2006

Freeness of Hopf algebras

Christopher David Walker

FREENESS OF HOPF ALGEBRAS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

Christopher David Walker

June 2006

FREENESS OF HOPF ALGEBRAS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

---

by

Christopher David Walker

June 2006

Approved by:

[redacted] 6/5/06

Davida Fischman, Committee Chair — Date

[redacted]

Gary Griffing, Committee Member

[redacted]

John Sarli, Committee Member

[redacted]

Peter Williams, Chair,
Department of Mathematics

[redacted]

J. Terry Hallett
Graduate Coordinator,
Department of Mathematics

Abstract

In 1975 Kaplansky discussed "Ten conjectures on Hopf algebras" during a lecture at the University of Chicago. The first of these conjectures concerned freeness of a Hopf algebra as a module over a subHopfalgebra. Specifically he conjectured that "a Hopf algebra is free as a module over any subHopfalgebra". Although this was quickly shown to be false in the infinite dimensional case, the finite dimensional case turned out to be true, and was proven 14 years later by Nichols and Zoeller. This result is the heart of this paper.

The Nichols-Zoeller freeness theorem states that a finite dimensional Hopf algebra is free as a module over any subHopfalgebra. We will prove this theorem, as well as the first significant generalization of this theorem, which was proven three years later. This generalization says that if the Hopf algebra is infinite dimensional, then the Hopf algebra is still free if the subHopfalgebra is finite dimensional and semisimple. We will also look at several other significant generalizations that have since been proven.

Acknowledgements

First an foremost, I would like to thank God for the abilities he has given me. It is my hope that I will use them for the purpose he intended.

I would next like to thank Dr. Davida Fischman for all of her guidance and insight on this project and throughout my studies at this university. I would also like to thank all of the professors in the department for sharing their mathematical knowledge with me over the years.

Finally, I would like to thank my family for all there support over the last three years as I worked toward my goals. Without them, none of this would be possible.

Table of Contents

Chapter 1

Introduction

In the last 20 years, the study of quantum physics has been revolutionized by the discovery of the connection between quantum groups and a previously obscure, abstract field of mathematics known as Hopf algebras. Mathematicians have known of Hopf algebras for over 60 years, but it was only with the connection to quantum groups that their application became apparent. Because of this connection, many advances in this field have become very important to several different disciplines in recent years. This paper will focus on the study of the ring theoretic properties of Hopf algebras.

The idea of a Hopf algebra originated in the work of Heinz Hopf. He discovered this structure in 1941 while working on homology and cohomology of topological groups [Hop41]. Several fundamental papers were written on the subject over the next two decades. Hopf algebras were developed in the context of topology in [Bor53] and [MM65], and then were developed in the context of algebraic geometry in [Car57] where they were called "hyperalgebras". It wasn't until 1969 that a text that gave a unifying development of the topic was published. It was during this year that Moss Sweedler published "Hopf Algebras", which is still considered the standard text on the basics of the subject today.

After the publication of Sweedler's book, mathematicians continued to develop the theory of Hopf algebras. In 1975 Irving Kaplansky stated ten conjectures on Hopf algebras during a lecture at the University of Chicago [Kap75]. Since then several of his conjectures have been settled, but the first one was the most important because of its extensive use. The first of these conjecture dealt with the relationship between Hopf algebras and their subHopfalgebras. It was already known that a Hopf algebra was a module over any of its subHopfalgebras, but it was not known when it was a free module. Kaplansky conjectured that "A Hopf algebra is free as a module over any of its subHopfalgebras". Immediately it was shown by Oberst and Schneider in [OS74] that this was not true in the infinite dimensional case. The finite diminsional case, however, remained open for 14 years until a former student of Kaplansky, Warren Nichols, along with one of his own students, M. Bettina Zoeller, settled the question in 1989. The celebrated Nichols-Zoeller theorem allowed researchers in Hopf algebras to deal with all finite dimensional Hopf algebras as free modules over any subHopfalebra of itself. As we will see in this paper, this is similar to working with a basis for a vector space, which simplifies many proofs and calculations within this field.

