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Quenching for degenerate semilinear parabolic problems with insulated boundary conditions

Bernard Iyawe

Quenching for Degenerate Semilinear Parabolic Problems with Insulated Boundary Conditions

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A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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Abstract

In this thesis I studied the existence, uniqueness, and quenching behavior of the solution to the degenerate equation

$$x^q u_t - u_{xx} = x^p f(u), \qquad \text{in } \Omega,$$

subject to the initial condition

$$u(x,0) = 0 \qquad \text{for } 0 \leq x \leq a,$$

and the second boundary conditions as follows:

$$u_x(0,t) = 0 = u_x(a,t), \text{ for } 0 < t < T.$$

Here, $0 < T \leq \infty$, $a > 0$, and $\Omega = (0,a) \times (0,T)$. It is assumed that p and q are any non-negative real numbers and $f > 0$, $f' > 0$, $f'' \geq 0$, and $\lim_{u \to c^-} f(u) = \infty$ for some positive constant c. This study shows that quenching occurs in the interval $[0,a]$ when $p = q$. Otherwise, quenching occurs only at the boundary $\{0\} \times (0,T)$ or $\{a\} \times (0,T)$ depending on $p > q$ or $p < q$. If $p > q$, then quenching occurs at $x = a$ and if $p < q$ then quenching occurs at $x = 0$. The Mean Value Theorem and the Maximum Principle are widely used throughout this study. The statements of these theorems are given in Appendix B.

Acknowledgements

It has always been a dream of mine to get a graduate degree in mathematics and I am very happy that this dream is finally coming to fruition. I am very grateful to the numerous people that were directly and indirectly instrumental in bringing this dream into fruition.

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Finally, I would like to thank all the members of my family for their patience and understanding throughout the period of this study. In particular, I would like to thank my father for showing me the value of education and encouraging me to go as far as my ability can take me.

Special thanks to my children for their understanding of my inability to spend enough time with them during this period of study. Thanks to Aisa, Brain and Bernard Jr. for maintaining their high academic performances despite my limited assistance.

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Chapter 1

Introduction

Let p and q be any non-negative real numbers and consider the degenerate equation

$$x^q u_t - u_{xx} = x^p f(u) \text{ in } (0,a) \times (0,T), \tag{1.1}$$

subject to the initial conditions

$$u(x,0) = 0, \text{ for } 0 \leq x \leq a,$$

and second boundary conditions as follows

$$u_x(0,t) = 0 = u_x(a,t), \text{ for } 0 < t < T$$

Here, it is assumed that $a > 0$, $0 < T \leq \infty$, $f > 0$, $f' > 0$, $f'' \geq 0$, and $\lim_{u \to c^-} f(u) = \infty$ for some constant c. Let $D = (0,a)$, $\Omega = D \times (0,T)$, $\Gamma_1 = \{0\} \times (0,T)$, and $\Gamma_2 = \{a\} \times (0,T)$. Let $Lu = x^q u_t - u_{xx}$, then equation (1.1) can be written as

$$Lu = x^p f(u) \text{ in } \Omega. \tag{1.2}$$

The solution u is said to quench if $\lim_{t \to T^-} \max_{0 \leq x \leq a} u(x,t) = c$.

In chapter 2, I gave a proof of the comparison theorem. This theorem, together with the Mean Value Theorem and the Maximum Principle, are frequently referred to in this study. I showed the existence of the solution to problem (1.1) and its unique properties. In chapter 3, I showed the conditions under which quenching occurs at each of the boundaries Γ_1 and Γ_2. Finally, I showed, in chapter 4, that complete quenching occurs when $p = q$.

Statements of theorems and definitions of some important terms are given in the Appendix.

Chapter 2

Existence and Uniqueness of the Solution

2.1 The Comparison Theorem

Since the comparison theorem will be widely used in the thesis, we will start by stating this theorem and giving its proof. We will then prove that the solution to problem (1.1) is unique and show its properties. We will also show that problem (1.1) has, for a given t, an upper solution which is dependent on the function f and constants p, q, and a. We will prove the local existence of the solution (Lemma 2.5) and then show global existence of the solution (Theorem 2.6) until quenching occurs.

The following theorem is similar to theorem 1 by C. Y. Chan and H. Liu in [[CL01]]

Theorem 2.1. *(The Comparison Theorem) For any $r \in (0, T)$, and any function $B(x,t)$ bounded on $\bar{D} \times [0, r]$, if*

$$\begin{aligned} (L - x^p B)u &\geq 0, && \text{in } D \times (0, r], \\ u(x, 0) &\geq 0, && \text{on } D, \\ u_x(0,t) \leq 0 \quad \text{and} \quad & u_x(a,t) \geq 0 && \text{for } 0 \leq t \leq r, \end{aligned} \tag{2.1}$$

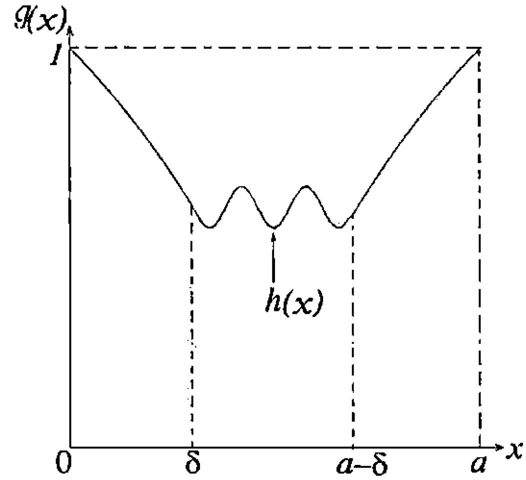

Figure 2.1: Sketch of the piecewise function $g(x) \in C^2([0,a])$

then $u \geq 0$ *on* $\bar{D} \times [0,r]$.

Proof. Let $m = \max_{\bar{D}\times[0,r]}|B|$, and $\delta = \min\left\{\frac{1}{2}, \frac{a}{2}\right\}$ be a positive number such that
$0 < -\frac{m}{2}\delta^2 - \delta + 1 < 1$. Let

$$g(x) = \begin{cases} -\frac{m}{2}x^2 - x + 1, & 0 \leq x \leq \delta, \\ h(x), & \delta < x < a - \delta, \\ -\frac{m}{2}(a-x)^2 - (a-x) + 1, & a - \delta \leq x \leq a, \end{cases} \tag{2.2}$$

where $h(x)$ is a positive C^∞ function less than 1 and chosen such that $g(x)$ is in $C^2(\bar{D})$. Therefore, $g(x)$ is defined in $[0,a]$ and its sketch is shown in figure 2.1.

Note: $h(x)$ is a positive C^∞ function means that $h(x)$ is a positive infinitely differentiable function.

Let η be a positive constant, and let

$$w(x,t) \quad = \quad u(x,t) + \eta e^{\beta t} g(x).$$

Note: The preceding expression can also be written without indicating the variables as shown below. Throughout this study, equations and expressions are written with or without the variables whenever it is convenient.

$$\begin{aligned} w &= u + \eta e^{\beta t} g, \\ w_x &= u_x + \eta e^{\beta t} g', \end{aligned}$$

where β is a positive constant to be determined. Since we have the second boundary condition, $u_x(0,t) = u_x(a,t) = 0$. Also, $g'(0) = -1$, and $g'(a) = 1$.

$$\begin{aligned} w_x(0,t) &= u_x(0,t) + \eta e^{\beta t} g'(0), \\ &= -\eta e^{\beta t}, \\ &< 0, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} w_x(a,t) &= u_x(a,t) + \eta e^{\beta t} g'(a), \\ &= \eta e^{\beta t}, \\ &> 0. \end{aligned} \tag{2.4}$$

Then

$$(L - x^p B)w = Lw - x^p Bw. \tag{2.5}$$

But

$$Lw = x^q w_t - w_{xx}, \tag{2.6}$$

and $w = u + \eta g e^{\beta t}$, then

$$w_t = u_t + \eta \beta g e^{\beta t}, \tag{2.7}$$

and

$$w_{xx} = u_{xx} + \eta e^{\beta t} g''. \tag{2.8}$$

Using equations (2.6), (2.7), and (2.8), equation (2.5) becomes

$$\begin{aligned} (L - x^p B)w &= x^q u_t + \eta \beta g x^q e^{\beta t} - u_{xx} - \eta e^{\beta t} g'' - x^p Bu - x^p B \eta g e^{\beta t} \\ &= x^q u_t - u_{xx} - x^p Bu + \eta \beta g x^q e^{\beta t} - x^p B \eta g e^{\beta t} - \eta e^{\beta t} g''. \end{aligned}$$

Since $Lu = x^q u_t - u_{xx}$, then

$$\begin{aligned}(L - x^p B)w &= Lu - x^p Bu + (\eta\beta x^q e^{\beta t} - x^p B\eta e^{\beta t})g - \eta e^{\beta t} g'' \\ &= (L - x^p B)u + \eta e^{\beta t}[(\beta x^q - x^p B)g - g''].\end{aligned} \tag{2.9}$$

But it is given in equation (2.1) that $(L - x^p B)u \geq 0$. We can drop $(L - x^p B)u$ from equation (2.9) in order to minimize its right hand side. This will result in the following:

$$\begin{aligned}(L - x^p B)w &\geq \eta e^{\beta t}[(\beta x^q - x^p B)g - g''] \\ &= x^q \eta\beta g e^{\beta t} - \eta e^{\beta t} g'' - x^p B\eta g e^{\beta t} \\ &= \eta e^{\beta t}(x^q \beta g - g'' - x^p B g) \\ &= \eta e^{\beta t}[(x^q \beta - x^p B)g - g''].\end{aligned}$$

Observe that $\dfrac{d}{dt}\eta g e^{\beta t} = \eta\beta g e^{\beta t}$ and $\dfrac{d^2}{dx^2}\eta g e^{\beta t} = \eta e^{\beta t} g''$. This implies that

$$\begin{aligned}x^q \eta\beta g e^{\beta t} - \eta e^{\beta t} g'' - x^p B\eta g e^{\beta t} &= L(\eta g e^{\beta t}) - x^p B(\eta g e^{\beta t}) \\ &= (L - x^p B)\eta g e^{\beta t}.\end{aligned}$$

Therefore,

$$(L - x^p B)w \geq (L - x^p B)\eta g e^{\beta t}.$$

Let us chose $\beta > \max\left(\dfrac{a^p m + \max_{\delta<x<a-\delta} h''}{\delta^q \min_{\delta<x<a-\delta} h}, \dfrac{m(a^p - 1)}{(a-\delta)^q\left(-\frac{m}{2}\delta^2 - \delta + 1\right)}\right)$.

