [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\
& +[b_1k][ee] + [b_1k][e(12)] + [b_1k][e(13)] + [b_1k][e(23)] + [b_1k][e(123)] + [b_1k][e(132)] \\
& +[b_2k][(12)e] + [b_2k][(12)(12)] + [b_2k][(12)(13)] + [b_2k][(12)(23)] + [b_2k][(12)(123)] \\
& +[b_2k][(12)(132)] \\
& +[b_3k][(13)e] + [b_3k][(13)(12)] + [b_3k][(13)(13)] + [b_3k][(13)(23)] + [b_3k][(13)(123)] \\
& +[b_3k][(13)(132)] \\
& +[b_4k][(23)e] + [b_4k][(23)(12)] + [b_4k][(23)(13)] + [b_4k][(23)(23)] + [b_4k][(23)(123)] \\
& +[b_4k][(23)(132)] \\
& +[b_5k][(123)e] + [b_5k][(123)(12)] + [b_5k][(123)(13)] + [b_5k][(123)(23)] \\
& +[b_5k][(123)(123)] + [b_5k][(123)(132)] \\
& +[b_6k][(132)e] + [b_6k][(132)(12)] + [b_6k][(132)(13)] + [b_6k][(132)(23)] \\
& +[b_6k][(132)(123)] + [b_6k][(132)(132)] \\
= \ & [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\
& +[b_1k]e + [b_1k](12) + [b_1k](13) + [b_1k](23) + [b_1k](123) + [b_1k](132) \\
& +[b_2k](12) + [b_2k]e + [b_2k](132) + [b_2k](123) + [b_2k](23) + [b_2k](13) \\
& +[b_3k](13) + [b_3k](123) + [b_3k]e + [b_3k](132) + [b_3k](12) + [b_3k](23)
\end{aligned}
$$

$$
\begin{aligned}
&+[b_4k](23)+[b_4k](132)+[b_4k](123)+[b_4k]e+[b_4k](13)+[b_4k](12)] \\
&+[b_5k](123)+[b_5k](13)+[b_5k](23)+[b_5k](12)+[b_5k](132)+[b_5k]e \\
&+[b_6k](132)+[b_6k](23)+[b_6k](12)+[b_6k](13)+[b_6k]e+[b_6k](123) \\
= \ & [k'e+k'(12)+k'(13)+k'(23)+k'(123)+k'(132)] \\
&+[b_1k]e+[b_1k](12)+[b_1k](13)+[b_1k](23)+[b_1k](123)+[b_1k](132) \\
&+[b_2k]e+[b_2k](12)+[b_2k](13)+[b_2k](23)+[b_2k](123)+[b_2k](132) \\
&+[b_3k]e+[b_3k](12)+[b_3k](13)+[b_3k](23)+[b_3k](123)+[b_3k](132) \\
&+[b_4k]e+[b_4k](12)+[b_4k](13)+[b_4k](23)+[b_4k](123)+[b_4k](132) \\
&+[b_5k]e+[b_5k](12)+[b_5k](13)+[b_5k](23)+[b_5k](123)+[b_5k](132) \\
&+[b_6k]e+[b_6k](12)+[b_6k](13)+[b_6k](23)+[b_6k](123)+[b_6k](132) \\
= \ & [k'e+k'(12)+k'(13)+k'(23)+k'(123)+k'(132)] \\
&+[b_1k+b_2k+b_3k+b_4k+b_5k+b_6k]e+[b_1k+b_2k+b_3k+b_4k+b_5k+b_6k](12) \\
&+[b_1k+b_2k+b_3k+b_4k+b_5k+b_6k](13)+[b_1k+b_2k+b_3k+b_4k+b_5k+b_6k](23) \\
&[b_1k+b_2k+b_3k+b_4k+b_5k+b_6k](123)+[b_1k+b_2k+b_3k+b_4k+b_5k+b_6k](132) \\
= \ & [k'e+k'(12)+k'(13)+k'(23)+k'(123)+k'(132)] \\
&+k[b_1+b_2+b_3+b_4+b_5+b_6]e+k[b_1+b_2+b_3+b_4+b_5+b_6](12) \\
&+k[b_1+b_2+b_3+b_4+b_5+b_6](13)+k[b_1+b_2+b_3+b_4+b_5+b_6](23) \\
&+k[b_1+b_2+b_3+b_4+b_5+b_6](123)+k[b_1+b_2+b_3+b_4+b_5+b_6](132) \\
= \ & k'e+k[b_1+b_2+b_3+b_4+b_5+b_6]e \\
&+k'(12)+k[b_1+b_2+b_3+b_4+b_5+b_6](12) \\
&+k'(13)+k[b_1+b_2+b_3+b_4+b_5+b_6](13) \\
&+k'(23)+k[b_1+b_2+b_3+b_4+b_5+b_6](23) \\
&+k'(123)+k[b_1+b_2+b_3+b_4+b_5+b_6](123) \\
&+k'(132)+k[b_1+b_2+b_3+b_4+b_5+b_6](132)
\end{aligned}
$$

Now $k \in \mathbb{Z}_2$, so $k = 0$ or $k = 1$. If $k = 0$, then we are left with

$$\begin{aligned}
&= k'e + 0[b_1+b_2+b_3+b_4+b_5+b_6]e + k'(12) + 0[b_1+b_2+b_3+b_4+b_5+b_6](12) \\
&\quad +k'(13) + 0[b_1+b_2+b_3+b_4+b_5+b_6](13) \\
&\quad +k'(23) + 0[b_1+b_2+b_3+b_4+b_5+b_6](23) \\
&\quad +k'(123) + 0[b_1+b_2+b_3+b_4+b_5+b_6](123) \\
&\quad +k'(132) + 0[b_1+b_2+b_3+b_4+b_5+b_6](132) \\
&= k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132) \\
&= 0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132) \text{ or} \\
&\quad 1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132) \\
&\quad \in A \text{ by definition of } A.
\end{aligned}$$

However, if $k = 1$ we have

$$\begin{aligned}
&= k'e + 1[b_1+b_2+b_3+b_4+b_5+b_6]e + k'(12) + 1[b_1+b_2+b_3+b_4+b_5+b_6](12) \\
&\quad +k'(13) + 1[b_1+b_2+b_3+b_4+b_5+b_6](13) \\
&\quad +k'(23) + 1[b_1+b_2+b_3+b_4+b_5+b_6](23) \\
&\quad +k'(123) + 1[b_1+b_2+b_3+b_4+b_5+b_6](123) \\
&\quad +k'(132) + 1[b_1+b_2+b_3+b_4+b_5+b_6](132) \\
&= k'e + [b_1+b_2+b_3+b_4+b_5+b_6]e + k'(12) + [b_1+b_2+b_3+b_4+b_5+b_6](12) \\
&\quad +k'(13) + [b_1+b_2+b_3+b_4+b_5+b_6](13) \\
&\quad +k'(23) + [b_1+b_2+b_3+b_4+b_5+b_6](23) \\
&\quad +k'(123) + [b_1+b_2+b_3+b_4+b_5+b_6](123) \\
&\quad +k'(132) + [b_1+b_2+b_3+b_4+b_5+b_6](132) \\
&= [k'+b_1+b_2+b_3+b_4+b_5+b_6]e + [k'+b_1+b_2+b_3+b_4+b_5+b_6](12) \\
&\quad +[k'+b_1+b_2+b_3+b_4+b_5+b_6](13) + [k'+b_1+b_2+b_3+b_4+b_5+b_6](23) \\
&\quad +[k'+b_1+b_2+b_3+b_4+b_5+b_6](123) + [k'+b_1+b_2+b_3+b_4+b_5+b_6](132)
\end{aligned}$$

It is important to note that each of the elements of S_3 has the same coefficients from $\mathbb{Z}_2$: $[k'+b_1+b_2+b_3+b_4+b_5+b_6]$. Then all that remains is to evaluate $[k'+b_1+b_2+b_3+$

$b_4 + b_5 + b_6]$. Since k' and each $b_i \in \mathbb{Z}_2$, the k' and b_i's are either equal to 0 or $1 \in \mathbb{Z}_2$. Then the sum of every coefficient could be equal to $0, 1, 2, 3, 4, 5, 6$, or 7. However, this sum is an element of $\mathbb{Z}_2$, so it will be congruent to either $0(mod2)$ or $1(mod2)$. Thus the sum $[k' + b_1 + b_2 + b_3 + b_4 + b_5 + b_6]$ is either equal to 0 or 1 and we have:

$$\begin{aligned} &= [k' + b_1 + b_2 + b_3 + b_4 + b_5 + b_6]e + [k' + b_1 + b_2 + b_3 + b_4 + b_5 + b_6](12) \\ &\quad +[k' + b_1 + b_2 + b_3 + b_4 + b_5 + b_6](13) + [k' + b_1 + b_2 + b_3 + b_4 + b_5 + b_6](23) \\ &\quad +[k' + b_1 + b_2 + b_3 + b_4 + b_5 + b_6](123) + [k' + b_1 + b_2 + b_3 + b_4 + b_5 + b_6](132) \\ &= 0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132), \end{aligned}$$

or

$$\begin{aligned} &= [k' + b_1 + b_2 + b_3 + b_4 + b_5 + b_6]e + [k' + b_1 + b_2 + b_3 + b_4 + b_5 + b_6](12) \\ &\quad +[k' + b_1 + b_2 + b_3 + b_4 + b_5 + b_6](13) + [k' + b_1 + b_2 + b_3 + b_4 + b_5 + b_6](23) \\ &\quad +[k' + b_1 + b_2 + b_3 + b_4 + b_5 + b_6](123) + [k' + b_1 + b_2 + b_3 + b_4 + b_5 + b_6](132) \\ &= 1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132). \end{aligned}$$

In either case, these elements belong to A since each element of S_3 has the same coefficient from $\mathbb{Z}_2 S_3$. Then A satisfies the Submodule Criterion and is a $\mathbb{Z}_2 S_3$-submodule of $\mathbb{Z}_2 S_3$.

We will attempt to find another submodule of $\mathbb{Z}_2 S_3$, different from A. This elements of this submodule have the form $A = \{a_1 e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132) \mid a_i = a_j$ for all $i, j\}$. Now we will test other sets to find $\mathbb{Z}_2$-submodules of $\mathbb{Z}_2 S_3$.

