Using non-Euclidean geometry in the Euclidean classroom

Kelli Jean Wasserman

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USING NON-EUCLIDEAN GEOMETRY IN THE EUCLIDEAN CLASSROOM

A Project
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Teaching:
Mathematics

by
Kelli Jean Wasserman
December 2009
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EUCLIDEAN CLASSROOM

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ABSTRACT

This study is designed to explore the ramifications of supplementing the basic Euclidean geometry curriculum, as it is currently taught in high schools, with spherical geometry, a non-Euclidean geometry curriculum. Due to the controversy surrounding Euclid's fifth postulate, other non-Euclidean geometries have evolved. Each of these geometries excludes Euclid's fifth postulate, replaces it with a new one, and thus has a very different structure. In this study, three high school geometry teachers incorporated a unit in spherical geometry that directly compared concepts of Euclidean geometry to that of spherical. Students were given a pre- and post-test to compare changes in students' understanding of Euclidean concepts. Additionally, pre- and post-surveys were administered along with interviews and student comments to measure changes in students attitudes. The goals are to determine if supplementing the current curriculum with spherical geometry curriculum will strengthen student understanding of the Euclidean geometry concepts, to see if exposure to higher mathematics will generate more interest in continuing mathematics education and to see if teachers who teach this unit will be inspired to implement more creative lessons plans in the classroom.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>iv</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>viii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>ix</td>
</tr>
<tr>
<td><strong>CHAPTER ONE: INTRODUCTION</strong></td>
<td></td>
</tr>
<tr>
<td>Mathematical Background</td>
<td>1</td>
</tr>
<tr>
<td>Implications for Teaching</td>
<td>3</td>
</tr>
<tr>
<td>Purpose of the Study</td>
<td>5</td>
</tr>
<tr>
<td>Goals of the Study</td>
<td>6</td>
</tr>
<tr>
<td>Theoretical Basis and Orientation</td>
<td>7</td>
</tr>
<tr>
<td>Limitations of the Study</td>
<td>8</td>
</tr>
<tr>
<td><strong>CHAPTER TWO: REVIEW OF THE LITERATURE</strong></td>
<td></td>
</tr>
<tr>
<td>History of Euclid's Fifth Postulate</td>
<td>9</td>
</tr>
<tr>
<td>Current Research on High School Geometry Curriculum</td>
<td>15</td>
</tr>
<tr>
<td>Statement of the Question</td>
<td>18</td>
</tr>
<tr>
<td><strong>CHAPTER THREE: METHODOLOGY</strong></td>
<td></td>
</tr>
<tr>
<td>Design of the Investigation</td>
<td>20</td>
</tr>
<tr>
<td>Population</td>
<td>21</td>
</tr>
<tr>
<td>Treatment</td>
<td>21</td>
</tr>
<tr>
<td>Instrumentation</td>
<td>29</td>
</tr>
<tr>
<td>Data Analysis Procedures</td>
<td>32</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
</tr>
<tr>
<td>--------------------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>Analyzing Student Test Data</td>
<td>32</td>
</tr>
<tr>
<td>Analyzing Student Survey Data</td>
<td>36</td>
</tr>
<tr>
<td>Analyzing the Qualitative Data</td>
<td>37</td>
</tr>
<tr>
<td>CHAPTER FOUR: RESULTS</td>
<td></td>
</tr>
<tr>
<td>Quantitative Analysis</td>
<td></td>
</tr>
<tr>
<td>Analyzing Student Test Data</td>
<td>38</td>
</tr>
<tr>
<td>The Rasch Analysis and T-test for Dependent Sample Data</td>
<td>38</td>
</tr>
<tr>
<td>Linear Regression Analyses</td>
<td>40</td>
</tr>
<tr>
<td>Analyzing Student Survey Data</td>
<td>48</td>
</tr>
<tr>
<td>The Rasch Analysis and T-test for Dependent Sample Data</td>
<td>48</td>
</tr>
<tr>
<td>Qualitative Analysis</td>
<td>50</td>
</tr>
<tr>
<td>CHAPTER FIVE: CONCLUSIONS, SUMMARY AND RECOMMENDATIONS</td>
<td></td>
</tr>
<tr>
<td>Conclusions</td>
<td>54</td>
</tr>
<tr>
<td>Student Understanding</td>
<td>54</td>
</tr>
<tr>
<td>Student Attitudes</td>
<td>56</td>
</tr>
<tr>
<td>Teacher Attitudes</td>
<td>57</td>
</tr>
<tr>
<td>Summary</td>
<td>60</td>
</tr>
<tr>
<td>Student Understanding</td>
<td>60</td>
</tr>
<tr>
<td>Student Attitudes</td>
<td>61</td>
</tr>
<tr>
<td>Teacher Attitudes</td>
<td>62</td>
</tr>
<tr>
<td>Limitations of the Study</td>
<td>62</td>
</tr>
</tbody>
</table>
LIST OF TABLES

Table 1. Implementation Schedule ........................................ 28

Table 2. T-test of Pre-post Test Analysis Using Measures
Obtained with Rasch Simple Logistic Model ......................... 39

Table 3. Coefficients for the Linear Regression Equation of
Matched Pair Pre-post Measures (EG+SG) ......................... 42

Table 4. Coefficients for the Linear Regression Equation of
Matched Pair Pre-post Measures (EG only) ...................... 45

Table 5. Descriptive Statistics of Linear Regression Analysis
of Pre-post Measures (EG only) ........................................ 46

Table 6. T-test of the Pre-post Survey Analysis Using Measures
Obtained with Rasch Simple Logistic Model ....................... 49
LIST OF FIGURES

Figure 1. Parallel Lines Cut by a Transversal, Visual Reference for Discussion of Ptolemy's Proof............... 13
Figure 2. Pre-Post Test Combined Data Variable Map................. 41
Figure 3. Post-Test Measures (EG+SG) Variable Map.................. 44
Figure 4. Post-Test Measures (EG Only) Variable Map................. 47
CHAPTER ONE
INTRODUCTION

"'Prove all things, hold fast that which is good,' does not mean demonstrate everything. From nothing assumed, nothing can be proved" (Lobachevski, 1914, p.5). Some have called geometry a science; others have called it a discipline. Regardless of its classification, all have agreed that plane geometry is a complex structure that is built on "the fewest and best controlled assumptions of the human intellect and of experience ..." (Callahan, 1931, p. 3). Euclid's greatness lies in his having stated and organized these assumptions and the propositions that follow them, thereby turning geometry into a science or discipline.

Mathematical Background

When one mentions geometry, most of the population automatically thinks about shapes, points and lines on a plane. The reason for this is that plane geometry was our geometry curriculum in school and it is still the standard high school curriculum today. This curriculum is based on Euclid's Elements, written around 300 B.C.E. It was Euclid who successfully organized the geometry understood by the Greeks (Callahan, 1931). He was the first to do so, which is the reason plane geometry is known as Euclidean geometry.
In *The Elements*, Euclid explicitly defined fundamental geometric terminology and then used this terminology to define basic concepts that he terms the "common notions." There are five common notions which are not restricted to geometry and are based only on real life experience such that all people, even non-mathematicians, will agree they are true. Euclid then stated five axioms, or postulates, which are concepts, assumed to be true without proof and are specific to geometry. His objective was to use the most basic ideas that were accepted as true based on logic and experience. The first four postulates are brief and simply stated and have remained unquestioned by mathematicians for more than twenty five centuries.

Contrary to the first four postulates, the fifth postulate is a much more complex statement that deals with the existence of parallel lines. Euclid understood the necessity of this statement, yet in his work he avoided using it if at all possible. This is an indication that Euclid himself may not have been satisfied with his stating it as a postulate. His first use of this postulate in the 29th proposition of *The Elements* was fundamental to the development of plane geometry as we know it today (Callahan, 1931). Why is it so important? Why is Euclidean geometry dependent on the acceptance of the fifth postulate?

The fifth postulate is known as the *parallel postulate* because it lays the foundation for work with parallel lines. In other words, there are several consequences to having lines on a plane with the property of being parallel. Using this postulate, we prove consequences including the congruence of
corresponding angles; the congruence of alternate interior angles; and interior angles on the same side of a transversal having a sum equal to two right angles. These consequences are needed to prove that, on a plane, the sum of the interior angles of a triangle is constant and equal to two right angles. These consequences also allow for the proof of triangle similarity, the Pythagorean Theorem and many other geometric statements (Ravindran, 2007).

Historically, the parallel postulate has generated much attention (Ravindran, 2007). Many excellent mathematicians have sought to prove the parallel postulate since it seemed too complex to be an assumption; they believed if it were true, it should be provable. These attempts have led to the discovery and development of many different types of geometry that assume a different postulate in place of Euclid’s fifth postulate. Some examples of such geometries are hyperbolic geometry, elliptical geometry, and spherical geometry, which are classified as non-Euclidean geometries, as well as other forms of Euclidean geometry, like absolute geometry which considers only the first four postulates.

Implications for Teaching

As reported by the California STAR testing website (2009), 47% of freshman (who represent approximately 32% of the state’s population of students enrolled in geometry) scored in the proficient or advanced categories on the CST Geometry test, but only 14% of sophomores and 7% of juniors (who together
represent approximately 63% of the state’s population enrolled in geometry) scored in the proficient or advanced categories. For the benefit of our students, it may be valuable to look for methods to enhance instruction of the geometry curriculum in high school classrooms. The curriculum currently taught in the standard high school classroom is based solely on Euclidean geometry. Non-Euclidean geometries are not included in the California Content Standards for Geometry. They are rarely introduced in high school geometry curriculum or in adopted textbooks and most high school geometry teachers lack understanding and experience with non-Euclidean geometries and so are not able to offer any enrichment using these fascinating extensions (Lenart, 1996). How would the incorporation of a non-Euclidean geometry curriculum affect student understanding of Euclidean geometry in the standard high school geometry classroom? What if teachers today addressed the controversy that surrounds the parallel postulate? How would teachers’ deeper understanding of the fifth postulate and its consequences impact the teaching of the current geometry curriculum in high schools? Would the introduction of non-Euclidean geometry enhance the teaching of the current Euclidean geometry curriculum taught in the high school classroom? Would teachers re-think their approach to teaching some of the standards? Would teachers and students gain a deeper understanding and appreciation of Euclidean geometry?

Spherical geometry is applicable to the world we live in. It is important for different professions including professional airline pilots, navigators, engineers,
astronomers, and more (Lenart, 1996). Therefore, exploration of the implications of the fifth postulate on the sphere would seem to be a suitable topic for discussion in the classroom. Students have a tendency to accept mathematical concepts as fact. Part of our role as math teachers is to inspire our students and teach them to question. The fifth postulate has been a source of question for mathematicians for thousands of years. Questioning is an important component of developing students' deductive reasoning skills and students' understanding of the need for proof. It is also important for igniting the minds of future mathematicians.

Purpose of the Study

The purpose of this project is to investigate the effects of incorporating a spherical geometry unit in a "standard" high school geometry classroom. A series of lessons was selected from the book Non-Euclidean Adventures on the Lenart Sphere, written by Istvan Lenart (1996), and organized into a cohesive unit. This unit included a series of 3-4 lessons (the first lesson was optional) that began with a navigational exploration on the sphere to obtain buy-in from the students. The unit gave students a brief preview of ways in which geometry is related to the real world. The parallel postulate and its equivalent statements hold true only in Euclidean geometry. For instance, 'given a line and a point not on the line there exists a unique line through the given point that is parallel to the given line,' (Playfair's axiom) is true only on the plane. In spherical geometry,
parallel lines do not exist. The unit explored this fact and its consequences and implications for some common Euclidean terms and concepts on the plane versus on the sphere.

Working with spherical geometry gave the high school students in the study an opportunity to explore some math that they would otherwise not be exposed to at this level of their education. Many concepts of spherical geometry are the same as that of Euclidean geometry, but often with different outcomes. The inclusion of spherical geometry offered "... students and teachers opportunities for learning that require creative thought, that allow for discovery, and that have applications in the real world" (Lenart, 1996, p. v). Participants in this project carried out their work on the Lenart sphere, a three dimensional spherical surface, which gave them a visual and kinesthetic experience with spherical geometry. This project examined different aspects of the impact of spherical geometry on the high school geometry classroom. These aspects defined the goals of the study.

Goals of the Study

Goal 1: To increase student understanding of Euclidean geometry, the basis for the California Content Standards for Geometry. The main focus of the project is to examine the effects of the incorporation of a spherical geometry unit on student understanding of the standard high school geometry curriculum. We expect that the inclusion of these lessons will make some fundamental Euclidean
concepts clearer for high school students, and give them a broader understanding of geometry and its place in the world.

Goal 2: To improve student attitudes regarding mathematics. The objective of this portion of the study is to measure changes in students' attitudes after students are presented with the interesting, real world applications of spherical geometry. We anticipate that the presentation will lead to an ignited interest in mathematics as a subject in high school or as a future topic of study.

Goal 3: To develop teachers' appreciation of non-standard curriculum. The intent for this aspect of the project is to provide teachers with a new method of approaching the instruction of Euclidean geometry. With the incorporation of a non-Euclidean geometry unit, we expect that the teachers will discover a renewed interest in mathematics and will use that energy to develop more creative lesson plans. We also expect that the teachers themselves will be motivated by the math driving them to explore and learn more about other non-Euclidean geometries, and to look for different ways to motivate and inspire the students in their own classrooms.

Theoretical Basis and Orientation

Several studies have been conducted using different non-Euclidean geometries in the Euclidean classroom. One study incorporated the use of dynamic geometry software in order to teach hyperbolic geometry in the classroom. Another study used fractals to demonstrate ways in which geometry
appears in nature. A third study considered taxicab geometry, which involves a new metric. To date, I have not found a study which uses spherical geometry as supplemental curriculum in the classroom. Nor have I found a study that offers statistical data regarding the use of non-Euclidean geometry in the classroom and its impact on student understanding of Euclidean geometry.

Limitations of the Study

The scope of the project was itself the main limitation. The amount of time available to teach the unit determined the number of lessons to be included. Since four lessons were the maximum allotted, we concluded that it was best to complete the unit in one week, rather than experiment with the small number of lessons over an extended period of time.

