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# A locus construction in the hyperbolic plane for elliptic curves with cross-ratio on the unit circle

Lyudmila Shved

# A Locus Construction in the Hyperbolic Plane for Elliptic Curves with Cross-Ratio on the Unit Circle

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A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

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In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

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by

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Approved by:

[redacted]

John Sarli, Committee Chair

7 MARCH 2011

Date

[redacted]

Corey Dunn, Committee Member

[redacted]

Susan Addington, Committee Member

[redacted]

Peter Williams, Chair,
Department of Mathematics

[redacted]

Charles Stanton
Graduate Coordinator,
Department of Mathematics

## Abstract

This paper demonstrates how an elliptic curve $f$ defined by invariance under two involutions can be represented by the locus of circumcenters of isosceles triangles in the hyperbolic plane, using any inversive model. An elliptic curve carries the structure of an abelian group, and we derive the product formula as a type of sum of the diameters of the circumcircles of the hyperbolic triangles whose centers are on the curve. We find that the curve has a cross-ratio $\chi$ of modulus 1. Because the cross-ratio is a birational invariant, it is unchanged by projective and inversive transformations. We use the fact that elliptic curves with the same cross-ratio are birationally equivalent to generalize the results obtained from curve $f$ to all its birationally equivalent loci in the hyperbolic plane.

## Acknowledgements

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# Chapter 1

# Introduction

## 1.1 Background

Elliptic curves are algebraic curves of genus 1 [MM99]. When algebraic curves are studied as projective varieties over the complex numbers, they are generally classified up to birational equivalence. That is, two curves are considered equivalent provided their respective fields of rational functions are isomorphic. It is well known that every elliptic curve is birationally equivalent to a non-singular cubic. In fact, typical representatives are often given in affine form by the polynomials

$$y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3),$$

where the $\alpha_j$ are distinct complex numbers [Bix06]. This form is useful when $y$ is viewed implicitly as a function of $x$.

However, elliptic curves often arise as solutions to the geometric locus problems where other forms of representation are more natural. In this project I will work with suitable representations of elliptic curves that occur as loci in the hyperbolic plane. The classes of curves that arise this way can be described as follows.

While the genus of a curve is a birational invariant, it is not sufficient to determine birational equivalence. In particular, there are many equivalence classes of elliptic curves, but their equivalence classes are determined by a single non-zero complex invariant called the *cross-ratio* of a curve. The cross-ratio $\chi$ is also a projective invariant because it is unchanged by the projective transformations. The value of $\chi$ can be computed from four collinear points in the projective plane, or, dually, from four concurrent lines.

Properties of cubics have been extensively studied, and there are many references to be found on this subject. In particular, the detailed exposition of this topic can be found in Robert Bix's *Conics and Cubics.* Suppose an elliptic curve is birationally equivalent to an irreducible cubic $f$. The theory of cubics tells us that $f$ has nine flexes, points on a curve that are also on its Hessian curve given by the determinant of its second partial derivatives. If $O$ is a flex, then there are four tangents to $f$ that pass through $O$, including the tangent at $O$. The cross-ratio of these four tangents is independent of which flex is chosen, and this value $\chi$ is the required cross-ratio of $f$, and of any elliptic curve birationally equivalent to $f$. Actually, the specific value of $\chi$ does depend on the order in which the tangents are taken, but the 24 permutations group themselves into the six possible values generated by the functions $\chi \mapsto \frac{1}{\chi}$ and $\chi \mapsto 1-\chi$. Thus the cross-ratio is actually a class of values $[\chi]$. If $\chi = -1$, then $[\chi] = \{-1, 2, \frac{1}{2}\}$, and this is the only cross-ratio with exactly three values in its class. In this case the curve is called *harmonic* [BEG99]. Harmonic curves are an example of elliptic curves whose cross-ratios contain a value of modulus 1. I will show that elliptic curves with cross-ratio on the unit circle are afforded by the following locus problem.

## 1.2 Locus Construction

Let $POQ$ be a triangle with $OP = OQ$, let $R$ be the intersection of the perpendicular bisectors of these equal sides, and let $\alpha$ be the oriented angle between the bisectors at $R$ (in the quadrilateral with vertices $O$ and $R$). Consider the collection of all isosceles triangles $P'OQ'$ with $OP' = OQ'$ such that $P'$ is collinear with $O$ and $P$, and such that the angle at $R'$, the intersection of the perpendicular bisectors of the equal sides, is $\alpha$. In the Euclidean plane, the locus of all such $R'$ is a straight line since all of these triangles are similar by dilation about $O$. In the hyperbolic plane, however, the locus is not a line. How can we characterize this locus? It may seem that any characterization depends on the model chosen for the hyperbolic plane. The problem itself suggests a conformal model because $\alpha$ must be constant. Some of the most useful conformal models are inversive models, for example, Poincaré disk and the upper half-plane. I will use the Poincaré disk in this project (though the result will be true for any inversive model) because the loci described above exhibit certain symmetries under Mobius transformations that permit them to be described very easily in this model. The main result is that such a locus consists of the real points on the elliptic curve with cross-ratio $[e^{2i\alpha}]$. Thus we obtain a characterization of elliptic curves whose cross-ratios are on the unit circle via an elementary locus problem in the hyperbolic plane [Sar08].

## 1.3 Product Structure

It is well-known that an irreducible and non-singular cubic curve carries the structure of an abelian group by the following construction [Bix06]. Let $O$ be any flex of the non-singular, irreducible cubic $F$. We are looking at $F$ as a projective complex curve, so the points of $F$ are homogeneous triples $(x, y, z)$ of complex numbers. Let $P$ and $Q$ be two points of $F$ that are not necessarily distinct. We define addition of two points $P$ and $Q$ in the following manner: let $S$ be the third point of intersection of line $PQ$ with $F$; then $P+Q$ is the third point of intersection of line $OS$ with $F$. Using this definition of addition of points on $F$, it is easy to show that the *commutative law* holds, $P+Q = Q+P$. Also, $P+O = P$, and, therefore, the flex $O$ is the *identity element*. For any point $P$ on $F$, there is the *additive inverse* $-P$ such that $P+(-P) = O$. Finally, it is possible to show that the *associative law* also holds for any three points of $F$, that is $(P+Q)+R = P+(Q+R)$ for points $P, Q, R$ on $F$. For a curve with cross-ratio on the unit circle, the abelian group described above has a natural subgroup that can be represented using the Poincaré disk.

If the vertex of the isosceles triangles in the locus construction described above is placed at $O$, the center of the disk, the resulting elliptic curve will be an irreducible cubic, and $O$ will be a flex, which we take as the identity element for the group product. Symmetry about the origin shows that the inverse of any point on the curve is its negative as a complex number. Following the product construction we find that product of any two points on the curve may fall outside of the disk. However, the curve is also invariant under the involution $f(z) = \frac{1}{z}$ so we can identify a point with the reciprocal of the complex number that represents it (in particular, the point $O$ is identified with the point at infinity in the completed complex plane). Thus the points on the curve within the disk are a quotient of the subgroup of real points on the curve by this involution. Since collineations of the hyperbolic plane induce birational transformations, it really does not matter where we place the vertex of the triangles. The resulting elliptic curve may not be cubic but the product structure can be carried over in a manner compatible with the transformation that takes the vertex back to $O$. We investigate how this product structure results in a type of sum on the diameters of the circumcircles of the triangles whose centers are the points on the curve. This provides an interpretation of the product structure that is independent of the projective representation of the curve.

Some extensive calculations needed for this paper as well as all graphics were produced using the *Scientific WorkPlace 5.5* designed by McKichan Software, Inc.

# Chapter 2

# Irreducible Cubic Polynomial

## 2.1 Introduction

As mentioned above, every elliptic curve is birationally equivalent to an irreducible cubic. We will work with a family of irreducible cubic curves symmetric with respect to the origin (in the Cartesian or complex coordinate system) that simplifies the computations of various relations. This choice of describing cubic curves provides a natural transition into the hyperbolic representation of the curves in the Poincaré Disk. We want to produce a general form of cubic irreducible curves that are invariant under two involutions: $z \mapsto -z$ and $z \mapsto \frac{1}{z}$ for a complex point $z = x + iy$ on the curve.

## 2.2 Deriving $f(x, y)$

Let $f(x, y)$ be an irreducible cubic polynomial with real coefficients of the form $a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} = 0$. Define the transformations $T_1(z) = -z$ and $T_2(z) = \frac{1}{z}$ for $z = x + iy$, where $x, y \in \mathbb{R}$. We need to find all irreducible cubic curves that are invariant under $T_1$ and $T_2$.

When we apply $T_1$ to the variety defined by $f(x, y)$, then we obtain variety represented by the polynomial $-a_{30}x^3 - a_{21}x^2y - a_{12}xy^2 - a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 - a_{10}x - a_{01}y + a_{00}$, which we want to be identically equal to $f(x, y)$. This will be true provided the affine variety is described by the equation $a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{10}x + a_{01}y = 0$. Let $g$ be the polynomial for this variety, thus $g(x, y) = a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{10}x + a_{01}y$.

Since $T_1$ and $T_2$ are involutions, the transformation by $T_2$ is given by $a_{10}x^5 - a_{01}x^4y + 2a_{10}x^3y^2 + a_{30}x^3 - 2a_{01}x^2y^3 - a_{21}x^2y + a_{10}xy^4 + a_{12}xy^2 - a_{01}y^5 - a_{03}y^3$. Let $h(x,y)$ represent this expression: $h(x,y) = a_{10}x^5 - a_{01}x^4y + 2a_{10}x^3y^2 + a_{30}x^3 - 2a_{01}x^2y^3 - a_{21}x^2y + a_{10}xy^4 + a_{12}xy^2 - a_{01}y^5 - a_{03}y^3$.

We took the third degree curve $g(x,y)$, applied transformation $T_2$ to it, and obtained a fifth degree curve, which will be equivalent to $g(x.y)$ as a curve if $h(x,y) = t(x,y)\,g(x,y)$, and $a_{10}x^5 - a_{01}x^4y + 2a_{10}x^3y^2 + a_{30}x^3 - 2a_{01}x^2y^3 - a_{21}x^2y + a_{10}xy^4 + a_{12}xy^2 - a_{01}y^5 - a_{03}y^3 = t(x,y)\left(a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{10}x + a_{01}y\right)$, where $t(x,y)$ is a polynomial of degree 2. In addition $t(x,y)$ must be of the form $\lambda x^2 + \mu xy + \nu y^2$, for some choice of $\lambda, \mu$, and $\nu$. Our claim holds if and only if the following equation is true:

$$a_{10}x^5 - a_{01}x^4y + 2a_{10}x^3y^2 + a_{30}x^3 - 2a_{01}x^2y^3 - a_{21}x^2y + a_{10}xy^4 + a_{12}xy^2 - a_{01}y^5 - a_{03}y^3 - (\lambda x^2 + \mu xy + \nu y^2)(a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{10}x + a_{01}y) = 0$$

or

$$x^3a_{30} + x^5a_{10} - y^3a_{03} - y^5a_{01} - 2x^2y^3a_{01} + 2x^3y^2a_{10} + xy^2a_{12} + xy^4a_{10} - x^2ya_{21} - x^4ya_{01} - x^3\lambda a_{10} - x^5\lambda a_{30} - y^3\nu a_{01} - y^5\nu a_{03} - x^2y^3\lambda a_{03} - x^3y^2\lambda a_{12} - x^2y^3\mu a_{12} - x^3y^2\mu a_{21} - x^2y^3\nu a_{21} - x^3y^2\nu a_{30} - x^2y\lambda a_{01} - xy^2\mu a_{01} - x^2y\mu a_{10} - xy^2\nu a_{10} - x^4y\lambda a_{21} - xy^4\mu a_{03} - x^4y\mu a_{30} - xy^4\nu a_{12} = 0$$

or

$$x^5(a_{10} - \lambda a_{30}) - x^4y(a_{01} + \lambda a_{21} + \mu a_{30}) - x^3(\lambda a_{10} - a_{30}) - x^3y^2(-2a_{10} + \lambda a_{12} + \mu a_{21} + \nu a_{30}) - x^2y(a_{21} + \lambda a_{01} + \mu a_{10}) - x^2y^3(2a_{01} + \lambda a_{03} + \mu a_{12} + \nu a_{21}) - xy^4(-a_{10} + \mu a_{03} + \nu a_{12}) - xy^2(\mu a_{01} - a_{12} + \nu a_{10}) - y^3(a_{03} + \nu a_{01}) - y^5(a_{01} + \nu a_{03}) = 0.$$

Comparing the coefficients of the corresponding terms, we get the following system of equations:

$$\begin{cases} a_{10} = \lambda a_{30} \\ a_{30} = \lambda a_{10} \\ a_{03} = -\nu a_{01} \\ a_{01} = -\nu a_{03} \\ a_{01} + \lambda a_{21} + \mu a_{30} = 0 \\ a_{21} + \lambda a_{01} + \mu a_{10} = 0 \\ a_{10} = \mu a_{03} + \nu a_{12} \\ a_{12} = \mu a_{01} + \nu a_{10} \\ -2a_{10} + \lambda a_{12} + \mu a_{21} + \nu a_{30} = 0 \\ 2a_{01} + \lambda a_{03} + \mu a_{12} + \nu a_{21} = 0 \end{cases}$$

In order to avoid the value of zero for $t(x, y)$ everywhere except the origin, $\lambda x^2 + \mu xy + \nu y^2$ must be a definite quadratic form. This puts some restriction on the values of $\lambda, \mu$, and $\nu$. We must have $4\lambda\nu - \mu^2 > 0$, which implies that $\lambda$ and $\nu$ must be either both positive or both negative. Note that $\lambda, \nu \neq 0$ otherwise the cubic will factor, which is a contradiction. Equations

$$\begin{aligned} a_{10} &= \lambda a_{30} \\ a_{30} &= \lambda a_{10} \\ a_{03} &= -\nu a_{01} \\ a_{01} &= -\nu a_{03} \end{aligned}$$

imply that $\lambda^2 = \nu^2 = 1$.