Define B to be a subset of $\mathbb{Z}_2 S_3$ such that

$$B = \{a_1 e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132) \mid \sum_{i=1}^{6} a_i = 1\}.$$

We may apply the Submodule Criterion to B. Then let $x = a_1 e + a_2(12) + a_e(13) + a_4(23) + a_5(123) + a_6(132)$ and $y = b_1 e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) \in B$ and let $\gamma = c_1 e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132) \in \mathbb{Z}_2 S_3$. We apply the

Submodule Criterion:

$$
\begin{aligned}
& x+\gamma y \\
= \; & [a_1 e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] \\
& + [c_1 e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132)] \cdot [b_1 e + b_2(12) + b_3(13) \\
& + b_4(23) + b_5(123) + b_6(132)] \\
= \; & [a_1 e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] \\
& + c_1 e [b_1 e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132)] \\
& + c_2(12) [b_1 e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132)] \\
& + c_3(13) [b_1 e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132)] \\
& + c_4(23) [b_1 e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132)] \\
& + c_5(123) [b_1 e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132)] \\
& + c_6(132) [b_1 e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132)] \\
= \; & [a_1 e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] \\
& + [c_1 b_1][ee] + [c_1 b_2][e(12)] + [c_1 b_3][e(13)] + [c_1 b_4][e(23)] + [c_1 b_5][e(123)] + [c_1 b_6][e(132)] \\
& + [c_2 b_1][(12)e] + [c_2 b_2][(12)(12)] + [c_2 b_3][(12)(13)] + [c_2 b_4][(12)(23)] \\
& + [c_2 b_5][(12)(123)] + [c_2 b_6][(12)(132)] \\
& + [c_3 b_1][(13)e] + [c_3 b_2][(13)(12)] + [c_3 b_3][(13)(13)] + [c_3 b_4][(13)(23)] \\
& + [c_3 b_5][(13)(123)] + [c_3 b_6][(13)(132)] \\
& + [c_4 b_1][(23)e] + [c_4 b_2][(23)(12)] + [c_4 b_3][(23)(13)] + [c_4 b_4][(23)(23)] \\
& + [c_4 b_5][(23)(123)] + [c_4 b_6][(23)(132)] \\
& + [c_5 b_1][(123)e] + [c_5 b_2][(123)(12)] + [c_5 b_3][(123)(13)] + [c_5 b_4][(123)(23)] \\
& + [c_5 b_5][(123)(123)] + [c_5 b_6][(123)(132)] \\
& + [c_6 b_1][(132)e] + [c_6 b_2][(132)(12)] + [c_6 b_3][(132)(13)] + [c_6 b_4][(132)(23)] \\
& + [c_6 b_5][(132)(123)] + [c_6 b_6][(132)(132)]
\end{aligned}
$$

$$
\begin{aligned}
&= [a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] \\
&\quad +[c_1b_1]e + [c_1b_2](12) + [c_1b_3](13) + [c_1b_4](23) + [c_1b_5](123) + [c_1b_6](132) \\
&\quad +[c_2b_1](12) + [c_2b_2]e + [c_2b_3](132) + [c_2b_4](123) + [c_2b_5](23) + [c_2b_6](13) \\
&\quad +[c_3b_1](13) + [c_3b_2](123) + [c_3b_3]e + [c_3b_4](132) + [c_3b_5](12) + [c_3b_6](23) \\
&\quad +[c_4b_1](23) + [c_4b_2](132) + [c_4b_3](123) + [c_4b_4]e + [c_4b_5](13) + [c_4b_6](12) \\
&\quad +[c_5b_1](123) + [c_5b_2](13) + [c_5b_3](23) + [c_5b_4](12) + [c_5b_5](132) + [c_5b_6]e \\
&\quad +[c_6b_1](132) + [c_6b_2](23) + [c_6b_3](12) + [c_6b_4](13) + [c_6b_5]e + [c_6b_6](123) \\
&= [a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] \\
&\quad +[c_1b_1]e + [c_1b_2](12) + [c_1b_3](13) + [c_1b_4](23) + [c_1b_5](123) + [c_1b_6](132) \\
&\quad +[c_2b_2]e + [c_2b_1](12) + [c_2b_6](13) + [c_2b_5](23) + [c_2b_4](123) + [c_2b_3](132) \\
&\quad +[c_3b_3]e + [c_3b_5](12) + [c_3b_1](13) + [c_3b_6](23) + [c_3b_2](123) + [c_3b_4](132) \\
&\quad +[c_4b_4]e + [c_4b_6](12) + [c_4b_5](13) + [c_4b_1](23) + [c_4b_3](123) + [c_4b_2](132) \\
&\quad +[c_5b_6]e + [c_5b_4](12) + [c_5b_2](13) + [c_5b_3](23) + [c_5b_1](123) + [c_5b_5](132) \\
&\quad +[c_6b_5]e + [c_6b_3](12) + [c_6b_4](13) + [c_6b_2](23) + [c_6b_6](123) + [c_6b_1](132) \\
&= [a_1 + c_1b_1 + c_2b_2 + c_3b_3 + c_4b_4 + c_5b_6 + c_6b_5]e \\
&\quad +[a_2 + c_1b_2 + c_2b_1 + c_3b_5 + c_4b_6 + c_5b_4 + c_6b_3](12) \\
&\quad +[a_3 + c_1b_3 + c_2b_6 + c_3b_1 + c_4b_5 + c_5b_2 + c_6b_4](13) \\
&\quad +[a_4 + c_1b_4 + c_2b_5 + c_3b_6 + c_4b_1 + c_5b_3 + c_6b_2](23) \\
&\quad +[a_5 + c_1b_5 + c_2b_4 + c_3b_2 + c_4b_3 + c_5b_1 + c_6b_6](123) \\
&\quad = [a_6 + c_1b_6 + c_2b_3 + c_3b_4 + c_4b_2 + c_5b_5 + c_6b_1](132)
\end{aligned}
$$

Again, each of these coefficients is in $\mathbb{Z}_2$. If they belong to B then their sums will be congruent to $1(mod2)$. This presents a significant number of cases to test the values of the a_i, b_j and c_k as well as their different combinations.

Additionally, note that the element $0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132) \notin B$, since at least one of the a_i's must be 1 in order for their sum to equal 1. Then B cannot be a submodule of $\mathbb{Z}_2S_3$. In fact, there are no other submodules of $\mathbb{Z}_2S_3$.

We know that $\mathbb{Z}_2S_3$ is not a simple $\mathbb{Z}_2$-module and that A is a submodule of $\mathbb{Z}_2S_3$.

Earlier we showed that $\mathbb{Z}_2S_3$ is semisimple as a $\mathbb{Z}_2$-module.

What else can we learn from the module $\mathbb{Z}_2S_3$?

5.8 The Radical of $\mathbb{Z}_2S_3$.

Recall from Definition 3.11 that the radical of a module is defined to be the intersection of the maximal submodules of M. The radical of a module is also defined to be the largest nilpotent ideal of the ring M. Recall that, as we did in Theorem 4.21, we may examine $\mathbb{Z}_2S_3$ as either a ring or as a module.

The ideal A is nilpotent. This means that there exists an $n \in \mathbb{N}$ such that $A^n = 0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)$. To see this, examine the elements α_0 and α_1 of A.

Consider each element of A. We can see that $\alpha_0 = 0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)$ is nilpotent, since $a_i = 0$ for all a_i.
Consider the case where $\alpha = \alpha_1$. Then

$$
\begin{aligned}
& (\alpha_1)^2 \\
= \; & [1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)][1e + 1(12) + 1(13) + 1(23) \\
& +1(123) + 1(132)] \\
= \; & 1e[1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)] \\
& +1(12)[1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)] \\
& +1(13)[1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)] \\
& +1(23)[1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)] \\
& +1(123)[1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)] \\
& +1(132)[1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)]
\end{aligned}
$$

$$\begin{aligned}
&= [1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)] + [1(12) + 1e + 1(132) \\
&\quad +1(123) + 1(23) + 1(13)] \\
&\quad +[1(13) + 1(123) + 1e + 1(132) + 1(12) + 1(23)] + [1(23) + 1(132) + 1(123) \\
&\quad +1e + 1(13) + 1(12)] \\
&\quad +[1(123) + 1(13) + 1(23) + 1(12) + 1(132) + 1e] + [1(132) + 1(23) + 1(12) \\
&\quad +1(13) + 1e + 1(123)] \\
&= 6e + 6(12) + 6(13) + 6(23) + 6(123) + 6(132) \\
&= 0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132).
\end{aligned}$$

So α_1 is nilpotent, since $(\alpha_1)^2 = 0$.
Thus, we may square the entire ideal $A^2 = 0$, and A is in fact nilpotent. In fact, A is the largest nilpotent ideal, since it's the unique $\mathbb{Z}_2$-submodule of $\mathbb{Z}_2S_3$. According to Definition 2.35, A is the radical of $\mathbb{Z}_2S_3$, or $Rad(\mathbb{Z}_2S_3) = A$.

The radical of a module is also defined to be the intersection of the maximal submodules. Then the intersection of the maximal submodules of $\mathbb{Z}_2S_3$ is equal to A.

5.9 The Submodules of $\mathbb{Z}_2S_3/Rad(\mathbb{Z}_2S_3)$.

Now recall the set $A = \{\alpha_0, \alpha_1\}$. Consider the quotient module $\mathbb{Z}_2S_3/Rad(\mathbb{Z}_2S_3) = \mathbb{Z}_2S_3/A$. We can find the number of elements of $\mathbb{Z}_2S_3/A$:

$$\begin{aligned}
|\mathbb{Z}_2S_3/A| &= |\mathbb{Z}_2S_3|/|A| \\
&= 64/2 \\
&= 32.
\end{aligned}$$

Is $\mathbb{Z}_2S_3/A$ semisimple? If it is semisimple, then it can be represented as a direct sum of simple submodules.

Let us form a sumbodule of $\mathbb{Z}_2S_3/A$ with the set $M = \{[a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] + A \mid \sum_{i=0}^{6} a_i = 0\}$.

For example, the following elements are in M, since the sum of their coefficients equal zero:

$$1e + 1(12) + 0(13) + 0(23) + 0(123) + 0(132) + A,$$

$$0e + 0(12) + 1(13) + 1(23) + 1(123) + 1(132) + A,$$

Notice that the elements of M have an even number of components whose coefficient a_i is 1. That is, each element of M has either

$$\begin{aligned} \text{zero } a_i = & \; 1, \\ \text{two } a_i = a_j = & \; 1, \\ \text{four } a_i = a_j = a_k = a_\ell = & \; 1, \\ \text{or six } a_i = a_j = a_k = a_\ell = a_n = a_m = & \; 1. \end{aligned}$$

Then what are the elements of M?

There is only one element in M with zero $a_i = 1$:

$$0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132) + A$$

$$\sum_{i=0}^{6} a_i = 0 + 0 + 0 + 0 + 0 + 0 = 0.$$

Now we will give the elements with two $a_i = a_j = 1$. Consider the following fifteen elements:

$$1e + 1(12) + 0(13) + 0(23) + 0(123) + 0(132) + A,$$

$$1e + 0(12) + 1(13) + 0(23) + 0(123) + 0(132) + A,$$

$$1e + 0(12) + 0(13) + 1(23) + 0(123) + 0(132) + A,$$

$$1e + 0(12) + 0(13) + 0(23) + 1(123) + 0(132) + A,$$

$$1e + 0(12) + 0(13) + 0(23) + 0(123) + 1(132) + A,$$

$$0e + 1(12) + 1(13) + 0(23) + 0(123) + 0(132) + A,$$

$$0e + 1(12) + 0(13) + 1(23) + 0(123) + 0(132) + A,$$

$$0e + 1(12) + 0(13) + 0(23) + 1(123) + 0(132) + A,$$

$$0e + 1(12) + 0(13) + 0(23) + 0(123) + 1(132) + A,$$

$$0e + 0(12) + 1(13) + 1(23) + 0(123) + 0(132) + A,$$

$$0e + 0(12) + 1(13) + 0(23) + 1(123) + 0(132) + A,$$

$$0e + 0(12) + 1(13) + 0(23) + 0(123) + 1(132) + A,$$

$$0e + 0(12) + 0(13) + 1(23) + 1(123) + 0(132) + A,$$

$$0e + 0(12) + 0(13) + 1(23) + 0(123) + 1(132) + A,$$

$$0e + 0(12) + 0(13) + 0(23) + 1(123) + 1(132) + A.$$

Without loss of generality, $\sum_{i=0}^{6} a_i = 1 + 1 + 0 + 0 + 0 + 0 = 0$ is in $\mathbb{Z}_2$ for each of the fifteen elements listed above.