Proving the Nichols-Zoeller theorem will be the main focus of this paper. In order to accomplish this goal we must first go through a rigorous study of quite a few topics in algebra. Among these is a look at algebras, coalgebras, bialgebras, and Hopf algebras in Chapter 2. We will next study modules, comodules and Hopf modules in Chapter 3. After all this is done we will then be able to prove all the important theorems that are neccessary to prove the Nichols-Zoeller Theroem. We will end the paper by giving the proof of one generalization of the theorem that was also proven by Nichols and Zoeller (by this time Richmond). We will also state several other generalizations, as well as mentioning a few open questions that remain in this field. Thoughout this paper it will be assumed that the reader has an understanding of basic abstract algebra and linear algebra.

Chapter 2

Algebras to Hopf Algebras

In the study of Hopf algebras, there are three key structures that we must understand before we can construct a Hopf algebra. In this chapter, we will begin with an algebra, dualize this to a coalgebra, and then discuss a combined structure called a bialgebra. From this point we can define a Hopf algebra based on a bialgebra. Even though the concepts introduced here may be unfamiliar in their full generality, we will see that there are many familiar sets which serve as good examples of each. This will be very helpful in building an understanding of these structures. We end the chapter with a discussion of a dual space to a vector space, and the duality relationships of the structures in this chapter.

2.1 Algebras

We start with an item that is used extensively throughout this field of study. A tensor product is similar to a cross product in that it makes pairs of elements from different sets, but that is where the similarities end. The basic difference is that a cross product is linear in both variables together, while a tensor product is linear in each variable separately.

Definition 2.1. Let V and W be k-vector spaces. The *tensor product* of V and W over k is the set $\{(v, w) \mid v \in V, \text{ and } w \in W\}$ along with the following three relations:

1. $(v_1, w) + (v_2, w) = (v_1 + v_2, w)$

2. $(v, w_1) + (v, w_2) = (v, w_1 + w_2)$

3. $\alpha(v, w) = (\alpha v, w) = (v, \alpha w)$

This tensor product is denoted $V \otimes_k W$.

It can be verified that the new set is also a vector space using the first and second properties for addition and the third property for scalar multiplication. When it is clear from the context what field we are talking about, we often drop k from the notation and simply write $V \otimes W$. There are a few subtleties in this definition that are easily missed. An important result of the third property is that zero does not come from just the combination of zero from each vector space. For any $v \in V$ and $w \in W$ then $(0, w) = (v, 0) = (0, 0)$. Thus zero only requires one side of the pair to be zero, but since we are working over a field there are no zero divisors. Thus we cannot get zero from a single element that did not have zero on one side to begin with. This tells us that zero is actually an equivalence class of elements. It turns out that all elements of the tensors product are also an equivalence class of elements. This is due to the fact that often when we add in this new vector space, items do not combine because properties one and two only allow elements with one side equal to be added. However, a variety of elements can be simplified to the same representative element. Thus a general element of this new vector space is actually a finite formal sum of elements. We denote the equivalence class of elements $[(v, w)]$ as $v \otimes w$. In set notation the new vector space can be rewritten as follows:

$$V \otimes W = \{\sum_i v_i \otimes w_i \mid v_i \in V, \text{ and } w_i \in W\}$$

We are now ready to define our first important structure. There are many ways to define an algebra, but the use of tensor products and commutative diagrams facilitates the dualization of an algebra to a coalgebra in the next section.

Definition 2.2. Let k be a field and A a ring and a vector space over k. Also let $u : k \to A$ (the unit) and $\mu : A \otimes A \to A$ (multiplication) be linear maps. (A, μ, u) is an *algebra* if the following diagrams commute:

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "id\otimes\mu"] \arrow[d, "\mu\otimes id"'] & A\otimes A \arrow[d, "\mu"] \\
A\otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
& A\otimes A \arrow[dd, "\mu"'] & \\
k\otimes A \arrow[ur, "u\oplus id"] \arrow[dr] & & A\otimes k \arrow[ul, "id\oplus u"'] \arrow[dl] \\
& A &
\end{tikzcd}$$

Each of these diagrams can be translated into one of the usual defining expression of an algebra. The first diagram gives us associativity of multiplication, or $(ab)c = a(bc)$ for all $a, b, c \in A$. The second diagram gives us the unit property, or that $(\alpha \cdot 1_A)a = a(\alpha \cdot 1_A)$. Depending on what we are doing, we may switch around from the diagram to the formulas as needed.

