2008

# Poincaré Duality

Christopher Michael Duran

# Poincaré Duality

---

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

---

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

Christopher Michael Duran

June 2008

POINCARÉ DUALITY

---

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

---

by

Christopher Michael Duran

June 2008

Approved by:

[redacted]

Dr. Corey Dunn, Committee Chair

6/3/08

Date

[redacted]

Dr. Zahid Hasan, Committee Member

[redacted]

Dr. Rolland Trapp, Committee Member

[redacted]

Dr. Peter Williams, Chair,
Department of Mathematics

[redacted]

Dr. Joseph Chavez,
Graduate Coordinator,
Department of Mathematics

## Abstract

This project is an expository study of the Poincaré duality theorem which equates the $k^{th}$ cohomology group with the $(n-k)^{th}$ homology group of a compact, orientable manifold of dimension $n < \infty$. We discuss homology, cohomology, and other algebraic and topological preliminaries before presenting a proof of the theorem. Subsequently, we illustrate the importance of the theorem by presenting some useful applications.

## Acknowledgments

I'd like to thank Dr. Dunn for his encouragement and guidance in studying Poincaré duality and helping me produce my thesis. I'd also like to thank my committee members, Dr. Trapp and Dr. Hasan, for their proof reading and helpful suggestions. Thank you to Dr. Chavez for your guidance throughout the M.A. program. And finally, I wish to thank Dr. John McCleary and Dr. Hayden Harker for their suggestions on applications of Poincaré duality, in particular, bilinear forms.

# Table of Contents

# List of Figures

# Chapter 1

# Introduction

The following work is an explanation of Poincaré duality for compact orientable manifolds. The Poincaré duality theorem says that for a compact, connected, orientable $n$-dimensional manifold $M$, the cohomology groups, $H^k(M)$, are isomorphic to the homology groups $H_{n-k}(M)$. As evident here, there are many algebraic and topological notions that must be understood before one can fully grasp what the theorem says. In the following chapters, we will define and prove some of these preliminary notions, including singular and simplicial homology and cohomology, Mayer-Vietoris sequences, and direct limits. Many of these topics can lead to extensive studies on their own. However, the goal here will be to discuss these notions in terms of how they build up to and are related to the Poincaré duality theorem.

The next phase of this work will be to use all of the preliminary notions to prove the theorem. The proof will be lengthy and will rely heavily on these preliminaries. The goal here will be for the reader to get an idea of how the theorem works, how it is proved, and an explanation of the algebraic machinery that is used.

The final chapter will be dedicated to discussing applications of the theorem. We will take time to explain several examples and how Poincaré duality is used in these situations, illuminating the importance of the theorem. In particular, we will use Poincaré duality to compute homology groups using the cap product. It will also be used to show that the Euler characteristic for an odd dimensional compact manifold is 0.

To understand the history of the Poincaré duality theorem, one must first start with the development of algebraic topology. Algebraic topology was first introduced and studied by Poincaré himself between 1900 and 1920 [Hat02]. His method was to view topological manifolds as $\Delta$-complex structures in order to extract information about the manifold (a $\Delta$-complex structure on a manifold is a way to build a smooth surface from smaller, more tractable pieces. See Chapter 3). Although Poincaré described these manifolds using algebraic objects, now called Betti numbers and torsion coefficients, he used an approach that is similar to simplicial homology today (homology will be described in Chapter 3). Instead of working with Betti numbers and torsion coefficients, algebraic topology began to be developed as we know it today in the late 1920's and early 1930's by such characters as Emmy Noether, Leopold Vietoris, Heinz Hopf, and Walther Mayer [Kat93]. This change in thinking was what led to the development of singular homology and cohomology. After some work by Samuel Eilenberg, James Alexander, and Solomon Lefschetz, the definition of singular homology as we know it today appeared in 1944 in a paper by Eilenberg [Hat02].

Since the time of these formal definitions of singular homology and cohomology, Poincaré's work, which was in terms of the alternate definitions of Betti numbers and torsion coefficients, has been translated into the new way of thinking. Now, Poincaré's work in algebraic topology is most often presented in terms of homology and cohomology. For example, the Poincaré duality theorem, which we briefly stated above in terms of homology and cohomology, would formally say that if $M$ is a compact oriented $n$-manifold without boundary, the $i^{th}$ Betti number of $M$ is the same as the $(n-i)^{th}$ Betti number for $0 \leq i \leq n$ [Vic94].

Although Poincaré's original work in algebraic topology has been translated into a new and modern formality, he was one of the original thinkers in the field and produced and proved many of the foundational results in the field. The Poincaré duality theorem is one of these foundational results. In fact, "Poincaré clearly considered that the climax of his work in topology was his famous duality theorem" [Die89]. In the following chapters, we intend to put the Poincaré duality theorem on display and in Chapter 7, show some of the reasons for the theorem's prominence.

The following is a brief outline of the paper. In Chapter 2 we define and give examples of the topological ideas necessary to understand homology and cohomology. Chapter 3 is devoted to defining singular and simplicial homology and cohomology as well as proving several important facts related to these. Chapter 4 is a combination of defining and proving more facts related to homology and cohomology groups. Also, we introduce a definition of orientability for manifolds and the fundamental class, two concepts that will lead to the isomorphism used in Poincaré duality. In Chapter 5, we state the remaining algebraic definitions and theorems that will be used in proving the Poincaré duality theorem. Several of these theorems are then proved. Next, we define the cap product and prove the Poincaré duality theorem for compact orientable manifolds, Theorem 6.3.1, in Chapter 6. Finally, Chapter 7 illustrates the use of Poincaré duality and explores several applications of the theorem.

# Chapter 2

# Topology

## 2.1 Homeomorphism and Homotopy

Poincaré duality is a notion that relies heavily on algebraic and topological concepts. It is important for the reader to be familiar with topological spaces, topological manifolds, homeomorphisms and homotopy. Throughout this chapter $X$ and $Y$ are topological spaces and all maps are assumed to be continous.

**Definition 2.1.1.** [Bre97] A map $F : X \to Y$ is called a *homeomorphism* if $F^{-1} : Y \to X$ exists (i.e., $F$ is one-one and onto) and both $F$ and $F^{-1}$ are continuous. The notation $X \approx Y$ will denote that $X$ is homeomorphic to $Y$.

A big part of topology is concerned with recognizing two different topological spaces as homeomorphic or not homeomorphic. We will be concerned with this skill in future chapters. An equivalence which is weaker than homeomorphism is homotopy. Two continous maps are said to be homotopic if they satisfy the following definition:

**Definition 2.1.2.** [Mas91] Two maps $F_0, F_1 : X \to Y$ are *homotopic* if and only if there exists a map $F : X \times I \to Y$ such that, for $x \in X$,

$$F(x, 0) = F_0(x),$$

$$F(x, 1) = F_1(x).$$

The notation $F_0 \simeq F_1$ will denote this relation, and $F$ is called a *homotopy* of the maps $F_0$ and $F_1$.

Note that if $f = g : X \to Y$, then $f \simeq g$. This leads to the most important concept we will be concerned with, homotopy of spaces.

**Definition 2.1.3.** [Bre97] A map $F : X \to Y$ is said to be a *homotopy equivalence* with homotopy inverse $G$ if there is a map $G : Y \to X$ such that $G \circ F \simeq 1_X$ and $F \circ G \simeq 1_Y$, where $1_Y(y) = y$, and $1_X(x) = x$. This relationship is denoted by $X \simeq Y$, and we would say that $X$ and $Y$ are homotopic or have the same homotopy type.

Notice that $\simeq$ and $\approx$ are equivalence relations. Also, a direct consequence of the definitions of homotopy and homeomorphism is the following:

**Remark 2.1.4.** If $X \approx Y$, then $X \simeq Y$.

*Proof.* Because two spaces being homeomorphic guarantees continuous mappings $F$ and $G = F^{-1}$, with $F \circ G = 1_Y$, and $G \circ F = 1_X$. Since $f = g \Rightarrow f \simeq g$ for any maps $f$ and $g$, we are done. □

The contrapositive of the above remark would say that if two spaces are not homotopic, then the spaces are also not homeomorphic. There are many classic examples of pairs of spaces that are homotopic. For one, the solid torus, $\mathbb{T}^2$, is homeomorphic (and hence homotopic) to a coffee mug with one handle (as pictured in Figure 2.1). Generally speaking, the idea comes from the fact that the coffee mug could be reshaped and transformed, without tearing or breaking it, until it becomes the torus. This is the basic idea of a homeomorphism of spaces, that one space can be transformed into the other space without tearing or breaking either space.

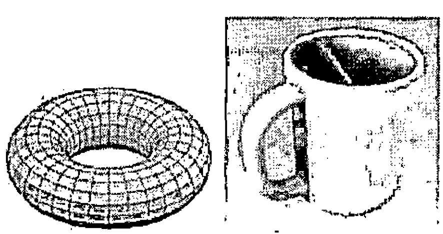


Figure 2.1: The Torus and Coffee Mug

**Example 2.1.5.** If

$$D^n = \{\vec{x} \in \mathbb{R}^n : |\vec{x}| \leq 1\},$$

and

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1\},$$

then $D^n - \{\vec{0}\} \simeq S^{n-1}$. This homotopy is achieved by defining $f : D^n - \{\vec{0}\} \to S^{n-1}$ as

$$f(\vec{x}) = \frac{\vec{x}}{|\vec{x}|}$$

and defining $g : S^{n-1} \to D^n - \{\vec{0}\}$ as $g(\theta) = \theta$ for $\theta \in S^{n-1}$.

That is, any disk minus a point in the interior can be stretched out or retracted to a sphere of that same dimension.

**Example 2.1.6.** $\mathbb{R}^n \simeq \{\vec{0}\}$ when given the maps $f : \mathbb{R}^n \to \{\vec{0}\}$ by $f(\vec{x}) = \vec{0}$ and $g : \{\vec{0}\} \to \mathbb{R}^n$ by $g(\vec{0}) = \vec{0}$.

This property of topological spaces occurs so frequently that it is given a special name.

**Definition 2.1.7.** [Bre97] A space is said to be *contractible* if it is homotopy equivalent to the one-point space.

## 2.2 Topological Manifolds

The final major topological concept that will be used will be the notion of a manifold.

**Definition 2.2.1.** [Hat02] A *(topological) manifold* of dimension $n$ is a Hausdorff space $M$ in which each point $x \in M$ has an open neighborhood homeomorphic to $\mathbb{R}^n$.

Many of the spaces mentioned previously in this chapter are topological manifolds. For example, Euclidean space, $\mathbb{R}^n$, of any dimension is a manifold, since every point in $\mathbb{R}^n$ has the open neighborhood $\mathbb{R}^n$ which is homeomorphic to $\mathbb{R}^n$. Also, the hollow torus, $\mathbb{T}^2 := S^1 \times S^1$, is a manifold, because any point on the torus has a neighborhood homeomorphic to $\mathbb{R}^2$. Similarly, a point on a sphere $S^n$ of any dimension $n$ has a neighborhood that is homeomorphic to $\mathbb{R}^n$ via stereographic projection. For detailed proofs of these facts, see [Sie92].

# Chapter 3

# Homology and Cohomology

## 3.1 Singular and Simplicial Homology

Homology theory is the study of topological spaces in terms of an algebraic language. In particular, it is concerned with expressing a topological space as a sequence of abelian groups. Although there are many types of homology known, two commonly used types of homology are *singular* and *simplicial* homology. In a certain sense, singular homology is the most general way of defining homology, but it is not necessarily the easiest form of homology to use when computing the homology of a particular topological space. Simplicial homology is the preferred form of homology to use in certain basic cases. It is equivalent to singular homology, but it allows one to compute the homology of some topological spaces combinatorially rather than topologically, as we will see. Both singular and simplicial homology will be defined here, and several differences between the two will be pointed out. The differences between singular and simplicial homology will not be emphasized, because the singular homology of a $\Delta$-complex $X$, $H_n(X)$, is isomorphic to the simplicial homology of the same space, $H_n^{\Delta}(X)$, for all $n$ [Hat02] (see Theorem 3.1.9). We now give a complete definition of homology.

We start by letting $\{e_0, e_1, ...\}$ be a basis for $\mathbb{R}^{\infty}$.

**Definition 3.1.1.** [KSW89] The basis elements $e_i$ are used to define the $n-$ *simplices* $\Delta_n$ as follows:

$$\Delta_n = \left\{ \sum_{i=0}^{n} t_i e_i \;\middle|\; t_i \geq 0 \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}.$$

Each $\Delta_n$ is called an $n-simplex$. The $0-simplex$, $\Delta_0$, is a single point. The $1-simplex$ $\Delta_1$ is a single line. The $2-simplex$ $\Delta_2$ is a triangle with three vertices, and so on. In each case, the $n+1$ vertices of the $n$-simplex are considered ordered by the ordering of the chosen basis $\{e_0, e_1, \ldots\}$ for $\mathbb{R}^\infty$. Now suppose $\sigma : \Delta_n \to X$, and if $P_i \in X$, with $P_i = \sigma(e_i)$, then $(P_0, \ldots, P_n)$ is a simple but incomplete way to describe $\sigma$. However, this notation will be useful. Now, let $C_n(X)$ be a free abelian group generated by the $n$-simplices, thus any $\theta \in C_n(X)$ is a finite formal sum defined by

$$\theta = \sum_{i=1}^{k} n_i \sigma_i$$

where $n_i \in \mathbb{Z}$.