**Case 2.1.** *Let* $\lambda = \nu = 1$

*Then*

$$\begin{cases} a_{10} = a_{30} \\ a_{01} = -a_{03} \\ a_{01} + a_{21} + \mu a_{30} = 0 \\ a_{21} + a_{01} + \mu a_{10} = 0 \\ a_{10} = \mu a_{03} + a_{12} \\ a_{12} = \mu a_{01} + a_{10} \\ a_{12} + \mu a_{21} = a_{30} \\ \mu a_{12} + a_{21} = a_{03} \end{cases}$$

*which leads to the following relations:*

$$\begin{cases} a_{10} = a_{30} \\ a_{01} = -a_{03} \\ a_{12} = \mu a_{01} + a_{10} = a_{30} \end{cases}$$

*and* $\mu = 0$. *By letting* $a_{30} = 1$, *we have* $f_{a_{03}}(x, y) = x^3 + a_{03}y^3 + a_{03}x^2y + xy^2 + x - a_{03}y = 0$.

**Case 2.2.** *Let* $\lambda = \nu = -1$

$$\begin{cases} a_{10} = -a_{30} \\ a_{01} = a_{03} \\ a_{01} - a_{21} + \mu a_{30} = 0 \\ a_{21} - a_{01} + \mu a_{10} = 0 \\ a_{10} = \mu a_{03} - a_{12} \\ a_{12} = \mu a_{01} - a_{10} \\ a_{12} - \mu a_{21} = a_{30} \\ a_{21} - \mu a_{12} = a_{03} \end{cases}$$

*and the following relations result:*

$$\begin{cases} a_{10} = -a_{30} \\ a_{01} = a_{03} \\ a_{12} = \mu a_{01} - a_{10} = a_{30} \end{cases}$$

*Thus,* $\mu = 0$ *in this case as well. Letting* $a_{30} = 1$, *we have* $f_{a_{03}}(x, y) = x^3 + a_{03}y^3 + a_{03}x^2y + xy^2 - x + a_{03}y = 0$ .

For simplicity of notation, let $q = a_{03}$. Thus the two cases yield these two irreducible cubics:

$$\tilde{f}_q(x, y) = x^3 + qy^3 + qx^2y + xy^2 + x - qy = 0$$

and

$$f_q(x, y) = x^3 + qy^3 + qx^2y + xy^2 - x + qy = 0.$$

## 2.3 Relation Between the Two Cases of $f$

In the process of finding the irreducible cubics $f$ invariant under the two transformations $T_1$ and $T_2$, we realized that there are two such families. Fortunately, we do not have to work with both of them. It happens that these two cubics are related by a simple projective transformation, and we will work only with one them. A transformation $T$ that relates the two expressions can be defined in the following way: $x \longleftrightarrow y, q \mapsto \frac{1}{q}$.

$$T\left(\tilde{f}_q(x,y)\right) = y^3 + \frac{1}{q}y^2x + yx^2 + \frac{1}{q}x^3 + y - \frac{1}{q}x = 0.$$

Multiplying both sides by $q$ and simplifying yields $x^3+qy^3+qx^2y+xy^2-x+qy = f_q(x,y)$. Therefore, we will consider only one of these cases, the second one. As a result, we have a one-parameter family of irreducible cubics

$$f_q(x,y) = x^3 + qy^3 + qx^2y + xy^2 - x + qy \tag{2.1}$$

that is invariant under the involutions $z \mapsto -z$ and $z \mapsto \frac{1}{z}$ for a point $(x, y)$ on the curve, represented by $z = x + iy$ in complex plane. Note also that $f_{-q}(x,y) = f_q(x,-y)$, so changing the sign of $q$ just reflects the curve in the $x$-axis. Thus, without loss of generality, we assume $q > 0$.

## 2.4 Special Case: $f_1(x,y)$

To illustrate the two cases obtained above, let $q = 1$. Then, we get the following two curves:

$$\tilde{f}_1(x,y) = x^3 + y^3 + x^2y + xy^2 + x - y = 0, \text{ for } \lambda = \nu = 1 \tag{2.2}$$

and

$$f_1(x,y) = x^3 + y^3 + x^2y + xy^2 - x + y = 0, \text{ for } \lambda = \nu = -1 \tag{2.3}$$

Figure 2.1 and Figure 2.2 show the graphs of these two curves. To transform 2.1 into 2.2, we rotate the curve by the angle of $\frac{\pi}{2}$ and reflect it in the $x$-axis.

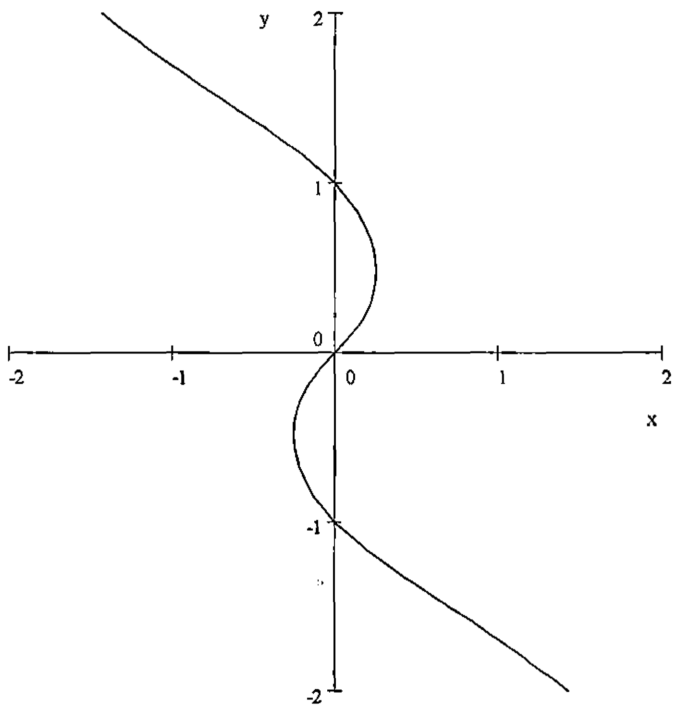


Figure 2.1: Graph of $\tilde{f}_1$ for $\lambda = \nu = 1$

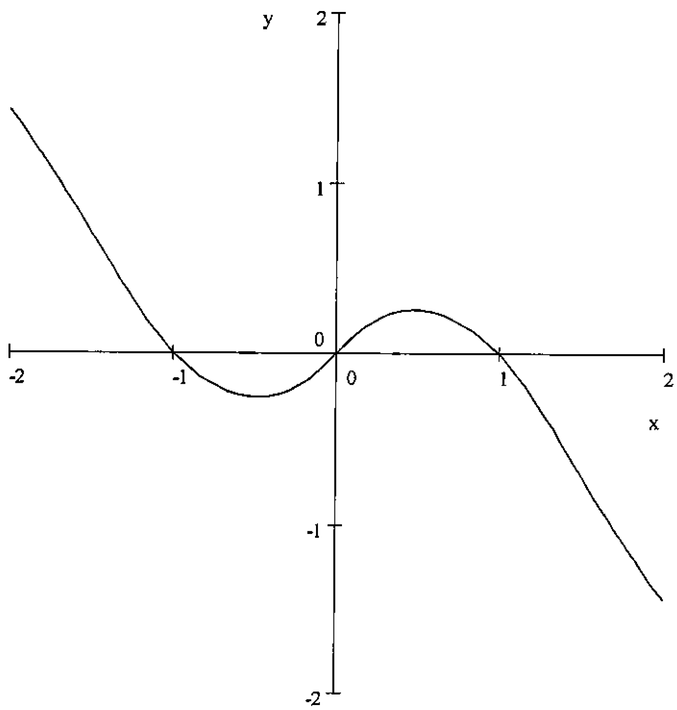


Figure 2.2: Graph of $f_1$ for $\lambda = \nu = -1$

# Chapter 3

# Cross-Ratio

## 3.1 Introduction

Computations with elliptic curves can be difficult to do. Fortunately, there are some ways of representing such curves by working with a suitable representation model. I will work with a representation of elliptic curves as loci in the hyperbolic plane. In addition, we can choose to consider equivalence classes of elliptic curves, which are determined by a single non-zero complex invariant called the *cross-ratio*. The cross-ratio $\chi$ is a projective invariant because it is unchanged by any projective transformation. I will use tools available in the projective geometry to evaluate $\chi$, which can be computed from four collinear points in the projective plane, or, dually, from four concurrent lines.

## 3.2 Introduction to the Projective Plane $\mathbb{R}P^2$

The choice of working in the projective plane to obtain $\chi$ was not accidental. Projective geometry has properties that make it much easier to get the job done. We define a homogeneous coordinate system similar to the usual three-dimensional system of coordinates with the $x$-, $y$-, and $z$-axes and the origin at $[0, 0, 0]$. The projective plane, usually denoted by $\mathbb{R}P^2$, is comprised of the Euclidean plane and so called *points at infinity* of the form $[a, b, 0]$ for $a, b \in \mathbb{R}$ [Bix06]. To distinguish between the objects in projective and Euclidean plane that have the same name but different meaning, we use capitalization of the first letter of the word to show that we identify a projective object. For example, a point in Euclidean plane is an ordered pair $(x, y)$ indicating its location in the $xy$-plane, while a projective Point $[a,b,c] \in \mathbb{R}P^2$ corresponds to a one-dimensional subspace $\mathbb{R}^2$spanned by $(a,b,c)$. A Line $< a,b,c > \in \mathbb{R}P^2$ corresponds to a normal vector $(a,b,c)$ defining a two-dimensional subspace of $\mathbb{R}^3$. The introduction of points at infinity yields some wonderful results in the projective plane. One of them is the Fundamental Theorem of Projective Geometry, which states that any two quadrangles in $\mathbb{R}P^2$ are congruent, and there exists a unique projective transformation that relates them [BEG99]. This theorem allows us to apply transformations and get figures congruent to the originally given figures, which significantly simplifies calculations in many cases. Another important result of projective geometry is the *principle of duality*, which allows us to interchange the roles of Points and Lines [BEG99]. Such exchange often helps to prove results that maybe hard to prove otherwise.

In this system of homogeneous coordinates, algebraic curves are obtained from the affine varieties in $x, y$ by homogenization, that is every term is made to be of the same degree by multiplying by appropriate powers of the third variable $z$. A curve $G(x,y,z)$ in the projective plane is said to be singular at a Point $P \in G(x,y,z)$ if every line through $P$ intersects $G(x,y,z)$ at least twice at that Point. If, on the other hand, there is a unique line intersecting $G(x,y,z)$ more than once at $P$, then $G(x,y,z)$ is said to be nonsingular at $P$ [Bix06]. This unique line is a tangent to the curve at $P$. There are Points on projective curves of special interest to us. A Point $P$ is a *flex* of $G(x,y,z)$ if $G$ is nonsingular at $P$ and if $G(x,y,z)$ intersects the tangent at $P$ at least three times [Bix06]. For example, a two-dimensional sketch of a cubic $y = x^3$ in Figure 3.1 shows a flex, or the inflection point, at the origin $O$. The origin is a flex because an infinitesimal rotation of the tangent through $O$ will intersect the curve $y = x^3$ three times.

## 3.3 Computation of the Cross-Ratio $\chi$

Consider a homogeneous irreducible cubic polynomial

$$f_q : x^3 + qy^3 + xy^2 + qx^2y - xz^2 + qyz^2 = 0, \text{ where } q > 0.$$

In order to calculate the cross-ratio of this curve, we need to find four collinear projective Points on this curve. Based on the principle of duality, the same result can be obtained by considering four concurrent lines that pass through the given Point $[0,0,1]$, which is the flex of this curve.