We also have a concept of algebra maps. We again use commutative diagrams in the definition in order to simplify dualization in the next section.

Definition 2.3. Let A and B be algebras. $f : A \to B$ is an *algebra map* if the following diagrams commute:

$$\begin{tikzcd}
A\otimes A \arrow[r, "f\otimes f"] \arrow[d, "\mu_A"'] & B\otimes B \arrow[d, "\mu_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "f"] & & B \\
& k \arrow[ul, "u_A"] \arrow[ur, "u_B"'] &
\end{tikzcd}$$

Translating these diagrams into formulas we also get the usual defining characteristics of a map. These expressions are $f(xy) = f(x)f(y)$ for all $x, y \in A$ (i.e. f is multiplicative) and $f(1_A) = 1_B$. This definition will be most important to our concept of a bialgebra later in the chapter.

Some important sets can be shown to be algebras with appropriate maps. Two familiar examples are $k[t]$ (polynomials over a field k) with the usual polynomial multiplication as μ and usual scalar multiplication as u, and $M_n(k)$ ($n \times n$ matrices over a field

k) with usual matrix multiplication as μ and usual scalar multiplication as u. We also look at three not so familiar examples. These are polynomials over $M_n(k)$ ($\mathcal{O}[M_n(k)]$), the group algebra kG, and Sweedlers 4-dimensional Hopf algebra (H_4). We will look at these five examples in relationship to algebras, coalgebras, and bialgebras, with three of them also qualifying as Hopf algebras. Unless otherwise noted, any set we consider is assumed to be a vector space over an appropriate field k as well as a ring. Also for each set A we define the unit $u : k \to A$ as $u(\alpha) = \alpha \cdot 1_A$ for all $\alpha \in k$.

Example 2.4. Consider the set of polynomials over a field k along with scalar multiplication inherited from its vector space structure and usual polynomial multiplication. This set can be defined as $k[t] = \{\sum_i \alpha_i t^i \mid \alpha_i \in k\}$. We know already that this set is a vector space and a ring, and thus only need to show that the two algebra diagrams commute. We begin by taking general elements and chasing them around the first diagram.

$$\begin{tikzcd}
\sum_i \alpha_i t^i \otimes \sum_j \beta_j t^j \otimes \sum_m \gamma_m t^m \arrow[r, "id \otimes \mu"] \arrow[dd, "\mu \otimes id"'] & \sum_i \alpha_i t^i \otimes \sum_{i,j} \beta_j \gamma_m t^{j+m} \arrow[d, "\mu"] \\
& \sum_{i,j,m} \alpha_i(\beta_j \gamma_m) t^{i+j+m} \\
\sum_{i,j} \alpha_i \beta_j t^{i+j} \otimes \sum_m \alpha_m t^m \arrow[r, "\mu"'] & \sum_{i,j,m} (\alpha_i \beta_j) \gamma_m t^{i+j+m}
\end{tikzcd}$$

The two paths yield the same result since multiplication in the field is associative. We can now chase a general element around the second diagram. Note that this diagram has two independent branches, so we treat them seperately.

$$\begin{tikzcd}
& (\beta \cdot 1_{k[t]}) \otimes \sum_i \alpha_i t^i \arrow[dd, "\mu"] & \sum_i \alpha_i t^i \otimes (\beta \cdot 1_{k[t]}) \arrow[dd, "\mu"'] & \\
\beta \otimes \sum_i \alpha_i t^i \arrow[ur, "u \oplus id"] \arrow[ddr] & & & \sum_i \alpha_i t^i \otimes \beta \arrow[ul, "id \oplus u"'] \arrow[ddl] \\
& \sum_i (\beta 1_k[t]) \alpha_i t^i & \sum_i \alpha_i (\beta 1_{k[t]}) t^i & \\
& \sum_i \beta \alpha_i t^i & \sum_i \alpha_i \beta t^i &
\end{tikzcd}$$

On the left side, we start with the fact that $1_{k[t]} = 1_k$. This gives us $\sum_i(\beta 1_{k[t]})\alpha_i t^i = \sum_i(\beta 1_k)\alpha_i t^i = \sum_i(\beta)\alpha_i t^i$ and, using the definition of scalar multiplication, we have our desired equality. Similarly, the right side holds when we include associativity in k. We therefore have that $(k[t], \mu, u)$ is an algebra. Also, since polynomial multiplication is always commutative, then $k[t]$ is a commutative algebra.