**Definition 3.1.2.** Let $\sigma = (P_0, \ldots, P_n) \in C_n(X)$. Then the *boundary operator* given by the map $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined by

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i (P_0, \ldots, \check{P}_i, \ldots, P_n).$$

The hat above $P_i$ means that it is removed, and

$$(P_0, \ldots, \check{P}_i, \ldots, P_n) = \sigma \mid_{span\{e_0, \ldots, \check{e}_i, \ldots, e_n\}} .$$

This map $\partial_n$ is sometimes written without the $n$ as $\partial$. We extend this map by linearity so that $\partial_n : C_n(X) \to C_{n-1}(X)$.

**Theorem 3.1.3.** If $\theta \in C_n(X)$, then

$$\partial_{n-1}\partial_n(\theta) = 0.$$

*Proof.* Let $\sigma = (P_0, P_1, \ldots, P_n)$ and $\theta \in C_n(X)$. Now,

$$\begin{aligned}
\partial_{n-1}(\partial_n \sigma) &= \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i (P_0, \ldots, \check{P}_i, \ldots, P_n)\right) \\
&= \sum_{i=0}^{n}(-1)^i \left(\partial_{n-1}(P_0, \ldots, \check{P}_i, \ldots, P_n)\right) \\
&= \sum_{i=0}^{n}(-1)^i \sum_{j=0}^{i-1}(-1)^j (P_0, \ldots, \check{P}_j, \ldots, \check{P}_i, \ldots, P_n) \\
&\quad + \sum_{i=0}^{n}(-1)^i \sum_{j=i+1}^{n}(-1)^{j-1}(P_0, \ldots, \check{P}_i, \ldots, \check{P}_j, \ldots, P_n) \\
&= 0.
\end{aligned}$$

It follows that $\partial_{n-1}(\partial_n\theta)$ for all $\theta \in C_n(\theta)$, since

$$\partial_{n-1}(\partial_n\theta) = \partial_{n-1}\left(\sum n_i(\partial_n\sigma_i)\right) = \sum n_i(\partial_{n-1}\partial_n(\sigma_i)) = \sum n_i(0) = 0.$$

□

**Definition 3.1.4.** The sequence of homomorphisms $\partial_n$ of the abelian groups $C_n(X)$ is called a *chain complex*.

The additional requirement that $\partial^2 = 0$ is the difference between a collection of abelian groups and a chain complex. Since we have $\partial_n : C_n(X) \to C_{n-1}(X)$ and $\partial_{n+1} : C_{n+1}(X) \to C_n(X)$, both $Ker(\partial_n)$ and $Im(\partial_{n+1})$ belong to $C_n(X)$. Also since $\partial^2 = 0$, and $C_n(X)$ is abelian, $Im(\partial_{n+1}) \trianglelefteq Ker(\partial_n)$.

**Definition 3.1.5.** [Hat02] The $n^{th}$ *homology group* of $X$ (with $\mathbb{Z}$ coefficients) is defined as

$$H_n(X) = \frac{Ker(\partial_n)}{Im(\partial_{n+1})}.$$

We point out that this definition is how we will denote homology of a space with $\mathbb{Z}$ coefficients, because homology may be defined with coefficients of any abelian group. That is, if $F$ is an abelian group, then $H_n(X; F)$ will denote the homology of $X$ with coefficients from $F$ by slightly altering our chain complex. Thus, our definition above could also be written as $H_n(X; \mathbb{Z})$. All of the calculations in this in this study will assume $\mathbb{Z}$ coefficients unless otherwise stated.

We now turn our attention to simplicial homology, defined nearly the same as singular homology. However, we must be more carful with the choice of $n$-simplices. The goal is to build any topological space by using some basic building blocks, and then to compute meaningful algebraic data from how the building blocks are glued together. For example, thinking about convex polygons in the plane as our spaces, any polygon could be made up of some number of triangles. Simply place a vertex at the center of the polygon and draw lines to the other vertices, triangulating the polygon. Therefore, any polygon can be built by some number of triangles. Similarly, if we think of any polyhedron as our space, it can be built by some number of triangular prisms. This idea can be generalized to say that an $n$-simplex will again be $\Delta_n$, and so the 0-simplex is a point, the 1-simplex is a straight line, the 2-simplex is a triangle with straight edges, the 3-simplex is a prism, and so on. But we assume our topological space has been built as a quotient space of disjoint unions of these objects glued together to respect the ordering of the vertices as before, and we call such spaces simplicial complexes. We will use a similar notation for an $n$-simplex $\sigma = [v_0, \ldots, v_n] \in C_n^{\Delta}(X)$, defined to be the free abelian group on the $n$-simplices of a simplicial complex $X$.

**Definition 3.1.6.** [Hat02] Given an $n$-simplex $\sigma = [v_0, \ldots, v_n] \in C_n^{\Delta}(X)$. The boundary operator is defined as

$$\partial_n^{\Delta}(\sigma) = \sum_{i=0}^{n}(-1)^i[v_0, \ldots, \check{v}_i, \ldots, v_n].$$

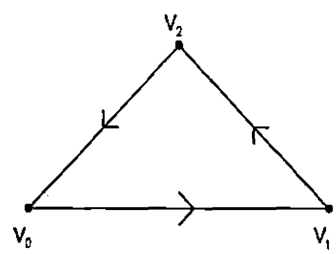



Figure 3.1: A 2-Simplex in $\mathbb{R}^2$

**Example 3.1.7.** We consider the 2-simplex $[v_0, v_1, v_2]$ in Figure 3.1 as an example. The arrows on each edge indicate the ordering of the vertices. Then,

$$\partial_2^{\Delta}[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1].$$

The simplicial homology groups are defined in a similar way to the singular homology groups.

**Definition 3.1.8.** [Hat02] The *simplicial homology groups* are defined as

$$H_n^{\Delta}(X) = \frac{Ker(\partial_n^{\Delta})}{Im(\partial_{n+1}^{\Delta})}.$$

Simplicial homology is handy because we can now represent many familiar topological spaces with a finite number of simplices, thus, for these spaces, $C_n^{\Delta}(X)$ will be finitely generated and the computation of its homology will be a purely combinatorial task. The manner in which certain simplices have been glued together to form a simplicial complex is called a $\Delta-$*complex* structure. The quick definition of a $\Delta$-complex is that it is a quotient space of a collection of disjoint simplices obtained by identifying certain of their faces via the canonical linear homeomorphisms that preserve the ordering of vertices. Constructing a $\Delta$-complex is like building something from a kit of pre-cut parts that only need to be snapped together following the instructions [Hat02]. Up to homotopy, many of these $\Delta$-complexes are familiar topological spaces. One $\Delta$-complex structure on $S^1$ would be one 0-simplex ($p$) and one 1-simplex (a) as in Figure 3.2. Another $\Delta$-complex

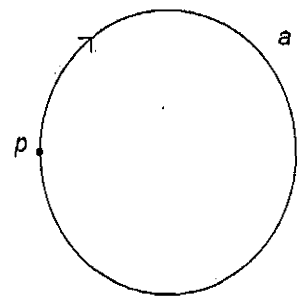



Figure 3.2: The 1-Dimensional Sphere, $S^1$

structure on $S^1$ would be two 0-simplices and two 1-simplices. One $\Delta$-complex structure on the torus would be one 0-simplex, three 1-simplices, and two 2-simplices as in Figure 3.3. The $b$ edges are first glued together according to their orientation, creating a tube. Then the $a$ edges are glued together according to their orientation, creating a space that is homotopic to the torus. There is an important relationship between $H_n^{\Delta}(X)$ and $H_n(X)$

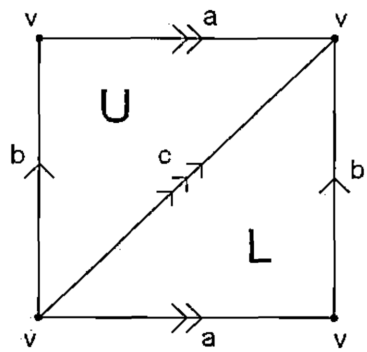



Figure 3.3: The Torus, $\mathbb{T}^2$

for simplicial complexes that will be of great use to us.

**Theorem 3.1.9.** [Hat02] Given a $\Delta$-complex structure on the space $X$,

$$H_n(X) \cong H_n^{\Delta}(X).$$

Therefore, simplicial and singular homologies are equivalent, and by computing the simplicial homology of a space with a $\Delta$-complex structure, we are also computing the singular homology of the same space.

**Example 3.1.10.** We compute $H_n^{\Delta}(S^1)$ using the $\Delta$-complex we put on $S^1$ in Figure 3.2. One 0-simplex, call it $p$, and one 1-simplex, call it $a$, are enough to construct $S^1$. The 1-simplex has both endpoints at $p$ and is oriented as drawn. Notice there is no gluing here that we cannot already express in our Figure 3.2, as compared to how the torus was created from its $\Delta$-complex structure. The corresponding chain complex for $S^1$ is

$$\cdots C_2^{\Delta}(S^1) \xrightarrow{\partial_2^{\Delta}} C_1^{\Delta}(S^1) \xrightarrow{\partial_1^{\Delta}} C_0^{\Delta}(S^1) \xrightarrow{\partial_0^{\Delta}} 0.$$

But $C_k^{\Delta}(S^1) = 0$ for all $k \geq 2$, since there are no simplices higher than the 1-simplex. Therefore, $H_k^{\Delta}(S^1) = \frac{0}{Im\partial_{k+1}^{\Delta}} = 0$ for all $k \geq 2$. Next, we will compute $H_1^{\Delta}(S^1)$ and $H_0^{\Delta}(S^1)$. First, $Im\partial_2^{\Delta} = 0$ since $\partial_2^{\Delta} : 0 \to C_1^{\Delta}(S^1)$. Next, $\partial_1^{\Delta}(a) = (p) - (p) = 0$. So $a \in Ker\partial_1^{\Delta}$. And $\partial_1^{\Delta}(na) = n\partial_1^{\Delta}(a) = 0$, where $n \in \mathbb{Z}$. So, $Ker(\partial_1^{\Delta}) = C_1^{\Delta}(S^1)$, and $C_1^{\Delta}(S^1) = \mathbb{Z}(a)$, where $\mathbb{Z}(a)$ is our notation for $\{na \mid n \in \mathbb{Z}\}$ and is isomorphic to $\mathbb{Z}$. Therefore, $H_1^{\Delta}(S^1) = \frac{\mathbb{Z}(a)}{(0)} \cong \mathbb{Z}(a) \cong \mathbb{Z}$. So $H_1^{\Delta}(S^1) \cong \mathbb{Z}$. Finally, $\partial_0^{\Delta} : C_0^{\Delta}(S^1) \to 0$, and $Ker\partial_0^{\Delta} = C_0^{\Delta}(S^1) = \langle p \rangle$. So, $C_0^{\Delta}(S^1) = \mathbb{Z}(p)$. Now, $Im\partial_1^{\Delta} = 0$ since $\partial_1^{\Delta}(C_1^{\Delta}(S^1)) = 0$. Therefore, $H_0^{\Delta}(S^1) = \frac{\mathbb{Z}(p)}{(0)} \cong \mathbb{Z}(p) \cong \mathbb{Z}$. So $H_0^{\Delta}(S^1) \cong \mathbb{Z}$. In conclusion,

$$H_n^{\Delta}(S^1) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Since $H_n^{\Delta}(X) \cong H_n(X)$, $H_n(S^1) \cong \mathbb{Z}$ when $n = 0, 1$ and $H_n(S^1) \cong 0$ elsewhere.

**Example 3.1.11.** Another common example is the homology of the torus, $\mathbb{T}^2$, and we suppress our calculations. Give $\mathbb{T}^2$ the same $\Delta$-complex structure as in Figure 3.3. The torus yields the following homology [Hat02]:

$$H_n^{\Delta}(\mathbb{T}^2) \cong \begin{cases} \mathbb{Z}(U - L) & \text{for } n = 2 \\ \mathbb{Z}(a) \oplus \mathbb{Z}(b) & \text{for } n = 1 \\ \mathbb{Z}(v) & \text{for } n = 0 \\ 0 & \text{for } n \geq 3 \end{cases}$$

There are several important but basic facts about homology that need to be stated here.

**Theorem 3.1.12.** [Vic94] If two topological spaces $X$ and $Y$ are homotopic, then

$$H_n(X) \cong H_n(Y).$$

The map that provides this isomorphism is $F_*$ that will be described in Definition 3.3.1 and is simply the induced map of homology on the homotopy map $F : X \to Y$.

**Lemma 3.1.13.** If $X = \mathbb{R}^n$, then $H_0(X) = \mathbb{Z}$ and $H_n(X) = 0$ for all $n$ greater than zero.

*Proof.* We start with the fact that $\mathbb{R}^n$ is contractable (homotopic) to the single point space, $\{x\}$. Next, it's easy to see that $H_0(\{x\}) = \mathbb{Z}$ and $H_n(\{x\}) = 0$ for $n \geq 1$. By Theorem 3.1.12, $H_n(\mathbb{R}^n) \cong H_n(\{x\})$ for all $n$. Therefore, $H_0(\mathbb{R}^n) = \mathbb{Z}$ and $H_n(\mathbb{R}^n) = 0$ for all $n$ greater than zero. □

## 3.2 Cohomology

In a certain sense, cohomology is the dual of homology and is defined by more or less dualizing the objects used to define homology. That is, the dual of $C_n(X)$ is $C^n(X)$ is defined to be the set of homomorphisms $\varphi : C_n(X) \to \mathbb{Z}$.