Below, we show that

$$(L - x^p B)w > 0. \tag{2.10}$$

Note that $h(x)$ in equation (2.2) is infinitely differentiable and that $g(x)$ is $C^2(\bar{D})$. Then, at δ,

$$\begin{aligned}g(\delta) &= h(\delta), \\ g'(\delta) &= h'(\delta) = \delta - 1, \\ \text{and} \qquad g''(\delta) &= h''(\delta) = -m.\end{aligned}$$

In the region $[0,\delta]$, $g'' = -m$. Since $\eta e^{\beta t} > 0$ and $(L - x^p B)u \geq 0$, we will have to show in equation (2.9) that $(\beta x^q - x^p B)g - g'' > 0$ for equation (2.10) to be satisfied. Now

$$\begin{aligned}(\beta x^q - x^p B)g - g'' &= (\beta x^q - x^p B)g + m \\ &\geq -x^p Bg + m.\end{aligned}$$

But $m = \max|B|$, $g \leq 1$, $\delta = \min\left\{\frac{1}{2}, \frac{a}{2}\right\}$, $x \in [0,\delta]$, and p is positive. Therefore, $x^p g \leq \left(\frac{1}{2}\right)^p$. Hence $x^p Bg \leq \left(\frac{1}{2}\right)^p m < m$. Therefore, $-x^p Bg + m > 0$. Hence,

$$(\beta x^q - x^p B)g - g'' > 0.$$

In the region $(\delta, a-\delta)$, we can write

$$\begin{aligned}(\beta x^q - x^p B)g - g'' &= \beta g x^q - g x^p B - g'' \\ &\geq \beta\delta^q \min_{\delta<x<a-\delta} g - a^p(\max_{\delta<x<a-\delta} B)(\max_{\delta<x<a-\delta} g) \\ &\quad -\max_{\delta<x<a-\delta} g''.\end{aligned}$$

But in this region, $\min_{\delta<x<a-\delta} g = \min_{\delta<x<a-\delta} h$, $\max_{\delta<x<a-\delta} g = 1$, $\max_{\delta<x<a-\delta} B = \max_{0\leq x\leq a}|B| = m$, and $\max_{\delta<x<a-\delta}|g''| = \max_{\delta<x<a-\delta}|h''|$.
Therefore,

$$\begin{aligned}(\beta x^q - x^p B)g - g'' &\geq \beta\delta^q \min_{\delta<x<a-\delta} h - a^p(\max_{\delta<x<a-\delta}|B|) - \max_{\delta<x<a-\delta} h'' \\ &= \beta\delta^q \min_{\delta<x<a-\delta} h - a^p m - \max_{\delta<x<a-\delta} h'' \\ &= \delta^q \min_{\delta<x<a-\delta} h \left(\beta - \frac{a^p m + (\max_{\delta<x<a-\delta} h''}{\delta^q \min_{\delta<x<a-\delta} h}\right) \\ &> 0.\end{aligned}$$

In the region $[a-\delta, a]$, we also have

$$\begin{aligned}(\beta x^q - x^p B)g - g'' &= \beta g x^q - g x^p B - g'' \\ &\geq \beta(a-\delta)^q \min_{a-\delta\leq x\leq a} g - a^p(\max_{a-\delta\leq x\leq a} B)(\max_{a-\delta\leq x\leq a} g) \\ &\quad -\max_{a-\delta\leq x\leq a} g''.\end{aligned}$$

But in this region, $\min_{a-\delta\leq x\leq a} g = g(a-\delta)$, $\max_{a-\delta\leq x\leq a} g = 1$, $\max_{a-\delta\leq x\leq a} B = \max_{0\leq x\leq a}|B| = m$, and $g'' = -m$.

Therefore,

$$\begin{aligned}(\beta x^q - x^p B)g - g'' &\geq \beta(a-\delta)^q\left(\frac{-m}{2}\delta^2 - \delta + 1\right) - a^p m + m \\ &= (a-\delta)^q\left(\frac{-m\delta^2}{2} - \delta + 1\right)\left(\beta - \frac{m(a^p-1)}{(a-\delta)^q\left(\frac{-m\delta^2}{2} - \delta + 1\right)}\right) \\ &> 0.\end{aligned}$$

Now, we will show that $w > 0$ on $\bar{D} \times [0, r]$. Suppose $w \leq 0$ somewhere on $\bar{D} \times [0, r]$, then the set $z_{\tilde{t}} = \{\tilde{t} : w(\tilde{x}, \tilde{t}) \leq 0 \text{ for some } \tilde{x} \in \bar{D}\}$ is non-empty. Let $\bar{t} = \inf z_{\tilde{t}}$. Since $w(x, 0) > 0$, we have $0 < \bar{t} \leq r$. Suppose there exist some $\bar{x} \in \bar{D}$ such that $w(\bar{x}, \bar{t}) = 0$. If $\bar{x} = 0$, then, from equation (2.3), we have $w_x(0, \bar{t}) < 0$. But

$$w_x(0,\bar{t}) = \lim_{x\to 0^+} \frac{w(x,\bar{t}) - w(0,\bar{t})}{x - 0}.$$

With $w(0, \bar{t}) = 0$ and $w(x, \bar{t}) \geq 0$ we have $w_x(0, \bar{t}) \geq 0$. This implies that $0 > w_x(0, \bar{t}) \geq 0$. This is a contradiction which shows that $\bar{x} \neq 0$.

Now, suppose that $\bar{x} = a$, then $w(a, \bar{t}) = 0$, and from equation (2.4), we have $w_x(a, \bar{t}) > 0$. But

$$w_x(a,\bar{t}) = \lim_{x\to a^-} \frac{w(a,\bar{t}) - w(x,\bar{t})}{a - x}.$$

With $w(a, \bar{t}) = 0$ and $w(x, \bar{t}) \geq 0$ we have

$$\lim_{x\to a^-} \frac{w(a,\bar{t}) - w(x,\bar{t})}{a - x} \leq 0.$$

Therefore, $0 < w_x(a, \bar{t}) \leq 0$. This is a contradiction which also shows that $\bar{x} \neq a$.

Therefore, $\bar{x} \in (0, a)$. On the other hand, since w attains its local minimum at $(\bar{x}, \bar{t})$, then $w_{xx}(\bar{x}, \bar{t}) \geq 0$. Also, $w_t(\bar{x}, \bar{t}) \leq 0$, hence $x^q w_t(\bar{x}, \bar{t}) \leq 0$. But

$$(L - \bar{x}^p B)w(\bar{x},\bar{t}) > 0.$$

Therefore,

$$\begin{aligned}(L - \bar{x}^p B)w(\bar{x},\bar{t}) &= Lw(\bar{x},\bar{t}) - \bar{x}^p Bw(\bar{x},\bar{t}) \\ &= \bar{x}^q w_t(\bar{x},\bar{t}) - w_{xx}(\bar{x},\bar{t}) - \bar{x}^p Bw(\bar{x},\bar{t}), \\ \bar{x}^q w_t(\bar{x},\bar{t}) &= (L - \bar{x}^p B)w(\bar{x},\bar{t}) + w_{xx}(\bar{x},\bar{t}) + \bar{x}^p Bw(\bar{x},\bar{t}).\end{aligned}$$

Since $w_{xx}(\bar{x},\bar{t}) \geq 0$ and $\bar{x}^p Bw(\bar{x},\bar{t}) = 0$, then

$$\bar{x}^q w_t(\bar{x},\bar{t}) \;\geq\; (L - \bar{x}^p B)w(\bar{x},\bar{t}).$$

But $\bar{x}^q \geq 0$ and $w_t(\bar{x},\bar{t}) \leq 0$ which implies that $\bar{x}^q w_t(\bar{x},\bar{t}) \leq 0$.
Therefore,

$$0 > \bar{x}^q w_t(\bar{x},\bar{t}) \geq (L - \bar{x}^p B)w(\bar{x},\bar{t}) > 0.$$

This is a contradiction which proves that $w > 0$ in $\bar{D} \times [0, r]$.
From $w = u + \eta g e^{\beta t}$, thus $u = w - \eta g e^{\beta t}$. As $\eta \to 0^+$, $u \to w$, therefore, $u \geq 0$. Thus, the theorem is proved.

□

2.2 Uniqueness and Properties of the Solution

The following lemma and its proof are similar to Lemma 2.2 of [Dya08].

Lemma 2.2. *The problem (1.1) has, at most, one solution u. The solution has the following properties:*

i. $u > 0$ in $\bar{D} \times (0, T)$;

ii. u is a strictly increasing function of t for all $x \in \bar{D}$.

Proof. Need to show that u is at most unique. Assume that there are two distinct solutions, u_1 and u_2, to the problem (1.1). Let $y = u_1 - u_2$ and let $z = u_2 - u_1$ such that $z = -y$. Since u_1 and u_2 are solutions to the problem, then $Lu_1 = x^q u_{1_t} - u_{1_{xx}} = x^p f(u_1)$ and $Lu_2 = x^q u_{2_t} - u_{2_{xx}} = x^p f(u_2)$. Therefore,

$$\begin{cases} x^q u_{1_t} &= u_{1_{xx}} + x^p f(u_1), \\ u_1(x,0) &= 0, \\ u_{1_x}(0,t) &= 0 \;=\; u_{1_x}(a,t), \end{cases} \tag{2.11}$$

and

$$\begin{cases} x^q u_{2_t} &= u_{2_{xx}} + x^p f(u_2), \\ u_2(x,0) &= 0, \\ u_{2_x}(0,t) &= 0 \;=\; u_{2_x}(a,t). \end{cases} \tag{2.12}$$

Subtracting equation (2.12) from equation (2.11), we have

$$\begin{cases} x^q(u_{1_t} - u_{2_t}) &= u_{1_{xx}} - u_{2_{xx}} + x^p(f(u_1) - f(u_2)), \\ u_1(x,0) - u_2(x,0) &= 0, \\ u_{1_x}(0,t) - u_{2_x}(0,t) &= 0 \;=\; u_{1_x}(a,t) - u_{2_x}(a,t). \end{cases}$$

By the Mean Value Theorem [Lay], $f(u_1) - f(u_2) = (u_1 - u_2)f'(\xi)$ with $\xi \in [u_1, u_2]$. Therefore,

$$\begin{cases} x^q y_t &= y_{xx} + x^p y f'(\xi), \\ y(x,t) &= 0, \\ y_x(0,t) &= 0 \;=\; y_x(a,t). \end{cases} \tag{2.13}$$

Equation (2.13) satisfies theorem 2.1 with $B(x,t) = f'(\xi(x,t))$, therefore, $y \geq 0$ in $\bar{D}$.

Subtracting equation (2.11) from equation (2.12), we have

$$\begin{cases} x^q(u_{2_t} - u_{1_t}) &= u_{2_{xx}} - u_{1_{xx}} + x^p(f(u_2) - f(u_1)), \\ u_2(x,0) - u_1(x,0) &= 0, \\ u_{2_x}(0,t) - u_{1_x}(0,t) &= 0 \;=\; u_{2_x}(a,t) - u_{1_x}(a,t). \end{cases}$$

Again by the Mean Value Theorem [Lay], $f(u_2) - f(u_1) = (u_2 - u_1)f'(\zeta)$ with $\zeta \in [u_1, u_2]$. Therefore,

$$\begin{cases} x^q z_t &= z_{xx} + x^p z f'(\zeta), \\ z(x,t) &= 0, \\ z_x(0,t) &= 0 \;=\; z_x(a,t). \end{cases} \tag{2.14}$$

Equation (2.14) satisfies theorem 2.1 with $B(x,t) = f'(\zeta(x,t))$, therefore $z \geq 0$ in $\bar{D}$.

We already had $y \geq 0$. With $z \geq 0$ and $z = -y$, which implies that $-y \geq 0$. The only possible conclusion is that $y = 0$, which implies that $u_1 = u_2$.

(i): We are going to show that $u > 0$ in $\bar{D} \times (0, T)$. Let $y = u - 0$. Since u is a solution, then

$$\begin{cases} x^q u_t - u_{xx} - x^p f(u) &= 0, \\ u(x,0) &= 0, \\ u_x(0,t) = 0 &= u_x(a,t), \end{cases} \tag{2.15}$$

and we know that $f(0) > 0$ for $x \in D$, therefore,

$$\begin{cases} 0 - 0 - x^p f(0) &< 0, \\ 0 &= 0, \\ 0 = 0 &= 0. \end{cases} \tag{2.16}$$

Subtracting equation (2.16) from equation (2.15), we have

$$\begin{cases} x^q u_t - u_{xx} - x^p (f(u) - f(0)) &> 0, \\ u(x,0) &= 0, \\ u_x(0,t) = 0 &= u_x(a,t). \end{cases}$$

By the Mean Value Theorem [Lay],

$$\begin{cases} x^q y_t - y_{xx} - x^p f'(\eta) y > 0, & \text{for some } \eta(x,t) \in [0, u]. \\ y(x,0) = 0, & \\ y_x(0,t) = 0 = y_x(a,t). & \end{cases} \tag{2.17}$$

Equation (2.17) satisfies theorem 2.1 with $f'(\eta) = B$. Therefore, $y \geq 0$.

If $y = 0$, then by the theorem 12 from Appendix B, we will have, from equation (2.17),

$$0 = x^q y_t - y_{xx} - x^p f(\eta) y > 0.$$

This is a contradiction, which proves that $y > 0$ in D.

Let us consider the boundary, $x = 0$ and $x = a$.
Suppose y attains its minimum value zero at $x = 0$ or $x = a$. By the parabolic

version of Hopf's Lemma [Lemma 7 in Appendix A], $y_x(0,t) > 0$ and $y_x(a,t) < 0$. This contradiction shows that $u > 0$ on $\bar{D}$.

(ii): For any $h \in (0,T)$, let u_h be defined in $\Omega_h = (0,a) \times (0, T-h)$ by $u_h(x,t) = u(x,t+h)$ and let $y = u_h - u$.