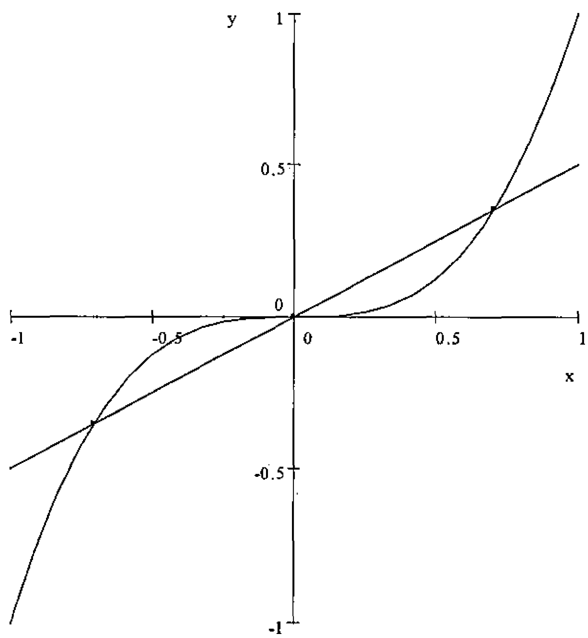


Figure 3.1: Flex of a cubic in $\mathbb{R}^2$

Let $[x_0, y_0, z_0] \in f_q$ and suppose the tangent at this Point passes through the flex $[0, 0, 1]$. Then

$$\nabla f_q = \begin{vmatrix} 3x^2 + y^2 + 2qxy - z^2 \\ 3qy^2 + 2xy + qx^2 + qz^2 \\ -2xz + 2qyz \end{vmatrix},$$

and $\nabla f_q(x_0, y_0, z_0) \cdot (x, y, z) = 0$ is the tangent at $[x_0, y_0, z_0]$. Thus,

$$\left(3x_0^2 + y_0^2 + 2qx_0y_0 - z_0^2\right) x + \left(3qy_0^2 + 2x_0y_0 + qx_0^2 + qz_0^2\right) y + (-2z_0)(x_0 - qy_0) z = 0.$$

Since $[0, 0, 1] \in f_q$, then this tangent contains this flex provided $z_0 (x_0 - qy_0) = 0$. The points on $f_q$ that satisfy this condition fall into two cases.

**Case 3.1.** *If* $z_0 = 0$,

$$\begin{aligned} x_0^3 + qy_0^3 + x_0y_0^2 + qx_0^2y_0 &= 0 \\ x_0\left(x_0^2 + y_0^2\right) + qy_0\left(x_0^2 + y_0^2\right) &= 0 \\ \left(x_0^2 + y_0^2\right)(x_0 + qy_0) &= 0 \end{aligned}$$

*which results in three points at infinity* $[1, i, 0], [i, 1, 0]$, *and* $[-q, 1, 0]$.

**Case 3.2.** *If* $z_0 \neq 0$, *say* $z_0 = 1$, *then* $x_0 = qy_0$. *Solutions to the latter equation are of the form* $[qy_0, y_0, 1]$. *We need to find the coordinates of this point such that* $[qy_0, y_0, 1] \in f_q$. *Then either* $2q = 0, q^2 + 1 = 0$, *or* $y_0^3 = 0$. *Since* $q$ *is a positive real number, the only possible solution is when* $y_0 = 0$. *Thus,* $[qy_0, y_0, 1]$ *is on the curve* $f_q$ *if is has coordinates* $[0, 0, 1]$.

Therefore, the four Points where Lines through the flex are tangent to $f_q$ are $[1, i, 0], [i, 1, 0], [-q, 1, 0]$, and $[0, 0, 1]$. Now we find equations of the four corresponding tangent Lines:

$$\begin{aligned}
[1, i, 0] &\Rightarrow (2 + 2qi)\, x + (-2q + 2i)\, y = 0 \Rightarrow \langle 1 + qi, -q + i, 0 \rangle, \\
[i, 1, 0] &\Rightarrow (-2 + 2qi)\, x + (2q + 2i)\, y = 0 \Rightarrow \langle -1 + qi, q + i, 0 \rangle, \\
[-q, 1, 0] &\Rightarrow \left(q^2 + 1\right) x + \left(q^3 + q\right) y = 0 \Rightarrow \langle 1, q, 0 \rangle, \\
[0, 0, 1] &\Rightarrow -x + qy = 0 \Rightarrow \langle -1, q, 0 \rangle.
\end{aligned}$$

The cross-ratio $\chi$ of the curve $f_q$ can be calculated as follows: $\chi = \frac{\frac{\beta}{\alpha}}{\frac{\delta}{\gamma}}$, where $\alpha, \beta, \gamma, \delta$ are solutions to the system of equations $\begin{cases} A = \alpha C + \beta D \\ B = \gamma C + \delta D \end{cases}$, and $A, B, C, D$ are four vectors representing the four concurrent Lines [Sar08]. Let $A = \langle 1 + qi, -q + i, 0 \rangle, B = \langle -1 + qi, q + i, 0 \rangle, C = \langle 1, q, 0 \rangle$, and $D = \langle -1, q, 0 \rangle$. Then,

$$\begin{cases} (1 + qi, -q + i, 0) = \alpha\,(1, q, 0) + \beta\,(-1, q, 0) \\ (-1 + qi, q + i, 0) = \gamma\,(1, q, 0) + \delta\,(-1, q, 0) \end{cases}$$

and

$$\alpha = \left(\frac{q^2 + 1}{2q}\right) i, \beta = \left(\frac{-q^2 + 1}{2q}\right) i - 1, \gamma = \left(\frac{q^2 + 1}{2q}\right) i, \delta = \left(\frac{-q^2 + 1}{2q}\right) i + 1.$$

$$\begin{aligned}
\chi &= \frac{\frac{\beta}{\alpha}}{\frac{\delta}{\gamma}} \\
\chi &= \frac{\frac{\left(\frac{-q^2+1}{2q}\right)i-1}{\left(\frac{q^2+1}{2q}\right)i}}{\frac{\left(\frac{-q^2+1}{2q}\right)i+1}{\left(\frac{q^2+1}{2q}\right)i}} \\
\chi &= \frac{\left(\frac{-q^2+1}{2q}\right)i-1}{\left(\frac{-q^2+1}{2q}\right)i+1} \\
\chi &= \frac{(q+i)^2}{(q-i)^2}.
\end{aligned}$$

Let $q = \tan\frac{\alpha}{2}$, where $\alpha \in (0, \pi)$ (this choice of $q$ is not accidental, and will appear in this project again). Then $\chi = \frac{(q-i)^2}{(q+i)^2} = \frac{\left(\tan\frac{\alpha}{2}-i\right)^2}{\left(\tan\frac{\alpha}{2}+i\right)^2}$. Note that $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$.

Simplifying, we get the following:

$$\chi = \frac{\left(\frac{\sin\alpha}{1+\cos\alpha} - i\right)^2}{\left(\frac{\sin\alpha}{1+\cos\alpha} + i\right)^2} = e^{2i\alpha}. \tag{3.1}$$

Since $\alpha \in (0, \pi)$, $\chi$ can take any value on a unit circle. Thus the curves $f_q$ represent all elliptic curves with cross-ratio of modulus 1.

# Chapter 4

# Locus Construction of $f_q$ in $D$

## 4.1 Introduction to the Inversive Model

At this point, we turn from the projective geometry to non-Euclidean geometry. More than one model can be used here. The best suitable model for this segment of my work, due to certain symmetries under Möbius transformations that will be easily described, is the inversive model called the Poincaré Disk. In this part of the project, our "world" will reduce to the unit disk $D = \{z : |z| < 1\}$. Lines in the disk are the arcs of generalized circles that meet the unit disk $D$ at right angles. We call such lines *hyperbolic*. Obviously, any diameter of $D$ is a hyperbolic line that is part of a Euclidean line.

It is very convenient to use Hermitian matrices when working with hyperbolic lines in an inversive model. Hermitian matrices correspond to planes that are secants, tangents, or exterior planes to $S^2$, depending whether the determinant is negative, zero, or positive, respectively. Hermitian matrices will help us to identify the type of the plane by its properties. First, we define a Hermitian matrix [Sch79]:

**Definition.** *A matrix $H$ is a Hermitian matrix if it is equal to its conjugate transpose denoted by $H^*$. If $H$ is Hermitian, it is of the form* $\begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}$, *for* $a, b, c, d \in \mathbb{R}$.

**Remark 4.1.** *Note that in this paper the asterisk * applied to the matrix indicates a conjugate transpose of that matrix, while asterisk * applied to a complex number $z$, indicates the complex conjugate commonly denoted by $\overline{z}$.*

It was shown by August Ferdinand Möbius that the value of the determinant of a Hermitian matrix $H$, which represents a plane, can tell us how this plane relates to the sphere. Much of this formalism can be found in the text written by Henry McKean and Victor Moll called *Elliptic Curves.*

**Theorem 4.2.** *If $H$ is a $2 \times 2$ Hermitian matrix, then $H$ represents:*

1. *a plane exterior to $S^2$ if $det(H) > 0$.*
2. *a plane tangent to $S^2$ if $det(H) = 0$.*
3. *a plane secant to $S^2$ if $det(H) < 0$.*

We can use these properties of the inversive plane by looking at Hermitian matrices as a projective vector space. In this case, we can utilize linear algebra tools. In particular, we define an inner product on the space of Hermitian matrices.

**Definition.** *An inner product on the space of two Hermitian matrices $H_1$ and $H_2$ is defined as $\langle H_1, H_2 \rangle = \frac{1}{2} Tr\left(H_1 \cdot H_2^{adj}\right)$.*

**Definition.** *If $H_1$ and $H_2$ are Hermitian matrices, then*

$$\cos\alpha = \frac{\langle H_1, H_2 \rangle}{\sqrt{\langle H_1, H_1 \rangle}\sqrt{\langle H_2, H_2 \rangle}} = \frac{\langle H_1, H_2 \rangle}{|H_1|\,|H_2|},$$

*where $\alpha$ is the angle between $H_1$ and $H_2$.*

If, for any given Hermitian matrices $H_1$ and $H_2$, $\cos\alpha = 0$, we conclude that $\alpha = \frac{\pi}{2}$, and $H_1 \perp H_2$. Remember that by definition, hyperbolic (inversive) lines are orthogonal to the unit disk $D$. Thus, the previous definition can be used to find the general form of a hyperbolic line.

First, note that the unit circle can be represented in the form of Hermitian matrix as $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

**Theorem 4.3.** *If an inversive line $H$ is perpendicular to $U$, then either*

1. *$H = \begin{pmatrix} 0 & b+ci \\ b-ci & 0 \end{pmatrix}$ is the line through the origin of slope $\frac{b}{c}$, or*

*2.* $H = \begin{pmatrix} 1 & b+ci \\ b-ci & 1 \end{pmatrix}$ *with* $b^2+c^2 > 1$ *is the circle of radius* $\sqrt{b^2+c^2-1}$ *centered at* $(-b+ci)$.

*Proof.* Let $H = \begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}$ and $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $H$ represents an inversive line if and only if $\frac{\langle H,U\rangle}{|H||U|} = 0$, which is possible only if $\langle H,U\rangle = 0$.

$$\begin{aligned}
\langle H,U\rangle &= \frac{1}{2}Tr\left(H \cdot U^{adj}\right) \\
&= \frac{1}{2}Tr\left(\begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{adj}\right) \\
&= \frac{1}{2}Tr\left(\begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \frac{1}{2}Tr\begin{pmatrix} -a & b+ic \\ -b+ic & d \end{pmatrix} \\
&= \frac{1}{2}(d-a).
\end{aligned}$$

$H$ is an inversive line if and only if $0 = d-a$ or $a = d$. There are two forms $H$ can take: $H = \begin{pmatrix} 0 & b+ic \\ b-ic & 0 \end{pmatrix}$ or $H = \begin{pmatrix} 1 & b+ic \\ b-ic & 1 \end{pmatrix}$. Let $z = x+iy$.

**Case 1** $H = \begin{pmatrix} 0 & b+ic \\ b-ic & 0 \end{pmatrix}$ with $\det(H) = -\left(b^2+c^2\right)$. Then,

$$\begin{aligned}
\begin{pmatrix} z & 1 \end{pmatrix}\begin{pmatrix} 0 & b+ic \\ b-ic & 0 \end{pmatrix}\begin{pmatrix} z^* \\ 1 \end{pmatrix} &= 0 \\
z^*(b-ic) + z(b+ic) &= 0 \\
2(bx-cy) &= 0 \\
y &= \frac{b}{c}x
\end{aligned} \tag{4.1}$$

and the inversive line is a line through the origin with a slope of $\frac{b}{c}$.

**Case 2** $H = \begin{pmatrix} 1 & b+ic \\ b-ic & 1 \end{pmatrix}$ with $\det(H) = 1 - \left(b^2 + c^2\right)$.

$$\begin{aligned} \begin{pmatrix} z & 1 \end{pmatrix} \begin{pmatrix} 1 & b+ic \\ b-ic & 1 \end{pmatrix} \begin{pmatrix} z^* \\ 1 \end{pmatrix} &= 0 \\ z^*\left(b - ic + z\right) + z\left(b + ic\right) + 1 &= 0 \\ \left(x - iy\right)\left(b - ic + x + iy\right) + \left(x + iy\right)\left(b + ic\right) + 1 &= 0 \\ x^2 + 2bx + y^2 - 2cy + 1 &= 0 \\ \left(x + b\right)^2 + \left(y - c\right)^2 &= b^2 + c^2 - 1. \end{aligned} \tag{4.2}$$

This inversive line is a circle centered at $(-b, c)$ with radius of $\sqrt{b^2 + c^2 - 1}$.