**Definition 3.2.1.** [Mas91] An $n$-cochain $\varphi \in C^n(X)$ is a homomorphism that sends an $n$-simplex $\sigma : \Delta^n \to X$ to a value $\varphi(\sigma) \in \mathbb{Z}$.

More generally, $C^n(X; G) = Hom(C_n(X; \mathbb{Z}); G)$ where $G$ is any abelian group, but the definition above will suffice for the remainder of this study where $G = \mathbb{Z}$. Next, define the coboundary map $d_n : C^n(X) \to C^{n+1}(X)$ as the adjoint of the boundary operator $\partial_n$ by $(d\varphi)(\theta) = \varphi(\partial\theta)$. For the cochain $\varphi \in C^n(X)$, its coboundary $d\varphi$ is the composition $C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\varphi} \mathbb{Z}$. So for an $(n+1)$-simplex $\sigma : \Delta^{n+1} \to X$ we have

$$d\varphi(\sigma) = \sum_i (-1)^i \varphi(\sigma \mid [v_0, \ldots, \check{v}_i, \ldots, v_{n+1}]). \tag{3.1}$$

Similar to the case of the boundary map, the coboundary map has the property that $d^2 = 0$.

**Theorem 3.2.2.** If $\theta \in C_{n-1}(X)$, then $(d_n d_{n-1}(\varphi))(\theta) = 0$.

*Proof.*

$$\begin{aligned}(d_n d_{n-1}(\varphi))(\theta) &= (d_n(d_{n-1}\varphi))(\theta)\\ &= (d_n\varphi)(\partial_{n-1}\theta)\\ &= \varphi(\partial_{n-1}\partial_n\theta)\\ &= \varphi(0)\\ &= 0.\end{aligned}$$

Because $d_n$ operates in the opposite direction that $\partial_n$ operates, the $n^{th}$ cohomology group is defined somewhat differently than the $n^{th}$ homology group. Notice again that $Im(d_{n-1}) \trianglelefteq Ker(d_n)$.

**Definition 3.2.3.** The $n^{th}$ *cohomology group* is defined as

$$H^n(X) = \frac{Ker(d_n)}{Im(d_{n-1})}.$$

*Simplicial cohomology* is similarly defined as

$$H^n_\Delta(X) = \frac{Ker(d^\Delta_n)}{Im(d^\Delta_{n-1})}. \tag{3.2}$$

The map $d^\Delta_n$ is defined identical to $d_n$ except it operates on the simplices of a $\Delta$-complex structure. It is true, as in homology, that $H^n_\Delta(X)$ is isomorphic to $H^n(X)$ [Hat02]. For an example of how to compute the cohomology groups of a topological space, we look to the Klein bottle.

**Example 3.2.4.** We can put the following $\Delta$-complex structure on the Klein bottle as in Figure 3.4. Notice to construct the Klein bottle the $a$ edges are first glued together accordingly. Then, the $b$ edges are glued together in such a way that their orientations agree. This is done by seemingly going through the bottle and attaching the edges. However, the Klein bottle has the characteristic of not intersecting itself.

We compute $H^2_\Delta(K) \cong H^2(K)$. First, notice

$$0 \rightarrow C^0_\Delta(K) \xrightarrow{d_0} C^1_\Delta(K) \xrightarrow{d_1} C^2_\Delta(K) \xrightarrow{d_2} 0. \tag{3.3}$$

The last map above, $d_2$, goes to 0 since there are no 3-simplices other than the 0-chain, by construction. Since $H^2_\Delta(K) = \frac{Ker\ d_2}{Im\ d_1}$, we need to know $Ker\ d_2$ and $Im\ d_1$. Now, we know $C^\Delta_2(K) = \langle U, L\rangle$. So if $\eta(U) = 1$, $\eta(L) = 0$, $\xi(U) = 0$, and $\xi(L) = 1$, then $C^2_\Delta(K) = \langle \eta, \xi\rangle$. Similarly, we know $C^\Delta_1 = \langle a, b, c\rangle$. So if $\alpha(a) = 1$, $\alpha(b) = 0$, $\alpha(c) = 0$,

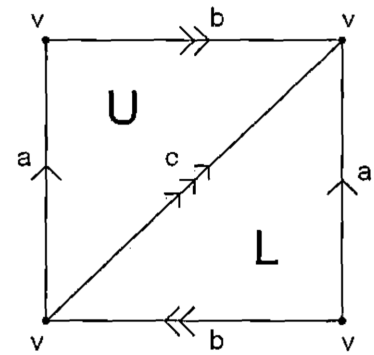



Figure 3.4: The Klein Bottle, $K$

$\beta(a) = 0$, $\beta(b) = 1$, $\beta(c) = 0$, $\gamma(a) = 0$, $\gamma(b) = 0$, and $\gamma(c) = 1$, then $C^1_\Delta(K) = \langle \alpha, \beta, \gamma \rangle$. Now, because $d_2 : C^2_\Delta \to 0$, we know $Ker\ d_2 = C^2_\Delta(K) = \langle \eta, \xi \rangle$. To find $Im\ d_1$, we need to know $d_\alpha$, $d_\beta$, and $d_\gamma$. Now, $(d(\alpha))(U) = \alpha(\partial U) = \alpha(b - c + a) = 1$ and $(d(\alpha))(L) = \alpha(\partial L) = \alpha(c - a + b) = -1$. So, $d_\alpha = \eta - \xi$. Next,

$$(d(\beta))(U) = \beta(\partial U) = \beta(b - c + a) = 1,$$

and

$$(d(\beta))(L) = \beta(\partial L) = \beta(c - a + b) = 1.$$

So, $d\beta = \eta + \xi$. Similarly, $(d(\gamma))(U) = \gamma(\partial U) = \gamma(b - c + a) = -1$, and

$$(d(\gamma))(L) = \gamma(\partial L) = \gamma(c - a + b) = 1.$$

So, $d\gamma = -\eta + \xi$. Therefore, $Im\ d_1 = \langle \eta - \xi, \eta + \xi, -\eta + \xi \rangle$ and

$$\begin{aligned} H^2_\Delta(K) &= \frac{\langle \eta,\xi \rangle}{\langle \eta-\xi,\eta+\xi,-\eta+\xi \rangle} &= \frac{\langle \eta,\xi \rangle}{\langle \eta-\xi;\eta+\xi \rangle} &= \frac{\langle \eta,\xi \rangle}{\langle 2\eta,\eta+\xi, \rangle} \\ &= \frac{\langle \eta,\eta+\xi \rangle}{\langle 2\eta,\eta+\xi, \rangle} &= \frac{\langle \eta \rangle}{\langle 2\eta \rangle} &= \mathbb{Z}_2(\eta). \end{aligned}$$

So, $H^2(K) \cong \mathbb{Z}_2$. The other cohomology groups of the Klein bottle, $H^1(K)$ and $H^0(K)$, can be found similarly.

## 3.3 Induced Homomorphisms

Given a continuous map $f : X \to Y$, there is a useful way to transfer information about the homology and cohomology of $X$ to the homology and cohomology of $Y$. Given any $n$-simplex on $X$, $\sigma : \Delta_n \to X$, the map $f \circ \sigma : \Delta_n \xrightarrow{\sigma} X \xrightarrow{f} Y$ is an $n$-simplex on $Y$. So one may extend this by linearity to the chain groups $C_n(X)$ and arrive at the following definition.

**Definition 3.3.1.** Define the mapping $f_* : H_n(X) \to H_n(Y)$ by $f_*[\alpha] = [f \circ \alpha]$, where $\alpha \in H_n(X)$.

One checks that this is a well-defined homomorphism $f_* : H_n(X) \to H_n(Y)$. Similarly, one uses this sort of construction to produce a mapping of cohomology groups as well.

**Definition 3.3.2.** If $f : X \to Y$, and $\theta \in \mathrm{Hom}(C_n(Y), \mathbb{Z}) = C^n(Y)$ is an $n$-cochain, then we may *pre-compose* $\theta$ with $f$ to define the object $f^*[\theta] = [\theta \circ f]$.

Here, we regard $f : C_n(X) \to C_n(Y)$, so that the cochain $f^*[\theta]$ is the cohomology class represented by the cochain $\theta \circ f$, a function which first sends chains on $X$ to chains on $Y$ (via $f$), and then produces an integer (using $\theta$). Again, one can check that this construction yields a well-defined homomorphism $f^* : H^n(Y) \to H^n(X)$. Both $f_*$ and $f^*$ are referred to as *induced homomorphisms.*

Notice that of the homomorphisms $f_*$ and $f^*$, the induced homomorphism in homology preserves direction, while the induced homomorphism in cohomology reverses direction. In addition, both homology and cohomology are objects which input topological spaces, and output a sequence of abelian groups. In this situation, mathematicians describe a process of this sort as a *functor*, and since $f_*$ preserves direction, homology is said to be a *covariant functor*, while since $f^*$ reverses direction, cohomology is said to be a *contravariant functor.* This discussion continues into more abstract notions of homology and cohomology theory, along with a long digression into category theory. So we content ourselves with the singular and simplicial homology and cohomology theories we have developed here, and avoid such a digression which is neither pertinent to our goal here, nor necessary. For further information on these broad topics, see [Sie92].

# Chapter 4

# Exact Sequences, Relative Homology, and Orientation

## 4.1 Exact Sequences

We are now ready to discuss exact sequences. Both long and short exact sequences will be used subsequently to examine some relationships between the homology of differnet spaces, subspaces, quotient spaces, and relative homology. We start with the definition of an exact sequence.

**Definition 4.1.1.** [Hat02] A sequence of abelian groups $\{A_n\}$, with homomorphisms $\alpha_n : A_n \to A_{n-1}$,

$$\cdots \to A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \to \cdots , \tag{4.1}$$

is said to be exact if $Ker \ \ \alpha_n = Im \ \ \alpha_{n+1}$ for each $n$.

Because $Im \ \ \alpha_{n+1} \subset Ker \ \ \alpha_n$, it is true that $\alpha_n \alpha_{n+1} = 0$. As a result of the definition of exact sequences, there are several nice characteristics that arise.

**Lemma 4.1.2.** [Hat02] If $A$, $B$, and $C$ are abelian groups, then the following hold:

1. $0 \to A \xrightarrow{\alpha} B$ is exact iff $\alpha$ is injective.

2. $A \xrightarrow{\alpha} B \to 0$ is exact iff $\alpha$ is surjective.

3. $0 \to A \xrightarrow{\alpha} B \to 0$ is exact iff $\alpha$ is an isomorphism.

4. $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact iff $\alpha$ is injective, $\beta$ is surjective, and $Ker\ \beta = Im\ \alpha$.

In 4 above, the sequence is called a short exact sequence. A long exact sequence is simply an exact sequence that is longer than the sequence in 4. The map $\beta$ induces an isomorphism $C \approx B/Im\ \alpha$, and if $A$ is a subgroup of $B$ and $\alpha$ is the inclusion map, then $C \approx B/A$. This fact will be very helpful later when developing the relationships between the homology of spaces, subspaces, and quotient spaces.

**Example 4.1.3.** The following sequence is a short exact sequence:

$$0 \to \mathbb{Z}_2 \xrightarrow{f} \mathbb{Z}_4 \xrightarrow{g} \mathbb{Z}_2 \to 0,$$

where $f([\alpha]_2) = [2\alpha]_4$ and $g([\beta]_4) = [\beta]_2$. The other maps, from 0 to $\mathbb{Z}_2$ and from $\mathbb{Z}_2$ to 0, are obvious. The main property to verify is that $Im\ f = Ker\ g$, since $f$ is injective and $g$ is surjective. This fact holds since the $Im\ f = \{[0]_4, [2]_4\} = Ker\ g$.

Another consequence of these exact sequences is the following lemma:

**Lemma 4.1.4.** [Bre97] In a commutative diagram of abelian groups as below, if the two rows are exact and $\alpha$, $\beta$, $\delta$, and $\epsilon$ are isomorphisms, then $\gamma$ is an isomorphism also.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\circ"] & B \arrow[r, "j"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circ"] & C \arrow[r, "k"] \arrow[d, "\gamma"] \arrow[dr, phantom, "\circ"] & D \arrow[r, "l"] \arrow[d, "\delta"] \arrow[dr, phantom, "\circ"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E'
\end{tikzcd}$$

*Proof.* First, we will show $\gamma$ is surjective. That is, $\forall c' \in C'$, $\exists x \in C$ such that $\gamma(x) = c'$. Now, let $c' \in C'$. That implies $k'(c') = \delta(d)$ for some $d \in D$, since $\delta$ is surjective. By commutativity, $\epsilon l(d) = l'\delta(d)$. Hence, $\epsilon l(d) = l'\delta(d) = l'k'(c')$. By exactness of the bottom row, $l'k'(c') \in Ker(l')$. Thus, $\epsilon l(d) = l'\delta(d) = l'k'(c') = 0$. But $\epsilon l(d) = 0$ implies $l(d) = 0$ since $\epsilon$ is injective. Now, $\exists c \in C$ such that $k(c) = d$ since $k(c) \in Im(k)$, $Im(k) = Ker(l)$ and $l(d) = 0$. So $l(d) = lk(c) = 0$. Next,

$$k'(c' - \gamma(c)) = k'(c') - k'\gamma(c) = k'(c') - \delta k(c) = k'(c') - \delta(d) = 0,$$

since $l'\delta(d) = l'k'(c')$ implies $l'(k'(c') - \delta(d)) = 0$. And since $l'$ is a homomorphism, $k'(c') - \delta(d) = 0$. If $k'(c' - \gamma(c)) = 0$, then $\exists b' \in B'$ such that $j'(b') = c' - \gamma(c)$, since $Im(j') = Ker(k')$. Now, since $\beta$ is surjective, $\exists b \in B$ such that $\beta(b) = b'$. So $j'\beta(b) = c' - \gamma(c)$. By commutativity, $j'\beta(b) = \gamma j(b)$. Hence, $\gamma j(b) = c' - \gamma(c)$ and $\gamma(j(b) + c) = c'$. Therefore, $\gamma$ is surjective.