Since u_h and u satisfy problem (1.1), we write

$$\begin{cases} x^q u_{h_t} - u_{h_{xx}} - x^p f(u_h) \;=\; 0, & \text{in } \Omega_h, \\ u_h(x,0) \;=\; u(x,h), & \text{on } \bar{D}, \\ u_{h_x}(0,t) \;=\; 0 \;=\; u_{h_x}(a,t), & 0 < t < T-h \end{cases} \tag{2.18}$$

also

$$\begin{cases} x^q u - u_{xx} - x^p f(u) \;\geq\; 0, & \text{in } \Omega, \\ u(x,0) \;=\; 0, & \text{on } \bar{D}, \\ u_x(0,t) \;=\; 0 \;=\; u_x(a,t), & 0 < t < T. \end{cases} \tag{2.19}$$

From part (i) above we showed that $u > 0$ in D. Therefore, $u(x,h) > 0$ for all $x \in \bar{D}$. This implies that $u(x,h) - u(x,0) > 0$.

Subtracting equation (2.19) from (2.18), we have

$$\begin{cases} x^q(u_h - u) - (u_{h_{xx}} - u_{xx}) - x^p(f(u_h) - f(u)) \;=\; 0, \\ u_h(x,0) - u(x,0) \;>\; 0, \\ u_{h_x}(0,t) - u_x(0,t) \;=\; 0 \;=\; u_{h_x}(a,t) - u_x(a,t). \end{cases}$$

By the Mean Value Theorem, $f(u_h) - f(u) = f'(\varsigma)y$ for some ς between u_h and u. Therefore,

$$\begin{cases} x^q y_t - y_{xx} - x^p f'(\varsigma) y \;=\; 0, & \text{in } \Omega_h, \\ y(x,0) \;>\; 0, & \text{in } \bar{D}, \\ y_x(0,t) \;=\; 0 \;=\; y_x(a,t), & 0 < t < T-h. \end{cases}$$

for some ς between u_h and u. By Theorem 2.1, $y \geq 0$. If $y = 0$ at some interior point $(x_3, t_3) \in (0,a) \times (0, T-h)$, then by the strong maximum principle $y = 0$ in $(0,a) \times (0, t_3]$. This contradicts the initial condition $y(x,0) > 0$ on $\bar{D}$. Therefore, $y > 0$ at any point in $(0,a)$. If $y = 0$ at some point, say $(0,t)$, then by the parabolic version of Hopf's Lemma [Lemma 7 in Appendix B], $y_x(0,t) > 0$. Similarly, if $y = 0$ at some point (a,t), then $y_x(a,t) < 0$. These contradict $y_x(0,t) = 0 = y_x(a,t)$, respectively. Thus, u is a strictly increasing function of t for $x \in \bar{D}$.

□

2.3 Construction of an Upper Solution

The following lemma and its proof were adopted from [Dya08].

Lemma 2.3. *There exist some positive constants t_0 ($\leq T$) and $\bar{c} \in (0,c)$ such that the problem (1.1) has an upper solution $\mu(x,t) \in C^{2,1}([0,a] \times [0,t_0])$, $\mu(x,t) \in (0,\bar{c}]$ and μ depends on f, a, p, and q.*

Let us define some constants that will be used in the proof of this theorem.

Definition 2.4. *Let us choose constants $\hat{m} > 0$, $0 < \gamma < min\left\{\frac{1}{2}, \frac{a}{2}\right\}$, and $K > \hat{m}$ so that*

$$\begin{array}{rcl} f(\hat{m}a^p(1+f(0))) & < & 1+f(0), \\ 0 & < & -(1/2)\gamma^2 - \gamma + \hat{m} < \hat{m}, \\ f(Ka^p(1+f(0))) & > & 1+f(0), \\ Ka^p(1+f(0)) & < & c. \end{array} \tag{2.20}$$

$\hat{m}$ is sufficiently small such that the growth of f is less than 1 when u varies from 0 to $\hat{m}a^p(1+f(0))$.

Let $0 < \epsilon < \gamma$ and $D_\epsilon = (\epsilon, a)$.

Proof. We will consider the following problem:

$$\left\{ \begin{array}{rcll} Lu_\epsilon & = & x^p f(u_\epsilon) & \text{in } D_\epsilon \times (0,t_0], \\ u_\epsilon(x,0) & = & 0 & \text{on } \bar{D}_\epsilon, \\ u_{\epsilon_x}(\epsilon,t) & = & 0 = u_{\epsilon_x}(a,t) & \text{for } 0 < t \leq t_0. \end{array} \right. \tag{2.21}$$

Let us construct an upper solution $\mu(x,t) \in C^{2,1} \times (\bar{D} \times [0,t_0])$ for all u_ϵ, where $\epsilon < \gamma$. Let

$$\theta(x) = \begin{cases} -\frac{1}{2}x^2 - x + \hat{m}, & 0 \le x \le \gamma, \\ \hat{h}(x), & \gamma < x < a-\gamma, \\ -\frac{1}{2}(a-x)^2 - (a-x) + \hat{m}, & a-\gamma \le x \le a, \end{cases}$$

where $\hat{h}(x)$ is a positive C^∞ function chosen such that $\theta(x)$ is in $C^2(\bar{D})$ and $\max_{\gamma<x<a-\gamma}\hat{h}(x) \le \hat{m}$.

Note that, for $0 \le x \le \gamma$, $\theta'(x) = -x-1$, $\theta'(x) < 0$ and $\theta'(0) = -1$.
For $a-\gamma \le x \le a$, $\theta'(x) = -(a-x)+1$ and $\theta'(a) = 1$. Also, $\max(\theta(x)) = \hat{m}$ and $\min_{0\le x\le\gamma}\theta(x) = -\dfrac{1}{2}\gamma^2 - \gamma + \hat{m}$. Since f is continuous, there exist t_1 such that the initial-value problem

$$\tau'(t) = \frac{(1+\max_{\gamma<x\le a}|\theta''|)a^p f(K\tau)}{\gamma^q(\min_{\gamma<x\le a}\theta)}, \quad \tau(0) = a^p(1+f(0)),$$

has a unique solution for $0 \le t \le t_1$.
Observe that $\tau(t)$ is an increasing function because $\tau'(t) > 0$. Since $\tau(0) = a^p(1+f(0))$, it follows from equation (2.20) that $f(\hat{m}\tau(0)) < 1 + f(0)$.

Now, let us choose some constant t_0 in $(0,t_1]$ such that

$$f(\hat{m}\tau(t_0)) \;\le\; 1 + f(0),$$

and

$$\tau(t_0) \;\le\; a^p f(Ka^p(1+f(0))) \le a^p f(K\tau).$$

Let $\mu(x,t) = \theta(x)\tau(t)$. Then, for any $x \in [0,\gamma]$ and $t \in (0,t_0]$, $x^q\theta\tau' \ge 0$ and $\theta'' = -1$ which is less than zero. Hence, $\mu_t = \theta\tau'$ and $\mu_{xx} = \theta''\tau = -\tau$.
Therefore,

$$\begin{aligned} L\mu - x^p f(\mu) &= x^q\mu_t - \mu_{xx} - x^p f(\mu) \\ &= x^q\theta\tau' - \tau\theta'' - x^p f(\theta\tau). \end{aligned} \tag{2.22}$$

The expression $x^q\theta\tau'$ is greater than zero, so it can be ignored in order to minimize the right hand side of equation (2.22). Since $\theta'' = -1$, then $-\tau\theta'' = \tau$. Therefore,

$$L\mu - x^p f(\mu) \geq \tau - x^p f(\theta\tau)$$

Since τ is an increasing function, it is minimum at $t = 0$ and maximum at $t = t_0$. Then, $\tau(t) \geq \tau(0)$ and $\tau(t) \leq \tau(t_0)$ for $0 \leq t \leq t_0$. Therefore,

$$L\mu - x^p f(\mu) \geq \tau(0) - x^p f(\theta(0)\tau(t_0)).$$

But $\tau(0) = a^p(1 + f(0))$ and $\theta(0) = \hat{m}$. Therefore,

$$\begin{aligned} L\mu - x^p f(\mu) &\geq a^p(1 + f(0)) - a^p f(\hat{m}\tau(t_0)) \\ &= a^p[(1 + f(0)) - f(\hat{m}\tau(t_0))]. \end{aligned}$$

But, $f(\hat{m}\tau(t_0)) < 1 + f(0)$ and $a^p > 0$. Hence,

$$a^p[(1 + f(0)) - f(\hat{m}\tau(t_0))] \geq 0.$$

Therefore, $L\mu - x^p f(\mu) \geq 0$ for $x \in [0, \gamma]$.

Now, for any $x \in (\gamma, a]$ and $t \in (0, t_0]$, we have

$$\begin{aligned} L\mu - x^p f(\mu) &= x^q\mu_t - \mu_{xx} - x^p f(\mu) \\ &= x^q\theta\tau' - \tau\theta'' - x^p f(\theta\tau) \\ &\geq \gamma^q \left(\min_{\gamma<x\leq a}\theta\right)\tau'(t) - \tau(t_0)\left(\max_{\gamma<x\leq a}|\theta''|\right) - a^p f(\theta\tau) \\ &\geq \gamma^q \left(\min_{\gamma<x\leq a}\theta\right)\tau'(t) - \tau(t_0)\left(\max_{\gamma<x\leq a}|\theta''|\right) - a^p f(\hat{m}\tau) \\ &\geq \gamma^q \left(\min_{\gamma<x\leq a}\theta\right)\tau'(t) - \tau(t_0)\left(\max_{\gamma<x\leq a}|\theta''|\right) - a^p f(K\tau) \\ &\geq \gamma^q \left(\min_{\gamma<x\leq a}\theta\right)\tau'(t) - a^p f(K\tau)\left(\max_{\gamma<x\leq a}|\theta''|\right) - a^p f(K\tau) \\ &= \gamma^q \left(\min_{\gamma<x\leq a}\theta\right)\left(\tau'(t) - \frac{a^p f(K\tau)\left(\max_{\gamma<x\leq a}|\theta''|\right) + a^p f(K\tau)}{\gamma^q(\min_{\gamma<x\leq a}\theta}\right) \\ &= 0. \end{aligned}$$

From construction, $\mu(x,0) = \theta(x)\tau(0) = \theta(x)a^p(1 + f(0)) \geq \theta(x) = u(x,0) = 0$. $\mu_x(0,t) = \theta'(0)\tau(t) < 0$, $\mu_x(a,t) = \theta'(a)\tau(t) > 0$, and $\mu(x,t) \in C^{2,1}(\bar{D} \times [0,t])$.

Let $y = \mu - u_\epsilon$. Then

$$\begin{cases} Ly - x^p f'(\vartheta)y & \geq & 0 \quad \text{in } D_\epsilon \times (0,t]_0, \\ y(0) & > & 0, x \in \bar{D}, \\ y_x(\epsilon,t) & < & 0, \quad y_x(a,t) \quad > \quad 0, \quad t \in [0,t_0], \end{cases}$$

where ϑ is between μ and u_ϵ for all $\epsilon < \gamma$. By theorem 2.1, function $y = \mu - u_\epsilon \geq 0$. We also observe that by construction $\mu(x,t)$ depends only on f, a, p, and q.