The two cases presented complete the proof of this theorem. □

There is another important definition we must include. It concerns the hyperbolic distance and its relation to the Euclidean distance.

**Definition.** *Let $z_1, z_2$ be two complex points in the unit disk. Then, the hyperbolic distance between $z_1$ and $z_2$ is defined as follows [BEG99]:*

$$d\left(z_1, z_2\right) = \tanh^{-1}\left(\left|\frac{z_2 - z_1}{1 - z_1^* z_2}\right|\right),$$

*and , consequently,*

$$d\left(0, z\right) = \tanh^{-1}\left(|z|\right).$$

*The Euclidean equivalent of the length between $O$ and a complex point $z \in D$ is given by*

$$|z| = \tanh d\left(0, z\right). \tag{4.3}$$

Note that $0 < d\left(0, z\right) < \infty$, while $0 \leq |z| < 1$.

## 4.2 The $\alpha$-Family of Isosceles Triangles

Let $POQ$ be an isosceles triangle with $OP = OQ$ and let $\theta$ be the angle at $O, 0 < \theta < \pi$. Let $R$ be the intersection of perpendicular bisectors of equal sides, and

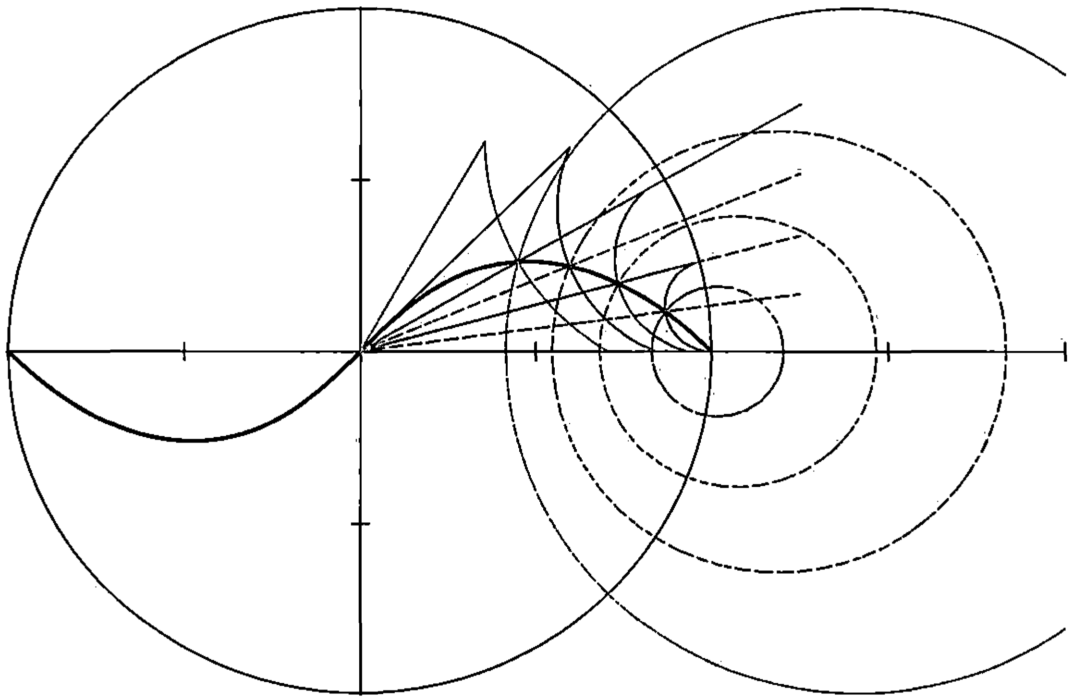

Figure 4.1: The $\alpha$-Family of Triangles for $\alpha = \frac{\pi}{2}$

let $\alpha$ be the oriented angle between the bisectors at $R$ (in the quadrilateral with vertices $O$ and $R$). Consider the collection of all isosceles triangles $P'OQ'$ with $OP' = OQ'$ such that $P'$ is collinear with $O$ and $P$, and such that the angle at $R'$, the intersection of the perpendicular bisectors of the equal sides, is $\alpha$. This collection of isosceles triangles $P'OQ'$ is an *$\alpha$-family* with a fixed angle $\alpha$ as defined above. Figure 4.1 shows a particular $\alpha$-family for $\alpha = \frac{\pi}{2}$. We will consider the locus of all points $R'$ in the hyperbolic plane using the Poincaré disk model of the hyperbolic plane. We will indicate the open unit disk by $D$.

Without loss of generality, consider the $\alpha$-family with the vertex $O = 0$ and the side $OP$ on the real axis. Let $s = d(O,P) = d(O,Q)$. Then, the Euclidean length of $OP$ is $\tanh s$. In order to find the point of intersection of perpendicular bisectors of the sides of the triangle $POQ$, point $R$, we need two equations representing two of the three perpendicular bisectors. First, consider the perpendicular bisector of $OP$. This hyperbolic line $L$ is an arc of the Euclidean circle centered on the real axis, and we can find the equation of this circle using Hermitian matrices [Sar08].

It can be proved that an inversion of a complex point $z$ in a circle centered at $(a, b)$ with radius $r$ can be represented by the transformation

$$T_1(z) = \frac{r^2}{(z-\gamma)^*} + \gamma, \tag{4.4}$$

for a complex number $\gamma = a + bi$, $z \neq \gamma$ [BEG99]. Now, consider the following construction. Let $D$ be the unit disk, and let $H$ be a circle centered at $C$ on the $x$-axis with radius $r$. Assume that $D$ and $H$ are orthogonal. This assumption makes the arc of $H$ interior to $D$ a hyperbolic line. Let $B$ denote one of the points of intersection of $U$ and $H$. It is obvious that the radii of the two circles are orthogonal. Thus, we can work with the right triangle $\triangle OBC$. By the Pythagorean Theorem,

$$1 + r^2 = a^2,$$

where $a = OC$.

To find the inversion of point $O$ in the circle $H$, we can use 4.4. In this case $z = 0$ and $\gamma = C = a + i0 = a$. Then,

$$\begin{aligned} T_1(0) &= \frac{r^2}{(0-a)^*} + a \\ T_1(0) &= \frac{r^2}{-a} + a \\ T_1(0) &= \frac{r^2 - a^2}{-a} \\ T_1(0) &= \frac{1}{a}. \end{aligned}$$

It it also known that an inversion of a complex number $z \neq 0$ in a unit circle is represented by this transformation $T_2(z) = \frac{1}{z^*}, z \neq 0$. Thus, for any $z = a + 0i, a \neq 0$, we have $T_2(a) = \frac{1}{a^*} = \frac{1}{a}$. Note that results of both inversions yield the same output. Let $C'\left(\frac{1}{a}, 0\right)$ denote the point resulting from both transformations. Since $T_1(0) = \frac{1}{a}$, then $M$ must be a midpoint of $OC'$ with the hyperbolic length of $\frac{1}{a}$, and $OM$ has the the hyperbolic length $\frac{1}{2a}$. This observation is illustrated in Figure 4.2.

We can use the result above as a tool to find an equation of the perpendicular bisector of $OP$ in the triangle $POQ$. Since $OP = s$, then there exists a unique circle orthogonal to $D$, whose center is of hyperbolic length $\frac{1}{s}$ from $O$. The important fact we

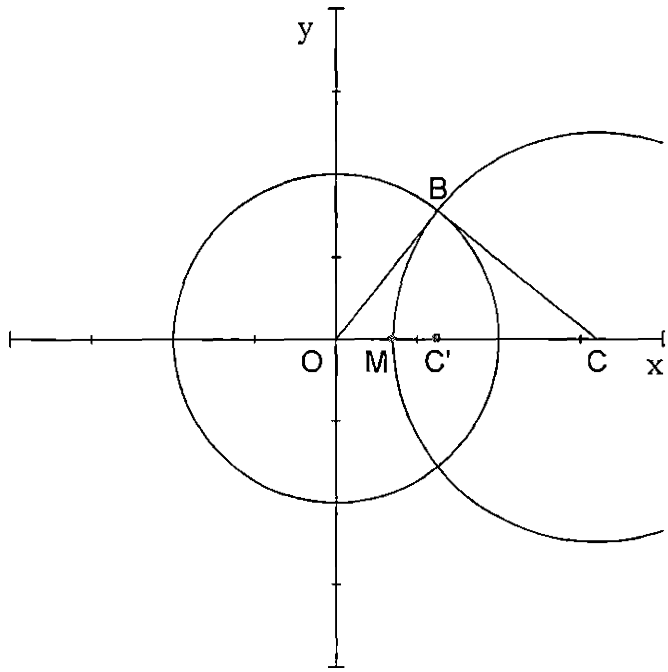


Figure 4.2: Inversions in $D$ and $H$

will use is that this circle bisects $OP$. However, $s$ is a hyperbolic length. What is its Euclidean equivalent? By 4.3, it is $\tanh s$. Then, the center of the circle we want is at $\left(\frac{1}{\tanh s}, 0\right)$ or $(\coth s, 0)$. Then, 4.2 insures that the equation of the circle whose arc is the perpendicular bisector of $OP$ has the Cartesian equation

$$(x - \coth s)^2 + y^2 = \coth^2 s - 1$$

or

$$(x - \coth s)^2 + y^2 = \operatorname{csch}^2 s. \tag{4.5}$$

Now consider the perpendicular bisector of $PQ$, which is also an angle bisector of $\angle POQ$. This line has equation

$$y = x \tan \frac{\theta}{2}. \tag{4.6}$$

We need to solve the system of equations 4.5 and 4.6:

$$\begin{cases} y = x \tan \frac{\theta}{2} \\ (x - \coth s)^2 + y^2 = \operatorname{csch}^2 s \end{cases}$$

$$\begin{aligned} (x - \coth s)^2 + \left(x \tan \frac{\theta}{2}\right)^2 &= \operatorname{csch}^2 s \\ x^2 \tan^2 \frac{\theta}{2} + x^2 - 2x \coth s + \coth^2 s - \operatorname{csch}^2 s &= 0 \\ \left(\tan^2 \frac{\theta}{2} + 1\right) x^2 + (-2 \coth s)\, x + \left(\coth^2 s - \frac{1}{\sinh^2 s}\right) &= 0. \end{aligned}$$

Using quadratic formula, we get:

$$\begin{aligned} x &= \frac{2 \coth s \pm \sqrt{4 \coth^2 s - 4 \left(\tan^2 \frac{\theta}{2} + 1\right) \left(\coth^2 s - \frac{1}{\sinh^2 s}\right)}}{2 \left(\tan^2 \frac{\theta}{2} + 1\right)} \\ x &= \frac{\coth s \pm \sqrt{\coth^2 s - \sec^2 \frac{\theta}{2} \left(\coth^2 s - \operatorname{csch}^2 s\right)}}{\sec^2 \frac{\theta}{2}} \\ x &= \left(\coth s \pm \sqrt{\coth^2 s - \sec^2 \frac{\theta}{2}}\right) \cos^2 \frac{\theta}{2}. \end{aligned}$$

Because $x = \left(\coth s + \sqrt{\coth^2 s - \sec^2 \frac{\theta}{2}}\right) \cos^2 \frac{\theta}{2}$ is outside of the unit disk, we will work with $x = \left(\coth s - \sqrt{\coth^2 s - \sec^2 \frac{\theta}{2}}\right) \cos^2 \frac{\theta}{2}$ as the only value of $x$ in the locus we seek.

Therefore,

$$x = \left(\coth s - \sqrt{\coth^2 s - \sec^2 \frac{\theta}{2}}\right) \cos^2 \frac{\theta}{2}, \quad (4.7)$$

$$y = \left(\coth s - \sqrt{\coth^2 s - \sec^2 \frac{\theta}{2}}\right) \cos \frac{\theta}{2} \sin \frac{\theta}{2}. \quad (4.8)$$

and for $x = r\cos\frac{\theta}{2}$ and $r = x\sec\frac{\theta}{2}$ , we have

$$r = \left(\coth s - \sqrt{\coth^2 s - \sec^2 \frac{\theta}{2}}\right) \cos \frac{\theta}{2}. \quad (4.9)$$

Parameters 4.7–4.9 describe the locus of circumcenters of the $\alpha$-family.