Next, to show $\gamma$ is injective we will first let $\gamma(c) = 0$ for some $c \in C$. By commutativity, $\delta k(c) = k'\gamma(c)$. Now, $\delta k(c) = k'\gamma(c) = k'(0) = 0$ since $k'$ is a homomorphism. Because $\delta$ is injective, $\delta k(c) = 0$ implies $k(c) = 0$. Now, since $c \in Ker(k)$, there exists $b \in B$ such that $j(b) = c$. By commutativity and the previous fact, $j'\beta(b) = \gamma j(b) = \gamma(c) = 0$. So $\beta(b) \in Ker(j')$. Hence, there exists $a' \in A'$ such that $i'(a') = \beta(b)$. Now since $\alpha$ is surjective, $\exists a \in A \ni \alpha(a) = a'$. Hence, $i'\alpha(a) = i'(a') = \beta(b)$. It follows that $\beta(i(a) - b) = \beta i(a) - \beta(b) = i'\alpha(a) - \beta(b) = 0$. Therefore, $\beta i(a) = \beta(b)$, and $i(a) = b$ since $\beta$ is injective. Recall that $j(b) = c$. Now by exactness, $j(b) = ji(a) = 0$. Therefore, $c = 0$ and $\gamma$ is injective. □

To better understand why the diagram needs to commute in the above lemma, we offer the following example. Consider the short exact sequence

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0,$$

and the short exact sequence of Example 4.1.3.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "a"] & \mathbb{Z}_2 \arrow[r] \arrow[d, "b"] & \mathbb{Z}_4 \arrow[r] \arrow[d, "c"] & \mathbb{Z}_2 \arrow[r] \arrow[d, "d"] & 0 \arrow[d, "e"] \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0
\end{tikzcd}$$

Now although the maps $a$, $b$, $d$, and $e$ are clearly isomorphisms, Lemma 4.1.4 does not ensure that the map $c$ is an isomorphism unless we could show that the squares commute. And if the squares do not commute, then we can't draw this conclusion from Lemma 4.1.4. According to Lemma 4.1.4, the map $c$ is an isomorphism if the squares commute. Obviously, there is no isomorphism $c : \mathbb{Z}_4 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

## 4.2 Relative Homology and Cohomology

Before discussing how homology groups are related to exact sequences, we must first discuss how chain complex groups are related to exact sequences. To begin the discussion, we define relative homology groups. Given a topological space $X$ and a subspace $A \subset X$, let $C_n(X, A)$ be the quotient group $C_n(X)/C_n(A)$. Therefore, any chain in the subspace $A$ is zero in $C_n(X, A)$. We have $C_n(A) \subsetneq C_n(X)$ and the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, it follows that there is the quotient boundary map $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$. In terms of a sequence we have

$$\cdots \to C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \to \cdots . \tag{4.2}$$

This sequence of groups is a chain complex since it has the property $\partial_n \partial_{n+1} = 0$. Therefore, we can define the relative homology groups

$$H_n(X, A) := \frac{Ker\ \partial_n}{Im\ \partial_{n+1}},$$

where $\partial$ here represents the quotient boundary operator. Because of this definition, an element in $H_n(X, A)$ is represented by an $n$-chain $\alpha \in C_n(X)$ such that $\partial(\alpha) \in C_{n-1}(A)$.

A key fact in discussing exact sequences of homology is that a short exact sequence of chain complexes gives rise to a long exact sequence in homology.

**Theorem 4.2.1.** [Vic94] If $A$, $B$, and $C$ are any spaces and

$$0 \to C_n(A) \xrightarrow{i} C_n(B) \xrightarrow{j} C_n(C) \to 0,$$

is a short exact sequence, then

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \to \cdots$$

is an induced long exact sequence.

Now that relative homology has been defined, we want to try to fit these groups into an exact sequence. This leads to the following result:

**Lemma 4.2.2.** [Hat02] If $A \subseteq X$, then the homology groups $H_n(X, A)$ where $n$ varies fit into the long exact sequence below. The mappings $i_*$ and $j_*$ are the inclusion and quotient maps respectively.

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \xrightarrow{j_*} H_{n-1}(X, A) \to \cdots$$
$$\cdots \to H_0(X, A) \to 0.$$

There is a property between long exact sequences of pairs of spaces called *naturality*.

**Definition 4.2.3.** [Hat02]

A long exact sequence of a pair is said to be *natural* if whenever there is a map $f : (X, A) \to (Y, B)$, then the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(X) \arrow[r, "j_*"] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r, "i_*"] & H_n(Y) \arrow[r, "j_*"] & H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

## 4.3 Mayer-Vietoris Sequences

The Mayer-Vietoris sequence is an exact sequence that often helps one to compute homology groups. Homology groups can be computed directly. However, these computations become complicated to deal with in many cases, and it is useful to have tools that allow one to compute homology groups from others that one already knows. The Mayer-Vietoris sequence is one of the most useful tools for this. In the following theorem, the maps $\Phi$, $\Psi$, and $\partial$ will be defined in the sketch of the proof.

**Theorem 4.3.1.** [Bre97] For a pair of subspaces $A, B \subseteq X$ such that the union of the interiors of $A$ and $B$ is the entire space $X$, there is an exact sequence (called a $Mayer - Vietoris$ *sequence*) of the form

$$\cdots \to H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots \to H_0(X) \to 0.$$

*Sketch of Proof.* This long exact sequence arises from the following short exact sequence of chain complexes:

$$0 \to C_n(A \cap B) \xrightarrow{\phi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \to 0,$$

where $C_n(A + B)$ is a subgroup of $C_n(X)$ that consists of chains that are sums of chains of $A$ and chains of $B$. The map $\phi$ and the map $\psi$ are defined as $\phi(x) = (x, -x)$ and $\psi(x, y) = x + y$. Notice that defining $\phi$ and $\psi$ this way means that $\psi\phi = 0$. Also, notice that $Im\ \phi = ([x], -[x])$ and $Ker\ \psi = ([x], -[x])$ since $\psi([x], -[x]) = [x] - [x] = 0$. Therefore, the above sequence is a short exact sequence, and this short exact sequence gives rise to the long exact sequence in Theorem 4.3.1. The difficult part in proving this theorem is showing $H_n(A+B) \cong H_n(A \cup B)$. To do this, Lemma 4.1.4 is used in addition to other machinery, to show that the maps $C_n(A+B) \to C_n(X)$ induce isomorphisms on homology. We refer the proof of this fact to [Hat02], where there is a detailed discussion.

## 4.4 Excision

From relative homology comes excision. The excision theorem says that the relative homology of a space $H_n(X, A)$ is unaffected by excising a subspace $Z \subset A$, where the closure of $Z$ is contained in the interior of $A$.

**Theorem 4.4.1.** [Hat02] Given subspaces $Z \subset A \subset X$ such that the closure of $Z$ is contained in the interior of $A$, then the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X - Z, A - Z) \to H_n(X, A)$ for all $n$.

This theorem provides some nice relationships that will be exposed later in some of the proofs leading up to Poincaré duality.

## 4.5 Orientable Manifolds

In Chapter 6 we will be proving Poincaré duality for compact orientable manifolds. A notion of orientability will be required, but how should an orientation be defined, and why should an orientation be needed? We want to define orientation in terms of the homology of a manifold in such a manner that the basic ideas of orientation remain. That is, whatever we define orientation to be, it should be preserved under rotations and reversed under reflections. The idea of rotations and reflections will have their own meaning in this context. Before defining orientation we must define the degree of $S^n$, the $n$-sphere.

**Definition 4.5.1.** [Hat02] Given the function $f : S^n \to S^n$, the induced function $f_* : H_n(S^n) \to H_n(S^n)$ is a homomorphism from an infinite cyclic group to itself and so must be of the form $f_*(\alpha) = d\alpha$ for some integer $d$ depending only on $f$. This integer $d$ is called the *degree* of $f$.

**Lemma 4.5.2.** [Hat02] If $f : S^n \to S^n$, then $\deg f = -1$ if $f$ is a reflection of $S^n$, and $\deg f = 1$ if $f$ is a rotation of $S^n$.

Now, we define orientability as follows:

**Definition 4.5.3.** [Hat02] A *local orientation* of an $n$-dimensional manifold $M$ at a point $x$ is a choice of generator $\mu_x$ of the infinite cyclic group $H_n(M, M - \{x\})$.

This definition satisfies the basic ideas of orientation since if $x \in M$, we have the isomorphisms $H_n(M, M - \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_{n-1}(\mathbb{R}^n - \{x\}) \cong H_{n-1}(S^{n-1})$, where $S^{n-1}$ is the sphere centered at $x$. And rotations of $S^{n-1}$ have degree 1, and reflections of $S^{n-1}$ have degree $-1$. The global orientation a manifold is then defined as follows:

**Definition 4.5.4.** [Hat02] An *orientation* of an $n$-dimensional manifold $M$ is a function $x \mapsto \mu_x$ assigning each $x \in M$ a local orientation $\mu_x \in H_n(M, M - \{x\})$, satisfying the condition that each $x \in M$ has a neighborhood $\mathbb{R}^n \subset M$ containing an open ball $B$ of finite radius about $x$ such that all local orientations $\mu_y$ at points $y \in B$ are the images of one generator $\mu_B$ of $H_n(M, M - B)$.

**Lemma 4.5.5.** [Hat02] Let $M$ be a manifold of dimension $n$ and let $A \subset M$ be a compact subset.

**(a)** Let $M$ be oriented, and let $x \mapsto \alpha_x$ be its orientation. Then there is a unique class $\alpha_A \in H_n(M, M - A)$ whose image in $H_n(M, M - \{x\})$ is $\alpha_x$ for all $x \in A$.

**(b)** $H_i(M, M - A) = 0$ for $i > n$.

This definition of orientability and the above lemma is what will lead the existence of the fundamental class (discussed in the next section), which will help us to define the isomorphism used in Poincaré duality in Chapter 6.

## 4.6 The Fundamental Class

In Poincaré duality in Chapter 6, we will be proving the theorem for any compact orientable manifold. The theorem is provided by the isomorphism given by the cap product. And the cap product is provided by the existence of a fundamental class, a property that exists if the manifold is compact and orientable. We proceed to show the existence of a fundamental class for such a manifold.

**Theorem 4.6.1.** [Hat02] Let $M$ be a closed connected $n$-manifold. If $M$ is $\mathbb{Z}$-orientable, the map $H_n(M) \to H_n(M, M - \{x\}) \cong \mathbb{Z}$ is an isomorphism for all $x \in M$.

**Definition 4.6.2.** [Hat02] An element of $H_n(M)$ whose image in $H_n(M, M - \{x\})$ is a generator for all $x$ is called a *fundamental class* for $M$ with coefficients in $\mathbb{Z}$. Thus, a fundamental class for a manifold is a choice of generator for the infinite cyclic group $H_n(M)$.

By Theorem 4.6.1, a fundamental class exists if $M$ is closed and $\mathbb{Z}$-orientable. This fact will be used in Chapter 6 when we define the map (called the cap product) that provides the isomorphisms in Poincaré duality.

# Chapter 5

# Some Algebraic Preliminaries

## 5.1 Cohomology with Compact Supports

In the proof of Poincaré duality, we will need to make use of the concept and consequences of a different cohomology theory known as cohomology with compact supports. Cohomology with compact supports is a way of defining cohomology with an added structure. This added structure leads to some nice results involving homology, cohomology and the relationships between the two. We begin with a definition.

**Definition 5.1.1.** [Mas91] A cochain $u \in C^k(X)$ has a *compact support* if and only if there exists a compact set $K \subset X$ such that $u \in C^k(X, X - K)$.

We will denote the set of cochains with compact supports by $C_c^k(X)$. An equivalent form of this definition is to define $C_c^k(X)$ as the set generated by functions that vanish at all but finitely many simplices. It is easy to see that using the same operator as before, we have $d : C_c^n(X) \to C_c^{n+1}(X)$ with $d^2 = 0$. Thus $(C_c^n(X), d)$ is a chain complex. We will then denote the $k^{th}$ cohomology group of this complex by $H_c^k(X)$. The cochains with compact supports are clearly a subgroup of $C^k(X)$. If the space $X$ is compact, then every cochain $u \in C^k(X)$ obviously lives in $C^k(X, X - K)$ where $K = X$, since $C^k(X, \emptyset) = C^k(X)$. Therefore, if $X$ is compact, then $C^k(X) = C_c^k(X)$ and

$$H^k(X) = H_c^k(X).$$

One property about singular homology and cohomology that does not transfer over to cohomology with compact supports is the homotopy property (Theorem 3.1.12). That is, two spaces $X$ and $Y$ being homotopic does not imply that $H_c^n(X) \cong H_c^n(Y)$. For example, one can compute $H_c^1(\mathbb{R}) = \mathbb{Z}$, while $\mathbb{R}$ is contractible $\mathbb{R} \simeq \{0\}$, and then $H_c^1(\{0\}) \cong H^1(\{0\}) \cong 0$ (see Example 2.1.6 and Lemma 3.1.13).

