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Homomorphic images of progenitors of order three

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Homomorphic Images of Progenitors of Order Three

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Faculty of

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in

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by

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Abstract

The main purpose of this thesis is to construct finite groups as homomorphic images of infinite semi-direct products, $2^{*n} : N$, $3^{*n} : N$, and $3^{*n} :_m N$, where 2^{*n} and 3^{*n} are free products of n copies of the cyclic group C_2 extended by N, a group of permutations on n letters. We constructed several finite homomorphic images of the semi-direct products $2^{*3} : S_3$, $3^{*4} : S_4$, and $3^{*4} :_m 2{\cdot}S_4^+$. We have constructed A_5 x D_6, $U_3 3$ x 3, $U_3(3)$, and M_{11} x 2.

Finite simple groups are the building blocks for constructing all finite groups. In order to take advantage of large finite groups, such as the Monster group, we need a method that allows us to represent the many elements in some short form that is more effective and manageable. The method of symmetric representation paves the way for us to represent elements of (simple) finite groups. It is very effective because it allows us to write these elements in a short form using the technique of double coset enumeration. Fortunately, this process can be utilized for all finite non-abelian simple groups and any other that has been constructed using double coset enumeration. Furthermore, symmetric representation gives a uniform way for us to construct finite groups, such as permutation groups or groups of matrices.

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Chapter 1

Introduction

There are two methods, permutations and matrices, that are commonly used to represent elements of finite groups. However, if the size of a group is large then none of the methods is completely satisfactory. In this thesis, we present an alternative method for dealing with finite groups. This method is based on the technique of double coset enumeration, explained below. Our method applies to all finite groups, and in particular to all non-abelian simple groups. For instance, this method is used in [?] to represent elements of the smallest Janko sporadic group group J_1, which are usually written as a permutations on 266 letters, as permutations on 11 letters of $L_2(11)$ followed by a word of length at most 4 in the symmetric generators.

1.1 Symmetric Generation of a Group

Let G be a group and and let $T = \{t_1, t_2, \ldots, t_n\} \subseteq G$ and $\overline{T} = \{T_1, T_2, \ldots, T_n\}$, with $T_i = \langle t_i \rangle$, and N the set normalizer of $\overline{T}$ in G .
If $G = \langle T \rangle$, and N acts transitively on $\overline{T}$, T is defined to be a symmetric generating set for G and N is called the control subgroup. In this case G is a homomorphic image of the (infinite) progenitor

$$m^{\star n} : N,$$

where $m^{\star n}$ is a free product of n cyclic groups C_m, where m is the order of t_i, and N is a group of automorphisms of $m^{\star n}$ which act, by conjugation, on the n cyclic subgroups. If $\pi \in N$, then $t_i^{\pi} = t_j^{r}$, where r is an integer and gcd(r,m)=1. Note that if $m = 2$

then N acts, by conjugation, as permutations of the n symmetric generators of order 2. In the case of the progenitor, every element can be represented uniquely as πw, where $\pi \in N$ and w is a word in the symmetric generators. We factor the progenitor by the relations of the type $\pi w(t_1, t_2, .., t_n)$, with $\pi \in N$ and w a word in the $t_i s$, to produce finite homomorphic images. In the next section we describe the process of recognition of these images.

1.2 Manual Double Coset Enumeration

We follow [?] and allow i to stand for the symmetric generator t_i in our expressions. We also represent the coset Nt_i by i, the coset Nt_it_j by ij. Every element of our group can be written as a permutation of N followed by a word in the t_is. Now the double coset NgN is given by $NgN = N\pi wN = NwN$, where $g \in G$, $\pi \in N$, and w is a word in the t_is. The double coset NwN is denoted by $[w]$. For example, the double coset [01] represents Nt_0t_1N. The double coset $NeN = \{N\}$, where e is the identity element, is denoted by $[\star]$. We need the following definitions to apply our technique of double coset enumeration.

The single point and two point stabilizers in N are given by $N^i = C_N(t_i)$ and $N^{ij} = C_N(\langle t_i, t_j \rangle)$ respectively. A k-point stabiliser is similarly defined. The coset stabilizing subgroup, $N^{(w)}$, of N is defined as

$N^{(w)} = \{\pi \in N : Nw\pi = Nw\}$, where w a word in the t_is. We note that N^w is a subgroup of $N^{(w)}$. The number of single cosets in the double coset $[w]$ is given by $\frac{|N|}{|N^{(w)}|}$ (see [?]). We use the process of manual double coset enumeration for G over N and find all of the double cosets $[w]$ and the number of single cosets in each double coset. This enables us to find the index of N in G. The double coset enumeration process is performed by obtaining the orbits of $N^{(w)}$ on the symmetric generators and for each double coset $[w]$, it suffices to recognize the double coset containing Nwt_i for one symmetric generator t_i from each orbit. The double coset enumeration partitions the image G as a union double cosets NgN, where $g \in 2^{*n} : N$ and gives a set $\{g_1, g_2, \ldots\}$ of elements of G such that $G = Ng_1N \cup Ng_2N \cup \ldots$. However, for each i, we have $g_i = \pi_i w_i$, where $\pi_i \in N$ and w_i is a word in the t_is, and so that the double coset deomposition is given by $G = N \cup Nw_2N \cup Nw_3N \cup \ldots$.

Chapter 2

$A_5 \times D_6$ as the Homomorphic Image of $2^{*3} : S_3$

We factor the progenitor $2^{*3} : S_3$ by the relations $[(012)t_0]^{10}$ and $[(01)t_0t_2t_0]^3$ to achieve the finite homomorphic image:

$$\mathrm{G} \cong \frac{2^{*3} : S_3}{[(012)t_0]^{10}, [(01)t_0t_2t_0]^3} \cong A_5 \times D_6.$$

We want to construct by hand the group $\mathrm{G} \cong A_5 \times D_6$.
A symmetric presentation for $2^{*3} : S_3$ is given by:
$< x, y, t \mid x^3 = y^2 = (yx)^2 = 1 = t^2 = [t, y] > \;\cong\; 2^{*3} : S_3$
where $\mathrm{N} \cong S_3$ and the action on x, y on the symmetric generators is given by

$x \sim (0\ 1\ 2)$

$y \sim (1\ 2)$

Our goal is to show that the order of G is at most 360.

Expanding our first relation $[(012)t_0]^{10}$, with $\pi = (012)$, yields:
$(\pi t_0)^{10} = \pi^{10} t_0^{\pi^9} t_0^{\pi^8} t_0^{\pi^7} t_0^{\pi^6} t_0^{\pi^5} t_0^{\pi^4} t_0^{\pi^3} t_0^{\pi^2} t_0^{\pi} t_0 = 1$
$\Rightarrow (012)\ t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0 = 1$

Our second relation $[(01)t_0t_2t_0]^3$, with $\pi = (01)$, yields:

$\pi^3(t_0t_2t_0)^{\pi^2}(t_0t_2t_0)^{\pi}t_0t_2t_0 = 1$

$\Rightarrow (01)\ t_0t_2t_0t_1t_2t_1t_0t_2t_0 = 1$

We begin the process of double coset enumeration of G over S_3.

We start with the double coset with representative word of length zero, namely $NeN = \{Nen|n \in N\} = \{Nnn^{-1}en|n \in N\} = \{Ne^n|n \in N\} =$ N. In our Cayley Diagram, we'll denote this double coset N by [*]. Now, N = S_3 is transitive on the set of symmetric generators $\{0,1,2\}$. That is, the double coset N, has only one orbit, namely $\{0,1,2\}$. It suffices to choose one representative from this orbit of N and ask to which double coset does Nt_i belong. Let us consider Nt_0. Because the first double coset is simply our control group S_3, Nt_0 is not represented within it. Hence, Nt_0 is an element of the new double coset Nt_0N, denoted [0] in our Cayley Diagram. In fact, because Nt_0, Nt_1, and Nt_2 are in the same orbit, they all are elements of [0].

Let us look at the double coset $Nt_0N =$ [0].
First, consider the coset stabilizing group $N^{(0)}$. The only elements of S_3 that will fix Nt_0N are the identity permutation and (12). Hence, the coset stabilizing group $N^{(0)} = N^0 = < e, (12) >$. The orbits of $N^{(0)}$ on $\{0,1,2\}$ are $\{0\}$, $\{1,2\}$. Using the fact that the number of distinct single cosets contained in a double coset is $\frac{|N|}{|N^{(i)}|}$, we find that [0] contains $\frac{6}{2} = 3$ distinct single cosets, namely Nt_0, Nt_1, and Nt_2. Now we choose one representative from each orbit and ask to which double coset does Nt_i belong. Since the order of t_i is 2, $Nt_0t_0 = NeN = N$ which is an element of [*]. However, Nt_0t_1 and Nt_0t_2 move forward together to the new double coset $Nt_0t_1N =$ [01], since neither Nt_0t_1 nor Nt_0t_2 are represented within [0].

Consider the double coset $Nt_0t_1N =$ [01].
$N^{(01)} = N^{01} = < e >$, so the double coset [01] contains $\frac{6}{1} = 6$ distinct single cosets. These 6 distinct single cosets are $\{N(t_0t_1)^n|n \in N\} = \{Nt_0t_1, Nt_1t_0, Nt_2t_1, Nt_0t_2, Nt_1t_2, Nt_2t_0\}$. The orbits of $N^{(01)}$ on $\{0,1,2\}$ are $\{0\}$, $\{1\}$, and $\{2\}$. We must consider each Nt_i from each orbit. $Nt_0t_1t_1 = Nt_0 \in$ [0], while $Nt_0t_1t_0$ and $Nt_0t_1t_2$ move forward to [010] and [012] respectively.

Following the same process, $N^{(010)} = N^{010} = < e >$, so the double coset [010] contains these 6 distinct single cosets $Nt_0t_1t_0N = \{Nt_0t_1t_0, Nt_1t_0t_1, Nt_2t_1t_2, Nt_0t_2t_0, Nt_1t_2t_1, Nt_2t_0t_2\}$. The orbits of $N^{(010)}$ on $\{0,1,2\}$ are $\{0\}$, $\{1\}$, and $\{2\}$. $Nt_0t_1t_0t_0 = Nt_0t_1 \in [01]$ while $Nt_0t_1t_0t_1$ and $Nt_0t_1t_0t_2$ move forward to [0101] and [0102] respectively.

Similarly, $N^{(012)} = N^{012} = < e >$, so we see that the double coset [012] also contains 6 distinct single cosets, namely $Nt_0t_1t_2N = \{Nt_0t_1t_2, Nt_1t_0t_2, Nt_2t_1t_0, Nt_0t_2t_1, Nt_1t_2t_0, Nt_2t_0t_1\}$. The orbits of $N^{(012)}$ on $\{0,1,2\}$ are $\{0\}$, $\{1\}$, and $\{2\}$. $Nt_0t_1t_2t_2 = Nt_0t_1 \in [01]$ while $Nt_0t_1t_2t_0$ and $Nt_0t_1t_2t_1$ are elements within [0120] and [0121] respectively.

Note that we have four double cosets of length four. We will see that the double cosets with words of length four have identical coset stabilizing groups. $N^{(0101)} = N^{(0102)} = N^{(0120)} = N^{(0121)} = < e >$, so each of the double cosets [0101], [0102], [0120], [0121] contains 6 distinct single cosets. The orbits of $N^{(0101)}$, $N^{(0102)}$, $N^{(0120)}$, and $N^{(0121)}$ on $\{0,1,2\}$ are $\{0\}$, $\{1\}$, and $\{2\}$.

We list each distinct single coset within each of the above mentioned double cosets:

$$\begin{aligned} Nt_0t_1t_0t_1N &= \{N(t_0t_1t_0t_1)^n | n \in N\} \\ &= \{Nt_0t_1t_0t_1, Nt_1t_0t_1t_0, Nt_2t_1t_2t_1, Nt_0t_2t_0t_2, Nt_1t_2t_1t_2, Nt_2t_0t_2t_0\}. \end{aligned}$$

$$\begin{aligned} Nt_0t_1t_0t_2N &= \{N(t_0t_1t_0t_2)^n | n \in N\} \\ &= \{Nt_0t_1t_0t_2, Nt_1t_0t_1t_2, Nt_2t_1t_2t_0, Nt_0t_2t_0t_1, Nt_1t_2t_1t_0, Nt_2t_0t_2t_1\}. \end{aligned}$$

$$\begin{aligned} Nt_0t_1t_2t_0N &= \{N(t_0t_1t_2t_0)^n | n \in N\} \\ &= \{Nt_0t_1t_2t_0, Nt_1t_0t_2t_1, Nt_2t_1t_0t_2, Nt_0t_2t_1t_0, Nt_1t_2t_0t_1, Nt_2t_0t_1t_2\}. \end{aligned}$$

$$\begin{aligned} Nt_0t_1t_2t_1N &= \{N(t_0t_1t_2t_1)^n | n \in N\} \\ &= \{Nt_0t_1t_2t_1, Nt_1t_0t_2t_0, Nt_2t_1t_0t_1, Nt_0t_2t_1t_2, Nt_1t_2t_0t_2, Nt_2t_0t_1t_0\}. \end{aligned}$$

We'll consider each of the above mentioned double cosets one at a time. We first consider double coset [0121].

Recall that the orbits of $N^{(0121)}$ on $\{0,1,2\}$ are $\{0\}$, $\{1\}$, and $\{2\}$.

It is clear that $Nt_0t_1t_2t_1t_1 = Nt_0t_1t_2 \in [012]$

We now would like to know to which double coset $Nt_0t_1t_2t_1t_0$ belongs.

Note that the second relation gives us: $(01)t_0t_2t_0t_1t_2t_1t_0t_2t_0 = 1$

$$\begin{aligned}
&\Rightarrow \quad (01)t_0t_2t_0t_1 = t_0t_2t_0t_1t_2 \\
&\Rightarrow \quad t_0(01)t_0t_2t_0t_1 = t_2t_0t_1t_2 \\
&\Rightarrow \quad (01)t_1t_0t_2t_0t_1 = t_2t_0t_1t_2 \\
&\Rightarrow \quad t_1t_0t_2t_0t_1 = (01)t_2t_0t_1t_2 \\
&\Rightarrow \quad (t_1t_0t_2t_0t_1)^{(01)} = ((01)t_2t_0t_1t_2)^{(01)} \\
&\Rightarrow \quad t_0t_1t_2t_1t_0 = (01)t_2t_1t_0t_2
\end{aligned}$$

We use this to show the following:

$$\begin{aligned}
t_0t_1t_2t_1t_0 &= (01)t_2t_1t_0t_2 \\
\Rightarrow t_0t_1t_2t_1t_0 &\in Nt_2t_1t_0t_2 \\
\Rightarrow t_0t_1t_2t_1t_0 &\in Nt_2t_1t_0t_2N \\
\Rightarrow t_0t_1t_2t_1t_0 &\in N(t_2t_1t_0t_2)^n, \quad \text{for n} = (02) \in N
\end{aligned}$$

$$\begin{aligned}
So, t_0t_1t_2t_1t_0 &\in N(t_2t_1t_0t_2)^{(02)} \\
\Rightarrow t_0t_1t_2t_1t_0 &\in Nt_0t_1t_2t_0 \\
\Rightarrow t_0t_1t_2t_1t_0 &\in Nt_0t_1t_2t_0N
\end{aligned}$$

Hence $Nt_0t_1t_2t_1t_0 \in [0120]$.

$Nt_0t_1t_2t_1t_2$ however, is a new double coset, namely [01212].
Now, $N^{(01212)} = < e >$, so the number of distinct single cosets in the double coset [01212] is 6. In addition, the orbits of $N^{(01212)}$ on $\{0,1,2\}$ are $\{0\}$, $\{1\}$ and $\{2\}$. $Nt_0t_1t_2t_1t_2t_2 = Nt_0t_1t_2t_1 \in [0121]$.
Let us consider to which double coset $Nt_0t_1t_2t_1t_2t_1$ belongs. Now, from our first relation $t_0t_1t_2t_1t_2t_1 = (102)(t_0t_1t_0t_1)^{(12)}$.
So:

$$\begin{aligned}
\Rightarrow t_0t_1t_2t_1t_2t_1 &= (102)(t_0t_2t_0t_2) \quad \text{(shown via MAGMA)} \\
\Rightarrow t_0t_1t_2t_1t_2t_1 &\in Nt_0t_2t_0t_2 \\
\Rightarrow t_0t_1t_2t_1t_2t_1 &\in Nt_0t_2t_0t_2N \\
\Rightarrow t_0t_1t_2t_1t_2t_1 &\in N(t_0t_2t_0t_2)^n, \quad \text{such that } n \in N \\
\Rightarrow t_0t_1t_2t_1t_2t_1 &\in N(t_0t_2t_0t_2)^{(12)} \\
\Rightarrow t_0t_1t_2t_1t_2t_1 &\in Nt_0t_1t_0t_1 \\
\Rightarrow t_0t_1t_2t_1t_2t_1 &\in Nt_0t_1t_0t_1N
\end{aligned}$$

So $Nt_0t_1t_2t_1t_2t_1 \in [0101]$.

It can also be shown via MAGMA that $Nt_0t_1t_2t_1t_2t_0$ does not form a new double coset. Rather, $Nt_0t_1t_2t_1t_2t_0 \in [01212]$.
Hence, the double coset [01212] is now closed under right multiplication.

Let us now look at [0120], the second double coset of length four. We've already seen that $N^{(0120)} = < e >$, which implies that [0120] has 6 distinct single cosets. In addition, we've shown that the orbits of $N^{(0120)}$ on $\{0,1,2\}$ are $\{0\}$, $\{1\}$, and $\{2\}$.
We know $Nt_0t_1t_2t_0t_0 = Nt_0t_1t_2 \in [012]$.

However, $Nt_0t_1t_2t_0t_2 \in [0121]$ since:

$$\begin{aligned} t_0t_1t_2t_1t_0 &= (01)(t_0t_1t_2t_0)^{(20)} \\ \Rightarrow t_0t_1t_2t_1t_0 &= (01)t_2t_1t_0t_2 \\ \Rightarrow t_0t_1t_2t_1 &= (01)t_2t_1t_0t_2t_0 \\ \Rightarrow t_0t_1t_2t_1 &\in Nt_2t_1t_0t_2t_0 \\ \Rightarrow t_0t_1t_2t_1 &\in Nt_2t_1t_0t_2t_0N \\ \Rightarrow t_0t_1t_2t_1 &\in N(t_2t_1t_0t_2t_0)^n, \quad \text{for n} = (02) \in N \end{aligned}$$

So, $t_0t_1t_2t_1 \in N(t_2t_1t_0t_2t_0)^{(02)}$
$\Rightarrow t_0t_1t_2t_1 \in Nt_0t_1t_2t_0t_2$
$\Rightarrow t_0t_1t_2t_1 \in Nt_0t_1t_2t_0t_2N$

Consequently, only $Nt_0t_1t_2t_0t_1$ belongs to a double coset, namely [01201], which has been considered so far.

$N^{(01201)} = <e, (12)>$ since:
$Nt_0t_1t_2t_0t_1(12) = Nt_0t_2t_1t_0t_2 = Nt_0t_1t_2t_0t_1$
According to our first relation, $N(012)t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0 = N$.

$$\begin{aligned} &\Rightarrow (012)t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0 = 1 \\ &\Rightarrow (012)t_0t_2t_1t_0t_2 = t_0t_1t_2t_0t_1 \\ &\Rightarrow N(012)t_0t_2t_1t_0t_2 = Nt_0t_1t_2t_0t_1 \\ &\Rightarrow Nt_0t_2t_1t_0t_2 = Nt_0t_1t_2t_0t_1 \end{aligned}$$

So $(12) \in N^{(01201)}$ and $N^{(01201)} = <(12)>$. Thus, [01201] contains 3 distinct single cosets and the orbits of $N^{(01201)}$ on $\{0,1,2\}$ are $\{0\}$ and $\{1,2\}$.
$Nt_0t_1t_2t_0t_1t_1 = Nt_0t_1t_2t_0 \in [0120]$ and $Nt_0t_1t_2t_0t_1t_2 \in [0120]$.
However, $Nt_0t_1t_2t_0t_1t_0 \in [012010]$, a new double coset.

Now $N^{(012010)} = <e, (12)>$ since:

$Nt_0t_1t_2t_0t_1t_0(12) = Nt_0t_2t_1t_0t_2t_0 = Nt_0t_1t_2t_0t_1t_0$ since, by our first relation, $Nt_0t_2t_1t_0t_2 = Nt_0t_1t_2t_0t_1$.

Therefore there are 3 distinct single cosets in [012010] and the orbits of $N^{(012010)}$ on $\{0,1,2\}$ are $\{0\}$ and $\{1,2\}$.
Now, $Nt_0t_1t_2t_0t_1t_0t_0 = Nt_0t_1t_2t_0t_1 \in [01201]$

$$\begin{aligned}
t_0t_1t_2t_0t_1t_0t_1 &= (12)(t_0t_1t_0t_2)^{(01)} \\
\Rightarrow t_0t_1t_2t_0t_1t_0t_1 &= (12)t_1t_0t_1t_2 \quad \text{(shown via MAGMA)} \\
\Rightarrow t_0t_1t_2t_0t_1t_0t_1 &\in N(12)t_1t_0t_1t_2 \\
\Rightarrow t_0t_1t_2t_0t_1t_0t_1 &\in Nt_1t_0t_1t_2N \\
\Rightarrow t_0t_1t_2t_0t_1t_0t_1 &\in N(t_1t_0t_1t_2)^n, \quad \text{such that } n \in N \\
\Rightarrow t_0t_1t_2t_0t_1t_0t_1 &\in N(t_1t_0t_1t_2)^{(01)} \\
\Rightarrow t_0t_1t_2t_0t_1t_0t_1 &\in Nt_0t_1t_0t_2 \\
\Rightarrow t_0t_1t_2t_0t_1t_0t_1 &\in Nt_0t_1t_0t_2N
\end{aligned}$$

$Nt_0t_1t_2t_0t_1t_0t_1 \in [0102]$.

Thus, $Nt_0t_1t_2t_0t_1t_0t_1$ and $Nt_0t_1t_2t_0t_1t_0t_2 \in [0102]$.
The double coset [012010] is now closed under right multiplication.

Let us now consider the double coset [0101], the third double coset of length four.
We already know that $N^{(0101)} = < e >$. Hence, there are 6 distinct single cosets in [0101].
The orbits of $N^{(0101)}$ on $\{0,1,2\}$ are $\{0\}$, $\{1\}$, and $\{2\}$.
Now, $Nt_0t_1t_0t_1t_1 = Nt_0t_1t_0 \in [010]$.

$$\begin{aligned}
t_0t_1t_2t_1t_2t_1 &= (102)(t_0t_1t_0t_1)^{(12)} \\
\Rightarrow t_0t_1t_2t_1t_2t_1 &= (102)t_0t_2t_0t_2 \quad \text{(shown earlier)} \\
\Rightarrow (t_0t_1t_2t_1t_2t_1)^{(12)} &= ((102)t_0t_2t_0t_2)^{(12)} \\
\Rightarrow t_0t_2t_1t_2t_1t_2 &= (201)t_0t_1t_0t_1 \\
\Rightarrow t_0t_2t_1t_2t_1 &= (201)t_0t_1t_0t_1t_2 \\
\Rightarrow Nt_0t_1t_0t_1t_2 &= Nt_0t_2t_1t_2t_1 \\
\Rightarrow Nt_0t_1t_0t_1t_2 &\in N(t_0t_2t_1t_2t_1)^n, \quad \text{such that } n \in N \\
\Rightarrow Nt_0t_1t_0t_1t_2 &\in N(t_0t_2t_1t_2t_1)^{(12)} \\
\Rightarrow Nt_0t_1t_0t_1t_2 &\in Nt_0t_1t_2t_1t_2N
\end{aligned}$$

We notice however, that $Nt_0t_1t_0t_1t_2 \in [01212]$.

Now $Nt_0t_1t_0t_1t_0 = N(t_0t_1t_0t_1t_0)^{(e)} = Nt_0t_1t_0t_1t_0$
$\Rightarrow Nt_0t_1t_0t_1t_0 \in [01010]$.
Notice $N^{(01010)} = < e, (021), (012) >$ from the following:

$Nt_0t_1t_0t_1t_0(021) = Nt_2t_0t_2t_0t_2 = Nt_0t_1t_0t_1t_0$ via MAGMA.
$Nt_0t_1t_0t_1t_0(012) = Nt_1t_2t_1t_2t_1 = Nt_0t_1t_0t_1t_0$
Both of these results are shown below:

$$\begin{aligned}
t_0t_1t_2t_1t_2t_1 &= (102)t_0t_2t_0t_2 \quad \text{(shown earlier)} \\
\Rightarrow t_1t_2t_1t_2t_1 &= t_0(102)t_0t_2t_0t_2 \\
\Rightarrow t_1t_2t_1t_2t_1 &= (102)t_2t_0t_2t_0t_2 \\
\Rightarrow (t_1t_2t_1t_2t_1)^{(201)} &= ((102)t_2t_0t_2t_0t_2)^{(201)} \\
\Rightarrow t_2t_0t_2t_0t_2 &= (210)t_0t_1t_0t_1t_0 \\
\Rightarrow Nt_2t_0t_2t_0t_2 &= Nt_0t_1t_0t_1t_0
\end{aligned}$$

$$\begin{aligned}
t_0t_1t_2t_1t_2t_1 &= (102)t_0t_2t_0t_2 \quad \text{(shown earlier)}\\
\Rightarrow t_1t_2t_1t_2t_1 &= t_0(102)t_0t_2t_0t_2\\
\Rightarrow t_1t_2t_1t_2t_1 &= (102)t_2t_0t_2t_0t_2\\
\Rightarrow Nt_1t_2t_1t_2t_1 &= Nt_2t_0t_2t_0t_2
\end{aligned}$$

So now we know that there are only 2 distinct single cosets within [01010]. In addition, the orbits of $N^{(01010)}$ on $\{0,1,2\}$ are $\{0,1,2\}$.
$Nt_0t_1t_0t_1t_0t_0 = Nt_0t_1t_0t_1 \in [0101]$.
This implies that $Nt_0t_1t_0t_1t_0t_1 \in [0101]$ and $Nt_0t_1t_0t_1t_0t_1 \in [0101]$.
Thus, the double coset [01010] is closed under right multiplication.

Finally, let's consider the double coset [0102], the final double coset of length four. As shown earlier $N^{(0102)} = < e >$, and the orbits of $N^{(0102)}$ on $\{0,1,2\}$ are $\{0\}$, $\{1\}$, and $\{2\}$.
First, we note that $Nt_0t_1t_0t_2t_2 = Nt_0t_1t_0 \in [010]$.
Secondly, $Nt_0t_1t_0t_2t_0 \in [012010]$ from the following:

$$\begin{aligned}
t_0t_1t_2t_0t_1t_0t_1 &= (12)(t_0t_1t_0t_2)^{(01)}\\
\Rightarrow t_0t_1t_2t_0t_1t_0t_1 &= (12)t_1t_0t_1t_2 \quad \text{(shown earlier)}\\
\Rightarrow t_0t_1t_2t_0t_1t_0 &= (12)t_1t_0t_1t_2t_1\\
\Rightarrow Nt_1t_0t_1t_2t_1 &\in Nt_0t_1t_2t_0t_1t_0\\
\Rightarrow N(t_1t_0t_1t_2t_1)^{(01)} &\in Nt_0t_1t_2t_0t_1t_0\\
\Rightarrow Nt_0t_1t_0t_2t_0 &\in Nt_0t_1t_2t_0t_1t_0
\end{aligned}$$

Finally, from relation 2 we find that $Nt_0t_1t_0t_2t_1 \in [0102]$:

$$\begin{aligned}
(01)t_0t_2t_0t_1t_2 &= t_0t_2t_0t_1 \\
\Rightarrow ((01)t_0t_2t_0t_1t_2)^{(12)} &= (t_0t_2t_0t_1)^{(12)} \\
\Rightarrow (02)t_0t_1t_0t_2t_1 &= t_0t_1t_0t_2 \\
\Rightarrow Nt_0t_1t_0t_2t_1 &= Nt_0t_1t_0t_2 \\
\Rightarrow Nt_0t_1t_0t_2t_1 &\in Nt_0t_1t_0t_2
\end{aligned}$$

This causes the double coset [0102] closed under right multiplication.
All of our work is summarized in the Caley Diagram of $A_5 \times D_6$ over S_3 below.

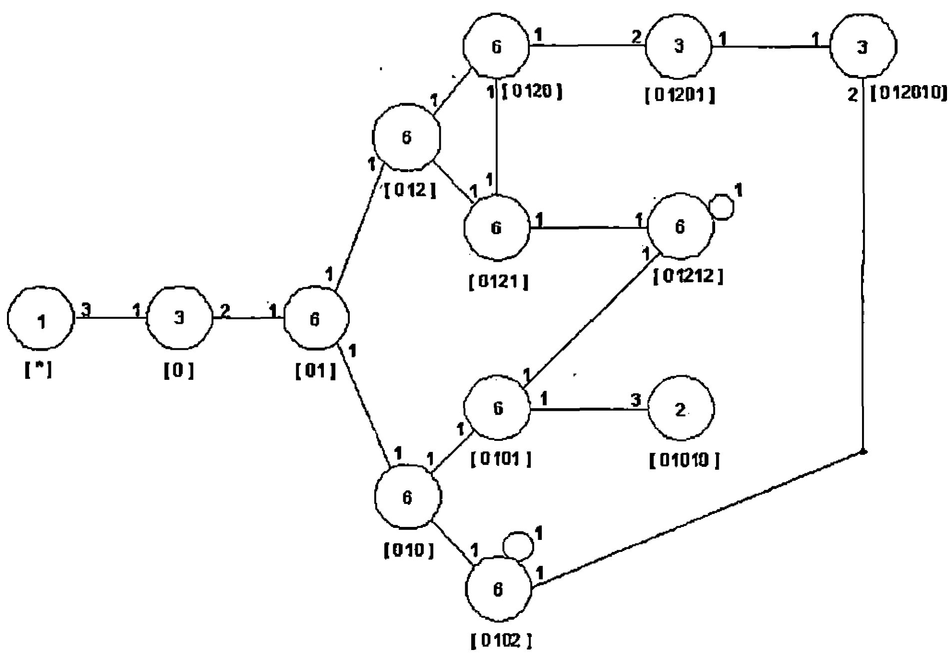

Figure 2.1: Cayley Diagram $A_5 \times D_6$ over S_3

Our argument shows that the order of G is at most $|N| \times 60 = 360$, where 60 is the number of single cosets shown in the diagram above. We now show that $|G|$ is at least 360. Now $G =< x, y, t >$ acts on X, the set of the single cosets mentioned above. Thus $\alpha : G \to S_X$ is a homomorphism. Since $N =< x, y >$ acts by conjugation and t acts

by right multiplication on the t_is, we compute the images xx, yy, and tt of x,y, and t, respectively, in S_X and verify that the additional relations hold in $< xx, yy, tt >$ within S_X and $| < xx, yy, tt > | = 360$. So $G/Ker\alpha \cong < xx, yy, tt >$. Hence $|G| \geq 360$.
Thus, $|G| = 360$.
We also verified that G satisfies a presentation of $\cong A_5 \times D_6$.

Chapter 3

$U_3(3) \times 3$ as the Homomorphic Image of $3^{*4} : S_4$

The general unitary group $GU_n(q)$, consisting of unitary matrices over F_q, is a subgroup of $GL_n(q^2)$. The center of $GL_n(q^2)$ is of order $q+1$ and $PGU_n(q)$ $=GU_n(q)/Z(GU_n(q))$ is called the projective unitary group. The subgroup of $GL_n(q^2)$, consisting of matrices of determinant 1 in $GU_n(q)$, of index $q+1$ in $GU_n(q)$, is denoted by $SU_n(q)$ and $PSU_n(q)$ is the factor group $SU_n(q)/Z(SU_n(q))$.

Iwasawa's Lemma: Let G a finite perfect group, whose action on a set ω is faithful and 2-transitive. If the point stabiliser H has a normal abelian subgroup A such that $G =< A^x | x \in G >$, then G is simple.

According to Iwasawa's Lemma, $PSU_n(q)$ is simple in the following cases.
(i) $n > 3$
(ii) $n = 3$ and $q > 2$
(iii) $n = 2$ and $q > 3$.
$U_2(2) = S_3$. Thus, $U_n(q)$ is simple except in the cases $U_2(3) = A_4$, $U_3(2) = 3^2 : Q_8$. In particular, $U_3(3)$ is simple. Our objective is to construct $U_3(3)$. We give a construction of $U_3(3) \times 3$ in this chapter and obtain $U_3(3)$, from this construction, in the next chapter.

Consider the progenitor $3^{*4} : S_4$, whose symmetric representation is given by:
$< x, y, t \mid x^4 = y^2 = (yx)^3 = 1 = t^3 = [t, y] = [t^x, y] >$
where N $= S_4$ and the action on the symmetric generators x, y is given by:

$x \sim (0\ 1\ 2\ 3)$

$y \sim (2\ 3)$

and t $\sim t_0$

The following relations may be used for the purpose of manual double coset enumeration:
R1. $[(0123)t_0]^{21} = 1$
R2. $[(012)t_0]^{24} = 1$
R3. $[(01)(23)t_0]^{12} = 1$
R4. $[(01)t_0]^{12} = 1$
R5. $[t_0^{-1}t_1]^4 = 1$
R6. $[(012)t_0t_1^{-1}]^4 = 1$
R7. $[(01)t_0t_2]^{24} = 1$
R8. $[(01)t_0t_2^{-1}]^7 = 1$
R9. $[t_0t_1]^3 = (23)$
R10. $[t_0^{-1}t_1]^2 = (23)$

If we factor $3^{*4} : S_4$ by relations 1-10, although R10 suffices, we will verify that its homomorphic image is a group isomorphic to $U_3(3) \times 3$. Thus it is our objective to demonstrate that $\dfrac{3^{*4} : S_4}{[t_0^{-1}t_1]^2 = (23)} \cong < x, y, t \mid x^4 = y^2 = (yx)^3 = 1 = t^3 = [t, y] = [t^x, y], (t^{-1}t^x)^2 y >$

is isomorphic to $U_3(3) \times 3$.

We now perform manual double coset enumeration of G $= \dfrac{3^{*4} : S_4}{[t_0^{-1}t_1]^2 = (23)}$ over S_4.

We start with the double coset with representative word of length zero, namely $NeN = \{Nen|n \in N\} = \{Nnn^{-1}en|n \in N\} = \{Ne^n|n \in N\} =$ N. In our Cayley Diagram, we'll denote this double coset N by [*]. Now, N $= S_4$ is transitive on the set of symmetric generators $\{0, 1, 2, 3\}$ and therefore on their inverses $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. Hence, the double coset N has orbits $\{0, 1, 2, 3\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. It suffices to choose one representative from each orbit and ask to which double coset does Nt_i belong. Let us consider Nt_0. Because the first double coset is simply our control group S_4, Nt_0 is not represented within it. Hence, Nt_0 is an element of the new double coset Nt_0N, denoted [0] in our Cayley Diagram. In fact, because Nt_0, Nt_1, Nt_2, and Nt_3 are in the same orbit, they all are elements of [0].

Similarly, $Nt_0^{-1}N = \{N(t_0^{-1})^n | n \in N\} = \{Nt_0^{-1}, Nt_1^{-1}, Nt_2^{-1}, Nt_3^{-1}\}$ denoted $[\bar{0}]$.

Now we determine, for [0] and $[\bar{0}]$, to which double coset Nt_0t_i and $Nt_0^{-1}t_i$ belong for one t_i from each orbit of N^0 and $N^{\bar{0}}$.

$N^0 = \{n \in N | (t_0)^n = t_0\} = \langle (12), (13), (23) \rangle \cong S_3$.

$N^0 = S_3$ has orbits $\{0\}$ and $\{1, 2, 3\}$.

Similarly, $N^{\bar{0}} = S_3$ has orbits $\{\bar{0}\}$ and $\{\bar{1}, \bar{2}, \bar{3}\}$.

So we need to consider the double cosets [00], [01], $[0\bar{0}]$, $[0\bar{1}]$, and $[\bar{0}0]$, $[\bar{0}1]$, $[\bar{0}\bar{0}]$, $[\bar{0}\bar{1}]$.

$00 = \bar{0} \Rightarrow Nt_0t_0 = Nt_0^{-1} \in Nt_0^{-1}N = [\bar{0}]$.

$\bar{0}\bar{0} = 0 \Rightarrow Nt_0^{-1}t_0^{-1} = Nt_0 \in Nt_0N = [0]$.

$0\bar{0} = \text{NeN} = \text{N} = \text{NeN} = \bar{0}0 \Rightarrow Nt_0t_0^{-1} = \text{N} = [*]$. Likewise, $Nt_0^{-1}t_0 = \text{N} = [*]$.

The following four double cosets that remain are new double cosets with representative words of length two: [01], $[0\bar{1}]$, $[\bar{0}1]$, and $[\bar{0}\bar{1}]$.

First let us look at [01] and $[\bar{0}\bar{1}]$.

$N^{(01)} = N^{(\bar{0}\bar{1})} = \{1, (23)\}$.

We know that the number of distinct single cosets within [01] and $[\bar{0}\bar{1}]$ is

$\frac{|N|}{|N^{(01)}|} = \frac{24}{2} = \frac{|N|}{|N^{(\bar{0}\bar{1})}|}$. Therefore, both [01] and $[\bar{0}\bar{1}]$ have 12 distinct single cosets.

The single cosets within [01] are:

$$Nt_0t_1$$
$$Nt_1t_0$$
$$Nt_2t_1$$
$$Nt_3t_1$$
$$Nt_0t_2$$

$$Nt_0t_3$$
$$Nt_1t_2$$
$$Nt_2t_0$$
$$Nt_1t_3$$
$$Nt_3t_0$$
$$Nt_2t_3$$
$$Nt_3t_2$$

The single cosets within $[\bar{0}\bar{1}]$ are:

$$Nt_0^{-1}t_1^{-1}$$
$$Nt_1^{-1}t_0^{-1}$$
$$Nt_2^{-1}t_1^{-1}$$
$$Nt_3^{-1}t_1^{-1}$$
$$Nt_0^{-1}t_2^{-1}$$
$$Nt_0^{-1}t_3^{-1}$$
$$Nt_1^{-1}t_2^{-1}$$
$$Nt_2^{-1}t_0^{-1}$$
$$Nt_1^{-1}t_3^{-1}$$
$$Nt_3^{-1}t_0^{-1}$$
$$Nt_2^{-1}t_3^{-1}$$
$$Nt_3^{-1}t_2^{-1}$$

By examining the coset stabilizing group for both $[01]$ and $[\bar{0}\bar{1}]$, we see that the orbits of $N^{(01)}$ and $N^{(\bar{0}\bar{1})}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

Now let us consider double cosets $[0\bar{1}]$ and $[\bar{0}1]$.
R10 gives: $[t_0^{-1}t_1]^2 = (23)$

$$\begin{aligned}
\Rightarrow Nt_0^{-1}t_1t_0^{-1}t_1 &= N(23) \\
\Rightarrow Nt_0^{-1}t_1 &= Nt_1^{-1}t_0 \\
\Rightarrow \bar{0}1 &= \bar{1}0
\end{aligned}$$

So then $N(t_0^{-1}t_1)^{(01)} = Nt_1^{-1}t_0 = Nt_0^{-1}t_1 \Rightarrow (01) \in N^{(\bar{0}1)}$.
This gives $N^{(\bar{0}1)} = \{1, (01), (23), (01)(23)\}$

In addition, R10 provides: $\bar{0}1 = \bar{1}0 \Rightarrow \bar{0}1\bar{0} = \bar{1}\underline{0\bar{0}} \Rightarrow \bar{0}1\bar{0} = \bar{1}$.
So then, $0\underline{\bar{1}} = \underline{0\bar{0}}1\bar{0} = 1\bar{0}$.
Therefore, $N(t_0t_1^{-1})^{(01)} = Nt_1t_0^{-1} = Nt_0t_1^{-1}$.
Hence, we can say $N^{(0\bar{1})} = \{1, (23), (01), (01)(23)\} = N^{(\bar{0}1)}$.
So, $[0\bar{1}]$ and $[\bar{0}1]$ have $\frac{24}{4} = 6$ distinct single cosets.

Below is a list of the six single cosets within $[0\bar{1}]$ along with their equal names. As a form of shorthand, we write each single coset with only their corresponding subscript.

$$0\bar{1} \sim 1\bar{0}$$
$$2\bar{1} \sim 1\bar{2}$$
$$3\bar{1} \sim 1\bar{3}$$
$$0\bar{2} \sim 2\bar{0}$$
$$0\bar{3} \sim 3\bar{0}$$
$$2\bar{3} \sim 3\bar{2}$$

Below is a list of the six single cosets within $[\bar{0}1]$ along with their equal names. As a form of shorthand, we write each single coset with only their corresponding subscript.

$$\bar{0}1 \sim \bar{1}0$$
$$\bar{2}1 \sim \bar{1}2$$
$$\bar{3}1 \sim \bar{1}3$$
$$\bar{0}2 \sim \bar{2}0$$
$$\bar{0}3 \sim \bar{3}0$$
$$\bar{2}3 \sim \bar{3}2$$

The coset stabilizing groups $N^{(0\bar{1})}$, $N^{(\bar{0}1)}$ show that their orbits on $\{0, 1, 2, 3\}$ are $\{0, 1\}$, $\{2, 3\}$, $\{\bar{0}, \bar{1}\}$, and $\{\bar{2}, \bar{3}\}$.

Expanding R9:

$$\begin{aligned} [t_0 t_1]^3 &= (23) \\ \Rightarrow N t_0 t_1 t_0 \underline{t_1 t_0 t_1} &= N(23) \\ \Rightarrow N t_0 t_1 t_0 &= N t_1^{-1} t_0^{-1} t_1^{-1} \\ \Rightarrow 010 &= \bar{1}\bar{0}\bar{1} \end{aligned}$$

Let us now determine to which double coset $N t_0 t_1 t_i$ belongs for one t_i from each orbit of $N^{(01)}$. Recall that the orbits of $N^{(01)}$ are $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

$0\underline{11} = 0\bar{1} \Rightarrow N t_0 t_1 t_1 = N t_0 t_1^{-1} \in N t_0 t_1^{-1} N = [0\bar{1}]$.
$0\underline{1\bar{1}} = 0 \Rightarrow N t_0 t_1 t_1^{-1} = N t_0 \in N t_0 N = [0]$.
$01\bar{0} = \bar{0}\bar{1}$ since $01\underline{\bar{0}} = \underline{0100} \underbrace{=}_{R9} \bar{1}0\underline{\bar{1}0} \underbrace{=}_{R10} \bar{1}\underline{\bar{0}\bar{0}}1 = \underline{\bar{1}01} \underbrace{=}_{R10} \bar{0}\underline{11} = \bar{0}\bar{1}$.

$\Rightarrow N t_0 t_1 t_0^{-1} = N t_0^{-1} t_1^{-1} \in N t_0^{-1} t_1^{-1} N = [\bar{0}\bar{1}]$.

We know that $N t_0 t_1 t_0$, $N t_0 t_1 t_2$, and $N t_0 t_1 t_2^{-1}$ are elements of the new double cosets [010], [012], and $[01\bar{2}]$, respectively. We'll consider these new double cosets after while.

We now determine to which double coset $N t_0^{-1} t_1^{-1} t_i$ belongs for one t_i from each orbit of $N^{(\bar{0}\bar{1})}$. Recall that the orbits of $N^{(\bar{0}\bar{1})}$ are $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

$\bar{0}\underline{\bar{1}\bar{1}} = \bar{0}1 \Rightarrow N t_0^{-1} t_1^{-1} t_1^{-1} = N t_0^{-1} t_1 \in N t_0^{-1} t_1 N = [\bar{0}1]$.
$\bar{0}\underline{\bar{1}1} = \bar{0} \Rightarrow N t_0^{-1} t_1^{-1} t_1 = N t_0^{-1} \in N t_0^{-1} N = [\bar{0}]$.
$\bar{0}\underline{\bar{1}0} \underbrace{=}_{R10} \underline{\bar{0}\bar{0}}1 = 01 \Rightarrow N t_0^{-1} t_1^{-1} t_0 = N t_0 t_1 \in N t_0 t_1 N = [01]$.

$(\bar{0}\bar{1}\bar{0})^{(01)} = \bar{1}\bar{0}\bar{1} \underbrace{=}_{R9} 010 \Rightarrow N t_0^{-1} t_1^{-1} t_0^{-1} = N t_0 t_1 t_0 \in N t_0 t_1 t_0 N = [010]$.

We know that $N t_0^{-1} t_1^{-1} t_2$ and $N t_0^{-1} t_1^{-1} t_2^{-1}$ are elements of the new double cosets $[\bar{0}\bar{1}2]$ and $[\bar{0}\bar{1}\bar{2}]$, respectively. We consider these new double cosets after further work.

We now determine to which double coset $Nt_0t_{\bar{1}}^{-1}t_i$ belongs for one t_i from each orbit of $N^{(0\bar{1})}$. Recall that the orbits of $N^{(0\bar{1})}$ are $\{0,1\}$, $\{2,3\}$, $\{\bar{0},\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

$0\underline{\bar{1}\bar{1}} = 01 \Rightarrow Nt_0t_{\bar{1}}^{-1}t_{\bar{1}}^{-1} = Nt_0t_1 \in Nt_0t_1N = [01]$.
$0\underline{\bar{1}1} = 0 \Rightarrow Nt_0t_{\bar{1}}^{-1}t_1 = Nt_0 \in Nt_0N = [0]$.