## 4.3 Relations Between the Parameters of $\alpha$-Family

Now, it is time to introduce some properties of hyperbolic geometry applicable to our construction of the $\alpha$–family. Let $ACB$ be a right triangle in the hyperbolic plane with sides of length $a, b$, and hypotenuse of length $c$. Then, the hyperbolic Pythagorean Theorem and Lobachevsky's Theorem state that

$$\cosh 2c = \cosh 2a \cdot \cosh 2b \quad (4.10)$$

and

$$\tan A = \frac{\tanh 2a}{\sinh 2b}, \quad (4.11)$$

respectively [BEG99]. Using 4.10 and 4.11, we can derive the Sine Formula

$$\sin A = \frac{\sinh 2a}{\sinh 2c}. \quad (4.12)$$

Returning to our construction, let $h$ be the altitude of $POQ$ from vertex $O$ to the base $PQ$, and let $PQ = 2a$, meaning that $h$ divides $PQ$ into two equal segments of length $a$. In the context of our construction of the collection of triangles $P'OQ'$, 4.10, 4.11, and 4.12 yield

$$\cosh 2s = \cosh 2a \cdot \cosh 2h, \quad (4.13)$$

$$\tan \frac{\theta}{2} = \frac{\tanh 2a}{\sinh 2h}, \quad (4.14)$$

$$\sin \frac{\theta}{2} = \frac{\sinh 2a}{\sinh 2s}. \quad (4.15)$$

Then

$$\begin{aligned}
\tan\frac{\theta}{2} &= \frac{\tanh 2a}{\sinh 2h} \\
\tan\frac{\theta}{2} &= \frac{\sinh 2a}{\cosh 2a}\cdot\frac{1}{\sinh 2h} \\
\tan\frac{\theta}{2} &= \frac{\sinh 2a}{1}\cdot\frac{\cosh 2h}{\cosh 2s}\cdot\frac{1}{\sinh 2h} \qquad (\text{ by 4.13}) \\
\tan\frac{\theta}{2} &= \frac{\sinh 2a}{1}\cdot\frac{\cosh 2h}{\cosh 2s}\cdot\frac{1}{\sinh 2h}.
\end{aligned}$$

Now, multiplying the left-hand side by $\frac{1}{\sin\frac{\theta}{2}}$ and right-hand side by $\frac{\sinh 2s}{\sinh 2a}$ (the two expressions are equal by 4.15), we get

$$\begin{aligned}
\left(\tan\frac{\theta}{2}\right)\frac{1}{\sin\frac{\theta}{2}} &= \left(\frac{\sinh 2a}{1}\cdot\frac{\cosh 2h}{\cosh 2s}\cdot\frac{1}{\sinh 2h}\right)\frac{\sinh 2s}{\sinh 2a} \\
\frac{1}{\cos\frac{\theta}{2}} &= \coth 2h\cdot\tanh 2s \\
\tanh 2h &= \cos\frac{\theta}{2}\cdot\tanh 2s.
\end{aligned}$$

Let $M$ denote the point of intersection of the perpendicular bisector of the side $OP$ and let $d$ denote the length of the diameter of the circumcircle centered at $R$. Figure 4.3 provides an example of the set of circles whose centers are intersections of the perpendicular bisectors of each triangle in the $\alpha$-family when $\alpha = \frac{\pi}{2}$. It follows that $OR = \frac{d}{2}$ and $OM = \frac{s}{2}$. By 4.12,

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \frac{\sinh\left(2\cdot\frac{s}{2}\right)}{\sinh\left(2\cdot\frac{d}{2}\right)} \\
\sinh s &= \sin\frac{\alpha}{2}\sinh d. \qquad (4.16)
\end{aligned}$$

This result describes the relation between the lengths $s$ and $d$ for a fixed angle $\alpha$.

Now consider a right triangle $OMR$. Let $u = d(R, M)$. Then, by 4.11,

$$\tan\frac{\theta}{2} = \frac{\tanh 2u}{\sinh s} \text{ and } \tan\frac{\alpha}{2} = \frac{\tanh s}{\sinh 2u}, \text{ or } \sinh^2 2u = \frac{\tanh^2 s}{\tan^2\frac{\alpha}{2}}.$$

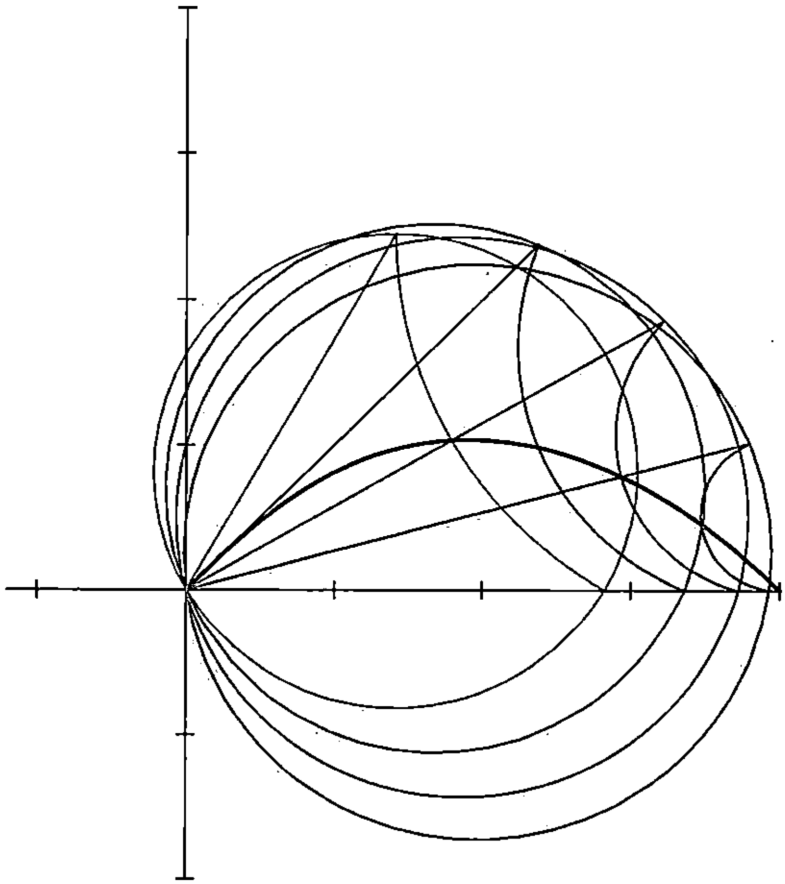

Figure 4.3: The Set of Circumcenters for the $\alpha$-Family when $\alpha = \frac{\pi}{2}$

Hyperbolic trigonometric identities state that $\tanh^2 2u = \frac{\sinh^2 2u}{\cosh^2 2u} = \frac{\sinh^2 2u}{1+\sinh^2 2u}$. Then,

$$\begin{aligned}
\tan^2\frac{\theta}{2} &= \frac{\tanh^2 2u}{\sinh^2 s} \\
\tan^2\frac{\theta}{2} &= \frac{\sinh^2 2u}{1+\sinh^2 2u}\cdot\frac{1}{\sinh^2 s} \\
\tan^2\frac{\theta}{2} &= \frac{\tanh^2 s}{\tan^2\frac{\alpha}{2}}\cdot\frac{1}{1+\frac{\tanh^2 s}{\tan^2\frac{\alpha}{2}}}\cdot\frac{1}{\sinh^2 s} \\
\tan^2\frac{\theta}{2} &= \frac{\tanh^2 s}{\tan^2\frac{\alpha}{2}}\cdot\frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\alpha}{2}+\tanh^2 s}\cdot\frac{1}{\sinh^2 s} \\
\tan^2\frac{\theta}{2} &= \frac{1}{\cosh^2 s\tan^2\frac{\alpha}{2}+\sinh^2 s} \\
\cot^2\frac{\theta}{2} &= \cosh^2 s\tan^2\frac{\alpha}{2}+\sinh^2 s \\
\cot^2\frac{\theta}{2} &= \cosh^2 s\sec^2\frac{\alpha}{2}-1 \\
\csc^2\frac{\theta}{2} &= \cosh^2 s\sec^2\frac{\alpha}{2} \\
\cos\frac{\alpha}{2} &= \sin\frac{\theta}{2}\cosh s. \qquad (4.17)
\end{aligned}$$

Expression 4.17 provides the relation between angles $\alpha$ and $\theta$.

## 4.4 Polynomial $f_q$ and $\alpha$-Family

Recall the relations 4.7 and 4.17 obtained in the previous sections:

$$x = \left(\coth s - \sqrt{\coth^2 s - \sec^2\frac{\theta}{2}}\right)\cos^2\frac{\theta}{2}$$

and

$$\cos\frac{\alpha}{2} = \sin\frac{\theta}{2}\cosh s,$$

respectively. We will use 4.17 in 4.7 to reduce the first relation to just one variable $s$ (because $\alpha$ is a fixed quantity). Changing 4.17 into expression containing $\cos\frac{\theta}{2}$ instead of $\sin\frac{\theta}{2}$ is useful in 4.7:

$$\begin{aligned}
\cos^2\frac{\alpha}{2} &= \sin^2\frac{\theta}{2}\cosh^2 s \\
\cos^2\frac{\alpha}{2} &= \left(1-\cos^2\frac{\theta}{2}\right)\cosh^2 s \\
\cos^2\frac{\alpha}{2}\operatorname{sech}^2 s &= 1-\cos^2\frac{\theta}{2} \\
\cos^2\frac{\theta}{2} &= 1-\cos^2\frac{\alpha}{2}\operatorname{sech}^2 s.
\end{aligned}$$

Now, 4.7 can be changed as follows:

$$\begin{aligned}
x &= \left(\coth s-\sqrt{\coth^2 s-\sec^2\frac{\theta}{2}}\right)\cos^2\frac{\theta}{2} \\
x &= \left(\coth s-\sqrt{\coth^2 s-\frac{1}{1-\cos^2\frac{\alpha}{2}\operatorname{sech}^2 s}}\right)\left(1-\cos^2\frac{\alpha}{2}\operatorname{sech}^2 s\right) \\
x &= \left(\coth s-\sqrt{\frac{\coth^2 s-\operatorname{csch}^2 s\cos^2\frac{\alpha}{2}-1}{1-\cos^2\frac{\alpha}{2}\operatorname{sech}^2 s}}\right)\left(1-\cos^2\frac{\alpha}{2}\operatorname{sech}^2 s\right) \\
x &= \coth s\left(1-\cos^2\frac{\alpha}{2}\operatorname{sech}^2 s\right)-\sqrt{\operatorname{csch}^2 s-\operatorname{csch}^2 s\cos^2\frac{\alpha}{2}}\sqrt{1-\cos^2\frac{\alpha}{2}\operatorname{sech}^2 s} \\
x &= \coth s-\cos^2\frac{\alpha}{2}\operatorname{sech} s\operatorname{csch} s-\operatorname{csch} s\sin\frac{\alpha}{2}\sqrt{1-\cos^2\frac{\alpha}{2}\operatorname{sech}^2 s}. \qquad (4.18)
\end{aligned}$$

Similarly,

$$\begin{aligned}
y &= \left(\coth s-\sqrt{\coth^2 s-\sec^2\frac{\theta}{2}}\right)\cos\frac{\theta}{2}\sin\frac{\theta}{2} \\
y &= \left(\coth s-\sqrt{\coth^2 s-\frac{1}{1-\cos^2\frac{\alpha}{2}\operatorname{sech}^2 s}}\right)\sqrt{1-\cos^2\frac{\alpha}{2}\operatorname{sech}^2 s}\left(\cos\frac{\alpha}{2}\operatorname{sech} s\right) \\
y &= \operatorname{csch} s\cos\frac{\alpha}{2}\sqrt{1-\cos^2\frac{\alpha}{2}\operatorname{sech}^2 s}-\operatorname{csch} s\operatorname{sech} s\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}. \qquad (4.19)
\end{aligned}$$

Recall the irreducible polynomial 2.1: $f_q = x^3+qy^3+xy^2+qx^2y-x+qy=0$. In its factored form this equation can be written as

$$\left(x^2+y^2\right)(x+qy) = x-qy. \qquad (4.20)$$

Our choice of $q$ should satisfy this equation. We claim that $q = \tan\frac{\alpha}{2}$. Using expressions 4.18 and 4.19 in 4.20, we will evaluate each element of this equation individually with this particular choice of $q$:

$$\begin{aligned}
&x + qy \\
&= \coth s - \cos^2\tfrac{\alpha}{2}\operatorname{sech} s \operatorname{csch} s - \operatorname{csch} s \sin\tfrac{\alpha}{2}\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s} \\
&+ \tan\tfrac{\alpha}{2}\operatorname{csch} s \cos\tfrac{\alpha}{2}\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s} - \tan\tfrac{\alpha}{2}\operatorname{csch} s \operatorname{sech} s \sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2} \\
&= \coth s - \cos^2\tfrac{\alpha}{2}\operatorname{sech} s \operatorname{csch} s - \operatorname{csch} s \operatorname{sech} s \sin^2\tfrac{\alpha}{2} \\
&= \tanh s
\end{aligned}$$

$$\begin{aligned}
&x - qy \\
&= \coth s - \cos^2\tfrac{\alpha}{2}\operatorname{sech} s \operatorname{csch} s - \operatorname{csch} s \sin\tfrac{\alpha}{2}\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s} \\
&- \tan\tfrac{\alpha}{2}\operatorname{csch} s \cos\tfrac{\alpha}{2}\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s} + \tan\tfrac{\alpha}{2}\operatorname{csch} s \operatorname{sech} s \sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2} \\
&= \coth s - \operatorname{sech} s \operatorname{csch} s \left(\cos^2\tfrac{\alpha}{2} - \sin^2\tfrac{\alpha}{2}\right) - 2\sin\tfrac{\alpha}{2}\operatorname{csch} s\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s}
\end{aligned}$$