The unit was designed to be implemented after students had experienced the standard geometry instruction to see if the inclusion of the non-standard unit improved student understanding of Euclidean geometry; this meant teaching the unit at the end of the school year when both teachers and administrators were test-focused. Additionally, teachers had minimal time to study the material and to prepare to teach the unit. These limitations are discussed further in Chapter 5.
CHAPTER TWO
REVIEW OF THE LITERATURE

What is mathematics? Courant and Robbins describe it as “...the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality” (1996, p. xv). Although application of mathematics to physical reality plays an important role in the development of mathematics throughout history, it began with the challenge of early mathematicians to define and prove everything. The tendency toward explicit definition began with Eudoxus' theory of the geometrical continuum in an attempt to overcome the perceived difficulties in natural mathematical concepts including motion, infinity, and measurement of arbitrary quantities (Callahan, 1931). However, it was Euclid who first postulated axioms in his development of plane, or Euclidean, geometry (Courant and Robbins, 1996). This is known as the axiomatic method, which is a method for proving results (Greenberg, 1972).

History of Euclid's Fifth Postulate

Euclid, who lived around 300 B.C.E., made at least two significant contributions to mathematics. His most famous work was a series of thirteen scrolls known as The Elements. The first four of these describe the fundamental concepts that we understand as plane geometry, while the next nine address topics including geometric and abstract algebra, number theory, circles, angles
and constructions (Joyce, 2003). In this series, Euclid systematically defined geometric terminology and concepts which continue to endure, practically untouched by mathematicians who came after him (Callahan, 1931). His “common notions”, as written by Mlodinow (2001, p. 35) (a somewhat modernized version of Heath’s translation), are stated as follows:

1. Two things which are both equal to a third thing are also equal to each other.
2. If equals are added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

These common notions are assumptions based on logic and experience and are simply stated such that everyone can agree that they are true. Euclid then stated his five postulates which are specific to geometry. It is the statement of these five postulates which lay the foundation for Euclidean geometry as we know it today. Euclid uses these five postulates as the roots of the 465 proofs that make up the text of The Elements. Euclid’s five postulates (also closely related to Heath’s translation) are (Mlodinow, 2001, p. 35):

1. Given any two points, a line segment can be drawn with those points as its endpoints.
2. Any line segment can extend indefinitely in either direction.
3. Given any point, a circle with any radius can be drawn with that point at its center.

4. All right angles are equal.

5. Given a line segment that crosses two lines in a way that the sum of inner angles on the same side is less than two right angles, then the two lines will eventually meet (on that side of the line segment).

The fifth postulate is different from the other four. It is not simple and straightforward, nor is it easily accepted based on experience. Euclid did not seem satisfied with it himself as he avoided using it to prove the first 28 propositions in *The Elements* (Callahan, 1931). This postulate has been the source of controversy for thousands of years, as described below.

The idea of parallel lines was not invented by Euclid. As a matter of fact, it was theorized by the Greeks for many years preceding him. Euclid merely inherited the problem of proving it. The Greeks recognized that structure of geometric science was weak, due to a circular reasoning (Callahan, 1931). Aristotle exemplifies an argument in reference to the theory of parallel lines by describing "... the case of a person who should demonstrate A through B, and B through C, while C was naturally adapted to be proved through A, for it happens that those who syllogize, prove A by itself" (Callahan, 1931, p. 9). In other words, no one could find a way to prove that lines were parallel without already assuming that parallel lines existed. The fifth postulate, which originated with Euclid, took the theory of parallels and stated it as fact. By declaring this theory
a postulate and using it in the 29th proposition of the first book, Euclid eliminated the problem of the circular reasoning (Callahan, 1931). It was a great assumption given no other mathematician has been able to prove or disprove the fifth postulate on the plane. This allowed Euclid to begin the organization of plane (Euclidean) geometry as we understand it today. However, the statement of this theory as fact allowed others to visualize the gap in the science of geometry and left Euclid open to criticism.

For more than 2500 years, Euclid's successors have been trying to close the gap. Some have claimed that “the more complicated nature of assertion made by the fifth postulate suggested that it should be a theorem rather than an assumption. . .” (Blumenthal, 1961, p.5). This led to numerous attacks on Euclid's geometric system pertaining to the fifth postulate. Many mathematicians have attempted to “remove this flaw” (Blumenthal, 1961, p.5) from Euclid's work. Some of these attempted proofs have led to the discovery of statements which are considered to be equivalent to the parallel postulate, such as the well known Playfair's axiom. Others have led to the discovery of non-Euclidean geometries, which replace Euclid’s fifth postulate with an alternative postulate. Every attempt to prove it has failed. The oldest attempt on record was made by Ptolemy (100-178 A.D.). His method was to devise a new proof for the 29th proposition and then to deduce the fifth postulate from that proof. Ptolemy's proof, as described by Callahan is approximately the following (see Figure 1 for visual reference):
Figure 1. Parallel Lines Cut by a Transversal, Visual Reference for Discussion of Ptolemy's Proof.

Given two parallel lines, AB and CD, and transversal FG which cuts both lines at F and G respectively, then three cases are possible: (1) The sum of the interior angles, AFG and FGC, is greater than two right angles; (2) The sum of the interior angles, AFG and FGC, is less than two right angles; or (3) The sum of the interior angles, AFG and FGC, on the same side of the transversal is equal to two right angles. In the first case, Ptolemy argues that if AFG and FGC are greater than two right angles, then BFG and FGD must also be greater than two right angles, but this is an impossibility since the four angles together make up four right angles, thus the first case is false. And similarly, in the second case he argues that if AFC and FGC are less than two right angles, then BFG and FGD must also be less than two right angles which also leads to an impossibility, and therefore is also false. Thus, the third case must be the true case, that the sums
of the interior angles must be equal to two right angles. This then allowed him to deduce the fifth postulate (Callahan, 1931).

Proclus objected to Ptolemy’s reasoning on the basis that Ptolemy cannot assume that what was true on one side of the transversal was also true on the other side. Therefore, he claimed that the argument was incomplete. Ptolemy should have listed the possibilities that on either side of the transversal, the angles are either greater than, less than, or equal to two right angles. This would have given six possible combinations and Ptolemy considered and rejected only two. Therefore, Proclus claimed that Ptolemy’s argument must be considered inconclusive. Proclus then made attempts to prove the fifth postulate himself. His argument used this fact that parallel lines remain at a constant distance from each other. As it turns out, the fifth postulate provides the justification for this implication rather than following from it (Blumenthal, 1961). Therefore, Proclus also failed to prove the fifth. Instead, Proclus’ attempt became known as one of the many equivalent forms of the fifth postulate. Over the years, many other mathematicians also attempted to prove the fifth postulate including Saccheri, Wallis, Lambert, and Legendre (Callahan, 1931). Some of these attempts led to equivalent statements of the fifth postulate, while others led to the development of non-Euclidean geometries. None, however, was able to prove the fifth postulate.

Gauss was apparently the first to really understand the essence of the problem of the parallels. He felt that the fifth postulate was independent of the
other four postulates and he began to wonder, what if it were possible to have more than one line parallel to a given line through a given point? This was a revolutionary idea! And, Gauss devoted thirty years to exploring it (Blumenthal, 1961). However, he thought it might be too controversial and never published his work. He did share the idea with his good friend Farkas Bolyai who later shared it with his son, Janos. Janos began to explore the idea and came out with a published work around the same time as Lobachevski in the late 1820’s (O’Connor and Robertson, 1996). These were the first real accounts of the development of non-Euclidean geometry, specifically hyperbolic geometry.

With the fifth postulate constantly being challenged, new ideas continued to emerge. Thus, the various attempts to prove the fifth postulate cannot be deemed failures as they have led to the understanding that a fifth postulate must be assumed, and the realization that if Euclid’s fifth is removed, another postulate must be put in its place. This understanding led to the development of additional types of non-Euclidean geometry. These include elliptical geometry, spherical geometry, hyperbolic geometry and taxicab geometry which are also classified as non-Euclidean geometries, as well as absolute geometry, which only assumes the first four postulates with no replacement for the fifth postulate.

Current Research on High School Geometry Curriculum

Part of what is beautiful in mathematics is the undeniable nature of its structure, which is that most things can be substantiated with proof. What role
does mathematical proof play in our schools today? Patricio Herbst (2002) wrote
a review on the development of proof in the curriculum of geometry in American
schools. He suggested that although using the two-column proof may have
afforded some stability to the coursework, it happened at the expense of
developing students' ability to create new ideas. In the effort of teaching
students "how" to prove, proof exercises have turned into drill exercises rather
than exercises that build on higher order thinking skills. Herbst (2002, p. 308)
warned that "...making proof a separate object of study will not empower children
to use proof as a means to know with, but will rather separate the practices of
proving from the practices of knowing."

So, how do we teach proof? Or better yet, what is the most effective
practice in developing students' geometrical deductive reasoning skills? Some
studies have theorized that teaching geometry in a dynamic geometry
environment (which means using some form of geometry software) provides
students the opportunity to experience the geometrical theories of Euclidean
geometry first hand (using click and drag and other features), bridging the gap
between construction and deduction (Jones, 2001). The study reported by Jones
(2001) considered middle school aged students with little formal geometric
experience. The evidence supported the use of dynamic geometry environments
to build a foundation on which deductive reasoning skills could later be
constructed. What is important to note here is that it was not the use of a
computer or specific software that built deductive reasoning, but rather what
opportunity the computer or the software allowed the student to control: the ability to manipulate, explore and discover geometrical theories in a geometrical environment.

Part of building deductive reasoning skills includes developing the students need to question. If early mathematicians had not questioned Euclid’s fifth postulate, other geometries may never have been discovered. Addressing the controversy behind the fifth postulate may be the push that geometry students need to discover other possibilities and further develop their deductive reasoning skills.

Recently, some studies have been conducted that introduced non-Euclidean geometry into the high school classroom. In one study entitled “Hyperbolic Geometry in the High School Classroom,” Christi Donald introduced hyperbolic geometry to her students with the goal that students would learn “… to analyze and evaluate versus remember and recall previously learned concepts” (2005, p. 39). She found that with the use of technology to help students visualize hyperbolic space, students were able to grasp an abstract idea that had previously been out of reach and led to interesting classroom discussions. Her data included identical pre- and post-tests in which students were asked to classify a list of 34 theorems as valid in Euclidean, hyperbolic, both or neither geometries. Her analysis reported that student scores increased between the pre- and post-test, which seems obvious, as students would not be able to classify theorems pertaining to hyperbolic geometry before they were
actually introduced to it. It would have been more helpful for her to compare the scores on the Euclidean theorems before and after the exposure to the hyperbolic geometry unit. Neither her results nor her conclusion discussed such a comparison. She concluded that her unit "...was effective in helping students learn about Euclidean Geometry versus Hyperbolic Geometry" (2005, p. 38).

Another study conducted by Christina Janssen used taxicab geometry to build problem-solving skills. This study was more of a personal journey for the researcher. She found out personally that by looking at geometry from a different perspective, she had to stop and re-evaluate what she had previously learned. "How can you not understand Euclidean Geometry better with that type of comparing and contrasting[?] My geometric understanding has grown beyond belief..." (2007, p. 60) and has ignited her own interest in learning. Janssen briefly introduces some of her work to her students, but gathers no quantitative or qualitative data. She suggests that taxicab geometry "...has the ability to encourage problem solving" (2007, p. 61), but offers no statistical data to support her findings.

Similarly, Alan Muenzenmay conducted a study with the goal of motivating and enhancing the high school geometry classroom. Muenzenmay used fractal geometry to "...bring math to 'life'" (1997, p. 4) by making various connections to the current curriculum and to different ideas not normally addressed in geometry. This was a qualitative study in which Muenzenmay used student surveys and his own observations to build his data. Unfortunately, neither his students nor his
colleagues were as excited about the introduction of fractals as he was. He suggested using technological supplements to help stimulate excitement and better motivate his students.

Statement of the Question

Non-Euclidean geometries have much to offer the high school student, from exposure to real world implications to the need for proof. Many students take for granted the concepts of Euclidean geometry, until they are faced with a circumstance that contradicts what they have been taught. How would the inclusion of non-Euclidean geometry impact the high school geometry classroom? This study is designed to provide empirical data regarding the impact of using non-Euclidean geometry on both student understanding of Euclidean concepts and students' and teachers' attitudes towards mathematics and the inclusion of non-standard curriculum in the high school classroom.
CHAPTER THREE
METHODOLOGY

The three goals of this project were: (1) To increase student understanding of Euclidean geometry. (2) To improve student attitudes about mathematics. (3) To develop teachers' appreciation of non-standard curriculum. Accomplishment of these goals required both quantitative and qualitative instruments for data collection.

Design of the Investigation

The items developed for the purposes of this project include a pre-test and post-test for students, pre-survey and post-survey for teachers, a pre-survey and post-survey for students, a unit incorporating both spherical and Euclidean geometry, as well as interview questions for students and teachers, respectively.

The pre- and post-tests for the students were used to measure changes in students' understanding of the content standards for geometry. This data served as the first portion of the quantitative data for this study. The pre- and post-surveys for students were used to measure changes in attitudes towards mathematics as a current or future topic of study. This data served as the second portion of quantitative data for this study. The pre- and post-surveys for teachers were intended to measure changes in attitudes regarding the use of non-conventional curriculum, as well as teachers' ideas about furthering their
own education and learning more about other non-Euclidean geometries. Additionally, one teacher and two students were selected from the sample to participate in interviews, following completion of the unit. The teacher selected was the teacher who had had previous experience with spherical geometry. The students selected were from this teacher’s classroom, were recommended by the teacher and were willing to participate. This data served as the qualitative data for this study.

Population

Three credentialed geometry teachers from two different public high schools in two different school districts in Southern California volunteered to participate in this project. By extension their 9th, 10th and 11th grade students enrolled in a regular geometry course, became the subjects of this study. This gave a total sample size of 105 students. Permission to implement new curriculum was sought and granted from the administrators at the two schools. One teacher did have some experience with spherical geometry, but the other two teachers did not. None of the students had prior knowledge of spherical geometry.