We know that $Nt_0t_{\bar{1}}^{-1}t_2$ and $Nt_0t_{\bar{1}}^{-1}t_2^{-1}$ are elements of the new double cosets $[0\bar{1}2]$ and $[0\bar{1}\bar{2}]$, respectively. We'll return to these new double cosets after additional work.

We now determine to which double coset $Nt_{\bar{0}}^{-1}t_1t_i$ belongs for one t_i from each orbit of $N^{(\bar{0}1)}$. Recall that the orbits of $N^{(\bar{0}1)}$ are $\{0,1\}$, $\{2,3\}$, $\{\bar{0},\bar{1}\}$, and $\{\bar{2},\bar{3}\}$.

$\bar{0}\underline{11} = \bar{0}\bar{1} \Rightarrow Nt_{\bar{0}}^{-1}t_1t_1 = Nt_{\bar{0}}^{-1}t_{\bar{1}}^{-1} \in Nt_{\bar{0}}^{-1}t_{\bar{1}}^{-1}N = [\bar{0}\bar{1}]$.
$\bar{0}\underline{1\bar{1}} = \bar{0} \Rightarrow Nt_{\bar{0}}^{-1}t_1t_{\bar{1}}^{-1} = Nt_{\bar{0}}^{-1} \in Nt_{\bar{0}}^{-1}N = [\bar{0}]$.

We know that $Nt_{\bar{0}}^{-1}t_1t_2$ and $Nt_{\bar{0}}^{-1}t_1t_2^{-1}$ are elements of the new double cosets $[\bar{0}12]$ and $[\bar{0}1\bar{2}]$.

Now we want to consider the double cosets with representative words of length three. Namely, double cosets: $[010]$, $[012]$, $[01\bar{2}]$, $[0\bar{1}2]$, $[0\bar{1}\bar{2}]$, $[\bar{0}12]$, $[\bar{0}1\bar{2}]$, $[\bar{0}\bar{1}2]$, $[\bar{0}\bar{1}\bar{2}]$.

Consider $[010]$.
First, note the following:

$$\begin{aligned}(010)^{(01)} &= \underline{1}01\\ &= \bar{1}\underline{\bar{1}0}1\\ &= \bar{1}0\underline{11}\\ &= \underline{\bar{1}0\bar{1}}\\ &= 010\end{aligned}$$

So then $N(t_0t_1t_0)^{(01)} = Nt_1t_0t_1 = Nt_0t_1t_0$. Hence, $N^{(010)} = \{e,(23),(01),(01)(23)\}$ and $[010]$ has six distinct single cosets. They are listed below with their equal names.

$$010 \sim 101$$
$$212 \sim 121$$
$$313 \sim 131$$
$$020 \sim 202$$
$$030 \sim 303$$
$$232 \sim 323$$

The orbits of $N^{(010)}$ on $\{0,1,2,3\}$ are $\{0,1\}$, $\{\bar{1},\bar{0}\}$, $\{2,3\}$, and $\{\bar{2},\bar{3}\}$. Let's find to which double coset $Nt_0t_1t_0t_i$ belongs for one t_i from each orbit of $N^{(010)}$. First, note the following:

$$\begin{aligned}(0101)^{(01)} &= 1\underline{010}\\ &= \underline{1\bar{1}}\bar{0}\bar{1}\\ &= \bar{0}\bar{1}\end{aligned}$$

Using what we've found, we can say the following:
$0101 = \bar{0}\bar{1} \Rightarrow Nt_0t_1t_0t_1 = Nt_0^{-1}t_1^{-1} \in Nt_0^{-1}t_1^{-1}N = [\bar{0}\bar{1}]$.
Also, $01\underline{0\bar{0}} = 01 \Rightarrow Nt_0t_1t_0t_0^{-1} = Nt_0t_1 \in Nt_0t_1N = [01]$.

We know that $Nt_0t_1t_0t_2$ and $Nt_0t_1t_0t_2^{-1}$ are elements of the new double cosets [0102] and $[010\bar{2}]$, respectively.

We continue to consider double cosets with representative words of length three.

Consider [012].
The coset stabilizing group $N^{(012)} = \{e\}$, which implies that there exists twenty-four distinct single cosets within [012]. All twenty-four single cosets are listed below with their corresponding subscript.

012	102	210	312
021	032	013	120
201	132	302	023
031	310	213	103
230	321	123	301
203	130	231	320

The orbits of $N^{(012)}$ on $\{0,1,2,3\}$ are $\{0\},\{1\},\{2\},\{3\},\{\bar{0}\},\{\bar{1}\},\{\bar{2}\}$, and $\{\bar{3}\}$. Let's find to which double coset $Nt_0t_1t_2t_i$ belongs for one t_i from each orbit of $N^{(012)}$.

$012\bar{2} = 01 \Rightarrow Nt_0t_1t_2t_2^{-1} = Nt_0t_1 \in Nt_0t_1N = [01]$.
$0122 = 01\bar{2} \Rightarrow Nt_0t_1t_2t_2 = Nt_0t_1t_2^{-1} \in Nt_0t_1t_2^{-1}N = [01\bar{2}]$.
$012\bar{0} = 010\bar{2} \Rightarrow Nt_0t_1t_2t_0^{-1} = Nt_0t_1t_0t_2^{-1} \in Nt_0t_1t_0t_2^{-1}N = [010\bar{2}]$.
$0121 = 0\bar{1}\bar{2}\bar{0} \Rightarrow Nt_0t_1t_2t_1 = Nt_0t_1^{-1}t_2^{-1}t_0^{-1} \in Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N = [0\bar{1}\bar{2}\bar{0}]$.
$012\bar{1} = 0\bar{1}\bar{2} \Rightarrow Nt_0t_1t_2t_1^{-1} = Nt_0t_1^{-1}t_2^{-1} \in Nt_0t_1^{-1}t_2^{-1}N = [0\bar{1}\bar{2}]$.

We know that $Nt_0t_1t_2t_0$, $Nt_0t_1t_2t_3$, and $Nt_0t_1t_2t_3^{-1}$ are elements of the new double cosets [0120], [0123], and $[012\bar{3}]$, respectively.

Consider $[01\bar{2}]$.
The coset stabilizing group $N^{(01\bar{2})} = \{e,(03)(12)\}$, which can be shown via MAGMA. We notice that there are twelve distinct single cosets within $[01\bar{2}]$ and list them below along with their equal names.

$01\bar{2} \sim 32\bar{1}$
$10\bar{2} \sim 32\bar{0}$
$21\bar{0} \sim 30\bar{1}$
$31\bar{2} \sim 02\bar{1}$
$02\bar{1} \sim 31\bar{2}$
$03\bar{2} \sim 12\bar{3}$
$01\bar{3} \sim 23\bar{1}$
$12\bar{0} \sim 30\bar{2}$
$20\bar{1} \sim 31\bar{0}$
$13\bar{2} \sim 02\bar{3}$

$$03\bar{1} \sim 21\bar{3}$$
$$10\bar{3} \sim 23\bar{0}$$

The orbits of $N^{(01\bar{2})}$ on $\{0,1,2,3\}$ are $\{0,3\}$, $\{1,2\}$, $\{\bar{0},\bar{3}\}$, and $\{\bar{1},\bar{2}\}$. Let's find to which double coset $Nt_0t_1t_2^{-1}t_i$ belongs for one t_i from each orbit of $N^{(01\bar{2})}$.

$01\bar{2}2 = 01 \Rightarrow Nt_0t_1t_2^{-1}t_2 = Nt_0t_1 \in Nt_0t_1N = [01]$.
$01\bar{2}\bar{2} = 012 \Rightarrow Nt_0t_1t_2^{-1}t_2^{-1} = Nt_0t_1t_2 \in Nt_0t_1t_2N = [012]$.
$01\bar{2}0 = \bar{0}\bar{1}2 \Rightarrow Nt_0t_1t_2^{-1}t_0 = Nt_0^{-1}t_1^{-1}t_2 \in Nt_0^{-1}t_1^{-1}t_2N = [\bar{0}\bar{1}2]$.

We know that $Nt_0t_1t_2^{-1}t_0^{-1}$ is an element of the new double coset $[01\bar{2}\bar{0}]$.

Consider $[0\bar{1}2]$.
The coset stabilizing group $N^{(0\bar{1}2)} = \{e, (01), (23), (03)(12), (02)(13), (01)(23), (0213), (0312)\}$, shown via MAGMA. We know that there are three distinct single cosets within $[0\bar{1}2]$. We list them and their equal names with their corresponding subscripts below.

$$0\bar{1}2 \sim 1\bar{0}2 \sim 0\bar{1}3 \sim 3\bar{2}1 \sim 2\bar{3}0 \sim 1\bar{0}3 \sim 2\bar{3}1 \sim 3\bar{2}0$$
$$2\bar{1}0 \sim 1\bar{2}0 \sim 2\bar{1}3 \sim 3\bar{0}1 \sim 0\bar{3}2 \sim 1\bar{2}3 \sim 0\bar{3}1 \sim 3\bar{0}2$$
$$3\bar{1}2 \sim 1\bar{3}2 \sim 3\bar{1}0 \sim 0\bar{2}1 \sim 2\bar{0}3 \sim 1\bar{3}0 \sim 2\bar{0}1 \sim 0\bar{2}3$$

The orbits of $N^{(0\bar{1}2)}$ on $\{0,1,2,3\}$ are $\{0,1,2,3\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$. We consider $Nt_0t_1^{-1}t_2t_i$ for one t_i from each orbit of $N^{(0\bar{1}2)}$ and find to which double coset it belongs.

$0\bar{1}2\bar{2} = 0\bar{1} \Rightarrow Nt_0t_1^{-1}t_2t_2^{-1} = Nt_0t_1^{-1} \in Nt_0t_1^{-1}N = [0\bar{1}]$.
$0\bar{1}22 = 0\bar{1}\bar{2} \Rightarrow Nt_0t_1^{-1}t_2t_2 = Nt_0t_1^{-1}t_2^{-1} \in Nt_0t_1^{-1}t_2^{-1}N = [0\bar{1}\bar{2}]$.

Consider $[0\bar{1}\bar{2}]$.
We know via MAGMA that the coset stabilizing group $N^{(0\bar{1}\bar{2})} = \{e, (01)\}$, implying that there exists twelve distinct single cosets within $[0\bar{1}\bar{2}]$. These single cosets and their equal names are listed below according to their corresponding subscripts.

$$0\bar{1}\bar{2} \sim 1\bar{0}\bar{2}$$

$$2\bar{1}\bar{0} \sim 1\bar{2}\bar{0}$$
$$3\bar{1}\bar{2} \sim 1\bar{3}\bar{2}$$
$$0\bar{2}\bar{1} \sim 2\bar{0}\bar{1}$$
$$0\bar{3}\bar{2} \sim 3\bar{0}\bar{2}$$
$$0\bar{1}\bar{3} \sim 1\bar{0}\bar{3}$$
$$0\bar{2}\bar{3} \sim 2\bar{0}\bar{3}$$
$$0\bar{3}\bar{1} \sim 3\bar{0}\bar{1}$$
$$3\bar{1}\bar{0} \sim 1\bar{3}\bar{0}$$
$$2\bar{1}\bar{3} \sim 1\bar{2}\bar{3}$$
$$2\bar{3}\bar{0} \sim 3\bar{2}\bar{0}$$
$$3\bar{2}\bar{1} \sim 2\bar{3}\bar{1}$$
$$1\bar{2}\bar{3} \sim 2\bar{1}\bar{3}$$

The orbits of $N^{(0\bar{1}\bar{2})}$ on $\{0,1,2,3\}$ are $\{0,1\}$, $\{\bar{0},\bar{1}\}$, $\{2\}$, $\{3\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0t_1^{-1}t_2^{-1}t_i$ for one t_i from each orbit of $N^{(0\bar{1}\bar{2})}$ and find to which double coset it belongs.

$0\bar{1}\bar{2}2 = 0\bar{1} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_2 = Nt_0t_1^{-1} \in Nt_0t_1^{-1}N = [0\bar{1}]$.
$0\bar{1}\bar{2}\bar{2} = 0\bar{1}2 \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_2^{-1} = Nt_0t_1^{-1}t_2 \in Nt_0t_1^{-1}t_2N = [0\bar{1}2]$.
$0\bar{1}\bar{2}0 = 012 \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_0 = Nt_0t_1t_2 \in Nt_0t_1t_2N = [012]$. (shown via MAGMA)

We know from MAGMA that $Nt_0t_1^{-1}t_2^{-1}t_3$, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}$, and $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}$ are elements of the new double cosets $[0\bar{1}\bar{2}3]$, $[0\bar{1}\bar{2}\bar{0}]$, and $[0\bar{1}\bar{2}\bar{3}]$, respectively.

Consider $[\bar{0}12]$.
We note that R10 provides: $\bar{0}1 = \bar{1}0$
$\Rightarrow \underline{\bar{0}1}2 = \bar{1}02$

So $N(t_0^{-1}t_1t_2)^{(01)} = Nt_1^{-1}t_0t_2 = Nt_0^{-1}t_1t_2$.
$\Rightarrow (01) \in N^{(\bar{0}12)}$.

Hence, $N^{(\bar{0}12)} = \{e, (01)\}$. There exists twelve distinct single cosets within $[\bar{0}12]$. We list them and their equal names below, according to their corresponding subscripts.

$$\bar{0}12 \sim \bar{1}02$$
$$\bar{2}10 \sim \bar{1}20$$
$$\bar{3}12 \sim \bar{1}32$$
$$\bar{0}21 \sim \bar{2}01$$
$$\bar{0}32 \sim \bar{3}02$$
$$\bar{0}23 \sim \bar{2}03$$
$$\bar{0}31 \sim \bar{3}01$$
$$\bar{3}10 \sim \bar{1}30$$
$$\bar{2}13 \sim \bar{1}23$$
$$\bar{0}13 \sim \bar{1}03$$
$$\bar{2}30 \sim \bar{3}20$$
$$\bar{3}21 \sim \bar{2}31$$

The orbits of $N^{(\bar{0}12)}$ on $\{0,1,2,3\}$ are $\{0,1\}$, $\{\bar{0},\bar{1}\}$, $\{2\}$, $\{3\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0^{-1}t_1t_2t_i$ for one t_i from each orbit of $N^{(\bar{0}12)}$ and find to which double coset it belongs.

$\bar{0}12\bar{2} = \bar{0}1 \Rightarrow Nt_0^{-1}t_1t_2t_2^{-1} = Nt_0^{-1}t_1 \in Nt_0^{-1}t_1N = [\bar{0}1]$.
$\bar{0}122 = \bar{0}1\bar{2} \Rightarrow Nt_0^{-1}t_1t_2t_2 = Nt_0^{-1}t_1t_2^{-1} \in Nt_0^{-1}t_1t_2^{-1}N = [\bar{0}1\bar{2}]$.
$\bar{0}12\bar{0} = \bar{0}\bar{1}\bar{2} \Rightarrow Nt_0^{-1}t_1t_2t_0^{-1} = Nt_0^{-1}t_1^{-1}t_2^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}N = [\bar{0}\bar{1}\bar{2}]$.

We know via MAGMA that $Nt_0^{-1}t_1t_2t_0$, $Nt_0^{-1}t_1t_2t_3$, and $Nt_0^{-1}t_1t_2t_3^{-1}$ are elements of the new double cosets $[\bar{0}120]$, $[\bar{0}123]$, and $[\bar{0}12\bar{3}]$, respectively.

Consider $[\bar{0}1\bar{2}]$.
MAGMA shows that the coset stabilizing group
$N^{(\bar{0}1\bar{2})} = \{e, (01), (23), (01)(23), (12)(03), (02)(13), (0312), (0213)\}$. So we know that there are three distinct single cosets within $[\bar{0}1\bar{2}]$. We list these single cosets and their equal names below, according to their corresponding subscripts.

$$\bar{0}1\bar{2} \sim \bar{1}0\bar{3} \sim \bar{3}2\bar{1} \sim \bar{2}3\bar{0} \sim \bar{3}2\bar{0} \sim \bar{2}3\bar{1} \sim \bar{1}0\bar{2} \sim \bar{0}1\bar{3}$$
$$\bar{2}1\bar{0} \sim \bar{1}2\bar{3} \sim \bar{3}0\bar{1} \sim \bar{0}3\bar{2} \sim \bar{3}0\bar{2} \sim \bar{0}3\bar{1} \sim \bar{1}2\bar{0} \sim \bar{2}1\bar{3}$$
$$\bar{3}1\bar{2} \sim \bar{1}3\bar{0} \sim \bar{0}2\bar{1} \sim \bar{2}0\bar{3} \sim \bar{0}2\bar{3} \sim \bar{2}0\bar{1} \sim \bar{1}3\bar{2} \sim \bar{3}1\bar{0}$$

The orbits of $N^{(\bar{0}1\bar{2})}$ on $\{0,1,2,3\}$ are $\{0,1,2,3\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$. We consider $Nt_0^{-1}t_1t_2^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}1\bar{2})}$ and find to which double coset it belongs.

$\bar{0}1\bar{2}2 = \bar{0}1 \Rightarrow Nt_0^{-1}t_1t_2^{-1}t_2 = Nt_0^{-1}t_1 \in Nt_0^{-1}t_1N = [\bar{0}1]$.
$\bar{0}1\bar{2}\bar{2} = \bar{0}12 \Rightarrow Nt_0^{-1}t_1t_2^{-1}t_2^{-1} = Nt_0^{-1}t_1t_2 \in Nt_0^{-1}t_1t_2N = [\bar{0}12]$.

Consider $[\bar{0}\bar{1}2]$.
MAGMA shows that the coset stabilizing group $N^{(\bar{0}\bar{1}2)} = \{e, (03)(12)\}$. Listed below are the twelve distinct single cosets within $[\bar{0}\bar{1}2]$ along with their equal names. They are listed according to their corresponding subscript.

$$\bar{0}\bar{1}2 \sim \bar{3}\bar{2}1$$
$$\bar{1}\bar{0}2 \sim \bar{3}\bar{2}0$$
$$\bar{3}\bar{1}2 \sim \bar{0}\bar{2}1$$
$$\bar{0}\bar{3}2 \sim \bar{1}\bar{2}3$$
$$\bar{0}\bar{1}3 \sim \bar{2}\bar{3}1$$
$$\bar{1}\bar{2}0 \sim \bar{3}\bar{0}2$$
$$\bar{2}\bar{0}1 \sim \bar{3}\bar{1}1$$
$$\bar{1}\bar{3}2 \sim \bar{0}\bar{2}3$$
$$\bar{0}\bar{3}1 \sim \bar{2}\bar{1}3$$
$$\bar{1}\bar{0}3 \sim \bar{2}\bar{3}0$$
$$\bar{3}\bar{0}1 \sim \bar{2}\bar{1}1$$
$$\bar{2}\bar{0}3 \sim \bar{1}\bar{3}0$$

The orbits of $N^{(\bar{0}\bar{1}2)}$ on $\{0,1,2,3\}$ are $\{0,3\}$, $\{1,2\}$, $\{\bar{0},\bar{3}\}$, and $\{\bar{1},\bar{2}\}$. We consider $Nt_0^{-1}t_1^{-1}t_2t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}2)}$ and find to which double coset it belongs.

$\bar{0}\bar{1}2\bar{2} = \bar{0}\bar{1} \Rightarrow Nt_0^{-1}t_1^{-1}t_2t_2^{-1} = Nt_0^{-1}t_1^{-1} \in Nt_0^{-1}t_1^{-1}N = [\bar{0}\bar{1}]$.
$\bar{0}\bar{1}22 = \bar{0}\bar{1}\bar{2} \Rightarrow Nt_0^{-1}t_1^{-1}t_2t_2 = Nt_0^{-1}t_1^{-1}t_2^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}N = [\bar{0}\bar{1}\bar{2}]$.
$\bar{0}\bar{1}2\bar{0} = 01\bar{2} \Rightarrow Nt_0^{-1}t_1^{-1}t_2t_0^{-1} = Nt_0t_1t_2^{-1} \in Nt_0t_1t_2^{-1}N = [01\bar{2}]$.
$\bar{0}\bar{1}20 = 01\bar{2}\bar{0} \Rightarrow Nt_0^{-1}t_1^{-1}t_2t_0 = Nt_0t_1t_2^{-1}t_0^{-1} \in Nt_0t_1t_2^{-1}t_0^{-1}N = [01\bar{2}\bar{0}]$.

Consider $[\bar{0}\bar{1}\bar{2}]$.

The coset stabilizing group $N^{(\bar{0}\bar{1}\bar{2})} = \{e\}$. So there are twenty-four distinct single cosets within $[\bar{0}\bar{1}\bar{2}]$. We list them below according to their corresponding subscript.

$\bar{0}\bar{1}\bar{2}$	$\bar{1}\bar{0}\bar{2}$	$\bar{2}\bar{1}\bar{0}$	$\bar{3}\bar{1}\bar{2}$	$\bar{0}\bar{2}\bar{1}$	$\bar{0}\bar{3}\bar{2}$
$\bar{0}\bar{1}\bar{3}$	$\bar{1}\bar{2}\bar{0}$	$\bar{2}\bar{0}\bar{1}$	$\bar{1}\bar{3}\bar{2}$	$\bar{3}\bar{0}\bar{2}$	$\bar{0}\bar{2}\bar{3}$
$\bar{0}\bar{3}\bar{1}$	$\bar{3}\bar{1}\bar{0}$	$\bar{2}\bar{1}\bar{3}$	$\bar{1}\bar{0}\bar{3}$	$\bar{2}\bar{3}\bar{0}$	$\bar{3}\bar{2}\bar{1}$
$\bar{1}\bar{2}\bar{3}$	$\bar{2}\bar{3}\bar{1}$	$\bar{3}\bar{0}\bar{1}$	$\bar{2}\bar{0}\bar{3}$	$\bar{1}\bar{3}\bar{0}$	$\bar{3}\bar{2}\bar{0}$

The orbits of $N^{(\bar{0}\bar{1}\bar{2})}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0^{-1}t_1^{-1}t_2^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}\bar{2})}$ and find to which double coset it belongs.

$\bar{0}\bar{1}\bar{2}2 = \bar{0}\bar{1} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_2 = Nt_0^{-1}t_1^{-1} \in Nt_0^{-1}t_1^{-1}N = [\bar{0}\bar{1}]$.
$\bar{0}\bar{1}\bar{2}\bar{2} = \bar{0}\bar{1}2 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_2^{-1} = Nt_0^{-1}t_1^{-1}t_2 \in Nt_0^{-1}t_1^{-1}t_2N = [\bar{0}\bar{1}2]$.
$\bar{0}\bar{1}\bar{2}0 = 0102 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0 = Nt_0t_1t_0t_2 \in Nt_0t_1t_0t_2N = [0102]$.
$\bar{0}\bar{1}\bar{2}1 = \bar{0}12 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_1 = Nt_0^{-1}t_1t_2 \in Nt_0^{-1}t_1t_2N = [\bar{0}12]$.
$\bar{0}\bar{1}\bar{2}\bar{1} = \bar{0}120 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_1^{-1} = Nt_0^{-1}t_1t_2t_0 \in Nt_0^{-1}t_1t_2t_0N = [\bar{0}120]$.

We know that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}$ are elements within the new double cosets $[\bar{0}\bar{1}\bar{2}3]$, $[\bar{0}\bar{1}\bar{2}\bar{0}]$, and $[\bar{0}\bar{1}\bar{2}\bar{3}]$, respectively.

Let us now point our attention to double cosets with representative words of length four. Namely, $[0102]$, $[010\bar{2}]$, $[0120]$, $[0123]$, $[012\bar{3}]$, $[01\bar{2}\bar{0}]$, $[0\bar{1}\bar{2}3]$, $[0\bar{1}\bar{2}\bar{0}]$, $[0\bar{1}\bar{2}\bar{3}]$, $[\bar{0}120]$, $[\bar{0}123]$, $[\bar{0}12\bar{3}]$, $[\bar{0}\bar{1}\bar{2}3]$, $[\bar{0}\bar{1}\bar{2}\bar{0}]$, and $[\bar{0}\bar{1}\bar{2}\bar{3}]$.

Consider $[0102]$.

Take note of the following:

$$
\begin{aligned}
\underline{010}2 \; & \underbrace{=}_{R9} \; \bar{1}\underline{\bar{0}}\bar{1}2 \\
& = \underline{\bar{1}0}0\bar{1}2 \\
& \underbrace{=}_{R10} (\bar{0}10\bar{1}2)^{(01)} \\
& = \bar{1}01\underline{\bar{0}}2 \\
& = \bar{1}\underline{010}02 \\
& \underbrace{=}_{R9} \underline{\bar{1}\bar{1}}\bar{0}\underline{\bar{1}0}2 \\
& \underbrace{=}_{R10} 1\underline{\bar{0}\bar{0}}12 \\
& = 1012
\end{aligned}
$$

Hence, $N(t_0t_1t_0t_2)^{(01)} = Nt_1t_0t_1t_2 = Nt_0t_1t_0t_2$ and the coset stabilizing group $N^{(0102)} = \{e, (01)\}$. We list the twelve distinct single cosets within [0102] below, along with their equal names.

$$
\begin{gathered}
0102 \sim 1012 \\
2120 \sim 1210 \\
3132 \sim 1312 \\
0201 \sim 2021 \\
0302 \sim 3032 \\
0103 \sim 1013 \\
0203 \sim 2023 \\
3130 \sim 1310 \\
2123 \sim 1213 \\
2320 \sim 3230 \\
3231 \sim 2321 \\
3031 \sim 0301
\end{gathered}
$$

The orbits of $N^{(0102)}$ on $\{0,1,2,3\}$ are $\{0,1\}$, $\{\bar{0}\bar{1}\}$, $\{2\}$, $\{3\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0t_1t_0t_2t_i$ for one t_i from each orbit of $N^{(0102)}$ and find to which double coset it belongs.

$01022 = 010\bar{2} \Rightarrow Nt_0t_1t_0t_2t_2 = Nt_0t_1t_0t_2^{-1} \in Nt_0t_1t_0t_2^{-1}N = [010\bar{2}]$.
$0102\bar{2} = 010 \Rightarrow Nt_0t_1t_0t_2t_2^{-1} = Nt_0t_1t_0 \in Nt_0t_1t_0N = [010]$.
$01020 = \bar{0}\bar{1}\bar{2}\bar{0} \Rightarrow Nt_0t_1t_0t_2t_0 = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{0}]$.
$0102\bar{0} = \bar{0}\bar{1}\bar{2} \Rightarrow Nt_0t_1t_0t_2t_0^{-1} = Nt_0^{-1}t_1^{-1}t_2^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}N = [\bar{0}\bar{1}\bar{2}]$.
$0102\bar{3} = \bar{0}12\bar{3} \Rightarrow Nt_0t_1t_0t_2t_3^{-1} = Nt_0^{-1}t_1t_2t_3^{-1} \in Nt_0^{-1}t_1t_2t_3^{-1}N = [\bar{0}12\bar{3}]$.

In addition to the last three statements, MAGMA provided that $Nt_0t_1t_0t_2t_3$ is an element of the new double coset $[01023]$.

Consider $[010\bar{2}]$.
MAGMA shows that the coset stabilizing group $N^{(010\bar{2})} = \{e, (01)\}$. Hence, there are twelve distinct single cosets within $[010\bar{2}]$, which are listed below with their equal names.

$$
\begin{gathered}
010\bar{2} \sim 101\bar{2} \\
212\bar{0} \sim 121\bar{0} \\
313\bar{2} \sim 131\bar{2} \\
020\bar{1} \sim 202\bar{1} \\
030\bar{2} \sim 303\bar{2} \\
010\bar{3} \sim 101\bar{3} \\
020\bar{3} \sim 202\bar{3} \\
030\bar{1} \sim 303\bar{1} \\
313\bar{0} \sim 131\bar{0} \\
212\bar{3} \sim 121\bar{3} \\
232\bar{0} \sim 323\bar{0} \\
323\bar{1} \sim 232\bar{1}
\end{gathered}
$$

The orbits of $N^{(010\bar{2})}$ on $\{0, 1, 2, 3\}$ are $\{0, 1\}$, $\{\bar{0}\bar{1}\}$, $\{2\}$, $\{3\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0t_1t_0t_2^{-1}t_i$ for one t_i from each orbit of $N^{(010\bar{2})}$ and find to which double coset it belongs.

$010\bar{2}2 = 010 \Rightarrow Nt_0t_1t_0t_2^{-1}t_2 = Nt_0t_1t_0 \in Nt_0t_1t_0N = [010]$.
$010\bar{2}\bar{2} = 0102 \Rightarrow Nt_0t_1t_0t_2^{-1}t_2^{-1} = Nt_0t_1t_0t_2 \in Nt_0t_1t_0t_2N = [0102]$.
$010\bar{2}0 = 012 \Rightarrow Nt_0t_1t_0t_2^{-1}t_0 = Nt_0t_1t_2 \in Nt_0t_1t_2N = [012]$.

$01\bar{0}\bar{2}\bar{0} = 0120 \Rightarrow Nt_0t_1t_0t_2^{-1}t_0^{-1} = Nt_0t_1t_2t_0 \in Nt_0t_1t_2t_0N = [0120]$.
$010\bar{2}3 = 0\bar{1}\bar{2}3 \Rightarrow Nt_0t_1t_0t_2^{-1}t_3 = Nt_0t_1^{-1}t_2^{-1}t_3 \in Nt_0t_1^{-1}t_2^{-1}t_3N = [0\bar{1}\bar{2}3]$.

In addition to the last three statements, MAGMA provides $Nt_0t_1t_0t_2^{-1}t_3^{-1}$ is an element of the new double coset $[010\bar{2}\bar{3}]$.

Consider [0120].
The coset stabilizing group $N^{(0120)} = \{e\}$, implying that there exists twenty-four distinct single cosets within [0120]. We list them below.

0120	1021	2102	3123	0210	0320
0130	1201	2012	1321	3023	0230
0310	3103	2132	1031	3203	3213
1231	3013	2032	1301	2312	3203

The orbits of $N^{(0120)}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0t_1t_2t_0t_i$ for one t_i from each orbit of $N^{(0120)}$ and find to which double coset it belongs.

$0120\bar{0} = 012 \Rightarrow Nt_0t_1t_2t_0t_0^{-1} = Nt_0t_1t_2 \in Nt_0t_1t_2N = [012]$.
$01200 = 012\bar{0} \Rightarrow Nt_0t_1t_2t_0t_0 = Nt_0t_1t_2t_0^{-1} \in Nt_0t_1t_2t_0^{-1}N = [012\bar{0}]$.
$0120\bar{2} = 01\bar{2}\bar{0} \Rightarrow Nt_0t_1t_2t_0t_2^{-1} = Nt_0t_1t_2^{-1}t_0^{-1} \in Nt_0t_1t_2^{-1}t_0^{-1}N = [01\bar{2}\bar{0}]$.
$01202 = \bar{0}\bar{1}\bar{2}\bar{0} \Rightarrow Nt_0t_1t_2t_0t_2 = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{0}]$.

In addition to the last two statements, MAGMA provides that $Nt_0t_1t_2t_0t_1$, $Nt_0t_1t_2t_0t_3$, $Nt_0t_1t_2t_0t_1^{-1}$, and $Nt_0t_1t_2t_0t_3^{-1}$ are elements within the new double cosets [01201], [01203], $[0120\bar{1}]$, and $[0120\bar{3}]$, respectively.

Consider [0123].
The coset stabilizing group $N^{(0123)} = \{e\}$, implying that there exists twenty-four distinct single cosets within [0123]. We list them below.

0123	1023	2103	3120	0213	0321
0132	1203	2013	1320	3021	0231
0312	3102	2130	1032	2301	3210
1230	3012	2031	1302	2310	3201

The orbits of $N^{(0123)}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0t_1t_2t_3t_i$ for one t_i from each orbit of $N^{(0123)}$ and find to which double coset it belongs.

$0123\bar{3} = 012 \Rightarrow Nt_0t_1t_2t_3t_3^{-1} = Nt_0t_1t_2 \in Nt_0t_1t_2N = [012]$.
$01233 = 012\bar{3} \Rightarrow Nt_0t_1t_2t_3t_3 = Nt_0t_1t_2t_3^{-1} \in Nt_0t_1t_2t_3^{-1}N = [012\bar{3}]$.
$0123\bar{2} = 01\bar{2}\bar{0} \Rightarrow Nt_0t_1t_2t_3t_2^{-1} = Nt_0t_1t_2^{-1}t_0^{-1} \in Nt_0t_1t_2^{-1}t_0^{-1}N = [01\bar{2}\bar{0}]$.
$0123\bar{0} = 0120\bar{3} \Rightarrow Nt_0t_1t_2t_3t_0^{-1} = Nt_0t_1t_2t_0t_3^{-1} \in Nt_0t_1t_2t_0t_3^{-1}N = [0120\bar{3}]$.

In addition to the last two statements, MAGMA shows that $Nt_0t_1t_2t_3t_1$, $Nt_0t_1t_2t_3t_2$, $Nt_0t_1t_2t_3t_0$, and $Nt_0t_1t_2t_3t_1^{-1}$ are elements within [01231], [01232], [01230], and $[0123\bar{1}]$, respectively.

Consider $[012\bar{3}]$.
MAGMA shows that the coset stabilizing group $N^{(012\bar{3})} = \{e, (01)(23), (02)(13), (03)(12)\}$. The six distinct single cosets within $[012\bar{3}]$ are listed below, along with their equal names.

$$012\bar{3} \sim 103\bar{2} \sim 321\bar{0} \sim 230\bar{1}$$
$$102\bar{3} \sim 013\bar{2} \sim 320\bar{1} \sim 231\bar{0}$$
$$210\bar{3} \sim 123\bar{0} \sim 301\bar{2} \sim 032\bar{1}$$
$$312\bar{0} \sim 130\bar{2} \sim 021\bar{3} \sim 203\bar{1}$$
$$120\bar{3} \sim 213\bar{0} \sim 302\bar{1} \sim 031\bar{2}$$
$$201\bar{3} \sim 023\bar{1} \sim 310\bar{2} \sim 132\bar{0}$$

The orbits of $N^{(012\bar{3})}$ on $\{0,1,2,3\}$ are $\{0,1,2,3\}$, and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$. We consider $Nt_0t_1t_2t_3^{-1}t_i$ for one t_i from each orbit of $N^{(012\bar{3})}$ and find to which double coset it belongs.

$012\bar{3}3 = 012 \Rightarrow Nt_0t_1t_2t_3^{-1}t_3 = Nt_0t_1t_2 \in Nt_0t_1t_2N = [012]$.

$01 2 \bar{3} \bar{3} = 0123 \Rightarrow Nt_0 t_1 t_2 t_3^{-1} t_3^{-1} = Nt_0 t_1 t_2 t_3 \in Nt_0 t_1 t_2 t_3 = [0123]$.

This double coset is now closed under right multiplication. That is, $Nt_0 t_1 t_2 t_3^{-1} t_i$ do not move forward to any double coset with representative word of length five.

Consider $[01\bar{2}\bar{0}]$.
The coset stabilizing group $N^{(01\bar{2}\bar{0})} = \{e\}$, implying that there exists twenty-four distinct single cosets within $[01\bar{2}\bar{0}]$. We list them below.

$01\bar{2}\bar{0}$	$10\bar{2}\bar{1}$	$21\bar{0}\bar{2}$	$31\bar{2}\bar{3}$	$02\bar{1}\bar{0}$	$03\bar{2}\bar{0}$
$01\bar{3}\bar{0}$	$12\bar{0}\bar{1}$	$20\bar{1}\bar{2}$	$13\bar{2}\bar{1}$	$02\bar{3}\bar{0}$	$03\bar{1}\bar{0}$
$31\bar{0}\bar{3}$	$30\bar{2}\bar{3}$	$21\bar{3}\bar{2}$	$10\bar{3}\bar{1}$	$23\bar{0}\bar{2}$	$32\bar{1}\bar{3}$
$12\bar{3}\bar{1}$	$30\bar{1}\bar{3}$	$20\bar{3}\bar{2}$	$13\bar{0}\bar{1}$	$23\bar{1}\bar{2}$	$32\bar{0}\bar{3}$

The orbits of $N^{(01\bar{2}\bar{0})}$ on $\{0, 1, 2, 3\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0 t_1 t_2^{-1} t_0^{-1} t_i$ for one t_i from each orbit of $N^{(01\bar{2}\bar{0})}$ and find to which double coset it belongs.

$01\bar{2}\bar{0}0 = 01\bar{2} \Rightarrow Nt_0 t_1 t_2^{-1} t_0^{-1} t_0 = Nt_0 t_1 t_2^{-1} \in Nt_0 t_1 t_2^{-1} N = [01\bar{2}]$.
$01\bar{2}\bar{0}\bar{0} = \bar{0}\bar{1}2 \Rightarrow Nt_0 t_1 t_2^{-1} t_0^{-1} t_0^{-1} = Nt_0^{-1} t_1^{-1} t_2 \in Nt_0^{-1} t_1^{-1} t_2 N = [\bar{0}\bar{1}2]$.
$01\bar{2}\bar{0}1 = 0123 \Rightarrow Nt_0 t_1 t_2^{-1} t_0^{-1} t_1 = Nt_0 t_1 t_2 t_3 \in Nt_0 t_1 t_2 t_3 N = [0123]$.
$01\bar{2}\bar{0}2 = 0124 \Rightarrow Nt_0 t_1 t_2^{-1} t_0^{-1} t_2 = Nt_0 t_1 t_2 t_4 \in Nt_0 t_1 t_2 t_4 = [0124]$.
$01\bar{2}\bar{0}\bar{2} = \bar{0}\bar{1}\bar{2}\bar{0} \Rightarrow Nt_0 t_1 t_2^{-1} t_0^{-1} t_2^{-1} = Nt_0^{-1} t_1^{-1} t_2^{-1} t_0^{-1} \in Nt_0^{-1} t_1^{-1} t_2^{-1} t_0^{-1} N = [\bar{0}\bar{1}\bar{2}\bar{0}]$.
$01\bar{2}\bar{0}\bar{1} = 01232 \Rightarrow Nt_0 t_1 t_2^{-1} t_0^{-1} t_1^{-1} = Nt_0 t_1 t_2 t_3 t_2 \in Nt_0 t_1 t_2 t_3 t_2 N = [01232]$.
$01\bar{2}\bar{0}\bar{3} = \bar{0}\bar{1}\bar{2}\bar{3} \Rightarrow Nt_0 t_1 t_2^{-1} t_0^{-1} t_3^{-1} = Nt_0^{-1} t_1^{-1} t_2^{-1} t_3^{-1} \in Nt_0^{-1} t_1^{-1} t_2^{-1} t_3^{-1} N = [\bar{0}\bar{1}\bar{2}\bar{3}]$.

In addition to the last six statements, MAGMA shows that $Nt_0 t_1 t_2^{-1} t_0^{-1} t_3$ is an element within $[01\bar{2}\bar{0}3]$.

Consider $[0\bar{1}\bar{2}\bar{0}]$.
MAGMA proves that the coset stabilizing group $N^{(0\bar{1}\bar{2}\bar{0})} = \{e, (02)\}$. We list the twelve distinct single cosets within $[0\bar{1}\bar{2}\bar{0}]$ along with their equal names.

$$0\bar{1}\bar{2}\bar{0} \sim 2\bar{1}\bar{0}\bar{2}$$
$$1\bar{0}\bar{2}\bar{1} \sim 2\bar{0}\bar{1}\bar{2}$$
$$3\bar{1}\bar{2}\bar{3} \sim 2\bar{1}\bar{3}\bar{2}$$
$$0\bar{2}\bar{1}\bar{0} \sim 1\bar{2}\bar{0}\bar{1}$$
$$0\bar{3}\bar{2}\bar{0} \sim 2\bar{3}\bar{0}\bar{2}$$
$$0\bar{1}\bar{3}\bar{0} \sim 3\bar{1}\bar{0}\bar{3}$$
$$1\bar{3}\bar{2}\bar{1} \sim 2\bar{3}\bar{1}\bar{2}$$
$$3\bar{0}\bar{2}\bar{3} \sim 2\bar{0}\bar{3}\bar{2}$$
$$0\bar{2}\bar{3}\bar{0} \sim 3\bar{2}\bar{0}\bar{3}$$
$$0\bar{3}\bar{1}\bar{0} \sim 1\bar{3}\bar{0}\bar{1}$$
$$1\bar{0}\bar{3}\bar{1} \sim 3\bar{0}\bar{1}\bar{3}$$
$$3\bar{2}\bar{1}\bar{3} \sim 1\bar{2}\bar{3}\bar{1}$$

The orbits of $N^{(0\bar{1}\bar{2}\bar{0})}$ on $\{0,1,2,3\}$ are $\{0,2\}$, $\{\bar{0},\bar{2}\}$, $\{1\}$, $\{3\}$, $\{\bar{1}\}$, and $\{\bar{3}\}$. We consider $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_i$ for one t_i from each orbit of $N^{(0\bar{1}\bar{2}\bar{0})}$ and find to which double coset it belongs.

$0\bar{1}\bar{2}\bar{0}0 = 0\bar{1}\bar{2} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_0 = Nt_0t_1^{-1}t_2^{-1} \in Nt_0t_1^{-1}t_2^{-1}N = [0\bar{1}\bar{2}]$.
$0\bar{1}\bar{2}\bar{0}\bar{0} = 012 \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_0^{-1} = Nt_0t_1t_2 \in Nt_0t_1t_2N = [012]$.
$0\bar{1}\bar{2}\bar{0}\bar{3} = 0120\bar{1} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1} = Nt_0t_1t_2t_0t_1^{-1} \in Nt_0t_1t_2t_0t_1^{-1}N = [0120\bar{1}]$.
$0\bar{1}\bar{2}\bar{0}\bar{1} = 0123\bar{1} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1} = Nt_0t_1t_2t_3t_1^{-1} \in Nt_0t_1t_2t_3t_1^{-1}N = [0123\bar{1}]$.

In addition to the last three statements, MAGMA provides that $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1$ and $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3$ are elements of the new double cosets $[0\bar{1}\bar{2}\bar{0}1]$ and $[0\bar{1}\bar{2}\bar{0}3]$, respectively.

Consider $[0\bar{1}\bar{2}3]$.
MAGMA proves that the coset stabilizing group $N^{(0\bar{1}\bar{2}3)} = \{e,(01),(23),(01)(23)\}$. We list the six distinct single cosets within $[0\bar{1}\bar{2}3]$ along with their equal names.

$$0\bar{1}\bar{2}3 \sim 1\bar{0}\bar{2}3 \sim 1\bar{0}\bar{3}2 \sim 0\bar{1}\bar{3}2$$

$$2\bar{1}\bar{0}3 \sim 1\bar{2}\bar{0}3 \sim 1\bar{2}\bar{3}0 \sim 2\bar{1}\bar{3}0$$
$$3\bar{1}\bar{2}0 \sim 1\bar{3}\bar{2}0 \sim 1\bar{3}\bar{0}2 \sim 3\bar{1}\bar{0}2$$
$$0\bar{2}\bar{1}3 \sim 2\bar{0}\bar{1}3 \sim 2\bar{0}\bar{3}1 \sim 0\bar{2}\bar{3}1$$
$$0\bar{3}\bar{2}1 \sim 3\bar{0}\bar{2}1 \sim 3\bar{0}\bar{1}2 \sim 0\bar{3}\bar{1}2$$
$$2\bar{3}\bar{0}1 \sim 3\bar{2}\bar{0}1 \sim 3\bar{2}\bar{1}0 \sim 2\bar{3}\bar{1}0$$

The orbits of $N^{(0\bar{1}\bar{2}3)}$ on $\{0,1,2,3\}$ are $\{0,1\}$, $\{\bar{0},\bar{1}\}$, $\{2,3\}$, and $\{\bar{2},\bar{3}\}$. We consider $Nt_0t_1^{-1}t_2^{-1}t_3t_i$ for one t_i from each orbit of $N^{(0\bar{1}\bar{2}3)}$ and find to which double coset it belongs.

$0\bar{1}\bar{2}3\bar{3} = 0\bar{1}\bar{2} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_3t_3^{-1} = Nt_0t_1^{-1}t_2^{-1} \in Nt_0t_1^{-1}t_2^{-1}N = [0\bar{1}\bar{2}]$.
$0\bar{1}\bar{2}33 = 0\bar{1}\bar{2}\bar{3} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_3t_3 = Nt_0t_1^{-1}t_2^{-1}t_3^{-1} \in Nt_0t_1^{-1}t_2^{-1}t_3^{-1} = [0\bar{1}\bar{2}\bar{3}]$.
$0\bar{1}\bar{2}3\bar{1} = 010\bar{2} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_3t_1^{-1} = Nt_0t_1t_0t_2^{-1} \in Nt_0t_1t_0t_2^{-1} = [010\bar{2}]$.
$0\bar{1}\bar{2}30 = 010\bar{2}\bar{3} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_3t_0 = Nt_0t_1t_0t_2^{-1}t_3^{-1} \in Nt_0t_1t_0t_2^{-1}t_3^{-1} = [010\bar{2}\bar{3}]$.

MAGMA shows the last two statements.