$$\begin{aligned}
&x^2 = \left(\coth s - \cos^2\tfrac{\alpha}{2}\operatorname{sech} s \operatorname{csch} s - \operatorname{csch} s \sin\tfrac{\alpha}{2}\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s}\right)^2 \\
&= \coth^2 s - \coth s \cos^2\tfrac{\alpha}{2}\operatorname{sech} s \operatorname{csch} s - \coth s \operatorname{csch} s \sin\tfrac{\alpha}{2}\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s} \\
&- \coth s \cos^2\tfrac{\alpha}{2}\operatorname{sech} s \operatorname{csch} s + \left(\cos^2\tfrac{\alpha}{2}\operatorname{sech} s \operatorname{csch} s\right)^2 \\
&+ \cos^2\tfrac{\alpha}{2}\operatorname{sech} s \operatorname{csch} s \operatorname{csch} s \sin\tfrac{\alpha}{2}\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s} \\
&- \coth s \operatorname{csch} s \sin\tfrac{\alpha}{2}\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s} + \cos^2\tfrac{\alpha}{2}\operatorname{sech} s \operatorname{csch} s \operatorname{csch} s \sin\tfrac{\alpha}{2} \\
&\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s} + \left(\operatorname{csch} s \sin\tfrac{\alpha}{2}\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s}\right)^2 \\
&= \coth^2 s - 2\cos^2\tfrac{\alpha}{2}\operatorname{csch}^2 s + \\
&\left(2\cos^2\tfrac{\alpha}{2}\operatorname{sech} s \operatorname{csch}^2 s \sin\tfrac{\alpha}{2} - 2\coth s \operatorname{csch} s \sin\tfrac{\alpha}{2}\right)\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s} \\
&+ \cos^4\tfrac{\alpha}{2}\operatorname{sech}^2 s \operatorname{csch}^2 s + \operatorname{csch}^2 s \sin^2\tfrac{\alpha}{2} - \cos^2\tfrac{\alpha}{2}\sin^2\tfrac{\alpha}{2}\operatorname{sech}^2 s \operatorname{csch}^2 s
\end{aligned}$$

$$\begin{aligned}
&y^2 = \left(\operatorname{csch} s \cos\tfrac{\alpha}{2}\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s} - \operatorname{csch} s \operatorname{sech} s \sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}\right)^2 \\
&= \left(\operatorname{csch} s \cos\tfrac{\alpha}{2}\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s}\right)^2 \\
&-2\left(\operatorname{csch} s \cos\tfrac{\alpha}{2}\sqrt{1-\cos^2\tfrac{\alpha}{2}\operatorname{sech}^2 s}\right)\left(\operatorname{csch} s \operatorname{sech} s \sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}\right) \\
&+ \left(\operatorname{csch} s \operatorname{sech} s \sin\tfrac{\alpha}{2}\cos\tfrac{\alpha}{2}\right)^2
\end{aligned}$$

$$= \operatorname{csch}^2 s \cos^2 \tfrac{\alpha}{2} - \cos^4 \tfrac{\alpha}{2} \operatorname{sech}^2 s \operatorname{csch}^2 s$$
$$-2 \operatorname{csch}^2 s \operatorname{sech} s \sin \tfrac{\alpha}{2} \cos^2 \tfrac{\alpha}{2} \sqrt{1 - \cos^2 \tfrac{\alpha}{2} \operatorname{sech}^2 s} + \operatorname{csch}^2 s \operatorname{sech}^2 s \sin^2 \tfrac{\alpha}{2} \cos^2 \tfrac{\alpha}{2}$$

$$\begin{aligned}
&x^2 + y^2 \\
&= \coth^2 s - 2\cos^2 \tfrac{\alpha}{2} \operatorname{csch}^2 s + 2\cos^2 \tfrac{\alpha}{2} \operatorname{sech} s \operatorname{csch}^2 s \sin \tfrac{\alpha}{2} \sqrt{1 - \cos^2 \tfrac{\alpha}{2} \operatorname{sech}^2 s} \\
&-2 \coth s \operatorname{csch} s \sin \tfrac{\alpha}{2} \sqrt{1 - \cos^2 \tfrac{\alpha}{2} \operatorname{sech}^2 s} + \cos^4 \tfrac{\alpha}{2} \operatorname{sech}^2 s \operatorname{csch}^2 s \\
&+ \operatorname{csch}^2 s \sin^2 \tfrac{\alpha}{2} - \cos^2 \tfrac{\alpha}{2} \sin^2 \tfrac{\alpha}{2} \operatorname{sech}^2 s \operatorname{csch}^2 s + \operatorname{csch}^2 s \cos^2 \tfrac{\alpha}{2} \\
&- \cos^4 \tfrac{\alpha}{2} \operatorname{sech}^2 s \operatorname{csch}^2 s - 2\cos^2 \tfrac{\alpha}{2} \operatorname{sech} s \operatorname{csch}^2 s \sin \tfrac{\alpha}{2} \sqrt{1 - \cos^2 \tfrac{\alpha}{2} \operatorname{sech}^2 s} \\
&+ \operatorname{csch}^2 s \operatorname{sech}^2 s \sin^2 \tfrac{\alpha}{2} \cos^2 \tfrac{\alpha}{2} \\
&= \coth^2 s + \operatorname{csch}^2 s \left(\sin^2 \tfrac{\alpha}{2} - \cos^2 \tfrac{\alpha}{2}\right) - 2 \coth s \operatorname{csch} s \sin \tfrac{\alpha}{2} \sqrt{1 - \cos^2 \tfrac{\alpha}{2} \operatorname{sech}^2 s}.
\end{aligned}$$

Summarizing the calculations above, we get:

$$\begin{aligned}
&(x + qy)\left(x^2 + y^2\right) \\
&= \tanh s \left(\coth^2 s + \operatorname{csch}^2 s \left(\sin^2 \tfrac{\alpha}{2} - \cos^2 \tfrac{\alpha}{2}\right) - 2 \coth s \operatorname{csch} s \sin \tfrac{\alpha}{2} \sqrt{1 - \cos^2 \tfrac{\alpha}{2} \operatorname{sech}^2 s}\right) \\
&= \tanh s \left(\coth^2 s - \cos^2 \tfrac{\alpha}{2} \operatorname{csch}^2 s - 2 \coth s \operatorname{csch} s \sin \tfrac{\alpha}{2} \sqrt{1 - \cos^2 \tfrac{\alpha}{2} \operatorname{sech}^2 s} + \sin^2 \tfrac{\alpha}{2} \operatorname{csch}^2 s\right) \\
&= \coth s - \cos^2 \tfrac{\alpha}{2} \operatorname{csch} s \operatorname{sech} s - 2 \operatorname{csch} s \sin \tfrac{\alpha}{2} \sqrt{1 - \cos^2 \tfrac{\alpha}{2} \operatorname{sech}^2 s} + \sin^2 \tfrac{\alpha}{2} \operatorname{csch} s \operatorname{sech} s \\
&= x - qy.
\end{aligned}$$

Therefore, we conclude that a locus of circumcenters of the $\alpha$-family in the hyperbolic plane is the same curve as the cubic obtained by the polynomial $f_q = x^3 + qy^3 + xy^2 + qx^2y - x + qy = 0$, when $q = \tan \frac{\alpha}{2}$ for a fixed $\alpha$.

# Chapter 5

# Product Structure

## 5.1 Introduction

It is well-known that an irreducible non-singular cubic curve carries the structure of an abelian group by the following construction. Let $O$ be any flex of the non-singular, irreducible cubic $F$. We are looking at $F$ as a projective complex curve, so the points of $F$ are homogeneous triples $(x, y, z)$ of complex numbers. Let $P$ and $Q$ be two points of $F$ that are not necessarily distinct. We define addition (also called "product") of two points $P$ and $Q$ in the following manner: let $S$ be the third point of intersection of line $PQ$ with $F$; then $P + Q$ is the third point of intersection of line $OS$ with $F$. Using this definition of addition of points on $F$, it is easy to show that the *commutative law* holds and $P + Q = Q + P$. Also, $P + O = P$, and, therefore, the flex $O$ is the *identity element.* For any point $P$ on $F$, there is the *additive inverse* $-P$ such that $P + (-P) = O$. Finally, it is possible to show that the *associative law* also holds for any three points of $F$, that is $(P + Q) + R = P + (Q + R)$ for points $P, Q, R$ on $F$. The abelian group described above has a natural subgroup that can be represented using the Poincaré disk.

If the vertex of the isosceles triangles in the locus construction described in Chapter 4 is placed at $O$, the center of the disk, the resulting elliptic curve will be an irreducible cubic, and $O$ will be a flex, which we take as the identity element for the group product. Symmetry about the origin shows that the inverse of any point on the curve is its negative as a complex number. Following the product construction we find that product of any two points on the curve may fall outside of the disk. However, the curve is also invariant under the involution $f(z) = \frac{1}{z}$ so we can identify a point with the reciprocal of the complex number that represents it (in particular, the point $O$ is identified with the point at infinity in the completed complex plane). Thus the points on the curve within the disk produce a quotient of the subgroup of real points on the curve by this involution. Since collineations of the hyperbolic plane induce birational transformations, it really does not matter where we place the vertex of the triangles. The resulting elliptic curve may not be cubic but the product structure can be carried over in a manner compatible with the transformation that takes the vertex back to $O$. We will investigate how this product structure results in a type of sum on the diameters of the circumcircles of the triangles whose centers are the points on the curve. This would provide an interpretation of the product structure that is independent of the projective representation of the curve.

## 5.2 Computing the Product Rule Using Complex Numbers

Consider the homogeneous irreducible cubic $f_q = x^3 + qy^3 + xy^2 + qx^2y - xz^2 + qyz^2$. Letting $z = 1$ to work in the complex plane, we get the curve $x^3 + qy^3 + xy^2 + qx^2y - x + qy = 0$. Once again, we want the factored form of this curve:

$$x^2 + y^2 = \frac{x - qy}{x + qy}.$$

Let $z = x + iy$ be the complex form of an arbitrary point on this curve. Note that for any complex number $z$, $z^* = x - iy$, the complex conjugate of $z$. Then, the factored form of the cubic curve presented above can be written in the complex form as following:

$$\begin{aligned}
zz^* &= \frac{\operatorname{Re} z - q \operatorname{Im} z}{\operatorname{Re} z + q \operatorname{Im} z} \\
zz^* &= \frac{\frac{1}{2}(z + z^*) - \frac{q}{2i}(z - z^*)}{\frac{1}{2}(z + z^*) + \frac{q}{2i}(z - z^*)} \\
zz^* &= \frac{z + z^* - iqz^* + iqz}{z + z^* + iqz^* - iqz} \\
zz^* &= \frac{(1 + qi)\, z + (1 - qi)\, z^*}{(1 - qi)\, z + (1 + qi)\, z^*} \\
zz^* &= \frac{\rho z + z^*}{z + \rho z^*},
\end{aligned} \tag{5.1}$$

where $\rho = \frac{1+qi}{1-qi}$.

If $w$ is another point on the curve $f_q$, then $ww^* = \frac{\rho w + w^*}{w + \rho w^*}$. Let $L(t) = t(z-w) + z$ be the Cartesian representation of the line through the points $z$ and $w$. This line has the third real point of intersection with $f_q$, say it is the point

$$L = t(z-w) + z. \tag{5.2}$$

Then

$$LL^* = \frac{\rho L + L^*}{L + \rho L^*}$$

or

$$LL^*(L + \rho L^*) - (\rho L + L^*) = 0.$$

Using the value 5.2 of $L$ to expand this relation, we get the following equation: $-t^3w^2w^* + t^3w^2z^* + 2t^3wzw^* - 2t^3wzz^* - \rho t^3w(w^*)^2 + 2\rho t^3ww^*z^* - \rho t^3w(z^*)^2 - t^3z^2 w^* + t^3z^2z^* + \rho t^3z(w^*)^2 - 2\rho t^3zw^*z^* + \rho t^3z(z^*)^2 + t^2w^2z^* + 2t^2wzw^* - 4t^2wzz^* + 2\rho t^2ww^* z^* - 2\rho t^2w(z^*)^2 - 2t^2z^2w^* + 3t^2z^2z^* + \rho t^2z(w^*)^2 - 4\rho t^2zw^*z^* + 3\rho t^2z(z^*)^2 - 2twzz^* - \rho tw(z^*)^2 + \rho tw - tz^2w^* + 3tz^2z^* - 2\rho tzw^*z^* + 3\rho tz(z^*)^2 - \rho tz + tw^* - tz^* + z^2z^* + \rho z(z^*)^2 - \rho z - z^* = 0$. Collecting the $t$-terms and factoring coefficients of each term, we get

$t^3((z-w)(z^*-w^*)(z-w+\rho(z^*-w^*)))$
$+t^2((z+\rho z^*)(z-w)(z^*-w^*) + (z-w+\rho(z^*-w^*))(z^*(z-w)+z(z^*-w^*)))$
$+t(w^*-z^*-\rho(z-w)+zz^*(z-w+\rho(z^*-w^*)) + (z+\rho z^*)(z^*(z-w)+z(z^*-w^*)))$
$+zz^*(z+\rho z^*) - z^* - z\rho = 0.$

The constant term $zz^*(z+\rho z^*) - z^* - z\rho = 0$ by 5.1. Therefore the polynomial above has no constant term. Let $F(t)$ denote this polynomial:

$F(t) = t^3((z-w)(z^*-w^*)(z-w+\rho(z^*-w^*)))$
$+t^2((z+\rho z^*)(z-w)(z^*-w^*) + (z-w+\rho(z^*-w^*))(z^*(z-w)+z(z^*-w^*)))$
$+t(w^*-z^*-\rho(z-w)+zz^*(z-w+\rho(z^*-w^*)) + (z+\rho z^*)(z^*(z-w)+z(z^*-w^*)))$.