Example 2.5. Consider the $n \times n$ matrices over the field k, denoted $M_n(k)$. In terms of the standard basis $\{E_{ij}\}$, where E_{ij} is the matrix with 1 in the ij^{th} position and 0 everywhere else, we get $M_n(k) = \{\sum_{ij} \alpha_{ij} E_{ij} \mid \alpha_i j \in k\}$. When considering $M_n(k)$, it becomes easier to translate the algebra diagrams into equations, and then check for equality. The diagrams translate into the following two equalities.

(1) $A(BC) = (AB)C$

Let $A, B \in M_n(k)$.

$$\begin{aligned}(AB)C &= ([a_{ij}][b_{ij}])[c_{ij}] \\ &= [\textstyle\sum_{k=1}^{n} a_{ik}b_{kj}][c_{ij}] \\ &= [\textstyle\sum_{g=1}^{n}(\sum_{k=1}^{n} a_{ik}b_{kg})c_{gj}] \\ &= [\textstyle\sum_{k=1}^{n} a_{ik}(\sum_{g=1}^{n} b_{kg}c_{gj})] \\ &= [a_{ij}][\textstyle\sum_{g=1}^{n} b_{ig}c_{gj}] \\ &= [a_{ij}]([b_{ij}][c_{ij}]) \\ &= A(BC)\end{aligned}$$

(2) $(\alpha 1_A)A = A(\alpha 1_A)$. Recall that $1_{M_n(k)} = I$.

$$\begin{aligned}(\alpha I)A &= (\alpha[1_{ii}])[a_{ij}] \\ &= [\alpha_{ii}][a_{ij}] \\ &= [\alpha a_{ij}] \\ &= [a_{ij}\alpha] \\ &= [a_{ij}][\alpha_{ii}] \\ &= [a_{ij}](\alpha[1_{ii}]) \\ &= A(\alpha I)\end{aligned}$$

Therefore, $(Mn(k), \mu, u)$ is an algebra. We consider commutativity by using two basis elements E_{ij} and E_{jk} with $i \neq j \neq k$. $E_{ij}E_{jk} = E_{ik}$ but $E_{jk}E_{ij} = 0$. Thus $M_n(k)$ is not commutative since matrix multiplication is not commutative.

Example 2.6. We can combine the properties of the two previous sets to get a new set which is also an algebra. Let $\mathcal{O}[M_n(k)] = k[X_{ij} \mid 1 \leq i, j \leq n]$ be the set of polynomial functions on the n^2 commuting variables X_{ij} where $X_{ij}(E_{kl}) = \delta^i_k \delta^j_l$ and $X_{ij}X_{ij} = X_{ij}^{\delta^i_j}$. The second property eleminates the need for exponents in our notation, so a general element in $\mathcal{O}[Mn(k)]$ would be $\sum \alpha_m \prod X_{ij}$ where the index m indicates which multiplicative combination of the variables we are dealing with. Using polynomial multiplication as in example 2.4, we can show that this is an algebra. We will not need the scalars since μ and u are linear. We begin by checking associativity.

$$\begin{tikzcd}
\sum\prod X_{ij} \otimes \sum\prod X_{ij} \otimes \sum\prod X_{ij} \arrow[r, "id \otimes \mu"] \arrow[dd, "\mu \otimes id"'] & \sum\prod X_{ij} \otimes \sum n^4 \prod X_{ij} \arrow[d, "\mu"] \\
 & \sum n^8 \prod X_{ij} \\
\sum n^4 \prod X_{ij} \otimes \sum\prod X_{ij} \arrow[r, "\mu"'] & \sum n^8 \prod X_{ij}
\end{tikzcd}$$

In each multiplication step we use the fact that the X_{ij}'s commute and that $X_{ij}X_{ij} = X_{ij}^{\delta^i_j}$ to get that for each two terms we multiply we get one of the original terms of the sum back. This gives us the coefficient $(n^2)^2 = n^4$ for each term in the resulting product. Next we chase a general element around the second diagram.