Treatment

In order for the teachers and students to grasp the spherical geometry concepts, it was important that the lessons be visual and activity based, since it is difficult for students to grasp three-dimensional concepts on a flat surface. This played a major role in the development of the spherical geometry unit. The
Lenart sphere became the focal manipulative for this project because it would allow the teachers and students the opportunity to explore the concepts of spherical geometry on a three-dimensional, tangible surface. The lessons in the unit were obtained from the book written by Istvan Lenart (1996), entitled *Non-Euclidean Adventures on the Lenart Sphere*, which is meant to be taught in conjunction with the use of the spheres. The unit was comprised of four lessons (or "adventures"), which were taken from different chapters of the book. Together the lessons introduced spherical geometry, addressed differences in definitions and axioms between Euclidean and spherical geometry, and correlated with the background of this project by exploring consequences of Euclid’s fifth postulate. Each of the adventures was an activity where the students completed items on both the plane and the sphere, and then made a comparison chart of the two.

The first lesson entitled “What color is the bear?” was given as an optional lesson, due to time constraints. Two of the three teachers chose to include it. In this lesson, students explored the differences between what the possibilities are if a bear travels due south, due east, and then due north on a plane versus on a sphere. Is it possible for the bear to end up where he started? If so, where would he be, and thus what color is the bear? The aim of this lesson was to spark interest in the subjects of this study.

In the second lesson, “Can you draw a straight line on a sphere?” students explored the differences in drawing a straight line on a plane versus on a sphere.
When a line segment is drawn on the surface of a sphere, it is really part of a circle, which is known as an arc; and a line drawn on the surface of a sphere is given the term a great circle because if one or both ends of a segment is continued on a sphere, it/they will always end up where it/they started, making it finite. It was in this lesson that students learned that parallel lines do not always exist, i.e. there is no such thing as parallel great circles. In the absence of parallels, Euclid's fifth postulate becomes mute, which is why it must be replaced and why spherical geometry is non-Euclidean.

The third lesson posed the question, "How many points can two lines share?" On the Euclidean plane, lines may either coincide having infinitely many points in common, may be parallel having no points in common or they may intersect in one point. On the sphere, as previously mentioned parallel great circles are non-existent, and since great circles completely encircle the sphere, there are only two possibilities. One possibility is that the circles coincide, intersecting in infinitely many points, or they intersect in precisely two points. This lesson pointed out differences in the basic concepts of Euclidean and spherical geometry, again a difference based on the fifth postulate.

The final lesson dealt with the sum of the angle measures of a triangle. On the plane, the sum of the interior angles is constant at 180°. On the sphere the angle sum of a triangle is not constant. The sum on a sphere varies between two degenerate triangles: One triangle with angle sum of 180°, whose points are collinear (which means they lie on the same great circle) with one side that lies
on top of the other two giving one 180° angle and two 0° angles; and the other triangle where the three vertices are also collinear (encompassing an entire great circle) with three 180° angles and have an angle sum of 540°. Constant sum on a plane is a consequence of the fifth postulate. This lesson demonstrated to students what might happen if the fifth postulate does not hold.

Additional supplemental discussion questions were composed and given to each teacher to help connect the lessons with the background of this project. The unit is included in Appendix A. This unit was designed with the California Content Standards for Geometry in mind. The standards addressed in this unit (taken directly from the Mathematics Content Standards for California Public Schools, 1999, p. 42-43) included:

1.0 Students demonstrate understanding by identifying and giving examples of undefined terms, axioms, theorems, and inductive and deductive reasoning.

7.0 Students prove and use theorems involving the properties of parallel lines cut by a transversal, the properties of quadrilaterals, and the properties of circles.

11.0 Students determine how changes in dimensions affect the perimeter, area, and volume of common geometric figures and solids.
12.0 Students find and use measure of sides and of interior and exterior angles of triangles and polygons to classify figures and solve problems.

13.0 Students prove relationships between angles in polygons by using properties of complementary, supplementary, vertical and exterior angles.

16.0 Students perform basic constructions with a straightedge and a compass, such as angle bisectors, perpendicular bisectors, and the line parallel to a given line through a point off the line.

To begin the study, appointments were made with each teacher for individual teacher training. At each individual session of about 60-90 minutes in length, the teachers were given a schedule with specific instructions for each day of the unit. Each day of the schedule outlined the activity (e.g. survey/test or teach Lesson 1.2), gave a list of required materials, made suggestions for discussions or homework, and gave reminders (e.g. “Have students write their identification number’s at the top of all surveys and tests”). A demonstration was given by me on use of the sphere and its tools, and each lesson was discussed emphasizing the necessary spherical geometry concepts in comparison to its corresponding Euclidean geometry concepts. Suggestions were made on how to teach the lessons; however, it was left to each teacher to decide if he or she wanted to teach the lessons using the discovery method or using the more structured lesson plans. The lesson plans were guidelines, not complete scripts,
so that each teacher had the freedom to implement the lessons in his or her own teaching style. At the end of each training session, each teacher was given the teacher pre-survey. In addition, each teacher was given my email address and phone number in case they had further questions.

Each of the three classes was given a class set of Lenart spheres for their explorations during the implementation of the spherical geometry unit. This included enough materials for the class to be broken into eight groups, plus an additional set of materials for the teacher's use. Each class set included nine of each of the following materials: the Lenart sphere, a torus (which is a base for the sphere), spherical protractor, spherical compass and center locator, spherical transparencies, a set of Vis-à-vis markers, along with a few other miscellaneous materials. Thanks to a generous donation, the spheres were purchased for the purposes of this project and are now the property of the California State University, San Bernardino Math Department. All materials were collected from each teacher at the completion of the unit.

During the implementation of the unit in the three classrooms, permission was sought from each teacher to observe one of their classes. Due to extenuating circumstances, I was only able to observe two of the three teachers' classes. During the first observation, I was asked to participate in teaching the lesson since I was more familiar with the material. I gladly offered my help. This particular observation was day 2 of the unit, although this teacher chose to teach the optional lesson, Lesson 0.1, which was supposed to be taught on day 1 if
teachers' had time. Students were asked to assemble in their groups and to
gather their necessary materials. There were some free explorations with the
sphere and its tools, but students quickly settled down and got to work in their
groups. Many students tried to guess the answer to the question, "What color is
the bear?" before the lesson had begun. This day was really about the students
becoming familiar with the tools and becoming intrigued with spherical geometry.
As the teacher and I circulated, I was pleased to see that all of the students were
actively engaged in the learning. One student even commented that she really
liked my project. Each group was successful in determining that the bear lived at
the North Pole, and therefore was white.

The observation of the second teacher occurred on day 3 of the unit, with
the lesson entitled "How many points can two lines share?" This class was a
little more chaotic. As in the first class, there was some playing with the tools,
but as this was the third day of the unit, it did not last long. The teacher began
the lesson by introducing new terms and concepts, but the students were quite
unruly so I am not sure all of the students received all of that information. The
class then broke into their groups to begin the assignment. Some students did
not even take out a piece of paper to do the assignment, they tried to squeeze all
of their answers, including the comparison chart, into the margin of the Student
Guide they were given or just complete the assignment by discussing it. As I
circulated, I offered to answer any questions students may have had and posed a
few of my own questions to certain students to help get them on the right track.
ended up working with a student that, I am guessing, did not regularly complete assignments in this class. However, with just a little coaxing, he borrowed a pencil from another student and we went through the lesson together.

The unit of study was designed to be completed in five days, including pre- and post-testing and pre- and post-surveying. On the first day of implementation, teachers were scheduled to administer the pre-survey and the pre-test, as the first step in data collection. If they had remaining time, they were to teach the first lesson, as it was optional. Each teacher was then scheduled to teach the next three lessons over the next three consecutive days, followed by administration of the post-test and post-survey on the fifth day. The implementation schedule was laid out as shown in Table 1 below.

Table 1.

*Implementation Schedule.*

<table>
<thead>
<tr>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-survey</td>
<td>Lesson 1.2:</td>
<td>Lesson 2.1:</td>
<td>Lesson 3.4:</td>
<td>Post-test</td>
</tr>
<tr>
<td>Pre-test</td>
<td>Can you draw</td>
<td>How many</td>
<td>What is the</td>
<td>Post-survey</td>
</tr>
<tr>
<td>Lesson 0.1:</td>
<td>a straight line</td>
<td>points can two</td>
<td>sum of the</td>
<td></td>
</tr>
<tr>
<td>What color is</td>
<td>on a sphere?</td>
<td>lines share?</td>
<td>angle</td>
<td></td>
</tr>
<tr>
<td>the bear?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(optional)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*Note:* A copy of the complete schedule is included in Appendix F.
The student work on the lessons was not to be included as part of the data. One teacher made the decision to have her students make a “packet” out of the unit, which included all of the students’ work for each lesson and a cover sheet for the packet. The cover sheet was to include comments about the unit. She gave credit to those students who did this assignment, although she assured them that credit would be given for any comments, whether positive or negative. These comments have been included as part of the qualitative data.

The next portion of data, the post-test and post-survey was collected on the last day of the unit. It must be stated that due to unforeseen circumstances, not all three teachers completed the unit in five consecutive work days, and one teacher did not teach the final lesson, Lesson 3.4 on the sum of the angles of a triangle.

Finally, after completion of the unit and after all quantitative data had been gathered, the three interviews were conducted on two different days. This completed the qualitative data for the study.

**Instrumentation**

The student pre-test and post-test (included in Appendix B) were both developed under the supervision of my advisors. The tests consisted of both open-ended/proof questions and multiple choice questions. The two tests were not completely identical, additional phrasing and a few additional questions regarding spherical geometry were added to the post-test. For example, the first question on the pre-test was, ‘Could a triangle have two right angles? Explain
why or why not.' On the post test, the first question was worded identically; however a part A for Euclidean geometry and a part B for spherical geometry were included in the answer space. For the pre-test, it was not necessary to specify Euclidean geometry, as the students had no knowledge of the existence of non-Euclidean geometries at that point. It was important to leave a space in the test booklet for a spherical geometry response as each lesson was a comparison of Euclidean and spherical geometry and the addition of questions regarding spherical geometry gave the students a feeling of justification for participating in the study. Questions 2, 4 and 5 of the pre-test were identical to those on the post-test, and question 6 of the pre-test was identical to question 7 on the post test. Question 3 differed between the two tests a bit. The concept being questioned was the same, but was a little more specific on the post test. There was a total of seven questions on the pre-test and a total of twelve questions on the post-test, including a, b and/or c parts of each question. Each question was developed to measure the students' mastery of the geometry standards before and after exposure to the unit. To remain on schedule, the tests and surveys needed to be completed in one class period (about 50 minutes). However, due to time constraints, one teacher, who was the same teacher that did not teach the last lesson, gave the post-test and post-survey in the time remaining after students completed the final exam for that course. Approximately one third (i.e. 11 students) of the class did not have enough time to take the post-test and 5 students did not complete the post-survey. All
students' pre-test data were included in the analyses to obtain item calibrations. The Rasch logistic measurement model used in the analysis was able to account for the missing data items. For the comparison analyses, students who did not have both pre- and post-test data were excluded.

Two sets of surveys were composed, one for the students and one for the teachers. Each of the pre-surveys was identical to its post-survey and each pair of surveys was intended to measure changes in students' and teachers' attitudes. Specifically, the student survey was designed to explore the students' interest in the topics presented and interest in mathematics, in general, and included questions regarding students' educational goals. The teacher's survey was designed to explore teachers' attitudes regarding the use of non-conventional curriculum, as well as their attitudes toward teaching geometry.

The student's survey consisted of twenty questions and the teacher's survey consisted of seventeen questions. Subjects were given a five point Likert Scale for each of the questions relating to attitudes toward mathematics, educational goals, mathematics curriculum, or teaching geometry, with response choices ranging from Strongly Disagree (1) to Strongly Agree (5).

Each of the surveys was composed under the supervision of my advisors and then went through a validation process. A panel of ten readers, all students in the Methods of Teaching Geometry course of the Master of Teaching Mathematics program at California State University, San Bernardino, was asked to read through the survey questions and comment on the readability of the
questions and whether or not the questions addressed attitudes toward mathematics appropriately. All comments were taken into consideration, discussed with advisors and necessary revisions were made prior to the distribution of the surveys to the treatment teachers. These surveys are included in Appendix C of this paper.

Interview questions were developed under the supervision of my advisors. There were eighteen questions developed for the student interviews and twenty-two questions developed for the teacher interview. Interview questions are included in Appendix D.

Data Analysis Procedures

Student identification numbers were used as opposed to project generated identification numbers on all collectable data, as they were easier for the students to remember between assessments. These identification numbers were shortened to the last five digits of the student’s id numbers. There were no duplicate numbers. The raw data were organized by class in ascending order of id number’s and then each student was assigned a project number beginning at 001 and ending at 105.

Analyzing Student Test Data. The first step in the quantitative analysis was the process of grading the pre- and post-tests. Before the tests were graded, a sample of tests and a copy of the grading rubric (included in Appendix E) were given to an impartial credentialed secondary math teacher with geometry experience, who was not one of the treatment teachers in this study. This was in
an effort to eliminate bias. That teacher graded ten of the pre-tests and ten of the post-tests. I then graded the same tests and we compared our scores. Any discrepancies were discussed and compared to the grading rubric until agreements were made regarding scores, before the remaining tests were graded. Open-ended and/or proof questions on the test were graded using the five-point (i.e. 0-4) grading rubric and the answers to the multiple choice questions were scored dichotomously.

The raw data from the pre- and post-tests were compiled in an Excel spreadsheet for analysis. (The raw data is not included, but is available upon request.) As stated previously, there were more questions on the post-test than there were on the pre-test. There were twelve items on the post-test and only seven on the pre-test. The items that were identical on both tests were stacked in single columns. The remaining cells on the pre-test portion of the spreadsheet were left blank. The complete data matrix included student scores from both the pre- and post-tests.