Note that if $t=0, L(0)=z$, and if $t=-1, L(-1)=w$. Since $z,w \in f_q$, $t=0,-1$ are the roots of $F(t)$. In order to find the third root (we represent it by $R$), let $G(t)$ be the polynomial of the form

$$G(t) = (z-w)(z^*-w^*)(z-w+\rho(z^*-w^*))\,t(t+1)(t-R).$$

We multiplied the product of three linear terms $t(t+1)(t-R)$ by the leading coefficient of $F(t)$ to equate $F(t)$ and $G(t)$ and solve for $R$. With the setup we have, we know that

equation $F(t) - G(t) = 0$ should hold. Expanding this equation, we get:
$tw^* - tz^* + tw\rho - tz\rho + t^2w^2w^* - t^2z^2w^* + 2t^2z^2z^* - tz^2w^* + 3tz^2z^* - tw\rho(z^*)^2 + 3t z\rho(z^*)^2 - Rt^2w^2w^* + Rt^2w^2z^* - Rt^2z^2w^* + Rt^2z^2z^* - 2twzz^* + t^2w\rho(w^*)^2 - t^2w\rho(z^*)^2 + 2t^2z\rho(z^*)^2 - Rtw^2w^* + Rtw^2z^* - Rtz^2w^* + Rtz^2z^* - 2t^2wzz^* - Rtw\rho(w^*)^2 - Rtw\rho(z^*)^2 + Rtz\rho(w^*)^2 + Rtz\rho(z^*)^2 + 2Rtwzw^* - 2Rtwzz^* - 2t^2z\rho w^*z^* - Rt^2w\rho(w^*)^2 - Rt^2w\rho(z^*)^2 + Rt^2 z\rho(w^*)^2 + Rt^2z\rho(z^*)^2 + 2Rt^2wzw^* - 2Rt^2wzz^* - 2tz\rho w^*z^* + 2Rt^2w\rho w^*z^* - 2Rt^2z\rho w^* z^* + 2Rtw\rho w^*z^* - 2Rtz\rho w^*z^* = 0.$

Collecting $t$-terms, we can simplify the coefficients using the relation 5.1, which holds for any $z, w \in f_q$. After additional simplification, the equation yields the following solution:
$R = \frac{w^*+2z^*+w\rho+2z\rho-z^2w^*-w\rho(z^*)^2-2wzz^*-2z\rho w^*z^*}{w^*-z^*+w\rho-z\rho-w^2z^*+z^2w^*+w\rho(z^*)^2-z\rho(w^*)^2-2wzw^*+2wzz^*-2w\rho w^*z^*+2z\rho w^*z^*}$. Since this expression represents one of the possible values $t$ can take, let $t = R$ in $L = t(z-w)+z$. We get $L(R) = z - \frac{(w-z)\left(w^*+2z^*+w\rho+2z\rho-z^2w^*-w\rho(z^*)^2-2wzz^*-2z\rho w^*z^*\right)}{-w^2z^*-2wzw^*+2wzz^*-2\rho ww^*z^*+\rho w(z^*)^2+\rho w+z^2w^*-\rho z(w^*)^2+2\rho zw^*z^*-\rho z+w^*-z^*}$.

If the product of $z$ and $w$, denoted by $z \circ w$, falls outside of the disk, we can identify any point on this curve with the reciprocal of the complex number that represents it, because $f_q$ is invariant under the involution $f_q(z) = \frac{1}{z}$. Then,
$z \circ w = \frac{-ww^*-2wz^*+2zw^*+zz^*-w^2\rho+z^2\rho+w^2\rho(z^*)^2-z^2\rho(w^*)^2-wz^2w^*+w^2zz^*}{-(w^*-z^*)(w-z)(w-z+\rho w^*-\rho z^*)}$.

Note that $z \circ O = O \circ z = z$ since $zz^* = \frac{\rho z+z^*}{z+\rho z^*}$ on $f_q$. However, it is not clear from this formula how to compute $z \circ z$. We will address this in Section 5.3.

## 5.3 Parametrization in Terms of $d$

At this point, we would like to develop another result from the locus of the $\alpha$-family that will help us to represent $z \circ w$ as a locus of circumcenters on $f_q$. Recall the result 4.17 which provided a relation between the length $s$ of the equal sides of an isosceles triangle and angle $\theta$ between the the equal sides. This relation is

$$\cos\frac{\alpha}{2} = \sin\frac{\theta}{2}\cosh s,$$

where $\alpha$ is a fixed oriented angle between the perpendicular bisectors of the triangle defining the $\alpha$-family. Now we can describe the locus of circumcenters of the $\alpha$-family

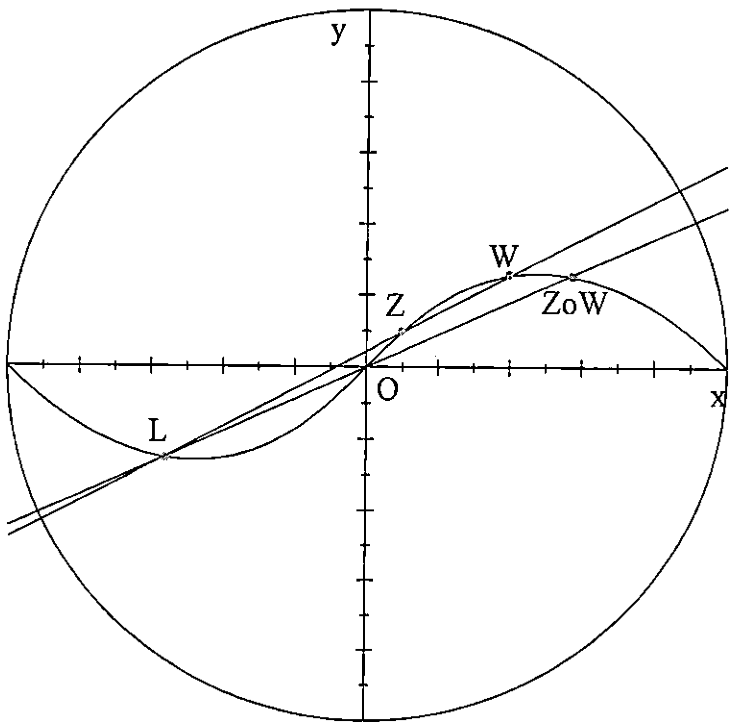


Figure 5.1: Product Structure of $f_q$

of triangles in terms of a parameter $d$, which is the diameter of the circumcircle centered at $z$, an element of the locus of circumcenters.

$$\begin{aligned}
\cos^2\frac{\alpha}{2} &= \sin^2\frac{\theta}{2}\cosh^2 s \\
\cos^2\frac{\alpha}{2} &= \sin^2\frac{\theta}{2}\left(1+\sin^2\frac{\alpha}{2}\sinh^2 d\right) \\
\cos^2\frac{\alpha}{2} &= \sin^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\sin^2\frac{\alpha}{2}\left(\cosh^2 d-1\right) \\
\cos^2\frac{\alpha}{2} &= \sin^2\frac{\theta}{2}\left(1-\sin^2\frac{\alpha}{2}\right)+\sin^2\frac{\theta}{2}\sin^2\frac{\alpha}{2}\cosh^2 d \\
\cos^2\frac{\alpha}{2} &= \sin^2\frac{\theta}{2}\cos^2\frac{\alpha}{2}+\sin^2\frac{\theta}{2}\sin^2\frac{\alpha}{2}\cosh^2 d \\
\cos^2\frac{\alpha}{2}\left(1-\sin^2\frac{\theta}{2}\right) &= \sin^2\frac{\theta}{2}\sin^2\frac{\alpha}{2}\cosh^2 d \\
\frac{\cos^2\frac{\alpha}{2}\cos^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2}\cos^2\frac{\alpha}{2}} &= \frac{\sin^2\frac{\theta}{2}\sin^2\frac{\alpha}{2}\cosh^2 d}{\sin^2\frac{\theta}{2}\cos^2\frac{\alpha}{2}} \\
\cot^2\frac{\theta}{2} &= \tan^2\frac{\alpha}{2}\cosh^2 d. \qquad (5.3)
\end{aligned}$$

When $q=\tan\frac{\alpha}{2}$, 5.3 yields $\cot\frac{\theta}{2}=q\cosh d$. Then,

$$\begin{aligned}
\cot^2\frac{\theta}{2}+1 &= 1+q^2\cosh^2 d \\
\csc^2\frac{\theta}{2} &= 1+q^2\cosh^2 d \\
\sin\frac{\theta}{2} &= \frac{1}{\sqrt{1+q^2\cosh^2 d}} \qquad (5.4)
\end{aligned}$$

and

$$\cot\frac{\theta}{2}+i=q\cosh d+i$$

If $z$ is the complex form of the center of the circumcircle, it has a form

$$z=x+iy=r\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right).$$

The length of the radius $r$ (the distance from the origin to $z$) can be expressed in terms of $d$ as $\tanh\frac{d}{2}$. It follows that,

$$\tanh\frac{d}{2}=\frac{\sinh d}{\cosh d+1}=\frac{\sqrt{\cosh^2 d-1}}{\cosh d+1}$$

and

$$z(d)=\frac{\sqrt{\cosh^2 d-1}\,(q\cosh d+i)}{(\cosh d+1)\sqrt{1+q^2\cosh^2 d}}.$$

In order to simplify the notation a little bit, let $H(d) = \frac{\sqrt{\cosh^2 d - 1}}{(\cosh d + 1)\sqrt{1 + q^2\cosh^2 d}}$. Then,

$$z(d) = H(d)(q\cosh d + i) = H(d)\, q\cosh d + iH(d)$$

is the one-parameter representation of a complex point $z \in f_q$, the center of a circumcircle, in terms of the diameter of the circle.

Let $d_1$ denote the diameter of a circle centered at $z$ and $d_2$ denote the diameter of a circle centered at $w$. Then,

$$z(d_1) = H(d_1)\, q\cosh d_1 + iH(d_1)$$

and

$$w(d_2) = H(d_2)\, q\cosh d_2 + iH(d_2).$$

If $z = x + iy$ and $w = u + iv$ are any complex numbers representing points of $f_q$, then we can write these number in terms of $d_1$ and $d_2$ using the following relations:

$$x = H(d_1)\, q\cosh d_1, \tag{5.5}$$

$$y = H(d_1), \tag{5.6}$$

$$u = H(d_2)\, q\cosh d_2, \tag{5.7}$$

$$v = H(d_2). \tag{5.8}$$

Recall the product rule we obtained earlier:
$z \circ w = \frac{-ww^* - 2wz^* + 2zw^* + zz^* - w^2\rho + z^2\rho + w^2\rho(z^*)^2 - z^2\rho(w^*)^2 - wz^2w^* + w^2zz^*}{-(w^* - z^*)(w - z)(w - z + \rho w^* - \rho z^*)}$ for any complex numbers $z, w \in f_q$. Letting $z = x + iy$, $w = u + iv$, and $\rho = \frac{1+qi}{1-qi}$ changes the product rule into a very lengthy result of the form $X + iY$, where $X$ and $Y$ are expressions in terms of $x, y, u, v$, and $q$. These, in their turn, can be replaced by 5.5-5.8. Now we calculate the ratio of $\frac{X}{Y}$. In this expression, we utilize 5.3, and for $q = \tan\frac{\alpha}{2}$ we have $\cot\frac{\theta}{2} = q\cosh d$. Simplification yields

$\cosh(d_1 \circ d_2) =$

$$\frac{q\left(-C_{d1}C_{d2} + C_{d1}^2 + C_{d2}^2 + q^2C_{d1}^2C_{d2}^2 - q^2C_{d1}C_{d2} - \sqrt{q^2C_{d1}^2 + 1}\sqrt{q^2C_{d2}^2 + 1}\sqrt{C_{d1} - 1}\sqrt{C_{d1} + 1}\sqrt{C_{d2} - 1}\sqrt{C_{d2} + 1} - 1\right)}{C_{d1}C_{d2} - q^2C_{d1}^2 - q^2C_{d2}^2 + q^2C_{d1}^2C_{d2}^2 + q^2C_{d1}C_{d2} - \sqrt{q^2C_{d1}^2 + 1}\sqrt{q^2C_{d2}^2 + 1}\sqrt{C_{d1} - 1}\sqrt{C_{d1} + 1}\sqrt{C_{d2} - 1}\sqrt{C_{d2} + 1} - 1},$$

where $C_{d1} = \cosh d_1, C_{d2} = \cosh d_2$. Notation $(d_1 \circ d_2)$ expresses the fact that the result has two parameters $d_1$ and $d_2$, as our expression is dependent only on $d_1$ and $d_2$.