Consider $[0\bar{1}\bar{2}\bar{3}]$.
MAGMA proves that the coset stabilizing group $N^{(0\bar{1}\bar{2}\bar{3})} = \{e, (01), (23), (01)(23)\}$. We list the six distinct single cosets within $[0\bar{1}\bar{2}\bar{3}]$ along with their equal names.

$$0\bar{1}\bar{2}\bar{3} \sim 1\bar{0}\bar{3}\bar{2} \sim 1\bar{0}\bar{2}\bar{3} \sim 0\bar{1}\bar{3}\bar{2}$$
$$2\bar{1}\bar{0}\bar{3} \sim 1\bar{2}\bar{3}\bar{0} \sim 1\bar{2}\bar{0}\bar{3} \sim 2\bar{1}\bar{3}\bar{0}$$
$$3\bar{1}\bar{2}\bar{0} \sim 1\bar{3}\bar{0}\bar{2} \sim 1\bar{3}\bar{2}\bar{0} \sim 3\bar{1}\bar{0}\bar{2}$$
$$0\bar{2}\bar{1}\bar{3} \sim 2\bar{0}\bar{3}\bar{1} \sim 2\bar{0}\bar{1}\bar{3} \sim 0\bar{2}\bar{3}\bar{1}$$
$$0\bar{3}\bar{2}\bar{1} \sim 3\bar{0}\bar{1}\bar{2} \sim 3\bar{0}\bar{2}\bar{1} \sim 0\bar{3}\bar{1}\bar{2}$$
$$2\bar{3}\bar{0}\bar{1} \sim 3\bar{2}\bar{1}\bar{0} \sim 3\bar{2}\bar{0}\bar{1} \sim 2\bar{3}\bar{1}\bar{0}$$

The orbits of $N^{(0\bar{1}\bar{2}\bar{3})}$ on $\{0,1,2,3\}$ are $\{0,1\}$, $\{\bar{0},\bar{1}\}$, $\{2,3\}$, and $\{\bar{2},\bar{3}\}$. We consider $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_i$ for one t_i from each orbit of $N^{(0\bar{1}\bar{2}\bar{3})}$ and find to which double coset it belongs.

$0\bar{1}\bar{2}\bar{3}3 = 0\bar{1}\bar{2} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_3 = Nt_0t_1^{-1}t_2^{-1} \in Nt_0t_1^{-1}t_2^{-1}N = [0\bar{1}\bar{2}]$.
$0\bar{1}\bar{2}\bar{3}\bar{3} = 0\bar{1}\bar{2}3 \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_3^{-1} = Nt_0t_1^{-1}t_2^{-1}t_3 \in Nt_0t_1^{-1}t_2^{-1}t_3N = [0\bar{1}\bar{2}3]$.
$0\bar{1}\bar{2}\bar{3}1 = 0\bar{1}\bar{2}\bar{0}3 \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1 = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3 \in Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N = [0\bar{1}\bar{2}\bar{0}3]$.

In addition to the last statement, MAGMA proves that $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}$ is an element of the new double coset $[0\bar{1}\bar{2}\bar{3}\bar{1}]$.

Consider $[\bar{0}120]$.
MAGMA proves that the coset stabilizing group $N^{(\bar{0}120)} = \{e, (02)\}$. We list the twelve distinct single cosets within $[\bar{0}120]$ along with their equal names.

$$\begin{gathered}
\bar{0}120 \sim \bar{2}102 \\
\bar{1}021 \sim \bar{2}012 \\
\bar{3}123 \sim \bar{2}132 \\
\bar{0}210 \sim \bar{1}201 \\
\bar{0}320 \sim \bar{2}302 \\
\bar{0}130 \sim \bar{3}103 \\
\bar{1}321 \sim \bar{2}312 \\
\bar{3}023 \sim \bar{2}032 \\
\bar{0}230 \sim \bar{3}203 \\
\bar{0}310 \sim \bar{1}301 \\
\bar{1}031 \sim \bar{3}013 \\
\bar{3}213 \sim \bar{1}231
\end{gathered}$$

The orbits of $N^{(\bar{0}120)}$ on $\{0,1,2,3\}$ are $\{0,2\}$, $\{\bar{0},\bar{2}\}$, $\{1\}$, $\{3\}$, $\{\bar{1}\}$, and $\{\bar{3}\}$. We consider $Nt_0^{-1}t_1t_2t_0t_i$ for one t_i from each orbit of $N^{(\bar{0}120)}$ and find to which double coset it belongs.

$\bar{0}120\bar{0} = \bar{0}12 \Rightarrow Nt_0^{-1}t_1t_2t_0t_0^{-1} = Nt_0^{-1}t_1t_2 \in Nt_0^{-1}t_1t_2N = [\bar{0}12]$.
$\bar{0}1200 = \bar{0}\bar{1}\bar{2} \Rightarrow Nt_0^{-1}t_1t_2t_0t_0 = Nt_0^{-1}t_1^{-1}t_2^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}N = [\bar{0}\bar{1}\bar{2}]$.

In addition to the last statement, MAGMA shows that $Nt_0^{-1}t_1t_2t_0t_1$, $Nt_0^{-1}t_1t_2t_0t_3$, $Nt_0^{-1}t_1t_2t_0t_1^{-1}$, and $Nt_0^{-1}t_1t_2t_0t_3^{-1}$ are elements within the new double cosets $[\bar{0}1201]$, $[\bar{0}1203]$, $[\bar{0}120\bar{1}]$, and $[\bar{0}120\bar{3}]$, respectively.

Consider $[\bar{0}123]$.
MAGMA proves that the coset stabilizing group $N^{(\bar{0}123)} = \{e, (01), (23), (01)(23)\}$. We list the six distinct single cosets within $[\bar{0}123]$ along with their equal names.

$$\begin{gathered}
\bar{0}123 \sim \bar{1}023 \sim \bar{0}132 \sim \bar{1}032 \\
\bar{2}103 \sim \bar{1}203 \sim \bar{2}130 \sim \bar{1}230 \\
\bar{3}120 \sim \bar{1}320 \sim \bar{3}102 \sim \bar{1}302 \\
\bar{0}213 \sim \bar{2}013 \sim \bar{0}231 \sim \bar{2}031 \\
\bar{0}321 \sim \bar{3}121 \sim \bar{0}312 \sim \bar{3}012 \\
\bar{2}310 \sim \bar{3}210 \sim \bar{2}301 \sim \bar{3}201
\end{gathered}$$

The orbits of $N^{(\bar{0}123)}$ on $\{0, 1, 2, 3\}$ are $\{0, 1\}$, $\{\bar{0}, \bar{1}\}$, $\{2, 3\}$, and $\{\bar{2}, \bar{3}\}$. We consider $Nt_0^{-1}t_1t_2t_3t_i$ for one t_i from each orbit of $N^{(\bar{0}123)}$ and find to which double coset it belongs.

$\bar{0}123\bar{3} = \bar{0}12 \Rightarrow Nt_0^{-1}t_1t_2t_3t_3^{-1} = Nt_0^{-1}t_1t_2 \in Nt_0^{-1}t_1t_2N = [\bar{0}123]$.
$\bar{0}1233 = \bar{0}12\bar{3} \Rightarrow Nt_0^{-1}t_1t_2t_3t_3 = Nt_0^{-1}t_1t_2t_3^{-1} \in Nt_0^{-1}t_1t_2t_3^{-1}N = [\bar{0}12\bar{3}]$.
$\bar{0}123\bar{1} = \bar{0}120\bar{3} \Rightarrow Nt_0^{-1}t_1t_2t_3t_1^{-1} = Nt_0^{-1}t_1t_2t_0t_3^{-1} \in Nt_0^{-1}t_1t_2t_0t_3^{-1}N = [\bar{0}120\bar{3}]$.

In addition to the last statement, MAGMA shows that $Nt_0^{-1}t_1t_2t_3t_1$ is an element of the new double coset $[\bar{0}1231]$.

Consider $[\bar{0}12\bar{3}]$.
MAGMA proves that the coset stabilizing group $N^{(\bar{0}12\bar{3})} = \{e, (01), (23), (01)(23)\}$. We list the six distinct single cosets within $[\bar{0}12\bar{3}]$ along with their equal names.

$$\bar{0}12\bar{3} \sim \bar{0}13\bar{2} \sim \bar{1}02\bar{3} \sim \bar{1}03\bar{2}$$
$$\bar{2}10\bar{3} \sim \bar{2}13\bar{0} \sim \bar{1}20\bar{3} \sim \bar{1}23\bar{0}$$
$$\bar{3}12\bar{0} \sim \bar{3}10\bar{2} \sim \bar{1}32\bar{0} \sim \bar{1}30\bar{2}$$
$$\bar{0}21\bar{3} \sim \bar{0}23\bar{1} \sim \bar{2}01\bar{3} \sim \bar{2}03\bar{1}$$
$$\bar{0}32\bar{1} \sim \bar{0}31\bar{2} \sim \bar{3}02\bar{1} \sim \bar{3}01\bar{2}$$
$$\bar{2}31\bar{0} \sim \bar{2}30\bar{1} \sim \bar{3}21\bar{0} \sim \bar{3}20\bar{1}$$

The orbits of $N^{(\bar{0}12\bar{3})}$ on $\{0,1,2,3\}$ are $\{0,1\}$, $\{\bar{0},\bar{1}\}$, $\{2,3\}$, and $\{\bar{2},\bar{3}\}$. We consider $Nt_0^{-1}t_1t_2t_3^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}12\bar{3})}$ and find to which double coset it belongs.

$\bar{0}12\bar{3}3 = \bar{0}12 \Rightarrow Nt_0^{-1}t_1t_2t_3^{-1}t_3 = Nt_0^{-1}t_1t_2 \in Nt_0^{-1}t_1t_2N = [\bar{0}12]$.
$\bar{0}12\bar{3}\bar{3} = \bar{0}123 \Rightarrow Nt_0^{-1}t_1t_2t_3^{-1}t_3^{-1} = Nt_0^{-1}t_1t_2t_3 \in Nt_0^{-1}t_1t_2t_3N = [\bar{0}123]$.
$\bar{0}12\bar{3}1 = 0102 \Rightarrow Nt_0^{-1}t_1t_2t_3^{-1}t_1 = Nt_0t_1t_0t_2 \in Nt_0t_1t_0t_2N = [0102]$.
$\bar{0}12\bar{3}\bar{1} = 01023 \Rightarrow Nt_0^{-1}t_1t_2t_3^{-1}t_1^{-1} = Nt_0t_1t_0t_2t_3 \in Nt_0t_1t_0t_2t_3N = [01023]$.

MAGMA shows the last two statements.

Consider $[\bar{0}\bar{1}\bar{2}3]$.
MAGMA proves that the coset stabilizing group $N^{(\bar{0}\bar{1}\bar{2}3)} = \{e, (01)(23), (02)(13), (03)(12)\}$.
We list the six distinct single cosets within $[\bar{0}\bar{1}\bar{2}3]$ along with their equal names.

$$\bar{0}\bar{1}\bar{2}3 \sim \bar{3}\bar{2}\bar{1}0 \sim \bar{1}\bar{0}\bar{3}2 \sim \bar{2}\bar{3}\bar{0}1$$
$$\bar{1}\bar{0}\bar{2}3 \sim \bar{3}\bar{2}\bar{0}1 \sim \bar{0}\bar{1}\bar{3}2 \sim \bar{2}\bar{3}\bar{1}0$$
$$\bar{2}\bar{1}\bar{0}3 \sim \bar{3}\bar{0}\bar{1}2 \sim \bar{1}\bar{2}\bar{3}0 \sim \bar{0}\bar{3}\bar{2}1$$
$$\bar{3}\bar{1}\bar{2}0 \sim \bar{0}\bar{2}\bar{1}3 \sim \bar{1}\bar{3}\bar{0}2 \sim \bar{2}\bar{0}\bar{3}1$$
$$\bar{1}\bar{2}\bar{0}3 \sim \bar{3}\bar{0}\bar{2}1 \sim \bar{2}\bar{1}\bar{3}0 \sim \bar{0}\bar{3}\bar{1}2$$
$$\bar{2}\bar{0}\bar{1}3 \sim \bar{3}\bar{1}\bar{0}2 \sim \bar{0}\bar{2}\bar{3}1 \sim \bar{1}\bar{3}\bar{2}0$$

The orbits of $N^{(\bar{0}\bar{1}\bar{2}3)}$ on $\{0,1,2,3\}$ are $\{0,1,2,3\}$, and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$. We consider $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}\bar{2}3)}$ and find to which double coset it belongs.

$\bar{0}\bar{1}\bar{2}3\bar{3} = \bar{0}\bar{1}\bar{2} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_3^{-1} = Nt_0^{-1}t_1^{-1}t_2^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}N = [\bar{0}\bar{1}\bar{2}]$.
$\bar{0}\bar{1}\bar{2}33 = \bar{0}\bar{1}\bar{2}\bar{3} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_3 = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{3}]$.

This double coset is now closed under right multiplication. That is, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3t_i$ do not move forward to any double coset with representative word of length five.

Consider $[\bar{0}\bar{1}\bar{2}\bar{0}]$.
The coset stabilizing group $N^{(\bar{0}\bar{1}\bar{2}\bar{0})} = \{e\}$, implying that there exists twenty-four distinct single cosets within $[\bar{0}\bar{1}\bar{2}\bar{0}]$. We list them below.

$\bar{0}\bar{1}\bar{2}\bar{0}$	$\bar{1}\bar{0}\bar{2}\bar{1}$	$\bar{2}\bar{1}\bar{0}\bar{2}$	$\bar{3}\bar{1}\bar{2}\bar{3}$	$\bar{0}\bar{2}\bar{1}\bar{0}$	$\bar{0}\bar{3}\bar{2}\bar{0}$
$\bar{0}\bar{1}\bar{3}\bar{0}$	$\bar{1}\bar{2}\bar{0}\bar{1}$	$\bar{2}\bar{0}\bar{1}\bar{2}$	$\bar{1}\bar{3}\bar{2}\bar{1}$	$\bar{3}\bar{0}\bar{2}\bar{3}$	$\bar{0}\bar{2}\bar{3}\bar{0}$
$\bar{0}\bar{3}\bar{1}\bar{0}$	$\bar{3}\bar{1}\bar{0}\bar{3}$	$\bar{2}\bar{1}\bar{3}\bar{2}$	$\bar{1}\bar{0}\bar{3}\bar{1}$	$\bar{2}\bar{3}\bar{0}\bar{2}$	$\bar{3}\bar{2}\bar{1}\bar{3}$
$\bar{1}\bar{2}\bar{3}\bar{1}$	$\bar{3}\bar{0}\bar{1}\bar{3}$	$\bar{2}\bar{0}\bar{3}\bar{2}$	$\bar{1}\bar{3}\bar{0}\bar{1}$	$\bar{2}\bar{3}\bar{1}\bar{2}$	$\bar{3}\bar{2}\bar{0}\bar{3}$

The orbits of $N^{(\bar{0}\bar{1}\bar{2}\bar{0})}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}\bar{2}\bar{0})}$ and find to which double coset it belongs.

$\bar{0}\bar{1}\bar{2}\bar{0}0 = \bar{0}\bar{1}\bar{2} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_0 = Nt_0^{-1}t_1^{-1}t_2^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}N = [\bar{0}\bar{1}\bar{2}]$.
$\bar{0}\bar{1}\bar{2}\bar{0}\bar{0} = 0102 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_0^{-1} = Nt_0t_1t_0t_2 \in Nt_0t_1t_0t_2N = [0102]$.
$\bar{0}\bar{1}\bar{2}\bar{0}\bar{2} = 0120 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_2^{-1} = Nt_0t_1t_2t_0 \in Nt_0t_1t_2t_0N = [0120]$.
$\bar{0}\bar{1}\bar{2}\bar{0}2 = 01\bar{2}\bar{0} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_2 = Nt_0t_1t_2^{-1}t_0^{-1} \in Nt_0t_1t_2^{-1}t_0^{-1}N = [01\bar{2}\bar{0}]$.
$\bar{0}\bar{1}\bar{2}\bar{0}1 = \bar{0}1203 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1 = Nt_0^{-1}t_1t_2t_0t_3 \in Nt_0^{-1}t_1t_2t_0t_3N = [\bar{0}1203]$.

In addition to the last statement, MAGMA shows that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3$, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}$, and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}$ are elements within the new double cosets $[\bar{0}\bar{1}\bar{2}\bar{0}3]$, $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}]$, and $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}]$, respectively.

Consider $[\bar{0}\bar{1}\bar{2}\bar{3}]$.
The coset stabilizing group $N^{(\bar{0}\bar{1}\bar{2}\bar{3})} = \{e\}$, implying that there exists twenty-four distinct single cosets within $[\bar{0}\bar{1}\bar{2}\bar{3}]$. We list them below.

$\bar{0}\bar{1}\bar{2}\bar{3}$	$\bar{1}\bar{0}\bar{2}\bar{3}$	$\bar{2}\bar{1}\bar{0}\bar{3}$	$\bar{3}\bar{1}\bar{2}\bar{0}$	$\bar{0}\bar{2}\bar{1}\bar{3}$	$\bar{0}\bar{3}\bar{2}\bar{1}$
$\bar{0}\bar{1}\bar{3}\bar{2}$	$\bar{1}\bar{2}\bar{0}\bar{3}$	$\bar{2}\bar{1}\bar{3}\bar{0}$	$\bar{2}\bar{0}\bar{1}\bar{3}$	$\bar{1}\bar{3}\bar{2}\bar{0}$	$\bar{3}\bar{0}\bar{2}\bar{1}$
$\bar{0}\bar{2}\bar{3}\bar{1}$	$\bar{0}\bar{3}\bar{1}\bar{2}$	$\bar{3}\bar{1}\bar{0}\bar{2}$	$\bar{1}\bar{0}\bar{3}\bar{2}$	$\bar{2}\bar{3}\bar{0}\bar{1}$	$\bar{3}\bar{2}\bar{1}\bar{0}$
$\bar{1}\bar{2}\bar{3}\bar{0}$	$\bar{3}\bar{0}\bar{1}\bar{2}$	$\bar{2}\bar{0}\bar{3}\bar{1}$	$\bar{1}\bar{3}\bar{0}\bar{2}$	$\bar{2}\bar{3}\bar{1}\bar{0}$	$\bar{3}\bar{2}\bar{0}\bar{1}$

The orbits of $N^{(\bar{0}\bar{1}\bar{2}\bar{3})}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}\bar{2}\bar{3})}$ and find to which double coset it belongs.

$\bar{0}\bar{1}\bar{2}\bar{3}3 = \bar{0}\bar{1}\bar{2} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_3 = Nt_0^{-1}t_1^{-1}t_2^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}N = [\bar{0}\bar{1}\bar{2}]$.
$\bar{0}\bar{1}\bar{2}\bar{3}\bar{3} = \bar{0}\bar{1}\bar{2}3 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_3^{-1} = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3 \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3N = [\bar{0}\bar{1}\bar{2}3]$.
$\bar{0}\bar{1}\bar{2}\bar{3}2 = 01\bar{2}\bar{0} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2 = Nt_0t_1t_2^{-1}t_0^{-1} \in Nt_0t_1t_2^{-1}t_0^{-1}N = [01\bar{2}\bar{0}]$.
$\bar{0}\bar{1}\bar{2}\bar{3}\bar{2} = 01\bar{2}\bar{0}3 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_2^{-1} = Nt_0t_1t_2^{-1}t_0^{-1}t_3 \in Nt_0t_1t_2^{-1}t_0^{-1}t_3N = [01\bar{2}\bar{0}3]$.
$\bar{0}\bar{1}\bar{2}\bar{3}1 = \bar{0}1201 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1 = Nt_0^{-1}t_1t_2t_0t_1 \in Nt_0^{-1}t_1t_2t_0t_1N = [\bar{0}1201]$.
$\bar{0}\bar{1}\bar{2}\bar{3}0 = \bar{0}\bar{1}\bar{2}\bar{0}3 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0 = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3 \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N = [\bar{0}\bar{1}\bar{2}\bar{0}3]$.

In addition to the last four statements, MAGMA shows that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}$ are elements within the new double cosets $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}]$ and $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}]$, respectively.

All of the double cosets with representative words of length four have been considered. We now investigate each double cosets with representative words of length five. Namely, $[01023]$, $[010\bar{2}\bar{3}]$, $[01201]$, $[01203]$, $[0120\bar{1}]$, $[0120\bar{3}]$, $[01231]$, $[01232]$, $[01230]$, $[0123\bar{1}]$, $[01\bar{2}\bar{0}3]$, $[0\bar{1}\bar{2}\bar{0}1]$, $[0\bar{1}\bar{2}\bar{0}3]$, $[0\bar{1}\bar{2}3\bar{1}]$, $[\bar{0}1201]$, $[\bar{0}1203]$, $[\bar{0}120\bar{1}]$, $[\bar{0}120\bar{3}]$, $[\bar{0}1231]$, $[\bar{0}\bar{1}\bar{2}\bar{0}3]$, $[\bar{0}\bar{1}\bar{2}0\bar{1}]$, $[\bar{0}\bar{1}\bar{2}0\bar{3}]$, $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}]$, and $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}]$.

Consider $[01023]$.
MAGMA shows that the coset stabilizing group $N^{(01023)} = \{e, (01)\}$, implying that there exists twelve distinct single cosets within $[01023]$. We list them below along with their equal names.

$$01023 \sim 10123$$
$$21203 \sim 12103$$
$$31320 \sim 13120$$
$$02013 \sim 20213$$
$$03021 \sim 30321$$
$$01032 \sim 10132$$
$$02031 \sim 20231$$
$$03012 \sim 30312$$
$$31302 \sim 13102$$
$$21230 \sim 12130$$
$$23201 \sim 32301$$
$$32310 \sim 23210$$

The orbits of $N^{(01023)}$ on $\{0,1,2,3\}$ are $\{0,1\}$, $\{\bar{0},\bar{1}\}$, $\{2\}$, $\{3\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0t_1t_0t_2t_3t_i$ for one t_i from each orbit of $N^{(01023)}$ and find to which double coset it belongs.

$010233̄ = 0102 \Rightarrow Nt_0t_1t_0t_2t_3t_3^{-1} = Nt_0t_1t_0t_2 \in Nt_0t_1t_0t_2N = [0102]$.
$010233 = \bar{0}12\bar{3} \Rightarrow Nt_0t_1t_0t_2t_3t_3 = Nt_0^{-1}t_1t_2t_3^{-1} \in Nt_0^{-1}t_1t_2t_3^{-1}N = [\bar{0}12\bar{3}]$.
$010231 = \bar{0}1231 \Rightarrow Nt_0t_1t_0t_2t_3t_1 = Nt_0^{-1}t_1t_2t_3t_1 \in Nt_0^{-1}t_1t_2t_3t_1N = [\bar{0}1231]$.
$01023\bar{1} = \bar{0}\bar{1}\bar{2}\bar{0}\bar{1} \Rightarrow Nt_0t_1t_0t_2t_3t_1^{-1} = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}]$.
$01023\bar{2} = 010\bar{2}\bar{3} \Rightarrow Nt_0t_1t_0t_2t_3t_2^{-1} = Nt_0t_1t_0t_2^{-1}t_3^{-1} \in Nt_0t_1t_0t_2^{-1}t_3^{-1}N = [010\bar{2}\bar{3}]$.

In addition to the last four statements, MAGMA shows that $Nt_0t_1t_0t_2t_3t_2$ is an element of the new double coset [010232].

Consider $[010\bar{2}\bar{3}]$.
MAGMA shows that the coset stabilizing group $N^{(010\bar{2}\bar{3})} = \{e, (01)\}$, implying that there exists twelve distinct single cosets within $[010\bar{2}\bar{3}]$. We list them below along with their equal names.

$$010\bar{2}\bar{3} \sim 101\bar{2}\bar{3}$$
$$212\bar{0}\bar{3} \sim 121\bar{0}\bar{3}$$
$$313\bar{2}\bar{0} \sim 131\bar{2}\bar{0}$$
$$020\bar{1}\bar{3} \sim 202\bar{1}\bar{3}$$
$$030\bar{2}\bar{1} \sim 303\bar{2}\bar{1}$$
$$010\bar{3}\bar{2} \sim 101\bar{3}\bar{2}$$
$$020\bar{3}\bar{1} \sim 202\bar{3}\bar{1}$$
$$030\bar{1}\bar{2} \sim 303\bar{1}\bar{2}$$
$$313\bar{0}\bar{2} \sim 131\bar{0}\bar{2}$$
$$212\bar{3}\bar{0} \sim 121\bar{3}\bar{0}$$
$$232\bar{0}\bar{1} \sim 323\bar{0}\bar{1}$$
$$323\bar{1}\bar{0} \sim 232\bar{1}\bar{0}$$

The orbits of $N^{(010\bar{2}\bar{3})}$ on $\{0,1,2,3\}$ are $\{0,1\}$, $\{\bar{0},\bar{1}\}$, $\{2\}$, $\{3\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0t_1t_0t_2^{-1}t_3^{-1}t_i$ for one t_i from each orbit of $N^{(010\bar{2}\bar{3})}$ and find to which double coset it belongs.

$010\bar{2}\bar{3}3 = 010\bar{2} \Rightarrow Nt_0t_1t_0t_2^{-1}t_3^{-1}t_3 = Nt_0t_1t_0t_2^{-1} \in Nt_0t_1t_0t_2^{-1}N = [010\bar{2}]$.
$010\bar{2}\bar{3}\bar{3} = 0\bar{1}\bar{2}3 \Rightarrow Nt_0t_1t_0t_2^{-1}t_3^{-1}t_3^{-1} = Nt_0t_1^{-1}t_2^{-1}t_3 \in Nt_0t_1^{-1}t_2^{-1}t_3N = [0\bar{1}\bar{2}3]$.
$010\bar{2}\bar{3}2 = 01023 \Rightarrow Nt_0t_1t_0t_2^{-1}t_3^{-1}t_2 = Nt_0t_1t_0t_2t_3 \in Nt_0t_1t_0t_2t_3N = [01023]$.
$010\bar{2}\bar{3}1 = 01201 \Rightarrow Nt_0t_1t_0t_2^{-1}t_3^{-1}t_1 = Nt_0t_1t_2t_0t_1 \in Nt_0t_1t_2t_0t_1N = [01201]$.
$010\bar{2}\bar{3}\bar{1} = 0\bar{1}\bar{2}\bar{3}\bar{1} \Rightarrow Nt_0t_1t_0t_2^{-1}t_3^{-1}t_1^{-1} = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1} \in Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N = [0\bar{1}\bar{2}\bar{3}\bar{1}]$.
$010\bar{2}\bar{3}\bar{2} = 010232 \Rightarrow Nt_0t_1t_0t_2^{-1}t_3^{-1}t_2^{-1} = Nt_0t_1t_0t_2t_3t_2 \in Nt_0t_1t_0t_2t_3t_2N = [010232]$.

MAGMA has shown the previous five statements.

Consider [01201].
The coset stabilizing group $N^{(01201)} = \{e\}$, so there are twenty-four distinct single cosets within [01201]. Below is the list of these single cosets.

01201	10210	21021	31231
02102	03203	01301	12012
20120	13213	30230	02302
03103	31031	21321	10310
23023	32132	12312	30130
20320	13013	23123	32032

The orbits of $N^{(01201)}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0t_1t_2t_0t_1t_i$ for one t_i from each orbit of $N^{(01201)}$ and find to which double coset it belongs.

$01201\bar{1} = 0120 \Rightarrow Nt_0t_1t_2t_0t_1t_1^{-1} = Nt_0t_1t_2t_0 \in Nt_0t_1t_2t_0N = [0120]$.
$012011 = 0120\bar{1} \Rightarrow Nt_0t_1t_2t_0t_1t_1^{-1} = Nt_0t_1t_2t_0t_1^{-1} \in Nt_0t_1t_2t_0t_1^{-1}N = [0120\bar{1}]$.
$012010 = 0\bar{1}\bar{2}\bar{3}\bar{1} \Rightarrow Nt_0t_1t_2t_0t_1t_0 = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1} \in Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N = [0\bar{1}\bar{2}\bar{3}\bar{1}]$.
$01201\bar{2} = \bar{0}\bar{1}\bar{2}\bar{0}\bar{3} \Rightarrow Nt_0t_1t_2t_0t_1t_2^{-1} = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}]$.
$01201\bar{3} = \bar{0}\bar{1}\bar{2}\bar{0}\bar{1} \Rightarrow Nt_0t_1t_2t_0t_1t_3^{-1} = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}]$.
$01201\bar{0} = 010\bar{2}\bar{3} \Rightarrow Nt_0t_1t_2t_0t_1t_0^{-1} = Nt_0t_1t_0t_2^{-1}t_3^{-1} \in Nt_0t_1t_0t_2^{-1}t_3^{-1}N = [010\bar{2}\bar{3}]$.

In addition to the last four statements, MAGMA shows that $Nt_0t_1t_2t_0t_1t_2$ and $Nt_0t_1t_2t_0t_1t_3$ are elements within the new double cosets [012012] and [012013], respectively.

Consider [01203].
MAGMA shows that the coset stabilizing group $N^{(01203)} = \{e, (03)(12)\}$, implying that there exists twelve distinct single cosets within [01203]. We list them below along with their equal names.

$01203 \sim 32130$
$10213 \sim 32031$
$21023 \sim 30132$
$31230 \sim 02103$
$03201 \sim 12310$
$01302 \sim 23120$

$$12013 \sim 30231$$
$$20123 \sim 31032$$
$$13210 \sim 02301$$
$$03102 \sim 21320$$
$$23021 \sim 10312$$
$$20321 \sim 13012$$

The orbits of $N^{(01203)}$ on $\{0,1,2,3\}$ are $\{0,3\}$, $\{\bar{0},\bar{3}\}$, $\{1,2\}$, and $\{\bar{1},\bar{2}\}$. We consider $Nt_0t_1t_2t_0t_3t_i$ for one t_i from each orbit of $N^{(01203)}$ and find to which double coset it belongs.

$012033\bar{3} = 0120 \Rightarrow Nt_0t_1t_2t_0t_3t_3^{-1} = Nt_0t_1t_2t_0 \in Nt_0t_1t_2t_0N = [0120]$.
$012033 = 0120\bar{3} \Rightarrow Nt_0t_1t_2t_0t_3t_3 = Nt_0t_1t_2t_0t_3^{-1} \in Nt_0t_1t_2t_0t_3^{-1}N = [0120\bar{3}]$.
$01203\bar{1} = \bar{0}\bar{1}\bar{2}\bar{0}\bar{1} \Rightarrow Nt_0t_1t_2t_0t_3t_1^{-1} = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}]$.

In addition to the last statement, MAGMA shows that $Nt_0t_1t_2t_0t_3t_1$ is an element of the new double coset [012031].

Consider $[0120\bar{1}]$.
MAGMA shows that the coset stabilizing group $N^{(0120\bar{1})} = \{e, (13)\}$, implying that there exists twelve distinct single cosets within $[0120\bar{1}]$. We list them below along with their equal names.

$$0120\bar{1} \sim 0320\bar{3}$$
$$1021\bar{0} \sim 1321\bar{3}$$
$$2102\bar{1} \sim 2302\bar{3}$$
$$3123\bar{1} \sim 3023\bar{0}$$
$$0210\bar{2} \sim 0310\bar{3}$$
$$0130\bar{1} \sim 0230\bar{2}$$
$$1201\bar{2} \sim 1301\bar{3}$$
$$2012\bar{0} \sim 2312\bar{3}$$

$$3103\bar{1} \sim 3203\bar{2}$$
$$2132\bar{1} \sim 2032\bar{0}$$
$$1031\bar{0} \sim 1231\bar{2}$$
$$3213\bar{2} \sim 3013\bar{0}$$

The orbits of $N^{(0120\bar{1})}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1,3\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1},\bar{3}\}$, and $\{\bar{2}\}$. We consider $Nt_0t_1t_2t_0t_1^{-1}t_i$ for one t_i from each orbit of $N^{(0120\bar{1})}$ and find to which double coset it belongs.

$0120\bar{1}1 = 0120 \Rightarrow Nt_0t_1t_2t_0t_1^{-1}t_1 = Nt_0t_1t_2t_0 \in Nt_0t_1t_2t_0N = [0120]$.
$0120\bar{1}\bar{1} = 01201 \Rightarrow Nt_0t_1t_2t_0t_1^{-1}t_1^{-1} = Nt_0t_1t_2t_0t_1 \in Nt_0t_1t_2t_0t_1N = [01201]$.
$0120\bar{1}2 = 01\bar{2}\bar{0}3 \Rightarrow Nt_0t_1t_2t_0t_1^{-1}t_2 = Nt_0t_1t_2^{-1}t_0^{-1}t_3 \in Nt_0t_1t_2^{-1}t_0^{-1}t_3N = [01\bar{2}\bar{0}3]$.
$0120\bar{1}0 = 0\bar{1}\bar{2}\bar{0} \Rightarrow Nt_0t_1t_2t_0t_1^{-1}t_0 = Nt_0t_1^{-1}t_2^{-1}t_0^{-1} \in Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N = [0\bar{1}\bar{2}\bar{0}]$.
$0120\bar{1}\bar{0} = 0\bar{1}\bar{2}\bar{3}1 \Rightarrow Nt_0t_1t_2t_0t_1^{-1}t_0^{-1} = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1 \in Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1N = [0\bar{1}\bar{2}\bar{3}1]$.

In addition to the last three statements, MAGMA shows that $Nt_0t_1t_2t_0t_1^{-1}t_2^{-1}$ is an element of the new double coset $[0120\bar{1}\bar{2}]$.

Consider $[0120\bar{3}]$.
MAGMA proves that the coset stabilizing group $N^{(0120\bar{3})} = \{e, (02)(13)\}$. The twelve distinct single cosets within $[0120\bar{3}]$ are listed below along with their equal names.

$$0120\bar{3} \sim 2302\bar{1}$$
$$1021\bar{3} \sim 2312\bar{0}$$
$$2102\bar{3} \sim 0320\bar{1}$$
$$3123\bar{0} \sim 2032\bar{1}$$
$$0210\bar{3} \sim 1301\bar{2}$$
$$0130\bar{2} \sim 3203\bar{1}$$
$$1201\bar{3} \sim 0310\bar{2}$$
$$2012\bar{3} \sim 1321\bar{0}$$
$$3023\bar{1} \sim 2132\bar{0}$$

$$0230\bar{1} \sim 3103\bar{2}$$
$$1031\bar{2} \sim 3213\bar{0}$$
$$1231\bar{0} \sim 3013\bar{2}$$

The orbits of $N^{(0120\bar{3})}$ on $\{0,1,2,3\}$ are $\{0,2\}$, $\{1,3\}$, $\{\bar{0},\bar{2}\}$, and $\{\bar{1},\bar{3}\}$. We consider $Nt_0t_1t_2t_0t_3^{-1}t_i$ for one t_i from each orbit of $N^{(0120\bar{3})}$ and find to which double coset it belongs.

$0120\bar{3}3 = 0120 \Rightarrow Nt_0t_1t_2t_0t_3^{-1}t_3 = Nt_0t_1t_2t_0 \in Nt_0t_1t_2t_0N = [0120]$.
$0120\bar{3}\bar{3} = 01203 \Rightarrow Nt_0t_1t_2t_0t_3^{-1}t_3^{-1} = Nt_0t_1t_2t_0t_3 \in Nt_0t_1t_2t_0t_3N = [01203]$.
$0120\bar{3}2 = 0123 \Rightarrow Nt_0t_1t_2t_0t_3^{-1}t_2 = Nt_0t_1t_2t_3 \in Nt_0t_1t_2t_3N = [0123]$.
$0120\bar{3}\bar{2} = 01230 \Rightarrow Nt_0t_1t_2t_0t_3^{-1}t_2^{-1} = Nt_0t_1t_2t_3t_0 \in Nt_0t_1t_2t_3t_0N = [01230]$.

MAGMA has shown us the previous two statements. This double coset is now closed under right multiplication. That is, $Nt_0t_1t_2t_0t_3^{-1}t_i$ do not map to any double coset with representative word of length six.

Consider [01231].
MAGMA proves that the coset stabilizing group $N^{(01231)} = \{e, (13)\}$. The twelve distinct single cosets within [01231] are listed below along with their equal names.

$$01231 \sim 03213$$
$$10230 \sim 13203$$
$$21031 \sim 23013$$
$$31201 \sim 30210$$
$$02132 \sim 03123$$
$$01321 \sim 02312$$
$$12032 \sim 13023$$
$$20130 \sim 23103$$
$$31021 \sim 32012$$
$$21301 \sim 20310$$

$$10320 \sim 12302$$
$$32102 \sim 30120$$

The orbits of $N^{(01231)}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1,3\}$, $\{2\}$, $\{\bar{0}\}$, $\{\bar{1},\bar{3}\}$, and $\{\bar{2}\}$. We consider $Nt_0t_1t_2t_3t_1t_i$ for one t_i from each orbit of $N^{(01231)}$ and find to which double coset it belongs.

$01231\bar{1} = 0123 \Rightarrow Nt_0t_1t_2t_3t_1t_1^{-1} = Nt_0t_1t_2t_3 \in Nt_0t_1t_2t_3N = [01231]$.
$012311 = 0123\bar{1} \Rightarrow Nt_0t_1t_2t_3t_1t_1 = Nt_0t_1t_2t_3t_1^{-1} \in Nt_0t_1t_2t_3t_1^{-1}N = [0123\bar{1}]$.

MAGMA shows that $Nt_0t_1t_2t_3t_1t_2$, $Nt_0t_1t_2t_3t_1t_0$, $Nt_0t_1t_2t_3t_1t_2^{-1}$, and $Nt_0t_1t_2t_3t_1t_0^{-1}$ are elements of the new double cosets [012312], [012310], [01231$\bar{2}$], and [01231$\bar{0}$], respectively.

Consider [01232].
MAGMA proves that the coset stabilizing group $N^{(01232)} = \{e, (23)\}$. The twelve distinct single cosets within [01232] are listed below along with their equal names.

$$01232 \sim 01323$$
$$10232 \sim 10323$$
$$21030 \sim 21303$$
$$31202 \sim 31020$$
$$02131 \sim 02313$$
$$03212 \sim 03121$$
$$12030 \sim 12303$$
$$20131 \sim 20313$$
$$13202 \sim 13020$$
$$30212 \sim 30121$$
$$23010 \sim 23101$$
$$32101 \sim 32010$$

The orbits of $N^{(01232)}$ on $\{0,1,2,3\}$ are $\{0\}$, $\{1\}$, $\{2,3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, and $\{\bar{2},\bar{3}\}$. We consider

$Nt_0t_1t_2t_3t_2t_i$ for one t_i from each orbit of $N^{(01232)}$ and find to which double coset it belongs.

$01232\bar{2} = 0123 \Rightarrow Nt_0t_1t_2t_3t_2t_2^{-1} = Nt_0t_1t_2t_3 \in Nt_0t_1t_2t_3N = [0123]$.
$012321 = \bar{0}1201 \Rightarrow Nt_0t_1t_2t_3t_2t_1 = Nt_0^{-1}t_1t_2t_0t_1 \in Nt_0^{-1}t_1t_2t_0t_1N = [\bar{0}1201]$.
$012320 = \bar{0}1203 \Rightarrow Nt_0t_1t_2t_3t_2t_0 = Nt_0^{-1}t_1t_2t_0t_3 \in Nt_0^{-1}t_1t_2t_0t_3N = [\bar{0}1203]$.
$012322 = 01\bar{2}\bar{0} \Rightarrow Nt_0t_1t_2t_3t_2t_2 = Nt_0t_1t_2^{-1}t_0^{-1} \in Nt_0t_1t_2^{-1}t_0^{-1}N = [01\bar{2}\bar{0}]$.
$01232\bar{1} = 01231\bar{2} \Rightarrow Nt_0t_1t_2t_3t_2t_1^{-1} = Nt_0t_1t_2t_3t_1t_2^{-1} \in Nt_0t_1t_2t_3t_1t_2^{-1}N = [01231\bar{2}]$.

In addition to the previous four statements, MAGMA shows that $Nt_0t_1t_2t_3t_2t_0^{-1}$ is an element of the new double coset $[01232\bar{0}]$.

Consider [01230].
MAGMA shows that the coset stabilizing group $N^{(01230)} = \{e, (03)\}$. We list the twelve distinct single cosets within [01230] below along with their equal names.

$01230 \sim 31203$
$10231 \sim 30213$
$21032 \sim 31023$
$02130 \sim 32103$
$03210 \sim 13201$
$01320 \sim 21302$
$12031 \sim 32013$
$20132 \sim 30123$
$02310 \sim 12301$
$10321 \sim 20312$
$23012 \sim 13021$
$23102 \sim 03120$

The orbits of $N^{(01230)}$ on $\{0, 1, 2, 3\}$ are $\{0, 3\}$, $\{1\}$, $\{2\}$, $\{\bar{0}, \bar{3}\}$, $\{\bar{1}\}$, and $\{\bar{2}\}$. We consider $Nt_0t_1t_2t_3t_0t_i$ for one t_i from each orbit of $N^{(01230)}$ and find to which double coset it belongs.

$01230\bar{0} = 0123 \Rightarrow Nt_0t_1t_2t_3t_0t_0^{-1} = Nt_0t_1t_2t_3 \in Nt_0t_1t_2t_3N = [0123]$.
$012303 = 0120\bar{3} \Rightarrow Nt_0t_1t_2t_3t_0t_3 = Nt_0t_1t_2t_0t_3^{-1} \in Nt_0t_1t_2t_0t_3^{-1}N = [0120\bar{3}]$.
$012302 = 012031 \Rightarrow Nt_0t_1t_2t_3t_0t_2 = Nt_0t_1t_2t_0t_3t_1 \in Nt_0t_1t_2t_0t_3t_1N = [012031]$.
$01230\bar{1} = 01231\bar{0} \Rightarrow Nt_0t_1t_2t_3t_0t_1^{-1} = Nt_0t_1t_2t_3t_1t_0^{-1} \in Nt_0t_1t_2t_3t_1t_0^{-1}N = [01231\bar{0}]$.
$01230\bar{2} = 01232\bar{0} \Rightarrow Nt_0t_1t_2t_3t_0t_2^{-1} = Nt_0t_1t_2t_3t_2t_0^{-1} \in Nt_0t_1t_2t_3t_2t_0^{-1}N = [01232\bar{0}]$.

In addition to the previous four statements, MAGMA shows that $Nt_0t_1t_2t_3t_0t_1$ is an element of the new double coset [012301].

Consider $[0123\bar{1}]$.
MAGMA shows that the coset stabilizing group $N^{(0123\bar{1})} = \{e, (01)\}$. Below is the list of the twelve distinct single cosets within $[0123\bar{1}]$, along with their equal names.

$0123\bar{1} \sim 1023\bar{0}$
$2103\bar{1} \sim 1203\bar{2}$
$3120\bar{1} \sim 1320\bar{3}$
$0213\bar{2} \sim 2013\bar{0}$
$0321\bar{3} \sim 3021\bar{0}$
$0132\bar{1} \sim 1032\bar{0}$
$0231\bar{2} \sim 2031\bar{0}$
$0312\bar{3} \sim 3012\bar{0}$
$3102\bar{1} \sim 1302\bar{3}$
$2301\bar{3} \sim 3201\bar{2}$
$3210\bar{2} \sim 2310\bar{3}$
$1230\bar{2} \sim 2130\bar{1}$

The orbits of $N^{(0123\bar{1})}$ on $\{0,1,2,3\}$ are $\{0,1\}$, $\{2\}$, $\{3\}$, $\{\bar{0},\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0t_1t_2t_3t_1^{-1}t_i$ for one t_i from each orbit of $N^{(0123\bar{1})}$ and find to which double coset it belongs.

$0123\bar{1}1 = 0123 \Rightarrow Nt_0t_1t_2t_3t_1^{-1}t_1 = Nt_0t_1t_2t_3 \in Nt_0t_1t_2t_3N = [0123]$.
$0123\bar{1}\bar{1} = 01231 \Rightarrow Nt_0t_1t_2t_3t_1^{-1}t_1^{-1} = Nt_0t_1t_2t_3t_1 \in Nt_0t_1t_2t_3t_1N = [01231]$.
$0123\bar{1}2 = 01\bar{2}\bar{0}3 \Rightarrow Nt_0t_1t_2t_3t_1^{-1}t_2 = Nt_0t_1t_2^{-1}t_0^{-1}t_3 \in Nt_0t_1t_2^{-1}t_0^{-1}t_3N = [01\bar{2}\bar{0}3]$.
$0123\bar{1}3 = 0\bar{1}\bar{2}\bar{0} \Rightarrow Nt_0t_1t_2t_3t_1^{-1}t_3 = Nt_0t_1^{-1}t_2^{-1}t_0^{-1} \in Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N = [0\bar{1}\bar{2}\bar{0}]$.
$0123\bar{1}\bar{3} = 0\bar{1}\bar{2}\bar{0}1 \Rightarrow Nt_0t_1t_2t_3t_1^{-1}t_3^{-1} = Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1 \in Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1N = [0\bar{1}\bar{2}\bar{0}1]$.

In addition to previous three statements, MAGMA shows that $Nt_0t_1t_2t_3t_1^{-1}t_2^{-1}$ is an element of the new double coset $[0123\bar{1}\bar{2}]$.