So far we have found sixteen elements of M, a subset of $\mathbb{Z}_2S_3/A$. However, M is defined similar to the way in which a quotient ideal is defined. That is, each element in M is added to the submodule $A = \{\alpha_0, \alpha_1\}$ of $\mathbb{Z}_2S_3$. This means that the element $0e+0(12)+0(13)+0(23)+0(123)+0(132)+A$ equals the element $1e+1(12)+1(13)+1(23)+1(123)+1(132)+A$, which is also an element of M since $\sum_{i=0}^{6} a_i = 1+1+1+1+1+1 = 0$ in $\mathbb{Z}_2$. Finding the elements with two $a_i = a_j = 1$ as we did above will automatically yield the elements with four $a_i = a_j = a_k = a_\ell = 1$ in the sense of equality on this quotient module. Thus, there is no need to list these additional elements, as we have already found equal elements.

Then we have found sixteen elements which belong to the subset M of $\mathbb{Z}_2S_3/A$. At the beginning of Section 5.9 we showed that there are thirty-two elements in $\mathbb{Z}_2S_3/A$.

Is it then the case that the remaining sixteen elements of $\mathbb{Z}_2S_3/A$ form another set N such that $N = \{[a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] + A \mid \sum_{i=0}^{6} a_i = 1\}$? For example, the elements

$$1e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132) + A, \text{ and}$$

$$0e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132) + A$$

belong to N since the sum of their coefficients equals one.

In the same way that the elements of M have an even number of components whose coefficient a_i is equal to one, the elements of N will have an odd number of components whose coefficient a_i is equal to one. That is, each element of N has either

$$\begin{aligned} \text{one } a_i = &\ 1, \\ \text{three } a_i = a_j = a_k = &\ 1, \\ \text{or five } a_i = a_j = a_k = a_\ell = a_n = &\ 1. \end{aligned}$$

The following are the elements of N.

$$\begin{gathered}
1e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132) + A, \\
0e + 1(12) + 0(13) + 0(23) + 0(123) + 0(132) + A, \\
0e + 0(12) + 1(13) + 0(23) + 0(123) + 0(132) + A, \\
0e + 0(12) + 0(13) + 1(23) + 0(123) + 0(132) + A, \\
0e + 0(12) + 0(13) + 0(23) + 1(123) + 0(132) + A, \\
0e + 0(12) + 0(13) + 0(23) + 0(123) + 1(132) + A, \\
1e + 1(12) + 1(13) + 0(23) + 0(123) + 0(132) + A, \\
1e + 1(12) + 0(13) + 1(23) + 0(123) + 0(132) + A, \\
1e + 1(12) + 0(13) + 0(23) + 1(123) + 0(132) + A, \\
1e + 1(12) + 0(13) + 0(23) + 0(123) + 1(132) + A, \\
0e + 1(12) + 1(13) + 1(23) + 0(123) + 0(132) + A, \\
0e + 1(12) + 1(13) + 0(23) + 1(123) + 0(132) + A, \\
0e + 1(12) + 1(13) + 0(23) + 0(123) + 1(132) + A, \\
0e + 0(12) + 1(13) + 1(23) + 1(123) + 0(132) + A, \\
0e + 0(12) + 1(13) + 1(23) + 0(123) + 1(132) + A, \\
0e + 0(12) + 0(13) + 1(23) + 1(123) + 1(132) + A.
\end{gathered}$$

Let us return to the subset M that we defined earlier. Does M form a $\mathbb{Z}_2S_3$-submodule of $\mathbb{Z}_2S_3/A$?

For some $m_1 \neq m_2 \in \mathbb{Z}_2S_3$, let $m_1 + A, m_2 + A$ be elements of the subset M such that $m_1 + A \neq m_2 + A$. This implies that $m_1 - m_2 \notin A$. We want to show that M is closed under addition by showing that $(m_1 + m_2) + A \in M$. As elements of $\mathbb{Z}_2S_3$, m_1 is of the form $a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)$. Similarly, m_2 is of the form $a'_1e + a'_2(12) + a'_3(13) + a'_4(23) + a'_5(123) + a'_6(132)$.

Consider the following case: without loss of generality, suppose that two coefficients a_i and a_j of $m_1 + A$ are each equal to 1 such that the remaining $a_k = 0$, for all $k \neq j \neq i$. Then, according to the structure of M, either $a'_i = 1$ or $a'_i = 0$ in $m_2 + A$.

Without loss of generality, consider the case where $a'_i = 1$. Then there will be only one other coefficient $a'_k = 1$, for $k \neq i$. If it happens that $a'_j = 1$, that is that $j = k$, then $m_1 = m_2$ and $m_1 + A = m_2 + A$. This is a contradiction. Then a'_j must be equal to 0 and some other a_k must be 1, for some $k \neq j \neq i$. Then $m_1 + m_2$ has a zero at $a_i + a'_i = 1 + 1 = 0$. Additionally, $a_j + a'_j = 1 + 0 = 1$ and $a_k + a'_k = 0 + 1 = 1$ implies that $(a_j + a'_j) + (a_k + a'_k) = 1 + 1 = 0$ in $\mathbb{Z}_2$. The sums of all remaining a_ℓ and a'_ℓ are zero, where $\ell \neq i \neq j \neq k$. Therefore, $m_1 + A + m_2 + A \in M$ and M is closed under addition.

To show that M is a submodule of $\mathbb{Z}_2S_3/A$, it remains to show that M is closed under the multiplication of elements of $\mathbb{Z}_2S_3/A$.

Let $r \in \mathbb{Z}_2S_3/A$ and let $m \in M$. Then $r = a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132) + A$ and $m = b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) + A$, where $\sum_{i=0}^{6} b_i = 0$. Consider

$$\begin{aligned}
& rm \\
= \; & [a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132) + A][b_1e + b_2(12) + b_3(13) \\
& + b_4(23) + b_5(123) + b_6(132) + A]
\end{aligned}$$

$$
\begin{aligned}
&= a_1e[b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) + A] \\
&\quad +a_2(12)[b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) + A] \\
&\quad +a_3(13)[b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) + A] \\
&\quad +a_4(23)[b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) + A] \\
&\quad +a_5(123)[b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) + A] \\
&\quad +a_6(132)[b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) + A] \\
&= [a_1b_1][ee] + [a_1b_2][e(12)] + [a_1b_3][e(13)] + [a_1b_4][e(23)] + [a_1b_5][e(123)] + [a_1b_6][e(132)] \\
&\quad +[a_2b_1][(12)e] + [a_2b_2][(12)(12)] + [a_2b_3][(12)(13)] + [a_2b_4][(12)(23)] \\
&\quad +[a_2b_5][(12)(123)] + [a_2b_6][(12)(132)] \\
&\quad +[a_3b_1][(13)e] + [a_3b_2][(13)(12)] + [a_3b_3][(13)(13)] + [a_3b_4][(13)(23)] \\
&\quad +[a_3b_5][(13)(123)] + [a_3b_6][(13)(132)] \\
&\quad +[a_4b_1][(23)e] + [a_4b_2][(23)(12)] + [a_4b_3][(23)(13)] + [a_4b_4][(23)(23)] \\
&\quad +[a_4b_5][(23)(123)] + [a_4b_6][(23)(132)] \\
&\quad +[a_5b_1][(123)e] + [a_5b_2][(123)(12)] + [a_5b_3][(123)(13)] + [a_5b_4][(123)(23)] \\
&\quad +[a_5b_5][(123)(123)] + [a_5b_6][(123)(132)] \\
&\quad +[a_6b_1][(132)e] + [a_6b_2][(132)(12)] + [a_6b_3][(132)(13)] + [a_6b_4][(132)(23)] \\
&\quad +[a_6b_5][(132)(123)] + [a_6b_6][(132)(132)] \\
&= [a_1b_1]e + [a_1b_2](12) + [a_1b_3](13) + [a_1b_4](23) + [a_1b_5](123) + [a_1b_6](132) \\
&\quad +[a_2b_1](12) + [a_2b_2]e + [a_2b_3](132) + [a_2b_4](123) + [a_2b_5](23) + [a_2b_6](13) \\
&\quad +[a_3b_1](13) + [a_3b_2](123) + [a_3b_3]e + [a_3b_4](132) + [a_3b_5](12) + [a_3b_6](23) \\
&\quad +[a_4b_1](23) + [a_4b_2](132) + [a_4b_3](123) + [a_4b_4]e + [a_4b_5](13) + [a_4b_6](12) \\
&\quad +[a_5b_1](123) + [a_5b_2](13) + [a_5b_3](23) + [a_5b_4](12) + [a_5b_5](132) + [a_5b_6]e \\
&\quad +[a_6b_1](132) + [a_6b_2](23) + [a_6b_3](12) + [a_6b_4](13) + [a_6b_5]e + [a_6b_6](123)
\end{aligned}
$$

$$
\begin{aligned}
&= [a_1b_1e + a_1b_2(12) + a_1b_3(13) + a_1b_4(23) + a_1b_5(123) + a_1b_6(132)] \\
&\quad +[a_2b_2e + a_2b_1(12) + a_2b_6(13) + a_2b_5(23) + a_2b_4(123) + a_2b_3(132)] \\
&\quad +[a_3b_3e + a_3b_5(12) + a_3b_1(13) + a_3b_6(23) + a_3b_2(123) + a_3b_4(132)] \\
&\quad +[a_4b_4e + a_4b_6(12) + a_4b_5(13) + a_4b_1(23) + a_4b_3(123) + a_4b_2(132)] \\
&\quad +[a_5b_6e + a_5b_4(12) + a_5b_2(13) + a_5b_3(23) + a_5b_1(123) + a_5b_5(132)] \\
&\quad +[a_6b_5e + a_6b_3(12) + a_6b_4(13) + a_6b_2(23) + a_6b_6(123) + a_6b_1(132) \\
&= a_1[b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) + A] \\
&\quad +a_2[b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) + A] \\
&\quad +a_3[b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) + A] \\
&\quad +a_4[b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) + A] \\
&\quad +a_5[b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) + A] \\
&\quad +a_6[b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132) + A]
\end{aligned}
$$