□

Lemma 2.5. *Let $0 < \epsilon_1 < \epsilon_2 < \gamma$ and suppose u_{ϵ_1} and u_{ϵ_2} are solutions of the problem (2.21) on $(0,t_0)$. If $p < q$, then $u_{\epsilon_x} < 0$ and $u_{\epsilon_1} > u_{\epsilon_2}$ in Ω_{ϵ_2}. If $p > q$, then $u_{\epsilon_x} > 0$ and $u_{\epsilon_1} < u_{\epsilon_2}$ in Ω_{ϵ_2}.*

Proof. We will prove the case where $p > q$.
Let $0 < \epsilon < \min\left\{\gamma, \dfrac{a}{2}\right\}$. Let u_ϵ be a solution to the regular problem

$$\begin{cases} Lu_\epsilon & = & x^p f(u_\epsilon), & (x,t) \in \Omega_\epsilon, \\ u_\epsilon(x,0) & = & 0, & \text{on } \bar{D}_\epsilon, \\ u_{\epsilon_x}(\epsilon,t) & = & 0 = u_{\epsilon_x}(a,t), & 0 < t < T, \end{cases} \tag{2.23}$$

and u_ϵ is positive in $\bar{D}_\epsilon \times (0,T)$.
Now

$$Lu_\epsilon = x^q u_{\epsilon_t} - u_{\epsilon_{xx}} = x^p f(u_\epsilon).$$

Differentiating with respect to x, we have

$$qx^{q-1}u_{\epsilon_t} + x^q u_{\epsilon_{tx}} - u_{\epsilon_{xxx}} = px^{p-1}f(u_\epsilon) + x^p u_{\epsilon_x} f'(u_\epsilon).$$

Observe that $x^q u_{\epsilon_{tx}} - u_{\epsilon_{xxx}} = Lu_{\epsilon_x}$. Therefore,

$$Lu_{\epsilon_x} - x^p u_{\epsilon_x} f'(u_{\epsilon_x}) = px^{p-1}f(u_\epsilon) - qx^{q-1}u_{\epsilon_t}.$$

But, from $x^q u_{\epsilon_t} - u_{\epsilon_{xx}} = x^p f(u_\epsilon)$, we have

$$\begin{aligned} u_{\epsilon_t} &= x^{p-q}f(u_\epsilon) + x^{-q}u_{\epsilon_{xx}}, \\ qx^{q-1}u_{\epsilon_t} &= qx^{p-1}f(u_\epsilon) + qx^{-1}u_{\epsilon_{xx}}. \end{aligned}$$

Therefore,

$$Lu_{\epsilon_x} - x^p u_{\epsilon_x} f'(u_\epsilon) \;=\; px^{p-1}f(u_\epsilon) - qx^{p-1}f(u_\epsilon) - qx^{-1}u_{\epsilon_{xx}}.$$

But $qx^{-1}u_{\epsilon_{xx}} = \frac{q}{x}\frac{\partial}{\partial x}u_{\epsilon_x}$. Therefore,

$$Lu_{\epsilon_x} + \frac{q}{x}\frac{\partial}{\partial x}u_{\epsilon_x} - x^p u_{\epsilon_x} f'(u_\epsilon) = (p-q)x^{p-1}f(u_\epsilon).$$

Therefore, differentiating problem (2.23) with respect to x gives the following problem:

$$\begin{cases} [L + \frac{q}{x}\frac{\partial}{\partial x} - x^p f'(u_\epsilon)]u_{\epsilon_x} \;=\; (p-q)x^{p-1}f(u_\epsilon), & (x,t) \in D_\epsilon \times (0,T), \\ u_{\epsilon_x}(x,0) \;=\; 0, & \text{on } [\epsilon, a], \\ u_{\epsilon_x}(\epsilon,t) \;=\; 0 \;=\; u_{\epsilon_x}(a,t), & 0 < t < T. \end{cases} \tag{2.24}$$

Since $\epsilon > 0$, problem (2.24) is regular, i.e. $x > 0$. And $f'(u_\epsilon)$ is bounded on D_ϵ. Therefore, we can apply theorem 12 (The Strong Maximum Principle) to the problem to determine the nature of u_ϵ.

Now, if $p > q$, then $(p-q)x^{p-1}f(u_\epsilon) > 0$. Therefore, by theorem 12, $u_{\epsilon_x} \geq 0$ for $(x,t) \in \bar{D}_\epsilon \times (0,T]$.

Similarly, if $p < q$, then $(p-q)x^{p-1}f(u_\epsilon) < 0$, and by theorem 12, $u_{\epsilon_x} \leq 0$ for $(x,t) \in \bar{D}_\epsilon \times (0,T]$.

However, $u_{\epsilon_x} \neq 0$ in D because if it is zero then, equation (2.24) will not hold.

Let $0 < \epsilon_1 < \epsilon_2 < \gamma$ and $p > q$. Then $u_{\epsilon_{1_x}}(\epsilon_2, t) > 0$. Let $y = u_{\epsilon_1} - u_{\epsilon_2}$. u_{ϵ_1} satisfies

$$\begin{cases} x^q u_{\epsilon_{1_t}} - u_{\epsilon_{1_{xx}}} - x^p f(u_{\epsilon_1}) \;=\; 0, & \text{in } \Omega_{\epsilon_2}, \\ u_{\epsilon_1}(x,0) \;=\; 0, & \text{on } [\epsilon_1, a], \\ u_{\epsilon_{1_x}}(\epsilon_2, t) \;>\; 0, & 0 < t < T, \\ u_{\epsilon_{1_x}}(a,t) \;=\; 0, & 0 < t < T. \end{cases} \tag{2.25}$$

Similarly, u_{ϵ_2} satisfies:

$$\left\{\begin{array}{rcll} x^q u_{\epsilon_{2_t}} - u_{\epsilon_{2_{xx}}} - x^p f(u_{\epsilon_2}) &=& 0, & \text{in } \Omega_{\epsilon_2}, \\ u_{\epsilon_2}(x,0) &=& 0, & \text{on } [\epsilon_2, a], \\ u_{\epsilon_{2_x}}(\epsilon_2, t) &=& 0, & 0 < t < T, \\ u_{\epsilon_{2_x}}(a,t) &=& 0, & 0 < t < T. \end{array}\right. \tag{2.26}$$

Subtracting equation (2.26) from equation (2.25), we have

$$\left\{\begin{array}{rcll} x^q(u_{\epsilon_{1_t}} - u_{\epsilon_{2_t}}) - (u_{\epsilon_{1_{xx}}} - u_{\epsilon_{2_{xx}}}) - x^p(f(u_{\epsilon_1}) - f(u_{\epsilon_2})) &=& 0, & \text{in } \Omega_{\epsilon_2}, \\ u_{\epsilon_1}(x,0) - u_{\epsilon_2}(x,0) &=& 0, & \text{on } [\epsilon_2, a], \\ u_{\epsilon_{1_x}}(\epsilon_2, t) - u_{\epsilon_{2_x}}(\epsilon_2, t) &>& 0, & 0 < t < T, \\ u_{\epsilon_{1_x}}(a,t) - u_{\epsilon_{2_x}}(a,t) &=& 0, & 0 < t < T. \end{array}\right.$$

$f(u_{\epsilon_1}) - f(u_{\epsilon_2}) = f'(\theta_1)(u_{\epsilon_1} - u_{\epsilon_2})$ in Ω_{ϵ_2} for some $\theta_1 \in [u_{\epsilon_1}, u_{\epsilon_2}]$. Therefore,

$$\left\{\begin{array}{rcll} x^q y_t - y_{xx} - x^p f'(\theta_1) y &=& 0, & \text{in } \Omega_{\epsilon_2}, \\ y(x,0) &=& 0, & \text{for } x \in [\epsilon_2, a], \\ y_x(\epsilon_2, t) &>& 0, & \text{for } 0 < t > T, \\ y_x(a,t) &=& 0, & \text{for } 0 < t > T. \end{array}\right.$$

Therefore, by the Comparison Theorem 2.1, $y \le 0$.

If $y = 0$ at some interior point $(x_4, t_4) \in (\epsilon_2, a) \times (0, T)$ then, by theorem 12 (The Strong Maximum Principle), $y = 0$ in $(\epsilon_2, a) \times (0, t_4]$. But $y_x(\epsilon_2, t) = u_{\epsilon_{1_x}} > 0$. This is a contradiction. Therefore, if $p > q$, then $u_{\epsilon_1} < u_{\epsilon_2}$ in Ω_{ϵ_2}.
Similarly, if $p < q$, then $u_{\epsilon_1} > u_{\epsilon_2}$ in Ω.

□

2.4 Existence of the Solution

The proof of the following result is a modification of that of Lemma 2 of [CL01].

Theorem 2.6. *Problem (1.1) has a classical solution* $u(x,t) \in C([0,a] \times [0,t_0]) \cap C^{2,1}((0,a] \times [0,t_0])$.

Proof. Equation (2.21) can be written as

$$u_{\epsilon_t} - \frac{1}{x^q} u_{xx} = \frac{x^p}{x^q} f(u_\epsilon). \tag{2.27}$$

i. The cylindrical domain of problem (2.27) is $D_\epsilon \times (0,T)$. Evidently, D_ϵ is a bounded and connected domain in R. i.e., $D_\epsilon = (\epsilon, a)$. The boundary of D_ϵ is $\delta D_\epsilon \in C^{2+\alpha}$, with $0 < \alpha < 1$.

ii. $\Omega_\epsilon = D_\epsilon \times [0, t_0]$.

iii. From problem (2.21), we have

$$\begin{aligned} x^q u_{\epsilon_t} - u_{\epsilon_{xx}} &= x^p f(u_\epsilon), \\ u_{\epsilon_t} - x^{-q} u_{\epsilon_{xx}} &= x^{p-q} f(u_\epsilon). \end{aligned}$$

with $x > 0$, x^{-q} and x^{p-q} are C^α continuous.

iv. Since $x > 0$, then $x^{-q} > 0$.

v. The boundary condition is $u_\epsilon(x, 0) = 0$ and $0 \in C^{2+\alpha}$.

vi. $u_{\epsilon_x}(\epsilon, t) = 0 \in C^{2+\alpha}(\bar{D}_\epsilon)$.

vii. Here $u_{\epsilon_x}(\epsilon, t) = u_{\epsilon_x}(a, t) = 0$.

The conclusion is that problem (2.27) satisfies [Wan97]'s hypothesis P and so, by [Wan97]'s theorem 3.1, this problem has a solution $u_\epsilon \in C^{2+\alpha, 1+\frac{\alpha}{2}}([\epsilon, a] \times [0, t_0])$. From lemma 2.5, which was proved previously, for $0 < \epsilon_1 < \epsilon_2 < \gamma$, if $p < q$ then $u_{\epsilon_1} > u_{\epsilon_2}$ in Ω_{ϵ_2}, while, if $p > q$ then $u_{\epsilon_1} < u_{\epsilon_2}$ in Ω_{ϵ_2}. Therefore, the sequence of u_ϵ's is monotone and bounded and so, there is a $\mathrm{Lim}_{\epsilon \to 0} u_\epsilon(x, t)$ for all $(x, t) \in \Omega$ and we call this limit $u(x, t)$.

Now, we have to show that $u(x, t)$ is a classical solution of problem (1.1). For any point $(x_6, t_6) \in (0, a) \times (0, t_0)$, there exist a set $Q = [b_1, b_2] \times [0, t_7] \subsetneqq \bar{D} \times [0, t_0]$ such that $0 < b_1 < x_6 < b_2 \leq a$ and $0 < t_6 < t_7 < t_0$.

From lemma 2.3, we have shown that $\mu(x,t)$ is an upper solution to the problem (1.1). Therefore, the solution $u_\epsilon \leq \mu$ and it is finite in Q.
Note: $||u_\epsilon||_{L_{\tilde{q}}}$ is the norm of u_ϵ in space $L_{\tilde{q}}$ as defined in [p. 154 [McO96]] and, also in Appendix A.
For $\mu \in L_{p(\Omega)}$, we may define

$$||\mu||_{L_p(\Omega)} = \left(\int_\Omega |\mu(x)|^p dx\right)^{\frac{1}{p}} \text{ for } 1 \leq p \leq \infty.$$

Therefore,

$$||u_\epsilon||_{L_{\tilde{q}}(Q)} = \left(\int_Q |u_\epsilon(x)|^{\tilde{q}} dx\right)^{\frac{1}{\tilde{q}}} \text{ for } 1 \leq \tilde{q} \leq \infty.$$

Since $u_\epsilon(x) \leq \mu(x)$, then

$$\left(\int_Q |u_\epsilon(x)|^{\tilde{q}} dx\right)^{\frac{1}{\tilde{q}}} \leq \left(\int_Q |\mu(x)|^{\tilde{q}} dx\right)^{\frac{1}{\tilde{q}}}.$$

But μ is finite, so, $\left(\int_\Omega |\mu(x)|^{\tilde{q}} dx\right)^{\frac{1}{\tilde{q}}}$ is less than some constant k_1. Therefore, we can write

$$||u_\epsilon||_{L_{\tilde{q}}} \leq ||\mu||_{L_{\tilde{q}}} \leq k_1.$$

Also,

$$||x^{p-q} f(u_\epsilon)||_{L_{\tilde{q}}(Q)} = \left(\int_Q |x^{p-q} f(u_\epsilon|^{\tilde{q}} dx\right)^{\frac{1}{\tilde{q}}}.$$