Consider $[01\bar{2}\bar{0}3]$.
MAGMA shows that the coset stabilizing group $N^{(01\bar{2}\bar{0}3)} = \{e, (03)\}$. We can list the twelve distinct single cosets within $[01\bar{2}\bar{0}3]$ along with their equal names.

$$
\begin{gathered}
01\bar{2}\bar{0}3 \sim 31\bar{2}\bar{3}0 \\
10\bar{2}\bar{1}3 \sim 30\bar{2}\bar{3}1 \\
21\bar{0}\bar{2}3 \sim 31\bar{0}\bar{3}2 \\
02\bar{1}\bar{0}3 \sim 32\bar{1}\bar{3}0 \\
03\bar{2}\bar{0}1 \sim 13\bar{2}\bar{1}0 \\
01\bar{3}\bar{0}2 \sim 21\bar{3}\bar{2}0 \\
12\bar{0}\bar{1}3 \sim 32\bar{0}\bar{3}1 \\
20\bar{1}\bar{2}3 \sim 30\bar{1}\bar{3}2 \\
02\bar{3}\bar{0}1 \sim 12\bar{3}\bar{1}0 \\
03\bar{1}\bar{0}2 \sim 23\bar{1}\bar{2}0 \\
10\bar{3}\bar{1}2 \sim 20\bar{3}\bar{2}1 \\
23\bar{0}\bar{2}1 \sim 13\bar{0}\bar{1}2
\end{gathered}
$$

The orbits of $N^{(01\bar{2}\bar{0}3)}$ on $\{0,1,2,3\}$ are $\{0,3\}$, $\{1\}$, $\{2\}$, $\{\bar{0},\bar{3}\}$, $\{\bar{1}\}$, and $\{\bar{2}\}$. We consider $Nt_0t_1t_2^{-1}t_0^{-1}t_3t_i$ for one t_i from each orbit of $N^{(01\bar{2}\bar{0}3)}$ and find to which double coset it belongs.

$$0\bar{1}\bar{2}\bar{0}3\bar{3} = 01\bar{2}\bar{0} \Rightarrow Nt_0t_1t_2^{-1}t_0^{-1}t_3t_3^{-1} = Nt_0t_1t_2^{-1}t_0^{-1} \in Nt_0t_1t_2^{-1}t_0^{-1}N = [01\bar{2}\bar{0}].$$
$$01\bar{2}\bar{0}33 = \bar{0}\bar{1}\bar{2}\bar{3} \Rightarrow Nt_0t_1t_2^{-1}t_0^{-1}t_3t_3 = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{3}].$$
$$01\bar{2}\bar{0}3\bar{1} = 0123\bar{1} \Rightarrow Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1^{-1} = Nt_0t_1t_2t_3t_1^{-1} \in Nt_0t_1t_2t_3t_1^{-1}N = [0123\bar{1}].$$
$$01\bar{2}\bar{0}3\bar{2} = 0120\bar{1} \Rightarrow Nt_0t_1t_2^{-1}t_0^{-1}t_3t_2^{-1} = Nt_0t_1t_2t_0t_1^{-1} \in Nt_0t_1t_2t_0t_1^{-1}N = [0120\bar{1}].$$
$$01\bar{2}\bar{0}32 = 0120\bar{1}\bar{2} \Rightarrow Nt_0t_1t_2^{-1}t_0^{-1}t_3t_2 = Nt_0t_1t_2t_0t_1^{-1}t_2^{-1} \in Nt_0t_1t_2t_0t_1^{-1}t_2^{-1}N = [0120\bar{1}\bar{2}].$$
$$01\bar{2}\bar{0}31 = 0123\bar{1}\bar{2} \Rightarrow Nt_0t_1t_2^{-1}t_0^{-1}t_3t_1 = Nt_0t_1t_2t_3t_1^{-1}t_2^{-1} \in Nt_0t_1t_2t_3t_1^{-1}t_2^{-1}N = [0123\bar{1}\bar{2}].$$

MAGMA shows the previous five statements.

Consider $[0\bar{1}\bar{2}\bar{0}1]$.
MAGMA proves that the coset stabilizing group $N^{(0\bar{1}\bar{2}\bar{0}1)} = \{e, (02), (13), (01)(23), (02)(13), (03)(12), (0123), (0321)\}$. There exists three distinct single cosets within $[0\bar{1}\bar{2}\bar{0}1]$, which we list below, along with their equal names.

$$0\bar{1}\bar{2}\bar{0}1 \sim 1\bar{2}\bar{3}\bar{1}2 \sim 0\bar{3}\bar{2}\bar{0}3 \sim 3\bar{0}\bar{1}\bar{3}0 \sim 2\bar{3}\bar{0}\bar{2}3 \sim 3\bar{2}\bar{1}\bar{3}2 \sim 1\bar{0}\bar{3}\bar{1}0 \sim 2\bar{1}\bar{0}\bar{2}1$$
$$1\bar{0}\bar{2}\bar{1}0 \sim 0\bar{2}\bar{3}\bar{0}2 \sim 1\bar{3}\bar{2}\bar{1}3 \sim 3\bar{1}\bar{0}\bar{3}1 \sim 2\bar{3}\bar{1}\bar{2}3 \sim 3\bar{2}\bar{0}\bar{3}2 \sim 0\bar{1}\bar{3}\bar{0}1 \sim 2\bar{0}\bar{1}\bar{2}0$$
$$3\bar{1}\bar{2}\bar{3}1 \sim 1\bar{2}\bar{0}\bar{1}2 \sim 3\bar{0}\bar{2}\bar{3}0 \sim 0\bar{3}\bar{1}\bar{0}3 \sim 2\bar{0}\bar{3}\bar{2}0 \sim 0\bar{2}\bar{1}\bar{0}2 \sim 1\bar{3}\bar{0}\bar{1}3 \sim 2\bar{1}\bar{3}\bar{2}1$$

The orbits of $N^{(0\bar{1}\bar{2}\bar{0}1)}$ on $\{0,1,2,3\}$ are $\{0,1,2,3\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$. We consider $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_i$ for one t_i from each orbit of $N^{(0\bar{1}\bar{2}\bar{0}1)}$ and find to which double coset it belongs.

$$0\bar{1}\bar{2}\bar{0}1\bar{1} = 0\bar{1}\bar{2}\bar{0} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_1^{-1} = Nt_0t_1^{-1}t_2^{-1}t_0^{-1} \in Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N = [0\bar{1}\bar{2}\bar{0}].$$
$$0\bar{1}\bar{2}\bar{0}11 = 0123\bar{1} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_1 = Nt_0t_1t_2t_3t_1^{-1} \in Nt_0t_1t_2t_3t_1^{-1}N = [0123\bar{1}].$$

MAGMA has shown the last statement above. This double coset is now closed under right multiplication. That is, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_1t_i$ do not move forward to any double coset with representative word of length six.

Consider $[0\bar{1}\bar{2}\bar{0}3]$.
MAGMA shows that the coset stabilizing group $N^{(0\bar{1}\bar{2}\bar{0}3)} = \{e, (02), (13), (02)(13)\}$. There are six distinct single cosets within $[0\bar{1}\bar{2}\bar{0}3]$ that we list below, along with their equal

names.

$$0\bar{1}\bar{2}\bar{0}3 \sim 0\bar{3}\bar{2}\bar{0}1 \sim 2\bar{1}\bar{0}\bar{2}3 \sim 2\bar{3}\bar{0}\bar{2}1$$
$$1\bar{0}\bar{2}\bar{1}3 \sim 1\bar{3}\bar{2}\bar{1}0 \sim 2\bar{0}\bar{1}\bar{2}3 \sim 2\bar{3}\bar{1}\bar{2}0$$
$$3\bar{1}\bar{2}\bar{3}0 \sim 3\bar{0}\bar{2}\bar{3}1 \sim 2\bar{1}\bar{3}\bar{2}0 \sim 2\bar{0}\bar{3}\bar{2}1$$
$$0\bar{2}\bar{1}\bar{0}3 \sim 0\bar{3}\bar{1}\bar{0}2 \sim 1\bar{2}\bar{0}\bar{1}3 \sim 1\bar{3}\bar{0}\bar{1}2$$
$$0\bar{1}\bar{3}\bar{0}2 \sim 0\bar{2}\bar{3}\bar{0}1 \sim 3\bar{1}\bar{0}\bar{3}2 \sim 3\bar{2}\bar{0}\bar{3}1$$
$$1\bar{0}\bar{3}\bar{1}2 \sim 1\bar{2}\bar{3}\bar{1}0 \sim 3\bar{0}\bar{1}\bar{3}2 \sim 3\bar{2}\bar{1}\bar{3}0$$

The cosets of $N^{(0\bar{1}\bar{2}\bar{0}3)}$ on $\{0,1,2,3\}$ are $\{0,2\}$, $\{1,3\}$, $\{\bar{0},\bar{2}\}$, and $\{\bar{1},\bar{3}\}$. We consider $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_i$ for one t_i from each orbit of $N^{(0\bar{1}\bar{2}\bar{0}3)}$ and find to which double coset it belongs.

$0\bar{1}\bar{2}\bar{0}3\bar{3} = 0\bar{1}\bar{2}\bar{0} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_3^{-1} = Nt_0t_1^{-1}t_2^{-1}t_0^{-1} \in Nt_0t_1^{-1}t_2^{-1}t_0^{-1}N = [0\bar{1}\bar{2}\bar{0}]$.
$0\bar{1}\bar{2}\bar{0}31 = 0120\bar{1} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_1 = Nt_0t_1t_2t_0t_1^{-1} \in Nt_0t_1t_2t_0t_1^{-1}N = [0120\bar{1}]$.
$0\bar{1}\bar{2}\bar{0}32 = 0\bar{1}\bar{2}\bar{3}\bar{1} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_2 = Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1} \in Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N = [0\bar{1}\bar{2}\bar{3}\bar{1}]$.
$0\bar{1}\bar{2}\bar{0}3\bar{2} = 0\bar{1}\bar{2}\bar{3} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_2^{-1} = Nt_0t_1^{-1}t_2^{-1}t_3^{-1} \in Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N = [0\bar{1}\bar{2}\bar{3}]$.

MAGMA shows the previous three statements. This double coset is now closed under right multiplication. That is, $Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3t_i$ do not move forward to any double coset with representative word of length six.

Consider $[0\bar{1}\bar{2}\bar{3}\bar{1}]$.
MAGMA shows that the coset stabilizing group $N^{(0\bar{1}\bar{2}\bar{3}\bar{1})} = \{e, (23)\}$. The twelve distinct single coset within $[0\bar{1}\bar{2}\bar{3}\bar{1}]$ are listed below along with their equal names.

$$0\bar{1}\bar{2}\bar{3}\bar{1} \sim 0\bar{1}\bar{3}\bar{2}\bar{1}$$
$$1\bar{0}\bar{2}\bar{3}\bar{0} \sim 1\bar{0}\bar{3}\bar{2}\bar{0}$$
$$2\bar{1}\bar{0}\bar{3}\bar{1} \sim 2\bar{1}\bar{3}\bar{0}\bar{1}$$

$$3\bar{1}\bar{2}\bar{0}\bar{1} \sim 3\bar{1}\bar{0}\bar{2}\bar{1}$$
$$0\bar{2}\bar{1}\bar{3}\bar{2} \sim 0\bar{2}\bar{3}\bar{1}\bar{2}$$
$$0\bar{3}\bar{2}\bar{1}\bar{3} \sim 0\bar{3}\bar{1}\bar{2}\bar{3}$$
$$1\bar{2}\bar{0}\bar{3}\bar{2} \sim 1\bar{2}\bar{3}\bar{0}\bar{2}$$
$$2\bar{0}\bar{1}\bar{3}\bar{0} \sim 2\bar{0}\bar{3}\bar{1}\bar{0}$$
$$1\bar{3}\bar{2}\bar{0}\bar{3} \sim 1\bar{3}\bar{0}\bar{2}\bar{3}$$
$$3\bar{0}\bar{2}\bar{1}\bar{0} \sim 3\bar{0}\bar{1}\bar{2}\bar{0}$$
$$2\bar{3}\bar{0}\bar{1}\bar{3} \sim 2\bar{3}\bar{1}\bar{0}\bar{3}$$
$$3\bar{2}\bar{1}\bar{0}\bar{2} \sim 3\bar{2}\bar{0}\bar{1}\bar{2}$$

The orbits of $N^{(0\bar{1}\bar{2}\bar{3}\bar{1})}$ on $\{0,1,2,3\}$ are $\{2,3\}$, $\{\bar{2},\bar{3}\}$, $\{0\}$, $\{1\}$, $\{\bar{0}\}$, and $\{\bar{1}\}$. We consider $Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_i$ for one t_i from each orbit of $N^{(0\bar{1}\bar{2}\bar{3}\bar{1})}$ and find to which double coset it belongs.

$$\begin{aligned} 0\bar{1}\bar{2}\bar{3}\bar{1}1 = 0\bar{1}\bar{2}\bar{3} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_1 &= Nt_0t_1^{-1}t_2^{-1}t_3^{-1} \\ &\in Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N \\ &= [0\bar{1}\bar{2}\bar{3}] \end{aligned}$$

$$\begin{aligned} 0\bar{1}\bar{2}\bar{3}\bar{1}\bar{1} = 0\bar{1}\bar{2}\bar{0}3 \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_1 &= Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3 \\ &\in Nt_0t_1^{-1}t_2^{-1}t_0^{-1}t_3N \\ &= [0\bar{1}\bar{2}\bar{0}3] \end{aligned}$$

$$\begin{aligned} 0\bar{1}\bar{2}\bar{3}\bar{1}2 = 010\bar{2}\bar{3} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2 &= Nt_0t_1t_0t_2^{-1}t_3^{-1} \\ &\in Nt_0t_1t_0t_2^{-1}t_3^{-1}N \\ &= [010\bar{2}\bar{3}] \end{aligned}$$

$$0\bar{1}\bar{2}\bar{3}\bar{1}\bar{2} = 01201 \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1} = Nt_0t_1t_2t_0t_1 \in Nt_0t_1t_2t_0t_1N = [01201]$$

$$0\bar{1}\bar{2}\bar{3}\bar{1}\bar{0} = 012012\bar{0} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1} = Nt_0t_1t_2t_0t_1t_2t_0^{-1} \in Nt_0t_1t_2t_0t_1t_2t_0^{-1}N = [012012\bar{0}]$$

$$0\bar{1}\bar{2}\bar{3}\bar{1}0 = 012012\bar{0}\bar{2} \Rightarrow Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0 = Nt_0t_1t_2t_0t_1t_2t_0^{-1}t_2^{-1} \in Nt_0t_1t_2t_0t_1t_2t_0^{-1}t_2^{-1}N = [012012\bar{0}\bar{2}]$$

MAGMA shows the previous five statements.

Consider $[\bar{0}1201]$.
MAGMA shows that the coset stabilizing group $N^{(\bar{0}1201)} = \{e, (02)\}$. We list the twelve distinct single cosets within $[\bar{0}1201]$ below, along with their equal names.

$\bar{0}1201 \sim \bar{2}1021$
$\bar{1}0210 \sim \bar{2}0120$
$\bar{3}1231 \sim \bar{2}1321$
$\bar{0}2102 \sim \bar{1}2012$
$\bar{0}3203 \sim \bar{2}3023$
$\bar{0}1301 \sim \bar{3}1031$
$\bar{1}3213 \sim \bar{2}3123$
$\bar{3}0230 \sim \bar{2}0320$

$$\bar{0}2302 \sim \bar{3}2032$$
$$\bar{0}3103 \sim \bar{1}3013$$
$$\bar{1}0310 \sim \bar{3}0130$$
$$\bar{3}2132 \sim \bar{1}2312$$

The orbits of $N^{(\bar{0}1201)}$ on $\{0,1,2,3\}$ are $\{0,2\}$, $\{\bar{0},\bar{2}\}$, $\{1\}$, $\{3\}$, $\{\bar{1}\}$, and $\{\bar{3}\}$. We consider $Nt_0^{-1}t_1t_2t_0t_1t_i$ for one t_i from each orbit of $N^{(\bar{0}1201)}$ and find to which double coset it belongs.

$\bar{0}1201\bar{1} = \bar{0}120 \Rightarrow Nt_0^{-1}t_1t_2t_0t_1t_1^{-1} = Nt_0^{-1}t_1t_2t_0 \in Nt_0^{-1}t_1t_2t_0N = [\bar{0}120]$.
$\bar{0}12011 = \bar{0}120\bar{1} \Rightarrow Nt_0^{-1}t_1t_2t_0t_1t_1 = Nt_0^{-1}t_1t_2t_0t_1^{-1} \in Nt_0^{-1}t_1t_2t_0t_1^{-1}N = [\bar{0}120\bar{1}]$.
$\bar{0}12012 = \bar{0}\bar{1}\bar{2}\bar{3}\bar{1} \Rightarrow Nt_0^{-1}t_1t_2t_0t_1t_2 = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}]$.
$\bar{0}1201\bar{2} = \bar{0}\bar{1}\bar{2}\bar{3} \Rightarrow Nt_0^{-1}t_1t_2t_0t_1t_2^{-1} = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{3}]$.
$\bar{0}1201\bar{3} = 01\bar{2}\bar{0}\bar{1} \Rightarrow Nt_0^{-1}t_1t_2t_0t_1t_3^{-1} = Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1} \in Nt_0t_1t_2^{-1}t_0^{-1}t_1^{-1}N = [01\bar{2}\bar{0}\bar{1}]$.
$\bar{0}12013 = 01231\bar{2} \Rightarrow Nt_0^{-1}t_1t_2t_0t_1t_3 = Nt_0t_1t_2t_3t_1t_2^{-1} \in Nt_0t_1t_2t_3t_1t_2^{-1}N = [01231\bar{2}]$.

MAGMA proves the last four statements.

Consider $[\bar{0}1203]$.
MAGMA shows that the coset stabilizing group $N^{(\bar{0}1203)} = \{e,(02)\}$. We list the twelve distinct single cosets within $[\bar{0}1203]$ below, along with their equal names.

$$\bar{0}1203 \sim \bar{2}1023$$
$$\bar{1}0213 \sim \bar{2}0123$$
$$\bar{3}1230 \sim \bar{2}1320$$
$$\bar{0}2103 \sim \bar{1}2013$$
$$\bar{0}3201 \sim \bar{2}3021$$
$$\bar{0}1302 \sim \bar{3}1032$$
$$\bar{1}3210 \sim \bar{2}3120$$
$$\bar{3}0231 \sim \bar{2}0321$$
$$\bar{0}2301 \sim \bar{3}2031$$

$$\bar{0}3102 \sim \bar{1}3012$$
$$\bar{1}0312 \sim \bar{3}0132$$
$$\bar{3}2130 \sim \bar{1}2310$$

The orbits of $N^{(\bar{0}1203)}$ on $\{0,1,2,3\}$ are $\{0,2\}$, $\{\bar{0},\bar{2}\}$, $\{1\}$, $\{3\}$, $\{\bar{1}\}$, and $\{\bar{3}\}$. We consider $Nt_0^{-1}t_1t_2t_0t_3t_i$ for one t_i from each orbit of $N^{(\bar{0}1203)}$ and find to which double coset it belongs.

$\bar{0}1203\bar{3} = \bar{0}120 \Rightarrow Nt_0^{-1}t_1t_2t_0t_3t_3^{-1} = Nt_0^{-1}t_1t_2t_0 \in Nt_0^{-1}t_1t_2t_0N = [\bar{0}120]$.
$\bar{0}12033 = \bar{0}120\bar{3} \Rightarrow Nt_0^{-1}t_1t_2t_0t_3t_3 = Nt_0^{-1}t_1t_2t_0t_3^{-1} \in Nt_0^{-1}t_1t_2t_0t_3^{-1}N = [\bar{0}120\bar{3}]$.
$\bar{0}12032 = \bar{0}\bar{1}\bar{2}\bar{0}\bar{1} \Rightarrow Nt_0^{-1}t_1t_2t_0t_3t_2 = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}]$.
$\bar{0}1203\bar{2} = \bar{0}\bar{1}\bar{2}\bar{0} \Rightarrow Nt_0^{-1}t_1t_2t_0t_3t_2^{-1} = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{0}]$.
$\bar{0}1203\bar{1} = 01232 \Rightarrow Nt_0^{-1}t_1t_2t_0t_3t_1^{-1} = Nt_0t_1t_2t_3t_2 \in Nt_0t_1t_2t_3t_2N = [01232]$.
$\bar{0}12031 = 01232\bar{0} \Rightarrow Nt_0^{-1}t_1t_2t_0t_3t_1 = Nt_0t_1t_2t_3t_2t_0^{-1} \in Nt_0t_1t_2t_3t_2t_0^{-1}N = [01232\bar{0}]$.

MAGMA has shown the previous four statements.

Consider $[\bar{0}120\bar{1}]$.
MAGMA shows that coset stabilizing group
$N^{(\bar{0}120\bar{1})} = \{e,(02),(13),(01)(23),(02)(13),(03)(12),(0123),(0321)\}$. Hence, there are three distinct single cosets within $[\bar{0}120\bar{1}]$. We list them below, along with their equal names.

$$\bar{0}120\bar{1} \sim \bar{3}213\bar{2} \sim \bar{2}102\bar{1} \sim \bar{0}320\bar{3} \sim \bar{1}031\bar{0} \sim \bar{1}231\bar{2} \sim \bar{3}013\bar{0} \sim \bar{2}302\bar{3}$$
$$\bar{1}021\bar{0} \sim \bar{3}203\bar{2} \sim \bar{2}012\bar{0} \sim \bar{1}321\bar{3} \sim \bar{0}130\bar{1} \sim \bar{0}230\bar{2} \sim \bar{3}103\bar{1} \sim \bar{2}312\bar{3}$$
$$\bar{3}123\bar{1} \sim \bar{0}210\bar{2} \sim \bar{2}132\bar{1} \sim \bar{3}023\bar{0} \sim \bar{1}301\bar{3} \sim \bar{1}201\bar{2} \sim \bar{0}310\bar{3} \sim \bar{2}032\bar{0}$$

The orbits of $N^{(\bar{0}120\bar{1})}$ on $\{0,1,2,3\}$ are $\{0,1,2,3\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$. We consider $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}120\bar{1})}$ and find to which double coset it belongs.

$\bar{0}120\bar{1}1 = \bar{0}120 \Rightarrow Nt_0^{-1}t_1t_2t_0t_1^{-1}t_1 = Nt_0^{-1}t_1t_2t_0 \in Nt_0^{-1}t_1t_2t_0N = [\bar{0}120]$.
$\bar{0}120\bar{1}\bar{1} = \bar{0}1201 \Rightarrow Nt_0^{-1}t_1t_2t_0t_1^{-1}t_1^{-1} = Nt_0^{-1}t_1t_2t_0t_1 \in Nt_0^{-1}t_1t_2t_0t_1N = [\bar{0}1201]$.

This double coset is now closed under right multiplication. That is, $Nt_0^{-1}t_1t_2t_0t_1^{-1}t_i$ do not move forward to any double coset with representative word of length six.

Consider $[\bar{0}120\bar{3}]$.
MAGMA shows that the coset stabilizing group $N^{(\bar{0}120\bar{3})} = \{e, (02), (13), (02)(13)\}$. This shows that there exists six distinct single cosets within $[\bar{0}120\bar{3}]$. We list these single cosets below, along with their equal names.

$$\begin{array}{l}
\bar{0}120\bar{3} \sim \bar{0}320\bar{1} \sim \bar{2}102\bar{3} \sim \bar{2}302\bar{1} \\
\bar{1}021\bar{3} \sim \bar{1}321\bar{0} \sim \bar{2}012\bar{3} \sim \bar{2}312\bar{0} \\
\bar{3}123\bar{0} \sim \bar{3}023\bar{1} \sim \bar{2}132\bar{0} \sim \bar{2}032\bar{1} \\
\bar{0}210\bar{3} \sim \bar{0}310\bar{2} \sim \bar{1}201\bar{3} \sim \bar{1}301\bar{2} \\
\bar{0}130\bar{2} \sim \bar{0}230\bar{1} \sim \bar{3}103\bar{2} \sim \bar{3}203\bar{1}
\end{array}$$

The orbits of $N^{(\bar{0}120\bar{3})}$ on $\{0,1,2,3\}$ are $\{0,2\}$, $\{1,3\}$, $\{\bar{0},\bar{2}\}$, and $\{\bar{1},\bar{3}\}$. We consider $Nt_0^{-1}t_1t_2t_0t_3^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}120\bar{3})}$ and find to which double coset it belongs.

$\bar{0}120\bar{3}3 = \bar{0}120 \Rightarrow Nt_0^{-1}t_1t_2t_0t_3^{-1}t_3 = Nt_0^{-1}t_1t_2t_0 \in Nt_0^{-1}t_1t_2t_0N = [\bar{0}120]$.
$\bar{0}120\bar{3}\bar{3} = \bar{0}1203 \Rightarrow Nt_0^{-1}t_1t_2t_0t_3^{-1}t_3^{-1} = Nt_0^{-1}t_1t_2t_0t_3 \in Nt_0^{-1}t_1t_2t_0t_3N = [\bar{0}1203]$.
$\bar{0}120\bar{3}2 = \bar{0}123 \Rightarrow Nt_0^{-1}t_1t_2t_0t_3^{-1}t_2 = Nt_0^{-1}t_1t_2t_3 \in Nt_0^{-1}t_1t_2t_3N = [\bar{0}123]$.
$\bar{0}120\bar{3}\bar{2} = \bar{0}1231 \Rightarrow Nt_0^{-1}t_1t_2t_0t_3^{-1}t_2^{-1} = Nt_0^{-1}t_1t_2t_3t_1 \in Nt_0^{-1}t_1t_2t_3t_1N = [\bar{0}1231]$.

MAGMA proves the previous two statements. This double coset is now closed under right multiplication. That is, $Nt_0^{-1}t_1t_2t_0t_3^{-1}t_i$ do not move forward to any double coset with representative word of length six.

Consider $[\bar{0}1231]$.

MAGMA shows that the coset stabilizing group $N^{(\bar{0}1231)} = \{e, (23)\}$. We know that there exists twelve distinct single cosets within $[\bar{0}1231]$. We list them below, along with their equal names.

$$\begin{aligned}
\bar{0}1231 &\sim \bar{0}1321\\
\bar{1}0230 &\sim \bar{1}0320\\
\bar{2}1031 &\sim \bar{2}1301\\
\bar{3}1201 &\sim \bar{3}1021\\
\bar{0}2132 &\sim \bar{0}2312\\
\bar{0}3213 &\sim \bar{0}3123\\
\bar{1}2032 &\sim \bar{1}2302\\
\bar{2}0130 &\sim \bar{2}0310\\
\bar{1}3203 &\sim \bar{1}3023\\
\bar{3}0210 &\sim \bar{3}0120\\
\bar{2}3013 &\sim \bar{2}3103\\
\bar{3}2102 &\sim \bar{3}2012
\end{aligned}$$

The orbits of $N^{(\bar{0}1231)}$ on $\{0, 1, 2, 3\}$ are $\{2, 3\}$, $\{\bar{2}, \bar{3}\}$, $\{0\}$, $\{1\}$, $\{\bar{0}\}$, and $\{\bar{1}\}$. We consider $Nt_0^{-1}t_1t_2t_3t_1t_i$ for one t_i from each orbit of $N^{(\bar{0}1231)}$ and find to which double coset it belongs.

$\bar{0}1231\bar{1} = \bar{0}123 \Rightarrow Nt_0^{-1}t_1t_2t_3t_1t_1^{-1} = Nt_0^{-1}t_1t_2t_3 \in Nt_0^{-1}t_1t_2t_3N = [\bar{0}123]$.
$\bar{0}12312 = \bar{0}\bar{1}\bar{2}\bar{0}\bar{1} \Rightarrow Nt_0^{-1}t_1t_2t_3t_1t_2 = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}]$.
$\bar{0}1231\bar{2} = 01023 \Rightarrow Nt_0^{-1}t_1t_2t_3t_1t_2^{-1} = Nt_0t_1t_0t_2t_3 \in Nt_0t_1t_0t_2t_3N = [01023]$.
$\bar{0}12311 = \bar{0}120\bar{3} \Rightarrow Nt_0^{-1}t_1t_2t_3t_1t_1 = Nt_0^{-1}t_1t_2t_0t_3^{-1} \in Nt_0^{-1}t_1t_2t_0t_3^{-1}N = [\bar{0}120\bar{3}]$.

In addition to the previous three statements, MAGMA shows that $Nt_0^{-1}t_1t_2t_3t_1t_0$ and $Nt_0^{-1}t_1t_2t_3t_1t_0^{-1}$ are elements of the new double cosets $[\bar{0}12310]$ and $[\bar{0}1231\bar{0}]$, respectively.

Consider $[\bar{0}\bar{1}\bar{2}\bar{0}3]$.

MAGMA shows that the coset stabilizing group $N^{(\bar{0}\bar{1}\bar{2}\bar{0}3)} = \{e, (02)(13)\}$. We list the twelve distinct single cosets within $[\bar{0}\bar{1}\bar{2}\bar{0}3]$ below, along with their equal names.

$$\begin{gathered}
\bar{0}\bar{1}\bar{2}\bar{0}3 \sim \bar{2}\bar{3}\bar{0}\bar{2}1 \\
\bar{1}\bar{0}\bar{2}\bar{1}3 \sim \bar{2}\bar{3}\bar{1}\bar{2}0 \\
\bar{2}\bar{1}\bar{0}\bar{2}3 \sim \bar{0}\bar{3}\bar{2}\bar{0}1 \\
\bar{1}\bar{3}\bar{2}\bar{1}0 \sim \bar{2}\bar{0}\bar{1}\bar{2}3 \\
\bar{3}\bar{0}\bar{2}\bar{3}1 \sim \bar{2}\bar{1}\bar{3}\bar{2}0 \\
\bar{1}\bar{0}\bar{3}\bar{1}2 \sim \bar{3}\bar{2}\bar{1}\bar{3}0 \\
\bar{1}\bar{2}\bar{3}\bar{1}0 \sim \bar{3}\bar{0}\bar{1}\bar{3}2 \\
\bar{2}\bar{0}\bar{3}\bar{2}1 \sim \bar{3}\bar{1}\bar{2}\bar{3}0 \\
\bar{1}\bar{3}\bar{0}\bar{1}2 \sim \bar{0}\bar{2}\bar{1}\bar{0}3 \\
\bar{3}\bar{2}\bar{0}\bar{3}1 \sim \bar{0}\bar{1}\bar{3}\bar{0}2 \\
\bar{1}\bar{2}\bar{0}\bar{1}3 \sim \bar{0}\bar{3}\bar{1}\bar{0}2 \\
\bar{3}\bar{0}\bar{2}\bar{3}1 \sim \bar{2}\bar{1}\bar{3}\bar{2}0
\end{gathered}$$

The orbits of $N^{(\bar{0}\bar{1}\bar{2}\bar{0}3)}$ on $\{0,1,2,3\}$ are $\{0,2\}$, $\{1,3\}$, $\{\bar{0},\bar{2}\}$, and $\{\bar{1},\bar{3}\}$. We consider $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}\bar{2}\bar{0}3)}$ and find to which double coset it belongs.

$$
\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{0}3\bar{3} = \bar{0}\bar{1}\bar{2}\bar{0} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_3^{-1} &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{0}] \\
\bar{0}\bar{1}\bar{2}\bar{0}33 = \bar{0}\bar{1}\bar{2}\bar{0}\bar{3} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_3 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}] \\
\bar{0}\bar{1}\bar{2}\bar{0}32 = \bar{0}\bar{1}\bar{2}\bar{3}\bar{0} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_2 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}] \\
\bar{0}\bar{1}\bar{2}\bar{0}3\bar{2} = \bar{0}\bar{1}\bar{2}\bar{3} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_2^{-1} &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{3}]
\end{aligned}
$$

MAGMA shows the previous two statements. This double coset is now closed under right multiplication. That is, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3t_i$ do not move forward to any double coset with representative word of length six.

Consider $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}]$.

The coset stabilizing group $N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{1})} = \{e\}$. So there exists twenty-four distinct single cosets within $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}]$, which we list below.

$\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}$	$\bar{1}\bar{0}\bar{2}\bar{1}\bar{0}$	$\bar{2}\bar{1}\bar{0}\bar{2}\bar{1}$	$\bar{3}\bar{1}\bar{2}\bar{3}\bar{1}$
$\bar{0}\bar{2}\bar{1}\bar{0}\bar{2}$	$\bar{0}\bar{3}\bar{2}\bar{0}\bar{3}$	$\bar{0}\bar{1}\bar{3}\bar{0}\bar{1}$	$\bar{1}\bar{2}\bar{0}\bar{1}\bar{2}$
$\bar{2}\bar{0}\bar{1}\bar{2}\bar{0}$	$\bar{1}\bar{3}\bar{2}\bar{1}\bar{3}$	$\bar{3}\bar{0}\bar{2}\bar{3}\bar{0}$	$\bar{0}\bar{2}\bar{3}\bar{0}\bar{2}$
$\bar{0}\bar{3}\bar{1}\bar{0}\bar{3}$	$\bar{3}\bar{1}\bar{0}\bar{3}\bar{1}$	$\bar{2}\bar{1}\bar{3}\bar{2}\bar{1}$	$\bar{1}\bar{0}\bar{3}\bar{1}\bar{0}$
$\bar{2}\bar{3}\bar{0}\bar{2}\bar{3}$	$\bar{3}\bar{2}\bar{1}\bar{3}\bar{2}$	$\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}$	$\bar{3}\bar{0}\bar{1}\bar{3}\bar{0}$
$\bar{2}\bar{0}\bar{3}\bar{2}\bar{0}$	$\bar{1}\bar{3}\bar{0}\bar{1}\bar{3}$	$\bar{2}\bar{3}\bar{1}\bar{2}\bar{3}$	$\bar{3}\bar{2}\bar{0}\bar{3}\bar{2}$

The orbits of $N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{1})}$ on $\{0, 1, 2, 3\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$. We consider $Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{1})}$ and find to which double coset it belongs.

$$\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}1 = \bar{0}\bar{1}\bar{2}\bar{0} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_1 = Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1} \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N = [\bar{0}\bar{1}\bar{2}\bar{0}]$$

$$\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}2 = 01203 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_2 = Nt_0t_1t_2t_0t_3 \in Nt_0t_1t_2t_0t_3N = [01203]$$

$$\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}3 = 01201 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3 = Nt_0t_1t_2t_0t_1 \in Nt_0t_1t_2t_0t_1N = [01201]$$

$$\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}0 = 01023 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0 = Nt_0t_1t_0t_2t_3 \in Nt_0t_1t_0t_2t_3N = [01023]$$

$$\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}\bar{1} = \bar{0}1203 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_1^{-1} = Nt_0^{-1}t_1t_2t_0t_3 \in Nt_0^{-1}t_1t_2t_0t_3N = [\bar{0}1203]$$

$$\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}\bar{0} = \bar{0}1231 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_0^{-1} = Nt_0^{-1}t_1t_2t_3t_1 \in Nt_0^{-1}t_1t_2t_3t_1N = [\bar{0}1231]$$

$$\begin{aligned}\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}\bar{3} = 012013 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_3^{-1} &= Nt_0t_1t_2t_0t_1t_3 \\ &\in Nt_0t_1t_2t_0t_1t_3N \\ &= [012013]\end{aligned}$$

$$\begin{aligned}\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}\bar{2} = 012031 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}t_2^{-1} &= Nt_0t_1t_2t_0t_3t_1 \\ &\in Nt_0t_1t_2t_0t_3t_1N \\ &= [012031]\end{aligned}$$

MAGMA has shown the previous seven statements.

Consider $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}]$.
MAGMA shows that the coset stabilizing group $N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{3})} = \{e, (03)(12)\}$. We list the twelve distinct single cosets within $[\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}]$ below, along with their equal names.

$$\begin{gathered}
\bar{0}\bar{1}\bar{2}\bar{0}\bar{3} \sim \bar{3}\bar{2}\bar{1}\bar{3}\bar{0} \\
\bar{1}\bar{0}\bar{2}\bar{1}\bar{3} \sim \bar{3}\bar{2}\bar{0}\bar{3}\bar{1} \\
\bar{2}\bar{1}\bar{0}\bar{2}\bar{3} \sim \bar{3}\bar{0}\bar{1}\bar{3}\bar{2} \\
\bar{3}\bar{1}\bar{2}\bar{3}\bar{0} \sim \bar{0}\bar{2}\bar{1}\bar{0}\bar{3} \\
\bar{0}\bar{3}\bar{2}\bar{0}\bar{1} \sim \bar{1}\bar{2}\bar{3}\bar{1}\bar{0} \\
\bar{0}\bar{1}\bar{3}\bar{0}\bar{2} \sim \bar{2}\bar{3}\bar{1}\bar{2}\bar{0} \\
\bar{1}\bar{2}\bar{0}\bar{1}\bar{3} \sim \bar{3}\bar{0}\bar{2}\bar{3}\bar{1} \\
\bar{2}\bar{0}\bar{1}\bar{2}\bar{3} \sim \bar{3}\bar{1}\bar{0}\bar{3}\bar{2} \\
\bar{1}\bar{3}\bar{2}\bar{1}\bar{0} \sim \bar{0}\bar{2}\bar{3}\bar{0}\bar{1} \\
\bar{1}\bar{0}\bar{3}\bar{1}\bar{2} \sim \bar{2}\bar{3}\bar{0}\bar{2}\bar{1} \\
\bar{2}\bar{0}\bar{3}\bar{2}\bar{1} \sim \bar{1}\bar{3}\bar{0}\bar{1}\bar{2}
\end{gathered}$$

The orbits of $N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{3})}$ on $\{0,1,2,3\}$ are $\{0,3\}$, $\{1,2\}$, $\{\bar{0},\bar{3}\}$, and $\{\bar{1},\bar{2}\}$. We consider

$Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}\bar{2}\bar{0}\bar{3})}$ and find to which double coset it belongs.

$$\begin{aligned}\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}3 = \bar{0}\bar{1}\bar{2}\bar{0} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_3 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1} \\ &\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}N \\ &= [\bar{0}\bar{1}\bar{2}\bar{0}]\end{aligned}$$

$$\begin{aligned}\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{3} = \bar{0}\bar{1}\bar{2}\bar{0}3 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_3^{-1} &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3 \\ &\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N \\ &= [\bar{0}\bar{1}\bar{2}\bar{0}3]\end{aligned}$$

$$\begin{aligned}\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}1 = 010\bar{2}\bar{3} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1 &= Nt_0t_1t_0t_2^{-1}t_3^{-1} \\ &\in Nt_0t_1t_0t_2^{-1}t_3^{-1}N \\ &= [010\bar{2}\bar{3}]\end{aligned}$$

$$\begin{aligned}\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1} = 012012 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}t_1^{-1} &= Nt_0t_1t_2t_0t_1t_2 \\ &\in Nt_0t_1t_2t_0t_1t_2N \\ &= [012012]\end{aligned}$$

MAGMA has shown the previous two statements.

Consider $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}]$.
MAGMA shows that the coset stabilizing group $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{1})} = \{e, (13)\}$. We list the twelve distinct single cosets within $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}]$ below, along with their equal names.

$$\bar{0}\bar{1}\bar{2}\bar{3}\bar{1} \sim \bar{0}\bar{3}\bar{2}\bar{1}\bar{3}$$
$$\bar{1}\bar{0}\bar{2}\bar{3}\bar{0} \sim \bar{1}\bar{3}\bar{2}\bar{0}\bar{3}$$
$$\bar{2}\bar{1}\bar{0}\bar{3}\bar{1} \sim \bar{2}\bar{3}\bar{0}\bar{1}\bar{3}$$
$$\bar{3}\bar{1}\bar{2}\bar{0}\bar{1} \sim \bar{3}\bar{0}\bar{2}\bar{1}\bar{0}$$
$$\bar{0}\bar{2}\bar{1}\bar{3}\bar{2} \sim \bar{0}\bar{3}\bar{1}\bar{2}\bar{3}$$
$$\bar{0}\bar{1}\bar{3}\bar{2}\bar{1} \sim \bar{0}\bar{2}\bar{3}\bar{1}\bar{2}$$
$$\bar{1}\bar{2}\bar{0}\bar{3}\bar{2} \sim \bar{1}\bar{3}\bar{0}\bar{2}\bar{3}$$
$$\bar{2}\bar{0}\bar{1}\bar{3}\bar{0} \sim \bar{2}\bar{3}\bar{1}\bar{0}\bar{3}$$
$$\bar{3}\bar{1}\bar{0}\bar{2}\bar{1} \sim \bar{3}\bar{2}\bar{0}\bar{1}\bar{2}$$
$$\bar{2}\bar{1}\bar{3}\bar{0}\bar{1} \sim \bar{2}\bar{0}\bar{3}\bar{1}\bar{0}$$
$$\bar{1}\bar{0}\bar{3}\bar{2}\bar{0} \sim \bar{1}\bar{2}\bar{3}\bar{0}\bar{2}$$
$$\bar{3}\bar{2}\bar{1}\bar{0}\bar{2} \sim \bar{3}\bar{0}\bar{1}\bar{2}\bar{0}$$

The orbits of $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{1})}$ on $\{0,1,2,3\}$ are $\{1,3\}$, $\{\bar{1},\bar{3}\}$, $\{0\}$, $\{2\}$, $\{\bar{0}\}$, and $\{\bar{2}\}$. We consider $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{1})}$ and find to which double coset it belongs.

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}1 = \bar{0}\bar{1}\bar{2}\bar{3} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_1 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{3}]
\end{aligned}$$

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{1} = \bar{0}1201 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_1^{-1} &= Nt_0^{-1}t_1t_2t_0t_1 \\
&\in Nt_0^{-1}t_1t_2t_0t_1N \\
&= [\bar{0}1201]
\end{aligned}$$

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{2} = 012312 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2^{-1} &= Nt_0t_1t_2t_3t_1t_2 \\
&\in Nt_0t_1t_2t_3t_1t_2N \\
&= [012312]
\end{aligned}$$

$$
\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}2 = 0123\bar{1}\bar{2} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_2 &= Nt_0t_1t_2t_3t_1^{-1}t_2^{-1} \\
&\in Nt_0t_1t_2t_3t_1^{-1}t_2^{-1}N \\
&= [0123\bar{1}\bar{2}]
\end{aligned}
$$

In addition to the previous three statements, MAGMA shows that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0$ and $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1}$ are elements of the new double cosets $[\bar{0}\bar{1}\bar{2}\bar{3}10]$ and $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{0}]$, respectively.

Consider $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}]$.
MAGMA shows that the coset stabilizing group $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{0})} = \{e, (03)\}$. We list the twelve distinct single cosets within $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}]$ below, along with their equal names.

$$
\begin{gathered}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{0} \sim \bar{3}\bar{1}\bar{2}\bar{0}\bar{3} \\
\bar{1}\bar{0}\bar{2}\bar{3}\bar{1} \sim \bar{3}\bar{0}\bar{2}\bar{1}\bar{3} \\
\bar{2}\bar{1}\bar{0}\bar{3}\bar{2} \sim \bar{3}\bar{1}\bar{0}\bar{2}\bar{3} \\
\bar{0}\bar{2}\bar{1}\bar{3}\bar{0} \sim \bar{3}\bar{2}\bar{1}\bar{0}\bar{3} \\
\bar{0}\bar{3}\bar{2}\bar{1}\bar{0} \sim \bar{1}\bar{3}\bar{2}\bar{0}\bar{1} \\
\bar{0}\bar{1}\bar{3}\bar{2}\bar{0} \sim \bar{2}\bar{1}\bar{3}\bar{0}\bar{2} \\
\bar{1}\bar{2}\bar{0}\bar{3}\bar{1} \sim \bar{3}\bar{2}\bar{0}\bar{1}\bar{3} \\
\bar{2}\bar{0}\bar{1}\bar{3}\bar{2} \sim \bar{3}\bar{0}\bar{1}\bar{2}\bar{3} \\
\bar{0}\bar{2}\bar{3}\bar{1}\bar{0} \sim \bar{1}\bar{2}\bar{3}\bar{0}\bar{1} \\
\bar{0}\bar{3}\bar{1}\bar{2}\bar{0} \sim \bar{2}\bar{3}\bar{1}\bar{0}\bar{2} \\
\bar{1}\bar{0}\bar{3}\bar{2}\bar{1} \sim \bar{2}\bar{0}\bar{3}\bar{1}\bar{2} \\
\bar{2}\bar{3}\bar{0}\bar{1}\bar{2} \sim \bar{1}\bar{3}\bar{0}\bar{2}\bar{1}
\end{gathered}
$$

The orbits of $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{0})}$ on $\{0, 1, 2, 3\}$ are $\{0, 3\}$, $\{\bar{0}, \bar{3}\}$, $\{1\}$, $\{2\}$, $\{\bar{1}\}$, and $\{\bar{2}\}$. We consider $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{0})}$ and find to which double coset it belongs.

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}0 = \bar{0}\bar{1}\bar{2}\bar{3} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_0 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{3}]
\end{aligned}$$

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{0} = \bar{0}\bar{1}\bar{2}\bar{0}3 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_0^{-1} &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3 \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3N \\
&= [\bar{0}\bar{1}\bar{2}\bar{0}3]
\end{aligned}$$

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{2} = 012012 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2^{-1} &= Nt_0t_1t_2t_0t_1t_2 \\
&\in Nt_0t_1t_2t_0t_1t_2N \\
&= [012012]
\end{aligned}$$

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}1 = \bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0 \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0N \\
&= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0]
\end{aligned}$$

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}2 = 0120\bar{1}\bar{2} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_2 &= Nt_0t_1t_2t_0t_1^{-1}t_2^{-1} \\
&\in Nt_0t_1t_2t_0t_1^{-1}t_2^{-1}N \\
&= [0120\bar{1}\bar{2}]
\end{aligned}$$

In addition to the previous four statements, MAGMA shows that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}$ is an element of the new double coset $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}]$.

We have considered all of the double cosets with representative words of length five. We now want to consider all of the double cosets with representative words of length six. Namely [010232], [012012], [012013], [012031], [$0120\bar{1}\bar{2}$], [012312], [012310], [$01231\bar{2}$], [$01231\bar{0}$], [$01232\bar{0}$], [012301], [$0123\bar{1}\bar{2}$], [$\bar{0}12310$], [$\bar{0}1231\bar{0}$], [$\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0$], [$\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{0}$], and [$\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}$].