Since we have already shown that M is closed under the addition of its elements, we may consider each line of the sum above separately. If each line in the sum above is an element of M, then the entire sum will be an element of M. Recall that the group S_3 is closed under the permutation of its elements, so each line will preserve all of the elements of S_3. We need only to consider the sums of the coefficients of each line. For example,

$$
\begin{aligned}
a_1[b_1 + b_2 + b_3 + b_4 + b_5 + b_6] &= a_1 \left[\sum_{i=1}^{6} b_i\right] \\
&= a_1[0], \text{ by definition of the element } m \text{ in } M \\
&= 0.
\end{aligned}
$$

Each of the coefficients from the sum above will behave in the same way:

$$
\begin{aligned}
a_2[b_1 + b_2 + b_3 + b_4 + b_5 + b_6] &= a_2 \left[\sum_{i=1}^{6} b_i\right] \\
&= a_2[0], \text{ by definition of the element } m \text{ in } M \\
&= 0.
\end{aligned}
$$

$$\begin{aligned}
a_3[b_1+b_2+b_3+b_4+b_5+b_6] &= a_3\left[\sum_{i=1}^{6} b_i\right] \\
&= a_3[0], \text{ by definition of the element } m \text{ in } M \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
a_4[b_1+b_2+b_3+b_4+b_5+b_6] &= a_4\left[\sum_{i=1}^{6} b_i\right] \\
&= a_4[0], \text{ by definition of the element } m \text{ in } M \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
a_5[b_1+b_2+b_3+b_4+b_5+b_6] &= a_5\left[\sum_{i=1}^{6} b_i\right] \\
&= a_5[0], \text{ by definition of the element } m \text{ in } M \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
a_6[b_1+b_2+b_3+b_4+b_5+b_6] &= a_6\left[\sum_{i=1}^{6} b_i\right] \\
&= a_6[0], \text{ by definition of the element } m \text{ in } M \\
&= 0.
\end{aligned}$$

Then we may say that the sum of the coefficients below

$$\begin{aligned}
= \; & a_1[b_1e+b_2(12)+b_3(13)+b_4(23)+b_5(123)+b_6(132)] \\
& +a_2[b_1e+b_2(12)+b_3(13)+b_4(23)+b_5(123)+b_6(132)] \\
& +a_3[b_1e+b_2(12)+b_3(13)+b_4(23)+b_5(123)+b_6(132)] \\
& +a_4[b_1e+b_2(12)+b_3(13)+b_4(23)+b_5(123)+b_6(132)] \\
& +a_5[b_1e+b_2(12)+b_3(13)+b_4(23)+b_5(123)+b_6(132)] \\
& +a_6[b_1e+b_2(12)+b_3(13)+b_4(23)+b_5(123)+b_6(132)]
\end{aligned}$$

will equal 0 and the element rm belongs to M.

Then M is a submodule of $\mathbb{Z}_2S_3/A$ since it is closed under the addition of its elements and it is closed under the action of elements from $\mathbb{Z}_2S_3/A$. Since M is a proper submodule of $\mathbb{Z}_2S_3/A$, $\mathbb{Z}_2S_3/A$ is not a simple $\mathbb{Z}_2S_3$-module.

Chapter 6

A Look at $\mathbb{Z}_5S_3$

6.1 The Left Module $\mathbb{Z}_5S_3$.

Now let us consider the left module $\mathbb{Z}_5S_3$. The elements of $\mathbb{Z}_5S_3$ are formed by taking formal sums of the elements of $\mathbb{Z}_5$, which are $\{0, 1, 2, 3, 4\}$, and the permutations of S_3, which are $\{e, (12), (13), (23), (123), (132)\}$. The elements of S_3 are discussed in more detail in Section 5.1. That is, we are taking formal sums of the elements

$$\begin{array}{ccccc}
0e, & 1e, & 2e, & 3e, & 4e, \\
0(12), & 1(12), & 2(12), & 3(12), & 4(12), \\
0(13) & 1(13) & 2(13) & 3(13) & 4(13) \\
0(23), & 1(23), & 2(23), & 3(23), & 4(23) \\
0(123), & 1(123), & 2(123), & 3(123), & 4(123) \\
0(132), & 1(132), & 2(132), & 3(132), & 4(132).
\end{array}$$

The elements of $\mathbb{Z}_5S_3$ are of the form:

$$a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132),$$

where a_k are elements of $\mathbb{Z}_5$.

For example, the element

$$4e + 3(12) + 3(13) + 0(23) + 2(123) + 1(132)$$

belongs to the module $\mathbb{Z}_5S_3$.

Addition " + " is defined as:

$$\begin{aligned} a_i g_1 + a_j g_1 &= (a_i + a_j) g_1, \text{ and} \\ a_i g_1 + a_j g_2 &= a_i g_1 + a_j g_2, \end{aligned}$$

where a_i, a_j are elements of $\mathbb{Z}_5$ and the g_k are elements of S_3.

Multiplication " $\cdot$ " is defined as:

$$(a_i g_1) \cdot (a_j g_2) = (a_i \cdot a_j)(g_1 \cdot g_2).$$

The additive identity is:

$$0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)$$

and the multiplicative identity is: $1e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)$.

6.2 The Simplicity and Semisimplicity of $\mathbb{Z}_5 S_3$.

First we will look at $\mathbb{Z}_5 S_3$ as an RG group ring. Recall from Chapter four that an RG-ring is a set of formal sums of the form

$$\sum_{g \in G} \alpha_g g, \alpha_g \in R.$$

By Example 2.24 we know that the characteristic of $\mathbb{Z}_5$ is 5, and the order of the group S_3 can be found by finding $3! = 6$. It is clear that 5 does not divide 6, so we may apply Theorem 4.22, Machke's Theorem. Since the characteristic of $\mathbb{Z}_5$ does not divide the order of S_3, this theorem tells us that $\mathbb{Z}_5 S_3$ is semisimple as a left $\mathbb{Z}_5 S_3$-module. This means that $\mathbb{Z}_5 S_3$ can be written as a direct sum of simple submodules.

Let us look at $\mathbb{Z}_5 S_3$ as a $\mathbb{Z}_5$-module. Is $\mathbb{Z}_5 S_3$ simple as a $\mathbb{Z}_5$-module?

Examine the submodules of $\mathbb{Z}_5S_3$ as a $\mathbb{Z}_5$-module. Does it have any proper submodules?

Recall the Submodule Criterion from Chapter 3: Let R be a ring and let M be a module. Then a subset N of M is a submodule of M if and only if

1. $N \neq \emptyset$, and

2. $x + \alpha y \in N$ for all $\alpha \in R$ and for all $x, y \in N$.

We are still working with S_3, so we will be referencing the properties from Chapter 5. Recall the subgroups of S_3 that were defined in Section 5.1 :

$$\begin{aligned} H_1 &= \{e, (12)\} \\ H_2 &= \{e, (13)\}, \\ H_3 &= \{e, (23)\}, \text{ and} \\ H_4 &= \{e, (123), (132)\}. \end{aligned}$$

Consider the submodule $\mathbb{Z}_5H_1 = \mathbb{Z}_5\{e, (12)\}$. This is closed in $\mathbb{Z}_5$, since

$$\begin{aligned} \alpha(a_1e + a_2(12)) &= \alpha a_1e + \alpha a_2(12) \\ &= c_1e + c_2(12) \\ &\in \mathbb{Z}_5H_1 \text{ for all } \alpha \in \mathbb{Z}_5, \end{aligned}$$

where $c_1 = ba_1, c_2 = ba_2$, and $c_1, c_2 \in \mathbb{Z}_5$, since $\mathbb{Z}_5$ is multiplicatively closed.

We now use the Submodule Criterion, as defined in Section 3.1. We can see that $\mathbb{Z}_5H_1$ is nonempty, since H_1 is nonempty. We now apply the second part of the Submodule Criterion and check to see if an element of the form $x + \alpha y$ belongs to $\mathbb{Z}_5S_3$, where x and y are elements of $\mathbb{Z}_5S_3$ and α is an element from $\mathbb{Z}_5$. Then

$$\begin{aligned} [a_1'e + a_2'(12)] + \alpha[a_1e + a_2(12)] &= [a_1'e + a_2'(12)] + [\alpha a_1e + \alpha a_2(12)] \\ &= [a_1'e + a_2'(12)] + [c_1e + c_2(12)], \text{ where } c_i = \alpha a_i \in \mathbb{Z}_5 \\ &= [a_1' + c_1]e + [a_2' + c_2](12) \end{aligned}$$

Since $\mathbb{Z}_5$ is closed under addition, $[a_i' + c_i] \in \mathbb{Z}_5$. Thus, according to the Submodule Criterion, $[a_1'e + a_2'(12)] + \alpha[a_1e + a_2(12)] \in \mathbb{Z}_5S_3$.

Then $\mathbb{Z}_5H_1$ is a proper submodule of $\mathbb{Z}_5S_3$ as a $\mathbb{Z}_5$-module. By Definition 3.16 a module is simple if its only submodules are 0 and the module itself. Since $\mathbb{Z}_5S_3$ has proper submodules as a $\mathbb{Z}_5$-module, it is not simple.

We have yet to check whether or not $\mathbb{Z}_5S_3$ is semisimple as a $\mathbb{Z}_5$-module. If $\mathbb{Z}_5S_3$ is semisimple then it is a direct sum of simple submodules.
We can represent our module as follows:

$$\mathbb{Z}_5S_3 \cong \mathbb{Z}_5H_1 \oplus \mathbb{Z}_5H_2 \oplus \mathbb{Z}_5H_3 \oplus \mathbb{Z}_5H_4,$$

where the H_i are the submodules of S_3 that were given in Section 6.2.

It is clear that $H_1 \cap H_2 \cap H_3 \cap H_4 = e$. Then the elements of $\mathbb{Z}_5S_3$ can be uniquely represented as an element of $\mathbb{Z}_5H_1 \oplus \mathbb{Z}_5H_2 \oplus \mathbb{Z}_5H_3 \oplus \mathbb{Z}_5H_4 \oplus$. For example, the element $1(12) + 4(23)$ can be represented as a sum of elements from $\mathbb{Z}_5H_1$ and $\mathbb{Z}_5H_3$:

$$1(12) + 4(23) = 1(12) + 0(13) + 4(23) + 0(123) + 0(132).$$

So $\mathbb{Z}_5S_3 \cong \mathbb{Z}_5H_1 \oplus \mathbb{Z}_5H_2 \oplus \mathbb{Z}_5H_3 \oplus \mathbb{Z}_5H_4$. However, it might be possible to find another, perhaps simpler, representation for $\mathbb{Z}_5S_3$ as a semisimple submodule of $\mathbb{Z}_5S_3$.

Consider $\mathbb{Z}_5S_3 \cong \mathbb{Z}_5\{e\} \oplus \mathbb{Z}_5\{(12)\} \oplus \mathbb{Z}_5\{(13)\} \oplus \mathbb{Z}_5\{(23)\} \oplus \mathbb{Z}_5\{(123)\} \oplus \mathbb{Z}_5\{(132)\}$. Each of these submodules are generated by a single element, which makes them simple submodules.