## 5.2 Direct Limits

Another important concept that will be useful for understanding Poincaré duality and its proof is the direct limit of groups. To begin the discussion of direct limits, we start with a directed set. By definition, a *directed set* $I$ is an ordered set having the property that for each pair $\alpha, \beta \in I$, there exists a $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Let $I$ be the index set on the abelian groups $G_\alpha$. Also, for each pair of elements $\alpha \leq \beta$ in $I$, there exists $f_{\alpha\beta} : G_\alpha \to G_\beta$ is a homomorphism. We also require that $f_{\alpha\alpha} = 1 \in G_\alpha$, and if $\alpha \leq \beta \leq \gamma$, then $f_{\alpha\gamma}$ is equal to the composition $f_{\beta\gamma} \circ f_{\alpha\beta}$. Now, since each $G_i$ is an abelian group, it follows that $\oplus_\alpha G_\alpha$ is also an abelian group. Let $R$ be the subgroup generated by elements of the form $a - f_{\alpha\beta}(a)$. Then the *direct limit group*, $\varinjlim G_\alpha$, is then defined as

$$\varinjlim G_\alpha = \frac{\oplus_\alpha G_\alpha}{R}.$$

A useful consequence of direct limits that nearly follows from the definition is that if we have a subset $J \subset I$ with the property that for each $\alpha \in I$ there exist a $\beta \in I$ with $\alpha \leq \beta$, then $\varinjlim G_\alpha$ is the same whether we compute it with $\alpha$ varying over $I$ or just over $J$. In particular, if $I$ has a maximal element $\gamma$, we can take $J = \{\gamma\}$ and then $\varinjlim G_\alpha = G_\gamma$ [Hat02]. We will illustrate this with an example.

**Example 5.2.1.** Let $I = \{1, 2, 3\}$, $G_1 = \mathbb{Z}$, $G_2 = \mathbb{Z} \oplus \mathbb{Z}$, and $G_3 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Then $\oplus_{\alpha \in I} G_\alpha = (\mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z}) \oplus (\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})$ and its elements are $(n_1, (n_2, n_3), (n_4, n_5, n_6))$. In $\varinjlim G_\alpha$, each element $(n_1, (n_2, n_3), (n_4, n_5, n_6))$ is equated with

$$(0, (0, 0), (n_1 + n_2 + n_4, n_3 + n_5, n_6)),$$

illustrating the isomorphism

$$\varinjlim G_\alpha \cong G_3 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

## 5.3 Some Consequences of Direct Limits and Compact Supports

Below are some final algebraic preliminaries to be used in the proof of the Poincaré duality theorem.

**Theorem 5.3.1.** If a space $X$ is the union of a directed set of subspaces $X_\alpha$ with the property that each compact set in $X$ is contained in some $X_\alpha$, then the natural map $\varinjlim H_n(X_\alpha) \to H_n(X)$ is an isomorphism for all $n$.

*Proof.* Let $f_\alpha = (i_\alpha)_*$, $f_{\alpha\beta} = (i_{\alpha\beta})_*$. Set $f : \left(\varinjlim H_n(X_\alpha)\right) \to H_n(X)$ as $f(\theta_{\alpha_1}, \ldots, \theta_{\alpha_n}) = \sum_{i=1}^{n} f_{\alpha_i}(\theta_{\alpha_i}) = \sum_{i=1}^{n} \theta_{\alpha_i}$. As $f$ is clearly linear, we must show that $f$ is bijective. First, let $\theta \in H_n(X)$. Then $\exists X_\alpha$ such that $Im(\theta) \subseteq X_\alpha$, since $Im(\theta)$ is compact in $X$. So, $\theta = i_\alpha \theta$, where $\theta \in H_n(X_\alpha)$. So, $\theta = f(\theta)$ and $f$ is surjective. Next, suppose $f(\theta) = 0$ where $\theta = \sum \theta_{\alpha_i}$. Then $f(\theta) = \partial\tau$ and $Im(\tau)$ is compact. Therefore, $\exists X_\alpha$ such that $Im(\partial\tau) \subseteq Im(\tau) \subseteq X_\alpha$. So, $\partial\tau = i_\alpha \partial\tau$, where $\partial\tau \in C_n(X_\alpha)$. Now each $\theta_{\alpha_i} \in H_n(X_{\alpha_i})$ and is compact. So, $\exists X_\beta$, such that $Im(\theta_{\alpha_i}) \subseteq X_\beta$ for all $i$, and $Im(\partial\tau) \subseteq X_\beta$. Then,

$$\theta = (\theta_{\alpha_1}, \ldots, \theta_{\alpha_n}) = (0, \ldots, 0, \sum \theta_{\alpha_i}) = (0, \ldots, 0, \partial\tau) = 0 \in \varinjlim H_n(X_\alpha).$$

Therefore, $f$ is also injective. □

Cohomology with compact supports can be defined in terms of direct limits.

**Theorem 5.3.2.** Let $X$ be a manifold of dimension $n$ and $K_i \subseteq X$ be compact subsets of $X$. Also, let $C_c^k(X)$ be the set of cochains with compact support. If the direct limit $\varinjlim H^k(X, X-K)$ is taken over all compact subsets $K$ of $X$, then the function

$$f : \varinjlim H^k(X, X-K) \to H_c^k(X)$$

given by $f(\theta_{K_1}, \ldots, \theta_{K_n}) = \sum \theta_{K_i}$ where $\theta_{K_i} \in C_c^k(X, X-K_i)$ is an isomorphism.

*Proof:* First, we need to show $f$ is well defined. If $K \subseteq L$, then the maps $f_{KL} : H^n(X, X-K) \to H^n(X, X-L)$ satisfy $f_{KL}(\theta) = \theta$, as inclusions. So, if $\theta \sim f_{KL}\theta$, then $f(\theta) = f(f_{KL}(\theta)) = f(\theta)$. So, $f$ is well defined. Next, let $\theta \in H_c^n(X)$, and let $\sigma_i$ be the finitely many simplices where $\theta(\sigma_i) \neq 0$. Then $K = \bigcup Im\sigma_i$ is compact, and

$\theta$ belongs to $H^n(X, X-K)$. So, $\theta = f(\theta)$ and $f$ is surjective. And finally, suppose $f(\theta_{K_1}, \ldots, \theta_{K_n}) = 0 \in H_c^n(X)$. Then $\sum \theta_{K_i} = \partial\tau$. Now $\bigcup K_i \bigcup Im\tau = L \supseteq K_i$. So in $C_c^n(X, X-L)$, $[\sum \theta_i] = [\partial\tau] = 0$, and

$$(\theta_{K_1}, \ldots, \theta_{K_n}) \sim (0, \ldots, 0, \sum \theta_{K_i}) = (0, \ldots, 0) \in \varinjlim H^n(X, X-K).$$

Therefore, $f$ is also injective and hence an isomorphism.

**Corollary 5.3.3.** If $M$ is compact, then $H_c^i(M) = H^i(M)$.

*Proof.* This lemma comes from the fact that there exists a unique largest compact set $K \subseteq M$, namely $M$ itself. And we know from the properties of direct limits that if there is a largest set $X_n$ in a directed set $X_1 \subset X_2 \subset \cdots \subset X_n = M$, then

$$\varinjlim H_c^k(X_i) \cong H^k(X_n) = H^k(M).$$

We conclude this chapter by stating one more result that will be directly referenced in Chapter 6 during the proof of Poincaré duality.

**Theorem 5.3.4.** [Hat02] Let $I$ be a directed set with $\alpha, \beta \in I$, and $\alpha \leq \beta$. Also, let $f_{\alpha\beta} : G_\alpha \to G_\beta$ and $h_{\alpha\beta} : H_\alpha \to H_\beta$. If there are isomorphisms $g_\alpha : G_\alpha \to H_\alpha$ $\forall \alpha \in I$, and if the following diagram

$$\begin{tikzcd} G_\alpha \arrow[r, "f_{\alpha\beta}"] \arrow[d, "\varphi_\alpha"'] & G_\beta \arrow[d, "\varphi_\beta"] \\ H_\alpha \arrow[r, "h_{\alpha\beta}"] & H_\beta \end{tikzcd}$$

commutes, then

$$\varinjlim G_\alpha \cong \varinjlim H_\alpha.$$

# Chapter 6

# Poincaré Duality

In this chapter we complete one of our goals by providing a complete proof of Poincaré duality for compact orientable manifolds. Poincaré duality says there are isomorphisms between each cohomology group of a compact orientable manifold and its complimentary homology group. The proof itself is as interesting as the result, as a new geometrical object, the cap product, is used. We begin by describing the cap product in detail.

## 6.1 Cap Product

We begin by describing a mapping that takes an element of $H^k(X)$ and sends it to $H_{n-k}(X)$, for a topological space $X$.

**Definition 6.1.1.** [Hat02] For an arbitrary space $X$, define the bilinear *cap product* $\frown : C_k(X) \times C^l(X) \to C_{k-l}(X)$ for $k \geq l$ by setting

$$\sigma \frown \varphi = \varphi(\sigma \mid [v_0, \ldots, v_l])\sigma \mid [v_l, \ldots, v_k]$$

for $\sigma : \Delta^k \to X$ and $\varphi \in C^l(X)$. We extend this map by linearity on $C_k(X)$.

This definition leads to an induced cap product in homology,

$$H_k(X) \times H^l(X) \xrightarrow{\frown} H_{k-l}(X),$$

and in our proof of Poincaré duality, we assert that the map $D_M : H^k(M) \to H_{n-k}(M)$ given by $D_M(\alpha) = [M] \frown \alpha$ is an isomorphism. Here $M$ is a compact oriented manifold of dimension $n$, and $[M]$ is its fundamental class. Recall that the fundamental class of $M$ is an element of $H_n(M)$ whose image in $H_n(M, M - \{x\})$ is a generator for each $x \in M$. The existence of this fundamental class is provided by Theorem 4.6.1.

## 6.2 Poincaré Duality for Noncompact Orientable Manifolds

Below is a discussion of duality for noncompact manifolds. Although in this chapter we are primarily concerned with proving the Poincaré duality theorem for compact manifolds, we will need a discussion of duality for noncompact manifolds in doing so. The coefficient ring will always be $\mathbb{Z}$ here, and all statements will be made relative to this ring.

Recall from Chapter 4 that an *orientation* for a manifold of dimension $n$ is an assignment $x \mapsto \mu_x \in H_n(M, M - \{x\})$, where $\mu_x$ is a generator for the $n$th homology group $H_n(M, M - \{x\}) \cong \mathbb{Z} \cdot \mu_x$. This isomorphism is evident from excision, and the long exact sequence of a pair, and the additional knowledge of the given generator is the extra assumption of orientability. In particular, the group will always have a generator regardless of orientability, but there are extra properties of this generator that orientability gives. We have the following lemma:

**Lemma 6.2.1.** [Hat02] Let $M$ be an oriented manifold of dimension $n$, let $x \in M$, and let $K \subseteq M$ be a compact subset of $M$. Denote $i : (M, M - K) \to (M, M - \{x\})$ as the inclusion map. Then there is a unique class $\mu_K \in H_n(M, M - K)$ so that $i_* \mu_K = \mu_x$, the orientation class.

Notice that the hypotheses did not exclude noncompact manifolds. Recall that we proved that

$$\varinjlim H^k(M, M - K) \cong H^k_c(M), \tag{6.1}$$

where the direct limit is taken over all compact subsets of $M$. We wish to describe a duality map $D_M : H^k_c(M) \to H_{n-k}(M)$ for any orientable manifold $M$ of dimension $n$.

We note, by the above lemma, that if $M$ is compact (using $K = M$), then there is a unique generator $\mu_M \in H_n(M, M - M) \cong H_n(M)$, and we define the duality map $D_M : H^k(M) \to H_{n-k}(M)$ as $D_M(\psi) = \mu_M \frown \psi$. And so the need for a suitable statement for a noncompact manifold is in order, and the appropriate generalization would be a map $D_M : H_c^k(M) \to H_{n-k}(M)$, as $H_c^k(M) \cong H^k(M)$ in the event $M$ is compact. And so we proceed as follows to produce such a map.

Consider the following commutative diagram exhibiting the naturality of the cap product [Hat02]. Here, $M$ may not be compact, and $K \subsetneq L \subseteq M$ are compact subsets.

$$\begin{tikzcd}
H_n(M, M-L) \arrow[dd, "i_*"'] & \times & H^k(M, M-L) \arrow[dr, "\frown"] & \\
& & & H_{n-k}(M) \\
H_n(M, M-K) & \times & H^k(M, M-K) \arrow[uu, "i^*"'] \arrow[ur, "\frown"'] &
\end{tikzcd}$$

**Lemma 6.2.2.** Let $M$ be an oriented manifold of dimension $n$. Let $K \subsetneq M$ be a compact subset of $M$, and let $\mu_K \in H_n(M, M-K)$ be the unique generator supplied by Lemma 4.5.5. The map $D_K : H^k(M, M-K) \to H_{n-k}(M)$ given as $\psi \mapsto \mu_K \frown \psi$ is a well-defined homomorphism.