Consider [010232].
MAGMA shows that the coset stabilizing group
$N^{(010232)} = \{e, (01), (23), (01)(23), (02)(13), (03)(12), (0213), (0312)\}$. We list the three distinct single cosets within [010232] below, along with their equal names.

$$010232 \sim 232010 \sim 101232 \sim 101323 \sim 323101 \sim 010323 \sim 323010 \sim 232101$$
$$212030 \sim 030212 \sim 121030 \sim 121303 \sim 303121 \sim 212303 \sim 303212 \sim 030121$$
$$313202 \sim 202313 \sim 131202 \sim 131020 \sim 020131 \sim 313020 \sim 020313 \sim 202131$$

The orbits of $N^{(010232)}$ on $\{0, 1, 2, 3\}$ are $\{0, 1, 2, 3\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$.
We consider $Nt_0t_1t_0t_2t_3t_2t_i$ for one t_i from each orbit of $N^{(010232)}$ and find to which double coset it belongs.

$0102322\bar{2} = 01023 \Rightarrow Nt_0t_1t_0t_2t_3t_2t_2^{-1} = Nt_0t_1t_0t_2t_3 \in Nt_0t_1t_0t_2t_3N = [01023]$.
$0102322 = 010\bar{2}\bar{3} \Rightarrow Nt_0t_1t_0t_2t_3t_2t_2 = Nt_0t_1t_0t_2^{-1}t_3^{-1} \in Nt_0t_1t_0t_2^{-1}t_3^{-1}N = [010\bar{2}\bar{3}]$.

MAGMA shows the previous statement. This double coset is now closed under right multiplication. That is, $Nt_0t_1t_0t_2t_3t_2t_i$ do not move forward to any double coset with representative word of length seven.

Consider [012012].
MAGMA shows that the coset stabilizing group $N^{(012012)} = \{e, (23)\}$. We list the twelve distinct single cosets within [012012] below, along with their equal names.

$$
\begin{aligned}
012012 &\sim 013013\\
102102 &\sim 103103\\
210210 &\sim 213213\\
312312 &\sim 310310\\
021021 &\sim 023023\\
032032 &\sim 031031\\
120120 &\sim 123123\\
201201 &\sim 203203\\
132132 &\sim 130130\\
302302 &\sim 301301\\
230230 &\sim 231231\\
321321 &\sim 320320
\end{aligned}
$$

The orbits of $N^{(012012)}$ on $\{0,1,2,3\}$ are $\{2,3\}$, $\{\bar{2},\bar{3}\}$, $\{0\}$, $\{1\}$, $\{\bar{0}\}$, and $\{\bar{1}\}$. We consider $Nt_0t_1t_2t_0t_1t_2t_i$ for one t_i from each orbit of $N^{(012012)}$ and find to which double coset it belongs.

$$
\begin{aligned}
012012\bar{2} = 01201 \Rightarrow Nt_0t_1t_2t_0t_1t_2t_2^{-1} &= Nt_0t_1t_2t_0t_1\\
&\in Nt_0t_1t_2t_0t_1N\\
&= [01201]
\end{aligned}
$$

$$
\begin{aligned}
0120122 = \bar{0}\bar{1}\bar{2}\bar{0}\bar{3} \Rightarrow Nt_0t_1t_2t_0t_1t_2t_2 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}\\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_3^{-1}N\\
&= [\bar{0}\bar{1}\bar{2}\bar{0}\bar{3}]
\end{aligned}
$$

$$
\begin{aligned}
0120121 = \bar{0}\bar{1}\bar{2}\bar{3}\bar{0} \Rightarrow Nt_0t_1t_2t_0t_1t_2t_1 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}\\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N\\
&= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}]
\end{aligned}
$$

$$
\begin{aligned}
0120121\bar{1} = 01201\bar{1}\bar{2} \Rightarrow Nt_0t_1t_2t_0t_1t_2t_1^{-1} &= Nt_0t_1t_2t_0t_1^{-1}t_2^{-1} \\
&\in Nt_0t_1t_2t_0t_1^{-1}t_2^{-1}N \\
&= [01201\bar{1}\bar{2}]
\end{aligned}
$$

In addition to the previous three statements, MAGMA shows that $Nt_0t_1t_2t_0t_1t_2t_0$ and $Nt_0t_1t_2t_0t_1t_2t_0^{-1}$ are elements within the new double cosets [0120120] and [012012$\bar{0}$], respectively.

Consider [012013].
MAGMA shows that the coset stabilizing group $N^{(012013)} = \{e, (01)(23), (02)(13), (03)(12)\}$. This double coset has six distinct single cosets, which we list below along with their equal names.

$$
\begin{gathered}
012013 \sim 103102 \sim 321320 \sim 230231 \\
102103 \sim 013012 \sim 320321 \sim 231230 \\
210213 \sim 123120 \sim 301302 \sim 032031 \\
312310 \sim 130132 \sim 021023 \sim 203201 \\
120123 \sim 213210 \sim 302301 \sim 031032 \\
201203 \sim 023021 \sim 310312 \sim 132130
\end{gathered}
$$

The orbits of $N^{(012013)}$ on $\{0, 1, 2, 3\}$ are $\{0, 1, 2, 3\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. We consider $Nt_0t_1t_2t_0t_1t_3t_i$ for one t_i from each orbit of $N^{(012013)}$ and find to which double coset it belongs.

$$
\begin{aligned}
0120133\bar{3} = 01201 \Rightarrow Nt_0t_1t_2t_0t_1t_3t_3^{-1} &= Nt_0t_1t_2t_0t_1 \\
&\in Nt_0t_1t_2t_0t_1N \\
&= [01201]
\end{aligned}
$$

$$
\begin{aligned}
0120133 = \bar{0}\bar{1}\bar{2}\bar{0}\bar{1} \Rightarrow Nt_0t_1t_2t_0t_1t_3t_3 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}]
\end{aligned}
$$

MAGMA shows the previous statement. This double coset is now closed under right multiplication. That is, $Nt_0t_1t_2t_0t_1t_3t_i$ do not move forward to any double coset with representative word of length seven.

Consider [012031].
MAGMA shows that the coset stabilizing group $N^{(012031)} = \{e, (13)\}$. The twelve distinct single cosets within [012031] are listed below, along with their equal names.

$012031 \sim 032013$
$102130 \sim 132103$
$210231 \sim 230213$
$312301 \sim 302310$
$021032 \sim 031023$
$013021 \sim 023012$
$120132 \sim 130123$
$201230 \sim 231203$
$310231 \sim 320312$
$213201 \sim 203210$
$103120 \sim 123102$
$321302 \sim 301320$

The orbits of $N^{(012031)}$ on $\{0,1,2,3\}$ are $\{1,3\}$, $\{\bar{1},\bar{3}\}$, $\{0\}$, $\{2\}$, $\{\bar{0}\}$, and $\{\bar{2}\}$. We consider $Nt_0t_1t_2t_0t_3t_1t_i$ for one t_i from each orbit of $N^{(012031)}$ and find to which double coset it belongs.

$$\begin{aligned} 012031\bar{1} = 01203 \Rightarrow Nt_0t_1t_2t_0t_3t_1t_1^{-1} &= Nt_0t_1t_2t_0t_3 \\ &\in Nt_0t_1t_2t_0t_3N \\ &= [01203] \end{aligned}$$

$$\begin{aligned} 0120311 = \bar{0}\bar{1}\bar{2}\bar{0}\bar{1} \Rightarrow Nt_0t_1t_2t_0t_3t_1t_1 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1} \\ &\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_0^{-1}t_1^{-1}N \\ &= [\bar{0}\bar{1}\bar{2}\bar{0}\bar{1}] \end{aligned}$$

$$\begin{aligned} 0120312 = \bar{0}12310 \Rightarrow Nt_0t_1t_2t_0t_3t_1t_2 &= Nt_0^{-1}t_1t_2t_3t_1t_0 \\ &\in Nt_0^{-1}t_1t_2t_3t_1t_0N \\ &= [\bar{0}12310] \end{aligned}$$

$$\begin{aligned} 0120310 = 01232\bar{0} \Rightarrow Nt_0t_1t_2t_0t_3t_1t_0 &= Nt_0t_1t_2t_3t_2t_0^{-1} \\ &\in Nt_0t_1t_2t_3t_2t_0^{-1}N \\ &= [01232\bar{0}] \end{aligned}$$

$$\begin{aligned} 012031\bar{0} = 01230 \Rightarrow Nt_0t_1t_2t_0t_3t_1t_0^{-1} &= Nt_0t_1t_2t_3t_0 \\ &\in Nt_0t_1t_2t_3t_0N \\ &= [01230] \end{aligned}$$

In addition to the previous four statements, MAGMA provides that $Nt_0t_1t_2t_0t_3t_1t_2^{-1}$ is an element of the new double coset $[012031\bar{2}]$.

Consider $[0120\bar{1}\bar{2}]$.
MAGMA shows that the coset stabilizing group $N^{(0120\bar{1}\bar{2})} = \{e, (02), (13), (02)(13)\}$. This double coset has six distinct single cosets, which we list below along with their equal names.

$$
\begin{array}{c}
0120\bar{1}\bar{2} \sim 2102\bar{1}\bar{0} \sim 0320\bar{3}\bar{2} \sim 2302\bar{3}\bar{0} \\
1021\bar{0}\bar{2} \sim 2012\bar{0}\bar{1} \sim 1321\bar{3}\bar{2} \sim 2312\bar{3}\bar{1} \\
3123\bar{1}\bar{2} \sim 2132\bar{1}\bar{3} \sim 3023\bar{0}\bar{2} \sim 2032\bar{0}\bar{3} \\
0210\bar{2}\bar{1} \sim 1201\bar{2}\bar{0} \sim 0310\bar{3}\bar{1} \sim 1301\bar{3}\bar{0} \\
0130\bar{1}\bar{3} \sim 3103\bar{1}\bar{0} \sim 0230\bar{2}\bar{3} \sim 3203\bar{2}\bar{0} \\
1031\bar{0}\bar{3} \sim 3013\bar{0}\bar{1} \sim 1231\bar{2}\bar{3} \sim 3213\bar{2}\bar{1}
\end{array}
$$

The orbits of $N^{(0120\bar{1}\bar{2})}$ on $\{0,1,2,3\}$ are $\{0,2\}$, $\{1,3\}$, $\{\bar{0},\bar{2}\}$, and $\{\bar{1},\bar{3}\}$. We consider $Nt_0t_1t_2t_0t_1^{-1}t_2^{-1}t_i$ for one t_i from each orbit of $N^{(0120\bar{1}\bar{2})}$ and find to which double coset it belongs.

$$
\begin{aligned}
0120\bar{1}\bar{2}2 = 0120\bar{1} \Rightarrow Nt_0t_1t_2t_0t_1^{-1}t_2^{-1}t_2 &= Nt_0t_1t_2t_0t_1^{-1} \\
&\in Nt_0t_1t_2t_0t_1^{-1}N \\
&= [0120\bar{1}]
\end{aligned}
$$

$$
\begin{aligned}
0120\bar{1}\bar{2}1 = 012012 \Rightarrow Nt_0t_1t_2t_0t_1^{-1}t_2^{-1}t_1 &= Nt_0t_1t_2t_0t_1t_2 \\
&\in Nt_0t_1t_2t_0t_1t_2N \\
&= [012012]
\end{aligned}
$$

$$0120\bar{1}\bar{2}\bar{1} = \bar{0}\bar{1}\bar{2}\bar{3}\bar{0} \Rightarrow Nt_0t_1t_2t_0t_1^{-1}t_2^{-1}t_1^{-1} = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}$$
$$\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N$$
$$= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}]$$

$$0120\bar{1}\bar{2}\bar{2} = 01\bar{2}\bar{0}\bar{3} \Rightarrow Nt_0t_1t_2t_0t_1^{-1}t_2^{-1}t_2^{-1} = Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}$$
$$\in Nt_0t_1t_2^{-1}t_0^{-1}t_3^{-1}N$$
$$= [01\bar{2}\bar{0}\bar{3}]$$

MAGMA has shown the previous three statements. This double coset is now closed under right multiplication. That is, $Nt_0t_1t_2t_0t_1^{-1}t_2^{-1}t_i$ do not move forward to any double coset with representative word of length seven.

Consider [012312].
MAGMA shows that the coset stabilizing group $N^{(012312)} = \{e, (02), (13), (02)(13)\}$. This double coset has six distinct single cosets, which we list below along with their equal names.

$$012312 \sim 230130 \sim 210310 \sim 032132$$
$$102302 \sim 231031 \sim 201301 \sim 132032$$
$$312012 \sim 203103 \sim 213013 \sim 302102$$
$$021321 \sim 130230 \sim 120320 \sim 031231$$
$$013213 \sim 320120 \sim 310210 \sim 023123$$

The orbits of $N^{(012312)}$ on $\{0, 1, 2, 3\}$ are $\{0, 2\}$, $\{1, 3\}$, $\{\bar{0}, \bar{2}\}$, and $\{\bar{1}, \bar{3}\}$. We consider $Nt_0t_1t_2t_3t_1t_2t_i$ for one t_i from each orbit of $N^{(012312)}$ and find to which double coset it belongs.

$$
\begin{aligned}
0123122\bar{2} = 01231 \Rightarrow Nt_0t_1t_2t_3t_1t_2t_2^{-1} &= Nt_0t_1t_2t_3t_1 \\
&\in Nt_0t_1t_2t_3t_1N \\
&= [01231]
\end{aligned}
$$

$$
\begin{aligned}
0123121 = \bar{0}\bar{1}\bar{2}\bar{3}\bar{1} \Rightarrow Nt_0t_1t_2t_3t_1t_2t_1 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}]
\end{aligned}
$$

$$
\begin{aligned}
0123122 = 012312\bar{2} \Rightarrow Nt_0t_1t_2t_3t_1t_2t_2 &= Nt_0t_1t_2t_3t_1t_2^{-1} \\
&\in Nt_0t_1t_2t_3t_1t_2^{-1}N \\
&= [01231\bar{2}]
\end{aligned}
$$

$$
\begin{aligned}
012312\bar{1} = 0123\bar{1}\bar{2} \Rightarrow Nt_0t_1t_2t_3t_1t_2t_1^{-1} &= Nt_0t_1t_2t_3t_1^{-1}t_2^{-1} \\
&\in Nt_0t_1t_2t_3t_1^{-1}t_2^{-1}N \\
&= [0123\bar{1}\bar{2}]
\end{aligned}
$$

MAGMA has shown the previous three statements. This double coset is now closed under right multiplication. That is, $Nt_0t_1t_2t_3t_1t_2t_i$ do not move forward to any double coset with representative word of length seven.

Consider [012310].

MAGMA shows that the coset stabilizing group

$N^{(012310)} = \{e, (02), (13), (01)(23), (02)(13), (03)(12), (0123), (0321)\}$. This double coset has three distinct single cosets, which we list below along with their equal names.

$$012310 \sim 123021 \sim 103201 \sim 230132 \sim 301203 \sim 032130 \sim 210312 \sim 321023$$
$$102301 \sim 023120 \sim 013210 \sim 231032 \sim 310213 \sim 132031 \sim 201302 \sim 320123$$
$$312013 \sim 120321 \sim 130231 \sim 203102 \sim 031230 \sim 302103 \sim 213012 \sim 021320$$

The orbits of $N^{(012310)}$ on $\{0,1,2,3\}$ are $\{0,1,2,3\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$.
We consider $Nt_0t_1t_2t_3t_1t_0t_i$ for one t_i from each orbit of $N^{(012310)}$ and find to which double coset it belongs.

$$\begin{aligned} 012310\bar{0} = 01231 \Rightarrow Nt_0t_1t_2t_3t_1t_0t_0^{-1} &= Nt_0t_1t_2t_3t_1 \\ &\in Nt_0t_1t_2t_3t_1N \\ &= [01231] \end{aligned}$$

$$\begin{aligned} 0123100 = 012310\bar{0} \Rightarrow Nt_0t_1t_2t_3t_1t_0t_0 &= Nt_0t_1t_2t_3t_1t_0^{-1} \\ &\in Nt_0t_1t_2t_3t_1t_0^{-1}N \\ &= [01231\bar{0}] \end{aligned}$$

This double coset is now closed under right multiplication. That is, $Nt_0t_1t_2t_3t_1t_0t_i$ do not move forward to any double coset with representative word of length seven.

Consider $[01231\bar{2}]$.
MAGMA shows that the coset stabilizing group $N^{(01231\bar{2})} = \{e, (02), (13), (02)(13)\}$. This double coset has six distinct single cosets, which we list below along with their equal names.

$$012312\bar{2} \sim 230130\bar{0} \sim 210310\bar{0} \sim 032132\bar{2}$$
$$102302\bar{2} \sim 231031\bar{1} \sim 201301\bar{1} \sim 132032\bar{2}$$
$$312012\bar{2} \sim 203103\bar{3} \sim 213013\bar{3} \sim 302102\bar{2}$$
$$021321\bar{1} \sim 130230\bar{0} \sim 120320\bar{0} \sim 031231\bar{1}$$
$$013213\bar{3} \sim 320120\bar{0} \sim 310210\bar{0} \sim 023123\bar{3}$$
$$103203\bar{3} \sim 321021\bar{1} \sim 301201\bar{1} \sim 123023\bar{3}$$

The orbits of $N^{(01231\bar{2})}$ on $\{0,1,2,3\}$ are $\{0,2\}$, $\{1,3\}$, $\{\bar{0},\bar{2}\}$, and $\{\bar{1},\bar{3}\}$. We consider $Nt_0t_1t_2t_3t_1t_2^{-1}t_i$ for one t_i from each orbit of $N^{(01231\bar{2})}$ and find to which double coset it belongs.

$$\begin{aligned} 01231\bar{2}2 = 01231 \Rightarrow Nt_0t_1t_2t_3t_1t_2^{-1}t_2 &= Nt_0t_1t_2t_3t_1 \\ &\in Nt_0t_1t_2t_3t_1N \\ &= [01231] \end{aligned}$$

$$\begin{aligned} 01231\bar{2}\bar{2} = 012312 \Rightarrow Nt_0t_1t_2t_3t_1t_2^{-1}t_2^{-1} &= Nt_0t_1t_2t_3t_1t_2 \\ &\in Nt_0t_1t_2t_3t_1t_2N \\ &= [012312] \end{aligned}$$

$$\begin{aligned} 01231\bar{2}1 = 01232 \Rightarrow Nt_0t_1t_2t_3t_1t_2^{-1}t_1 &= Nt_0t_1t_2t_3t_2 \\ &\in Nt_0t_1t_2t_3t_2N \\ &= [01232] \end{aligned}$$

$$\begin{aligned} 01231\bar{2}\bar{1} = \bar{0}1201 \Rightarrow Nt_0t_1t_2t_3t_1t_2^{-1}t_1^{-1} &= Nt_0^{-1}t_1t_2t_0t_1 \\ &\in Nt_0^{-1}t_1t_2t_0t_1N \\ &= [\bar{0}1201] \end{aligned}$$

MAGMA has shown the previous two statements. This double coset is now closed under right multiplication. That is, $Nt_0t_1t_2t_3t_1t_2^{-1}t_i$ do not move forward to any double coset with representative word of length seven.

Consider $[01231\bar{0}]$.
MAGMA shows that the coset stabilizing group $N^{(01231\bar{0})} = \{e, (02), (13), (02)(13)\}$. We list the six distinct single cosets within $[01231\bar{0}]$ below, along with their equal names.

$$\begin{array}{c}
01231\bar{0} \sim 21031\bar{2} \sim 23013\bar{2} \sim 03213\bar{0} \\
10230\bar{1} \sim 20130\bar{2} \sim 23103\bar{2} \sim 13203\bar{1} \\
31201\bar{3} \sim 21301\bar{2} \sim 20310\bar{2} \sim 30210\bar{3} \\
02132\bar{0} \sim 12032\bar{1} \sim 13023\bar{1} \sim 03123\bar{0} \\
01321\bar{0} \sim 31021\bar{3} \sim 32012\bar{3} \sim 02312\bar{0} \\
10320\bar{1} \sim 30121\bar{3} \sim 32102\bar{3} \sim 12303\bar{1}
\end{array}$$

The orbits of $N^{(01231\bar{0})}$ on $\{0,1,2,3\}$ are $\{0,2\}$, $\{1,3\}$, $\{\bar{0},\bar{2}\}$, and $\{\bar{1},\bar{3}\}$. We consider $Nt_0t_1t_2t_3t_1t_0^{-1}t_i$ for one t_i from each orbit of $N^{(01231\bar{0})}$ and find to which double coset it belongs.

$$\begin{aligned}
01231\bar{0}0 = 01231 \Rightarrow Nt_0t_1t_2t_3t_1t_0^{-1}t_0 &= Nt_0t_1t_2t_3t_1 \\
&\in Nt_0t_1t_2t_3t_1N \\
&= [01231]
\end{aligned}$$

$$\begin{aligned}
01231\bar{0}\bar{0} = 012310 \Rightarrow Nt_0t_1t_2t_3t_1t_0^{-1}t_0^{-1} &= Nt_0t_1t_2t_3t_1t_0 \\
&\in Nt_0t_1t_2t_3t_1t_0N \\
&= [012310]
\end{aligned}$$

$$
\begin{aligned}
0123 1\bar{0}1 = 01230 \Rightarrow N t_0 t_1 t_2 t_3 t_1 t_0^{-1} t_1 &= N t_0 t_1 t_2 t_3 t_0 \\
&\in N t_0 t_1 t_2 t_3 t_0 N \\
&= [01230]
\end{aligned}
$$

$$
\begin{aligned}
0123 1\bar{0}\bar{1} = 012301 \Rightarrow N t_0 t_1 t_2 t_3 t_1 t_0^{-1} t_1^{-1} &= N t_0 t_1 t_2 t_3 t_0 t_1 \\
&\in N t_0 t_1 t_2 t_3 t_0 t_1 N \\
&= [012301]
\end{aligned}
$$

MAGMA has shown the previous two statements. This double coset is now closed under right multiplication. That is, $N t_0 t_1 t_2 t_3 t_1 t_0^{-1} t_i$ do not move forward to any double coset with representative word of length seven.

Consider $[01232\bar{0}]$.
MAGMA shows that the coset stabilizing group $N^{(01232\bar{0})} = \{e, (01), (23), (01)(23)\}$. This double coset has six distinct single cosets, which we list below along with their equal names.

$$
\begin{gathered}
01232\bar{0} \sim 01323\bar{0} \sim 10232\bar{1} \sim 10323\bar{1} \\
21030\bar{2} \sim 21303\bar{2} \sim 12030\bar{1} \sim 12303\bar{1} \\
31202\bar{3} \sim 31020\bar{3} \sim 13202\bar{1} \sim 13020\bar{3} \\
02131\bar{0} \sim 02313\bar{0} \sim 20131\bar{2} \sim 20313\bar{2} \\
03212\bar{0} \sim 03121\bar{0} \sim 30212\bar{3} \sim 30121\bar{3} \\
23010\bar{2} \sim 23101\bar{2} \sim 32010\bar{3} \sim 32101\bar{3}
\end{gathered}
$$

The orbits of $N^{(01232\bar{0})}$ on $\{0, 1, 2, 3\}$ are $\{0, 1\}$, $\{2, 3\}$, $\{\bar{0}, \bar{1}\}$, and $\{\bar{2}, \bar{3}\}$. We consider $N t_0 t_1 t_2 t_3 t_2 t_0^{-1} t_i$ for one t_i from each orbit of $N^{(01232\bar{0})}$ and find to which double coset it belongs.

$$\begin{aligned}
0123\bar{2}\bar{0}0 = 01232 \Rightarrow Nt_0t_1t_2t_3t_2t_0^{-1}t_0 &= Nt_0t_1t_2t_3t_2 \\
&\in Nt_0t_1t_2t_3t_2N \\
&= [01232]
\end{aligned}$$

$$\begin{aligned}
0123\bar{2}\bar{0}2 = 01230 \Rightarrow Nt_0t_1t_2t_3t_2t_0^{-1}t_2 &= Nt_0t_1t_2t_3t_0 \\
&\in Nt_0t_1t_2t_3t_0N \\
&= [01230]
\end{aligned}$$

$$\begin{aligned}
0123\bar{2}\bar{0}\bar{0} = \bar{0}1203 \Rightarrow Nt_0t_1t_2t_3t_2t_0^{-1}t_0^{-1} &= Nt_0^{-1}t_1t_2t_0t_3 \\
&\in Nt_0^{-1}t_1t_2t_0t_3N \\
&= [\bar{0}1203]
\end{aligned}$$

$$\begin{aligned}
0123\bar{2}\bar{0}\bar{2} = 012031 \Rightarrow Nt_0t_1t_2t_3t_2t_0^{-1}t_2^{-1} &= Nt_0t_1t_2t_0t_3t_1 \\
&\in Nt_0t_1t_2t_0t_3t_1N \\
&= [012031]
\end{aligned}$$

MAGMA shows the previous three statements. This double coset is now closed under right multiplication. That is, $Nt_0t_1t_2t_3t_2t_0^{-1}t_i$ do not move forward to any double coset with representative word of length seven.

Consider [012301].
MAGMA shows that the coset stabilizing group $N^{(012301)} = \{e, (01), (03), (13), (013), (031)\}$. This double coset has four distinct single cosets, which we list below along with their equal names.

$$
\begin{array}{c}
012301 \sim 312031 \sim 302130 \sim 102310 \sim 032103 \sim 132013 \\
210321 \sim 310231 \sim 320132 \sim 120312 \sim 230123 \sim 130213 \\
021302 \sim 321032 \sim 301230 \sim 201320 \sim 031203 \sim 231023 \\
013201 \sim 213021 \sim 203120 \sim 103210 \sim 023102 \sim 123012
\end{array}
$$

The orbits of $N^{(012301)}$ on $\{0,1,2,3\}$ are $\{0,1,3\}$, $\{\bar{0},\bar{1},\bar{3}\}$, $\{2\}$, and $\{\bar{2}\}$. We consider $Nt_0t_1t_2t_3t_0t_1t_i$ for one t_i from each orbit of $N^{(012301)}$ and find to which double coset it belongs.

$$
\begin{aligned}
01230 1\bar{1} = 01230 \Rightarrow Nt_0t_1t_2t_3t_0t_1t_1^{-1} &= Nt_0t_1t_2t_3t_0 \\
&\in Nt_0t_1t_2t_3t_0N \\
&= [01230]
\end{aligned}
$$

$$
\begin{aligned}
0123011 = 01231\bar{0} \Rightarrow Nt_0t_1t_2t_3t_0t_1t_1 &= Nt_0t_1t_2t_3t_1t_0^{-1} \\
&\in Nt_0t_1t_2t_3t_1t_0^{-1}N \\
&= [01231\bar{0}]
\end{aligned}
$$

$$
\begin{aligned}
012301\bar{2} = 012031\bar{2} \Rightarrow Nt_0t_1t_2t_3t_0t_1t_2^{-1} &= Nt_0t_1t_2t_0t_3t_1t_2^{-1} \\
&\in Nt_0t_1t_2t_0t_3t_1t_2^{-1}N \\
&= [012031\bar{2}]
\end{aligned}
$$

In addition to the previous two statements, MAGMA shows that $Nt_0t_1t_2t_3t_0t_1t_2$ is an element of the new double coset [0123012].

Consider $[0123\bar{1}\bar{2}]$.
MAGMA shows that the coset stabilizing group $N^{(0123\bar{1}\bar{2})} = \{e, (01), (23), (01)(23)\}$. We list the six distinct single cosets within $[0123\bar{1}\bar{2}]$ below, along with their equal names.

$$\begin{array}{l}
0123\bar{1}\bar{2} \sim 1023\bar{0}\bar{2} \sim 0132\bar{1}\bar{3} \sim 1032\bar{0}\bar{3} \\
2103\bar{1}\bar{0} \sim 1203\bar{2}\bar{0} \sim 2130\bar{1}\bar{3} \sim 1230\bar{2}\bar{3} \\
3120\bar{1}\bar{2} \sim 1320\bar{3}\bar{2} \sim 3102\bar{1}\bar{0} \sim 1302\bar{3}\bar{0} \\
0213\bar{2}\bar{1} \sim 2013\bar{0}\bar{1} \sim 0231\bar{2}\bar{3} \sim 2031\bar{0}\bar{3} \\
0321\bar{3}\bar{2} \sim 3021\bar{0}\bar{2} \sim 0312\bar{3}\bar{1} \sim 3012\bar{0}\bar{1} \\
2301\bar{3}\bar{0} \sim 3201\bar{2}\bar{0} \sim 2310\bar{3}\bar{1} \sim 3210\bar{2}\bar{1}
\end{array}$$

The orbits of $N^{(0123\bar{1}\bar{2})}$ on $\{0,1,2,3\}$ are $\{0,1\}$, $\{2,3\}$, $\{\bar{0},\bar{1}\}$, and $\{\bar{2},\bar{3}\}$. We consider $Nt_0t_1t_2t_3t_1^{-1}t_2^{-1}t_i$ for one t_i from each orbit of $N^{(0123\bar{1}\bar{2})}$ and find to which double coset it belongs.

$$\begin{aligned}
0123\bar{1}\bar{2}2 = 0123\bar{1} \Rightarrow Nt_0t_1t_2t_3t_1^{-1}t_2^{-1}t_2 &= Nt_0t_1t_2t_3t_1^{-1} \\
&\in Nt_0t_1t_2t_3t_1^{-1}N \\
&= [0123\bar{1}]
\end{aligned}$$

$$\begin{aligned}
0123\bar{1}\bar{2}1 = 012312 \Rightarrow Nt_0t_1t_2t_3t_1^{-1}t_2^{-1}t_1 &= Nt_0t_1t_2t_3t_1t_2 \\
&\in Nt_0t_1t_2t_3t_1t_2N \\
&= [012312]
\end{aligned}$$

$$0123\bar{1}\bar{2}\bar{1} = \bar{0}\bar{1}\bar{2}\bar{3}\bar{1} \Rightarrow Nt_0t_1t_2t_3t_1^{-1}t_2^{-1}t_1^{-1} \begin{array}{l} = Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1} \\ \in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N \\ = [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}] \end{array}$$

$$0123\bar{1}\bar{2}\bar{2} = 01\bar{2}\bar{0}3 \Rightarrow Nt_0t_1t_2t_3t_1^{-1}t_2^{-1}t_2^{-1} \begin{array}{l} = Nt_0t_1t_2^{-1}t_0^{-1}t_3 \\ \in Nt_0t_1t_2^{-1}t_0^{-1}t_3N \\ = [01\bar{2}\bar{0}3] \end{array}$$

MAGMA shows the previous three statements. This double coset is now closed under right multiplication. That is, $Nt_0t_1t_2t_3t_1^{-1}t_2^{-1}t_i$ do not move forward to any double coset with representative word of length seven.

Consider $[\bar{0}12310]$.
MAGMA shows that the coset stabilizing group $N^{(\bar{0}12310)} = \{e, (01), (23), (01)(23)\}$. We list the six distinct single cosets within $[\bar{0}12310]$ below, along with their equal names.

$$\begin{array}{c}
\bar{0}12310 \sim \bar{1}02301 \sim \bar{0}13210 \sim \bar{1}03201 \\
\bar{2}10312 \sim \bar{1}20321 \sim \bar{2}13012 \sim \bar{1}23021 \\
\bar{3}12013 \sim \bar{1}32031 \sim \bar{3}10213 \sim \bar{1}30231 \\
\bar{0}21320 \sim \bar{2}01302 \sim \bar{0}23120 \sim \bar{2}03102 \\
\bar{0}32130 \sim \bar{3}02103 \sim \bar{0}31230 \sim \bar{3}01203 \\
\bar{2}30132 \sim \bar{3}20123 \sim \bar{2}31032 \sim \bar{3}21023
\end{array}$$

The orbits of $N^{(\bar{0}12310)}$ on $\{0, 1, 2, 3\}$ are $\{0, 1\}$, $\{2, 3\}$, $\{\bar{0}, \bar{1}\}$, and $\{\bar{2}, \bar{3}\}$. We consider $Nt_0^{-1}t_1t_2t_3t_1t_0t_i$ for one t_i from each orbit of $N^{(\bar{0}12310)}$ and find to which double coset it belongs.

$$\bar{0}12310\bar{0} = \bar{0}1231 \Rightarrow Nt_0^{-1}t_1t_2t_3t_1t_0t_0^{-1} \begin{array}{l} = Nt_0^{-1}t_1t_2t_3t_1 \\ \in Nt_0^{-1}t_1t_2t_3t_1N \\ = [\bar{0}1231] \end{array}$$

$$\bar{0}123100 = \bar{0}12310\bar{0} \Rightarrow Nt_0^{-1}t_1t_2t_3t_1t_0t_0 \begin{array}{l} = Nt_0^{-1}t_1t_2t_3t_1t_0^{-1} \\ \in Nt_0^{-1}t_1t_2t_3t_1t_0^{-1}N \\ = [\bar{0}12310\bar{0}] \end{array}$$

$$\bar{0}12310\bar{2} = 012031 \Rightarrow Nt_0^{-1}t_1t_2t_3t_1t_0t_2^{-1} \begin{array}{l} = Nt_0t_1t_2t_0t_3t_1 \\ \in Nt_0t_1t_2t_0t_3t_1N \\ = [012031] \end{array}$$

$$\bar{0}123102 = 012031\bar{2} \Rightarrow Nt_0^{-1}t_1t_2t_3t_1t_0t_2 \begin{array}{l} = Nt_0t_1t_2t_0t_3t_1t_2^{-1} \\ \in Nt_0t_1t_2t_0t_3t_1t_2^{-1}N \\ = [012031\bar{2}] \end{array}$$

MAGMA shows the previous two statements.

Consider $[\bar{0}12310\bar{0}]$.
MAGMA shows that the coset stabilizing group
$N^{(\bar{0}12310\bar{0})} = \{e, (01), (23), (01)(23), (02)(13), (03)(12), (0213), (0312)\}$. There are three distinct single cosets within $[\bar{0}12310\bar{0}]$, which we list below with their equal names.

$$\bar{0}1231\bar{0} \sim \bar{2}3013\bar{2} \sim \bar{0}1321\bar{0} \sim \bar{3}2102\bar{3} \sim \bar{2}3103\bar{2} \sim \bar{1}0320\bar{1} \sim \bar{1}0230\bar{1} \sim \bar{3}2012\bar{3}$$
$$\bar{2}1031\bar{2} \sim \bar{0}3213\bar{0} \sim \bar{2}1301\bar{2} \sim \bar{3}0120\bar{3} \sim \bar{0}3123\bar{0} \sim \bar{1}2302\bar{1} \sim \bar{1}2032\bar{1} \sim \bar{3}0210\bar{3}$$
$$\bar{3}1201\bar{3} \sim \bar{2}0310\bar{2} \sim \bar{3}1021\bar{3} \sim \bar{0}2132\bar{0} \sim \bar{2}0130\bar{2} \sim \bar{1}3023\bar{1} \sim \bar{1}3203\bar{1} \sim \bar{0}2312\bar{0}$$

The orbits of $N^{(\bar{0}1231\bar{0})}$ on $\{0,1,2,3\}$ are $\{0,1,2,3\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$.
We consider $Nt_0^{-1}t_1t_2t_3t_1t_0^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}1231\bar{0})}$ and find to which double coset it belongs.

$$\begin{aligned}
\bar{0}1231\bar{0}0 = \bar{0}1231 \Rightarrow Nt_0^{-1}t_1t_2t_3t_1t_0^{-1}t_0 &= Nt_0^{-1}t_1t_2t_3t_1 \\
&\in Nt_0^{-1}t_1t_2t_3t_1N \\
&= [\bar{0}1231]
\end{aligned}$$

$$\begin{aligned}
\bar{0}1231\bar{0}\bar{0} = \bar{0}12310 \Rightarrow Nt_0^{-1}t_1t_2t_3t_1t_0^{-1}t_0^{-1} &= Nt_0^{-1}t_1t_2t_3t_1t_0 \\
&\in Nt_0^{-1}t_1t_2t_3t_1t_0N \\
&= [\bar{0}12310]
\end{aligned}$$

This double coset is now closed under right multiplication. That is, $Nt_0^{-1}t_1t_2t_3t_1t_0^{-1}t_i$ do not move forward to any double coset with representative word of length seven.

Consider $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0]$.
MAGMA shows that the coset stabilizing group $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0)} = \{e, (02), (13), (02)(13)\}$. We list the six distinct single cosets within $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0]$ below, along with their equal names.

$$\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0 \sim \bar{2}\bar{3}\bar{0}\bar{1}\bar{3}2 \sim \bar{2}\bar{1}\bar{0}\bar{3}\bar{1}2 \sim \bar{0}\bar{3}\bar{2}\bar{1}\bar{3}0$$
$$\bar{1}\bar{0}\bar{2}\bar{3}\bar{0}1 \sim \bar{2}\bar{3}\bar{1}\bar{0}\bar{3}2 \sim \bar{2}\bar{0}\bar{1}\bar{3}\bar{0}2 \sim \bar{1}\bar{3}\bar{2}\bar{0}\bar{3}1$$
$$\bar{3}\bar{1}\bar{2}\bar{0}\bar{1}3 \sim \bar{2}\bar{0}\bar{3}\bar{1}\bar{0}2 \sim \bar{2}\bar{1}\bar{3}\bar{0}\bar{1}2 \sim \bar{3}\bar{0}\bar{2}\bar{1}\bar{0}3$$

$$\bar{0}\bar{2}\bar{1}\bar{3}\bar{2}0 \sim \bar{1}\bar{3}\bar{0}\bar{2}\bar{3}1 \sim \bar{1}\bar{2}\bar{0}\bar{3}\bar{2}1 \sim \bar{0}\bar{3}\bar{1}\bar{2}\bar{3}0$$
$$\bar{0}\bar{1}\bar{3}\bar{2}\bar{1}0 \sim \bar{3}\bar{2}\bar{0}\bar{1}\bar{2}3 \sim \bar{3}\bar{1}\bar{0}\bar{2}\bar{1}3 \sim \bar{0}\bar{2}\bar{3}\bar{1}\bar{2}0$$
$$\bar{1}\bar{0}\bar{3}\bar{2}\bar{0}1 \sim \bar{3}\bar{2}\bar{1}\bar{0}\bar{2}3 \sim \bar{3}\bar{0}\bar{1}\bar{2}\bar{0}3 \sim \bar{1}\bar{2}\bar{3}\bar{0}\bar{2}1$$

The orbits of $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0)}$ on $\{0,1,2,3\}$ are $\{0,2\}$, $\{1,3\}$, $\{\bar{0},\bar{2}\}$, and $\{\bar{1},\bar{3}\}$. We consider $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0)}$ and find to which double coset it belongs.

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0\bar{0} = \bar{0}\bar{1}\bar{2}\bar{3}\bar{1} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_0^{-1} &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}]
\end{aligned}$$

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}00 = \bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{0} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_0 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{0}]
\end{aligned}$$

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0\bar{1} = \bar{0}\bar{1}\bar{2}\bar{3}\bar{0} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1^{-1} &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}]
\end{aligned}$$

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}01 = \bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_1 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}]
\end{aligned}$$

MAGMA shows the previous two statements. This double is now closed under right multiplication. That is, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0t_i$ do not move forward to any double coset with representative word of length seven.

Consider $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{0}]$.
MAGMA shows that the coset stabilizing group
$N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{0})} = \{e, (02), (13), (01)(23), (02)(13), (03)(12), (0123), (0321)\}$. There are three distinct single cosets within $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{0}]$, which we list below with their equal names.

$$\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{0} \sim \bar{0}\bar{3}\bar{2}\bar{1}\bar{3}\bar{0} \sim \bar{1}\bar{2}\bar{3}\bar{0}\bar{2}\bar{1} \sim \bar{2}\bar{3}\bar{0}\bar{1}\bar{3}\bar{2} \sim \bar{3}\bar{2}\bar{1}\bar{0}\bar{2}\bar{3} \sim \bar{2}\bar{1}\bar{0}\bar{3}\bar{1}\bar{2} \sim \bar{3}\bar{0}\bar{1}\bar{2}\bar{0}\bar{3} \sim \bar{1}\bar{0}\bar{3}\bar{2}\bar{0}\bar{1}$$
$$\bar{1}\bar{0}\bar{2}\bar{3}\bar{0}\bar{1} \sim \bar{1}\bar{3}\bar{2}\bar{0}\bar{3}\bar{1} \sim \bar{0}\bar{2}\bar{3}\bar{1}\bar{2}\bar{0} \sim \bar{2}\bar{3}\bar{1}\bar{0}\bar{3}\bar{2} \sim \bar{3}\bar{2}\bar{0}\bar{1}\bar{2}\bar{3} \sim \bar{2}\bar{0}\bar{1}\bar{3}\bar{0}\bar{2} \sim \bar{3}\bar{1}\bar{0}\bar{2}\bar{1}\bar{3} \sim \bar{0}\bar{1}\bar{3}\bar{2}\bar{1}\bar{0}$$
$$\bar{3}\bar{1}\bar{2}\bar{0}\bar{1}\bar{3} \sim \bar{3}\bar{0}\bar{2}\bar{1}\bar{0}\bar{3} \sim \bar{1}\bar{2}\bar{0}\bar{3}\bar{2}\bar{1} \sim \bar{2}\bar{0}\bar{3}\bar{1}\bar{0}\bar{2} \sim \bar{0}\bar{2}\bar{1}\bar{3}\bar{2}\bar{0} \sim \bar{2}\bar{1}\bar{3}\bar{0}\bar{1}\bar{2} \sim \bar{0}\bar{3}\bar{1}\bar{2}\bar{3}\bar{0} \sim \bar{1}\bar{3}\bar{0}\bar{2}\bar{3}\bar{1}$$

The orbits of $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{0})}$ on $\{0, 1, 2, 3\}$ are $\{0, 1, 2, 3\}$ and $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$.
We consider $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{0})}$ and find to which double coset it belongs.

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{0}0 = \bar{0}\bar{1}\bar{2}\bar{3}\bar{1} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1}t_0 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}]
\end{aligned}$$

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}\bar{0}\bar{0} = \bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1}t_0^{-1} &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0 \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0N \\
&= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0]
\end{aligned}$$

This double coset is now closed under right multiplication. That is, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0^{-1}t_i$ do not move forward to any double coset with representative word of length seven.

Consider $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}]$.
MAGMA shows that the coset stabilizing group $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1})} = \{e, (01), (03), (13), (013), (031)\}$. There exists four distinct single cosets within $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}]$. We list them below along with their equal names.

$$\begin{aligned}
&\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1} \sim \bar{0}\bar{3}\bar{2}\bar{1}\bar{0}\bar{3} \sim \bar{3}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1} \sim \bar{1}\bar{3}\bar{2}\bar{0}\bar{1}\bar{3} \sim \bar{3}\bar{0}\bar{2}\bar{1}\bar{3}\bar{0} \sim \bar{1}\bar{0}\bar{2}\bar{3}\bar{1}\bar{0}\\
&\bar{2}\bar{1}\bar{0}\bar{3}\bar{2}\bar{1} \sim \bar{2}\bar{3}\bar{0}\bar{1}\bar{2}\bar{3} \sim \bar{3}\bar{1}\bar{0}\bar{2}\bar{3}\bar{1} \sim \bar{1}\bar{3}\bar{0}\bar{2}\bar{1}\bar{3} \sim \bar{3}\bar{2}\bar{0}\bar{1}\bar{3}\bar{2} \sim \bar{1}\bar{2}\bar{0}\bar{3}\bar{1}\bar{2}\\
&\bar{0}\bar{2}\bar{1}\bar{3}\bar{0}\bar{2} \sim \bar{0}\bar{3}\bar{1}\bar{2}\bar{0}\bar{3} \sim \bar{3}\bar{2}\bar{1}\bar{0}\bar{3}\bar{2} \sim \bar{2}\bar{3}\bar{1}\bar{0}\bar{2}\bar{3} \sim \bar{3}\bar{0}\bar{1}\bar{2}\bar{3}\bar{0} \sim \bar{2}\bar{0}\bar{1}\bar{3}\bar{2}\bar{0}\\
&\bar{0}\bar{1}\bar{3}\bar{2}\bar{0}\bar{1} \sim \bar{0}\bar{2}\bar{3}\bar{1}\bar{0}\bar{2} \sim \bar{2}\bar{1}\bar{3}\bar{0}\bar{2}\bar{1} \sim \bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2} \sim \bar{2}\bar{0}\bar{3}\bar{1}\bar{2}\bar{0} \sim \bar{1}\bar{0}\bar{3}\bar{2}\bar{1}\bar{0}
\end{aligned}$$

The orbits of $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1})}$ on $\{0, 1, 2, 3\}$ are $\{0, 1, 3\}$, $\{\bar{0}, \bar{1}, \bar{3}\}$, $\{2\}$, and $\{\bar{2}\}$. We consider $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1})}$ and find to which double coset it belongs.

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}1 = \bar{0}\bar{1}\bar{2}\bar{3}\bar{0} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_1 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}\\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}N\\
&= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}]
\end{aligned}$$

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{1} = \bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_1^{-1} &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0\\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}t_0N\\
&= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{1}0]
\end{aligned}$$

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}2 = 0120120 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_2 &= Nt_0t_1t_2t_0t_1t_2t_0\\
&\in Nt_0t_1t_2t_0t_1t_2t_0N\\
&= [0120120]
\end{aligned}$$

In addition to the previous two statements, MAGMA shows that $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_2^{-1}$ is an element of the new double coset $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2}]$.

We have considered all of the double cosets with representative words of length six. We will now consider all of the double cosets with representative words of length seven. Namely, [0120120], $[012012\bar{0}]$, $[012031\bar{2}]$, [0123012], and $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2}]$.