Then $\mathbb{Z}_5S_3$ is said to be completely reducible, since all of the submodules are are irreducible, or simple.

Additionally, $\mathbb{Z}_5S_3$ is finite, so it satisfies the Descending Chain Condition, or we say it is Artinian as a $\mathbb{Z}_5$-module. Since the module $\mathbb{Z}_5S_3$ satisfies the Descending Chain Condition and the module is completely reducible, it is therefore semisimple as a $\mathbb{Z}_5$-module.

We may further examine the semisimplicity of $\mathbb{Z}_5S_3$ by looking at its radical. By Definition 3.11, the radical of $\mathbb{Z}_5S_3$ will be the intersection of the maximal submodules of

$\mathbb{Z}_5S_3$. Let us examine the maximal submodules of $\mathbb{Z}_5S_3$ as a $\mathbb{Z}_5$-module. These maximal submodules are formed by removing one of the elements of S_3 from the module $\mathbb{Z}_5S_3$. These maximal submodules are:

$$\begin{aligned}
\mathbb{Z}_5S_3\backslash 1e &= \mathbb{Z}_5\{1(12), 1(13), 1(23), 1(123), 1(132)\} \\
\mathbb{Z}_5S_3\backslash 1(12) &= \mathbb{Z}_5\{1e, 1(13), 1(23), 1(123), 1(132)\} \\
\mathbb{Z}_5S_3\backslash 1(13) &= \mathbb{Z}_5\{1e, 1(12), 1(23), 1(123), 1(132)\} \\
\mathbb{Z}_5S_3\backslash 1(23) &= \mathbb{Z}_5\{1e, 1(12), 1(13), 1(123), 1(132)\} \\
\mathbb{Z}_5S_3\backslash 1(123) &= \mathbb{Z}_5\{1e, 1(12), 1(13), 1(23), 1(132)\} \\
\mathbb{Z}_5S_3\backslash 1(132) &= \mathbb{Z}_5\{1e, 1(12), 1(13), 1(23), 1(123)\}
\end{aligned}$$

The intersection of these maximal submodules is zero. Thus the radical of $\mathbb{Z}_5S_3$ is zero. According to Theorem 3.29, $\mathbb{Z}_5S_3$ is semisimple as a $\mathbb{Z}_5$-module if and only if $\mathbb{Z}_5S_3 \cdot Rad(\mathbb{Z}_5S_3) = 0$.

In this case, $Rad(\mathbb{Z}_5S_3) = 0$, so we have

$$\begin{aligned}
\mathbb{Z}_5S_3 \cdot Rad(\mathbb{Z}_5S_3) &= \mathbb{Z}_5S_3 \cdot 0 \\
&= 0.
\end{aligned}$$

Additionally recall that the radical of $\mathbb{Z}_5S_3$ is also defined to be the largest nilpotent ideal of $\mathbb{Z}_5S_3$.

Since $\mathbb{Z}_5$ is a field, $\mathbb{Z}_5$ is not nilpotent. Furthermore, $(S_3)^n \neq 0$ for any $n \in \mathbb{N}$. Therefore 0 is the largest nilpotent ideal of $\mathbb{Z}_5S_3$ and the radical of $\mathbb{Z}_5S_3$ is 0. This confirms what Theorem 3.29 told us earlier, that $\mathbb{Z}_5S_3$ is semisimple.

Thus by Theorem 4.22, $\mathbb{Z}_5S_3$ is semisimple as a $\mathbb{Z}_5S_3$-module and by Theorem 3.29 $\mathbb{Z}_5S_3$ is semisimple as a $\mathbb{Z}_5$-module.

6.3 Submodules of $\mathbb{Z}_5S_3$.

Now let us examine all of the submodlues of $\mathbb{Z}_5S_3$. So far we have only examined maximal submodules. Once again we will recall the Submodule Criterion, which we defined in

Definition 3.4: Let R be a ring and let M be a module. Then a subset N of M is a submodule of M if and only if

1. $N \neq \emptyset$, and
2. $x + \alpha y \in N$ for all $\alpha \in R$ and for all $x, y \in N$.

Examine a subset N of $\mathbb{Z}_5S_3$ such that $N = \mathbb{Z}_5 \cdot e \subset \mathbb{Z}_5S_3$. Now, $N \neq 0$ since $0 \cdot e \in N$. Let $ae + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)$ and $be + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)$ be elements of N and let β be an element of $\mathbb{Z}_5S_3$ where $\beta = c_1e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132)$ and c_i is an element of $\mathbb{Z}_5$ for $i = 0, 1, ..., 6$. For simplicity, we will refer to the element $ae + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)$ as ae and $be + 0(12) + 0(13) + 0(23) + 0(123) + 0(132)$ as be.
Consider

$$\begin{aligned}
ae &+ [c_1e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132)]be \\
&= ae + [c_1e][be] + [c_2(12)][be] + c[_3(13)][be] + [c_4(23)][be] + [c_5(123)][be] + [c_6(132)][be] \\
&= ae + [c_1b][ee] + [c_2b][(12)e] + [c_3b][(13)e] + [c_4b][(23)e] + [c_5b][(123)e] + [c_6b][(132)e] \\
&= ae + [c_1b]e + [c_2b](12) + [c_3b](13) + [c_4b](23) + [c_5b](123) + [c_6b](132) \\
&= [a + c_1b]e + [c_2b](12) + [c_3b](13) + [c_4b](23) + [c_5b](123) + [c_6b](132) \\
&\notin N, \text{ unless } c_2b, c_3b, c_4b, c_5b, \text{ and } c_6b \text{ are all zero.}
\end{aligned}$$

So the Submodule Criterion does not hold for all $\beta = c_1e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132) \in \mathbb{Z}_5S_3$. Then $\mathbb{Z}_5 \cdot e$ is not a $\mathbb{Z}_5S_3$-submodule of $\mathbb{Z}_5S_3$.

However, if $\mathbb{Z}_5S_3$ is semisimple as a $\mathbb{Z}_5S_3$-module, it remains to show that it is a direct sum of simple modules:

$$\mathbb{Z}_5S_3 = M_1 \oplus M_2 \oplus \cdots \oplus M_k,$$

where the M_i are simple submodules of $\mathbb{Z}_5S_3$ and for some $k \in \mathbb{N}$.
We then want to find the simple submodules of $\mathbb{Z}_5S_3$ as a $\mathbb{Z}_5S_3$-module.

Let A be a subset of $\mathbb{Z}_5S_3$ such that an element a of A is of the form:

$$a = ke + k(12) + k(13) + k(23) + k(123) + k(132),$$

where k is an element of $\mathbb{Z}_5$. This is the same way that we defined the submodule A in Chapter 3. Then A will satisfy the Submodule Criterion. Let $x = ke + k(12) + k(13) + k(23) + k(123) + k(132)$ and $y = k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)$ be elements of the subset A and let $\beta = b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132)$ be an element from $\mathbb{Z}_5S_3$.
Then consider

$$
\begin{aligned}
& x + \beta y \\
= \ & [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\
& + [b_1e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132)] \cdot [ke + k(12) + k(13) \\
& + k(23) + k(123) + k(132)] \\
= \ & [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\
& + b_1e[ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\
& + b_2(12)[ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\
& + b_3(13)[ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\
& + b_4(23)[ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\
& + b_5(123)[ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\
& + b_6(132)[ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\
= \ & [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\
& + [b_1k][ee] + [b_1k][e(12)] + [b_1k][e(13)] + [b_1k][e(23)] + [b_1k][e(123)] + [b_1k][e(132)] \\
& + [b_2k][(12)e] + [b_2k][(12)(12)] + [b_2k][(12)(13)] + [b_2k][(12)(23)] + [b_2k][(12)(123)] \\
& + [b_2k][(12)(132)] \\
& + [b_3k][(13)e] + [b_3k][(13)(12)] + [b_3k][(13)(13)] + [b_3k][(13)(23)] + [b_3k][(13)(123)] \\
& + [b_3k][(13)(132)] \\
& + [b_4k][(23)e] + [b_4k][(23)(12)] + [b_4k][(23)(13)] + [b_4k][(23)(23)] + [b_4k][(23)(123)] \\
& + [b_4k][(23)(132)]
\end{aligned}
$$

$$
\begin{aligned}
& +[b_5k][(123)e] + [b_5k][(123)(12)] + [b_5k][(123)(13)] + [b_5k][(123)(23)] \\
& +[b_5k][(123)(123)] + [b_5k][(123)(132)] \\
& +[b_6k][(132)e] + [b_6k][(132)(12)] + [b_6k][(132)(13)] + [b_6k][(132)(23)] \\
& +[b_6k][(132)(123)] + [b_6k][(132)(132)] \\
= \quad & [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\
& +[b_1k]e + [b_1k](12) + [b_1k](13) + [b_1k](23) + [b_1k](123) + [b_1k](132) \\
& +[b_2k](12) + [b_2k]e + [b_2k](132) + [b_2k](123) + [b_2k](23) + [b_2k](13) \\
& +[b_3k](13) + [b_3k](123) + [b_3k]e + [b_3k](132) + [b_3k](12) + [b_3k](23) \\
& +[b_4k](23) + [b_4k](132) + [b_4k](123) + [b_4k]e + [b_4k](13) + [b_4k](12)] \\
& +[b_5k](123) + [b_5k](13) + [b_5k](23) + [b_5k](12) + [b_5k](132) + [b_5k]e \\
& +[b_6k](132) + [b_6k](23) + [b_6k](12) + [b_6k](13) + [b_6k]e + [b_6k](123) \\
= \quad & [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\
& +[b_1k]e + [b_1k](12) + [b_1k](13) + [b_1k](23) + [b_1k](123) + [b_1k](132) \\
& +[b_2k]e + [b_2k](12) + [b_2k](13) + [b_2k](23) + [b_2k](123) + [b_2k](132) \\
& +[b_3k]e + [b_3k](12) + [b_3k](13) + [b_3k](23) + [b_3k](123) + [b_3k](132) \\
& +[b_4k]e + [b_4k](12) + [b_4k](13) + [b_4k](23) + [b_4k](123) + [b_4k](132) \\
& +[b_5k]e + [b_5k](12) + [b_5k](13) + [b_5k](23) + [b_5k](123) + [b_5k](132) \\
& +[b_6k]e + [b_6k](12) + [b_6k](13) + [b_6k](23) + [b_6k](123) + [b_6k](132) \\
= \quad & [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\
& +[b_1k + b_2k + b_3k + b_4k + b_5k + b_6k]e + [b_1k + b_2k + b_3k + b_4k + b_5k + b_6k](12) \\
& +[b_1k + b_2k + b_3k + b_4k + b_5k + b_6k](13) + [b_1k + b_2k + b_3k + b_4k + b_5k + b_6k](23) \\
& [b_1k + b_2k + b_3k + b_4k + b_5k + b_6k](123) + [b_1k + b_2k + b_3k + b_4k + b_5k + b_6k](132) \\
= \quad & [k'e + k'(12) + k'(13) + k'(23) + k'(123) + k'(132)] \\
& +k[b_1 + b_2 + b_3 + b_4 + b_5 + b_6]e + k[b_1 + b_2 + b_3 + b_4 + b_5 + b_6](12) \\
& +k[b_1 + b_2 + b_3 + b_4 + b_5 + b_6](13) + k[b_1 + b_2 + b_3 + b_4 + b_5 + b_6](23) \\
& +k[b_1 + b_2 + b_3 + b_4 + b_5 + b_6](123) + k[b_1 + b_2 + b_3 + b_4 + b_5 + b_6](132)
\end{aligned}
$$