*Proof.* The only assertion to prove is that the map is well-defined, as it is defined to be a linear. Let $\psi \in H^k(M, M-K)$, and let $\mu_K \in H_n(M, M-K)$. Notice the element $\mu_K \in C_n(M)/C_n(M-K)$ and is thus represented by the coset $\mu + \theta$, where $\mu \in C_n(M)$, and $\theta \in C_n(M-K)$, and $\mu_K - \mu = \theta \in C_n(M-K)$. Notice

$$\psi \in C^k(M, M-K) = \mathrm{Hom}(C_n(M, M-K), \mathbb{Z}),$$

which is not a quotient group, and so we need not see if the outcome of $D_K(\psi)$ depends on some coset representative of $\psi$. Rather, we need to show that $\mu_K \frown \psi = \mu \frown \psi$. We consider the cap product $\mu_K \frown \psi = (\mu + \theta) \frown \psi = \mu \frown \psi + \theta \frown \psi$, and show that $\theta \frown \psi = 0$. Notice

$$\psi \in \mathrm{Hom}(C_n(M, M-K), \mathbb{Z}) = \mathrm{Hom}\left(\frac{C_n(M)}{C_n(M-K)}, \mathbb{Z}\right),$$

so that if $\sigma$ is any chain in $M - K$, it is 0 in the group $C_n(M-K)$, and thus we have $\psi(\sigma) = \psi(0) = 0$. It follows that $\theta \frown \psi = 0$, and the result follows. $\square$

Thus we have a well-defined homomorphism $D_K : H^k(M, M-K) \to H_{n-k}(M)$ for each compact subset $K$ of $M$. We define a map $D_M : H_c^k(M) \to H_{n-k}(M)$ as follows. Since $H_c^k(M) = \varinjlim H^k(M, M-K)$ (see Equation (6.1)), we define this map on the direct limit group, which is a quotient of the group $\oplus H^k(M, M-K)$, where the sum is over all compact subsets $K \subseteq M$. Define, for $\psi_{K_i} \in H^k(M, M-K_i)$ and $K_i \subset M$ a compact subset,

$$D_M(\psi_{K_1}, \ldots, \psi_{K_p}) = \sum_{i=1}^{p} D_{K_i}(\psi_{K_i}). \tag{6.2}$$

**Lemma 6.2.3.** The map $D_M : H_c^k(M) \to H_{n-k}(M)$ defined in Equation (6.2) is a well-defined group homomorphism.

*Proof.* By Lemma 6.2.2, the image of this map is in $H_{n-k}(M)$. Since each of the $D_K$ are linear, the map $D_M$ is linear. We must only show it is well-defined to conclude that it is a homomorphism. By definition, the element $(\psi_{K_1}, \ldots, \psi_{K_p}) \sim (\tilde{\psi}_{L_1}, \ldots, \tilde{\psi}_{L_k})$ if and only if there is a correspondence $\psi_{K_i} \leftrightarrow \tilde{\psi}_{L_j}$, where $K_i \subseteq L_j$, and $\tilde{\psi}_{L_j} = i^*\psi_{K_i}$, where $i : (M, M-L_j) \to (M, M-K_i)$ is the inclusion map. So the result will follow if we can show that $D_K(\psi_K) = D_L(\psi_L)$, when $\psi_L = i^*\psi_K$ for compact subsets $K \subseteq L$ of $M$, as the result is simply $\max\{k, p\}$ applications of this fact.

So suppose $K \subseteq L$, and that $i^*\psi_K = \psi_L$, where $i : (M, M-L) \to (M, M-K)$. We recall that $i_*\mu_L = \mu_K$ by the uniqueness part of Lemma 4.5.5. Then using naturality in the middle equality, we complete the proof by noting

$$D_L(\psi_L) = \mu_L \frown \psi_L = \mu_L \frown i^*\psi_K = i_*\mu_L \frown \psi_K = \mu_K \frown \psi_K = D_K(\psi_K).$$

□

**Lemma 6.2.4.** [Hat02] If $M$ is the union of two open sets $U$ and $V$, then there is a diagram of Mayer-Vietoris sequences, commutative up to sign:

$$\begin{tikzcd}
\cdots \arrow[r] & H_c^k(U \cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H_c^k(U) \oplus H_c^k(V) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H_c^k(M) \arrow[r] \arrow[d, "D_M"] & H_c^{k+1}(U \cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U \cap V) \arrow[r] & H_{n-k}(U) \oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k-1}(U \cap V) \arrow[r] & \cdots
\end{tikzcd}$$

## 6.3 Poincaré Duality for Compact Orientable Manifolds

We now can accomplish one of our goals, proving the Poincaré duality theorem for compact orientable manifolds. Recall that the fundamental class $[M] \in H_n(M)$ exists if the manifold $M$ is compact and orientable. The theorem is formally stated here.

**Theorem 6.3.1.** If $M$ is a compact orientable $n$-manifold with its fundamental class $[M] \in H_n(M)$, then the map $D : H^k(M) \to H_{n-k}(M)$ defined by $D(\alpha) = [M] \frown \alpha$ is an isomorphism for all $k$.

The following proof relies on many of the theorems, lemmas, and definitions in the previous chapter. It is broken up into five parts below. These parts cover all the possible cases for a compact orientable manifold $M$. The arguments used in parts **(2)** and **(3)** are inductive. Although we want to prove $H^k(M) \cong H_{n-k}(M)$, this is equivalent to proving $H^k_c(M) \cong H_{n-k}(M)$ for a compact manifold $M$, since if $M$ is compact, $H^k_c(M) \cong H^k(M)$ (see Section 5.1). The map $D_X : H^k(X) \to H_{n-k}(X)$ will define the duality map for any oriented manifold $X$, possibly non-compact. For example, we will use this map where $X$ is an open subset of $M$ (hence, a manifold in its own right).

**(A)** If $M$ is the union of open sets $U$ and $V$, and if $D_U$, $D_V$, and $D_{U\cap V}$ are isomorphisms, then so is $D_M$.

*Proof.* Since $M = U \cup V$, the diagram in Lemma 6.2.4 commutes. Therefore, we have the map $D_M : H^k_c(M) \to H_{n-k}(M)$. By Lemma 4.1.4, since $D_U$, $D_V$, and $D_{U\cap V}$ are isomorphisms, $D_M : H^k_c(M) \to H_{n-k}(M)$ is an isomorphism. Also, $H^k_c(M) = H^k(M)$, since $M$ is compact. Therefore, $H^k(M) \cong H_{n-k}(M)$ for all $k$.

**(B)** If $M = \bigcup_{i=1}^{\infty} U_i$ where $U_i$ is open $\forall i$, $U_1 \subseteq U_2 \subseteq \cdots$, and $D_{U_i}$ is an isomorphism $\forall i$, then $D_M$ is an isomorphism.

*Proof.* First, we note that by Theorem 5.3.1, $H_{n-k}(M) = \varinjlim H_{n-k}(U_i)$. Next, we want a map $f_i : H^k_c(U_i) \to H^k_c(U_{i+1})$ such that $\varinjlim H^k_c(U_i)$ exists, and that from there $\varinjlim H^k_c(U_i) \cong H^k_c(M)$. To see that such a map exists, let $K$ be a compact subset of $U_i$. Then by Theorem 5.3.2 where the direct limit is taken over all compact subsets $K$ of $U_i$,

$$H^k_c(U_i) = \varinjlim H^k(U_i, U_i - K).$$

By excision, $H^k(M, M - K) \cong H^k(U_i, U_i - K)$. The same argument could be made to show that $H^k(M, M - K) \cong H^k(U_{i+1}, U_{i+1} - K)$, since the latter is the direct limit over all compact subsets of $U_{i+1}$, which includes all compact subsets of $U_i$. Therefore, $H^k(U_i, U_i - K) \cong H^k(U_{i+1}, U_{i+1} - K)$, and there must exist a map $f_i$ between $H^k_c(U_i)$ and $H^k_c(U_{i+1})$. From there it is clear that

$$\varinjlim H^k_c(U_i) \cong H^k_c(M).$$

Next, since the diagram

$$\begin{tikzcd} H^k_c(U_i) \arrow[r, "f_i"] \arrow[d, "D_{U_i}"'] & H^k_c(U_{i+1}) \arrow[d, "D_{U_{i+1}}"] \\ H_{n-k}(U_i) \arrow[r, "h_i"] & H_{n-k}(U_{i+1}) \end{tikzcd}$$

commutes, $\varinjlim H^k_c(U_i) \cong \varinjlim H_{n-k}(U_i)$ by Theorem 5.3.4. That is,

$$D_M : H^k_c(M) \to H_{n-k}(M)$$

is an isomorphism for all $k$.

**(1)** If $M \approx \mathbb{R}^n$, then the map $D_{\mathbb{R}^n} : H^k_c(\mathbb{R}^n) \to H_{n-k}(\mathbb{R}^n)$ is an isomorphism for all $k$, and all $n \geq 1$.

*Proof.* We denote the map $D_{\mathbb{R}^n} = D$ for simplicity. Below, we recall some basic facts that we will use for the proof. In addition, we take this opportunity to establish notation.

1. Since $\mathbb{R}^n$ is contractible, the only nonzero homology group $H_{n-k}(\mathbb{R}^n) = \mathbb{Z}$ when $k = n$. This fact was Lemma 3.1.13.

2. Any compact set $K \subseteq \mathbb{R}^n$ of Euclidean space (with the standard metric space topology) is bounded. Thus, for each compact set $K$, there exists a closed ball $B$ of finite radius (which is compact), centered at the origin, so that $K \subseteq B$. In addition, there exists a circumscribed simplex $S_d := \Delta^n_d$ of diameter $d$, which is homeomorphic to the standard simplex $\Delta_n$ embedded into $\mathbb{R}^n$. Thus, by Theorem 5.3.2 we have the following isomorphism of direct limit groups:

$$\begin{aligned} H^k_c(\mathbb{R}^n) &\cong \varinjlim H^k(\mathbb{R}^n, \mathbb{R}^n - K) \\ &\cong \varinjlim H^k(\mathbb{R}^n, \mathbb{R}^n - B) \\ &\cong \varinjlim H^k(\mathbb{R}^n, \mathbb{R}^n - S_d). \end{aligned}$$

3. Since each ball and enlarged simplex $S_d$ is contractible, it follows directly from the long exact sequence of a pair provided by Lemma 4.2.2 that the only nonzero cohomology groups $H^k(\mathbb{R}^n, \mathbb{R}^n - B) = H^k(\mathbb{R}^n, \mathbb{R}^n - S_d) = \mathbb{Z}$ for $k = n$.

4. There is a well-defined map $D_B : H^k(\mathbb{R}^n, \mathbb{R}^n - B) \to H_{n-k}(\mathbb{R}^n)$, given to us by $\psi \mapsto \mu_B \cap \psi$, similarly for $D_d$, the map which caps with the fundamental class $\mu_{S_d}$ of $H_n(\mathbb{R}^n, \mathbb{R}^n - S_d)$.

We list how these facts help us in the proof. If $n$ is fixed, then the only value of $k$ for which we need to study is $k = n$, as the other groups for varying $k$ all vanish, so $D$ is trivially an isomorphism.

The proof will be complete if we can prove the following assertions:

(i) The duality map $D_\Delta : H^n(\Delta^n, \partial\Delta^n) \to H_0(\Delta^n)$ is an isomorphism.

(ii) If $D_\Delta$ is an isomorphism, then $D_d$ is an isomorphism for any $d > 0$.

(iii) If every $D_d$ is an isomorphism, then $D$ is an isomorphism.

Starting with the first assertion, consider the $\Delta$-complex structure on the space $\Delta^n$ which is only one $n$-simplex, $\sigma : \Delta^n \to \Delta^n$, where $\sigma = [v_0, \dots, v_n]$. Then it follows (by considering the corresponding chain complex) that $H_n^\Delta(\Delta_n, \partial\Delta^n) = \langle\sigma\rangle$, and

$$H^n_\Delta(\Delta^n, \partial\Delta^n) = \mathrm{Hom}_{\mathbb{Z}}(H_n^\Delta(\Delta_n, \partial\Delta^n), \mathbb{Z}) = \langle\sigma^*\rangle,$$

where $\sigma^*$ is the dual to $\sigma$. Then up to a change of sign (as the only generators of the cyclic group $H^n_\Delta(\Delta^n, \partial\Delta^n)$ are $\pm\sigma^*$, and the only generators of the cyclic group $H_n^\Delta(\Delta^n, \partial\Delta^n)$ are $\pm\sigma$), the map $D_\Delta$ is completely determined by $\sigma \frown \sigma^*$. And if $\sigma = [v_0, \dots, v_n]$, then $\sigma \frown \sigma^* = \sigma^*(\sigma)[v_n] = [v_n]$, which is an isomorphism to $H_0(\Delta^n)$. This proves assertion (i).