The raw data from the student pre- and post-tests were analyzed using Georg Rasch's simple logistic measurement model (1960, 1980) for partial credit scoring (Masters, 1982). The Rasch measurement model was chosen for multiple reasons. One reason was that the Rasch analyses provide linear, interval scale data which are preferable to the ordinal raw score data. The Rasch model is explained by the following equation:
This equation provides the probability of success \((X_{ni} = 1)\) by person \(i\) with ability \(\beta_i\) on item \(n\) with difficulty \(\delta_i\). The unit of measurement on the Rasch scale is the ‘logit’. Other benefits of using the Rasch model include its ability to accommodate for missing data and its ability to estimate item parameters independent of the persons who took the assessment. The analysis assumes that the data fit the model. Deviations from the model can be detected by examining the output parameters.

To begin the analysis, the pre- and post- raw scores were stacked and the whole data set run at the same time. The resulting logit measures were then transformed to a user friendly scale that fit between 0-100 units. The data reported included the students' pre- and post-test means, standard deviations, and person separation reliability coefficient for the combined (i.e. the stacked) data set. The Winsteps program version 3.68.1 (Linacre, 2009) used to conduct the Rasch analysis also reported the Cronbach Alpha, which is the raw score reliability index. However, the Cronbach Alpha is approximate due to missing data (i.e. there were fewer items on the pre-test than on the post-test and not all students answered all questions), and is therefore not reported in the results of this study.

The Rasch model assumes that there is a unidimensional construct underlying the content of each item, in this case "geometry." Spherical and
Euclidean geometry together form the underlying construct of geometry. This model does not account for student understanding of the two geometries included in this project separately. The logit measures obtained in the Rasch analysis were used to conduct tests of statistical significance, including a t-test for dependent samples and two separate regression analyses. The t-test was run to test the null hypothesis that there was no difference between the students' pre-test and post-test means ($H_0: \mu_{post} = \mu_{pre}; H_1: \mu_{post} \neq \mu_{pre}; \alpha = .05$). Acceptance of the null hypothesis would indicate that it is quite likely for the sample means to come from the same population. The acceptance of the alternative hypothesis would indicate that the difference between pre- and post-means is an unlikely occurrence. The absence of threats to internal validity would support the conclusion that the inclusion of the spherical geometry unit improved students' understanding of Euclidean geometry.

One regression analysis was conducted to determine how well students' pre-measures in Euclidean geometry predicted measures on the post-test that included both Euclidean and spherical geometry. Matched pairs of the pre- and post-test logit measures obtained from the Rasch analysis were used to run a simple regression analysis with the pre-test measures in Euclidean geometry as the independent variable and the post-test measures as the dependent variable. (This model resulted in a slightly smaller sample size of 85 students, as we could only use the data of the students who took both the pre- and post-tests.) Additionally, the Pearson correlation coefficient ($R$) was computed for the
matched pairs, which indicated the strength of the association between the variables. The $R^2$ was used to examine the amount of variation in the post-test measures that is accounted for by the pre-test measures.

A second regression analysis was conducted to determine how well students' pre-measures in Euclidean geometry predicted post-measures for Euclidean geometry only. The difference between this analysis and the one described above was that this analysis included only items that assessed performance in Euclidean geometry. The items pertaining to spherical geometry were left out, giving a total of seven items for this analysis. A simple regression analysis of the matched pairs of the pre- and post-tests were used. The raw scores used in the Rasch analysis provided logit measures. The logit measures were then transformed to the user-friendly units used for the regression analysis.

**Analyzing Student Survey Data.** Survey data were also compiled on an Excel spreadsheet for analysis. The pre- and post-surveys were identical, so the stacked data matrix for the survey data was nearly complete. The missing data on this matrix was due to student absence on the day the surveys were administered or a lack of student responses. The survey data were also analyzed using Georg Rasch's simple logistic measurement model. The pre- and post- raw scores were stacked and the data sets run at the same time. The statistics reported include the students' pre- and post-survey means, standard deviations, and person separation reliability coefficient for the combined (i.e. the stacked) data set. The obtained logit measures were transformed to a user-
friendly scale and used in the analysis of the pre- and post-surveys. A t-test for statistical significance was conducted to compare the pre- and post-means and to test the null hypothesis \((H_0: \mu_{post} = \mu_{pre}; H_1: \mu_{post} \neq \mu_{pre}; \alpha = .05)\).

Acceptance of the null hypothesis would indicate that incorporation of the spherical geometry unit had no effect on students' attitudes toward mathematics. The alternative hypothesis stated the post-test mean was greater than the pre-test mean, which would indicate that the inclusion of the spherical geometry unit improved students' attitudes about mathematics.

Analyzing the Qualitative Data. The qualitative data consisted of three interviews and various student comments from the teacher generated packets. The teacher interview was conducted after school in a quiet classroom chosen by the teacher. Two students from the same class were selected for interview, as it was important to select students who had had a similar experience to be able to follow trends in their responses. Each student interview was conducted in the mathematics office at the school, during regular school hours, with other adults present. The interviews were recorded using a recording device and later transcribed for analysis.
CHAPTER FOUR

RESULTS

Two separate Rasch analyses were run, one for the student test data collected which measured the changes in students’ understanding of Euclidean and spherical geometry, and one for the student survey data collected that measured changes in students’ attitudes toward mathematics. The measures obtained by the Rasch simple logistic measurement model were then used to run tests of significance on each set of data. Additionally, three interviews and student comments compiled the qualitative data for this study.

Quantitative Analysis

Analyzing Student Test Data

The Rasch Analysis and T-test for Dependent Sample Data. The first analysis using Georg Rasch’s simple logistic measurement model was for the student pre- and post-test data. The pre-test included only Euclidean geometry items while the post-test included both Euclidean and spherical geometry items. Logit measures from the analysis of the stacked pre-post data were transformed to a user-friendly scale ranging from 0 to 100. The user-friendly mean (Umean) and unit scale (Uscale) were 58.76 and 7.86, respectively. A t-test for dependent sample data was conducted by comparing the pre- and post-means for matched pairs of data. The mean for the pre-test was reported at 54.14 with
a standard deviation of 9.08, and the mean for the post-test was reported at
57.22 with a standard deviation of 10.14. The t-test indicated that the mean of
the post-measure was statistically significantly greater than the mean of the pre-
measure (t=2.80, df=84, p<0.01). Since the p value was less than 0.05, we
rejected the null hypothesis and accepted the alternative hypothesis. Table 2
below provides the information for the t-test for correlated sample data.

Table 2.

*T-test of Pre-post Test Analysis Using Measures Obtained with Rasch Simple
Logistic Model*

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>t-statistic (df)</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre</td>
<td>54.14</td>
<td>9.08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Post</td>
<td>57.22</td>
<td>10.14</td>
<td>2.80 (84)</td>
<td>&lt;0.01</td>
</tr>
</tbody>
</table>

Mean Post-Mean Pre 3.08

Pooled standard deviation = 9.62  
Umean = 58.76  
Uscale = 7.86  
Person separation reliability coefficient = 0.74

Note. Differences in pre-measures are due to the amount data used by the Rasch analysis in estimating parameters.

To measure the effect size, Cohen's D was computed using the formula:

\[
\text{Cohen's } D = \frac{(Mean_{post} - Mean_{pre})}{SD_{pooled}}
\]
From the Rasch analysis measures, a pooled standard deviation of 9.62 was obtained and used to compute the Cohen’s D. The Cohen’s D value was reported at 0.32. Thus, the Cohen’s D indicated a moderately large positive educational effect, implying that the students’ post-test scores increased after the incorporation of the spherical geometry unit by 0.32 pooled standard deviation units. The person separation reliability coefficient for the combined data set was reported at 0.74.

Figure 2 below represents the variable map for the combined (i.e. stacked) pre- and post-test data (n=188) reported by the Rasch analysis. The left side of the dotted line represents the measures on the user-friendly scale of students’ ability in geometry. The “M” immediately to the left and toward the middle of the dotted line represents the mean ability level of the students taking the test. The right side of the map represents the calibration of the test questions. The “M” immediately to the right and toward the middle of the dotted line indicates the mean difficulty level (i.e. calibration) of the test questions. This map shows that the mean ability level of the students taking the test is slightly lower than the mean difficulty of the test questions. The item analysis obtained by the Rasch model for the test data indicated that the data fit the model.

**Linear Regression Analyses.** Two separate linear regression analyses were conducted on the student test measures using the user-friendly units obtained in the Rasch analysis. The simple regression equation used for this analysis was: \( y_{(EG+SG)post} = b_0 + b_1 \times x_{pre} \), where \( b_0 \) represented the intercept (i.e.
constant) and $b_1$ represented the coefficient of the independent variable (i.e. the pre-test). The purpose of this regression analysis was to evaluate how well the
pre-measures predicted cumulative post-measures on Euclidean and spherical geometry test items. The data set included a total of 85 matched pairs of pre- and post-data measures, with pre-test measures on Euclidean geometry used as the independent variable. Table 3 below illustrates the coefficients needed for the linear regression analysis which resulted in the following equation:

\[ y_{(EG+SG)\text{post}} = 35.48 + (0.40)x_{\text{pre}}. \]

Table 3.

Coefficients for the Linear Regression Equation of Matched Pair Pre-post Measures (EG+SG)

<table>
<thead>
<tr>
<th>Model</th>
<th>Unstd. Coeff.</th>
<th>Std. Coeff.</th>
<th>Beta</th>
<th>t-statistic (df)</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (Constant)</td>
<td>35.48</td>
<td>4.84</td>
<td>0.45</td>
<td>7.33 (84)</td>
<td>&lt;0.01</td>
</tr>
<tr>
<td>Pre EG</td>
<td>0.40</td>
<td>0.09</td>
<td>4.57</td>
<td>(84)</td>
<td>&lt;0.01</td>
</tr>
</tbody>
</table>

a. Predictors: (Constant), Pre EG  
b. Dependent Variable: Post EG+SG  
Pearson correlation \((R) = 0.45\)  
\(R^2 = 0.20\)

The pre- and post- matched student measures were analyzed using SPSS 17.0. The pre- and post-means reported were the same as those reported in the t-test. The Pearson correlation \((R)\) was reported at 0.45 with an \(R^2\) of 0.20. The
$R^2$ value indicated that approximately 20% of the variance in post-test measures was associated with the variance in the pre-test measures.

Figure 3 below represents the variable map for the Euclidean and spherical geometry (EG+SG) post-test data. The left side of the dotted line represents the post-test measures of students’ ability in geometry (EG+SG) given in the user-friendly units. The “M” immediately to the left and toward the middle of the dotted line represents the mean ability level of the students taking the test. The right side of the map represents the calibration of the test questions. The “M” immediately to the right and toward the middle of the dotted line indicates the mean difficulty level of the test questions. This map indicates that the mean ability level of the students taking the test is the same as the mean difficulty of the test questions. It is also important to note here that seven of the twelve questions lie above the mean, but more importantly that many students performed above this mean as well with only a few students performing below the easiest questions. This implies that all levels of students were appropriately challenged by the post-test.

The second linear regression represented a comparison of the students’ pre- and post-test measures in Euclidean geometry. The pre-test measures that assessed student abilities in Euclidean geometry were used as measures on the independent variable in the regression equation. The post-test measures from the Euclidean geometry (EG only) items were used as the measures on the dependent variable. Data pertaining to the spherical geometry items were not
Figure 3. Post-Test Measures (EG+SG) Variable Map

included in this analysis. The regression equation that was used to determine how well the pre-assessment measures in Euclidean geometry predicted performance in Euclidean geometry on the post test read (see Table 4 below):

\[ Y_{EG_{post}} = 36.85 + (0.39)X_{pre} \]
Table 4.

*Coefficients for the Linear Regression Equation of Matched Pair Pre-post Measures (EG only)*

<table>
<thead>
<tr>
<th>Model</th>
<th>Unstd. Coeff.</th>
<th>Std. Coeff.</th>
<th>Beta</th>
<th>t-statistic (df)</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (Constant)</td>
<td>36.85</td>
<td>5.16</td>
<td>0.43</td>
<td>7.12 (84)</td>
<td>&lt;0.01</td>
</tr>
<tr>
<td>Pre EG</td>
<td>0.39</td>
<td>0.09</td>
<td>4.31 (84)</td>
<td>&lt;0.01</td>
<td></td>
</tr>
</tbody>
</table>

*a. Predictors: (Constant), Pre EG  b. Dependent Variable: Post EG only*

The mean and standard deviation for the pre-test items in this analysis are reported at 55.65 and 10.34, respectively. The mean for the post-test items regarding EG only was reported at 58.78 with a standard deviation of 9.53. (The variation in the means for the pre-test in this analysis and the analysis above is due to the amount of data considered in the analysis. The Rasch analysis gives an estimate based on the available data, and this analysis is for Euclidean items only, which offers substantially less data.) The regression gave a $t$-value of 2.71 with 84 degrees of freedom. The Pearson correlation ($R$) for this sample was reported at 0.43 with an $R^2$ value of 0.18. The Pearson correlation and $R^2$ values indicated that there was approximately 18% variance in the post-test measures that was associated with the variance in the pre-test measures (see Table 5 below).
Table 5.

Descriptive Statistics of Linear Regression Analysis of Pre-post Measures (EG only)

<table>
<thead>
<tr>
<th>Assessment</th>
<th>N</th>
<th>Mean</th>
<th>Std. Dev</th>
<th>t-statistic (df)</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre EG</td>
<td>85</td>
<td>55.65</td>
<td>10.34</td>
<td>2.71 (84)</td>
<td>&lt;0.01</td>
</tr>
<tr>
<td>Post EG only</td>
<td>85</td>
<td>58.78</td>
<td>9.53</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Pearson correlation ($R$) = 0.43  $R^2 = 0.19$

Note. Differences in pre-measures are due to the amount data used by the Rasch analysis in estimating parameters.

Figure 4 below is the variable map showing the location of the persons and items on the Euclidean geometry (EG only) post-test. The map shows a total of 7 questions on Euclidean geometry. The left side represents the students’ ability in Euclidean geometry given in the user-friendly units. As in the previous variable map, the "M" directly to the left and toward the middle of the dotted line represents the mean ability level of the students taking the test. The “M” on the right side of the map and toward the middle of the dotted line indicates the mean difficulty level of the test questions. It is important to note that on this map, several students performed above the most difficult test items (i.e. Q2, 4a and 4b) and only a few students demonstrated abilities below the easiest test items. In this case, the test items were not well targeted for the more able students (after the inclusion of the spherical geometry unit). The test would have been
better targeted with additional questions that were more challenging for the students.