$$\begin{aligned}
&= k'e + k[b_1 + b_2 + b_3 + b_4 + b_5 + b_6]e \\
&\quad +k'(12) + k[b_1 + b_2 + b_3 + b_4 + b_5 + b_6](12) \\
&\quad +k'(13) + k[b_1 + b_2 + b_3 + b_4 + b_5 + b_6](13) \\
&\quad +k'(23) + k[b_1 + b_2 + b_3 + b_4 + b_5 + b_6](23) \\
&\quad +k'(123) + k[b_1 + b_2 + b_3 + b_4 + b_5 + b_6](123) \\
&\quad +k'(132) + k[b_1 + b_2 + b_3 + b_4 + b_5 + b_6](132) \\
&= [k' + kb_1 + kb_2 + kb_3 + kb_4 + kb_5 + kb_6]e \\
&\quad +[k' + kb_1 + kb_2 + kb_3 + kb_4 + kb_5 + kb_6](12) \\
&\quad +[k' + kb_1 + kb_2 + kb_3 + kb_4 + kb_5 + kb_6](13) \\
&\quad +[k' + kb_1 + kb_2 + kb_3 + kb_4 + kb_5 + kb_6](23) \\
&\quad +[k' + kb_1 + kb_2 + kb_3 + kb_4 + kb_5 + kb_6](123) \\
&\quad +[k' + kb_1 + kb_2 + kb_3 + kb_4 + kb_5 + kb_6](132)
\end{aligned}$$

Then each element of S_3 that is listed above shares the same coefficient: $[k' + kb_1 + kb_2 + kb_3 + kb_4 + kb_5 + kb_6] \in \mathbb{Z}_5$. Thus the sum above may be rewritten as

$$\ell e + \ell(12) + \ell(13) + \ell(23) + \ell(123) + \ell(132),$$

where $\ell = [k' + kb_1 + kb_2 + kb_3 + kb_4 + kb_5 + kb_6] \in \mathbb{Z}_5$.
According to the Submodule Criterion, A is a $\mathbb{Z}_5S_3$-submodule of $\mathbb{Z}_5S_3$. Since A is a submodule, it is closed under the addition and the $\mathbb{Z}_5S_3$ action on its elements.

Now, consider a second subset of $\mathbb{Z}_5S_3$:

$$C = \{a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132) \mid \sum_{i=0}^{6} a_i = 0\}.$$

We want to show that C is a submodule of the $\mathbb{Z}_5S_3$-module $\mathbb{Z}_5S_3$. Recall the Submodule Criterion that we introduced in Chapter 3.
We know that C is nonempty since $0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132) \in C$.

Now consider $x = a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)$ and $y = c_1e + c_2(12) +$

$c_3(13) + c_4(23) + c_5(123) + c_6(132)$ be elements of C. Note that this means $\sum_{i=1}^{6} a_i = 0$ and $\sum_{i=1}^{6} c_i = 0$ in $\mathbb{Z}_5$. We also let $\beta = b_1 e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132)$ be an element of $\mathbb{Z}_5 S_3$. Then

$$
\begin{aligned}
& x + \beta y \\
= \; & [a_1 e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] \\
& + [b_1 e + b_2(12) + b_3(13) + b_4(23) + b_5(123) + b_6(132)] \cdot [c_1 e + c_2(12) \\
& + c_3(13) + c_4(23) + c_5(123) + c_6(132)] \\
= \; & [a_1 e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] \\
& + b_1 e[c_1 e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132)] \\
& + b_2(12)[c_1 e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132)] \\
& + b_3(13)[c_1 e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132)] \\
& + b_4(23)[c_1 e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132)] \\
& + b_5(123)[c_1 e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132)] \\
& + b_6(132)[c_1 e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132)] \\
= \; & [a_1 e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] \\
& + [b_1 c_1][ee] + [b_1 c_2][e(12)] + [b_1 c_3][e(13)] + [b_1 c_4][e(23)] + [b_1 c_5][e(123)] \\
& + [b_1 c_6][e(132)] \\
& + [b_2 c_1][(12)e] + [b_2 c_2][(12)(12)] + [b_2 c_3][(12)(13)] + [b_2 c_4][(12)(23)] \\
& + [b_2 c_5][(12)(123)] + [b_2 c_6][(12)(132)] \\
& + [b_3 c_1][(13)e] + [b_3 c_2][(13)(12)] + [b_3 c_3][(13)(13)] + [b_3 c_4][(13)(23)] \\
& + [b_3 c_5][(13)(123)] + [b_3 c_6][(13)(132)] \\
& + [b_4 c_1][(23)e] + [b_4 c_2][(23)(12)] + [b_4 c_3][(23)(13)] + [b_4 c_4][(23)(23)] \\
& + [b_4 c_5][(23)(123)] + [b_4 c_6][(23)(132)] \\
& + [b_5 c_1][(123)e] + [b_5 c_2][(123)(12)] + [b_5 c_3][(123)(13)] + [b_5 c_4][(123)(23)] \\
& + [b_5 c_5][(123)(123)] + [b_5 c_6][(123)(132)]
\end{aligned}
$$

$$
\begin{aligned}
& +[b_6c_1][(132)e] + [b_6c_2][(132)(12)] + [b_6c_3][(132)(13)] + [b_6c_4][(132)(23)] \\
& +[b_6c_5][(132)(123)] + [b_6c_6][(132)(132)] \\
= \; & [a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] \\
& +[b_1c_1]e + [b_1c_2](12) + [b_1c_3](13) + [b_1c_4](23) + [b_1c_5](123) + [b_1c_6](132) \\
& +[b_2c_1](12) + [b_2c_2]e + [b_2c_3](132) + [b_2c_4](123) + [b_2c_5](23)] + [b_2c_6](13) \\
& +[b_3c_1](13) + [b_3c_2](123) + [b_3c_3]e + [b_3c_4](132) + [b_3c_5](12) + [b_3c_6](23) \\
& +[b_4c_1](23) + [b_4c_2](132) + [b_4c_3](123) + [b_4c_4]e + [b_4c_5](13) + [b_4c_6](12) \\
& +[b_5c_1](123) + [b_5c_2](13) + [b_5c_3](23) + [b_5c_4](12) + [b_5c_5](132) + [b_5c_6]e \\
& +[b_6c_1](132) + [b_6c_2](23)] + [b_6c_3](12) + [b_6c_4][(13) + [b_6c_5]e + [b_6c_6](123) \\
= \; & [a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] \\
& +[b_1c_1]e + [b_1c_2](12) + [b_1c_3](13) + [b_1c_4](23) + [b_1c_5](123) + [b_1c_6](132) \\
& +[b_2c_2]e + [b_2c_1](12) + [b_2c_6](13) + [b_2c_5](23) + [b_2c_4](123) + [b_2c_3](132) \\
& +[b_3c_3]e + [b_3c_5](12) + [b_3c_6](23) + [b_3c_1](13) + [b_3c_2](123) + [b_3c_4](132) \\
& +[b_4c_4]e + [b_4c_6](12) + [b_4c_5](13) + [b_4c_1](23) + [b_4c_3](123) + [b_4c_2](132) \\
& +[b_5c_6]e + [b_5c_4](12) + [b_5c_2](13) + [b_5c_3](23) + [b_5c_1](123) + [b_5c_5](132) \\
& +[b_6c_5]e + [b_6c_3](12) + [b_6c_4][(13) + [b_6c_2](23) + [b_6c_6](123) + [b_6c_1](132) \\
= \; & [a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] \\
& +b_1[c_1e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132)] \\
& +b_2[c_2e + c_1(12) + c_6(13) + c_5(23) + c_4(123) + c_3(132)] \\
& +b_3[c_3e + c_5(12) + c_6(23) + c_1(13) + c_2(123) + c_4(132)] \\
& +b_4[c_4e + c_6(12) + c_5(13) + c_1(23) + c_3(123) + c_2(132)] \\
& +b_5[c_6e + c_4(12) + c_2(13) + c_3(23) + c_1(123) + c_5(132)] \\
& +b_6[c_5e + c_3(12) + c_4(13) + c_2(23) + c_6(123) + c_1(132)]
\end{aligned}
$$

Note that, as it was in Chapter 5, the last six elements in the sum above are of the form $bj[c_1e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132)]$. The coefficients of each of these elements may be represented as

$$b_i\left[\sum_{i=1}^{6} c_i\right].$$

Then

$$\begin{aligned}
& b_1[c_1e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132)] \\
= \; & b_1\left[\sum_{i=1}^{6} c_i\right] \\
= \; & b_1[0], \text{ by definition of the element } y \text{ in } C \\
= \; & 0.
\end{aligned}$$

We may continue to treat the other elements of this sum in a similar fashion:

$$\begin{aligned}
& b_2[c_2e + c_1(12) + c_6(13) + c_5(23) + c_4(123) + c_3(132)] \\
= \; & b_2\left[\sum_{i=1}^{6} c_i\right] \\
= \; & b_2[0], \text{ by definition of the element } y \text{ in } C \\
= \; & 0.
\end{aligned}$$

$$\begin{aligned}
& b_3[c_3e + c_5(12) + c_6(23) + c_1(13) + c_2(123) + c_4(132)] \\
= \; & b_3\left[\sum_{i=1}^{6} c_i\right] \\
= \; & b_3[0], \text{ by definition of the element } y \text{ in } C \\
= \; & 0.
\end{aligned}$$

$$\begin{aligned}
& b_4[c_4e + c_6(12) + c_5(13) + c_1(23) + c_3(123) + c_2(132)] \\
= \; & b_4\left[\sum_{i=1}^{6} c_i\right] \\
= \; & b_4[0], \text{ by definition of the element } y \text{ in } C \\
= \; & 0.
\end{aligned}$$

$$\begin{aligned}
& b_5[c_6e + c_4(12) + c_2(13) + c_3(23) + c_1(123) + c_5(132)] \\
= \; & b_5\left[\sum_{i=1}^{6} c_i\right] \\
= \; & b_5[0], \text{ by definition of the element } y \text{ in } C \\
= \; & 0.
\end{aligned}$$

$$\begin{aligned}
& b_6[c_5e + c_3(12) + c_4(13) + c_2(23) + c_6(123) + c_1(132)] \\
= \; & b_6\left[\sum_{i=1}^{6} c_i\right] \\
= \; & b_6[0], \text{ by definition of the element } y \text{ in } C \\
= \; & 0.
\end{aligned}$$

Then we have

$$\begin{aligned}
= \; & [a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] \\
& +b_1[0] + b_2[0] + b_3[0] + b_4[0] + b_1[0] \\
= \; & [a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] + 0 \\
= \; & [a_1e + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132)] \\
& \in C
\end{aligned}$$