If $p < q$, the largest value of x^{p-q} will be at $x = b_1$. Therefore,

$$\begin{aligned}\left(\int_Q |x^{p-q} f(u_\epsilon)|^{\tilde{q}} dx\right)^{\frac{1}{\tilde{q}}} &\leq \left(\int_Q |b_1^{p-q} f(u_\epsilon)|^{\tilde{q}} dx\right)^{\frac{1}{\tilde{q}}} \\ &= b_1^{p-q} \left(\int_Q |f(u_\epsilon)|^{\tilde{q}} dx\right)^{\frac{1}{\tilde{q}}}.\end{aligned}$$

Since $u_\epsilon(x) \leq \mu(x)$, then

$$\begin{aligned}\left(\int_Q |x^{p-q} f(u_\epsilon)|^{\tilde{q}} dx\right)^{\frac{1}{\tilde{q}}} &\leq b_1^{p-q} \left(\int_Q |f(\mu)|^{\tilde{q}} dx\right)^{\frac{1}{\tilde{q}}} \\ &= b_1^{p-q} ||f(\mu)||_{L_{\tilde{q}}(Q)}.\end{aligned}$$

If $p > q$, the largest value of x^{p-q} will be at $x = b_2$. By analogy, we have

$$||x^{p-q}f(u_\epsilon)||_{L_{\tilde{q}}(Q)} \le b_2^{p-q}||f(\mu)||_{L_{\tilde{q}}(Q)}.$$

Therefore, for any constant $\tilde{q} > 1$ we have,

i.

$$||u_\epsilon||_{L_{\tilde{q}}} \le ||\mu||_{L_{\tilde{q}}(Q)} \le k_1.$$

ii.

$$||x^{p-q}f(u_\epsilon)||_{L_{\tilde{q}}(Q)} \le b_1^{p-q}||f(\mu)||_{L_{\tilde{q}}(Q)}, \text{ if } p < q.$$

iii.

$$||x^{p-q}f(u_\epsilon)||_{L_{\tilde{q}}(Q)} \le b_2^{p-q}||f(\mu)||_{L_{\tilde{q}}(Q)}, \text{ if } p > q.$$

By [[LSU68], p 341-342, 351], $u_\epsilon \in W_{\tilde{q}}^{2,1}(Q)$. By the embedding theorem [[LSU68], p 61 & 80], $W_{\tilde{q}}^{2,1}(Q) \hookrightarrow H^{\alpha,\frac{\alpha}{2}}(Q)$, with $0 < \alpha < 1$, and $\tilde{q} >$max$\left\{3, \dfrac{2}{1-\alpha}\right\}$. Then $||u_\epsilon||_{H^{\alpha,\frac{\alpha}{2}}(Q)} \le k_2$ for some positive constant k_2.
Using the definition of $H^{\alpha,\alpha/2}(Q)$ given on [p.155,156 [McO96]], we have

$$\begin{aligned}
||x^{p-q}f(u_\epsilon)||_{H^{\alpha,\alpha/2}(Q)} &= ||x^{p-q}f(u_\epsilon)||_\infty \\
&+ \text{Sup}_{(x,t)(\breve{x},t)\in Q}\frac{|x^{p-q}f(u_\epsilon(x,t)) - \breve{x}^{p-q}f(u_\epsilon(\breve{x},t))|}{|x-\breve{x}|^\alpha} \\
&+ \text{Sup}_{(x,t)(x,\breve{t})\in Q}\frac{|x^{p-q}f(u_\epsilon(x,t)) - \breve{x}^{p-q}f(u_\epsilon(x,\breve{t}))|}{|t-\breve{t}|^{\alpha/2}}.
\end{aligned}$$

Since u_ϵ is bounded above by μ and x^{p-q} is bounded above by b_1^{p-q} (because $p < q$), then

$$||x^{p-q}f(u_\epsilon)||_\infty \le b_1^{p-q}||f(\mu)||_\infty.$$

Now

$$\begin{aligned}
&\text{Sup}_{(x,t)(\breve{x},t)\in Q}\frac{|x^{p-q}f(u_\epsilon(x,t)) - \breve{x}^{p-q}f(u_\epsilon(\breve{x},t))|}{|x-\breve{x}|^\alpha} \\
&= \text{Sup}_{(x,t)(\breve{x},t)\in Q}\frac{|x^{p-q}f(u_\epsilon(x,t)) - x^{p-q}f(u_\epsilon(\breve{x},t)) + x^{p-q}f(u_\epsilon(\breve{x},t)) - \breve{x}^{p-q}f(u_\epsilon(\breve{x},t))|}{|x-\breve{x}|^\alpha} \\
&= \text{Sup}_{(x,t)(\breve{x},t)\in Q}\frac{|x^{p-q}f(u_\epsilon(x,t)) - x^{p-q}f(u_\epsilon(\breve{x},t))|}{|x-\breve{x}|^\alpha} \\
&+ \text{Sup}_{(x,t)(\breve{x},t)\in Q}\frac{f(u_\epsilon(\breve{x},t))|x^{p-q} - \breve{x}^{p-q}|}{|x-\breve{x}|^\alpha}.
\end{aligned}$$

By the Mean Value Theorem [Lay], $f(u_\epsilon(x,t)) - f(u_\epsilon(\check{x},t)) = f'(\xi)(u_\epsilon(x,t) - u_\epsilon(\check{x},t))$, where $\xi \in [u_\epsilon(x,t), u_\epsilon(\check{x},t)]$. Note that x^{p-q} is bounded above by b_1^{p-q} when $p < q$. Therefore,

$$\begin{aligned}
\mathrm{Sup}_{(x,t),(\check{x},t)\in Q}\frac{x^{p-q}|f(u_\epsilon(x,t)) - f(u_\epsilon(\check{x},t))|}{|x-\check{x}|^\alpha} & \\
+ \ \mathrm{Sup}_{(x,t),(x,\check{t})\in Q}\frac{|x^{p-q}f(u_\epsilon(x,t)) - \check{x}^{p-q}f(u_\epsilon(x,\check{t}))|}{|t-\check{t}|^{\alpha/2}} & \\
= \ \mathrm{Sup}_{(x,t),(\check{x},t)\in Q}\frac{x^{p-q}|f'(\xi)||u_\epsilon(x,t) - u_\epsilon(\check{x},t)|}{(x-\check{x})^\alpha} & \\
+ \ \mathrm{Sup}_{(x,t),(x,\check{t})\in Q}\frac{x^{p-q}|f(u_\epsilon(x,t)) - f(u_\epsilon(x,\check{t}))}{(t-\check{t})^{\alpha/2}}. & \\
\leq \ ||f'(\mu)||_\infty b_1^{p-q}||u_\epsilon||_{H^{\alpha,\alpha/2}(Q)}. &
\end{aligned}$$

$$\mathrm{Sup}_{(x,t),(\check{x},t)\in Q}\frac{f(u_\epsilon(x,t))|x^{p-q} - \check{x}^{p-q}|}{|x-\check{x}|^\alpha} \leq ||f(\mu)||_\infty||x_{p-q}||_{H^\alpha(Q)}.$$

Putting all together, we have

$$\begin{aligned}
||x^{p-q}f(u_\epsilon)||_{H^{\alpha,\alpha/2}(Q)} \ \leq \ & b_1^{p-q}||f(\mu)||_\infty + b_1^{p-q}||f'(\mu)||_\infty||u_\epsilon||_{H^{\alpha,\alpha/2}(Q)} \\
+ \ & ||f(\mu)||_\infty||x^{p-q}||_{H^{\alpha,\alpha/2}(Q)}.
\end{aligned} \tag{2.28}$$

Since the components on the right hand side of equation (2.28) are bounded, we can say that

$$||x^{p-q}f(u_\epsilon)||_{H^{\alpha,\alpha/2}(Q)} \leq k_3,$$

for some positive k_3 which is independent of ϵ.

Going through the same analysis for $p > q$ we have

$$||x^{p-q}f(u_\epsilon)||_{H^{\alpha,\alpha/2}(Q)} \leq \check{k}_3,$$

for some positive $\check{k}_3$ which is also independent of ϵ.

By theorem 10 of [LSU68], pp 351 and 352], we have

$$||u_\epsilon||_{H^{2+\alpha,1+\alpha/2}(\Omega_\epsilon)} \leq k_4,$$

for some constant k_4 which is independent of ϵ.
Since we have the space $H^{2+\alpha,1+\alpha/2}$, then, u_ϵ, u_{ϵ_x}, $u_{\epsilon_{xx}}$, u_{ϵ_t} are equicontinuous in Q.

By Ascoli-Arzela theorem [Eva98], $||u||_{H^{2+\alpha,1+\alpha/2}(Q)} \leq k_4$ and the partial derivatives of u are the limits of the corresponding derivatives of u_ϵ. Thus, $u(x,t) \in C(\bar{D}) \cap C^{2,1}((0,a] \times [0,t_0])$.

□

2.5 Unboundedness of $f(u)$

If T is the supremum over t_0 for which the problem (1.1) has a unique solution $u(x,t) \in C(\bar{D}) \cap C^{2,1}((0,a] \times [0,t_0])$ so that $u < c$. Then, there is a unique solution $u(x,t) \in C(\bar{D} \times [0,T)) \cap C^{2,1}((0,a] \times [0,T))$, where $u < c$.

The proof of this theorem is similar to that in [Flo91].

Theorem 2.7. *If $T < \infty$, then $f(u)$ is unbounded in Ω.*

Proof. Let's assume that $f(u)$ is bounded above by some positive constant M in Ω. From theorem 2.6, we know that there is a unique solution $u < c$ where c is where the solution quenches. Since $f(u)$ is bounded above, there exist a unique number $c^* > 0$ such that $u \leq c^* < c$.
We would like to show that $f(u)$ can be continued into a time interval $[0, T + \tilde{t}_1]$ for some $\tilde{t}_1 > 0$. To do this, we want to show that the problem (1.1) has an upper solution $\tilde{\mu}(x,t) \in C^{2,1}([0,a] \times [T, T + \tilde{t}_1])$.
Let us chose constants as follows:

i. τ_0 small enough so that

$$f(0.5(c + c^*)\tau_0) < 1 + f(0).$$

ii. K^* large enough so that

$$\tau_0 < a^p f(K^* \tau_0).$$

iii. $K^* > \dfrac{c + c^*}{2}$

iv. $0 < \tilde{\gamma} < \min\left\{\dfrac{a}{2}, \dfrac{1}{2}\right\}$ is such that $\dfrac{-K^*}{2}\tilde{\gamma}^2 - \tilde{\gamma} + \dfrac{c + c^*}{2} > c^*$.

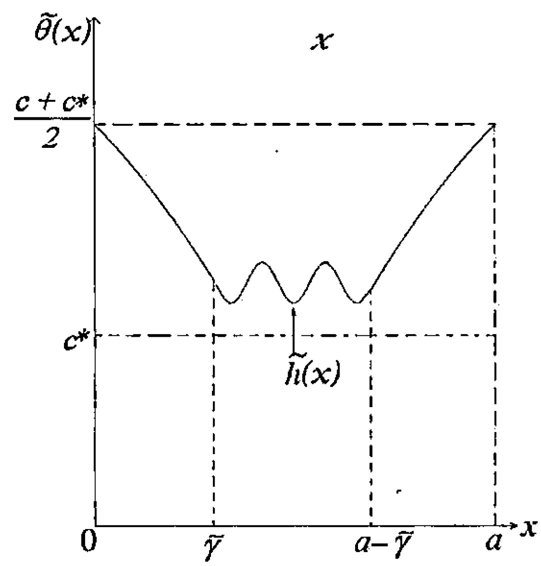

Figure 2.2: Sketch of $\tilde{\theta}(x)$

Let

$$\tilde{\theta}(x) = \begin{cases} -\dfrac{K^*}{2}x^2 - x + \dfrac{c+c^*}{2}, & 0 \le x \le \tilde{\gamma}, \\ \tilde{h}(x), & \tilde{\gamma} < x < a - \tilde{\gamma}, \\ -\dfrac{K^*}{2}(a-x)^2 - (a-x) + \dfrac{c+c^*}{2}, & a - \tilde{\gamma} \le x \le a. \end{cases} \tag{2.29}$$

Where $\tilde{h}(x)$ is a C^∞ function chosen such that $c^* < \tilde{h}(x) \le \dfrac{c+c^*}{2}$ and $\tilde{\theta}(x)$ is in $C^2(D)$.
Note that for $0 \le x \le \tilde{\gamma}$, $\tilde{\theta}'(x) < 0$, and $\tilde{\theta}'(0) = -K^*$.
A sketch of $\tilde{\theta}(x)$ is shown in fig 2.2. Since f is continuous, the initial-value problem

$$\tilde{\tau}'(t-T) = \frac{(a^p f(K^*\tilde{\tau}(t))(\max_{\tilde{\gamma}<x\le a}|\tilde{\theta}''| + 1)}{\tilde{\gamma}^q \min_{\tilde{\gamma}<x\le a}\tilde{\theta}}, \qquad \tilde{\tau}(0) = \tau_0$$

has a unique solution for $0 \le t \le \tilde{t}_2$. Observe that $\tilde{\tau}'(t)$ is positive, so, $\tilde{\tau}(t)$ is an increasing function.