Figure 4. Post-Test Measures (EG Only) Variable Map
To address concerns related to internal validity, the following precautions were taken during the design and implementation of the spherical geometry unit. Volunteer teachers were solicited from various school districts in Southern California, of those solicited, three teachers volunteered. Every student in each of these classes was asked to participate in this study. Each class was a regular geometry class and received no special treatment prior to or in conjunction with the incorporation of the spherical geometry unit (i.e. this was the students' only geometry class, and no other instruction in geometry occurred during the course of the unit). Instruction occurred in a controlled environment. The study was designed to take place in a short period of time (5 days). The assessments used were a combination of short answer/proof questions and multiple choice questions and though the two tests had some identical questions, the tests as a whole were not identical. There were no control groups, only treatment groups and they were not aware of each other. There was some attrition as some students did not complete both the pre- and post-assessments.

Analyzing Student Survey Data

The Rasch Analysis and T-test for Dependent Sample Data. Georg Rasch's simple logistic measurement model was used once again to construct interval scale measures for the Likert-type student survey data. The measures were used to compare the students' pre- and post-survey mean performance. Logit measures from the analysis of the stacked pre-post survey data were transformed to a user-friendly scale ranging from 0 to 100. The user-friendly
mean and unit scale were 35.36 and 6.32, respectively. Table 6 below shows a mean measure for the students on the pre-survey of 34.48 with a standard deviation of 3.43 and the mean measure for the post-survey of 34.65 with a standard deviation of 3.26.

Table 6.

T-test of the Pre-post Survey Analysis Using Measures Obtained with Rasch Simple Logistic Model

<table>
<thead>
<tr>
<th>Survey</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>t-statistic (df)</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre</td>
<td>34.31</td>
<td>3.51</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Post</td>
<td>34.74</td>
<td>3.22</td>
<td>2.13 (91)</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Mean Post-Mean Pre 0.43

Pooled standard deviation = 3.35 Cohen's D = 0.13
Umean = 35.36 Uscale = 6.32
Person separation reliability coefficient = 0.71

A t-test for dependent sample data was run on these measures. The t-test indicated that the post mean was statistically significantly greater than the mean of the pre-measure (t=2.13, df=91, p=0.04). Since the p value was less than 0.05, we rejected the null hypothesis and accepted the alternative hypothesis.
The Cohen's D of 0.13 indicated a small difference between the means in terms of the pooled standard deviation. Although the analysis for the student survey data indicated a statistically significant difference regarding student’s attitudes toward mathematics after incorporation of the spherical geometry unit as measured by the pre- and post-surveys, the practical significance as measured by the effect size was small.

Additionally, the item analysis for this data set indicated that some of the survey items were not a good fit for the model. Further investigation of the misfitting items is needed to determine why they misfitted (e.g. they create noise in the data or are redundant), however this information will not affect the data reported in this study. If this study were to be conducted again, revision of the survey items would be necessary.

Qualitative Analysis

The qualitative data in this study included three interviews, one interview of a teacher (who is labeled Teacher B), and two student interviews. The students were chosen from Teacher B’s class. The three subjects were chosen from the same environment to see if there were trends in their responses. In other words, the responses were compared to see if the teacher’s interpretations regarding student participation, attitudes and comprehension of Euclidean geometry concepts as compared to spherical geometry concepts was reflected in the student responses. This was the same teacher who had had her students
make packets of the spherical geometry unit with written student comments given on a student designed cover sheet. These comments were included as part of the qualitative data as well.

Regarding participation during the spherical geometry unit, Teacher B reported that her class remained focused and on-task. As she monitored her class, she commented on how each of the groups of students responded appropriately to her questions and was pleased when they continued to talk “. . . about things in discovery, things on their own. It was really fun to watch. It was a very excited classroom!” (personal communication, June 10, 2009). She also reported she had chosen this class specifically to participate in the unit as this was a “good period.”

Most of the student comments revolved around the differences between spherical and Euclidean geometry along with statements regarding enjoyment of participating in the unit. There were only two student comments regarding participation, and one came from a low performing student. This student commented,

I think it was very interesting working with my group. Everyone had different opinions and it was creative. I enjoyed talking to my group about the sphere. My group was well-matched, I like how they think. I learned how to cooperate with others. My group really listened and we did argue a lot because we wanted to see who was right at what (see Student Comments, Appendix G).
One additional student commented, "Everyone enjoyed working with S.G. [spherical geometry] and using the sphere" (see Student Comments Appendix G).

The teacher interview revealed that this teacher had herself struggled with math in school. It was two of her teachers whose influence turned her around and inspired her to pursue math education as a profession. She was looking forward to implementing the spherical geometry unit. She expressed it was something that she, "... really wanted to do. For them [the students] to be able to touch the math and see it work I think turns them [the students] around into liking math more also" (Teacher B, personal communication, June 10, 2009). This was important to note because motivating students is an important part of teaching. This teacher understood that first hand and was looking for new methods to get her students excited about learning math.

Regarding student attitudes, of the 26 packets of student comments turned in, all comments were positive. Eleven of the comments began with the statements, "I enjoyed..." or "I liked..." Sixteen students made specific reference to comparing the plane and the sphere (e.g. "...I learned that construction on a plane is very different than construction on a sphere..."), eleven of the student comments made reference to specific spherical concepts from the unit (e.g. "...the bear was white") and five students made positive statements regarding the hands-on use of the sphere. Both student interview
subjects commented they liked doing the spherical geometry unit and were interested in learning more about spherical geometry.

The teacher reported the material was comprehensible for the different student levels. Regarding the lessons included in the unit, she stated, "...[they were] well written for the variety of students that I have in my classroom." Both student interview subjects commented the spherical material was easy in the end and the comparisons to Euclidean geometry made the Euclidean concepts clearer for them (Participants 1 and 2, personal communication, June 11, 2009).

The quantitative data compiled for the pre- and post-tests showed a positive correlation between students' understanding of Euclidean geometry and incorporation of the spherical geometry unit, causation will be discussed in the next chapter. Although the survey data indicated a small positive effect on students' attitudes about mathematics, the qualitative data, including the interviews and student comments, supported this indication. These results will be discussed further in the next chapter, as well.
CHAPTER FIVE
CONCLUSIONS, SUMMARY AND RECOMMENDATIONS

The foundation for the design of this project was the need for empirical data that describes the effects of incorporating a non-Euclidean geometry in the standard high school geometry classroom. Three different aspects of these effects defined the goals of this study.

Goal 1: To increase student understanding of Euclidean geometry.
Goal 2: To improve student attitudes about mathematics.
Goal 3: To develop teachers’ appreciation of non-standard curriculum.

Conclusions

Student Understanding

Three statistical analyses were run on the student test data. The Rasch probabilistic measurement model was used to construct linear interval measures before running statistical procedures. The raw data from the students’ pre- and post-tests were stacked in columns in an Excel spreadsheet and analyzed together. This analysis yielded mean measures for the pre- and post-tests. These measures were used to run a t-test for statistical significance. The p-value calculated during this analysis led to the rejection of the null hypothesis and acceptance of the alternative hypothesis. Thus, the t-test indicated an increase in student ability in the underlying construct of geometry (including both
Euclidean and spherical geometry). Since the threats to internal validity were minimal, there is strong evidence that suggests that the increase in student understanding of Euclidean geometry was due to the incorporation of the spherical geometry unit. In other words, in one week with teachers who were inexperienced with the spherical geometry curriculum, students performed better on the post-test items after inclusion of the spherical geometry unit, whether they were items regarding Euclidean geometry or spherical geometry.

Variable maps (Figures 2, 3 and 4) obtained in the Rasch analysis showed a progression of an increase in student ability. The mean student ability went from below the mean item calibration in the combined data map, to equivalent to the mean item calibration in the complete post-test (which included both Euclidean and non-Euclidean test items), to above the mean item calibration on the post-test data of Euclidean items only. Again, this was accomplished in one week, with instruction of only three or four lessons of spherical geometry.

Additionally, the simple linear regression analyses reported that the pre-test measure was a good predictor of the post-test measures, for both the Euclidean geometry and spherical geometry model (the whole post-test) and for the Euclidean geometry only model (using only the Euclidean geometry related test items). This meant that the linear relationship between the variables in the analysis, the pre- and post-test measures, had a positive correlation (i.e. a positive slope on a linear graph). The Pearson correlation coefficient was reported at 0.45 (Table 3) when pre-test measures were correlated with
measures on the combined Euclidean and spherical geometry test; and it was reported at 0.43 (Table 5) when pre-test measures were correlated with measures on the Euclidean only test. These values also indicated a positive relationship between the variables. In other words, since the $r$ values reported lie between 0 and 1, the indication is that the two variables (i.e. the pre- and post-test measures) will either increase or decrease together. In this case, they increase together.

The students in the sample generally performed better on the multiple choice items, as opposed to the open-ended or proof type questions on the assessments. With the data provided by this study, it was difficult to predict if the reason for this was that students have not developed their deductive reasoning skills, or if they lacked experience with open-ended/proof type questions that require justification.

**Student Attitudes**

The statistical analysis of the student survey data indicated that there was a statistically significant difference in the pre- and post-survey means, although the Cohen's D indicated a small effect size. The item analysis indicated that misfitting survey items could be problematic in the measurement of changes in student attitudes regarding mathematics. Therefore, the evaluation of the qualitative data was helpful in drawing conclusions about the impact spherical geometry had on students' attitudes towards mathematics. It is significant to note that of the 26 comments given by students on the teacher generated
packets, there were 25 comments about learning the non-standard curriculum, and each was positive. These comments were solicited by only one of the treatment teachers who participated in the study. My observation of this class confirmed that the students were actively engaged during the implementation of the observed lesson. Student motivation is linked to student performance; if a student likes what he/she is learning, he/she is more likely to learn it, as indicated in Kirsten Olson's study on improving student attitudes and performance in mathematics (1998). Additional research should be done on attitudes and learning in the geometry classroom, yet based on the student comments from this class, the majority of students had a positive learning experience with the spherical geometry unit. This was confirmed during the two student interviews. The comments made by the students interviewed corresponded to the comments made on the student packets that the spherical geometry unit was interesting and that they had had a positive learning experience. Both students commented that learning about spherical geometry made some Euclidean concepts easier for them because of the comparisons they did with the plane and the sphere. They also commented that they would be interested in learning more about spherical geometry.

Teacher Attitudes

As previously mentioned, no statistical analysis on the teacher survey data was run since the sample size was too small to produce a reliable statistical analysis. However, some of the raw data items from the surveys provided insight
into the teachers' attitudes about implementing the spherical geometry unit in their classrooms. Teachers are labeled A, B and C for ease of reference and each is discussed separately.

Teacher A was the teacher whose class I did not have the opportunity to observe. Teacher A had no previous experience with spherical geometry. He was the teacher who opted not to include the first lesson, “What color is the bear.” This teacher did complete all of the mandatory lessons in the unit and did have the majority of his students complete all items for data collection. On completion of the study, Teacher A verbally commented that he would like to try the spherical geometry again when he had more time. On both the surveys, this teacher responded with a 5 (Strongly Agree) to the statement ‘I enjoy teaching mathematics.’ Also, to the statement, ‘I have an in depth understanding of Euclid’s fifth postulate,’ this teacher responded with a 3 (Undecided) on the pre-survey, and a 4 (Agree) on the post-survey. Finally, this teacher responded with a 5 (Strongly Agree) to the statement, ‘Adding some non-Euclidean geometry curriculum to the current high school geometry curriculum may be beneficial to students' understanding of Euclidean geometry’ on the post-survey.

Teacher B was the only teacher who reported having prior experience with spherical geometry. This was the teacher who ran out of time, so she did not teach the last lesson in the unit and her students were administered the post-test and post-survey after their final exam on the last day of school. Teacher B was also the teacher who generated the packets which included the student
comments regarding the incorporation of the spherical geometry unit, which became a valuable part of the qualitative data for this study. Similar to Teacher A, Teacher B teacher responded with a 5 (Strongly Agree) to the statement ‘I enjoy teaching mathematics’ on both surveys; and to the statement, ‘I have an in depth understanding of Euclid’s fifth postulate,’ Teacher B responded with a 3 (Undecided) on the pre-survey, and a 4 (Agree) on the post-survey.

During the teacher interview, which was with Teacher B, this teacher explained that she was excited to teach this unit for this study, and that she planned on teaching it again in the future. Based on Teacher B’s observations of her class and discussions with her students, this teacher believed that her students had a clearer understanding of the Euclidean concepts after completing the comparison activities. She explained in the interview that sometimes the introduction of something new can be intimidating, for both the student and the teacher, but the sphere is a ball and “…a ball is friendly...so it’s not so brand new and not intimidating.” Her students had fun. This teacher believed that adding some non-Euclidean curriculum to the current curriculum could be beneficial to her future students’ understanding of Euclidean geometry. This statement was confirmed with her response of 5 (Strongly Agree) on the post-survey.

Teacher C completed all of the lessons in the unit, and had the majority of his class complete all items for data collection. Teacher C responded with a 2 (Disagree) to the statement, ‘I enjoy teaching mathematics’ on both the pre- and post-surveys. To the statement, ‘I have an in depth understanding of Euclid’s 5th
Postulate' this teacher circled 5 (Strongly Agree) on the pre-survey and 2 (Disagree) on the post-survey. This was an important response. This teacher learned that he did not know as much about the fifth postulate as he had previously thought, which was an indication of growth in this teacher's understanding of the fifth postulate. Finally, this teacher responded with a 4 (Agree) to the statement, 'Adding some non-Euclidean geometry curriculum to the current high school geometry curriculum could be beneficial to students' understanding of Euclidean geometry.'

In conclusion, these responses indicated that the teachers' attitudes about teaching mathematics did not change after teachers incorporated the spherical geometry unit. All three responses indicated that the teachers had a better understanding of the fifth postulate and all three teachers agreed that incorporating a non-Euclidean geometry could be beneficial to geometry instruction in the classroom.