Consider [0120120].
MAGMA shows that the coset stabilizing group
$N^{(0120120)} = \{e, (02), (03), (23), (023), (032)\}$. We list the four distinct single cosets within [0120120] below, along with their equal names.

$$0120120 \sim 0130130 \sim 2102102 \sim 2132132 \sim 3103103 \sim 3123123$$
$$1021021 \sim 1031031 \sim 2012012 \sim 2032032 \sim 3013013 \sim 3023023$$
$$0210210 \sim 0230230 \sim 1201201 \sim 1231231 \sim 3203203 \sim 3213213$$
$$0320320 \sim 0310310 \sim 2302302 \sim 2312312 \sim 1301301 \sim 1321321$$

The orbits of $N^{(0120120)}$ on $\{0,1,2,3\}$ are $\{0,2,3\}$, $\{\bar{0},\bar{2},\bar{3}\}$, $\{1\}$, and $\{\bar{1}\}$. We consider $Nt_0t_1t_2t_0t_1t_2t_0t_i$ for one t_i from each orbit of $N^{(0120120)}$ and find to which double coset it belongs.

$$\begin{aligned} 0120120\bar{0} = 012012 \Rightarrow Nt_0t_1t_2t_0t_1t_2t_0t_0^{-1} &= Nt_0t_1t_2t_0t_1t_2 \\ &\in Nt_0t_1t_2t_0t_1t_2N \\ &= [012012] \end{aligned}$$

$$\begin{aligned} 01201200 = 012012\bar{0} \Rightarrow Nt_0t_1t_2t_0t_1t_2t_0t_0 &= Nt_0t_1t_2t_0t_1t_2t_0^{-1} \\ &\in Nt_0t_1t_2t_0t_1t_2t_0^{-1}N \\ &= [012012\bar{0}] \end{aligned}$$

$$\begin{aligned} 01201201 = \bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2} \Rightarrow Nt_0t_1t_2t_0t_1t_2t_0t_1 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_2^{-1} \\ &\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_2^{-1}N \\ &= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2}] \end{aligned}$$

$$\begin{aligned} 0120120\bar{1} = \bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1} \Rightarrow Nt_0t_1t_2t_0t_1t_2t_0t_1^{-1} &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1} \\ &\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N \\ &= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}] \end{aligned}$$

MAGMA shows the previous two statements. This double coset is now closed under right multiplication. That is, $Nt_0t_1t_2t_0t_1t_2t_0t_i$ do not move forward to any double coset with representative word of length eight.

Consider $[012012\bar{0}]$.
MAGMA shows that the coset stabilizing group
$N^{(012012\bar{0})} = \{e, (01), (23), (01)(23)\}$. We list the six distinct single cosets within $[012012\bar{0}]$ below, along with their equal names.

$$\begin{gathered} 012012\bar{0} \sim 103103\bar{1} \sim 102102\bar{1} \sim 013013\bar{0} \\ 210210\bar{2} \sim 123123\bar{1} \sim 120120\bar{1} \sim 213213\bar{2} \\ 312312\bar{3} \sim 130130\bar{1} \sim 132132\bar{1} \sim 310310\bar{3} \\ 021021\bar{0} \sim 203203\bar{2} \sim 201201\bar{2} \sim 023023\bar{0} \\ 032032\bar{0} \sim 301301\bar{3} \sim 302302\bar{3} \sim 031031\bar{0} \\ 230230\bar{2} \sim 321321\bar{3} \sim 320320\bar{3} \sim 231231\bar{2} \end{gathered}$$

The orbits of $N^{(0120120\bar{0})}$ on $\{0,1,2,3\}$ are $\{0,1\}$, $\{2,3\}$, $\{\bar{0},\bar{1}\}$, and $\{\bar{2},\bar{3}\}$. We consider $Nt_0t_1t_2t_0t_1t_2t_0^{-1}t_i$ for one t_i from each orbit of $N^{(0120120\bar{0})}$ and find to which double coset it belongs.

$$\begin{aligned} 012012\bar{0}0 = 012012 \Rightarrow Nt_0t_1t_2t_0t_1t_2t_0^{-1}t_0 &= Nt_0t_1t_2t_0t_1t_2 \\ &\in Nt_0t_1t_2t_0t_1t_2N \\ &= [012012] \end{aligned}$$

$$\begin{aligned} 012012\bar{0}\bar{0} = 0120120 \Rightarrow Nt_0t_1t_2t_0t_1t_2t_0^{-1}t_0^{-1} &= Nt_0t_1t_2t_0t_1t_2t_0 \\ &\in Nt_0t_1t_2t_0t_1t_2t_0N \\ &= [0120120] \end{aligned}$$

$$\begin{aligned} 012012\bar{0}2 = 0\bar{1}\bar{2}\bar{3}\bar{1} \Rightarrow Nt_0t_1t_2t_0t_1t_2t_0^{-1}t_2 &= Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1} \\ &\in Nt_0t_1^{-1}t_2^{-1}t_3^{-1}N \\ &= [0\bar{1}\bar{2}\bar{3}\bar{1}] \end{aligned}$$

In addition to the previous statement, MAGMA shows that $Nt_0t_1t_2t_0t_1t_2t_0^{-1}t_2^{-1}$ is an element of the new double coset $[012012\bar{0}\bar{2}]$.

Consider $[012031\bar{2}]$.

MAGMA shows that the coset stabilizing group $N^{(012031\bar{2})} = \{e, (12), (13), (23), (123), (132)\}$. We list the four distinct single cosets within $[012031\bar{2}]$ below, along with their equal names.

$$012031\bar{2} \sim 023012\bar{3} \sim 021032\bar{1} \sim 031023\bar{1} \sim 032013\bar{2} \sim 013021\bar{3}$$
$$102130\bar{2} \sim 123102\bar{3} \sim 120132\bar{0} \sim 130123\bar{0} \sim 132103\bar{2} \sim 103120\bar{3}$$

$$2102310\bar{0} \sim 2032103\bar{3} \sim 2012301\bar{1} \sim 2312031\bar{1} \sim 2302130\bar{0} \sim 2132013\bar{3}$$
$$3123012\bar{2} \sim 3203120\bar{0} \sim 3213021\bar{1} \sim 3013201\bar{1} \sim 3023102\bar{2} \sim 3103210\bar{0}$$

The orbits of $N^{(012031\bar{2})}$ on $\{0,1,2,3\}$ are $\{1,2,3\}$, $\{\bar{1},\bar{2},\bar{3}\}$, $\{0\}$, and $\{\bar{0}\}$. We consider $Nt_0t_1t_2t_0t_3t_1t_2^{-1}t_i$ for one t_i from each orbit of $N^{(012031\bar{2})}$ and find to which double coset it belongs.

$$\begin{aligned} 012031\bar{2}2 = 012031 \Rightarrow Nt_0t_1t_2t_0t_3t_1t_2^{-1}t_2 &= Nt_0t_1t_2t_0t_3t_1N \\ &\in Nt_0t_1t_2t_0t_3t_1N \\ &= [012031] \end{aligned}$$

$$\begin{aligned} 012031\bar{2}\bar{2} = \bar{0}12310 \Rightarrow Nt_0t_1t_2t_0t_3t_1t_2^{-1}t_2^{-1} &= Nt_0^{-1}t_1t_2t_3t_1t_0N \\ &\in Nt_0^{-1}t_1t_2t_3t_1t_0N \\ &= [\bar{0}12310] \end{aligned}$$

$$\begin{aligned} 012031\bar{2}0 = 012301 \Rightarrow Nt_0t_1t_2t_0t_3t_1t_2^{-1}t_0 &= Nt_0t_1t_2t_3t_0t_1N \\ &\in Nt_0t_1t_2t_3t_0t_1N \\ &= [012301] \end{aligned}$$

$$\begin{aligned} 012031\bar{2}\bar{0} = 0123012 \Rightarrow Nt_0t_1t_2t_0t_3t_1t_2^{-1}t_0^{-1} &= Nt_0t_1t_2t_3t_0t_1t_2N \\ &\in Nt_0t_1t_2t_3t_0t_1t_2N \\ &= [0123012] \end{aligned}$$

MAGMA shows the previous three statements. This double coset is now closed under right multiplication. That is, $Nt_0t_1t_2t_0t_3t_1t_2^{-1}t_i$ do not move forward to any double coset with representative word of length eight.

Consider [0123012].
MAGMA shows that the coset stabilizing group $N^{(0123012)} = \langle e, (01), (02), (03) \rangle \cong S_4$. There is only one distinct single coset within this double coset. However, this single coset has twenty-four equal names. We identify this single coset, along with its equal names, below.

$$\begin{array}{l}
0123012 \sim 1203120 \sim 2130213 \sim 0321032 \sim 1320132 \sim 0231023 \sim \\
2301230 \sim 0213021 \sim 3102310 \sim 1302130 \sim 3201320 \sim 3021302 \sim \\
0312031 \sim 3012301 \sim 1023102 \sim 2103210 \sim 1230123 \sim 3210321 \sim \\
2013201 \sim 0132013 \sim 3120312 \sim 2310231 \sim 1032103 \sim 2031203 \sim
\end{array}$$

The orbits of $N^{(0123012)}$ on $\{0,1,2,3\}$ are $\{0,1,2,3\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$.
We consider $Nt_0t_1t_2t_3t_0t_1t_2t_i$ for one t_i from each orbit of $N^{(0123012)}$ and find to which double coset it belongs.

$$\begin{aligned}
0123012\bar{2} = 012301 \Rightarrow Nt_0t_1t_2t_3t_0t_1t_2t_2^{-1} &= Nt_0t_1t_2t_3t_0t_1 \\
&\in Nt_0t_1t_2t_3t_0t_1N \\
&= [012301]
\end{aligned}$$

$$\begin{aligned}
01230122 = 012031\bar{2} \Rightarrow Nt_0t_1t_2t_3t_0t_1t_2t_2 &= Nt_0t_1t_2t_0t_3t_1t_2^{-1} \\
&\in Nt_0t_1t_2t_0t_3t_1t_2^{-1}N \\
&= [012031\bar{2}]
\end{aligned}$$

MAGMA shows the previous statement. This double coset is now closed under right multiplication. That is, $Nt_0t_1t_2t_3t_0t_1t_2t_i$ do not move forward to any double coset with representative word of length eight.

Consider $[\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2}]$.
MAGMA shows that the coset stabilizing group $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2})} = \langle e, (01), (02), (03)\rangle \cong S_4$. There is only one distinct single coset within this double coset. However, this single coset has twenty-four equal names. We identify this single coset, along with its equal names, below.

$$\begin{gathered}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2} \sim \bar{3}\bar{2}\bar{0}\bar{1}\bar{3}\bar{2}\bar{0} \sim \bar{2}\bar{0}\bar{1}\bar{3}\bar{2}\bar{0}\bar{1} \sim \bar{3}\bar{1}\bar{0}\bar{2}\bar{3}\bar{1}\bar{0} \sim \bar{0}\bar{1}\bar{3}\bar{2}\bar{0}\bar{1}\bar{3} \sim \bar{1}\bar{3}\bar{2}\bar{0}\bar{1}\bar{3}\bar{2} \sim \\
\bar{0}\bar{3}\bar{2}\bar{1}\bar{0}\bar{3}\bar{2} \sim \bar{1}\bar{0}\bar{3}\bar{2}\bar{1}\bar{0}\bar{3} \sim \bar{1}\bar{2}\bar{0}\bar{3}\bar{1}\bar{2}\bar{0} \sim \bar{1}\bar{3}\bar{0}\bar{2}\bar{1}\bar{3}\bar{0} \sim \bar{3}\bar{0}\bar{2}\bar{1}\bar{3}\bar{0}\bar{2} \sim \bar{2}\bar{1}\bar{3}\bar{0}\bar{2}\bar{1}\bar{3} \sim \\
\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2}\bar{3} \sim \bar{0}\bar{3}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1} \sim \bar{3}\bar{1}\bar{2}\bar{0}\bar{3}\bar{1}\bar{2} \sim \bar{3}\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1} \sim \bar{0}\bar{2}\bar{1}\bar{3}\bar{0}\bar{2}\bar{1} \sim \bar{0}\bar{2}\bar{3}\bar{1}\bar{0}\bar{2}\bar{3} \sim \\
\bar{2}\bar{1}\bar{0}\bar{3}\bar{2}\bar{1}\bar{0} \sim \bar{2}\bar{0}\bar{3}\bar{1}\bar{2}\bar{0}\bar{3} \sim \bar{2}\bar{3}\bar{1}\bar{0}\bar{2}\bar{3}\bar{1} \sim \bar{2}\bar{3}\bar{0}\bar{1}\bar{2}\bar{3}\bar{0} \sim \bar{1}\bar{0}\bar{2}\bar{3}\bar{1}\bar{0}\bar{2} \sim \bar{3}\bar{2}\bar{1}\bar{0}\bar{3}\bar{2}\bar{1} \sim
\end{gathered}$$

The orbits of $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2})}$ on $\{0,1,2,3\}$ are $\{0,1,2,3\}$ and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$.
We consider $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_2^{-1}t_i$ for one t_i from each orbit of $N^{(\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2})}$ and find to which double coset it belongs.

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2}2 = \bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1} \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_2^{-1}t_2 &= Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1} \\
&\in Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}N \\
&= [\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}]
\end{aligned}$$

$$\begin{aligned}
\bar{0}\bar{1}\bar{2}\bar{3}\bar{0}\bar{1}\bar{2}\bar{2} = 0120120 \Rightarrow Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_2^{-1}t_2^{-1} &= Nt_0t_1t_2t_0t_1t_2t_0 \\
&\in Nt_0t_1t_2t_0t_1t_2t_0N \\
&= [0120120]
\end{aligned}$$

MAGMA shows the previous statement. This double coset is now closed under right multiplication. That is, $Nt_0^{-1}t_1^{-1}t_2^{-1}t_3^{-1}t_0^{-1}t_1^{-1}t_2^{-1}t_i$ do not move forward to any double coset with representative word of length eight.

Consider $[012012\bar{0}\bar{2}]$.
MAGMA shows that the coset stabilizing group
$N^{(012012\bar{0}\bar{2})} = \{e, (01), (23), (01)(23), (02)(13), (03)(12), (0213), (0312)\}$. We list the three distinct single cosets within $[012012\bar{0}\bar{2}]$ below, along with their equal names.

$012012\bar{0}\bar{2} \sim 231231\bar{2}\bar{1} \sim 102102\bar{1}\bar{2} \sim 320320\bar{3}\bar{0} \sim 103103\bar{1}\bar{3} \sim 321321\bar{3}\bar{1} \sim 013013\bar{0}\bar{3} \sim 230230\bar{2}\bar{0}$

$210210\bar{2}\bar{0} \sim 031031\bar{0}\bar{1} \sim 120120\bar{1}\bar{0} \sim 302302\bar{3}\bar{2} \sim 123123\bar{1}\bar{3} \sim 301301\bar{3}\bar{1} \sim 213213\bar{2}\bar{3} \sim 032032\bar{0}\bar{2}$

$312312\bar{3}\bar{2} \sim 201201\bar{2}\bar{1} \sim 132132\bar{1}\bar{2} \sim 023023\bar{0}\bar{3} \sim 130130\bar{1}\bar{0} \sim 021021\bar{0}\bar{1} \sim 310310\bar{3}\bar{0} \sim 203203\bar{2}\bar{3}$

The orbits of $N^{(012012\bar{0}\bar{2})}$ on $\{0,1,2,3\}$ are $\{0,1,2,3\}$, and $\{\bar{0},\bar{1},\bar{2},\bar{3}\}$. We consider $Nt_0t_1t_2t_0t_1t_2t_0^{-1}t_2^{-1}t_i$ for one t_i from each orbit of $N^{(012012\bar{0}\bar{2})}$ and find to which double coset it belongs.

$$\begin{aligned} 012012\bar{0}\bar{2}2 = 012012\bar{0} \Rightarrow Nt_0t_1t_2t_0t_1t_2t_0^{-1}t_2^{-1}t_2 &= Nt_0t_1t_2t_0t_1t_2t_0^{-1} \\ &\in Nt_0t_1t_2t_0t_1t_2t_0^{-1}N \\ &= [012012\bar{0}] \end{aligned}$$

$$\begin{aligned} 012012\bar{0}\bar{2}\bar{2} = 0\bar{1}\bar{2}\bar{3}\bar{1} \Rightarrow Nt_0t_1t_2t_0t_1t_2t_0^{-1}t_2^{-1}t_2^{-1} &= Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1} \\ &\in Nt_0t_1^{-1}t_2^{-1}t_3^{-1}t_1^{-1}N \\ &= [0\bar{1}\bar{2}\bar{3}\bar{1}] \end{aligned}$$

MAGMA shows the previous statement. This double coset is now closed under right multiplication. That is, $Nt_0t_1t_2t_0t_1t_2t_0^{-1}t_2^{-1}t_i$ do not move forward to any double coset with representative word of length nine.

All of our work is summarized in the Cayley Diagram of $U_3(3) \times 3$ over S_4 given below.

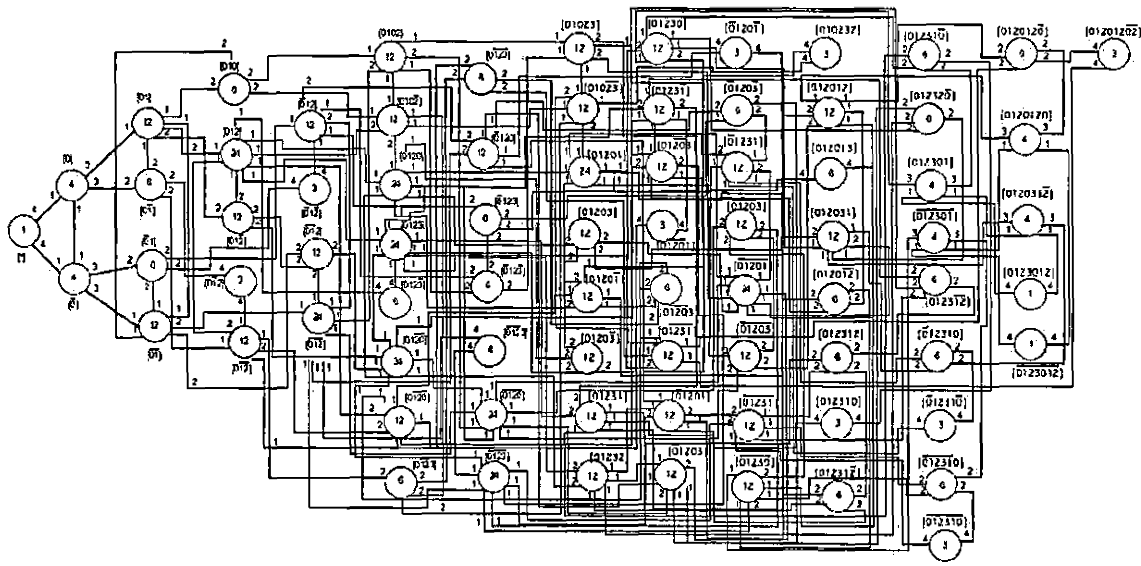

Figure 3.1: Cayley Diagram $U_3(3) \times 3$ over S_4

Our argument shows that the order of G is at most $|N| \times 756 = 18,144$, where 756 is the number of single cosets shown in the diagram above. We now show that $|G|$ is at least 18,144. Now $G =< x, y, t >$ acts on X, the set of the single cosets mentioned above. Thus $\alpha : G \to S_X$ is a homomorphism. Since $N =< x, y >$ acts by conjugation and t acts by right multiplication on the t_is, we compute the images xx, yy, and tt of x,y, and t, respectively, in S_X and verify that the additional relations hold in $< xx, yy, tt >$ within S_X and $| < xx, yy, tt > | = 18,144$. So $G/Ker\alpha \cong < xx, yy, tt >$. Hence $|G| \geq 18,144$.
Thus, $|G| = 18,144$.
We also verified that G satisfies a presentation of $\cong U_3(3) \times 3$.

Due to the complexity of the above Cayley diagram, we split the image into sections that allow for better review.

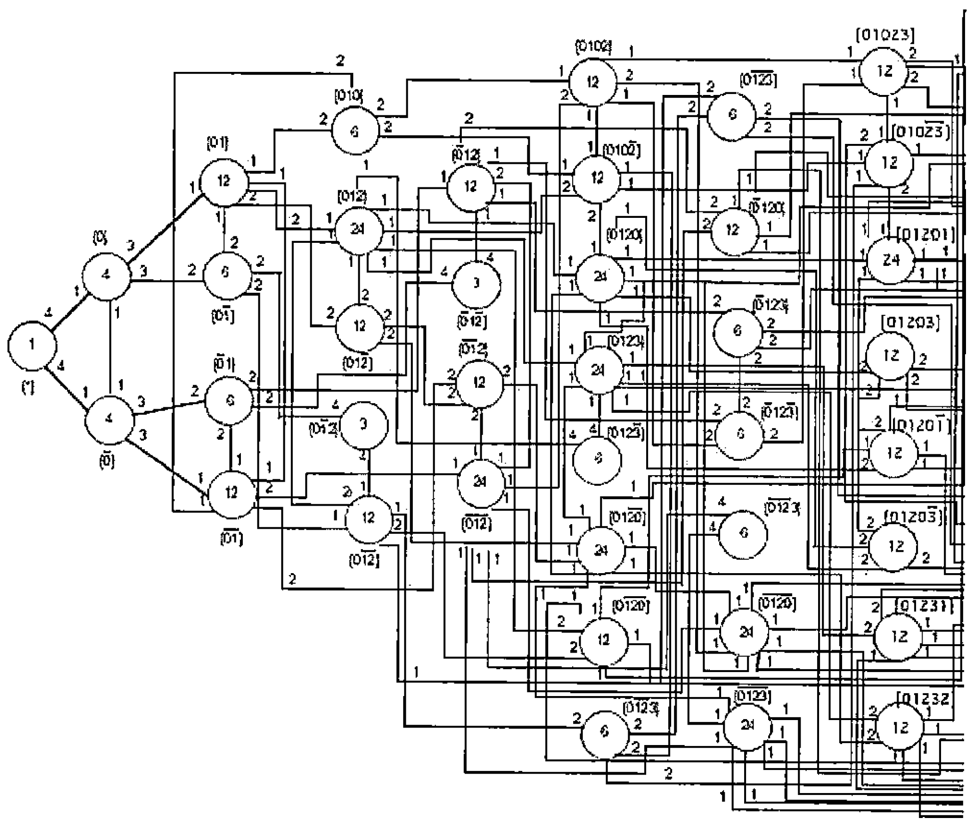

Figure 3.2: Cayley Diagram (left section) $U_3(3) \times 3$ over S_4

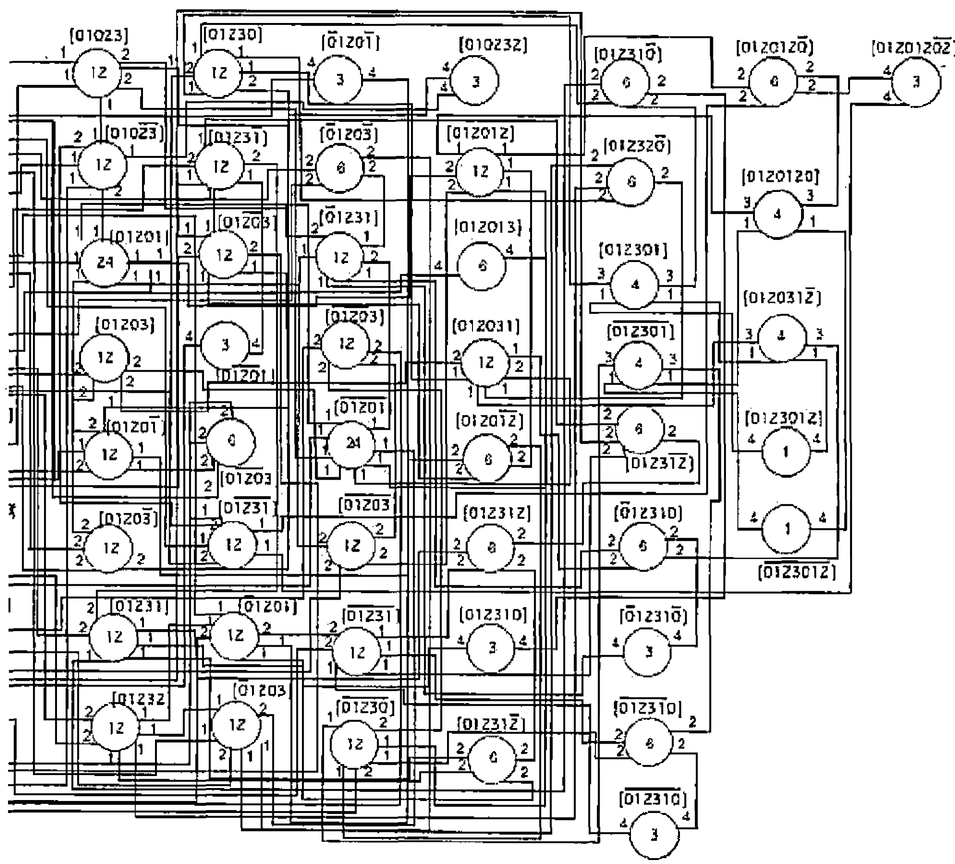

Figure 3.3: Cayley Diagram (right section) $U_3(3) \times 3$ over S_4

Chapter 4

$U_3(3)$ as the Homomorphic Image of $3^{*4} : S_4$

Now the center of $U_3(3) \times 3$ is $\mathbb{Z}_3 = < (xt^{x^2})^7 >$.
We factor $U_3(3) \times 3$ by its center to obtain $U_3(3)$. The details of the double coset enumeration are summarized in the Cayley diagram given below. Because of its size, we split the Cayley diagram into two sections for better review.

Our argument shows that the order of G is at most $|N| \times 252 = 6,048$, where 252 is the number of single cosets shown in the diagram above. We now show that $|G|$ is at least 6,048. Now $G =< x, y, t >$ acts on X, the set of the single cosets mentioned above. Thus $\alpha : G \to S_X$ is a homomorphism. Since $N =< x, y >$ acts by conjugation and t acts by right multiplication on the t_is, we compute the images xx, yy, and tt of x,y, and t, respectively, in S_X and verify that the additional relations hold in $< xx, yy, tt >$ within S_X and $| < xx, yy, tt > | = 6,048$. So $G/Ker\alpha \cong < xx, yy, tt >$. Hence $|G| \geq 6,048$. Thus, $|G| = 6,048$.

A presentation for $U_3(3)$ is $< a, b | a^2 = b^6 = (ab)^7 = [a, (ab^2)^3] = b^3(b^2, ab^3a)^2 = 1 >$. Now $a = (x^2y)^2)t_2t_3^{-1}$ and $b = xyx^2t_1^{-1}t_2t_3$ belong to G and satisfy the above presentation of $U_3(3)$. Thus $U_3(3) \leq G$. But $|U_3(3)| = 6048 = |G|$

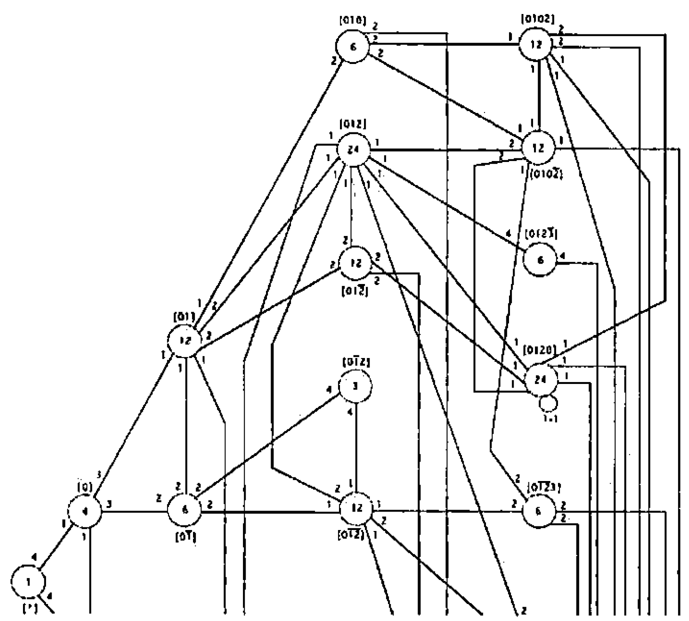

Figure 4.1: Cayley Diagram (top section) $U_3(3)$ over S_4

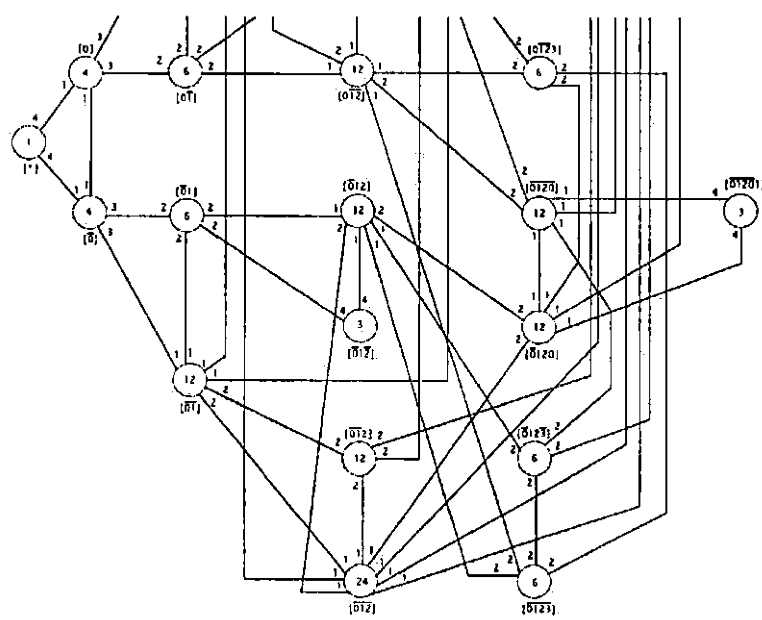

Figure 4.2: Cayley Diagram (bottom section) $U_3(3)$ over S_4

Chapter 5

$M_{11} \times 2$ as the Homomorphic Image of $3^{*4} :_m 2^{\cdot}S_4^+$

There are twenty-six sporadic simple groups. In his three papers that were published in 1860, 1861, and 1873, Mathieu proved the existence of five of the twenty six, namely M_{11}, M_{12}, M_{22}, M_{23}, and M_{24}. Now M_{11}, M_{12}, M_{22}, and M_{23} are subgroups of M_{24}. A Steiner system S(4,5,11) is a set of 5-element subsets (pentads) with the property that no two pentads have 4 or more elements in common. M_{11} is the group of automorphisms of a Steiner system S(4,5,11), which has $\binom{11}{4}\binom{5}{4} = 66$ pentads. Since M_{12} is 5-transitive, the point stabiliser M_{11} is 4-transitive on 11 points. Now M_{11} is a subgroup of M_{24}, and within M_{24} it has orbits of lengths 1, 11, and 12. Thus M_{11} has a transitive action on 12 letters. Moreover, this action is 3-transitive.

Now consider the progenitor $3^{*4} :_m 2^{\cdot}S_4^+$, whose symmetric representation is given by:
$< x, y, t \mid x^4 = (yx)^3 = y^2 = 1 = t^3 = (t, y) = (yt^x)^2 = (xt)^{22} = (yxt)^5 = (x^2t)^6 = (x^3yt)^6 = (yt^{x^6})^{10} >$
where N = ${}_m2^{\cdot}S_4^+$ is a double cover of S_4 and the action on the symmetric generators x, y is given by:

$x \sim (0\ \bar{1}\ \bar{2}\ 3\ \bar{0}\ 1\ 2\ \bar{3})$

$y \sim (1\ \bar{1})(2\ 3)(\bar{2}\ \bar{3})$

and t $\sim t_0$

The following relations may be used for the purpose of manual double coset enumeration:

R1. $(xt)^{22} = 1$

R2. $(yxt)^5 = 1$

R3. $(x^2t)^6 = 1$

R4. $(x^3yt)^6 = 1$

R5. $(yt^{x^6})^{10} = 1$

If we factor $3^{*4} :_m 2^{\cdot}S_4^+$ by relations 1-5, although R2 suffices, we will verify that its homomorphic image is a group isomorphic to $M_{11} \times 2$.

Thus it is our objective to demonstrate that $\dfrac{3^{*4} :_m 2^{\cdot}S_4^+}{(yxt)^5 = 1}$

$\cong < x, y, t \mid x^4 = (yx)^3 = y^2 = 1 = t^3 = (t, y) = (yt^x)^2 = (xt)^{22} =$

$(yxt)^5 = (x^2t)^6 = (x^3yt)^6 = (yt^{x^6})^{10} >$ is isomorphic to $M_{11} \times 2$.

We now perform manual double coset enumeration of G $= \dfrac{3^{*4} :_m 2^{\cdot}S_4^+}{(yxt)^5 = 1}$ over ${}_m2^{\cdot}S_4^+$.

We start with the double coset with representative word of length zero, namely $NeN = \{Nen | n \in N\} = \{Nnn^{-1}en | n \in N\} = \{Ne^n | n \in N\} =$ N. In our Cayley Diagram, we'll denote this double coset N by [*]. The order of our control group is of order 48 and we list each element below:

e, $(12\bar{3}0\bar{1}\bar{2}3\bar{0})$, $(1\bar{0}3\bar{2}\bar{1}0\bar{3}2)$, $(1\bar{3}\bar{1}3)(20\bar{2}\bar{0})$, $(13\bar{0}\bar{1}\bar{3}0)(2\bar{2})$, $(13\bar{1}\bar{3})(2\bar{0}\bar{2}0)$, $(1032\bar{1}\bar{0}\bar{3}\bar{2})$, $(1\bar{2})(2\bar{1})(0\bar{0})$, $(1\bar{1})(23)(\bar{2}\bar{3})$, $(1\bar{2}0\bar{1}2\bar{0})(3\bar{3})$, $(10\bar{3}\bar{1}\bar{0}3)(2\bar{2})$, $(130\bar{2}\bar{1}\bar{3}\bar{0}2)$, $(1\bar{3}2\bar{0}\bar{1}3\bar{2}0)$, $(1\bar{3}02\bar{1}3\bar{0}\bar{2})$, $(1\bar{0}\bar{3})(3\bar{1}0)$, $(123\bar{1}\bar{2}\bar{3})(0\bar{0})$, $(1\bar{2}3)(2\bar{3}\bar{1})$, $(1320\bar{1}\bar{3}\bar{2}\bar{0})$, $(1\bar{1})(3\bar{0})(0\bar{3})$, $(1\bar{0}2\bar{1}0\bar{2})(3\bar{3})$, $(120)(\bar{1}\bar{2}\bar{0})$, $(102)(\bar{1}\bar{0}\bar{2})$, $(2\bar{2})(30)(\bar{3}\bar{0})$, $(2\bar{3})(3\bar{2})(0\bar{0})$, $(12)(3\bar{3})(\bar{1}\bar{2})$, $(1\bar{3}\bar{0})(30\bar{1})$, $(2\bar{0}3)(0\bar{3}\bar{2})$, $(10\bar{2}3\bar{1}\bar{0}2\bar{3})$, $(1\bar{2}\bar{3}\bar{0}\bar{1}230)$, $(1\bar{0}\bar{2}\bar{3}\bar{1}023)$, $(1\bar{1})(203\bar{2}\bar{0}\bar{3})$, $(1\bar{1})(2\bar{2})(3\bar{3})(0\bar{0})$, $(1\bar{3})(2\bar{2})(3\bar{1})$, $(10\bar{1}\bar{0})(23\bar{2}\bar{3})$, $(12\bar{1}\bar{2})(30\bar{3}\bar{0})$, $(1\bar{1})(2\bar{0})(0\bar{2})$, $(10)(3\bar{3})(\bar{1}\bar{0})$, $(20)(3\bar{3})(\bar{2}\bar{0})$, $(1\bar{2}\bar{1}2)(3\bar{0}\bar{3}0)$, $(1\bar{3}\bar{2}\bar{1}32)(0\bar{0})$, $(13\bar{2})(2\bar{1}\bar{3})$, $(1\bar{0})(2\bar{2})(0\bar{1})$, $(1\bar{2}\bar{0}3\bar{1}20\bar{3})$, $(23\bar{0})(0\bar{2}\bar{3})$, $(1\bar{0}\bar{1}0)(2\bar{3}\bar{2}3)$, $(1\bar{1})(2\bar{3}\bar{0}\bar{2}30)$, $(12\bar{0}3\bar{1}\bar{2}03)$, $(13)(0\bar{0})(\bar{1}\bar{3})$.

Now, our control group is transitive on the set of symmetric generators $\{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. Hence, the double coset N has only orbit, namely $\{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. It suffices to choose one representative from this orbit and ask to which double coset does Nt_i belong. Let us consider Nt_0. Because the first double coset Nen = [*] consists of only our control group, Nt_0 is not represented within it. Hence, Nt_0 is an element of the new double coset Nt_0N, denoted [0] in our Cayley Diagram. In fact, because Nt_0, Nt_1, Nt_2, Nt_3, Nt_0^{-1}, Nt_1^{-1}, Nt_2^{-1}, and Nt_3^{-1} are in the same orbit, they all are elements of [0].

Thus, $Nt_0N = \{N(t_0)^n | n \in N\} = \{Nt_0, Nt_1, Nt_2, Nt_3, Nt_0^{-1}, Nt_1^{-1}, Nt_2^{-1}, Nt_3^{-1}\}$ denoted by [0].

Now we determine to which double coset Nt_0t_i belongs for one t_i from each orbit of $N^{(0)}$.

$$\begin{aligned} N^{(0)} = N^0 &= \{n \in N | N(t_0)^n = Nt_0\} \\ &= \{e, (1\bar{1})(23)(\bar{2}\bar{3}), (1\bar{2}3)(2\bar{3}\bar{1}), (12)(3\bar{3})(\bar{1}\bar{2}), (1\bar{3})(2\bar{2})(3\bar{1}), (13\bar{2})(2\bar{1}\bar{3}\} \end{aligned}$$

So $N^{(0)}$ has orbits $\{0\}$, $\{\bar{0}\}$, and $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$ on $\{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$.
It suffices to choose one representative from each orbit and ask to which double coset does Nt_0t_i belong.

$0\bar{0} = e \Rightarrow Nt_0t_0^{-1} = \text{Ne} = \text{N} \in \text{NeN} = [*]$.
$00 = \bar{0} \Rightarrow Nt_0t_0 = Nt_0^{-1} \in Nt_0N = [0]$.
Nt_0t_1 is neither represented in [*] nor [0]. Therefore Nt_0t_1N, denoted [01], is a new double coset.

Now [01] is a new double coset with representative word of length two.
We notice that $N^{(01)} = \{n \in N | N(t_0t_1)^n = Nt_0t_1\} = \{e\}$.
We know that the number of distinct single cosets within [01] is $\frac{|N|}{|N^{(01)}|} = \frac{48}{1}$.
Therefore [01] has 48 distinct single cosets.
Each distinct single coset is given by $Nt_0t_1N = \{N(t_0t_1)^n | n \in N\}$ and is listed below:

$$Nt_0t_1, Nt_1^{-1}t_2, Nt_3^{-1}t_0^{-1}, Nt_2^{-1}t_3^{-1}, Nt_1t_3, Nt_2t_3$$
$$Nt_3t_0, Nt_0^{-1}t_2^{-1}, Nt_0t_1^{-1}, Nt_1^{-1}t_2^{-1}, Nt_3^{-1}t_0, Nt_2^{-1}t_3$$
$$Nt_1t_3^{-1}, Nt_2t_3^{-1}, Nt_3t_0^{-1}, Nt_0^{-1}t_2, Nt_0t_2^{-1}, Nt_1^{-1}t_3$$
$$Nt_3^{-1}t_1^{-1}, Nt_2^{-1}t_0^{-1}, Nt_1t_2, Nt_2t_0, Nt_3t_1, Nt_0^{-1}t_1$$
$$Nt_0t_2, Nt_1^{-1}t_3^{-1}, Nt_3^{-1}t_1, Nt_2^{-1}t_0, Nt_1t_2^{-1}, Nt_2t_0^{-1}$$
$$Nt_3t_1^{-1}, Nt_0^{-1}t_1^{-1}, Nt_0t_3^{-1}, Nt_1^{-1}t_0, Nt_3^{-1}t_2, Nt_2^{-1}t_1^{-1}$$
$$Nt_1t_0, Nt_2t_1, Nt_3t_2^{-1}, Nt_0^{-1}t_3^{-1}, Nt_0t_3, Nt_1^{-1}t_0^{-1}$$
$$Nt_3^{-1}t_2^{-1}, Nt_2^{-1}t_1, Nt_1t_0^{-1}, Nt_2t_1^{-1}, Nt_3t_2, Nt_0^{-1}t_3$$

Let us consider $N^{(01)}$ and then determine to which double coset $Nt_0t_1t_i$ belongs for one t_i from each orbit of $N^{(01)}$.

By examining the coset stabilizing group for [01], we see that the orbits of $N^{(01)}$ on $\{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.

It suffices to choose one representative from each orbit and ask to which double coset $Nt_0t_1t_i$ belongs.

MAGMA shows the third statement.

$01\bar{1} = 0 \Rightarrow Nt_0t_1t_1^{-1} = Nt_0 \in Nt_0N = [0]$.
$011 = 0\bar{1} \Rightarrow Nt_0t_1t_1 = Nt_0t_1^{-1} \in Nt_0t_1N = [01]$.
$013 = 01 \Rightarrow Nt_0t_1t_3 = Nt_0t_1 \in Nt_0t_1N = [01]$.

MAGMA also shows that [010], [012], $[01\bar{0}]$, $[01\bar{2}]$, and $[01\bar{3}]$ are new double cosets that have not been represented thus far.

Let us consider [010].

We note that $N^{(010)} = \{e\}$, implying that there exists 48 distinct single cosets within [010].

Each distinct single coset is given by $Nt_0t_1t_0N = \{N(t_0t_1t_0)^n | n \in N\}$ and is listed below:

$$Nt_0t_1t_0, Nt_{\bar{1}}^{-1}t_2t_{\bar{1}}^{-1}, Nt_{\bar{3}}^{-1}t_{\bar{0}}^{-1}t_{\bar{3}}^{-1}, Nt_{\bar{2}}^{-1}t_{\bar{3}}^{-1}t_{\bar{2}}^{-1}, Nt_1t_3t_1, Nt_2t_3t_2$$
$$Nt_3t_0t_3, Nt_{\bar{0}}^{-1}t_{\bar{2}}^{-1}t_{\bar{0}}^{-1}, Nt_0t_{\bar{1}}^{-1}t_0, Nt_{\bar{1}}^{-1}t_{\bar{2}}^{-1}t_{\bar{1}}^{-1}, Nt_{\bar{3}}^{-1}t_0t_{\bar{3}}^{-1}, Nt_{\bar{2}}^{-1}t_3t_{\bar{2}}^{-1}$$
$$Nt_1t_{\bar{3}}^{-1}t_1, Nt_2t_{\bar{3}}^{-1}t_2, Nt_3t_{\bar{0}}^{-1}t_3, Nt_{\bar{0}}^{-1}t_2t_{\bar{0}}^{-1}, Nt_0t_{\bar{2}}^{-1}t_0, Nt_{\bar{1}}^{-1}t_3t_{\bar{1}}^{-1}$$
$$Nt_{\bar{3}}^{-1}t_{\bar{1}}^{-1}t_{\bar{3}}^{-1}, Nt_{\bar{2}}^{-1}t_{\bar{0}}^{-1}t_{\bar{2}}^{-1}, Nt_1t_2t_1, Nt_2t_0t_2, Nt_3t_1t_3, Nt_{\bar{0}}^{-1}t_1t_{\bar{0}}^{-1}$$
$$Nt_0t_2t_0, Nt_{\bar{1}}^{-1}t_{\bar{3}}^{-1}t_{\bar{1}}^{-1}, Nt_{\bar{3}}^{-1}t_1t_{\bar{3}}^{-1}, Nt_{\bar{2}}^{-1}t_0t_{\bar{2}}^{-1}, Nt_1t_{\bar{2}}^{-1}t_1, Nt_2t_{\bar{0}}^{-1}t_2$$
$$Nt_3t_{\bar{1}}^{-1}t_3, Nt_{\bar{0}}^{-1}t_{\bar{1}}^{-1}t_{\bar{0}}^{-1}, Nt_0t_{\bar{3}}^{-1}t_0, Nt_{\bar{1}}^{-1}t_0t_{\bar{1}}^{-1}, Nt_{\bar{3}}^{-1}t_2t_{\bar{3}}^{-1}, Nt_{\bar{2}}^{-1}t_{\bar{1}}^{-1}t_{\bar{2}}^{-1}$$
$$Nt_1t_0t_1, Nt_2t_1t_2, Nt_3t_{\bar{2}}^{-1}t_3, Nt_{\bar{0}}^{-1}t_{\bar{3}}^{-1}t_{\bar{0}}^{-1}, Nt_0t_3t_0, Nt_{\bar{1}}^{-1}t_{\bar{0}}^{-1}t_{\bar{1}}^{-1}$$
$$Nt_{\bar{3}}^{-1}t_{\bar{2}}^{-1}t_{\bar{3}}^{-1}, Nt_{\bar{2}}^{-1}t_1t_{\bar{2}}^{-1}, Nt_1t_{\bar{0}}^{-1}t_1, Nt_2t_{\bar{1}}^{-1}t_2, Nt_3t_2t_3, Nt_{\bar{0}}^{-1}t_3t_{\bar{0}}^{-1}$$

Let us consider $N^{(010)}$ and then determine to which double coset $Nt_0t_1t_0t_i$ belongs for one t_i from each orbit of $N^{(010)}$.
Since $N^{(010)} = \{e\}$, we see that the orbits of $N^{(010)}$ on $\{0,1,2,3,\bar{0},\bar{1},\bar{2},\bar{3}\}$ are $\{0\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{\bar{0}\}$, $\{\bar{1}\}$, $\{\bar{2}\}$, and $\{\bar{3}\}$.
It suffices to choose one representative from each orbit and ask to which double coset $Nt_0t_1t_0t_i$ belongs.