$$\begin{tikzcd}
 & (\beta \cdot 1) \otimes \sum\prod X_{ij} \arrow[dd, "\mu"] & & \sum\prod X_{ij} \otimes (\beta \cdot 1) \arrow[dd, "\mu"'] & \\
\beta \otimes \sum\prod X_{ij} \arrow[ur, "u \oplus id"] \arrow[dddr] & & . & & \sum\prod X_{ij} \otimes \beta \arrow[ul, "id \oplus u"'] \arrow[dddl] \\
 & & & & \\
 & \sum(\beta \cdot 1) \prod X_{ij} & & \sum\prod X_{ij} (\beta \cdot 1) & \\
 & \sum \beta \prod X_{ij} & & \sum \beta \prod X_{ij} &
\end{tikzcd}$$

Using the fact that $1_k = 1_{\mathcal{O}[M_n(k)]}$ and the definition of multiplication we see that the both sides are equal, and thus the second diagram commutes. Therefore $(\mathcal{O}[M_n(k)], \mu, u)$ is an algebra. Just like in example 2.4, $\mathcal{O}[M_n(k)]$ is commutative since polynomial multiplication over commuting variables is always commutative.

Example 2.7. Let k be a field and G be a group. Define a group algebra as the set of formal sums $kG = \{\sum_{i=1}^{n} \alpha_i g_i \mid \alpha_i \in k, g_i \in G\}$. kG is defined as a vector space with basis $\{g \mid g \in G\}$, and is an algebra via multiplication defined over the finite formal sums as $(\sum \alpha_i g_i)(\sum \beta_j h_j) = \sum_{f=g_i h_j} \alpha_i \beta_j f$. We now need to show the two algebra diagrams commute. We begin by chasing general elements around the first diagram.

$$\begin{tikzcd}
\sum_i \alpha_i g_i \otimes \sum_j \beta_j g_j \otimes \sum_m \gamma_m g_m \arrow[r, "id \otimes \mu"] \arrow[dd, "\mu \otimes id"'] & \sum_i \alpha_i t_i \otimes \sum_{h=g_j g_m} \beta_j \gamma_m h \arrow[d, "\mu"] \\
 & \sum_{f=g_i(g_j g_m)} \alpha_i(\beta_j \gamma_m) f \\
\sum_{e=g_i g_j} \alpha_i \beta_j e \otimes \sum_m \gamma_m g_m \arrow[r, "\mu"'] & \sum_{d=(g_i g_j)g_m} (\alpha_i \beta_j)\gamma_m d
\end{tikzcd}$$

By associativity in the field and in the group we have that the two paths are equivalent. Now we chase a general element around the second diagram.

$$\begin{tikzcd}
 & (\beta \cdot 1_{kG} \otimes \sum_i \alpha_i g_i \arrow[dd, "\mu"] & & \sum_i \alpha_i g_i \otimes (\beta \cdot 1_{kG}) \arrow[dd, "\mu"'] & \\
\beta \otimes \sum_i \alpha_i g_i \arrow[ur, "u \oplus id"] \arrow[ddr] & & & & \sum_i \alpha_i g_i \otimes \beta \arrow[ul, "id \oplus u"'] \arrow[ddl] \\
 & \sum_i (\beta \cdot 1_k G)\alpha_i g_i & & \sum_i \alpha_i g_i (\beta \cdot 1_{kG}) & \\
 & \sum_i \beta \alpha_i g_i & & \sum_i \alpha_i \beta g_i &
\end{tikzcd}$$

By definition of multiplication in kG the right side becomes $\sum_i \alpha_i g_i(\beta \cdot 1_{kG}) = \sum_i \alpha_i \beta g_i 1_{kG} = \sum_i \alpha_i \beta g_i$, and so the right side commutes. Similiarly on the left side we have $\sum_i (\beta \cdot 1_{kG})\alpha_i g_i = \sum_i \beta \alpha_i 1_{kG} g_i = \sum_i \beta \alpha_i g_i$, and thus the right side commutes. Therefore (kG, μ, u) is an algebra.

Whether or not kG is commutative depends on G itself. In fact kG is commutative iff G is an abelian group.

One reason this is an important set to use while building up the notation is that the Nichols-Zoeller theorem was first shown to be true in the specific case kG [NZ89b].

Another reason will be seen in the next section when we describe what we call the group-like elements of a Hopf algebra.