Then C satisfies the Submodule Criterion and is a $\mathbb{Z}_5S_3$-submodule of $\mathbb{Z}_5S_3$.
Now we would like to show that $\mathbb{Z}_5S_3 = A \oplus C$. This means that every element of $\mathbb{Z}_5S_3$ can be represented as a sum of elements from A and C. Take $a = ke + k(12) + k(13) +$

$k(23)+k(123)+k(132) \in A$ and $c = c_1e+c_2(12)+c_3(13)+c_4(23)+c_5(123)+c_6(132) \in C$ such that $\sum_{i=1}^{6} c_i = 0$. Then

$$\begin{aligned} a+c &= [ke + k(12) + k(13) + k(23) + k(123) + k(132)] \\ &\quad +[c_1e + c_2(12) + c_3(13) + c_4(23) + c_5(123) + c_6(132)] \\ &= (k+c_1)e + (k+c_2)(12) + (k+c_3)(13) + (k+c_4)(23) + (k+c_5)(123) + \\ &\quad (k+c_6)(132) \end{aligned}$$

The sum of the k and $c_i, i = 1, ..., 6$ are in $\mathbb{Z}_5$ and can be represented as

$$\begin{aligned} \sum_{i=1}^{6}(k+c_i) &= \sum_{i=1}^{6} k + \sum_{i=1}^{6} c_i \\ &= \sum_{i=1}^{6} k + 0 \\ &= \sum_{i=1}^{6} k \\ &= 6k \end{aligned}$$

Since $6k$ is an element of $\mathbb{Z}_5$, it is equal to one of the elements of the set $\{0, 1, 2, 3, 4\}$. If $k = 0$, then we have

$$\begin{aligned} \sum_{i=1}^{6}(k+c_i) &= \sum_{i=1}^{6}(0+c_i) \\ &= \sum_{i=1}^{6} 0 + \sum_{i=1}^{6} c_i \\ &= 0+0 \\ &= 0. \end{aligned}$$

If $k = 1$, then we have

$$\begin{aligned}
\sum_{i=1}^{6}(k + c_i) &= \sum_{i=1}^{6}(1 + c_i) \\
&= \sum_{i=1}^{6} 1 + \sum_{i=1}^{6} c_i \\
&= 6 + 0 \\
&= 6 \\
&= 1 \text{ in } \mathbb{Z}_5 S_3.
\end{aligned}$$

If $k = 2$, then we have

$$\begin{aligned}
\sum_{i=1}^{6}(k + c_i) &= \sum_{i=1}^{6}(2 + c_i) \\
&= \sum_{i=1}^{6} 2 + \sum_{i=1}^{6} c_i \\
&= 12 + 0 \\
&= 12 \\
&= 2 \text{ in } \mathbb{Z}_5 S_3.
\end{aligned}$$

If $k = 3$, then we have

$$\begin{aligned}
\sum_{i=1}^{6}(k + c_i) &= \sum_{i=1}^{6}(3 + c_i) \\
&= \sum_{i=1}^{6} 3 + \sum_{i=1}^{6} c_i \\
&= 18 + 0 \\
&= 18 \\
&= 3 \text{ in } \mathbb{Z}_5 S_3.
\end{aligned}$$

If $k = 4$, then we have

$$\begin{aligned}
\sum_{i=1}^{6}(k + c_i) &= \sum_{i=1}^{6}(4 + c_i) \\
&= \sum_{i=1}^{6} 4 + \sum_{i=1}^{6} c_i \\
&= 24 + 0 \\
&= 24 \\
&= 4 \text{ in } \mathbb{Z}_5 S_3.
\end{aligned}$$

If $\mathbb{Z}_5 S_3 \cong A \oplus C$, then we must be able to represent each element of $\mathbb{Z}_5 S_3$ as a unique sum of elements from A and C.

Is it possible to write any $\alpha \in \mathbb{Z}_5 S_3$ in the form $a + c$, where $a \in A$ and $c \in C$?

For example, choose an arbitrary element α of $\mathbb{Z}_5 S_3$. We will try to represent this element α as a sum $a + c$, where $a \in A$ and $c \in C$.

Let $\alpha = 4e + 1(12) + 0(13) + 3(23) + 2(123) + 1(132) \in \mathbb{Z}_5 S_3$. Then choose $a = 1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132) \in A$ and $c = 3e + 0(12) + 4(13) + 2(23) + 1(123) + 0(132) \in C$, since $\sum_{i=0}^{6} c_i = 3 + 0 + 4 + 2 + 1 + 0 = 10 = 0 \in \mathbb{Z}_5 S_3$.
Then

$$\begin{aligned}
a + c &= [1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)] \\
&\quad + [3e + 0(12) + 4(13) + 2(23) + 1(123) + 0(132)] \\
&= (1+3)e + (1+0)(12) + (1+4)(13) + (1+2)(23) + (1+1)(123) + (1+0)(132) \\
&= 4e + 1(12) + 0(13) + 3(23) + 2(123) + 1(132) \\
&= \alpha.
\end{aligned}$$

If we add the a_i of α, we have

$$\begin{aligned}
\sum_{i=0}^{6} a_i &= 4 + 1 + 0 + 3 + 2 + 1 \\
&= 11 \\
&= 1 \text{ in } \mathbb{Z}_5.
\end{aligned}$$

As we demonstrated in the cases above, the sum of the a_i of α is equal to the value of k, the coefficient of the components of a. That is, both $k = 1$ and $\sum_{i=0}^{6} a_i = 1$.

Consider a second example. Let β be another element of $\mathbb{Z}_5S_3, \beta$ different from α. We wish to represent β as $a' + c'$, where $a' \in A$ and $c' \in C$.

Let $\beta = 3e + 1(12) + 3(13) + 4(23) + 4(123) + 2(132) \in \mathbb{Z}_5S_3$. We may then choose $a' = 2e + 2(12) + 2(13) + 2(23) + 2(123) + 2(132) \in A$ and $c' = 1e + 4(12) + 1(13) + 2(23) + 2(123) + 0(132) \in C$, since

$$\sum_{i=0}^{6} c_i' = 1 + 4 + 1 + 2 + 2 + 0 = 10 = 0 \in \mathbb{Z}_5S_3$$

.

Then

$$\begin{aligned}
a' + c' &= [2e + 2(12) + 2(13) + 2(23) + 2(123) + 2(132)] \\
&\quad + [1e + 4(12) + 1(13) + 2(23) + 2(123) + 0(132)] \\
&= (2+1)e + (2+4)(12) + (2+1)(13) + (2+2)(23) + (2+2)(123) + (2+0)(132) \\
&= 3e + 1(12) + 3(13) + 4(23) + 4(123) + 2(132) \\
&= \beta.
\end{aligned}$$

If we add the a_i' of β, we have

$$\begin{aligned}
\sum_{i=0}^{6} a_i &= 3 + 1 + 3 + 4 + 4 + 2 \\
&= 17 \\
&= 2 \text{ in } \mathbb{Z}_5.
\end{aligned}$$

Once again, the sum of the a_i of β is equal to the value of k. Both are equal to 2.

Now we want to see if it is possible to represent each of the single elements of $\mathbb{Z}_5S_3$. Consider the element $1e$. We want to represent $1e$ as the sum of some $a + c$ where $a \in A$ and $c \in C$. Then let $a = 1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)$ and let $c = 0e + 4(12) + 4(13) + 4(23) + 4(123) + 4(132) \in C$, since

$$\sum_{i=0}^{6} c_i = 0 + 4 + 4 + 4 + 4 + 4 = 20 = 0 \in \mathbb{Z}_5.$$

So

$$\begin{aligned}
a+c &= [1e+1(12)+1(13)+1(23)+1(123)+1(132)] \\
&\quad +[0e+4(12)+4(13)+4(23)+4(123)+4(132)] \\
&= 1e+5(12)+5(13)+5(23)+5(123)+5(132) \\
&= 1e+0(12)+0(13)+0(23)+0(123)+0(132) \text{ in } \mathbb{Z}_5S_3 \\
&= 1e.
\end{aligned}$$

Now try to represent the element $2e \in \mathbb{Z}_5S_3$ in a similar fashion. We would like to find $a \in A$ and $c \in C$ so that $a+c=2e$. Let $a=2e+2(12)+2(13)+2(23)+2(123)+2(132)$ and let $c=0e+3(12)+3(13)+3(23)+3(123)+3(132)$. The element c belongs to C since

$$\sum_{i=0}^{6} c_i = 0+3+3+3+3+3=15=0 \in \mathbb{Z}_5.$$

Then

$$\begin{aligned}
a+c &= [2e+2(12)+2(13)+2(23)+2(123)+2(132)] \\
&\quad +[0e+3(12)+3(13)+3(23)+3(123)+3(132)] \\
&= 2e+5(12)+5(13)+5(23)+5(123)+5(132) \\
&= 2e+0(12)+0(13)+0(23)+0(123)+0(132) \text{ in } \mathbb{Z}_5S_3 \\
&= 2e.
\end{aligned}$$

Let us now write the element $3e \in \mathbb{Z}_5S_3$ as a sum of $a \in A$ and $c \in C$. Let $a=3e+3(12)+3(13)+3(23)+3(123)+3(132)$ and let $c=0e+2(12)+2(13)+2(23)+2(123)+2(132)$. This element c is in C since

$$\sum_{i=0}^{6} c_i = 0+2+2+2+2+2=10=0 \in \mathbb{Z}_5.$$

Then

$$\begin{aligned}
a+c &= [3e+3(12)+3(13)+3(23)+3(123)+3(132)] \\
&\quad +[0e+2(12)+2(13)+2(23)+2(123)+2(132)] \\
&= 3e+5(12)+5(13)+5(23)+5(123)+5(132) \\
&= 3e+0(12)+0(13)+0(23)+0(123)+0(132) \text{ in } \mathbb{Z}_5S_3 \\
&= 3e.
\end{aligned}$$

Finally let us represent $4e$ as a sum of $a+c \in A+C$. Let $a = 4e+4(12)+4(13)+4(23)+4(123)+4(132)$ and $c = 0e+1(12)+1(13)+1(23)+1(123)+1(132)$. Now $c \in C$ because

$$\sum_{i=0}^{6} c_i = 0+1+1+1+1+1 = 5 = 0 \in \mathbb{Z}_5 S_3.$$

So

$$\begin{aligned}
a+c &= [4e+4(12)+4(13)+4(23)+4(123)+4(132)] \\
&\quad +[0e+1(12)+1(13)+1(23)+1(123)+1(132)] \\
&= 4e+5(12)+5(13)+5(23)+5(123)+5(132) \\
&= 4e+0(12)+0(13)+0(23)+0(123)+0(132) \text{ in } \mathbb{Z}_5 S_3 \\
&= 4e.
\end{aligned}$$