Summary

Student Understanding

The measurement of the changes in student understanding of Euclidean geometry was evaluated from three different perspectives. Each analysis indicated that students' understanding of Euclidean geometry increased after the incorporation of the spherical geometry unit. The Rasch analysis provided the ordinal scale data that was used to run tests of statistical significance. A t-test
indicated that the post-mean was statistically significantly greater than the pre-mean. The two separate regression analyses were run on the test measures. These analyses indicated that the pre-test means were good predictors of both the post-test means for the cumulative test items and the Euclidean geometry only test items, respectively. All of these analyses make a strong argument in favor of including a non-Euclidean geometry curriculum to improve student understanding of Euclidean geometry.

**Student Attitudes**

The measurement of changes in students' attitudes regarding mathematics was evaluated both quantitatively and qualitatively. Quantitatively, there was a small statistically significant difference regarding attitudes measured by the pre- and post-surveys. The raw scores obtained from administering the surveys were transformed into interval scale measures using Rasch's probabilistic measurement theory. The Rasch analysis, however, indicated that some of the survey questions may not have been a good fit for the model. This was an indication that some of responses to the survey items may have been too predictable or included too much error variance. Since the findings had minimal practical significance, evaluation of the qualitative data including the two student interviews and the student comments from the teacher-generated packets became more important. Verbal and written comments from the interviews and packets indicated a positive reception of the non-standard curriculum (i.e. students *liked* working with spherical geometry).
Teacher Attitudes

The measurement of changes in teachers' attitudes regarding the use of non-standard curriculum was evaluated by teacher pre- and post-surveys. Although no statistical analysis was conducted, evaluation of the raw data suggested that the treatment teachers all agreed that inclusion of some non-Euclidean geometry would be beneficial to student understanding of Euclidean geometry.

Improving student attitudes about mathematics and improving teacher appreciation for the use of non-standard curriculum is a valuable consequence of the inclusion of the spherical geometry unit. Ultimately, measuring the changes in student understanding of Euclidean geometry was the most important aspect of this project. An indication that the incorporation of a non-Euclidean geometry has a positive educational effect could have major implications on the methods of teaching geometry in the future.

Limitations of the Study

As discussed in the introduction, the scope of the project was the main limitation. Testing the impact of a non-standard curriculum on student understanding of Euclidean geometry thoroughly requires a long-term large-scale study; it was done on a very small scale for this project. The best fit for this study was to include a unit of spherical geometry after students had received the majority of the standard instruction of Euclidean geometry, at the end of the
school year. One consequence of this plan was that both teachers and administrators were very “test-focused,” as schools were just completing the California state standardized testing and classes were beginning to prepare for final exams. Thus, this timing afforded teachers minimal time to prepare for this unit, which was unfortunate since only one of the teachers had some prior experience with spherical geometry. Teacher training sessions were generally fit into a prep period or lunch hour.

Additionally, teachers did not have much instructional time left in the school year for implementation which restricted the window for instruction of the spherical geometry unit. A maximum of five days was allotted for instruction, which restricted the number of lessons to four. Two of the three teachers were only able to teach 3 of the 4 lessons included in the unit, which may have affected student scores on the post-tests. Yet, even under these circumstances, all of the statistical analyses reflected an increase in students’ understanding of Euclidean geometry.

Implications and Recommendations for Future Research

Non-Euclidean geometries have much to offer the high school teacher and student, from understanding real world applications to understanding and internalizing the need for proof. The unit planned for this project was designed on a very small scale. It would be valuable to expand the project regarding content and to implement it on a larger scale to obtain additional significant data.
In the search for studies that used non-Euclidean geometry in the classroom, the studies obtained for the background of this project did not give statistical information on how the inclusion of non-Euclidean geometry impacted students' understanding of Euclidean geometry. This is an important aspect for research since California (as well as the other states) school curriculum revolves around defined geometry content standards, all of which are founded upon Euclidean geometry. Non-Euclidean geometries are not included in the standards and therefore, they receive very little, if any, attention in adopted textbooks and curriculum. Personal experience indicates that most district pacing guides do not include lessons using non-Euclidean geometries either.

It would be valuable to explore the impact of teaching spherical geometry, or one of the other non-Euclidean geometries, in conjunction with Euclidean geometry for an entire course. A researcher could compare test scores at the end of the course of a treatment group that incorporated non-Euclidean geometry as part of their normal curriculum, with a control group that did not. The unit used for this project included only four lessons regarding spherical geometry and even so yielded a positive correlation on student understanding of Euclidean geometry. Comparison lessons, like those presented in this unit, taught throughout an entire course could have a major impact on student understanding of geometry (both Euclidean and non-Euclidean) as a whole. This could lead to a positive change regarding how geometry is taught in public high schools in the
future. I know that I am looking forward to implementing spherical geometry in my own future classrooms.

Another important aspect for research is how the addition of a non-Euclidean geometry component in the curriculum impacts student attitudes toward mathematics. Before we can teach our students, we must motivate and inspire them. The survey data indicated a positive (although small) effect on students' attitudes towards mathematics. The positive student comments in both the interviews and the student packets supported this indication.

A valuable aspect of the spherical geometry lessons incorporated in this study was that they directly compared the spherical geometry concepts to the analogous Euclidean concepts. The results of this study provide additional support for Christina Janssen's (2007) conclusion that direct comparison can greatly impact students' understanding of geometry, as she mentioned in her personal journey through Taxicab geometry. Due to the structure of these lessons as direct comparisons, if teachers wanted to include these lessons in their standard curriculum, it would be possible for them to cover the same amount of material in conjunction with spherical geometry, without needing additional time. Additionally, the Lenart sphere and its tools make these lessons easy to implement. However, the teachers need to be convinced of the value of this addition to the curriculum. Very few high school mathematics teachers have experience with non-Euclidean geometry; this is especially true for those who do not hold a Bachelor's degree in mathematics. Unless research indicates that
non-Euclidean geometry can positively impact either students' understanding of Euclidean geometry, or students' attitudes towards learning mathematics, it is not likely that math teachers, districts or schools will add non-Euclidean geometry to their curriculum.

Teachers' inexperience with non-Euclidean geometry leads us to the next important recommendation: To develop a seminar, or a series of seminars where math, in particular geometry, teachers can learn about non-Euclidean geometries for professional growth. Teachers need an appropriate environment to learn about and experience the benefits of incorporating non-Euclidean geometry themselves. Once teachers gain experience, further testing of the ideas presented in this project will be possible.
APPENDIX A

SPHERICAL GEOMETRY UNIT
What color is the bear?

You may be familiar with the following riddle:
A wandering bear leaves home and walks 100 kilometers south. After a rest, she turns west and walks straight ahead for 100 kilometers. Then she turns again and walks north. To her surprise she finds that she arrives back home again. What color is the bear?

Construction on the Plane
Sketch on a sheet of paper a drawing of the bear's trip.

Investigate
1. Is it possible for the bear to end up at the same place she started?

Construction on the Sphere
Sketch on your sphere a drawing of the bear's trip.

Investigate
2. Is it possible for the bear to end up at the same place she started?
3. Where does the bear live?
4. What color is the bear?

Welcome to the World of Spherical Geometry
In this adventure you noticed that the story of the bear has a different ending depending on what kind of surface the bear uses for her travels. Geometry can change quite a bit when you draw and study it on two different surfaces!

In this book you will investigate geometry on the surface of a sphere. You will draw and experiment on your Lénárt Sphere, just as you use a flat piece of paper or a flat computer screen to experiment with geometry on a plane.

You are already familiar with many aspects of plane geometry. In these adventures you will often compare what you know is true on the plane with what you discover on the sphere.

5. Which surface do you prefer?
6. Which makes a simpler geometry?
7. Which geometric system is more intriguing?

As you discover geometry on a surface shaped like our planet earth, enjoy your explorations!
Can you draw a straight line on a sphere?

Let's consider the point to be the simplest shape on the plane and on the sphere.

- Describe the simplest, shortest path between two points on the plane.
- Describe the simplest, shortest path between two points on the sphere.
- Describe the shape you get when you extend each of these two paths.

Construction on the Plane

**Step 1** Draw two different points on the plane. Label them A and B.

**Step 2** Connect points A and B with three different lines or curves.

**Step 3** Draw the shortest path between points A and B if you have not yet done so. Use a taut string to show that you have really drawn the shortest path.

**Step 4** Use a straightedge to extend your shortest path until it reaches the edges of your paper.

Investigate

1. If you could extend the ends of your line forever, would they ever meet?
2. How is the shortest path between points A and B different from the other paths you drew?
3. a. Into how many sections do points A and B divide your line?
   b. How many of these sections are finite?
   c. How many are infinitely long?
4. How many different straight lines can you draw through one point on a plane?
5. How many different straight lines can you draw through two points on a plane?

Make a Guess

6. What shape will you get when you connect two points on a sphere with the shortest possible path?
7. What will happen when you extend this path in both directions around the sphere?

Construction on the Sphere

**Step 1** Draw two different points on your sphere. Label them A and B.

**Step 2** Stretch a piece of string on the sphere between the two points to find the shortest path between them. Have your partner draw a line along the taut string with a marker.

**Step 3** Pick either of the two ruled edges of your spherical ruler and try to align it with your line on the sphere. What do you observe?
Step 4: Continue drawing the line along the spherical ruler and extend it as far as possible in both directions.

Investigate
8. You have just created a great circle on a sphere. Describe it.
9. a. Into how many arcs do points A and B divide your great circle?
   b. How many of these arcs are finite?
   c. How many are infinitely long?
10. Determine which edges of your spherical ruler trace arcs of great circles and which don't.
11. How many great circles can you draw through one point on a sphere?
   a. How many great circles can you draw through two points on a sphere?
   b. Is your answer true for any two points on a sphere?

Compare the Plane and the Sphere
13. See how many observations you can make about straight lines on the plane and great circles on the sphere. Record them on a comparison chart like the one at right. Add as many rows as you need.
14. Decide which you think is simpler: straight lines on the plane or great circles on the sphere. Why?
15. Now try to reverse your argument. Give reasons why lines are simpler on the surface you didn't choose above.

Explore More
16. a. Put a drop of water onto a tilted flat surface and allow the drop to run down the surface. Describe the path of the drop of water.
   b. Put a drop of water near the top of your sphere and allow the drop to run down the surface. Describe the path of the drop of water. Does it follow a great circle?
17. Use a globe that depicts the earth.
   a. Find two places on the globe between which there is more than one shortest route.
   b. Find another such pair of places.
18. An airplane flies from San Francisco, California, to Moscow, Russia.
   a. Use a globe and describe the shortest route for the flight.
   b. Explain why there is only one shortest route.
   c. Follow the same route on the planar map at right. Does the route appear to be straight?
19. The great in great circle means large. What's so great about a great circle?
How many points can two lines share?

When two distinct lines intersect on the plane or on the sphere, they meet at one or more points.

- Investigate the points of intersection of two straight lines on the plane.
- Investigate the points of intersection of two great circles on the sphere.
- Explain your observations about parallel lines on the plane and on the sphere.

**Construction on the Plane**

**Step 1**  Draw a straight line. Label it \( l \).

**Step 2**  Try to draw another straight line that has no point in common with line \( l \).
Label it \( a \).

**Step 3**  Try to draw a straight line that has exactly one point in common with line \( l \). Label it \( b \).

**Step 4**  Try to draw a straight line that has exactly two points in common with line \( l \). Label it \( c \).

**Step 5**  Try to draw a straight line that has more than two points in common with line \( l \). Label it \( d \).

**Investigate**

1. Which constructions were possible on the plane?
2. Which of your lines are parallel? Why?
3. Describe all the different ways in which two distinct lines can intersect on the plane.

**Make a Guess**

4. Will your conclusions be the same for great circles on a sphere?

**Construction on the Sphere**

5. Perform the same steps on the sphere that you performed on the plane, replacing the straight lines with great circles. Keep track of which constructions are possible on the sphere.

**Investigate**

6. Describe all the ways in which two distinct great circles can intersect on the sphere.

7. Can two great circles ever be parallel?

**Compare the Plane and the Sphere**

8. See how many observations you can make about the intersection of two straight lines on the plane and the intersection of two great circles on the sphere. Record them on a comparison chart like the one at right. Add as many rows as you need.
9. Do you think the intersection of two lines is simpler on the plane or on the sphere? Which case is more intriguing? Why?

10. Now try to reverse your argument. Give reasons why the intersection of two lines is simpler or more intriguing on the surface you didn’t choose above.

**Explore More**

11. Imagine that it is possible for a pair of railroad tracks to extend all the way around the earth. Can the railroad tracks represent parallel lines?

12. Parallel lines on the plane are always the same distance apart. Draw a great circle on your sphere. Then draw a different figure that is always the same distance from your great circle.
   a. Describe this figure.
   b. Decide if the figure could be a great circle.

13. A boat travels in a such a way that it is always 50 km from the equator. Explain why the boat is not traveling in the most direct path between two points.

14. Euclid was a mathematician from ancient Greece who is famous for being one of the first to organize the ideas of geometry. In his treatise titled *Elements*, Euclid lists a set of axioms for geometry. Euclid’s axioms were statements that he believed were so obviously true that he was willing to accept them without proof. Almost two and a half thousand years later, we still base plane geometry on Euclid’s axioms. However, his last axiom, commonly called the parallel postulate, has always been open to debate. Here is one form of Euclid’s parallel postulate: Given a straight line and a point not on this straight line, you can draw only one straight line through the given point that is parallel to the given straight line.
   a. On a piece of paper draw a straight line and a point not on the line. Draw as many lines as you can through the point that are parallel to the first line. Use your drawing to explain why Euclid’s parallel postulate makes sense on the plane.
   b. Rewrite the parallel postulate for the sphere by replacing the words *straight line* with *great circle*. Then make a construction on your sphere similar to the construction you just made on the plane. Now explain why Euclid’s parallel postulate does not make sense on the sphere.
   c. Write your own parallel postulate that is true for geometry on a sphere.

15. Describe all the ways that three distinct great circles can intersect.
What is the sum of the angle measures of a triangle?