Now, in the locus construction, $s$ denotes the length of the two equal sides of isosceles triangle. Relation 4.16 will become very useful now. It states that $\sinh s = \sin\beta \sinh d$, where $\beta = \frac{\alpha}{2}$. Then,

$$\begin{aligned} q^2 \cosh^2 d + 1 &= \tan^2 \beta \cosh^2 d + 1 \\ &= \frac{\left(1 + \sinh^2 d\right)}{\cos^2 \beta} \frac{\sin^2 \beta}{1} + \frac{\cos^2 \beta}{\cos^2 \beta} \\ &= \frac{\cosh^2 s}{\cos^2 \beta}. \end{aligned}$$

Therefore

$$\sqrt{q^2 \cosh^2 d + 1} = \frac{\cosh s}{\cos \beta}. \tag{5.9}$$

We also will use the following hyperbolic trigonometric property:

$$\sqrt{(\cosh d_1 - 1)(\cosh d_1 + 1)}\sqrt{(\cosh d_2 - 1)(\cosh d_2 + 1)} = \sinh d_1 \sinh d_2. \tag{5.10}$$

Let $C_{s1} = \cosh s_1, C_{s2} = \cosh s_2, S_{d1} = \sinh d_1, S_{d2} = \sinh d_2, S_{s1} = \sinh s_1$, and $S_{s2} = \sinh s_2$. Then 5.9 and 5.10 yield

$$\sqrt{q^2 C_d^2 + 1} = \frac{C_s}{\cos\beta}$$

and

$$\sqrt{C_{d1} - 1}\sqrt{C_{d1} + 1}\sqrt{C_{d2} - 1}\sqrt{C_{d2} + 1} = S_{d1} S_{d2}.$$

Using these relations, we can eliminate the square roots from $\cosh(d_1 \circ d_2)$. In addition, we can substitute $q$ by $\tan\beta$ :

$$\cosh(d_1 \circ d_2) = \frac{(\tan\beta)\left(-C_{d1}C_{d2} + C_{d1}^2 + C_{d2}^2 - C_{d1}C_{d2}\tan^2\beta + C_{d1}^2 C_{d2}^2 \tan^2\beta - \frac{C_{s1}C_{s2}S_{d1}S_{d2}}{\cos^2\beta} - 1\right)}{C_{d1}C_{d2} - C_{d1}^2\tan^2\beta - C_{d2}^2\tan^2\beta + C_{d1}C_{d2}\tan^2\beta + C_{d1}^2 C_{d2}^2\tan^2\beta - \frac{C_{s1}C_{s2}S_{d1}S_{d2}}{\cos^2\beta} - 1}.$$

Repeated use of the relation $\sinh s = \sin\beta \sinh d$ in this expression, simplifies it even further to

$$\cosh(d_1 \circ d_2) = \frac{(S_{s1})\left(-C_{d1}C_{d2} + S_{d1}^2 + S_{d2}^2 - C_{s1}C_{s2}S_{d1}S_{d2} + S_{d1}S_{d2}S_{s1}S_{s2} + 1\right)}{\sqrt{S_{d1}^2 - S_{s1}^2}\left(S_{d1}^2 S_{s2}^2 + C_{d1}C_{d2} - C_{s1}C_{s2}S_{d1}S_{d2} - 1\right)}.$$

It can be shown that this relation is equivalent to the following expression [Sar08]:

$$\cosh\left(d_1 \circ d_2\right) = \frac{C_{d1}C_{d2} + S_{d1}S_{d2}C_{s1}C_{s2}}{\left|\left(S_{s1}S_{d2} - 1\right)\left(S_{d1}S_{s2} + 1\right)\right|}$$

or

$$\cosh\left(d_1 \circ d_2\right) = \frac{\cosh d_1 \cosh d_2 + \sinh d_1 \sinh d_2 \cosh s_1 \cosh s_2}{\left|\left(\sinh s_1 \sinh d_2 - 1\right)\left(\sinh d_1 \sinh s_2 + 1\right)\right|}. \tag{5.11}$$

This is our product formula in terms of the diameters of the circumcircles at the points $z$ and $w$ on the curve. Note that, as $\alpha \to 0$, the formula reduces to

$$\cosh\left(d_1 \circ d_2\right) = \cosh d_1 \cosh d_2 + \sinh d_1 \sinh d_2 = \cosh\left(d_1 + d_2\right).$$

# Chapter 6

# Birational Equivalence of Elliptic Curves

It is well known that any elliptic curve (degree does not have to be three) can be transformed into an irreducible cubic using a birational transformation. We say that these curves are birationally equivalent As it was stated before, the classes of the irreducible cubics depend on the value of the cross-ratio of each class. Using birational equivalence, we can attribute this cross-ratio any elliptic curve.

**Theorem 6.1.** *The locus of circumcenters of an $\alpha$-family of isosceles triangles in any inversive model of the hyperbolic plane is the real points of an elliptic curve whose cross-ratio is $\left[e^{2i\alpha}\right]$. Conversely, the real points of any elliptic curve whose cross-ratio is on unit circle can be produced by such a locus.*

*Proof.* Without loss of generality, we may use the Poincaré disk model of the hyperbolic plane, because any other model can be obtained from an inversive transformation that afford an equivalent measure of distance. If we transform the $\alpha$-family with vertex at the origin and axis the real line to any other vertex and axis, then the new family of triangles will still be an $\alpha$-family, and the new locus of circumcenters will remain an elliptic curve birationally equivalent to the original one because linear fractional transformations induce birational transformations. In particular, its cross-ratio will still be $e^{2i\alpha}$. By 3.1, we know that $f_q$ has the cross-ratio on the unit circle. On the other hand, at the end of Section 4.4 we demonstrated that a locus of circumcenters of $\alpha$-family in the hyperbolic plane is the curve $f_q$ when $q = \tan\frac{\alpha}{2}$. □

Therefore, the argument above insures that the product rule 5.11 obtained by working with specially designed cubic $f_q$ is valid for any locus of circumcenters in the hyperbolic plane.

Let $f^{\alpha}(0, L_{\phi})$ be the locus of circumcenters for the constant angle $\alpha$, with vertex at 0 and axis $L_{\phi}$, the rotation of the real line by $\phi$. Then $f^{\alpha}(0, L_{\phi})$ is a cubic curve for any $\phi$, and $f^{-\alpha}(0, L_{\phi})$ is its reflection in $L_{\phi}$. Since any transformation is a hyperbolic translation $T_m(Z) = \frac{Z+m}{m^* Z+1}$ followed by a rotation about 0, we look at the effect of $T_m$ on $f^{\alpha}(0, L_0)$. Let $L = T_m(L_0)$. If $m = a+ib$ we find that $f^{\alpha}(m, L)$ is described by the equation $f_{-q}(a,b)\left(x^2+y^2\right)^2 + g_m(x,y) = 0$, where $g_m$ is cubic. It follows that $T_m\left(f^{\alpha}(0, L_0)\right)$ is cubic precisely when $m$ is on $f^{-\alpha}(0, L_0)$, which generalizes to the following conclusion [Sar08].

**Theorem 6.2.** *The curve $T_m\left(f^{\alpha}(0, L_{\phi})\right)$ is cubic if and only if $m$ is on $f^{-\alpha}(0, L_{\phi})$.*

**Corollary 6.3.** *If $m \neq 0$ there is a unique axis $L$ for which $f^{\alpha}(m, L)$ is a cubic. If $M$ is the line through 0 and $m$, and $\psi$ is the angle of parallelism for the distance $s = s(0, m)$, then $L = R^m_{\phi}(M)$, the hyperbolic rotation of $M$ about $m$, where $\tan\phi = \cot\frac{\alpha}{2}\sin\psi$.*

For example, consider a transformation $T(z) = \frac{z+m}{m^* z+1}$ for $z = x+iy$. Note that $T(0) = m$, which means that $T$ maps vertex $O$ to some point $m \in D, m = a+ib$. How will the locus of circumcenters look after this transformation? The original locus had the equation $f_q(x,y) = x^3 + qy^3 + qx^2y + xy^2 - x + qy$. Without loss of generality, let $q = 1$. Then, $f_1(x,y) = x^3 + y^3 + x^2y + xy^2 - x + y$. We can find $T(f_1)$ by applying $T^{-1}$ to the variables $x, y$ and substituting into $f_q$.

$$T^{-1}(z) = \frac{z-m}{-m^* z+1} = \left(\frac{(b-y)(ay-bx)+(a-x)(ax+by-1)}{(-ay+bx)^2+(-ax-by+1)^2}\right) + i\left(\frac{(b-y)(ax+by-1)-(a-x)(ay-bx)}{(-ay+bx)^2+(-ax-by+1)^2}\right).$$

Using these new values of $x$ and $y$ in the curve $f_1$, we get the following new equation in the polar form:

$$
\begin{aligned}
&f(r,\theta) = \\
&-r^4\left(a+b-a^3+b^3-ab^2+a^2b\right) + r^3(\cos\theta)\left(6ab-2b^2-a^4+b^4+2ab^3+2a^3b+1\right) \\
&+r^3(\sin\theta)\left(2ab+6b^2-a^4+b^4-2ab^3-2a^3b+1\right) \\
&-2r^2\left(2b\left(a^2+b^2+1\right)+a\left(3b^2-a^2+1\right)\sin 2\theta + b\left(3a^2-b^2-1\right)\cos 2\theta\right) \\
&-r(\sin\theta)\left(6b^2-2ab-a^4+b^4+2ab^3+2a^3b+1\right) \\
&+r(\cos\theta)\left(b^4-2b^2-a^4-6ab-2ab^3-2a^3b+1\right) \\
&+\left(a-b-a^3-b^3-ab^2-a^2b\right).
\end{aligned}
$$

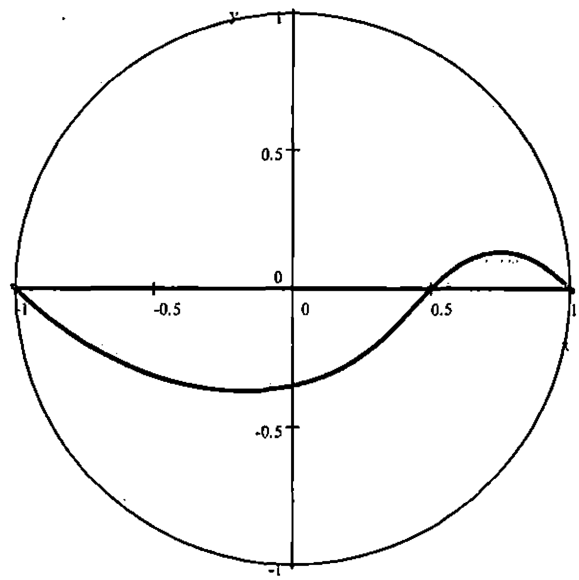


Figure 6.1: $f_1$ for $m = \frac{1}{2}$

Such quartics are called bicircular because its fourth degree term is of the form $(x^2 + y^2)^2$. Such curves have genus 0 or 1 because they have two ordinary singularities at infinity [MM99].

Since $m = a + ib$ is the vertex of the curve, we can experiment with it and move it from $O$ (originally used as our vertex in this project) to another point in the units disk. We can find the equation of $f_1$ for this particular choice of $m$.

Let $m = \frac{1}{2}$. This choice of the vertex shifts this point from the origin to the point $(\frac{1}{2}, 0)$ on the real axis. The equation resulting is

$$5y - 5x - 8xy + 5x^3 - 2x^4 + 5y^3 - 2y^4 + 5xy^2 + 5x^2y - 4x^2y^2 + 2 = 0.$$

Figure 6.1 shows the resulting locus of circumcenters.

Consider another example when the vertex $O$ is moved away from the real axis. This time, let $m = \frac{1}{2} + i\frac{1}{2}$. The resulting equation after this transformation is

$$5x + 9y - 12xy - 10x^2 + 9x^3 - 14y^2 - 4x^4 + 11y^3 - 4y^4 + 9xy^2 + 11x^2y - 8x^2y^2 - 2 = 0,$$

and Figure 6.2 presents the resulting curve in this case.

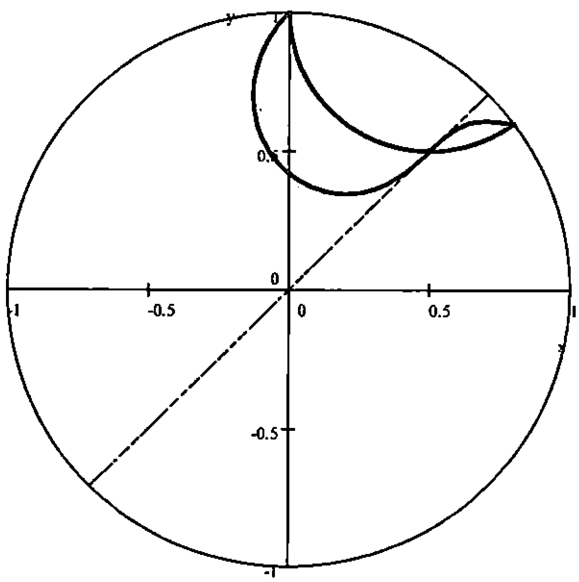


Figure 6.2: $f_1$ for $m = \frac{1}{2} + i\frac{1}{2}$

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