$010\bar{0} = 01 \Rightarrow Nt_0t_1t_0t_{\bar{0}}^{-1} = Nt_0t_1 \in Nt_0t_1N = [01]$.

MAGMA shows the following four statements.
$0100 = 01\bar{0} \Rightarrow Nt_0t_1t_0t_0 = Nt_0t_1t_{\bar{0}}^{-1} \in Nt_0t_1t_{\bar{0}}^{-1}N = [01\bar{0}]$.
$010\bar{1} = 010 \Rightarrow Nt_0t_1t_0t_{\bar{1}}^{-1} = Nt_0t_1t_0 \in Nt_0t_1t_0N = [010]$.
$010\bar{3} = 010 \Rightarrow Nt_0t_1t_0t_{\bar{3}}^{-1} = Nt_0t_1t_0 \in Nt_0t_1t_0N = [010]$.
$0103 = 0121 \Rightarrow Nt_0t_1t_0t_3 = Nt_0t_1t_2t_1 \in Nt_0t_1t_2t_1N = [0121]$.

MAGMA also shows [0101], [0102], and [010$\bar{2}$] as new double cosets.

Let us consider [012].
MAGMA gives that $N^{(012)} = \{e, (1\bar{1})(2\bar{0})(0\bar{2})\}$, implying that there exists 24 distinct single cosets within [012]. Each distinct single coset is given by $Nt_0t_1t_2N = \{N(t_0t_1t_2)^n | n \in N\}$ and is listed below, along with its equivalent name:

$$
\begin{gathered}
012 \sim \bar{2}\bar{1}\bar{0} \\
\bar{1}2\bar{3} \sim 3\bar{2}1 \\
\bar{3}\bar{0}1 \sim \bar{1}03 \\
\bar{2}\bar{3}0 \sim \bar{0}32 \\
13\bar{2} \sim 2\bar{3}\bar{1} \\
23\bar{0} \sim 0\bar{3}\bar{2} \\
30\bar{1} \sim 1\bar{0}\bar{3} \\
\bar{0}\bar{2}\bar{1} \sim 120 \\
0\bar{1}3 \sim \bar{3}1\bar{0} \\
\bar{1}\bar{2}\bar{0} \sim 021 \\
\bar{3}0\bar{2} \sim 2\bar{0}3 \\
\bar{2}31 \sim \bar{1}\bar{3}2 \\
1\bar{3}\bar{0} \sim 03\bar{1} \\
3\bar{0}2 \sim \bar{2}0\bar{3} \\
\bar{0}23 \sim \bar{3}\bar{2}0 \\
0\bar{2}\bar{3} \sim 32\bar{0} \\
\bar{1}30 \sim \bar{0}\bar{3}1 \\
\bar{3}\bar{1}2 \sim \bar{2}13 \\
\bar{2}\bar{0}\bar{1} \sim 102 \\
201 \sim \bar{1}\bar{0}\bar{2} \\
31\bar{2} \sim 2\bar{1}\bar{3} \\
\bar{0}1\bar{3} \sim 3\bar{1}0 \\
1\bar{2}3 \sim \bar{3}2\bar{1} \\
\bar{0}\bar{1}\bar{2} \sim 210
\end{gathered}
$$

Let us consider $N^{(012)}$ and then determine to which double coset $Nt_0t_1t_2t_i$ belongs for one t_i from each orbit of $N^{(012)}$. Since $N^{(012)} = \{e, (1\bar{1})(2\bar{0})(0\bar{2})\}$, we see that the orbits of $N^{(012)}$ on $\{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ are $\{3\}$, $\{\bar{3}\}$, $\{1, \bar{1}\}$, $\{2, \bar{0}\}$, and $\{0, \bar{2}\}$. It suffices to choose one representative from each orbit and ask to which double coset $Nt_0t_1t_2t_i$ belongs.

$012\bar{2} = 01 \;\Rightarrow Nt_0t_1t_2t_2^{-1} = Nt_0t_1 \in Nt_0t_1N = [01]$.
$0122 = 01\bar{2} \;\Rightarrow Nt_0t_1t_2t_2 = Nt_0t_1t_2^{-1} \in Nt_0t_1t_2^{-1}N = [01\bar{2}]$.

MAGMA shows the following statement.

$0123 = 01\bar{3} \;\Rightarrow Nt_0t_1t_2t_3 = Nt_0t_1t_3^{-1} \in Nt_0t_1t_3^{-1}N = [01\bar{3}]$.

MAGMA also gives that [0121] and [$012\bar{3}$] are new double cosets.

Let us now consider double coset [$01\bar{0}$].
We note that $N^{(01\bar{0})} = \{e, (1\bar{0})(2\bar{2})(0\bar{1})\}$. Thus there exists 24 distinct single cosets within [$01\bar{0}$]. Each coset is given by $Nt_0t_1t_0^{-1}N = \{N(t_0t_1t_0^{-1})^n | n \in N\}$ and is listed below, along with its equivalent name:

$01\bar{0} \sim \bar{1}\bar{0}1$
$\bar{1}21 \sim \bar{2}12$
$\bar{3}\bar{0}3 \sim 03\bar{0}$
$\bar{2}\bar{3}2 \sim 32\bar{3}$
$13\bar{1} \sim \bar{3}\bar{1}3$
$23\bar{2} \sim \bar{3}\bar{2}3$
$30\bar{3} \sim \bar{0}\bar{3}3$
$\bar{0}\bar{2}0 \sim 20\bar{2}$
$0\bar{1}\bar{0} \sim 1\bar{0}\bar{1}$
$\bar{1}\bar{2}1 \sim 21\bar{2}$
$\bar{3}03 \sim \bar{0}30$
$\bar{2}32 \sim \bar{3}23$
$1\bar{3}\bar{1} \sim 3\bar{1}\bar{3}$
$2\bar{3}\bar{2} \sim 3\bar{2}\bar{3}$
$3\bar{0}\bar{3} \sim 0\bar{3}\bar{0}$
$\bar{0}20 \sim \bar{2}02$
$0\bar{2}\bar{0} \sim 2\bar{0}\bar{2}$
$\bar{1}31 \sim \bar{3}13$

$$\bar{2}\bar{0}2 \sim 02\bar{0}$$
$$12\bar{1} \sim \bar{2}\bar{1}2$$
$$31\bar{3} \sim \bar{1}\bar{3}1$$
$$\bar{0}10 \sim \bar{1}01$$
$$1\bar{2}\bar{1} \sim 2\bar{1}\bar{2}$$
$$\bar{0}\bar{1}0 \sim 10\bar{1}$$

Let us consider $N^{(01\bar{0})}$ and then determine to which double coset $Nt_0t_1t_0^{-1}t_i$ belongs for one t_i from each orbit of $N^{(01\bar{0})}$. Since $N^{(01\bar{0})} = \{e, (1\bar{0})(2\bar{2})(0\bar{1})\}$, we see that the orbits of $N^{(01\bar{0})}$ on $\{0,1,2,3,\bar{0},\bar{1},\bar{2},\bar{3}\}$ are $\{3\}$, $\{\bar{3}\}$, $\{1,\bar{0}\}$, $\{2,\bar{6}\}$, and $\{0,\bar{1}\}$. It suffices to choose one representative from each orbit and ask to which double coset $Nt_0t_1t_0^{-1}t_i$ belongs.

$01\bar{0}0 = 01 \Rightarrow Nt_0t_1t_0^{-1}t_0 = Nt_0t_1 \in Nt_0t_1N = [01]$.
$01\bar{0}\bar{0} = 010 \Rightarrow Nt_0t_1t_0^{-1}t_0^{-1} = Nt_0t_1t_0 \in Nt_0t_1t_0N = [010]$.

MAGMA shows the following three statements.

$01\bar{0}3 = 01\bar{2} \Rightarrow Nt_0t_1t_0^{-1}t_3 = Nt_0t_1t_2^{-1} \in Nt_0t_1t_2^{-1}N = [01\bar{2}]$.
$01\bar{0}2 = 01\bar{0} \Rightarrow Nt_0t_1t_0^{-1}t_2 = Nt_0t_1t_0^{-1} \in Nt_0t_1t_0^{-1}N = [01\bar{0}]$.
$01\bar{0}\bar{3} = 0102 \Rightarrow Nt_0t_1t_0^{-1}t_3^{-1} = Nt_0t_1t_0t_2 \in Nt_0t_1t_0t_2N = [0102]$.

Let us consider double coset $[01\bar{2}]$.
MAGMA shows that $N^{(01\bar{2})} = \{e, (20)(3\bar{3})(\bar{2}\bar{0})\}$, implying that there exists 24 distinct single cosets within $[01\bar{2}]$. Each coset is given by $Nt_0t_1t_2^{-1}N = \{N(t_0t_1t_2^{-1})^n | n \in N\}$ and is listed below, along with its equivalent name:

$$01\bar{2} \sim 21\bar{0}$$
$$\bar{1}23 \sim \bar{3}21$$
$$\bar{3}0\bar{1} \sim 1\bar{0}3$$
$$\bar{2}3\bar{0} \sim 0\bar{3}2$$
$$132 \sim \bar{2}3\bar{1}$$
$$230 \sim \bar{0}3\bar{2}$$

$$301 \sim \bar{1}0\bar{3}$$
$$\bar{0}\bar{2}1 \sim \bar{1}\bar{2}0$$
$$0\bar{1}\bar{3} \sim 3\bar{1}\bar{0}$$
$$\bar{3}02 \sim \bar{2}03$$
$$1\bar{3}0 \sim \bar{0}\bar{3}\bar{1}$$
$$2\bar{3}1 \sim \bar{1}\bar{3}\bar{2}$$
$$3\bar{0}\bar{2} \sim 2\bar{0}\bar{3}$$
$$\bar{0}2\bar{3} \sim 320$$
$$0\bar{2}3 \sim \bar{3}\bar{2}\bar{0}$$
$$\bar{1}3\bar{0} \sim 031$$
$$\bar{3}\bar{1}\bar{2} \sim 2\bar{1}3$$
$$\bar{2}\bar{0}1 \sim \bar{1}\bar{0}2$$
$$12\bar{0} \sim 02\bar{1}$$
$$20\bar{1} \sim 10\bar{2}$$
$$312 \sim \bar{2}1\bar{3}$$
$$\bar{0}13 \sim \bar{3}10$$
$$1\bar{2}\bar{3} \sim 3\bar{2}\bar{1}$$
$$\bar{0}\bar{1}2 \sim \bar{2}\bar{1}0$$

Let us consider $N^{(01\bar{2})}$ and then determine to which double coset $Nt_0t_1t_2^{-1}t_i$ belongs for one t_i from each orbit of $N^{(01\bar{2})}$. Since $N^{(01\bar{2})} = \{e, (20)(3\bar{3})(\bar{2}\bar{0})\}$, we see that the orbits of $N^{(01\bar{2})}$ on $\{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ are $\{1\}$, $\{\bar{1}\}$, $\{2, 0\}$, $\{\bar{2}, \bar{0}\}$, and $\{3, \bar{3}\}$. It suffices to choose one representative from each orbit and ask to which double coset $Nt_0t_1t_2^{-1}t_i$ belongs.

$01\bar{2}2 = 01 \;\Rightarrow Nt_0t_1t_2^{-1}t_2 = Nt_0t_1 \in Nt_0t_1N = [01]$.
$01\bar{2}\bar{2} = 012 \;\Rightarrow Nt_0t_1t_2^{-1}t_2^{-1} = Nt_0t_1t_2 \in Nt_0t_1t_2N = [012]$.

MAGMA shows the following two statements.

$01\bar{2}\bar{1} = 01\bar{0} \;\Rightarrow Nt_0t_1t_2^{-1}t_1^{-1} = Nt_0t_1t_0^{-1} \in Nt_0t_1t_0^{-1}N = [01\bar{0}]$.
$01\bar{2}1 = 0102 \;\Rightarrow Nt_0t_1t_2^{-1}t_1 = Nt_0t_1t_0t_2 \in Nt_0t_1t_0t_2N = [0102]$.

MAGMA also shows that $[01\bar{2}3]$ is a new double coset.

Let us consider double coset $[01\bar{3}]$.
We have $N^{(01\bar{3})} = \{e, (10)(3\bar{3})(\bar{1}\bar{0})\}$. So, there exists 24 distinct single cosets within $[01\bar{3}]$. Each coset is given by $Nt_0t_1t_3^{-1}N = \left\{N(t_0t_1t_3^{-1})^n | n \in N\right\}$ and is listed below, along with its equivalent name:

$01\bar{3} \sim 103$
$\bar{1}20 \sim 2\bar{1}\bar{0}$
$\bar{3}\bar{0}2 \sim \bar{0}\bar{3}\bar{2}$
$\bar{2}\bar{3}\bar{1} \sim \bar{3}\bar{2}1$
$130 \sim 31\bar{0}$
$231 \sim 32\bar{1}$
$30\bar{2} \sim 032$
$\bar{0}\bar{2}\bar{3} \sim \bar{2}\bar{0}3$
$0\bar{1}\bar{2} \sim \bar{1}02$
$\bar{1}\bar{2}3 \sim \bar{2}\bar{1}\bar{3}$
$\bar{3}0\bar{1} \sim 0\bar{3}1$
$\bar{2}3\bar{0} \sim 3\bar{2}0$
$1\bar{3}2 \sim \bar{3}1\bar{2}$
$2\bar{3}0 \sim \bar{3}2\bar{0}$
$3\bar{0}1 \sim \bar{0}3\bar{1}$
$\bar{0}21 \sim 2\bar{0}\bar{1}$
$0\bar{2}\bar{1} \sim \bar{2}01$
$\bar{1}3\bar{2} \sim 3\bar{1}2$
$\bar{3}\bar{1}0 \sim \bar{1}\bar{3}\bar{0}$
$12\bar{3} \sim 213$
$20\bar{3} \sim 023$
$\bar{0}12 \sim 1\bar{0}\bar{2}$
$1\bar{2}\bar{0} \sim \bar{2}10$
$3\bar{1}2 \sim \bar{1}3\bar{2}$

Let us consider $N^{(01\bar{3})}$ and then determine to which double coset $Nt_0t_1t_3^{-1}t_i$ belongs for one t_i from each orbit of $N^{(01\bar{3})}$. Since $N^{(01\bar{3})} = \{e, (10)(3\bar{3})(\bar{1}\bar{0})\}$, we see that the orbits of $N^{(01\bar{3})}$ on $\{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ are $\{0, 1\}$, $\{\bar{0}, \bar{1}\}$, $\{3, \bar{3}\}$, $\{2\}$, and $\{\bar{2}\}$. It suffices to choose one representative from each orbit and ask to which double coset $Nt_0t_1t_3^{-1}t_i$ belongs.

$01\bar{3}3 = 01 \Rightarrow Nt_0t_1t_3^{-1}t_0 = Nt_0t_1 \in Nt_0t_1N = [01]$.

MAGMA shows the following four statements.
$01\bar{3}\bar{2} = 012 \Rightarrow Nt_0t_1t_3^{-1}t_2^{-1} = Nt_0t_1t_2 \in Nt_0t_1t_2N = [012]$.
$01\bar{3}\bar{1} = 01\bar{3} \Rightarrow Nt_0t_1t_3^{-1}t_1^{-1} = Nt_0t_1t_3^{-1} \in Nt_0t_1t_3^{-1}N = [01\bar{3}]$.
$01\bar{3}2 = 012\bar{3} \Rightarrow Nt_0t_1t_3^{-1}t_2 = Nt_0t_1t_2t_3^{-1} \in Nt_0t_1t_2t_3^{-1}N = [012\bar{3}]$.
$01\bar{3}1 = 010\bar{2} \Rightarrow Nt_0t_1t_3^{-1}t_1 = Nt_0t_1t_0t_2^{-1} \in Nt_0t_1t_0t_2^{-1}N = [010\bar{2}]$.

Let us consider double coset [0121].
Now $N^{(0121)} = \{e, (1\bar{0})(2\bar{2})(0\bar{1})\}$. Thus, there exists 24 distinct single cosets within [0121]. Each coset is given by $Nt_0t_1t_2t_1N = \{N(t_0t_1t_2t_1)^n | n \in N\}$ and is listed below, along with its equivalent name:

$$
\begin{gathered}
0121 \sim \bar{1}\bar{0}\bar{2}\bar{0} \\
\bar{1}2\bar{3}2 \sim \bar{2}131 \\
\bar{3}01\bar{0} \sim 03\bar{1}3 \\
\bar{2}30\bar{3} \sim 32\bar{0}2 \\
13\bar{2}3 \sim \bar{3}\bar{1}2\bar{1} \\
23\bar{0}3 \sim \bar{3}\bar{2}0\bar{2} \\
30\bar{1}0 \sim \bar{0}\bar{3}1\bar{3} \\
\bar{0}\bar{2}\bar{1}\bar{2} \sim 2010 \\
0\bar{1}3\bar{1} \sim 1\bar{0}\bar{3}\bar{0} \\
\bar{1}2\bar{0}2 \sim 2102 \\
\bar{3}0\bar{2}0 \sim \bar{0}323 \\
\bar{2}313 \sim \bar{3}2\bar{1}2 \\
1\bar{3}0\bar{3} \sim 3\bar{1}0\bar{1} \\
2\bar{3}\bar{1}\bar{3} \sim 3\bar{2}1\bar{2}
\end{gathered}
$$

$$3\bar{0}2\bar{0} \sim 0\bar{3}\bar{2}\bar{3}$$
$$\bar{0}232 \sim \bar{2}0\bar{3}0$$
$$0\bar{2}\bar{3}\bar{2} \sim 2\bar{0}3\bar{0}$$
$$\bar{1}303 \sim \bar{3}1\bar{0}1$$
$$\bar{2}\bar{0}\bar{1}\bar{0} \sim 0212$$
$$1202 \sim \bar{2}\bar{1}\bar{0}\bar{1}$$
$$31\bar{2}1 \sim \bar{1}\bar{3}2\bar{3}$$
$$\bar{0}1\bar{3}1 \sim \bar{1}030$$
$$1\bar{2}3\bar{2} \sim 2\bar{1}\bar{3}\bar{1}$$
$$\bar{0}\bar{1}\bar{2}\bar{1} \sim 1020$$

Let us consider $N^{(0121)}$ and then determine to which double coset $Nt_0t_1t_2t_1t_i$ belongs for one t_i from each orbit of $N^{(0121)}$. Since $N^{(0121)} = \{e, (1\bar{0})(2\bar{2})(0\bar{1})\}$, we see that the orbits of $N^{(0121)}$ on $\{0,1,2,3,\bar{0},\bar{1},\bar{2},\bar{3}\}$ are $\{0,\bar{1}\}$, $\{\bar{0},1\}$, $\{2,\bar{2}\}$, $\{3\}$, and $\{\bar{3}\}$. It suffices to choose one representative from each orbit and ask to which double coset $Nt_0t_1t_2t_1t_i$ belongs.

$0121\bar{1} = 012 \Rightarrow Nt_0t_1t_2t_1t_1^{-1} = Nt_0t_1t_2 \in Nt_0t_1t_2N = [012]$.

MAGMA shows the following four statements.
$01213 = 01\bar{2}3 \Rightarrow Nt_0t_1t_2t_1t_3 = Nt_0t_1t_2^{-1}t_3 \in Nt_0t_1t_2^{-1}t_3N = [01\bar{2}3]$.
$0121\bar{3} = 010\bar{2} \Rightarrow Nt_0t_1t_2t_1t_3^{-1} = Nt_0t_1t_0t_2^{-1} \in Nt_0t_1t_0t_2^{-1}N = [010\bar{2}]$.
$01211 = 0121 \Rightarrow Nt_0t_1t_2t_1t_1 = Nt_0t_1t_2t_1 \in Nt_0t_1t_2t_1N = [0121]$.
$01212 = 010 \Rightarrow Nt_0t_1t_2t_1t_2 = Nt_0t_1t_0 \in Nt_0t_1t_0N = [010]$.

Let us consider double coset $[012\bar{3}]$.
$N^{(012\bar{3})} = \{e, (1\bar{1})(2\bar{0})(0\bar{2})\}$ implies that there exists 24 distinct single cosets within $[012\bar{3}]$. Each coset is given by $Nt_0t_1t_2t_3^{-1}N = \{N(t_0t_1t_2t_3^{-1})^n | n \in N\}$ and is listed below, along with its equivalent name:

$$012\bar{3} \sim \bar{2}\bar{1}\bar{0}\bar{3}$$
$$\bar{1}2\bar{3}0 \sim 3\bar{2}10$$

$$\bar{3}\bar{0}12 \sim \bar{1}032$$
$$\bar{2}\bar{3}0\bar{1} \sim \bar{0}32\bar{1}$$
$$13\bar{2}0 \sim 2\bar{3}\bar{1}0$$
$$23\bar{0}1 \sim 0\bar{3}\bar{2}1$$
$$30\bar{1}\bar{2} \sim 1\bar{0}\bar{3}\bar{2}$$
$$\bar{0}\bar{2}\bar{1}\bar{3} \sim 120\bar{3}$$
$$0\bar{1}3\bar{2} \sim \bar{3}1\bar{0}\bar{2}$$
$$\bar{1}\bar{2}\bar{0}3 \sim 0213$$
$$\bar{3}0\bar{2}\bar{1} \sim 2\bar{0}3\bar{1}$$
$$\bar{2}31\bar{0} \sim \bar{1}\bar{3}2\bar{0}$$
$$1\bar{3}\bar{0}2 \sim 03\bar{1}2$$
$$3\bar{0}21 \sim \bar{2}0\bar{3}1$$
$$\bar{0}231 \sim \bar{3}\bar{2}01$$
$$0\bar{2}\bar{3}\bar{1} \sim 32\bar{0}\bar{1}$$
$$\bar{1}30\bar{2} \sim \bar{0}\bar{3}1\bar{2}$$
$$\bar{3}\bar{1}20 \sim \bar{2}130$$
$$\bar{2}\bar{0}\bar{1}3 \sim 1023$$
$$201\bar{3} \sim \bar{1}\bar{0}\bar{2}\bar{3}$$
$$31\bar{2}\bar{0} \sim 2\bar{1}\bar{3}\bar{0}$$
$$\bar{0}1\bar{3}2 \sim 3\bar{1}02$$
$$1\bar{2}3\bar{0} \sim \bar{3}2\bar{1}\bar{0}$$
$$\bar{0}\bar{1}\bar{2}3 \sim 2103$$

Let us consider $N^{(012\bar{3})}$ and then determine to which double coset $Nt_0t_1t_2t_3^{-1}t_i$ belongs for one t_i from each orbit of $N^{(012\bar{3})}$. Since $N^{(012\bar{3})} = \{e, (1\bar{1})(2\bar{0})(0\bar{2})\}$, we see that the orbits of $N^{(012\bar{3})}$ on $\{0,1,2,3,\bar{0},\bar{1},\bar{2},\bar{3}\}$ are $\{0,\bar{2}\}$, $\{\bar{0},2\}$, $\{1,\bar{1}\}$, $\{3\}$, and $\{\bar{3}\}$. It suffices to choose one representative from each orbit and ask to which double coset $Nt_0t_1t_2t_3^{-1}t_i$ belongs.

$012\bar{3}3 = 012 \;\Rightarrow Nt_0t_1t_2t_3^{-1}t_3 = Nt_0t_1t_2 \in Nt_0t_1t_2N = [012]$.

MAGMA shows the following four statements.

$01\bar{2}\bar{3}1 = 012\bar{3} \Rightarrow Nt_0t_1t_2t_3^{-1}t_1 = Nt_0t_1t_2t_3^{-1} \in Nt_0t_1t_2t_3^{-1}N = [012\bar{3}]$.
$012\bar{3}2 = 0102 \Rightarrow Nt_0t_1t_2t_3^{-1}t_2 = Nt_0t_1t_0t_2 \in Nt_0t_1t_0t_2N = [0102]$.
$012\bar{3}0 = 012\bar{3} \Rightarrow Nt_0t_1t_2t_3^{-1}t_0 = Nt_0t_1t_2t_3^{-1} \in Nt_0t_1t_2t_3^{-1}N = [012\bar{3}]$.
$012\bar{3}\bar{3} = 01\bar{3} \Rightarrow Nt_0t_1t_2t_3^{-1}t_3^{-1} = Nt_0t_1t_3^{-1} \in Nt_0t_1t_3^{-1}N = [01\bar{3}]$.

Let us consider double coset [0101].
$N^{(0101)} = \{e, (13\bar{2})(2\bar{1}\bar{3}), (1\bar{3})(2\bar{2})(3\bar{1}), (1\bar{2}3)(2\bar{3}\bar{1}), (1\bar{1})(23)(\bar{2}\bar{3}), (12)(3\bar{3})(\bar{1}\bar{2})\}$, implies that there exists 8 distinct single cosets within [0101]. Each coset is given by $Nt_0t_1t_0t_1N = \{N(t_0t_1t_0t_1)^n | n \in N\}$ and is listed below, along with its equivalent names:

$$0101 \sim 0303 \sim 0\bar{3}0\bar{3} \sim 0\bar{2}0\bar{2} \sim 0\bar{1}0\bar{1} \sim 0202$$
$$\bar{1}2\bar{1}2 \sim \bar{1}\bar{0}\bar{1}\bar{0} \sim \bar{1}0\bar{1}0 \sim \bar{1}3\bar{1}3 \sim \bar{1}\bar{2}\bar{1}\bar{2} \sim \bar{1}\bar{3}\bar{1}\bar{3}$$
$$\bar{3}0\bar{3}0 \sim \bar{3}\bar{2}\bar{3}\bar{2} \sim \bar{3}2\bar{3}2 \sim \bar{3}\bar{1}\bar{3}\bar{1} \sim \bar{3}\bar{0}\bar{3}\bar{0} \sim \bar{3}1\bar{3}1$$
$$\bar{2}\bar{3}\bar{2}\bar{3} \sim \bar{2}1\bar{2}1 \sim \bar{2}\bar{1}\bar{2}\bar{1} \sim \bar{2}\bar{0}\bar{2}\bar{0} \sim \bar{2}3\bar{2}3 \sim \bar{2}0\bar{2}0$$
$$1313 \sim 1\bar{0}1\bar{0} \sim 1010 \sim 1212 \sim 1\bar{3}1\bar{3} \sim 1\bar{2}1\bar{2}$$
$$2323 \sim 2\bar{1}2\bar{1} \sim 2121 \sim 2020 \sim 2\bar{3}2\bar{3} \sim 2\bar{0}2\bar{0}$$
$$3030 \sim 3232 \sim 3\bar{2}3\bar{2} \sim 3131 \sim 3\bar{0}3\bar{0} \sim 3\bar{1}3\bar{1}$$
$$\bar{0}\bar{2}\bar{0}\bar{2} \sim \bar{0}3\bar{0}3 \sim \bar{0}\bar{3}\bar{0}\bar{3} \sim \bar{0}1\bar{0}1 \sim \bar{0}2\bar{0}2 \sim \bar{0}\bar{1}\bar{0}\bar{1}$$

Let us consider $N^{(0101)}$ and then determine to which double coset $Nt_0t_1t_0t_1t_i$ belongs for one t_i from each orbit of $N^{(0101)}$.
Since $N^{(0101)} = \{e, (13\bar{2})(2\bar{1}\bar{3}), (1\bar{3})(2\bar{2})(3\bar{1}), (1\bar{2}3)(2\bar{3}\bar{1}), (1\bar{1})(23)(\bar{2}\bar{3}), (12)(3\bar{3})(\bar{1}\bar{2})\}$, we see that the orbits of $N^{(0101)}$ on $\{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ are $\{0\}$, $\{\bar{0}\}$, and $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$. It suffices to choose one representative from each orbit and ask to which double coset $Nt_0t_1t_0t_1t_i$ belongs.

$0101\bar{1} = 010 \Rightarrow Nt_0t_1t_0t_1t_1^{-1} = Nt_0t_1t_0 \in Nt_0t_1t_0N = [010]$.

MAGMA shows the following statement:
$0101\bar{0} = 0101 \Rightarrow Nt_0t_1t_0t_1t_0^{-1} = Nt_0t_1t_0t_1 \in Nt_0t_1t_0t_1N = [0101]$.

MAGMA also shows that [01010] is a new double coset.

Let us consider double coset [0102].

MAGMA provides $N^{(0102)} = \{e, (20)(3\bar{3})(\bar{2}\bar{0})\}$, implying that there exists 24 distinct single cosets within [0102]. Each coset is given by $Nt_0t_1t_0t_2N = \{N(t_0t_1t_0t_2)^n | n \in N\}$ and is listed below, along with its equivalent names:

$$0102 \sim 2120$$
$$\bar{1}2\bar{1}\bar{3} \sim \bar{3}2\bar{3}\bar{1}$$
$$\bar{3}\bar{0}\bar{3}1 \sim 1\bar{0}1\bar{3}$$
$$\bar{2}\bar{3}\bar{2}0 \sim 0\bar{3}0\bar{2}$$
$$131\bar{2} \sim \bar{2}3\bar{2}1$$
$$232\bar{0} \sim \bar{0}3\bar{0}2$$
$$303\bar{1} \sim \bar{1}0\bar{1}3$$
$$\bar{0}\bar{2}\bar{0}\bar{1} \sim \bar{1}\bar{2}\bar{1}\bar{0}$$
$$0\bar{1}03 \sim 3\bar{1}30$$
$$\bar{3}0\bar{3}\bar{2} \sim \bar{2}0\bar{2}\bar{3}$$
$$1\bar{3}1\bar{0} \sim \bar{0}\bar{3}\bar{0}1$$
$$2\bar{3}2\bar{1} \sim \bar{1}\bar{3}\bar{1}2$$
$$3\bar{0}32 \sim 2\bar{0}23$$
$$\bar{0}2\bar{0}3 \sim 323\bar{0}$$
$$0\bar{2}0\bar{3} \sim \bar{3}\bar{2}\bar{3}0$$
$$\bar{1}3\bar{1}0 \sim 030\bar{1}$$
$$\bar{3}\bar{1}\bar{3}2 \sim 2\bar{1}2\bar{3}$$
$$\bar{2}\bar{0}\bar{2}\bar{1} \sim \bar{1}\bar{0}\bar{1}\bar{2}$$
$$1210 \sim 0201$$
$$2021 \sim 1012$$
$$313\bar{2} \sim \bar{2}1\bar{2}3$$
$$\bar{0}1\bar{0}\bar{3} \sim \bar{3}1\bar{3}\bar{0}$$
$$1\bar{2}13 \sim 3\bar{2}31$$
$$\bar{0}\bar{1}\bar{0}\bar{2} \sim \bar{2}\bar{1}\bar{2}\bar{0}$$

Let us consider $N^{(0102)}$ and then determine to which double coset $Nt_0t_1t_0t_2t_i$ belongs for one t_i from each orbit of $N^{(0102)}$. Since $N^{(0102)} = \{e, (20)(3\bar{3})(\bar{2}\bar{0})\}$, we see that the orbits of $N^{(0102)}$ on $\{0,1,2,3,\bar{0},\bar{1},\bar{2},\bar{3}\}$ are $\{0,2\}$, $\{\bar{0},\bar{2}\}$, $\{3,\bar{3}\}$, $\{1\}$, and $\{\bar{1}\}$. It suffices to choose one representative from each orbit and ask to which double coset $Nt_0t_1t_0t_2t_i$ belongs.

$0102\bar{2} = 010 \Rightarrow Nt_0t_1t_0t_2t_2^{-1} = Nt_0t_1t_0 \in Nt_0t_1t_0N = [010]$.
$01022 = 010\bar{2} \Rightarrow Nt_0t_1t_0t_2t_2 = Nt_0t_1t_0t_2^{-1} \in Nt_0t_1t_0t_2^{-1}N = [010\bar{2}]$.

MAGMA shows the following three statements.
$01023 = 012\bar{3} \Rightarrow Nt_0t_1t_0t_2t_3 = Nt_0t_1t_2t_3^{-1} \in Nt_0t_1t_2t_3^{-1}N = [012\bar{3}]$.
$01021 = 01\bar{0} \Rightarrow Nt_0t_1t_0t_2t_1 = Nt_0t_1t_0^{-1} \in Nt_0t_1t_0^{-1}N = [01\bar{0}]$.
$0102\bar{1} = 01\bar{2} \Rightarrow Nt_0t_1t_0t_2t_1^{-1} = Nt_0t_1t_2^{-1} \in Nt_0t_1t_2^{-1}N = [01\bar{2}]$.

Let us consider double coset $[010\bar{2}]$.
Since $N^{(010\bar{2})} = \{e, (1\bar{1})(2\bar{0})(0\bar{2})\}$, there exists 24 distinct single cosets within $[010\bar{2}]$. Each coset is given by $Nt_0t_1t_0t_2^{-1}N = \{N(t_0t_1t_0t_2^{-1})^n | n \in N\}$ and is listed below, along with its equivalent names:

$$010\bar{2} \sim \bar{2}\bar{1}\bar{2}0$$
$$\bar{1}2\bar{1}3 \sim 3\bar{2}3\bar{1}$$
$$\bar{3}\bar{0}\bar{3}\bar{1} \sim \bar{1}0\bar{1}\bar{3}$$
$$\bar{2}\bar{3}\bar{2}\bar{0} \sim \bar{0}3\bar{0}\bar{2}$$
$$1312 \sim 2\bar{3}21$$
$$2320 \sim 0\bar{3}02$$
$$3031 \sim 1\bar{0}13$$
$$\bar{0}\bar{2}\bar{0}1 \sim 121\bar{0}$$
$$0\bar{1}0\bar{3} \sim \bar{3}1\bar{3}0$$
$$\bar{1}\bar{2}\bar{1}0 \sim 020\bar{1}$$
$$\bar{3}0\bar{3}2 \sim 2\bar{0}2\bar{3}$$
$$\bar{2}3\bar{2}\bar{1} \sim \bar{1}\bar{3}\bar{1}\bar{2}$$
$$1\bar{3}10 \sim 0301$$
$$3\bar{0}3\bar{2} \sim \bar{2}0\bar{2}3$$

$$\bar{0}2\bar{0}\bar{3} \sim \bar{3}\bar{2}\bar{3}\bar{0}$$
$$0\bar{2}03 \sim 3230$$
$$\bar{1}3\bar{1}\bar{0} \sim \bar{0}\bar{3}\bar{0}\bar{1}$$
$$\bar{3}\bar{1}\bar{3}\bar{2} \sim \bar{2}\bar{2}\bar{3}$$
$$\bar{2}\bar{0}\bar{2}1 \sim 101\bar{2}$$
$$202\bar{1} \sim \bar{1}\bar{0}\bar{1}2$$
$$3132 \sim 2\bar{1}23$$
$$\bar{0}1\bar{0}3 \sim 3\bar{1}3\bar{0}$$
$$1\bar{2}1\bar{3} \sim \bar{3}2\bar{3}1$$
$$\bar{0}\bar{1}\bar{0}2 \sim 212\bar{0}$$

Let us consider $N^{(010\bar{2})}$ and then determine to which double coset $Nt_0t_1t_0t_2^{-1}t_i$ belongs for one t_i from each orbit of $N^{(010\bar{2})}$. Since $N^{(010\bar{2})} = \{e, (1\bar{1})(2\bar{0})(0\bar{2})\}$, we see that the orbits of $N^{(010\bar{2})}$ on $\{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ are $\{0, \bar{2}\}$, $\{\bar{0}, 2\}$, $\{1, \bar{1}\}$, $\{3\}$, and $\{\bar{3}\}$. It suffices to choose one representative from each orbit and ask to which double coset $Nt_0t_1t_0t_2^{-1}t_i$ belongs.

$010\bar{2}2 = 010 \;\Rightarrow Nt_0t_1t_0t_2^{-1}t_2 = Nt_0t_1t_0 \in Nt_0t_1t_0N = [010]$.
$010\bar{2}\bar{2} = 0102 \;\Rightarrow Nt_0t_1t_0t_2^{-1}t_2^{-1} = Nt_0t_1t_0t_2 \in Nt_0t_1t_0t_2N = [0102]$.

MAGMA shows the following three statements.
$010\bar{2}3 = 01\bar{2}3 \;\Rightarrow Nt_0t_1t_0t_2^{-1}t_3 = Nt_0t_1t_2^{-1}t_3 \in Nt_0t_1t_2^{-1}t_3N = [01\bar{2}3]$.
$010\bar{2}\bar{3} = 0121 \;\Rightarrow Nt_0t_1t_0t_2^{-1}t_3^{-1} = Nt_0t_1t_2t_1 \in Nt_0t_1t_2t_1N = [0121]$.
$010\bar{2}1 = 01\bar{3} \;\Rightarrow Nt_0t_1t_0t_2^{-1}t_1 = Nt_0t_1t_3^{-1} \in Nt_0t_1t_3^{-1}N = [01\bar{3}]$.

Let us consider double coset $[01\bar{2}3]$.
MAGMA provides $N^{(01\bar{2}3)} = \{e, (1\bar{3})(2\bar{2})(3\bar{1})\}$, implying that there exists 24 distinct single cosets within $[01\bar{2}3]$. Each coset is given by $Nt_0t_1t_2^{-1}t_3N = \{N(t_0t_1t_2^{-1}t_3)^n | n \in N\}$ and is listed below, along with its equivalent names:

$$01\bar{2}3 \sim 0\bar{3}2\bar{1}$$
$$\bar{1}23\bar{0} \sim \bar{1}0\bar{3}\bar{2}$$

$$\bar{3}\bar{0}\bar{1}\bar{2} \sim \bar{3}210$$
$$\bar{2}\bar{3}\bar{0}1 \sim \bar{2}\bar{1}03$$
$$132\bar{0} \sim 10\bar{2}\bar{3}$$
$$230\bar{1} \sim 21\bar{0}\bar{3}$$
$$3012 \sim 3\bar{2}\bar{1}\bar{0}$$
$$\bar{0}\bar{2}13 \sim \bar{0}\bar{3}\bar{1}2$$
$$0\bar{1}\bar{3}2 \sim 0\bar{2}31$$
$$\bar{1}\bar{2}0\bar{3} \sim \bar{1}3\bar{0}2$$
$$\bar{3}021 \sim \bar{3}\bar{1}\bar{2}\bar{0}$$
$$\bar{2}3\bar{1}0 \sim \bar{2}\bar{0}1\bar{3}$$
$$1\bar{3}0\bar{2} \sim 12\bar{0}3$$
$$2\bar{3}1\bar{0} \sim 20\bar{1}3$$
$$3\bar{0}\bar{2}\bar{1} \sim 3120$$
$$\bar{0}2\bar{3}\bar{1} \sim \bar{0}13\bar{2}$$
$$0\bar{2}31 \sim 0\bar{1}\bar{3}2$$
$$\bar{0}2\bar{1}\bar{3} \sim \bar{0}31\bar{2}$$
$$\bar{1}\bar{3}\bar{2}0 \sim \bar{1}\bar{0}23$$
$$\bar{3}102 \sim \bar{3}\bar{2}\bar{0}\bar{1}$$
$$\bar{2}03\bar{1} \sim \bar{2}1\bar{3}\bar{0}$$
$$1\bar{2}\bar{3}0 \sim 1\bar{0}32$$
$$2\bar{0}\bar{3}1 \sim 2\bar{1}30$$
$$3\bar{1}\bar{0}\bar{2} \sim 3201$$

Let us consider $N^{(01\bar{2}3)}$ and then determine to which double coset $Nt_0t_1t_2^{-1}t_3t_i$ belongs for one t_i from each orbit of $N^{(01\bar{2}3)}$. Since $N^{(01\bar{2}3)} = \{e, (1\bar{3})(2\bar{2})(3\bar{1})\}$, we see that the orbits of $N^{(01\bar{2}3)}$ on $\{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ are $\{1, \bar{3}\}$, $\{\bar{1}, 3\}$, $\{2, \bar{2}\}$, $\{0\}$, and $\{\bar{0}\}$. It suffices to choose one representative from each orbit and ask to which double coset $Nt_0t_1t_2^{-1}t_3t_i$ belongs.

$01\bar{2}3\bar{3} = 01\bar{2} \Rightarrow Nt_0t_1t_2^{-1}t_3t_3^{-1} = Nt_0t_1t_2^{-1} \in Nt_0t_1t_2^{-1}N = [01\bar{2}]$.

MAGMA shows the following four statements.

$01\bar{2}33 = 01\bar{2}3 \Rightarrow Nt_0t_1t_2^{-1}t_3t_3 = Nt_0t_1t_2^{-1}t_3 \in Nt_0t_1t_2^{-1}t_3N = [01\bar{2}3]$.
$01\bar{2}30 = 0121 \Rightarrow Nt_0t_1t_2^{-1}t_3t_0 = Nt_0t_1t_2t_1 \in Nt_0t_1t_2t_1N = [0121]$.
$01\bar{2}3\bar{0} = 010\bar{2} \Rightarrow Nt_0t_1t_2^{-1}t_3t_0^{-1} = Nt_0t_1t_0t_2^{-1} \in Nt_0t_1t_0t_2^{-1}N = [010\bar{2}]$.
$01\bar{2}32 = 01\bar{2}3 \Rightarrow Nt_0t_1t_2^{-1}t_3t_2 = Nt_0t_1t_2^{-1}t_3 \in Nt_0t_1t_2^{-1}t_3N = [01\bar{2}3]$.

Let us consider double coset [01010].
MAGMA provides $N^{(01010)}$ is the entire control group, the double cover of S_4:

e, $(12\bar{3}0\bar{1}\bar{2}3\bar{0})$, $(1\bar{0}3\bar{2}\bar{1}0\bar{3}2)$, $(1\bar{3}\bar{1}3)(20\bar{2}\bar{0})$, $(13\bar{0}\bar{1}\bar{3}0)(2\bar{2})$, $(13\bar{1}\bar{3})(2\bar{0}\bar{2}0)$, $(1032\bar{1}\bar{0}\bar{3}\bar{2})$,
$(1\bar{2})(2\bar{1})(0\bar{0})$, $(1\bar{1})(23)(\bar{2}\bar{3})$, $(1\bar{2}0\bar{1}2\bar{0})(3\bar{3})$, $(10\bar{3}\bar{1}\bar{0}3)(2\bar{2})$, $(130\bar{2}\bar{1}\bar{3}\bar{0}2)$, $(1\bar{3}\bar{2}0\bar{1}32\bar{0})$,
$(1\bar{3}02\bar{1}3\bar{0}\bar{2})$, $(1\bar{0}\bar{3})(3\bar{1}0)$, $(123\bar{1}\bar{2}\bar{3})(0\bar{0})$, $(1\bar{2}3)(2\bar{3}\bar{1})$, $(1320\bar{1}\bar{3}\bar{2}\bar{0})$, $(1\bar{1})(3\bar{0})(0\bar{3})$,
$(1\bar{0}2\bar{1}0\bar{2})(3\bar{3})$, $(120)(\bar{1}\bar{2}\bar{0})$, $(102)(\bar{1}\bar{0}\bar{2})$, $(2\bar{2})(30)(\bar{3}\bar{0})$, $(2\bar{3})(3\bar{2})(0\bar{0})$, $(12)(3\bar{3})(\bar{1}\bar{2})$,
$(1\bar{3}\bar{0})(30\bar{1})$, $(2\bar{0}3)(0\bar{3}\bar{2})$, $(10\bar{2}3\bar{1}\bar{0}2\bar{3})$, $(1\bar{2}\bar{3}0\bar{1}230)$, $(1\bar{0}\bar{2}\bar{3}\bar{1}023)$, $(1\bar{1})(203\bar{2}\bar{0}\bar{3})$,
$(1\bar{1})(2\bar{2})(3\bar{3})(0\bar{0})$, $(1\bar{3})(2\bar{2})(3\bar{1})$, $(10\bar{1}\bar{0})(23\bar{2}\bar{3})$, $(12\bar{1}\bar{2})(30\bar{3}\bar{0})$, $(1\bar{1})(2\bar{0})(0\bar{2})$, $(10)(3\bar{3})(\bar{1}\bar{0})$,
$(20)(3\bar{3})(\bar{2}\bar{0})$, $(1\bar{2}\bar{1}2)(3\bar{0}\bar{3}0)$, $(1\bar{3}\bar{2}\bar{1}32)(0\bar{0})$, $(13\bar{2})(2\bar{1}\bar{3})$, $(1\bar{0})(2\bar{2})(0\bar{1})$, $(1\bar{2}\bar{0}3\bar{1}20\bar{3})$,
$(23\bar{0})(0\bar{2}\bar{3})$, $(1\bar{0}\bar{1}0)(2\bar{3}\bar{2}3)$, $(1\bar{1})(2\bar{3}\bar{0}\bar{2}30)$, $(12\bar{0}3\bar{1}\bar{2}03)$, $(13)(0\bar{0})(\bar{1}\bar{3})$.