Let us chose $\tilde{t}_1$ in $(0, \tilde{t}_2]$ small enough such that

$$f(0.5(c+c^*)\tilde{\tau}(\tilde{t}_1)) \le 1 + f(0),$$

and

$$\tilde{\tau}(\tilde{t}_1) \leq a^p f(K^*\tau_0)) \leq a^p f(K^*\tilde{\tau}).$$

Let $\tilde{\mu}(x,t) = \tilde{\theta}(x)\tilde{\tau}(t-T)$. Then, for any $x \in [0,\tilde{\gamma}]$, and $t \in (T, T+\tilde{t}_2]$, $x^q\tilde{\theta}\tilde{\tau}' \geq 0$ and $\tilde{\theta}'' = -K^* < 0$. Therefore,

$$\begin{aligned} L\tilde{\mu} - x^p f(\tilde{\mu}) &= x^q\tilde{\mu}_t - \tilde{\mu}_{xx} - x^p f(\tilde{\mu}) \\ &= x^q\tilde{\theta}\tilde{\tau}' - \tilde{\tau}\tilde{\theta}'' - x^p f(\tilde{\theta}\tilde{\tau}). \end{aligned}$$

We have $\min_{0\leq x\leq\tilde{\gamma}} x^q\tilde{\theta}\tilde{\tau}' = 0$ and $\max_{0\leq x\leq\tilde{\gamma}}\tilde{\tau}\tilde{\theta}'' = \tilde{\tau}(0)$ because $\tilde{\theta}'' = -K^*$. Therefore,

$$\begin{aligned} L\tilde{\mu} - x^p f(\tilde{\mu}) &\geq K^*\tilde{\tau}(0) - a^p f(\tilde{\theta}(0)\tilde{\tau}(\tilde{t}_0)) \\ &\geq K^* a^p(1+f(0)) - a^p f(0.5(c+c^*)\tilde{\tau}(\tilde{t}_0) \\ &= a^p[K^*(1+f(0)) - f(0.5(c+c^*)\tilde{\tau}(\tilde{t}_0)] \\ &\geq 0. \end{aligned}$$

For any $x \in (\tilde{\gamma}, a]$, we have

$$\begin{aligned} L\tilde{\mu} - x^p f(\tilde{\mu}) &= x^q\tilde{\mu}_t - \tilde{\mu}_{xx} - x^p f(\tilde{\mu}) \\ &= x^q\tilde{\theta}\tilde{\tau}' - \tilde{\tau}\tilde{\theta}'' - x^p f(\tilde{\theta}\tilde{\tau}) \\ &\geq \tilde{\gamma}^p\tilde{\tau}'(t)(\min_{\tilde{\gamma}<x\leq a}\tilde{\theta}) - \tilde{\tau}(\max_{\tilde{\gamma}<x\leq a}|\tilde{\theta}''|) - a^p f(0.5(c+c^*)\tilde{\tau}(t)). \end{aligned}$$

But $\tilde{\tau}(\tilde{t}_1) \leq a^p f(K^*\tilde{\tau}(t))$ and since f is an increasing function and by definition of K^*, $f(0.5(c+c^*)\tilde{\tau}(t)) \leq f(K^*\tilde{\tau}(t))$,

$$\begin{aligned} L\tilde{\mu} - x^p f(\tilde{\mu}) &\geq \tilde{\gamma}^q\tilde{\tau}'(t-T)(\min_{\tilde{\gamma}<x\leq a}\tilde{\theta}) - a^p f(K^*\tilde{\tau}(\tilde{t}_0))\max_{\tilde{\gamma}<x\leq a}|\tilde{\theta}''|) - a^p f(K^*\tilde{\tau}(\tilde{t}_1)) \\ &= \tilde{\gamma}^q(\min_{\tilde{\gamma}<x\leq a}\tilde{\theta})[\tilde{\tau}'(t-T) - \frac{(\max_{\tilde{\gamma}<x\leq a}|\tilde{\theta}''|+1)a^p f(K^*\tilde{\tau}(t))}{\tilde{\gamma}^q\min_{\tilde{\gamma}<x\leq a}\tilde{\theta}}] \\ &= 0. \end{aligned}$$

By theorem 2.1, $\tilde{\mu}(x,t)$ is an upper solution of u on $\bar{D} \times [T, T+\tilde{t}_1]$. As in lemma 2.5 and theorem 2.6, it can be shown that the problem (2.1) has a unique solution $u(x,t) \in C(\bar{D}\times[0,T+\tilde{t}_1]) \cap C^{2,1}((0,a]\times[0,T+\tilde{t}_1])$. This contradicts the definition of T, and hence, the theorem is proved. □

2.6 Influence of Constants p and q on $u_x(x,t)$

This lemma and its proof are similar to Lemma 2.7 of [Dya08].

Lemma 2.8. *For any $(x,t) \in \Omega$, if $p > q$; then $u_x(x,t) > 0$, while if $p < q$, then $u_x(x,t) < 0$.*

Proof. From lemma 2.5, we know that if $p < q$, then $u_{\epsilon_x} < 0$ in Ω_{ϵ_2} while if $p > q$, then $u_{\epsilon_x} > 0$ in Ω_{ϵ_2}. From lemma 2.6, a solution of problem (2.1) is $u = \lim_{\epsilon \to 0} u_\epsilon$. Therefore, when $p > q$, $u_x \geq 0$ and when $p < q$, $u_x \leq 0$ in Ω_{ϵ_2}. We have, in Ω

$$\begin{aligned}(L - x^p f'(u))u_x &= Lu_x - x^p f'(u)u_x, \\ &= x^q u_{xt} - u_{xxx} - x^p f'(u)u_x.\end{aligned}$$

But

$$x^p f(u) = x^q u_t - u_{xx}. \tag{2.30}$$

Differentiating both sides of equation (2.30) with respect to x, we have

$$\begin{aligned}px^{p-1}f(u) + x^p f'(u)u_x &= qx^{q-1}u_t + x^q u_{tx} - u_{xxx}, \\ x^p f'(u)u_x &= -px^{p-1}f(u) + qx^{q-1}u_t + x^q u_{tx} - u_{xxx}.\end{aligned}$$

Therefore,

$$(L - x^p f'(u))u_x = px^{p-1}f(u) - qx^{q-1}u_t.$$

From equation (2.30), we have

$$Lu_x + \frac{q}{x}u_x - x^p f'(u)u_x = (p-q)x^{p-1}f(u).$$

Therefore, we can write

$$\begin{cases} \left(L + \dfrac{q}{x}\dfrac{d}{dx} - x^p f'(u)\right) u_x = (p-q)x^{p-1}f(u), & (x,t) \in D \times (0,T), \\ u_x(x,0) = 0, & \text{for } 0 < x < a, \\ u_{xx}(0,t) = 0 = u_{xx}(a,t), & 0 < t < T. \end{cases} \tag{2.31}$$

As in the proof of Lemma 2.5, we obtain $u_x > 0$ if $p > q$, while $u_x < 0$ if $p < q$.

□

Chapter 3

Quenching At The Boundary

The lemma and theorem in this chapter show that the solution quenches only at the boundaries. They are respectively adopted from lemma 3.1 and theorem 3.2 of [Dya08].

Lemma 3.1. *The following holds:*

i. Let $0 < \tilde{x}_1 < \tilde{x}_2 < a$. Let $p > q$ and the positive number $T_0 < T$ be such that $u_x(x,t) > 0$ in $(\tilde{x}_1, \tilde{x}_2) \times (T_0, T)$. Then there is no quenching point in $(\tilde{x}_1, \tilde{x}_2)$.

ii. Let $0 < \hat{x}_1 < \hat{x}_2 < a$. Let $p < q$ and the positive number $T_0 < T$ be such that $u_x(x,t) < 0$ in $(\hat{x}_1, \hat{x}_2) \times (T_0, T)$. Then there is no quenching point in $(\hat{x}_1, \hat{x}_2)$.

Proof. (i) $p > q$: We will prove this part of the theorem by showing a contradiction. Suppose that there exist some $x_0 \in (\tilde{x}_1, \tilde{x}_2)$ such that u quenches at $x = x_0$. By lemma 2.2, $u_t \geq 0$. From lemma 2.8, $\lim_{t \to T} u(x,t) = c$ for $x_0 < x < \tilde{x}_2$. For $x_0 \leq \tilde{x}_3 \leq \tilde{x}_4 \leq \tilde{x}_2$, let

$$z(x,t) = u_x(x,t) - \epsilon \tilde{h}(x) \text{ in } (\tilde{x}_3, \tilde{x}_4) \times (T_0, T),$$

where $\tilde{h}(x) = \sin\left(\dfrac{(x-\tilde{x}_3)\pi}{\tilde{x}_4-\tilde{x}_3}\right)+1$, and ϵ is a positive constant to be determined.

$$\begin{aligned}
\tilde{h}' &= \frac{\pi}{\tilde{x}_4-\tilde{x}_3}\cos\left(\frac{(x-\tilde{x}_3)\pi}{\tilde{x}_4-\tilde{x}_3}\right).\\
\tilde{h}'' &= -\left(\frac{\pi}{\tilde{x}_4-\tilde{x}_3}\right)^2\sin\left(\frac{(x-\tilde{x}_3)\pi}{\tilde{x}_4-\tilde{x}_3}\right).\\
z_x &= u_{xx}-\epsilon\tilde{h}'.
\end{aligned}$$

Now, $Lz = Lu_x - L\epsilon\tilde{h}$ and $L\epsilon\tilde{h} = x^p\epsilon\dfrac{d\tilde{h}}{dt} - \epsilon\dfrac{d^2\tilde{h}}{dx^2}$. Since $\tilde{h}$ is only a function of x, $\dfrac{d\tilde{h}}{dt}=0$, leaving us with $L\epsilon\tilde{h} = -\epsilon\dfrac{d^2\tilde{h}}{dx^2}$. From equation (2.31), $Lu_x = x^pf'(u)u_x + (p-q)x^{p-1}f(u) - qx^{-1}u_{xx}$. Therefore,

$$\begin{aligned}
Lz &= x^pf'(u)u_x+(p-q)x^{p-1}f(u)-qx^{-1}u_{xx}+\epsilon\tilde{h}'',\\
\frac{q}{x}\frac{dz}{dx} &= qx^{-1}u_{xx}-qx^{-1}\epsilon\tilde{h}',\\
\text{and } x^pf'(u)z &= x^pf'(u)u_x-x^pf'(u)\epsilon\tilde{h}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(L+\frac{q}{x}\frac{d}{dx}-x^pf'(u)\right)z &= x^pf'(u)u_x+(p-q)x^{p-1}f(u)-qx^{-1}u_{xx}+\epsilon\tilde{h}''\\
&\quad + qx^{-1}u_{xx}-qx^{-1}\epsilon\tilde{h}'-x^pf'(u)u_x+x^pf'(u)\epsilon\tilde{h}\\
&= -qx^{-1}\epsilon\tilde{h}'+x^pf'\epsilon\tilde{h}+(p-q)x^{p-1}f+\epsilon\tilde{h}''.
\end{aligned}$$