If you add the measures of the angles of a triangle, do you always get the same sum?
- Investigate the sum of the angle measures of planar triangles.
- Investigate the sum of the angle measures of spherical triangles.

Construction on the Plane

Draw two triangles, one completely inside the other.
Measure the interior angles of each triangle.

Investigate
1. Explain why the sum of the measures of the interior angles of a planar triangle is always the same.

Make a Guess
2. What is the sum of the measures of the interior angles of a spherical triangle?

Construction on the Sphere

Draw three triangles, the first triangle completely inside the second and the second completely inside the third.
Measure the interior angles of each triangle.

Investigate
3. Find the sum of the measures of the angles of each of your spherical triangles.
4. Explain why you get different answers for different triangles.
5. a. What do you think is the smallest possible sum of angle measures for a spherical triangle?
   b. What is the largest? Explain your reasoning.

Compare the Plane and the Sphere

6. See how many observations you can make about the sum of the angle measures of a triangle on the plane and on the sphere. Record them on a comparison chart like the one at right. Add as many rows as you need.

<table>
<thead>
<tr>
<th>Sum of the angle measures of a triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>On the plane</td>
</tr>
</tbody>
</table>

7. Do you think the sum of the angle measures of a triangle is simpler on the plane or on the sphere? Why? Which case is more interesting? Which case is the more likely to inspire connections between angle measurement and other properties of triangles?

8. Now try to reverse your argument. Give reasons why the surface you didn't choose above is simpler or more interesting.
Student's Guide to Adventure 3.4

Explore More

9. Construct a triangle with three right angles. Explain why this is possible.

10. Draw a small triangle on your sphere. Suppose we allowed the interior of a spherical triangle to be the larger triangular region "around the back." Now what is the greatest possible sum of the interior angles of a spherical triangle?

11. Investigate sums of the measures of the angles of quadrilaterals on the sphere.

12. Investigate sums of the measures of the angles of other polygons on the sphere. What is the sum of the measures of the angles of a spherical polygon with \( n \) sides?
Lesson 1.2:
1. Sum it all up with the question: What does straight mean in Euclidean geometry? In spherical geometry?

Goal: For students to realize that straight is not necessarily a line that you draw using a straightedge. Straight represents the shortest distance. On a plane, straight is a line, but on a sphere straight is an arc. There are other differences for straightness in different geometries, i.e. Taxicab geometry.

Lesson 2.1:
1. List some of the consequences of the existence of parallel lines on the plane.

Goal: On the plane, with parallel lines we can achieve different types of congruent angles, i.e. alternate interior angles and corresponding angles, which we need in order to prove that the sum of the interior angles is 180° and constant.

2. Are these consequences the same on the sphere?

Goal: No. There are no consequences of parallel lines on the sphere because parallel lines do not exist in spherical geometry.

3. We know that in Euclidean geometry there is one and only one line through a given point parallel to a given line. We know that the possibility of having no parallels exists in spherical geometry. What other possibilities can you think of regarding parallel lines?

Goal: That there is the possibility of having infinitely many parallel lines, as in hyperbolic geometry.

4. Do we really need to know if lines are parallel?

Goal: If we are talking about Euclidean geometry we do! By Euclid stating the Parallel Postulate, he ultimately defined Euclidean geometry or geometry on the plane. It is required for many proofs on the plane. It is the absence of this postulate that opens doors for three dimensional geometric exploration, which is how the first non-Euclidean geometry was discovered.
Lesson 3.4:
1. What is a triangle?
   a. Mark 3 collinear points on your sphere and label them A, B and C.
   a. Connect each point on the sphere with a segment of a different color.
   b. Do these segments form a triangle?
   c. What is the sum of the angles of △ABC?

Goal: To help students construct question 5b on the adventure. The sum of the angles of this triangle is exactly 540°. To construct the degenerate triangle with angle sum of 180°, the construction is similar to the above where the points are collinear, but the third side lies on top of the other two.
APPENDIX B

STUDENT PRE- AND POST-TESTS
Geometry Project Pre-Test

*Answer each question completely to the best of your ability. Formal and informal proofs are both acceptable. Answer all the questions in the context of Euclidean (plane) geometry.*

1. Could a triangle have two right angles? Explain why or why not.

2. Given △ABC and \( \overline{PQ} \) through vertex A that is parallel to \( \overline{BC} \). Prove that the sum of the angles of the triangle is 180°, that is:
   
   a. Prove: \( m \angle ABC + m \angle BAC + m \angle ACB = 180° \)

3. What geometric facts are most crucial in your proof of #2? List no more than 3 such facts. You do not need to prove these facts, just state them clearly.
4. Use the figure below to answer the following questions:
   Given: △ABC with point D as the midpoint of \(\overline{BC}\) and congruent sides \(\overline{AD} \cong \overline{DE}\).
   \[\text{Diagram of \triangle ABC with point D as the midpoint of \overline{BC} and congruent sides \overline{AD} \cong \overline{DE}.}\]

   a. Prove that \(\triangle ACD \cong \triangle EBD\).

   b. What can you say about the relationship between segments \(\overline{AC}\) and \(\overline{BE}\)? List as many facts as you can think of. For each fact you list, state briefly why it is true.

   For the following questions, circle the correct choice:
   5. In Euclidean geometry, there is/are...
      a. No lines through a given point that are parallel to a given line.
      b. One and only one line through a given point that is parallel to a given line.
      c. More than one line (possibly infinitely many lines) through a given point that are parallel to a given line.

   6. For any triangle in a plane, what are the possible options for the sum of its interior angles?
      a. The sum is less than 180°
      b. The sum is equal to 180°
      c. The sum is greater than 180°
Geometry Project Post-Test

Answer each question completely to the best of your ability. Formal and informal proofs are both acceptable.

1. Could a triangle have two right angles? Explain why or why not.
   a. EUCLIDEAN GEOMETRY:
   b. SPHERICAL GEOMETRY:

2. Given \( \triangle ABC \), on a plane, and \( \overline{PQ} \) through vertex A that is parallel to \( \overline{BC} \). Prove that the sum of the angles of the triangle is 180°, that is:
   a. Prove: \( m \angle ABC + m \angle BAC + m \angle ACB = 180° \)

3. The existence of a unique line \( \overline{PQ} \) parallel to \( \overline{BC} \) and through vertex A is crucial to the proof of #2.
   a. What does the word "unique" mean?
   b. What do you think might happen if there were no such line \( \overline{PQ} \)?
   c. What do you think might happen if there were more than one such line \( \overline{PQ} \)?
4. Use the figure below to answer the following questions:

Given: \( \triangle ABC \) with point D as the midpoint of \( BC \) and congruent sides \( AD \cong DE \).

- a. Prove that \( \triangle ACD \cong \triangle EBD \).

- b. What can you say about the relationship between segments \( AC \) and \( BE \)?
   List as many facts as you can think of. For each fact you list, state briefly why it is true.

For the following questions, circle the correct choice:

5. In EUCLIDEAN geometry, there is/are:
   a. No lines through a given point that are parallel to a given line.
   b. One and only one line through a given point that is parallel to a given line.
   c. More than one line (possibly infinitely many lines) through a given point that are parallel to a given line.

6. In SPHERICAL geometry, there is/are...
   a. No lines through a given point that are parallel to a given line.
   b. One and only one line through a given point that is parallel to a given line.
   c. More than one line (possibly infinitely many lines) through a given point that are parallel to a given line.

7. For any triangle on a PLANE, what do you know about the sum of its interior angles?
   a. The sum is between 0° and 180°
   b. The sum is equal to 180°
   c. The sum is greater than or equal 180° and less than or equal to 540°
8. For any triangle on a SPHERE, what do you know about the sum of its interior angles?
   a. The sum is between 0° and 180°
   b. The sum is equal to 180°
   c. The sum is greater than or equal 180° and less than or equal to 540°
APPENDIX C

PRE- AND POST-SURVEYS
STUDENT PRE- (AND POST-) SURVEY

Directions: Please respond to the following statements accurately and to the best of your ability by circling the appropriate choice.

1. Gender:   M    F

2. Grade:   9    10    11    12

3. I like mathematics.
   Strongly Disagree    Disagree    Undecided    Agree    Strongly Agree
   1                     2            3                4                5

4. I enjoy hands on mathematics activities.
   Strongly Disagree    Disagree    Undecided    Agree    Strongly Agree
   1                     2            3                4                5

5. I think the best way to learn is from the text.
   Strongly Disagree    Disagree    Undecided    Agree    Strongly Agree
   1                     2            3                4                5

6. Learning mathematics is a waste of time.
   Strongly Disagree    Disagree    Undecided    Agree    Strongly Agree
   1                     2            3                4                5

7. Understanding math is not necessary for success in real life activities.
   Strongly Disagree    Disagree    Undecided    Agree    Strongly Agree
   1                     2            3                4                5

8. Math is my favorite subject.
   Strongly Disagree    Disagree    Undecided    Agree    Strongly Agree
   1                     2            3                4                5

9. I am interested in studying mathematics in college.
   Strongly Disagree    Disagree    Undecided    Agree    Strongly Agree
   1                     2            3                4                5

10. Doing well in mathematics is essential for success in my future goals.
    Strongly Disagree    Disagree    Undecided    Agree    Strongly Agree
     1                     2            3                4                5
11. I enjoy learning different types of math (i.e. algebra, geometry, trig, etc.).

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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12. I am interested in learning how math relates to the world.

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<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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13. So far, geometry is my favorite branch of mathematics.

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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14. Mathematics does not scare me at all.

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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15. Studying mathematics makes me feel nervous.

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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16. My mind goes blank and I am unable to think clearly when working on mathematics.

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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</table>

17. I enjoy studying mathematics in school.

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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18. I am happier in a math class than in any other class.

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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19. I plan on taking as much mathematics as I can during my education.

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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</table>

20. The challenges of mathematics appeal to me.

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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TEACHER PRE- (AND POST-) SURVEY

This survey is being conducted to better understand the high school geometry teacher’s perspectives on controversial issues in geometry and their thoughts on approaches to teaching geometry. Thank you for your participation in this survey.

1. In which district do you teach? ____________________________________________

2. What type of credential do you have? ______________________________________

3. Do you have a Bachelor’s degree in mathematics? (circle one)
   Yes     No
   a. If yes, go to number 4, if no, do you have a minor in mathematics? (circle one)
      Yes     No

4. Do you have a Master’s degree in mathematics? (circle one)
   Yes     No

5. How long have you been teaching math? (circle one)
   0-2 years  3-5 years  6-10 years  11-20 years  20+ years

6. Of those years, how many were spent teaching Geometry?
   0-2 years  3-5 years  6-10 years  11-20 years  20+ years

Use the scale provided to rate your level of agreement for each statement. Please approach each statement with your feelings about teaching math and how you teach in your classroom everyday.

7. I enjoy teaching mathematics.
   Strongly Disagree Disagree Undecided Agree Strongly Agree
   1  2  3  4  5

8. I teach strictly from the textbook, following district pacing guides.
   Strongly Disagree Disagree Undecided Agree Strongly Agree
   1  2  3  4  5

9. I enjoy exploring and implementing different types of math curriculum.
   Strongly Disagree Disagree Undecided Agree Strongly Agree
   1  2  3  4  5

10. I believe in being a lifelong learner.
    Strongly Disagree Disagree Undecided Agree Strongly Agree
    1  2  3  4  5

Additional comments regarding teaching math are welcomed:

__________________________________________________________________________

__________________________________________________________________________
For each of the following statements, please rate your level of agreement using the scale provided.

11. I have an in depth understanding of Euclidean (plane) geometry.

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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</table>

12. I have an in depth understanding of Euclid’s 5th Postulate (also known as the Parallel Postulate).

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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</table>

13. Many mathematicians have stated that the Parallel Postulate is not obvious enough to be accepted as a postulate, rather it should be a theorem to be proven. I agree that the postulate needs to be proven.

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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14. I have read about and understand the controversy surrounding Euclid’s 5th Postulate.

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<thead>
<tr>
<th>Strongly Disagree</th>
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<th>Agree</th>
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</table>

15. I am interested in learning more about Euclid’s 5th postulate and the controversy surrounding it.

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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16. Adding some non-Euclidean geometry curriculum to the current high school geometry curriculum may be beneficial to student’s understanding of Euclidean geometry.

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
<th>Undecided</th>
<th>Agree</th>
<th>Strongly Agree</th>
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</table>

17. I am interested in learning more about spherical and other non-Euclidean geometries.

<table>
<thead>
<tr>
<th>Strongly Disagree</th>
<th>Disagree</th>
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<th>Agree</th>
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Additional comments regarding your background and interest in teaching Euclidean and non-Euclidean geometry are welcomed: ____________________________________________

________________________________________________________

________________________________________________________

87
APPENDIX D

INTERVIEW QUESTIONS
Teacher Interview Questions

1. Why did you become a math teacher?

2. What is it about teaching mathematics that you enjoy? What is it about teaching geometry that you enjoy? What was it about teaching spherical geometry that you enjoyed?

3. How prepared did you feel to teach the spherical geometry unit to your class? What could we have done differently to help you feel more prepared?

4. How successful do you feel your presentations of the spherical geometry lessons were? How do you measure your successfulness?

5. How do you feel your students responded to the lessons? Was this a typical response? How was it different? Why do you think it was different?

6. How engaged were your students? What could have been done differently to actively engage all of your students?

7. How comprehensible was the material to the different student levels in your classroom? How could it be made more comprehensible?

8. How likely are you to teach non-conventional curriculum again? Describe how you felt about teaching something that you didn't already know.

9. How has teaching some spherical geometry made the concepts you teach in plane geometry clearer/less clear to you?

10. What impact has the introduction of spherical geometry made as to how you view mathematics and on how you think mathematics is relevant in the world?

11. What did you find was helpful in approaching the standards from a different perspective? What do you think hindered your students when using this approach?
Student Interview Questions

1. Compared to other subjects, how would you describe the difficulty of learning math? Compared to other subjects, how would you describe your proficiency in math? Compared to other areas of mathematics, how would you describe your experiences in your geometry class?