This implies that there exists 1 distinct single coset within [01010], which is given by $Nt_0t_1t_0t_1t_0N = \{N(t_0t_1t_0t_1t_0)^n | n \in N\}$. This single coset has 48 equivalent names, which we list below:

$$01010 \sim \bar{1}2\bar{1}2\bar{1} \sim \bar{3}\bar{0}\bar{3}\bar{0}\bar{3} \sim \bar{2}\bar{3}\bar{2}\bar{3}\bar{2} \sim 13131 \sim 23232 \sim 30303 \sim \bar{0}\bar{2}\bar{0}\bar{2}\bar{0} \sim 0\bar{1}0\bar{1}0 \sim \bar{1}\bar{2}\bar{1}\bar{2}\bar{1} \sim$$
$$\bar{3}0\bar{3}0\bar{3} \sim \bar{2}3\bar{2}3\bar{2} \sim 1\bar{3}1\bar{3}1 \sim 2\bar{3}2\bar{3}2 \sim 3\bar{0}3\bar{0}3 \sim \bar{0}2\bar{0}2\bar{0} \sim 0\bar{2}0\bar{2}0 \sim \bar{1}3\bar{1}3\bar{1} \sim \bar{3}\bar{1}\bar{3}\bar{1}\bar{3} \sim \bar{2}\bar{0}\bar{2}\bar{0}\bar{2} \sim$$
$$12121 \sim 20202 \sim 31313 \sim \bar{0}1\bar{0}1\bar{0} \sim 02020 \sim \bar{1}\bar{3}\bar{1}\bar{3}\bar{1} \sim \bar{3}1\bar{3}1\bar{3} \sim \bar{2}0\bar{2}0\bar{2} \sim 1\bar{2}1\bar{2}1 \sim 2\bar{0}2\bar{0}2 \sim$$
$$3\bar{1}3\bar{1}3 \sim \bar{0}\bar{1}\bar{0}\bar{1}\bar{0} \sim 0\bar{3}0\bar{3}0 \sim \bar{1}0\bar{1}0\bar{1} \sim \bar{3}2\bar{3}2\bar{3} \sim \bar{2}\bar{1}\bar{2}\bar{1}\bar{2} \sim 10101 \sim 21212 \sim 3\bar{2}3\bar{2}3 \sim \bar{0}\bar{3}\bar{0}\bar{3}\bar{0} \sim$$
$$03030 \sim \bar{1}\bar{0}\bar{1}\bar{0}\bar{1} \sim \bar{3}\bar{2}\bar{3}\bar{2}\bar{3} \sim \bar{2}1\bar{2}1\bar{2} \sim 1\bar{0}1\bar{0}1 \sim 2\bar{1}2\bar{1}2 \sim 32323 \sim \bar{0}3\bar{0}3\bar{0}$$

Let us consider $N^{(01010)}$ and then determine to which double coset $Nt_0t_1t_0t_1t_0t_i$ belongs for one t_i from each orbit of $N^{(01010)}$. Since $N^{(01010)}$ is the entire control group, we see that the orbit of $N^{(01010)}$ on $\{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ is $\{0, 1, 2, 3, \bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. It suffices to choose one representative from this orbit and ask to which double coset $Nt_0t_1t_0t_1t_0t_i$ belongs.

$010100\bar{0} = 0101 \Rightarrow Nt_0t_1t_0t_1t_0t_0^{-1} = Nt_0t_1t_0t_1 \in Nt_0t_1t_0t_1N = [0101]$.

This double coset is now closed under right multiplication. That is, $Nt_0t_1t_0t_1t_0t_i$ does not move forward to any double coset with representative word of length six.

All of our work is summarized in the Cayley Diagrams of $M_{11} \times 2$ over $2^{\cdot}S_4^+$ below.

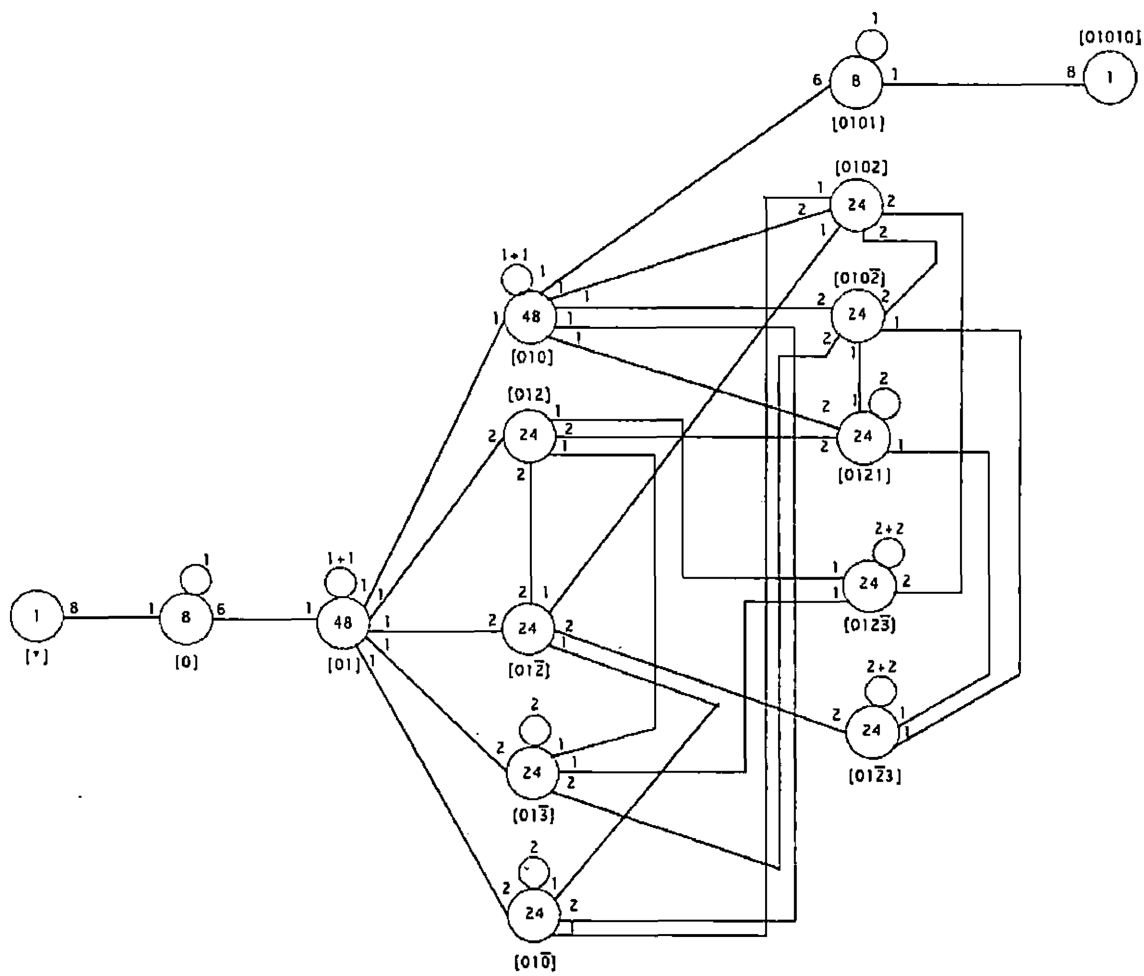

Figure 5.1: Cayley Diagram $M_{11} \times 2$ over $2^{\cdot}S_4^+$

Our argument shows that the order of G is at most $|N| \times 330 = 15,840$, where 330 is the number of single cosets shown in the diagram above. We now show that $|G|$ is at least 15,840. Now $G =< x, y, t >$ acts on X, the set of the single cosets mentioned above. Thus $\alpha : G \to S_X$ is a homomorphism. Since $N =< x, y >$ acts by conjugation and t acts by right multiplication on the t_is, we compute the images xx, yy, and tt of x,y, and t, respectively, in S_X and verify that the additional relations hold in $< xx, yy, tt >$ within S_X and $| < xx, yy, tt > | = 15,840$. So $G/Ker\alpha \cong < xx, yy, tt >$. Hence $|G| \geq 15,840$. Thus, $|G| = 15,840$.

We also verified that G satisfies a presentation of $\cong U_3(3) \times 3$.

5.1 MAGMA Code

MAGMA code for M_{11} x 2

```
S:=Sym(8);
xx:=S!(4,5,6,3,8,1,2,7);
yy:=S!(1,5)(2,3)(6,7);
N:=sub<S|xx,yy>;
G<x,y,t>:=Group<x,y,t|x^4=(y*x)^3,y^2,t^3,(t,y),
(y*t^x)^2,(x*t)^22,
(y*x*t)^5,(x^2*t)^6,(x^3*y*t)^6,(y*t^(x^6))^10>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
IN:=sub<G1|f(x),f(y)>;
Index(G,sub<G|x,y>);

/*330*/

#N;

/*48*/

#G;

/*15,840*/

ts:=[Id(G1) : i in [1..8]];
ts[4]:=f(t); ts[1]:=f(t^(x^5));
ts[2]:=f(t^(x^6));ts[3]:=f(t^(x^3));
ts[8]:=(ts[4])^-1; ts[5]:=(ts[1])^-1;
ts[6]:=(ts[2])^-1;
ts[7]:=(ts[3])^-1;
cst := [null : i in [1 .. 330]] where null is [Integers() | ];
prodim := function(pt, Q, I)
/*
 Return the image of pt under permutations
 Q[I] applied sequentially.
  */
  v := pt;
for i in I do
  v := v^(Q[i]);
end for;
return v;
end function;
for i := 1 to 4 do
cst[prodim(1, ts, [i])] := [i];
```

```
end for;
m:=0;
 for i in [1..330] do if cst[i] ne []
 then m:=m+1; end if; end for; m;
for i := 5 to 8 do
cst[prodim(1, ts, [i])] := [i];
end for;
m:=0;
 for i in [1..330] do if cst[i] ne []
 then m:=m+1; end if; end for; m;


/*Now we want to find the coset stabilising group for [4].
This in turn will give us the orbits of [4].*/

N4:=Stabiliser(N,4);
Orbits(N4);

/*{4}, {8}, {1, 2, 3, 5, 6, 7}*/

/*We now know that {1, 2, 3, 5, 6, 7}
are in the same orbit.
However, both 4 & 8 are in their own orbit.
I now want to know the order of the coset
stabilizing group of [4].
Once I know the order, I divide it into the order
of my control group.
I know it will be 48/6 = 8, which represents the
# of distinct single cosets within [4].*/

#N4;

/*6*/

/*The following code will keep track of my index,
which is the number of single cosets I have.
When I total 330, I know I am done.*/

T4:=Transversal(N,N4);
for i in [1..#T4] do
  ss:=[4]^T4[i];
  cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;
```

```
 for i in [1..330] do if cst[i] ne []
 then m:=m+1; end if; end for; m;

/*8*/

/*So double coset [4] has 8 distinct single cosets.*/

/*Examining double coset [4].
Recall that the orbits are {4}, {8}, {1, 2, 3, 5, 6, 7}.
I know that 48 = 1, so (t_0)^-1 takes me from [4] to [*].*/

/*The following code shows that 44 = 4, so t_0
takes me from [4] to [4].*/

for n in IN do for m in IN do
if ts[4]*ts[4] eq n*(ts[4])^m
then m, n;
end if; end for; end for;

/*Now I take one representative from the following orbit
to see to which double coset t_i belongs: {1, 2, 3, 5, 6, 7}.
Because none of my relations take a word of length 2 to a word
of length 1, I know that [4,1] is my new double
coset.  Therefore, all six single cosets move
forward to the new double coset [41].*/


/*Now we handle double coset [4,1].*/

N41:=Stabiliser(N4,1);
N41s:=sub<N|N41>;

/*Now I use the following code to see
what "common" names we have
in [4,1].*/

S:={[[4],[1]]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[1]
eq g*ts[Rep(SSS[i])[1][1]]*ts[Rep(SSS[i])[2][1]]
then print SSS[i];
end if; end for; end for;
```

```
/*The above code provided that there are no
common names within [41].*/

Orbits(N41s);

/*The orbits are are {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}.

/*Now I can find the order of the coset stabilising group [4,1].
This number, when divided into the order of my control group N,
which is 48/1 = 48, gives the # of distinct single cosets within
the double coset [4,1].*/

#N41s;

/*1*/

/*Finding the index*/

T41:=Transversal(N,N41s);
for i in [1..#T41] do
  ss:=[4,1]^T41[i];
  cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;
 for i in [1..330] do if cst[i] ne []
 then m:=m+1; end if; end for; m;

/*56*/

/*I now want to know to which double
coset Nt_0t_1t_i belongs.
I recall that the orbits are
{1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}.*/
/*I know 415 = 4, so (t_1)^-1 takes
me back to [4].*/
/*The code below shows that 411 = 41,
so t_1 takes me from 41 back to 41.
The code also shows that 413 = 41,
so t_3 takes me from 41 back to 41.*/

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[1] eq n*(ts[4]*ts[1])^m
then m, n;
```

```
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[3] eq n*(ts[4]*ts[1])^m
then m, n;
end if; end for; end for;

/*The same code shows that 412, 414, 416, 417,
and 418 are new double cosets.*/


/*Now we handle double coset [4,1,2].*/

N412:=Stabiliser(N41,2);
N412s:=sub<N|N412>;

/*Now I use the following code to see
what "common" names we have
in [4,1,2].*/

S:={[[4],[1],[2]]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1][1]]*ts[Rep(SSS[i])[2][1]]
*ts[Rep(SSS[i])[3][1]]
then print SSS[i];
end if; end for; end for;

/*The above code provided that 658 = 412 within [41].*/

for g in N do if 4^g eq 6 and 1^g eq 5 and 2^g eq 8
then N412s:=sub<N|N412s,g>;
end if; end for;

Orbits(N412s);

/*The orbits are are {3}, {7}, {1, 5}, {2, 8}, {4, 6}.

/*Now I can find the order of the coset
stabilising group [4,1,2].
This number, when divided into the order
of my control group N,
```

```
which is 48/2 = 24, gives the # of distinct
single cosets within
the double coset [4,1,2].*/

#N412s;

/*2*/

/*Finding the index*/

T412:=Transversal(N,N412s);
for i in [1..#T412] do
  ss:=[4,1,2]^T412[i];
  cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;
 for i in [1..330] do if cst[i] ne []
 then m:=m+1; end if; end for; m;

/*80*/

/*Now I want to know to which double
coset does Nt_0t_1t_2t_i belongs.
I recall that the orbits are
{3}, {7}, {1, 5}, {2, 8}, {4, 6}.*/
/*I know that 4126 = 41,
so both (t_2)^-1 and t_0 take me
from 412 to 41.
Also, 4122 = 416, so both t_2 and (t_0)^-1
take me from 412 to 416.*/
/*The following code shows that 4123 = 417,
so t_3 takes me from 412 to 417.*/

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[2]*ts[3] eq n*(ts[4]*ts[1]*ts[7])^m
then m, n;
end if; end for; end for;

/*The same code shows that the following
double cosets are new: 4121 and 4127.*/


/*Now we handle double coset [4,1,4].*/
```

```
N414:=Stabiliser(N41,4);
N414s:=sub<N|N414>;

/*Now I use the following code to see what
"common" names we have
in [4,1,4].*/

S:={[[4],[1],[4]]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[1]*ts[4]
eq g*ts[Rep(SSS[i])[1][1]]*ts[Rep(SSS[i])[2][1]]
*ts[Rep(SSS[i])[3][1]]
then print SSS[i];
end if; end for; end for;

/*The above code provided that there are no
common names within [414].*/

Orbits(N414s);

/*The orbits are are {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}.

/*Now I can find the order of the coset stabilising group [4,1,4].
This number, when divided into the order of my control group N,
which is 48/1 = 48, gives the # of distinct single cosets within
the double coset [4,1,4].*/

#N414s;

/*1*/

/*Finding the index*/

T414:=Transversal(N,N414s);
for i in [1..#T414] do
  ss:=[4,1,4]^T414[i];
  cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;
 for i in [1..330] do if cst[i] ne []
 then m:=m+1; end if; end for; m;
```

```
/*128*/

/*Now I want to know to which double coset
does Nt_0t_1t_0t_i belongs.
I recall that the orbits are
{1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}.*/
/*I know that 4148 = 41, so (t_0)^-1 takes
me from 414 to 41.
Also, 4144 = 418, so t_0 takes me from 414 to 418.*/
/*The code below shows that 4145 = 414, so (t_1)^-1
takes me from 414 to 414.
Also, 4147 = 414, so (t_3)^-1 takes me from 414 to 414.
Also, 4143 = 4121, so t_3 takes me from 414 to 4121.*/

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[4]*ts[5] eq n*(ts[4]*ts[1]*ts[4])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[4]*ts[7] eq n*(ts[4]*ts[1]*ts[4])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[4]*ts[3] eq n*(ts[4]*ts[1]*ts[2]*ts[1])^m
then m, n;
end if; end for; end for;

/*The same code shows that the following are new:
4141, 4142, 4146.*/


/*Now we handle double coset [4,1,6].*/

N416:=Stabiliser(N41,6);
N416s:=sub<N|N416>;

/*Now I use the following code to see what
"common" names we have
in [4,1,6].*/

S:={[[4],[1],[6]]};
SS:=S^N;
```

```
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[1]*ts[6]
eq g*ts[Rep(SSS[i])[1][1]]*ts[Rep(SSS[i])[2][1]]
*ts[Rep(SSS[i])[3][1]]
then print SSS[i];
end if; end for; end for;

/*The above code provided that 416 = 218 within [414].*/

for g in N do if 4^g eq 2 and 1^g eq 1 and 6^g eq 8
then N416s:=sub<N|N416s,g>;
end if; end for;

Orbits(N416s);

/*The orbits are are {1}, {5}, {2, 4}, {3, 7}, {6, 8}.

/*Now I can find the order of the coset
stabilising group [4,1,6].
This number, when divided into the order of
my control group N,
which is 48/2 = 24, gives the # of distinct
single cosets within
the double coset [4,1,6].*/

#N416s;

/*2*/

/*Finding the index*/

T416:=Transversal(N,N416s);
for i in [1..#T416] do
  ss:=[4,1,6]^T416[i];
  cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;
 for i in [1..330] do if cst[i] ne []
 then m:=m+1; end if; end for; m;

/*152*/

/*I now want to know to which double coset
```

```
does Nt_0t_1t_2^-1t_i belongs.
I recall that the orbits are
{1}, {5}, {2, 4}, {3, 7}, {6, 8}.*/
/*I know that 4162 = 41, so t_2 and t_0
take me from 416 to 41.
Also, 4166 = 412, so (t_2)^-1 and (t_0)^-1
take me from 416 to 412.*/
/*The following code shows that 4165 = 418,
so (t_1)^-1 takes me from 416 to 418.
Also, 4161 = 4142, so t_1 takes me from 416 to 4142.*/

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[6]*ts[5] eq n*(ts[4]*ts[1]*ts[8])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[6]*ts[1] eq n*(ts[4]*ts[1]*ts[4]*ts[2])^m
then m, n;
end if; end for; end for;

/*The same code shows that the following are new: 4163.*/


/*Now we handle double coset [4,1,7].*/

N417:=Stabiliser(N41,7);
N417s:=sub<N|N417>;

/*Now I use the following code to see what
"common" names we have
in [4,1,7].*/

S:={[[4],[1],[7]]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[1]*ts[7]
eq g*ts[Rep(SSS[i])[1][1]]*ts[Rep(SSS[i])[2][1]]
*ts[Rep(SSS[i])[3][1]]
then print SSS[i];
end if; end for; end for;

/*The above code provided that 417 = 143 within [417].*/
```

```
for g in N do if 4^g eq 1 and 1^g eq 4 and 7^g eq 3
then N417s:=sub<N|N417s,g>;
end if; end for;

Orbits(N417s);

/*The orbits are are {2}, {6}, {1, 4}, {3, 7}, {5, 8}.

/*Now I can find the order of the coset stabilising group [4,1,7].
This number, when divided into the order of my control group N,
which is 48/2 = 24, gives the # of distinct single cosets within
the double coset [4,1,7].*/

#N417s;

/*2*/

/*Finding the index*/

T417:=Transversal(N,N417s);
for i in [1..#T417] do
  ss:=[4,1,7]^T417[i];
  cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;
 for i in [1..330] do if cst[i] ne []
 then m:=m+1; end if; end for; m;

/*176*/

/*I now want to know to which double coset
does Nt_0t_1t_3^-1t_i belongs.
I recall that the orbits are
{2}, {6}, {1, 4}, {3, 7}, {5, 8}.*/
/*I know that 4173 = 41, so t_3 and (t_3)^-1
take me from 417 to 41.
/*The following code shows that 4176 = 412,
so (t_2)^-1 takes me from 417 to 412.
Also, 4175 = 417, so (t_1)^-1 and (t_0)^-1
take me from 417 to 417.
Also, 4172 = 4127, so t_2 takes me from 417 to 4127.
Also, 4171 = 4146, so t_1 and t_0 take me from 417 to 4146.*/
```

```
for n in IN do for m in IN do
if ts[4]*ts[1]*ts[7]*ts[6] eq n*(ts[4]*ts[1]*ts[2])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[7]*ts[5] eq n*(ts[4]*ts[1]*ts[7])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[7]*ts[2] eq n*(ts[4]*ts[1]*ts[2]*ts[7])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[7]*ts[1] eq n*(ts[4]*ts[1]*ts[4]*ts[6])^m
then m, n;
end if; end for; end for;


/*Now we handle double coset [4,1,8].*/

N418:=Stabiliser(N41,8);
N418s:=sub<N|N418>;

/*Now I use the following code to see what
"common" names we have
in [4,1,8].*/

S:={[[4],[1],[8]]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[1]*ts[8]
eq g*ts[Rep(SSS[i])[1][1]]*ts[Rep(SSS[i])[2][1]]
*ts[Rep(SSS[i])[3][1]]
then print SSS[i];
end if; end for; end for;

/*The above code provided that 418 = 581
within [418].*/

for g in N do if 4^g eq 5 and 1^g eq 8 and 8^g eq 1
```

```
then N418s:=sub<N|N418s,g>;
end if; end for;

Orbits(N418s);

/*The orbits are are {3}, {7}, {1, 8}, {2, 6}, {4, 5}.

/*Now I can find the order of the coset
stabilising group [4,1,8].
This number, when divided into the
order of my control group N,
which is 48/2 = 24, gives the # of distinct
single cosets within
the double coset [4,1,8].*/

#N418s;

/*2*/

/*Finding the index*/

T418:=Transversal(N,N418s);
for i in [1..#T418] do
  ss:=[4,1,8]^T418[i];
  cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;
 for i in [1..330] do if cst[i] ne []
 then m:=m+1; end if; end for; m;

/*200*/

/*I now want to know to which double
coset Nt_0t_1t_0^-1t_i belongs.
I recall that the orbits are
{3}, {7}, {1, 8}, {2, 6}, {4, 5}.*/
/*I know that 4184 = 41, so t_0 and
(t_1)^-1 take me from 418 to 41.
Also, 4188 = 414, so (t_0)^-1 and t_1
take me from 418 to 414.*/
/*The following code shows that 4183 = 416,
so t_3 takes me from 418 to 416.
Also, 4182 = 418, so t_2 and (t_2)^-1
take me from 418 to 418.
```

```
Also, 4187 = 4142, so (t_3)^-1 takes
me from 418 to 4142.*/

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[8]*ts[3] eq n*(ts[4]*ts[1]*ts[6])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[8]*ts[2] eq n*(ts[4]*ts[1]*ts[8])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[8]*ts[7] eq n*(ts[4]*ts[1]*ts[4]*ts[2])^m
then m, n;
end if; end for; end for;


/*Now we handle double coset [4,1,2,1].*/

N4121:=Stabiliser(N412,1);
N4121s:=sub<N|N4121>;

/*Now I use the following code to see what
"common" names we have
in [4,1,2,1].*/

S:={[[4],[1],[2],[1]]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[1]*ts[2]*ts[1]
eq g*ts[Rep(SSS[i])[1][1]]*ts[Rep(SSS[i])[2][1]]
*ts[Rep(SSS[i])[3][1]]
*ts[Rep(SSS[i])[4][1]]
then print SSS[i];
end if; end for; end for;

/*The above code provided that 4121 = 5868
within [4121].*/

for g in N do if 4^g eq 5 and 1^g eq 8 and 2^g eq 6
and 1^g eq 8
```

```
then N4121s:=sub<N|N4121s,g>;
end if; end for;

Orbits(N4121s);

/*The orbits are are {3}, {7}, {1, 8}, {2, 6}, {4, 5}.

/*Now I can find the order of the coset
stabilising group [4,1,2,1].
This number, when divided into the order
of my control group N,
which is 48/2 = 24, gives the # of distinct
single cosets within
the double coset [4,1,2,1].*/

#N4121s;

/*2*/

/*Finding the index*/

T4121:=Transversal(N,N4121s);
for i in [1..#T4121] do
  ss:=[4,1,2,1]^T4121[i];
  cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;
 for i in [1..330] do if cst[i] ne [] then m:=m+1;
 end if; end for; m;

/*224*/

/*I now want to know to which double coset
Nt_0t_1t_2t_1t_i belongs.
I recall that the orbits are {3}, {7}, {1, 8}, {2, 6}, {4, 5}.*
/*I know that 41215 = 412, so (t_1)^-1 and t_0 take
me from 4121 to 412.*/
/*The following code shows that 41213 = 4163.
So t_3 takes me from 4121 to 4163.
Also, 41217 = 4146, so (t_3)^-1 takes me from
4121 to 4146.
Also, 41211 = 4121, so t_1 and (t_0)^-1 take me
from 4121 to 4121.
Also, 41212 = 414, so t_2 and (t_2)^-1 take me
```

```
from 4121 to 414.*/

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[2]*ts[1]*ts[3] eq n*(ts[4]*ts[1]*ts[6]*ts[3])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[2]*ts[1]*ts[7] eq n*(ts[4]*ts[1]*ts[4]*ts[6])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[2]*ts[1]*ts[1] eq n*(ts[4]*ts[1]*ts[2]*ts[1])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[2]*ts[1]*ts[2] eq n*(ts[4]*ts[1]*ts[4])^m
then m, n;
end if; end for; end for;


/*Now we handle double coset [4,1,2,7].*/

N4127:=Stabiliser(N412,7);
N4127s:=sub<N|N4127>;

/*Now I use the following code to see what
"common" names we have
in [4,1,2,7].*/

S:={[[4],[1],[2],[7]]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[1]*ts[2]*ts[7]
eq g*ts[Rep(SSS[i])[1][1]]*ts[Rep(SSS[i])[2][1]]
*ts[Rep(SSS[i])[3][1]]
*ts[Rep(SSS[i])[4][1]]
then print SSS[i];
end if; end for; end for;

/*The above code provided that 4127 = 6587
```

```
within [4127].*/

for g in N do if 4^g eq 6 and 1^g eq 5 and 2^g
eq 8 and 7^g eq 7
then N4127s:=sub<N|N4127s,g>;
end if; end for;

Orbits(N4127s);

/*The orbits are are {3}, {7}, {1, 5}, {2, 8}, {4, 6}.

/*Now I can find the order of the coset
stabilising group [4,1,2,7].
This number, when divided into the order of
my control group N,
which is 48/2 = 24, gives the # of distinct
single cosets within
the double coset [4,1,2,7].*/

#N4127s;

/*2*/

/*Finding the index*/

T4127:=Transversal(N,N4127s);
for i in [1..#T4127] do
  ss:=[4,1,2,7]^T4127[i];
  cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;
 for i in [1..330] do if cst[i] ne [] then m:=m+1;
 end if; end for; m;

/*248*/

/*I now want to know to which double coset
Nt_0t_1t_2t_3^-1t_i belongs.
I recall that the orbits are
{3}, {7}, {1, 5}, {2, 8}, {4, 6}.*/
/*I know that 41273 = 412, so t_3 takes
me from 4127 to 412.*/
/*The following code shows that 41271 = 4127.
So t_1 and (t_1)^-1 take me from 4127 to 4127.
```

```
Also, 41272 = 4142, so t_2 and (t_0)^-1
take me from 4127 to 4142.
Also, 41274 = 4127, so t_0 and (t_2)^-1
take me from 4127 to 4127.
Also, 41277 = 417, so (t_3)^-1 takes me
from 4127 to 417.*/

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[2]*ts[7]*ts[1] eq n*(ts[4]*ts[1]*ts[2]*ts[7])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[2]*ts[7]*ts[2] eq n*(ts[4]*ts[1]*ts[4]*ts[2])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[2]*ts[7]*ts[4] eq n*(ts[4]*ts[1]*ts[2]*ts[7])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[2]*ts[7]*ts[7] eq n*(ts[4]*ts[1]*ts[7])^m
then m, n;
end if; end for; end for;


/*Now we handle double coset [4,1,4,1].*/

N4141:=Stabiliser(N414,1);
N4141s:=sub<N|N4141>;

/*Now I use the following code to see what
"common" names we have
in [4,1,4,1].*/

S:={[[4],[1],[4],[1]]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[1]*ts[4]*ts[1]
eq g*ts[Rep(SSS[i])[1][1]]*ts[Rep(SSS[i])[2][1]]
*ts[Rep(SSS[i])[3][1]]
```

```
*ts[Rep(SSS[i])[4][1]]
then print SSS[i];
end if; end for; end for;

/*The above code provided that 4141 = 4343 =
4747 = 4646 = 4545 = 4242
within [4127].*/

for g in N do if 4^g eq 4 and 1^g eq 3 and 4^g eq
4 and 1^g eq 3
then N4141s:=sub<N|N4141s,g>;
end if; end for;

for g in N do if 4^g eq 4 and 3^g eq 7 and 4^g eq
4 and 3^g eq 7
then N4141s:=sub<N|N4141s,g>;
end if; end for;

for g in N do if 4^g eq 4 and 7^g eq 6 and 4^g eq
4 and 7^g eq 6
then N4141s:=sub<N|N4141s,g>;
end if; end for;

for g in N do if 4^g eq 4 and 6^g eq 5 and 4^g eq
4 and 6^g eq 5
then N4141s:=sub<N|N4141s,g>;
end if; end for;

for g in N do if 4^g eq 4 and 5^g eq 2 and 4^g eq
4 and 5^g eq 2
then N4141s:=sub<N|N4141s,g>;
end if; end for;

Orbits(N4141s);

/*The orbits are are {4}, {8}, {1, 2, 3, 5, 6, 7}.

/*Now I can find the order of the coset
stabilising group [4,1,4,1].
This number, when divided into the order
of my control group N,
which is 48/6 = 8, gives the # of distinct
single cosets within
the double coset [4,1,4,1].*/
```

```
#N4141s;

/*6*/


/*Finding the index*/

T4141:=Transversal(N,N4141s);
for i in [1..#T4141] do
  ss:=[4,1,4,1]^T4141[i];
  cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;
 for i in [1..330] do if cst[i] ne []
 then m:=m+1; end if; end for; m;

/*256*/

/*I now want to know to which double
coset Nt_0t_1t_0t_1t_i belongs.
I recall that the orbits are
{4}, {8}, {1, 2, 3, 5, 6, 7}.*/
/*I know that 41415 = 414.  So each t_i
in the orbit of (t_1)^-1 takes me
from 4141 to 414.*/
/*The following code shows that 41418 = 4141.
So (t_0)^-1 takes me from 4141 to 4141.*/

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[4]*ts[1]*ts[8] eq
n*(ts[4]*ts[1]*ts[4]*ts[1])^m
then m, n;
end if; end for; end for;

/*The same code shows that the following
are new: 41414.*/


/*Now we handle double coset [4,1,4,2].*/

N4142:=Stabiliser(N414,2);
N4142s:=sub<N|N4142>;
```

```
/*Now I use the following code to see what
"common" names we have
in [4,1,4,2].*/

S:={[[4],[1],[4],[2]]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[1]*ts[4]*ts[2]
eq g*ts[Rep(SSS[i])[1][1]]*ts[Rep(SSS[i])[2][1]]
*ts[Rep(SSS[i])[3][1]]
*ts[Rep(SSS[i])[4][1]]
then print SSS[i];
end if; end for; end for;

/*The above code provided that 4142 = 2124
within [4127].*/

for g in N do if 4^g eq 2 and 1^g eq 1 and 4^g
eq 2 and 2^g eq 4
then N4142s:=sub<N|N4142s,g>;
end if; end for;

Orbits(N4142s);

/*The orbits are are {1}, {5}, {2, 4}, {3, 7}, {6, 8}.

/*Now I can find the order of the coset
stabilising group [4,1,4,2].
This number, when divided into the
order of my control group N,
which is 48/2 = 24, gives the # of distinct
single cosets within
the double coset [4,1,4,2].*/

#N4142s;

/*2*/

/*Finding the index*/

T4142:=Transversal(N,N4142s);
for i in [1..#T4142] do
  ss:=[4,1,4,2]^T4142[i];
```

```
 cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..330] do if cst[i] ne []
then m:=m+1; end if; end for; m;

/*280*/

/*I now want to know to which double
coset Nt_0t_1t_0t_2t_i belongs.
I recall that the orbits are
{1}, {5}, {2, 4}, {3, 7}, {6, 8}*/
/*I know that 41426 = 414.
So (t_2)^-1 and (t_0)^-1 take
me from 4142 to 414.
Also, 41422 = 4146.  So t_2 and
t_0 take me from 4142 to 4146.*/

/*The following code shows that
41423 = 4127.  So t_3 and (t_3)^-1
move me from 4142 to 4127.
Also, 41421 = 418, so t_1 takes me
from 4142 to 418.
Also, 41425 = 416, so (t_1)^-1 takes
me from 4142 to 416.*/

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[4]*ts[2]*ts[3] eq
n*(ts[4]*ts[1]*ts[2]*ts[7])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[4]*ts[2]*ts[1] eq
n*(ts[4]*ts[1]*ts[8])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[4]*ts[2]*ts[5] eq
n*(ts[4]*ts[1]*ts[6])^m
then m, n;
end if; end for; end for;
```

```
/*Now we handle double coset [4,1,4,6].*/

N4146:=Stabiliser(N414,6);
N4146s:=sub<N|N4146>;

/*Now I use the following code to see
what "common" names we have
in [4,1,4,6].*/

S:={[[4],[1],[4],[6]]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[1]*ts[4]*ts[6]
eq g*ts[Rep(SSS[i])[1][1]]*ts[Rep(SSS[i])[2][1]]
*ts[Rep(SSS[i])[3][1]]
*ts[Rep(SSS[i])[4][1]]
then print SSS[i];
end if; end for; end for;

/*The above code provided that 4146 = 6564
within [4146].*/

for g in N do if 4^g eq 6 and 1^g eq 5 and 4^g
eq 6 and 6^g eq 4
then N4146s:=sub<N|N4146s,g>;
end if; end for;

Orbits(N4146s);

/*The orbits are are
{3}, {7}, {1, 5}, {2, 8}, {4, 6}.

/*Now I can find the order of the
coset stabilising group [4,1,4,6].
This number, when divided into the
order of my control group N,
which is 48/2 = 24, gives the # of
distinct single cosets within
the double coset [4,1,4,6].*/

#N4146s;
```

```
/*2*/

/*Finding the index*/

T4146:=Transversal(N,N4146s);
for i in [1..#T4146] do
  ss:=[4,1,4,6]^T4146[i];
  cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;
 for i in [1..330] do if cst[i] ne []
 then m:=m+1; end if; end for; m;

/*304*/

/*I now want to know to which double coset
Nt_0t_1t_0t_2^-1t_i belongs.
I recall that the orbits are
{3}, {7}, {1, 5}, {2, 8}, {4, 6}*/
/*I know that 41462 = 414.  So t_2 and
(t_0)^-1 take me from 4146 to 414.
Also, 41466 = 4142, so (t_2)^-1 and t_0
take me from 4146 to 4142.*/
/*The following code shows that 41463 = 4163,
so t_3 takes me from 4146 to 4163.
Also, 41467 = 4121, so (t_3)^-1 takes me
from 4146 to 4121.
Also, 41461 = 417, so t_1 and (t_1)^-1 take
me from 4146 to 417.*/

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[4]*ts[6]*ts[3] eq
n*(ts[4]*ts[1]*ts[6]*ts[3])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[4]*ts[6]*ts[7] eq
n*(ts[4]*ts[1]*ts[2]*ts[1])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[4]*ts[6]*ts[1] eq
```

```
n*(ts[4]*ts[1]*ts[7])^m
then m, n;
end if; end for; end for;


/*Now we handle double coset [4,1,6,3].*/

N4163:=Stabiliser(N416,3);
N4163s:=sub<N|N4163>;

/*Now I use the following code to see
what "common" names we have
in [4,1,6,3].*/

S:={[[4],[1],[6],[3]]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[1]*ts[6]*ts[3]
eq g*ts[Rep(SSS[i])[1][1]]*ts[Rep(SSS[i])[2][1]]
*ts[Rep(SSS[i])[3][1]]
*ts[Rep(SSS[i])[4][1]]
then print SSS[i];
end if; end for; end for;

/*The above code provided that 4163 = 4725
within [4163].*/

for g in N do if 4^g eq 4 and 1^g eq 7 and
6^g eq 2 and 3^g eq 5
then N4163s:=sub<N|N4163s,g>;
end if; end for;

Orbits(N4163s);

/*The orbits are are {4}, {8}, {1, 7}, {2, 6}, {3, 5}.

/*Now I can find the order of the coset
stabilising group [4,1,6,3].
This number, when divided into the order
of my control group N,
which is 48/2 = 24, gives the # of distinct
single cosets within
the double coset [4,1,6,3].*/
```

```
#N4163s;

/*2*/

/*Finding the index*/

T4163:=Transversal(N,N4163s);
for i in [1..#T4163] do
  ss:=[4,1,6,3]^T4163[i];
  cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;
 for i in [1..330] do if cst[i] ne []
 then m:=m+1; end if; end for; m;

/*328*/

/*I now want to know to which double coset
Nt_0t_1t_2^-1t_3t_i belongs.
I recall that the orbits are
{4}, {8}, {1, 7}, {2, 6}, {3, 5}.*/
/*I know that 41637 = 416, so (t_3)^-1 and
t_1 take me from 4163 to 416.*/
/*The following code shows that 41633 = 4163,
so t_3 and (t_1)^-1
take me from 4163 to 4163.
Also, 41634 = 4121, so t_0 takes me from
4163 to 4121.
Also, 41638 = 4146, so (t_0)^-1 takes me from
 4163 to 4146.
Also, 41632 = 4163, so t_2 and (t_2)^-1 take me
 from 4163 to 4163.*/

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[6]*ts[3]*ts[3] eq
n*(ts[4]*ts[1]*ts[6]*ts[3])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[6]*ts[3]*ts[4] eq
n*(ts[4]*ts[1]*ts[2]*ts[1])^m
then m, n;
```

```
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[6]*ts[3]*ts[8] eq
n*(ts[4]*ts[1]*ts[4]*ts[6])^m
then m, n;
end if; end for; end for;

for n in IN do for m in IN do
if ts[4]*ts[1]*ts[6]*ts[3]*ts[2] eq
n*(ts[4]*ts[1]*ts[6]*ts[3])^m
then m, n;
end if; end for; end for;


/*Now we handle double coset [4,1,4,1,4].*/

N41414:=Stabiliser(N4141,4);
N41414s:=sub<N|N41414>;

/*Now I use the following code to see what
"common" names we have
in [4,1,4,1,4].*/

S:={[[4],[1],[4],[1],[4]]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[4]*ts[1]*ts[4]*ts[1]*ts[4]
eq g*ts[Rep(SSS[i])[1][1]]*ts[Rep(SSS[i])[2][1]]
*ts[Rep(SSS[i])[3][1]]
*ts[Rep(SSS[i])[4][1]]*ts[Rep(SSS[i])[5][1]]
then print SSS[i];
end if; end for; end for;

/*The above code provided that 41414 = 16161 =
68686 = 45454 =
34343 = 47474 = 63636 = 14141 = 64646 =
13131 = 71717 = 74747 =
23232 = 54545 = 28282 = 78787 = 42424 =
 21212 = 38383 = 24242 =
46464 = 12121 = 86868 = 56565 = 65656 =
61616 = 31313 = 52525 =
18181 = 27272 = 43434 = 81818 = 82828 =
```

```
17171 = 58585 = 35353 =
53535 = 75757 = 85858 = 32323 = 67676 =
25252 = 87878 = 36363 =
57575 = 72727 = 76767 = 83838 within [41414].*/

for g in N do if 4^g eq 1 and 1^g eq 6
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 1^g eq 6 and 6^g eq 8
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 6^g eq 4 and 8^g eq 5
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 4^g eq 3 and 5^g eq 4
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 3^g eq 4 and 4^g eq 7
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 4^g eq 6 and 7^g eq 3
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 6^g eq 1 and 3^g eq 4
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 1^g eq 6 and 4^g eq 4
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 6^g eq 1 and 4^g eq 3
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 1^g eq 7 and 3^g eq 1
then N41414s:=sub<N|N41414s,g>;
end if; end for;
```

```
for g in N do if 7^g eq 7 and 1^g eq 4
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 7^g eq 2 and 4^g eq 3
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 2^g eq 5 and 3^g eq 4
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 5^g eq 2 and 4^g eq 8
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 2^g eq 7 and 8^g eq 8
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 7^g eq 4 and 8^g eq 2
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 4^g eq 2 and 2^g eq 1
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 2^g eq 3 and 1^g eq 8
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 3^g eq 2 and 8^g eq 4
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 2^g eq 4 and 4^g eq 6
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 4^g eq 1 and 6^g eq 2
then N41414s:=sub<N|N41414s,g>;
end if; end for;
```

```
for g in N do if 1^g eq 8 and 2^g eq 6
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 8^g eq 5 and 6^g eq 6
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 5^g eq 6 and 6^g eq 5
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 6^g eq 6 and 5^g eq 1
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 6^g eq 3 and 1^g eq 1
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 3^g eq 5 and 1^g eq 2
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 5^g eq 1 and 2^g eq 8
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 1^g eq 2 and 8^g eq 7
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 2^g eq 4 and 7^g eq 3
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 4^g eq 8 and 3^g eq 1
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 8^g eq 8 and 1^g eq 2
then N41414s:=sub<N|N41414s,g>;
end if; end for;
```

```
for g in N do if 8^g eq 1 and 2^g eq 7
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 1^g eq 5 and 7^g eq 8
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 5^g eq 3 and 8^g eq 5
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 3^g eq 5 and 5^g eq 3
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 5^g eq 7 and 3^g eq 5
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 7^g eq 8 and 5^g eq 5
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 8^g eq 3 and 5^g eq 2
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 3^g eq 6 and 2^g eq 7
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 6^g eq 2 and 7^g eq 5
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 2^g eq 8 and 5^g eq 7
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 8^g eq 3 and 7^g eq 6
then N41414s:=sub<N|N41414s,g>;
end if; end for;
```

```
for g in N do if 3^g eq 5 and 6^g eq 7
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 5^g eq 7 and 7^g eq 2
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 7^g eq 7 and 2^g eq 6
then N41414s:=sub<N|N41414s,g>;
end if; end for;

for g in N do if 7^g eq 8 and 6^g eq 3
then N41414s:=sub<N|N41414s,g>;
end if; end for;

Orbits(N41414s);

/*The orbits are are {1, 2, 3, 4, 5, 6, 7, 8}.

/*Now I can find the order of the coset
stabilising group [4,1,4,1,4].
This number, when divided into the
order of my control group N,
which is 48/48 = 1, gives the # of distinct
single cosets within
the double coset [4,1,4,1,4].*/

#N41414s;

/*48*/

/*Finding the index*/

T41414:=Transversal(N,N41414s);
for i in [1..#T41414] do
  ss:=[4,1,4,1,4]^T41414[i];
  cst[prodim(1, ts, ss)] := ss;
 end for;
 m:=0;
 for i in [1..330] do if cst[i] ne []
 then m:=m+1; end if; end for; m;
```

```
/*329*/

/*I now want to know to which double
coset does Nt_0t_1t_0t_1t_0t_i belongs.
I recall that the orbits are
{1, 2, 3, 4, 5, 6, 7, 8}.*/
/*I know that 414148 = 4141, so
each t_i in the orbit of (t_0)^-1 takes me
from 41414 to 4141.
This double coset is now closed under
right multiplication.*/


/*I know that we are done because my
index has reached 329 + 1
(from double coset [*] ) = 330.*/
```

Chapter 6

M_{11} as the Homomorphic Image of $3^{*4} :_m 2^{\cdot}S_4^+$

Now the center of $M_{11} \times 2$ is $= \; < x^2yx^{-2}tt^{(x^5)}tt^{(x^5)}t >$.
We factor $M_{11} \times 2$ by its center to obtain M_{11}. The details of the double coset enumeration are summarized in the Cayley diagram given below.

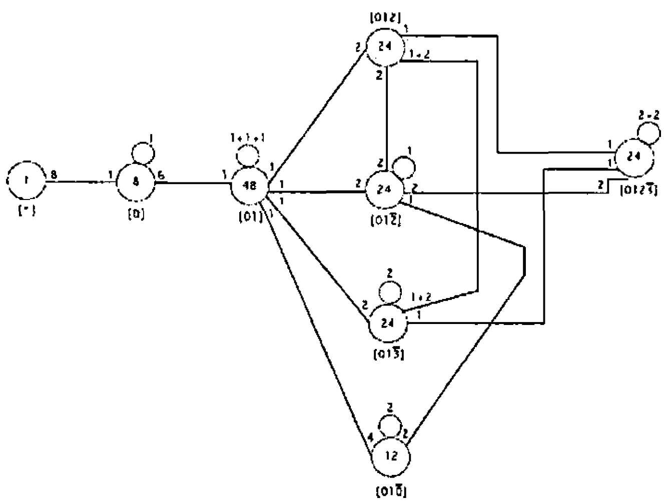

Figure 6.1: Cayley Diagram M_{11} over $2^{\cdot}S_4^+$

Our argument shows that the order of G is at most $|N| \times 165 = 7,920$, where 165 is the number of single cosets shown in the diagram above. We now show that $|G|$ is at least 7,920. Now $G =< x, y, t >$ acts on X, the set of the single cosets mentioned above. Thus $\alpha : G \rightarrow S_X$ is a homomorphism. Since $N =< x, y >$ acts by conjugation and t acts by right multiplication on the t_is, we compute the images xx, yy, and tt of x,y, and t, respectively, in S_X and verify that the additional relations hold in $< xx, yy, tt >$ within S_X and $| < xx, yy, tt > | = 7,920$. So $G/Ker\alpha \cong < xx, yy, tt >$. Hence $|G| \geq 7,920$.
Thus, $|G| = 7,920$.
We also verified that G satisfies a presentation of $\cong U_3(3) \times 3$.

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