$\mathrm{Max}_{\tilde{x}_3\le x\le\tilde{x}_4}(qx^{-1}\epsilon\tilde{h}') = q\tilde{x}_3^{-1}\epsilon(\mathrm{Max}_{\tilde{x}_3\le x\le\tilde{x}_4}\tilde{h}') = q\tilde{x}_3^{-1}\epsilon\left(\dfrac{\pi}{\tilde{x}_4-\tilde{x}_3}\right)$.
Since $\mathrm{Min}_{\tilde{x}_3\le x\le\tilde{x}_4}\tilde{h} = 1$, then $x^pf'\epsilon\tilde{h} \ge \tilde{x}_3^pf'\epsilon$. Since $p>q$, $(p-q)x^{p-1}f>0$, so, ignoring $(p-q)x^{p-1}f$ will help minimize the expression. $\mathrm{Min}_{\tilde{x}_3\le x\le\tilde{x}_4}\tilde{h}'' = -\left(\dfrac{\pi}{\tilde{x}_4-\tilde{x}_3}\right)^2$. Therefore,

$$\begin{aligned}
\left(L+\frac{q}{x}\frac{d}{dx}-x^pf'\right)z &\ge \frac{-q\tilde{x}_3^{-1}\epsilon\pi}{\tilde{x}_4-\tilde{x}_3}+\tilde{x}_3^pf'\epsilon-\frac{\epsilon\pi^2}{(\tilde{x}_4-\tilde{x}_3)^2}\\
&= \epsilon\left[\tilde{x}_3^pf'-\frac{q\tilde{x}_3^{-1}\pi}{\tilde{x}_4-\tilde{x}_3}-\left(\frac{\pi}{\tilde{x}_4-\tilde{x}_3}\right)^2\right].
\end{aligned} \tag{3.1}$$

Since $u\to c$ as $t\to T$ in $(\tilde{x}_3,\tilde{x}_4)$, there exist some $T_1\ge T_0$ such that

$$f'(u) \ge \frac{1}{\tilde{x}_3^p}\left[\frac{q\pi}{\tilde{x}_3(\tilde{x}_4-\tilde{x}_3)}+\left(\frac{\pi}{\tilde{x}_4-\tilde{x}_3}\right)^2\right]. \tag{3.2}$$

Since $u_x > 0$, ϵ can be chosen so small that $z(x, T_1) > 0$ for $x \in [\tilde{x}_3, \tilde{x}_4]$. At $x = \tilde{x}_3$, and at $x = \tilde{x}_4$, $z > 0$. By the Maximum Principle, $z > 0$ in $[\tilde{x}_3, \tilde{x}_4] \times [T_1, T)$. Thus

$$\begin{aligned} u_x(x,t) &> \epsilon\tilde{h}(x) \\ &= \epsilon \sin\left(\frac{(x-\tilde{x}_3)\pi}{\tilde{x}_4 - \tilde{x}_3}\right) + \epsilon \quad \text{in } [\tilde{x}_3, \tilde{x}_4] \times [T_1, T). \end{aligned}$$

Integrating the above inequality from $\tilde{x}_3$ to $\tilde{x}_4$, we have

$$\begin{aligned} u(\tilde{x}_4, t) - u(\tilde{x}_3, t) &> \left[-\epsilon\frac{\tilde{x}_4 - \tilde{x}_3}{\pi}\cos\left(\frac{x-\tilde{x}_3)\pi}{\tilde{x}_4 - \tilde{x}_3}\right) + \epsilon x\right]_{\tilde{x}_3}^{\tilde{x}_4} \\ &= \frac{2\epsilon(\tilde{x}_4 - \tilde{x}_3)}{\pi} + \epsilon(\tilde{x}_4 - \tilde{x}_3) \\ &= \left(\frac{2}{\pi} + 1\right)\epsilon(\tilde{x}_4 - \tilde{x}_3). \end{aligned}$$

As $t \to T$, the left hand side tends to $c - c = 0$ while the right hand side remains positive. This contradiction shows that there is no quenching point in $(\tilde{x}_1, \tilde{x}_2)$.

(ii) $p < q$: We will also prove this part of the theorem by showing a contradiction. Suppose that there exist some $x_0 \in (\hat{x}_1, \hat{x}_2)$ such that u quenches at $x = x_0$. By lemma 2.2, $u_t \geq 0$. From lemma 2.8, $\lim_{t\to T} u(x,t) = c$ for $x_0 < x < \hat{x}_2$. For $x_0 \leq \hat{x}_3 \leq \hat{x}_4 \leq \hat{x}_2$, let

$$z(x,t) = u_x(x,t) - \epsilon\hat{h}(x) \text{ in } (\hat{x}_3, \hat{x}_4) \times (T_0, T),$$

where $\hat{h}(x) = \sin\left(\frac{(x-\hat{x}_4)\pi}{\hat{x}_4 - \hat{x}_3}\right) - 1$, and ϵ is a positive constant to be determined.

$$\begin{aligned} \hat{h}' &= \frac{\pi}{\hat{x}_4 - \hat{x}_3}\cos\left(\frac{(x-\hat{x}_4)\pi}{\hat{x}_4 - \hat{x}_3}\right). \\ \hat{h}'' &= -\left(\frac{\pi}{\hat{x}_4 - \hat{x}_3}\right)^2 \sin\left(\frac{(x-\hat{x}_4)\pi}{\hat{x}_4 - \hat{x}_3}\right). \\ z_x &= u_{xx} - \epsilon\hat{h}'. \end{aligned}$$

Now, $Lz = Lu_x - L\epsilon\hat{h}$ and $L\epsilon\hat{h} = x^p\epsilon\frac{d\hat{h}}{dt} - \epsilon\frac{d^2\hat{h}}{dx^2}$. Since $\hat{h}$ is only a function of x, $\frac{d\hat{h}}{dt} = 0$, leaving us with $L\epsilon\hat{h} = -\epsilon\frac{d^2\hat{h}}{dx^2}$. From equation (2.31), $Lu_x = x^p f'(u)u_x +$

$(p-q)x^{p-1}f(u) - qx^{-1}u_{xx}$. Therefore,

$$\begin{aligned}
Lz &= x^p f'(u)u_x + (p-q)x^{p-1}f(u) - qx^{-1}u_{xx} + \epsilon\hat{h}'', \\
\frac{q}{x}\frac{dz}{dx} &= qx^{-1}u_{xx} - qx^{-1}\epsilon\hat{h}', \\
\text{and } x^p f'(u)z &= x^p f'(u)u_x - x^p f'(u)\epsilon\hat{h}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(L + \frac{q}{x}\frac{d}{dx} - x^p f'(u)\right) z &= x^p f'(u)u_x + (p-q)x^{p-1}f(u) - qx^{-1}u_{xx} + \epsilon\hat{h}'' \\
&\quad + qx^{-1}u_{xx} - qx^{-1}\epsilon\hat{h}' - x^p f'(u)u_x + x^p f'(u)\epsilon\hat{h} \\
&= -qx^{-1}\epsilon\hat{h}' + x^p f'\epsilon\hat{h} + (p-q)x^{p-1}f + \epsilon\hat{h}''.
\end{aligned}$$

$\mathrm{Max}_{\hat{x}_3 \le x \le \hat{x}_4}(-qx^{-1}\epsilon\hat{h}') = q\hat{x}_3^{-1}\epsilon\left(\dfrac{\pi}{\hat{x}_4 - \hat{x}_3}\right)$.
Since $\mathrm{Max}_{\hat{x}_3 \le x \le \hat{x}_4}\hat{h} = -1$, then $x^p f'\epsilon\hat{h} \le \hat{x}_3^p f'\epsilon$. Since $p < q$, $(p-q)x^{p-1}f < 0$, so, ignoring $(p-q)x^{p-1}f$ will help maximize the expression. $\mathrm{Max}_{\hat{x}_3 \le x \le \hat{x}_4}\hat{h}'' = \left(\dfrac{\pi}{\hat{x}_4 - \hat{x}_3}\right)^2$. Therefore,

$$\begin{aligned}
\left(L + \frac{q}{x}\frac{d}{dx} - x^p f'\right) z &\le \frac{q\hat{x}_3^{-1}\epsilon\pi}{\hat{x}_4 - \hat{x}_3} - \hat{x}_3^p f'\epsilon + \frac{\epsilon\pi^2}{(\hat{x}_4 - \hat{x}_3)^2} \\
&= -\epsilon\left[\hat{x}_3^p f' - \frac{q\hat{x}_3^{-1}\pi}{\hat{x}_4 - \hat{x}_3} - \left(\frac{\pi}{\hat{x}_4 - \hat{x}_3}\right)^2\right].
\end{aligned} \tag{3.3}$$

Since $u \to c$ as $t \to T$ in $(\hat{x}_3, \hat{x}_4)$, there exist some $T_1 \ge T_0$ such that

$$f'(u) \ge \frac{1}{\hat{x}_3^p}\left[\frac{q\pi}{\hat{x}_3(\hat{x}_4 - \hat{x}_3)} + \left(\frac{\pi}{\hat{x}_4 - \hat{x}_3}\right)^2\right]. \tag{3.4}$$

Since $u_x < 0$, ϵ can be chosen small enough so that $z(x, T_1) < 0$ for $x \in [\hat{x}_3, \hat{x}_4]$. At $x = \hat{x}_3$, and at $x = \hat{x}_4$, $z < 0$. By the Maximum Principle, $z < 0$ in $[\hat{x}_3, \hat{x}_4] \times [T_1, T)$. Thus,

$$\begin{aligned}
u_x(x,t) &< \epsilon\hat{h}(x) \\
&= \epsilon\sin\left(\frac{(x - \hat{x}_4)\pi}{\hat{x}_4 - \hat{x}_3}\right) - \epsilon \quad \text{in } [\hat{x}_3, \hat{x}_4] \times [T_1, T).
\end{aligned}$$

Integrating the above inequality from $\hat{x}_3$ to $\hat{x}_4$, we have

$$\begin{aligned} u(\hat{x}_4,t)-u(\hat{x}_3,t) &< \left[-\epsilon\frac{\hat{x}_4-\hat{x}_3}{\pi}\cos\left(\frac{(x-\hat{x}_4)\pi}{\hat{x}_4-\hat{x}_3}\right)-\epsilon x\right]_{\hat{x}_3}^{\hat{x}_4} \\ &= -\frac{2\epsilon(\hat{x}_4-\hat{x}_3)}{\pi}-\epsilon(\hat{x}_4-\hat{x}_3) \\ &= -\left(\frac{2}{\pi}+1\right)\epsilon(\hat{x}_4-\hat{x}_3). \end{aligned}$$

As $t \to T$, the left hand side tends to $c-c=0$ while the right hand side remains negative. This contradiction shows that there is no quenching point in $(\hat{x}_1, \hat{x}_2)$.

□

Theorem 3.2. *Suppose u quenches,*

i. If $p > q$, $x = a$ is the only quenching point.

ii. If $p < q$, $x = 0$ is the only quenching point.

Proof. i) By lemma 2.8, $u_x > 0$ in Ω. Therefore, $x = a$ is a quenching point. By lemma 3.1, there is no quenching point in D. Therefore, $x = 0$ is not the quenching point.

ii) By lemma 2.8, $u_x < 0$ in Ω. Therefore, $x = 0$ is a quenching point. By lemma 3.1, there is no quenching point in D. Therefore, $x = a$ is not the quenching point.

□

Chapter 4

Complete Quenching

The following theorem demonstrates that if $p = q$, then the solution quenches on $\bar{D}$.

Theorem 4.1. *Let $p = q$. If u quenches, then the quenching set for the solution of (2.1) is $\bar{D}$.*

Proof. We had

$$\begin{aligned} Lu &= x^q u_t - u_{xx} &= x^p f(u), \\ x^q u_t &= x^p f(u) + u_{xx}. \end{aligned}$$

Since $q = p$, we have

$$u_t = f(u) + \frac{u_{xx}}{x^q}.$$

Below, $v(t)$ is a solution of the initial value problem;

$$\begin{cases} v_t = f(v) & \text{in } (0, \tilde{t}), \\ v(0) = 0. \end{cases} \tag{4.1}$$

Then $v_x = 0$ and, therefore, $v_{xx} = 0$. $x^q v_t - v_{xx} = x^p f(v)$ and $x^q v_t = x^p f(v) + v_{xx}$. But $q = p$, therefore $v_t = f(v)$. v is a unique solution of problem (2.1). Quenching of (4.1) occurs since $\lim_{v \to c^-} f(v) = \infty$ for some constant c. Since the function does not depend on x, the quenching is on $\bar{D}$.