We may represent the remaining elements of $\mathbb{Z}_5 S_3$ in a similar fashion:

$$\begin{aligned}
1(12) &= [1e+1(12)+1(13)+1(23)+1(123)+1(132)] \\
&\quad +[4e+0(12)+4(13)+4(23)+4(123)+4(132), \\
2(12) &= [2e+2(12)+2(13)+2(23)+2(123)+2(132)] \\
&\quad +[3e+0(12)+3(13)+3(23)+3(123)+3(132), \\
3(12) &= [3e+3(12)+3(13)+3(23)+3(123)+3(132)] \\
&\quad +[2e+0(12)+2(13)+2(23)+2(123)+2(132), \\
4(12) &= [4e+4(12)+4(13)+4(23)+4(123)+4(132)] \\
&\quad +[1e+0(12)+1(13)+1(23)+1(123)+1(132),
\end{aligned}$$

$$\begin{aligned}
1(13) &= [1e+1(12)+1(13)+1(23)+1(123)+1(132)] \\
&\quad +[4e+4(12)+0(13)+4(23)+4(123)+4(132), \\
2(13) &= [2e+2(12)+2(13)+2(23)+2(123)+2(132)] \\
&\quad +[3e+3(12)+0(13)+3(23)+3(123)+3(132), \\
3(13) &= [3e+3(12)+3(13)+3(23)+3(123)+3(132)] \\
&\quad +[2e+2(12)+0(13)+2(23)+2(123)+2(132), \\
4(13) &= [4e+4(12)+4(13)+4(23)+4(123)+4(132)] \\
&\quad +[1e+1(12)+0(13)+1(23)+1(123)+1(132),
\end{aligned}$$

$$
\begin{aligned}
1(23) &= [1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)] \\
&\quad +[4e + 4(12) + 4(13) + 0(23) + 4(123) + 4(132), \\
2(23) &= [2e + 2(12) + 2(13) + 2(23) + 2(123) + 2(132)] \\
&\quad +[3e + 3(12) + 3(13) + 0(23) + 3(123) + 3(132), \\
3(23) &= [3e + 3(12) + 3(13) + 3(23) + 3(123) + 3(132)] \\
&\quad +[2e + 2(12) + 2(13) + 0(23) + 2(123) + 2(132), \\
4(23) &= [4e + 4(12) + 4(13) + 4(23) + 4(123) + 4(132)] \\
&\quad +[1e + 1(12) + 1(13) + 0(23) + 1(123) + 1(132),
\end{aligned}
$$

$$
\begin{aligned}
1(123) &= [1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)] \\
&\quad +[4e + 4(12) + 4(13) + 4(23) + 0(123) + 4(132), \\
2(123) &= [2e + 2(12) + 2(13) + 2(23) + 2(123) + 2(132)] \\
&\quad +[3e + 3(12) + 3(13) + 3(23) + 0(123) + 3(132), \\
3(123) &= [3e + 3(12) + 3(13) + 3(23) + 3(123) + 3(132)] \\
&\quad +[2e + 2(12) + 2(13) + 2(23) + 0(123) + 2(132), \\
4(123) &= [4e + 4(12) + 4(13) + 4(23) + 4(123) + 4(132)] \\
&\quad +[1e + 1(12) + 1(13) + 1(23) + 0(123) + 1(132),
\end{aligned}
$$

$$
\begin{aligned}
1(132) &= [1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132)] \\
&\quad +[4e + 4(12) + 4(13) + 4(23) + 4(123) + 0(132), \\
2(132) &= [2e + 2(12) + 2(13) + 2(23) + 2(123) + 2(132)] \\
&\quad +[3e + 3(12) + 3(13) + 3(23) + 3(123) + 0(132), \\
3(132) &= [3e + 3(12) + 3(13) + 3(23) + 3(123) + 3(132)] \\
&\quad +[2e + 2(12) + 2(13) + 2(23) + 2(123) + 0(132), \\
4(132) &= [4e + 4(12) + 4(13) + 4(23) + 4(123) + 4(132)] \\
&\quad +[1e + 1(12) + 1(13) + 1(23) + 1(123) + 0(132),
\end{aligned}
$$

If we can prove that these representations are unique, then we may say that $\mathbb{Z}_5 S_3 = A \oplus C$, and that $\mathbb{Z}_5 S_3$ is semisimple. This supports the findings of Theorem 4.22, Machke's Theorem.

Chapter 7

Future Research

The overall goal of this research has been to explore the structure of $\mathbb{Z}2S_3$ and $\mathbb{Z}5S_3$. There is still much to be learned about these two modules. This chapter contains the goals that we have set for future research and study of the modules $\mathbb{Z}_2S_3$ and $\mathbb{Z}_5S_3$.

First we would like to find the simple submodules of $\mathbb{Z}_2S_3/\text{Rad}(\mathbb{Z}_2S_3)$ which make $\mathbb{Z}_2S_3/\text{Rad}(\mathbb{Z}_2S_3)$ a semisimple module. We will consider it both as a $\mathbb{Z}_2S_3$-submodule as well as a $\mathbb{Z}_2S_3/\text{Rad}(\mathbb{Z}_2S_3)$-submodule in $\mathbb{Z}_2S_3$. The progress that has been made thusfar is given below in Section 7.1.

In Chapter 3 we gave the definition of the socle and socle series of a module. We aim to find the socle of the modules $\mathbb{Z}_5S_3$ and $\mathbb{Z}_2S_3$. To do this, we will find the largest semisimple submodule of each module, which is called the socle, or soc$M$. Then we will find the quotient modules that result from dividing each submodule by the socle of the module. These resulting quotient modules will yield a sequence of submodules which cause the module M to be a Noetherian module as well as an Artinian module.

7.1 The Simple Submodules of $\mathbb{Z}_2S_3/Rad(\mathbb{Z}_2S_3)$.

We have started looking for the simple submodules which cause $\mathbb{Z}_2S_3/\text{Rad}(\mathbb{Z}2S_3)$ to be semisimple. Below is some of the work that has been made towards this goal.

Consider a subset M_1 of $\mathbb{Z}_2S_3/A$:

$$M_1 = \{A, 1e + 1(12) + 1(123) + A\}.$$

Since $A = \{\alpha_0, \alpha_1\}$, we may define the elements of M_1 in two ways:

$$\begin{aligned} A &= 1e + 1(12) + 1(13) + 1(23) + 1(123) + 1(132) + A \\ &= 0e + 0(12) + 0(13) + 0(23) + 0(123) + 0(132) + A \\ 1e + 1(12) + 1(123) + A &= 1(13) + 1(23) + 1(132) + A \end{aligned}$$

M_1 is also closed under the multiplication of elements of $\mathbb{Z}_2S_3$:
We can see that $1e \cdot A = A$, as it was with the two previous submodules.

$$\begin{aligned} & [1e + 1(12) + 1(13)] \cdot [1e + 1(12) + 1(13) + A] \\ = \; & 1e1e + 1e1(12) + 1e1(13) + 1(12)1e + 1(12)1(12) + 1(12)1(13) + 1(13)1e \\ & +1(13)1(12) + 1(13)1(13) + A \\ = \; & 1e + 1(12) + 1(123) + 1(12) + 1e + 1(23) + 1(123) + 1(13) + 1(132) + A \\ = \; & 2e + 2(12) + 1(13) + 1(23) + 2(123) + 1(132) + A \\ = \; & 1(13) + 1(23) + 1(132) + A \\ = \; & 1e + 1(12) + 1(123) + A \\ & \in M_1. \end{aligned}$$

We can prove that M_1 is a submodule of $\mathbb{Z}_2S_3/A$ and that it is generated by its element $< 1e + 1(12) + 1(123) + A >$. Therefore it is a simple submodule of $\mathbb{Z}_2S_3/A$.

So in the future we would like to find other simple submodules of $\mathbb{Z}_2S_3/A$ and prove its semisimplicity.

Chapter 8

Conclusion

In Chapters 2 and 3, we have demonstrated the connections that exist between Ring Theory and Module Theory. The aim of these two chapters has been both to create the necessary foundation for understanding modules as well as for helping the reader to see the similarities between the two fields of study. In Chapter 4 we explored Representation Theory. This particular field of study has many different aspects to it. Our focus has remained within the representations of a group using modules.

In Chapter 5 we examined the module $\mathbb{Z}_2S_3$. After examining the elements of $\mathbb{Z}_2S_3$ and the action that they have on each other, the module $\mathbb{Z}_2S_3$ was examined both as a $\mathbb{Z}_2$-module and as a $\mathbb{Z}_2S_3$-module. We have shown that the module $\mathbb{Z}_2S_3$ is not semisimple as a $\mathbb{Z}_2S_3$-module. As a direct result it is not simple as a $\mathbb{Z}_2S_3$-module either. The subset $A = \{\alpha_0, \alpha_1\}$, which was defined in Section 5.4, has been shown to be both a $\mathbb{Z}_2S_3$-submodule of $\mathbb{Z}_2S_3$ and a $\mathbb{Z}_2$-submodule of $\mathbb{Z}_2S_3$. Because A is a proper submodule, the module $\mathbb{Z}_2S_3$ is not simple, neither as a $\mathbb{Z}_2S_3$-module or as a $\mathbb{Z}_2$-module. The submodule A is the radical of $\mathbb{Z}_2S_3$. We then used the submodule A to form a quotient module $\mathbb{Z}_2S_3/A$. Finally we showed that the subset M forms a proper $\mathbb{Z}_2S_3$-submodule of the quotient module $\mathbb{Z}_2S_3/A$.

In Chapter 6 we looked at the structure and properties of the module $\mathbb{Z}_5S_3$. We examined its elements and their actions in relation to one another in a similar fashion to the way we examined those of the module $\mathbb{Z}_2S_3$ in Chapter 5. Because the order of the module $\mathbb{Z}_5S_3$ is larger than the order of the module $\mathbb{Z}_5S_3$, the structure of $\mathbb{Z}_5S_3$ differs from the structure of $\mathbb{Z}_2S_3$. We then considered $\mathbb{Z}_5S_3$ as a $\mathbb{Z}_5$-module as well as a $\mathbb{Z}_5S_3$-

module. Upon examination we concluded that, according to Machke's Theorem, $\mathbb{Z}_5S_3$ is semisimple as a $\mathbb{Z}_5S_3$-module. It is also semisimple as a $\mathbb{Z}_5$-module, since we found that the radical of $\mathbb{Z}_5S_3$ is equal to zero. The module is also semisimple because $\mathbb{Z}_5S_3$ can be viewed as a vector space over $\mathbb{Z}_5$. However, $\mathbb{Z}_5S_3$ is not simple as a $\mathbb{Z}_5$-module because we found that $\mathbb{Z}_5S_3$ has proper submodules. In Section 6.3, we found two $\mathbb{Z}_5S_3$-submodules of $\mathbb{Z}_5S_3$ and named them A and C. We then gave evidence that $\mathbb{Z}_5S_3 \cong A \oplus C$.

There is still much that remains to be explored within the structures of $\mathbb{Z}_2S_3$ and $\mathbb{Z}_5S_3$. Chapter seven is dedicated to future research. Our future goal is to find the simple submodules of $\mathbb{Z}_2S_3/A$ which make it semisimple as a $\mathbb{Z}_5S_3$-module as well as a $\mathbb{Z}_5S_3/A$-module.

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