2. Describe your thoughts and feelings when you were asked to learn geometry that was not from your textbook.

3. Which aspects of the spherical geometry activities did you enjoy? Which did you not enjoy?

4. How challenging were the spherical geometry activities?

5. How would you describe the difficulty of the spherical geometry concepts? What made the concepts difficult to understand? What do you think might have helped you better understand the concepts?

6. What connections do you see between the spherical geometry you just learned and the plane geometry you have been studying all year long?

7. How has learning some spherical geometry made the concepts you learned in plane geometry clearer/less clear?

8. What impact has the introduction of spherical geometry made as to how you view mathematics and on how you think mathematics is relevant in the world?

9. How interested are you in studying spherical geometry in greater detail?

10. How interested are you in learning about other non-Euclidean geometries?

11. Are you planning on going to college?

12. What are your career goals?

13. Have you considered majoring in science or mathematics in college?
APPENDIX E

GRADING RUBRIC
### Grading Rubric: Mathematics

<table>
<thead>
<tr>
<th>Level</th>
<th>Conceptual Understanding</th>
<th>Mathematical Reasoning and Problem-Solving Strategies</th>
<th>Communication</th>
</tr>
</thead>
<tbody>
<tr>
<td>Superior -- 4</td>
<td>• The solution shows a deep conceptual understanding of the problem, including the ability to apply the correct mathematical concepts and the necessary information for its solution.</td>
<td>• Uses a well-organized and high-level strategy that leads directly to a solution of the problem. Uses complex reasoning. Uses correct procedures to solve the problem and validate the solution.</td>
<td>• Explanation is clear and comprehensive. All details are represented to solve the problem. All steps and procedures are incorporated so that the explanation of the solution is understandable to the reader. Precise mathematical representation used to communicate concepts related to the problem's solution. The use of precise mathematical language, terminology and notation is applied throughout the solution of the problem.</td>
</tr>
<tr>
<td>Satisfactory -- 3</td>
<td>• The solution indicates that the student has more than a basic comprehension of the problem and the main concepts necessary for its solution.</td>
<td>• Uses a sound strategy that leads to a solution of the problem. Uses mathematical reasoning correctly. Mathematical procedures are applied.</td>
<td>• The explanation is clear. Mathematical representation used is accurate and appropriate. Mathematical terminology and notation is used effectively.</td>
</tr>
<tr>
<td>Approaching Satisfactory -- 2</td>
<td>• Incomplete solution, i.e., showing parts of the problem are not recognized.</td>
<td>• Uses a strategy that is somewhat useful, leading toward an incomplete solution. Some indication of mathematical strategies. Incomplete mathematical procedures.</td>
<td>• There is an incomplete and unclear explanation. There is minimal use of correct mathematical representation. There is minimal use of mathematical terminology and notation appropriate to the problem.</td>
</tr>
<tr>
<td>Unsatisfactory -- 1</td>
<td>• No solution exists, or the solution is not connected to the test question. Skills and concepts used are inconsistent and do not apply to the test question.</td>
<td>• No evidence of a problem-solving strategy. No plan or use of a strategy, or uses a procedure that does not help solve the problem. No indication of mathematical reasoning. Too many mathematical errors so that the problem could not be solved.</td>
<td>• Solution is not explained, or the explanation is incomprehensible or not connected to the problem. No mathematical representations (e.g., figures, diagrams, graphs, tables, etc.) are used. Usage of mathematical terms is incorrect.</td>
</tr>
</tbody>
</table>
APPENDIX F

TEACHER SCHEDULE
### SCHEDULE AT A GLANCE

<table>
<thead>
<tr>
<th>DAY 1</th>
<th>DAY 2</th>
<th>DAY 3</th>
<th>DAY 4</th>
<th>DAY 5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>What to do:</strong> Pre-test Pre-Survey Lesson 0.1 (if time)</td>
<td><strong>What to do:</strong> Lesson 1.2</td>
<td><strong>What to do:</strong> Lesson 2.1</td>
<td><strong>What to do:</strong> Lesson 3.4 Wrap-up</td>
<td><strong>What to do:</strong> Post-test Post-survey Interview</td>
</tr>
<tr>
<td><strong>What you need:</strong> Included: Lenart Spheres Overhead Pens Paper towels Group packets</td>
<td><strong>What you need:</strong> Included: Lenart Spheres Overhead pens Paper towels String Eye droppers Paper cups Water Not included: Straightedges</td>
<td><strong>What you need:</strong> Included: Lenart Spheres Overhead pens Paper towels String</td>
<td><strong>What you need:</strong> Included: Lenart spheres Overhead pens Paper towels</td>
<td><strong>What you need:</strong></td>
</tr>
<tr>
<td><strong>Homework:</strong> Adventure Card 1.2</td>
<td><strong>Homework:</strong> Supplemental question from 1.2 Adventure Card 2.1</td>
<td><strong>Homework:</strong> Supplemental questions from 2.1 Adventure Card 3.4</td>
<td><strong>Homework:</strong> Review their work from this week Test tomorrow!</td>
<td><strong>Homework:</strong> None, you are done!</td>
</tr>
<tr>
<td><strong>Remember to have students label their surveys and tests with the last four digits of their Student ID!</strong></td>
<td></td>
<td></td>
<td></td>
<td><strong>Remember to have students label their surveys and tests with the last four digits of their Student ID!</strong></td>
</tr>
</tbody>
</table>

- Remember to have students label their surveys and tests with the last four digits of their Student ID!
APPENDIX G

TABLE OF STUDENT COMMENTS
## Spherical Geometry Project

### Student Comments

<table>
<thead>
<tr>
<th>ID#</th>
<th>Sex</th>
<th>Year</th>
<th>S1 Gr</th>
<th>Q4 Gr</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>21302</td>
<td>M</td>
<td>10</td>
<td>A</td>
<td>C-</td>
<td>I enjoyed studying the spherical geometry. The bear was white.</td>
</tr>
<tr>
<td>11344</td>
<td>M</td>
<td>10</td>
<td>C-</td>
<td>B+</td>
<td>I love it. It was fun doing hands on. I like the sphere, it's big and round.</td>
</tr>
<tr>
<td>37070</td>
<td>M</td>
<td>10</td>
<td>A+</td>
<td>A+</td>
<td>This project was a great experience for me. I learned that construction on a plane is very different from construction on a sphere.</td>
</tr>
<tr>
<td>9087</td>
<td>M</td>
<td>11</td>
<td>D+</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>1428</td>
<td>F</td>
<td>9</td>
<td>A+</td>
<td>A</td>
<td>I enjoyed spherical geometry because I was able to learn differences between a plane and a sphere. For example, I learned that the ends of a line could actually meet on a sphere if extended in opposite directions while they could not meet on a plane.</td>
</tr>
<tr>
<td>56592</td>
<td>F</td>
<td>10</td>
<td>C-</td>
<td>C+</td>
<td></td>
</tr>
<tr>
<td>32929</td>
<td>M</td>
<td>9</td>
<td>B-</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>3950</td>
<td>M</td>
<td>9</td>
<td>B</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>8511</td>
<td>F</td>
<td>10</td>
<td>C+</td>
<td>A-</td>
<td>This spherical geometry helped a lot by understanding more about different line segments. Now I see planes differently than I used to.</td>
</tr>
<tr>
<td>24881</td>
<td>F</td>
<td>11</td>
<td>C-</td>
<td>A+</td>
<td>In spherical we learned that we had to think outside the box because things that were not possible in plane geometry were possible in spherical geometry and vice versa.</td>
</tr>
<tr>
<td>18637</td>
<td>M</td>
<td>9</td>
<td>A+</td>
<td>A</td>
<td>Spherical geometry can be used in every day life. There was really a lot of enjoyment of learning this. Great problem solving. Thank you.</td>
</tr>
<tr>
<td>1457</td>
<td>F</td>
<td>9</td>
<td>A</td>
<td>A+</td>
<td></td>
</tr>
<tr>
<td>5278</td>
<td>F</td>
<td>10</td>
<td>C</td>
<td>B-</td>
<td>Everyone enjoyed working with S.G. and using the sphere. Most of my group understood it and thought it was an easy assignment.</td>
</tr>
<tr>
<td>2527</td>
<td>M</td>
<td>9</td>
<td>B-</td>
<td>A-</td>
<td>I like how plane and sphere are very different from each other and I learned a lot.</td>
</tr>
<tr>
<td>44631</td>
<td>F</td>
<td>10</td>
<td>C-</td>
<td>D-</td>
<td></td>
</tr>
<tr>
<td>15449</td>
<td>F</td>
<td>10</td>
<td>C-</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>77509</td>
<td>F</td>
<td>9</td>
<td>A+</td>
<td>A+</td>
<td>The spherical geometry helped me in tying in what I have learned previously this year. It's mind boggling to see differences from plane to spherical geometry. It was especially evident in the “What color is the Bear?” On a regular plane, going west and north you don’t end up where you started. With the spherical geometry however, if you go west and back north you do end up where you started. It was also fun to see how on a plane two lines cross at one point; on a sphere it crosses at 2 points.</td>
</tr>
<tr>
<td>82969</td>
<td>M</td>
<td>10</td>
<td>B</td>
<td>A-</td>
<td>I enjoyed drawing lines on the sphere. You can get answers on the sphere that you can’t get on plane geometry. You can apply sphere geometry in different types of problems. It really was fun finding out how a bear can be white because the sphere is like the world in that problem.</td>
</tr>
<tr>
<td>70492</td>
<td>F</td>
<td>10</td>
<td>A-</td>
<td>B</td>
<td>I learned that spherical geometry is much more easier and different than plane geometry.</td>
</tr>
<tr>
<td>30943</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>I think it was very interesting working with my group. Everyone had different opinions and it was creative. I enjoyed talking to my group about the sphere. My group was well-matched, I like how they think. I learned how to cooperate</td>
</tr>
<tr>
<td>ID</td>
<td>Gender</td>
<td>Grade</td>
<td>Grade</td>
<td>Comment</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>--------</td>
<td>-------</td>
<td>-------</td>
<td>---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>1215</td>
<td>M</td>
<td>9</td>
<td>A+</td>
<td>A+</td>
<td>Comparing spherical geometry to regular geometry was fun because they both have different functions in daily life. Spherical geometry makes someone think outside the box. Spherical geometry helped better understand segments and how to think in 3D shapes.</td>
</tr>
<tr>
<td>42426</td>
<td>F</td>
<td>9</td>
<td>B+</td>
<td>B</td>
<td>I really enjoyed learning about spheres because of the way they relate to, but are different from planes. I love learning new things, and having fun while doing it is a great advantage.</td>
</tr>
<tr>
<td>12839</td>
<td>F</td>
<td>9</td>
<td>A+</td>
<td>A+</td>
<td>Spherical geometry has shown the differences between the lines on a plane and sphere. On a sphere I understood how many points can reach only two points anywhere on the sphere to intersect, like a plane diagram it could be looked at as an asterisk. Great circles are also the equator because it goes around the sphere at its biggest diameter and I understood why it would be called the “Great” circle.</td>
</tr>
<tr>
<td>9473</td>
<td>F</td>
<td>10</td>
<td>A</td>
<td>A</td>
<td>Spherical geometry uses your mind more, to think in a different way other than plane. To me spherical geometry was pretty simple but at times I did have to think a bit.</td>
</tr>
<tr>
<td>4811</td>
<td>F</td>
<td>12</td>
<td>C+</td>
<td>A-</td>
<td>Doing the spherical geometry was very interesting. I liked working with the sphere because it was more hands on. I learned about great circles and how lines can only intersect at one point.</td>
</tr>
<tr>
<td>9491</td>
<td>F</td>
<td>10</td>
<td>A+</td>
<td>A+</td>
<td>It was fun to work with the sphere and to learn how they relate to math.</td>
</tr>
<tr>
<td>26797</td>
<td>M</td>
<td>10</td>
<td>A+</td>
<td>A+</td>
<td>I really enjoyed taking the time to study spherical geometry. I realized there are many differences between plane geometry and spherical geometry. I am glad that I learned something that will be useful to my education in the future I can apply it to my drawings and other types of math. I learned that although lines are very much alike, they can give different results on a plane and on a sphere. The bear is white!</td>
</tr>
<tr>
<td>9415</td>
<td>M</td>
<td>10</td>
<td>B</td>
<td>B-</td>
<td>I enjoyed studying spherical geometry! BIG difference between plane &amp; spherical geometry! The lines are different on the plane tan on the sphere. 2 lines can cross on 2 points on a sphere. 2 lines can cross on 1 point on a plane. You can apply it to other types of math. The bear was white. It was hard, but fun!</td>
</tr>
<tr>
<td>41221</td>
<td>F</td>
<td>10</td>
<td>A</td>
<td>A+</td>
<td>It was good but at times confusing because you were on your own. It was great learning a new way of geometry. Interesting working on a sphere.</td>
</tr>
<tr>
<td>52604</td>
<td>F</td>
<td>11</td>
<td>B+</td>
<td>C-</td>
<td>The spherical geometry was much different than the geometry I was learning. The bear was white.</td>
</tr>
<tr>
<td>15042</td>
<td>M</td>
<td>11</td>
<td>B</td>
<td>A-</td>
<td>This experiment with the globe was great! Hands on projects really attract more kids attention and with it being visual it also helps us understand. I learned about another name for Equator; great circle. Every line below and above gradually gets smaller. Also with the bear project there is no found way that the bear could end up home. These different projects for the sphere are wonderful and perfect to learn and see how to work out problems.</td>
</tr>
<tr>
<td>12795</td>
<td>M</td>
<td>9</td>
<td>A-</td>
<td>A</td>
<td>I liked the spherical geometry a lot because it was something new to me and very interesting. I enjoyed learning spherical geometry better than geometry because there was a lot more hands on activities. 2 lines cross on 2 points on a sphere.</td>
</tr>
<tr>
<td>10547</td>
<td>M</td>
<td>10</td>
<td>C</td>
<td>F</td>
<td>I liked the spherical geometry a lot because it was something new to me and very interesting. I enjoyed learning spherical geometry better than geometry because there was a lot more hands on activities. 2 lines cross on 2 points on a sphere.</td>
</tr>
</tbody>
</table>
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