Next, using excision and the set $U = S_d^c$, inclusion of the appropriate spaces induces an isomorphism $H_n(S_d, \partial S_d) \to H_n(\mathbb{R}^n, \mathbb{R}^n - S_d)$, and as all of the $S_d$ are homotopic (via a homotopy which preserves the correct corresponding subspaces), there is a well-defined isomorphism $H_n(\Delta^n, \partial\Delta^n) \to H_n(S_d, \partial S_d)$. Denote this composite isomorphism

$$\varphi : H_n(\Delta^n, \partial\Delta^n) \to H_n(\mathbb{R}^n, \mathbb{R}^n - S_d).$$

These induce isomorphisms in cohomology as well, so by an abuse of notation, denote $\varphi$ as the induced isomorphism between the corresponding cohomology groups as well. As both of these groups are infinite cyclic groups, $\varphi(\sigma)$ must be a generator for $H_n(\mathbb{R}^n, \mathbb{R}^n - S_d)$, and this generator is the fundamental class $\mu_{S_d}$ for $H_n(\mathbb{R}^n, \mathbb{R}^n - S_d)$ up to a possible sign change. So up to a sign, using naturality, we have the map

$$D_d(\psi) = \pm\varphi(\sigma) \frown \psi = \pm\sigma \frown \varphi(\psi).$$

So $D_d = D_\Delta \circ \varphi$, a composition of isomorphisms. This proves assertion **(ii)**.

Finally, to prove the last assertion, we note that if

$$\theta = (\theta_{d_1}, \dots, \theta_{d_k}) \in \varinjlim H^n(\mathbb{R}^n, \mathbb{R}^n - S_d),$$

then there exists a large enough diameter $\bar{d}$ so that $\theta_{d_i}$ are chains in $S_{\bar{d}}$. So the induced homomorphism $D$ on the direct limit is just $D = D_{\bar{d}}$, which is an isomorphism.

**(2)** If $M$ is an arbitrary open set in $\mathbb{R}^n$, then $D_M$ is an isomorphism.

*Proof.* We use two inductive arguments to establish this assertion. Let $M$ be an arbitrary open set in $\mathbb{R}^n$. First, since the topological space $\mathbb{R}^n$ is second countable, $M$ is a countable union of convex open sets $U_i$ (for example, open balls of finite radius), and let $V_i = \bigcup_{j<i} U_j$. If $M$ is the union of one convex open set $U$, then $U$ is homeomorphic to $\mathbb{R}^n$ and $D_M$ is an isomorphism by **(1)**. Next, assume that if $M$ is the union of $i-1$ convex open sets, then $D_M$ is an isomorphism. Now, suppose $M$ is the union of $i$ convex open sets, $M = U_1 \cup U_2 \cup \cdots \cup U_{i-1} \cup U_i$. So for $V_i = U_1 \cup U_2 \cup \cdots \cup U_{i-1}$, and $V_i \cap U_i = (U_1 \cup U_2 \cup \cdots \cup U_{i-1}) \cap U_i = (U_i \cap U_1) \cup \cdots \cup (U_i \cap U_{i-1})$. We have $V_i \cap U_i$ is now the union of $i-1$ convex open sets, and by the assumption, then $D_{V_i}$ and $D_{V_i \cap U_i}$ are isomorphisms. Because $U_i$ is a single convex open set homeomorphic to $\mathbb{R}^n$, $D_{U_i}$ is an isomorphism by **(1)**. Also, because $M = U_i \cup V_i$ and $D_{V_i}$, $D_{V_i \cap U_i}$, and $D_{U_i}$ are isomorphisms, $D_M$ is an isomorphism by **(A)**. Therefore, by induction, if $M$ is a finite union of convex open sets, then $D_M$ is an isomorphism.

Now, if $M = U_1 \cup U_2 \cup U_3 \cup \cdots$, then $V_1 \subset V_2 \subset V_3 \subset \cdots$ and $M = V_1 \cup V_2 \cup V_3 \cup \cdots$. Since each $V_i$ is the union of a finite number of convex open sets, each $D_{V_i}$ is an isomorphism. By **(B)**, $D_M$ is an isomorphism.

**(3)** If $M$ is a compact manifold of dimension $n$, then $D_M$ is an isomorphism.

*Proof.* First, since $M$ is a manifold, every point $x \in M$ has a open neighborhood that is homeomorphic to $\mathbb{R}^n$, and the collection of these open neighborhoods creates an open cover for $M$. And since $M$ is compact, there exists a finite subcover $\{\bigcup U_i\}_{i=1}^k$ such that $M = \bigcup_{i=1}^k U_i$, and each $U_i \approx \mathbb{R}^n$. Now, we want to induct on the number of open sets homeomorphic to $\mathbb{R}^n$ it takes to cover $M$. To start, if $k = 1$, then $M \approx \mathbb{R}^n$ and $D_M$ is an isomorphism by **(1)**. Next, suppose that any time we have a manifold covered by $k-1$ open sets homeomorphic to $\mathbb{R}^n$, the corresponding duality map is an isomorphism. Now suppose $M = \bigcup_{i=1}^k (U_i)$, $U_i \approx \mathbb{R}^n$ for all $i$, and the set $V_k = \bigcup_{i=1}^{k-1}(U_i)$ where $k \geq 2$. Then $M = V_k \cup U_k$.

Notice that $V_k$ is the union of $k-1$ open sets homeomorphic to $\mathbb{R}^n$. By the induction hypothesis, $D_{V_k}$ is an isomorphism. Also, notice that $U_k \approx \mathbb{R}^n$. By **(1)**, $D_{U_k}$ is an isomorphism. So, if we could show $D_{U_k \cap V_k}$ is an isomorphism, then by **(A)**, $D_{U_k \cup V_k}$ is an isomorphism, and we would be done. Now,

$$U_k \cap V_k = \left( \bigcup_{i=1}^{k-1} (U_i) \right) \cap U_k = \bigcup_{i=1}^{k-1} (U_i \cap U_k). \tag{6.3}$$

Therefore, $U_k \cap V_k$ is the union of $k-1$ open sets but not necessarily open sets homeomorphic to $\mathbb{R}^n$. So we can't use the induction hypothesis to show that $D_{U_k \cap V_k}$ is an isomorphism. However, $U_k \approx \mathbb{R}^n$ means there exists a homeomorphism $f : U_k \to \mathbb{R}^n$ and $f^{-1} : \mathbb{R}^n \to U_k$. By **(2)**,

$$D_{f(U_i \cap U_k)} : H_c^l(f(U_i \cap U_k)) \to H_l(f(U_i \cap U_k)) \tag{6.4}$$

is an isomorphism for all $l$ since $f(U_i \cap U_k)$ is open in $\mathbb{R}^n$. Now, let

$$\mathfrak{U} = f\left( \bigcup_{i=1}^{k-1} (U_i \cap U_k) \right) = \bigcup_{i=1}^{k-1} f\,(U_i \cap U_k)\,. \tag{6.5}$$

Then by **(2)**, $D_{\mathfrak{U}} : H_c^l(\mathfrak{U}) \to H_l(\mathfrak{U})$ is an isomorphism for all $l$ since $\mathfrak{U}$ is open in $\mathbb{R}^n$. Now, since $f : V_k \cap U_k \to \mathfrak{U} \subseteq \mathbb{R}^n$ and $f$ is a homeomorphism, $f^*$ induces an isomorphism $H_c^l(\mathfrak{U}) \to H_c^l(V_k \cap U_k)$ and $f_*$ induces an isomorphism $H_{n-l}(V_k \cap U_k) \to H_{n-l}(\mathfrak{U})$. We now have the following diagram.

$$\begin{tikzcd}
H_c^l(V_k \cap U_k) \arrow[r, "D_{V_k \cap U_k}"] \arrow[d, shift left, "(f^{-1})^*"] & H_{n-l}(V_k \cap U_k) \arrow[d, shift left, "f_*"] \\
H_c^l(\mathfrak{U}) \arrow[r, "D_{\mathfrak{U}}"'] \arrow[u, shift left, "f^*"] & H_{n-l}(\mathfrak{U}) \arrow[u, shift left, "(f^{-1})_*"]
\end{tikzcd}$$

By the naturality of the cap product, it follows that the above diagram commutes. That is, if $[\varphi] \in H_c^l(V_k \cap U_k)$, then

$$D_{V_k \cap U_k}[\varphi] = \left(f^{-1}\right)_* D_{\mathfrak{U}} \left(f^{-1}\right)^* [\varphi]. \tag{6.6}$$

Because the right hand side is a composition of isomorphisms, the left side, $D_{V_k \cap U_k}$, is also an ismorphism. By **(A)**, $D_{U_k \cup V_k} = D_M$ is an isomorphism. □

This concludes the proof of the Poincaré duality theorem for compact orientable manifolds. Poincaré duality does exist for noncompact and nonorientable manifolds. There are also other forms of duality for manifolds, namely Poincaré - Lefschetz duality and Alexander duality. For a treatment of some of these topics see [Bre97, Vic94, Mas80]. We will now look at some applications of Poincaré duality for compact orientable manifolds.

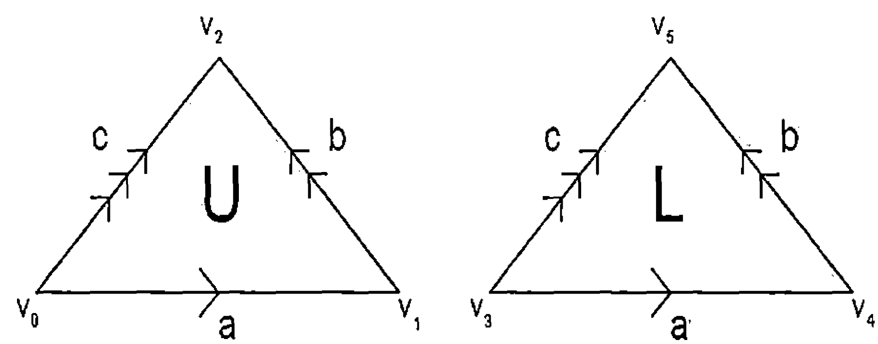
# Chapter 7

# Applications of Poincaré Duality

## 7.1 Illustrating How Poincaré Duality Works

Before looking at some applications of Poincaré duality, we use it to compute the homology and cohomology of several spaces.

**Example 7.1.1.** Give $S^2$ the following $\Delta$-complex structure:




Here $U = [v_0, v_1, v_2]$, $L = [v_3, v_4, v_5]$, and the corresponding edges are glued to each other as illustrated to form $S^2$. In particular, $[v_0] = [v_3]$, $[v_1] = [v_4]$, and $[v_2] = [v_5]$.

The goal here will be to first compute the cohomology of $S^2$, then use Poincaré duality to find the homology of $S^2$. First, as a generator of $H_2(S^2)$, the fundamental class of $S^2$, $[S^2] = [U - L]$, and $v_0^*$ will represent the dual of $v_0$, $v_1^*$ the dual of $v_1$, $a^*$ the dual of $a$ and so on. We recall how the cap product is defined. If $\sigma = [v_0, \dots, v_l] \in C_l(X)$, and $\psi \in C^k(X)$, the cap product is defined as $\sigma \frown \psi = (\psi[v_0, \dots, v_k]) \cdot [v_k, \dots, v_l]$.

Now, to compute the cohomology of $S^2$, we need to know $Ker\ d_0$, $Im\ d_0$, $Ker\ d_1$, $Im\ d_1$, and $Ker\ d_2$. First, $d_0v_0^*(a) = v_0^*\partial(a) = v_0^*(v_1 - v_0) = 0 - 1 = -1$. Also, $d_0v_0^*(b) = 0$, and $d_0v_0^*(c) = -1$. Therefore, $d_0v_0^* = -a^* - c^*$. By similar calculations, $d_0v_1^* = a^* - b^*$, and $d_0v_2^* = b^* + c^*$. So,

$$Ker\ d_0 = \langle v_0^* + v_1^* + v_2^* \rangle,$$

and

$$Im\ d_0 = \langle -a^* - c^*, a^* - b^*, b^* + c^* \rangle.$$

Next, we want to know $Ker\ d_1$ and $Im\ d_1$. We have

$$d_1a^*(U) = a^*(\partial U) = a^*([v_1, v_2] - [v_0, v_2] + [v_0, v_1]) = 1,$$

and $d_1a^*(L) = a^*(\partial L) = a^*([v_4, v_5] - [v_3, v_5] + [v_3, v_4]) = 1$. So, $d_1a^* = U^* + L^*$. Similarly, $d_1b^* = U^* + L^*$, and $d_1c^* = -U^* - L^*$. Therefore,

$$Ker\ d_1 = \langle a^* + c^*, b^* + c^* \rangle,$$

and

$$Im\ d_1 = \langle U^* + L^* \rangle.$$

Lastly, $d_2U^* = d_2L^* = 0$. Therefore, $Ker\ d_2 = \langle U^*, L^* \rangle$. So,

$$H^0(S^2) = \frac{Ker\ d_0}{\langle 0 \rangle} = \langle v_0^* + v_1^* + v_2^* \rangle \cong \mathbb{Z}.$$

$$H^1(S^2) = \frac{Ker\ d_1}{Im\ d_0} = \frac{\langle a^* + c^*, b^* + c^* \rangle}{\langle -a^* - c^*, a^* - b^*, b^* + c^* \rangle} \cong \frac{\langle a^* + c^* \rangle}{\langle a^* + c^*, a^* - b^* \rangle} \cong \langle 0 \rangle.$$

$$H^2(S^2) = \frac{Ker\ d_2}{Im\ d_1} = \frac{\langle U^*, L^* \rangle}{\langle U^* + L^* \rangle} \cong \langle U^* \rangle \cong \mathbb{Z}.$$