□

Example. Let us consider the Ordinary Differential Equation (ODE) with initial condition:

$$\begin{cases} u' = \dfrac{1}{1-u}, \\ \quad u(0) = 0, \end{cases}$$

where u is a function of the independent variable t. Let us solve for u using separation of variables method:

$$\begin{aligned} \frac{du}{dt} &= \frac{1}{1-u}, \\ (1-u)du &= dt. \end{aligned}$$

Integrating both sides of the preceding equation will produce

$$u - \frac{1}{2}u^2 = t + c.$$

The given initial condition was that $u(0) = 0$ which means that when $t = 0$, u is 0. This implies that the constant of integration c is zero. Therefore

$$\begin{aligned} u - \frac{1}{2}u^2 &= t, \\ u^2 - 2u + 2t &= 0. \end{aligned}$$

Solving for u using quadratic formula, we obtain

$$u = 1 - \sqrt{1-2t}.$$

Now, at $t = 0.5$, $u = 1$, and $u' = \infty$. Therefore quenching occurs at $t = 0.5$. The function $u(t) = 1 - \sqrt{1-2t}$ is the solution of the initial value problem

$$\begin{cases} u_t &= \dfrac{1}{1-u}, \\ u(0) &= 0. \end{cases}$$

On the other hand, $u(t)$ formally satisfies the following problem:

$$\begin{cases} x^q u_t = u_{xx} + \dfrac{x^q}{1-u} & \text{in } (0,a) \times (0, 0.5), \\ u(0,t) = 0 & \text{for any } 0 \le x \le a, \\ u_x(0,t) = 0 = u_x(a,t) & \text{for } 0 < t < 0.5, \end{cases} \tag{4.2}$$

since $u_x(t) = 0$, $u_{xx}(t) = 0$, and $x^q u_t = u_{xx} + \dfrac{x^q}{1-u}$. When $t = 0.5$, $u(0.5) = 1$ and $u_t = \dfrac{1}{1-u}$ becomes unbounded. Therefore, solution $u(t)$ of (4.2) quenches in finite time $t = 0.5$ and the quenching set is $[0, a]$.

Appendix A

1. Definitions of some basic function spaces.

Definition 1. *Let X denote a real linear space. A mapping $|| \; || : X \to [0, \infty)$ is called a norm if*

(a) $||u + v|| \leq ||u|| + ||v||$ *for all* $u, v \in X$.

(b) $||\lambda u|| = |\lambda| ||u||$ *for all* $u \in X, \lambda \in R$.

(c) $||u|| = 0$ *if and only if* $u = 0$.

Hereafter we assume X is a normed linear space.

Definition 2. *We say a sequence $\{u_k\}_{k=1}^{\infty} \subset X$ converges to $u \in X$, written*

$$u_k \to u$$

if

$$\lim_{k \to \infty} ||u_k - u|| = 0.$$

Definition 3. *(a) A sequence $\{u_k\}_{k=1}^{\infty} \subset X$ is called a Cauchy sequence provided for each $\epsilon > 0$ there exists $N > 0$ such that*

$$||u_k - u_l|| < \epsilon \qquad \text{for all } k, l \geq N.$$

(b) X is complete if each Cauchy sequence in X converges; that is, whenever $\{u_k\}_{k=1}^{\infty}$ is a Cauchy sequence, there exists $u \in X$ such that $\{u_k\}_{k=1}^{\infty}$ converges to u.

(c) A Banach space X is a complete, normed linear space.

$L_q(\Omega)$ is the Banach space consisting of all functions on Ω with the norm

$$||u||_{q,Q} = \left(\int_\Omega |u(x)|^q dx \right)^{\frac{1}{q}}.$$

Also,

$$||u||_{\infty,\Omega} = \mathrm{vrai}_\Omega \mathrm{max}|u|.$$

$L_{q,r}(Q_T)$ is the Banach space consisting of all functions on Q_T with a finite norm

$$||u||_{q,r,Q_T} = \left(\int_0^T \left(\int_\Omega |u(x,t)|^q dx \right)^{\frac{r}{q}} dt \right)^{\frac{1}{r}},$$

where $q \geq 1$ and $r \geq 1$.

Generalized derivatives are to be understood in the way that is now customary in the majority of papers on differential equations.

$W_q^l(\Omega)$ for l integral is the Banach space consisting of all elements of $L_q(\Omega)$ having generalized derivatives of all forms up to order l inclusively, that are qth-power summable on Ω. The norm in $W_q^l(\Omega)$ is defined by the equality

$$||u||_{q,\Omega}^{(l)} \;=\; \sum_{j=0}^{l} \langle\langle u \rangle\rangle_{q,\Omega}^{(l)}, \tag{A.1}$$

where

$$\langle\langle u \rangle\rangle_{q,\Omega}^{(l)} \;=\; \sum_{(j)} \left|\left| D_x^j u \right|\right|_{q,\Omega}. \tag{A.2}$$

The symbol D_x^j denotes any derivative of $u(x)$ with respect to x of order j, while $\sum_{(j)}$ denotes summation over possible derivatives of u of order j. For domains with "not too bad" boundaries $W_q^l(\Omega)$ coincides with the closure in norm (A.1) of the set of all functions that are infinitely differentiable in $\bar{\Omega}$.

This will be true, for example, for domains with piecewise-smooth boundaries. Sometimes W_q^l is written in place of $W_q^l(\Omega)$, particularly if the domain Ω is subject to further refinement.

For $u \in L^p(\Omega)$, we may define

$$||u||_p = \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}} \quad \text{for } 1 \leq p \leq \infty.$$

We proceed to define the Hölder spaces $H^l(\bar{D})$ and $H^{l,l/2}(\bar{\Omega})$, where l is always a non-integral positive number.

Definition 4. *$H^l(\bar{D})$ is the Banach space whose elements are continuous functions $u(x)$ in $\bar{D}$ having in $\bar{D}$ continuous derivatives up to order $[l]$ inclusively and a finite value for the quantity*

$$|u|_D^{(l)} = \langle u\rangle_D^{(l)} + \sum_{j=0}^{|l|} \langle u\rangle_D^{(j)}, \tag{A.3}$$

where

$$\langle u\rangle_D^{(0)} = |u|_D^{(0)} = max_D|u|.$$

$$\begin{aligned}\langle u\rangle_D^{(j)} &= \textstyle\sum_{(j)} |D_x^j u|_D^{(0)}. \\ \langle u\rangle_D^{(l)} &= \textstyle\sum_{(|l|)} \left\langle D_x^{|l|} u\right\rangle_D^{(l-|l|)}.\end{aligned}$$

Equality (A.3) defines the norm $|u|_D^{(l)}$ in $H^l(D)$.

Definition 5. *$H^{l,l/2}(\bar{\Omega})$ is the Banach space of functions $u(x,t)$ that are continuous in $\bar{\Omega}$, together with all derivatives of the form $D_t^r D_x^s$ for $2r + s < l$, and have a finite norm*

$$|u|_\Omega^{(l)} = \langle u\rangle_\Omega^{(l)} + \sum_{j=0}^{|l|} \langle u\rangle_\Omega^{(j)}, \tag{A.4}$$

where

$$\begin{aligned}
\langle u\rangle^{(0)}_{\Omega} &\equiv |u|^{(0)}_{\Omega} = max_{\Omega}|u|.\\
\langle u\rangle^{(j)}_{\Omega} &= \textstyle\sum_{(2r+s-j)} |D^r_t D^s_x u|^{(0)}_{\Omega}.\\
\langle u\rangle^{(l)}_{\Omega} &= \langle u\rangle^{(l)}_{x,\Omega} + \langle u\rangle^{(l/2)}_{t,\Omega}.\\
\langle u\rangle^{(l)}_{x,\Omega} &= \textstyle\sum_{(2r+s-|j|)} \langle D^r_t D^s_x u\rangle^{(l-|l|)}_{x,\Omega}.\\
\langle u\rangle^{(l/2)}_{x,\Omega} &= \textstyle\sum_{0<l-2r-s<2} \langle D^r_t D^s_x u\rangle^{\frac{l-2r-s}{2}}_{t,\Omega}.
\end{aligned}$$

2. **Hopf's Lemma**

Definition 6. *The inside Strong Sphere Property. Let $P^0 = (x^0, t^0)$ be a point on the boundary $\partial\Omega$ of a domain Ω. If there exists a closed ball B with center $(\bar{x}, \bar{t})$ such that $B \subset \bar{\Omega}$, $B \cap \partial\Omega = \{P^0\}$, and if $\bar{x} \neq x^0$, then we say that P^0 has the inside strong sphere property.*

Lemma 7. *Let the foregoing assumptions be satisfied and let P^0 have the inside strong sphere property. Assume further that, for some neighborhood V of P^0, $u < M$ in $D \cap V$. Then, for any non-tangential inward direction τ,*

$$\frac{\partial u}{\partial \tau} \equiv \lim_{\Delta\tau\to 0} inf \frac{\Delta u}{\Delta \tau} < 0 \qquad at\ P^0.$$

By a non-tangential inward direction we mean direction pointing from P^0 into the interior of the ball B whose boundary touches ∂D at P^0.

3. **Ascoli-Arzela Compactness Criterion** Suppose that $\{f_k(x)\}_{k=1}^{\infty}$ is a sequence of real-valued functions defined on R^n such that

$$|f_k(x)| \leq M \quad (k = 1, ..., x \in R^n)$$

for some constant M, and the $\{f_k\}_{k=1}^{\infty}$ are uniformly *equicontinuous.* Then there exist a subsequence $\{f_{k_j}\}_{j=1}^{\infty} \subseteq \{f_k\}_{k=1}^{\infty}$ and a continuous function f, such that

$$f_{k_j} \to f \text{ uniformly on compact subsets of } R^n.$$

To say the $\{f_k\}_{k=1}^{\infty}$ are uniformly equicontinuous means that for each $\epsilon > 0$, there exists $\delta > 0$, such that $|x - y| < \delta$ implies $|f_k(x) - f_k(y)| < \epsilon$, for $x, y \in R^n$, $k = 1, \ldots.$

4. Other Definition.

Definition 8. $\mu(x,t) \in C^{2,1}([0,a] \times [0,T])$ *is an upper solution for problem (1.1) if it satisfies the following:*

$$\begin{aligned} L\mu - x^p f(\mu) &\geq 0, & \text{in } \Omega, \\ \mu(x,0) &\geq u(x,0), & 0 \leq x \leq a, \\ \mu_x(0,t) &\leq 0, & 0 \leq t \leq T, \\ \mu_x(a,t) &\geq 0, & 0 \leq t \leq T. \end{aligned}$$

Appendix B

1. **The Mean Value Theorem**

   Theorem 9. *(Mean Value Theorem) Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exist at least one point $c \in (a, b)$ such that*
   $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

2. **The Maximum Principle**

   Consider the operator
   $$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t} \tag{B.1}$$
   in an $(n+1)$-dimensional domain Ω with the following assumptions:

   (a) L is parabolic in Ω, i.e., for every $(x,t) \in \Omega$ and for any real vector $\xi \neq 0$, $\sum a_{ij}(x,t)\xi_i\xi_j > 0$;

   (b) the coefficients of L are continuous functions in Ω;

   (c) $c(x,t) \leq 0$ in Ω.

   The functions u in (B.1) are always assumed to have two continuous x-derivatives and one continuous $t-$derivative in Ω.

Definition 10. *Notation. For any point $P^0 = (x^0, t^0)$ in Ω, we denote by $S(P^0)$ the set of all points Q in Ω which can be connected to P^0 by a simple* *continuous curve in Ω along which the $t-$coordinate is nondecreasing from Q to P^0. By $C(P^0)$, we denote the component (in $t = t^0$) of $\Omega \cap \{t = t^0\}$ which contains P^0. Note that $S(P^0) \supset C(P^0)$.*

Theorem 11. *Let (2a), (2b), (2c) hold. If $Lu \geq 0$ ($Lu \leq 0$) in Ω and if u has in D a positive maximum (negative minimum) which is attained at a point $P^0(x^0, t^0)$, then $u(P) = u(P^0)$ for all $P \in S(P^0)$.*

3. Extensions of the Maximum Principle

Theorem 12. *Let (2a), (2b) hold. If $u \leq 0$ ($u \geq 0$) in $S(P^0)$, $Lu \geq 0$ ($Lu \leq 0$) in $S(P^0)$ and $u(P^0) = 0$, then $u \equiv 0$ in $S(P^0)$.*

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