Using the cap product and Poincaré duality we can now compute the homology of $S^2$. We compute $(U - L) \frown (v_0^* + v_1^* + v_2^*) =$

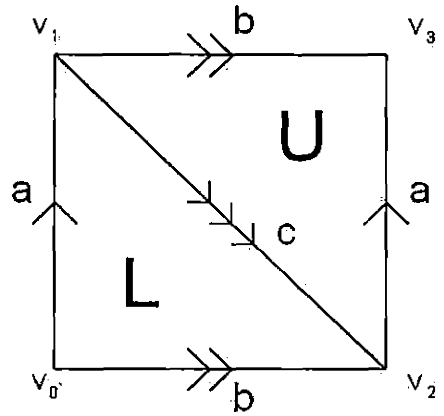
$$(U \frown v_0^*) + (U \frown v_1^*) + (U \frown v_2^*) - (L \frown v_0^*) - (L \frown v_1^*) - (L \frown v_2^*).$$

Now $U \frown v_0^* = v_0^*([v_0])(v_0, v_1, v_2) = (v_1, v_2)$. The others are $U \frown v_1^* = U \frown v_2^* = 0$. Similarly, $L \frown v_0^* = L \frown v_1^* = L \frown v_2^* = 0$. So,

$$(U - L) \frown (v_0^* + v_1^* + v_2^*) = (v_0, v_1, v_2).$$

By Poincaré duality, $H_{2-0}(S^2) \cong \mathbb{Z}(v_1, v_2) \cong \mathbb{Z}$ since $H^0(S^2) =< v_0^* >$, an infinite cycle, and $D_{S^2}$ is an isomorphism. So $H_2(S^2)$ is cyclic with generator $D_{S^2}(v_0^*)$. This happens because isomorphisms send generators to generators. Now, to find $H_1(S^2)$ we need to compute $(U - L) \frown \langle 0 \rangle$. But $(U - L) \frown \langle 0 \rangle = 0$. So $H_{2-1}(S^2) \cong \langle 0 \rangle$. Finally, we compute $(U-L) \frown (U^*)$. We have $U \frown U^* = U^*([v_0, v_1, v_2])[v_2] = [v_2]$ and $L \frown U^* = 0$. Therefore, $H_{2-2}(S^2) \cong \mathbb{Z}(v_2) \cong \mathbb{Z}$, again since generators are sent to generators. That is, $H_0(S^2) \cong \mathbb{Z}$.

**Example 7.1.2.** Let the torus, $\mathbb{T}^2$, have the same $\Delta$-complex structure as section 3.1.




So $U = [v_1, v_2, v_3]$, $L = [v_0, v_1, v_2]$, and $[v_0] = [v_1] = [v_2] = [v_3]$.

First off, as a generator for $H_2(\mathbb{T}^2)$ the fundamental class of the torus is $[U - L]$. Like the previous example, we would like to know what $H^0(\mathbb{T}^2)$, $H^1(\mathbb{T}^2)$, and $H^2(\mathbb{T}^2)$ are. Here,

$$H^2(\mathbb{T}^2) = \frac{\langle U^*, L^* \rangle}{\langle U^* + L^* \rangle} \cong \frac{\langle U^* + L^*, L^* \rangle}{\langle U^* + L^* \rangle} \cong \langle L^* \rangle.$$

Now, we could use cap product to compute $(U - L) \frown (L^*) = U \frown L^* - L \frown L^*$ and find $H_0(\mathbb{T}^2)$.

$$U \frown L^* = L^*([v_1, v_2, v_3]) \cdot [v_3] = 0$$

and

$$L \frown L^* = L^*([v_0, v_1, v_2]) \cdot [v_2] = [v_2].$$

Therefore, $(U - L) \frown (L^*) = -[v_2]$, and $H_0(\mathbb{T}^2) \cong \mathbb{Z}(-[v_2]) \cong \mathbb{Z}$. Next, it's true that $H^1(\mathbb{T}^2) \cong \langle a^* + b^*, b^* + c^* \rangle$. To find $H_1(\mathbb{T}^2)$ we can compute $(U - L) \frown (a^* + b^*)$ and $(U - L) \frown (b^* + c^*)$. Then

$$(U - L) \frown (a^* + b^*) = U \frown a^* + U \frown B^* - L \frown a^* - L \frown b^*.$$

Computing these separately we have

$$U \frown a^* = a^*([v_1, v_2])[v_2, v_3] = 0,$$

$$U \frown b^* = b^*([v_1, v_2])[v_2, v_3] = 0,$$

$L \frown a^* = a^*([v_0, v_1])[v_1, v_2] = c$, and $L \frown b^* = b^*([v_0, v_1])[v_1, v_2] = 0$. Therefore,

$$(U - L) \frown (a^* + b^*) = -c.$$

Similarly, when we compute $(U - L) \frown (b^* + c^*)$ we find that only $U \frown c^* = a$ is nonzero. That is, $(U - L) \frown (b^* + c^*) = a$. Therefore, $H_1(\mathbb{T}^2) \cong \langle a, -c \rangle \cong \mathbb{Z}(a) \oplus \mathbb{Z}(c)$. Similarly, we could use $H^2(\mathbb{T}^2)$ and Poincaré duality to find $H_0(\mathbb{T}^2) \cong \mathbb{Z}$.

## 7.2 The Klein Bottle

From Example 3.2.4 in Section 3.2 we obtained $H^2(K) \cong \mathbb{Z}_2$. If the Klein bottle $K$ has the same $\Delta$-complex structure as in Example 3.2.4, then we could compute

$$H_0(K) = \frac{Ker\ (\partial_0)}{Im\ (\partial_1)}$$

to find $H_0(K) \cong \mathbb{Z}$, since $Ker\ \partial_0 = C_0(K) = \mathbb{Z}(v)$ where $v$ is the only vertex of the $\Delta$-complex structure, and $Im\ \partial_1 = 0$. Notice that $H^2(K) \ncong H_0(K)$. This fact seems to contradict the Poincaré duality theorem. However, this illustrates that the Klein bottle is not $\mathbb{Z}$-orientable and cannot make use of this version of the Poincaré duality theorem. Although $K$ is not $\mathbb{Z}$-orientable, it is $\mathbb{Z}_2$-orientable, and $H^k(K, \mathbb{Z}_2) \cong H_{2-k}(K, \mathbb{Z}_2)\ \forall k$. This example illustrates the importance of the manifold being orientable.

## 7.3 Euler Characteristic

**Definition 7.3.1.** [Mas91] If $X$ is a compact manifold given by a finite $\Delta$-complex structure, then denote the number of $n$-cells of $X$ by $c_n < \infty$. Then the *Euler characteristic* of $X$ is defined to be the integer

$$\chi(X) = \sum_{n \geq 0} (-1)^n c_n.$$

This is a generalization of Euler's familiar formula of *vertices - edges + faces* for 2-dimensional complexes. The following result shows that $\chi(X)$ can be defined in terms of homology of $X$ and depends only on the homotopy type of $X$.

**Theorem 7.3.2.** [Mas91] If $X$ is a compact manifold whose homology groups are finitely generated, and given by a $\Delta$-complex structure, then

$$\chi(X) = \sum_{n\geq 0}(-1)^n rank(H_n(X)),$$

where $rank(H_n(X))$ is the number of $\mathbb{Z}$ summands in the finitely generated abelian group $H_n(X)$.

**Example 7.3.3.** Recall from Chapter 3 that $H_n(S^1) = \mathbb{Z}$ when $n = 0, 1$ and is zero elsewhere. Then $\chi(S^1) = rank(H_0(S^1)) - rank(H_1(S^1)) = 1 - 1 = 0$.

**Example 7.3.4.** We compute $\chi(\mathbb{T}^2)$. Recall the homology of the torus from Chapter 3. Now, $\chi(\mathbb{T}^2) = rank(H_0(\mathbb{T}^2)) - rank(H_1(\mathbb{T}^2)) + rank(H_2(\mathbb{T}^2)) = 1 - 2 + 1 = 0$.

**Example 7.3.5.** Recall from Example 7.1.1 that $H_0(S^2) \cong H_2(S^2) \cong \mathbb{Z}$ and $H_n(S^2) = 0$ for $n \neq 0, 2$. This trend continues for any sphere $S^n$. That is, $H_0(S^n) \cong H_n(S^n) \cong \mathbb{Z}$ and $H_k(S^n) = 0$ for $k \neq 0, n$. It follows that

$$\begin{aligned} \chi(S^n) &= rank(H_0(S^n)) - rank(H_1(S^n)) + \cdots + (-1)^n rank(H_n(S^n)) \\ &= 1 - 0 + \cdots + (-1)^n. \end{aligned}$$

Therefore, $\chi(S^n) = 2$ if $n$ is even, and $\chi(S^n) = 0$ if $n$ is odd.

Notice in the previous three examples that all the odd dimensional manifolds have Euler characteristic zero, but not all the even dimensional manifolds have Euler characteristic zero. We wish to prove in Corollary 7.3.7 that any closed manifold of odd dimension has Euler characteristic zero. To do so we will need the following fact that follows from the *universal coefficient theorem.*

**Theorem 7.3.6.** [Hat02] If $M$ is a topological space, then

$$rank(H^{n-i}(M)) = rank(H_{n-i}(M)).$$

**Corollary 7.3.7.** [Hat02] A closed orientable manifold of odd dimension $n$ has Euler characteristic zero.

*Proof.* First, since $M$ is odd and $H_k(M) = 0$ for $k > n$ [Hat02], there are an even number of $H_n(M)$ that are possibly not trivial. By Poincaré duality, each $H^k(M)$ is isomorphic to $H_{n-k}(M)$. We have by Theorem 7.3.6, that $rank(H^k(M)) = rank(H_k(M))$. Therefore, we have $rank(H_k(M)) = rank(H_{n-k}(M))$. And since $M$ is odd dimensional, $(-1)^k rank(H_k(M))$ and $(-1)^{n-k} rank(H_{n-k}(M))$ have opposite signs and will cancel each other out in pairs when computing $\chi(M)$, making $\chi(M) = 0$. □

There is a similar argument involving the universal coefficient theorem if $M$ is not orientable that follows from a version of Poincaré duality using $\mathbb{Z}_2$ coefficients rather than $\mathbb{Z}$ coefficients.

## 7.4 Bilinear Forms

**Theorem 7.4.1.** Given a $\mathbb{Z}$-oriented $2n$-dimensional manifold $M$, there are bilinear maps from $H^n(M) \times H^n(M) \to \mathbb{Z}$ given by

$$\langle \varphi, \theta \rangle \mapsto \theta(D_M(\varphi)) \text{ and } \langle \varphi, \theta \rangle \mapsto \varphi(D_M(\theta)),$$

where $\varphi, \theta \in H^n(M)$.

The Poincaré duality theorem provides these mappings since $\varphi \in H^n(M)$ implies that $D_M(\varphi) \in H_{2n-n}(M) = H_n(M)$. And $\theta(x) \in \mathbb{Z}$ for some $x = D_M(\varphi) \in H_n(M)$.

*Proof.* To check that each map is bilinear we notice that if $\varphi_1, \varphi_2, \theta \in H^n(M)$, then $\langle \varphi_1 + \varphi_2, \theta \rangle = \theta(D_M(\varphi_1) + D_M(\varphi_2)) = \theta(D_M(\varphi_1)) + \theta(D_M(\varphi_2)) = \langle \varphi_1, \theta \rangle + \langle \varphi_2, \theta \rangle$. Similarly, if $\varphi, \theta_1, \theta_2 \in H^n(M)$, then

$$\langle \varphi, \theta_1 + \theta_2 \rangle = (\theta_1 + \theta_2)(D_M(\varphi)) = \theta_1(D_M(\varphi)) + \theta_2(D_M(\varphi)) = \langle \varphi, \theta_1 \rangle + \langle \varphi, \theta_2 \rangle.$$

□

Throughout this study, we have intended to illustrate the significance of the Poincaré duality theorem by displaying and defining the topics used to arrive at Poincaré duality and displaying the abundance of applications of the theorem. As shown in the first five chapters of the study, Poincaré duality is built up from homology and cohomology and makes use of many other topics in algebraic topology. Chapter 6 was dedicated to the proof of the Poincaré duality theorem. And Chapter 7 finished by examining mathematical applications of the theorem.

# Bibliography

[Bre97] Glen E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. Corrected third printing of the 1993 original.

[Die89] Jean Dieudonné. *A history of algebraic and differential topology. 1900–1960.* Birkhäuser Boston Inc., Boston, MA, 1989.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry.* Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[Kat93] Victor J. Katz. *A history of mathematics.* HarperCollins College Publishers, New York, 1993. An introduction.

[KSW89] Fima C. Klebaner, Aidan Sudbury, and G. A. Watterson. The volumes of simplices, or, find the penguin. *J. Austral. Math. Soc. Ser. A*, 47(2):263–268, 1989.

[Mas80] William S. Massey. *Singular homology theory*, volume 70 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1980.

[Mas91] William S. Massey. *A basic course in algebraic topology*, volume 127 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[Sie92] Allan J. Sieradski. *An Introduction to Topology and Homotopy.* The Prindle, Weber & Schmidt Series in Advanced Mathematics. PWS-KENT Publishing Co., Boston, MA, 1992.

[Vic94] James W. Vick. *Homology theory*, volume 145 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1994. An introduction to algebraic topology.