Example 2.8. Sweedler first described the unique 4-dimensional Hopf algebra H_4 [Swe69]. It is the smallest non-commutative, non-cocommutative Hopf algebra. It is defined as $H_4 = k\{1, g, x, gx \mid g^2 = 1, x^2 = 0, xg = -gx\}$ for any field k such that char$k \neq 2$. In terms of this notation, a general element of this set is $\alpha_1 1 + \alpha_g g + \alpha_x x + \alpha_g xgx$, or to simplify $\sum_{h\in\{1,x,g,gx\}} \alpha_h h$. This allows us to define multiplication as follows:

$$\left(\sum_{h\in\{1,x,g,gx\}} \alpha_h h\right)\left(\sum_{f\in\{1,x,g,gx\}} \alpha_f f\right) = \sum_{y=hf} \alpha_h \alpha f y$$

We begin here by proving H_4 is an algebra. Chasing the second form of the general elements around the first diagram we get:

$$\begin{tikzcd}
\displaystyle\sum_{a\in\{1,x,g,gx\}} \alpha_a a \otimes \sum_{b\in\{1,x,g,gx\}} \alpha_b b \otimes \sum_{c\in\{1,x,g,gx\}} \alpha_c c \arrow[r, "id \otimes \mu"] \arrow[dd, "\mu \otimes id"'] & \displaystyle\sum_{a\in\{1,x,g,gx\}} \alpha_a a \otimes \sum_{e=bc} \alpha_b \alpha_c e \arrow[d, "\mu"] \\
 & \displaystyle\sum_{h=a(bc)} \alpha_a(\alpha_b\alpha_c)h \\
\displaystyle\sum_{d=ab} \alpha_a\alpha_b d \otimes \sum_{c\in\{1,x,g,gx\}} \alpha_c c \arrow[r, "\mu"'] & \displaystyle\sum_{f=(ab)c} (\alpha_a\alpha_b)\alpha_c f
\end{tikzcd}$$

By associativity of k we know that the scalars are equal, so we only will need to check associativity of the sum of the basis elements in H_4.

$[(1 + g + x + gx)(1 + g + x + gx)](1 + g + x + gx)$
$= (1 + g + x + gx + g + 1 + gx + x + x - gx + gx - x)(1 + g + x + gx)$
$= (2 + 2g + 2x + 2gx)(1 + g + x + gx)$
$= 2(1 + g + x + gx)(1 + g + x + gx)$

$(1 + g + x + gx)[(1 + g + x + gx)(1 + g + x + gx)]$
$= (1 + g + x + gx)(1 + g + x + gx + g + 1 + gx + x + x - gx + gx - x)$
$= (1 + g + x + gx)(2 + 2g + 2x + 2gx)$
$= 2(1 + g + x + gx)(1 + g + x + gx)$

Since the end results of the two equations are equal, then H_4 is associative. Now we chase the general element around the second diagram, beginning with the left branch. Here we use the first form of the general element.

$$\begin{tikzcd}
 & (\beta\cdot 1)\otimes(\alpha_1 1+\alpha_g g+\alpha_x x+\alpha_{gx} gx) \arrow[d, "\mu"] \\
\beta\otimes(\alpha_1 1+\alpha_g g+\alpha_x x+\alpha_{gx} gx) \arrow[ur, "u\otimes id"] \arrow[dr] & (\beta\cdot 1)(\alpha_1 1+\alpha_g g+\alpha_x x+\alpha_{gx} gx) \\
 & \beta(\alpha_1 1+\alpha_g g+\alpha_x x+\alpha_{gx} gx)
\end{tikzcd}$$

and by definition of multiplication, we have

$(\beta\cdot 1)(\alpha_1 1+\alpha_g g+\alpha_x x+\alpha_{gx} gx)$

$=\beta\alpha_1 1+\beta\alpha_g g+\beta\alpha_x x+\beta\alpha_{gx} gx$

$=\beta(\alpha_1 1+\alpha_g g+\alpha_x x+\alpha_{gx} gx)$

Which means the left side commutes. Similiarly, on the right side we have:

$$\begin{tikzcd}
(\alpha_1 1+\alpha_g g+\alpha_x x+\alpha_{gx} gx)\otimes(\beta\cdot 1) \arrow[d, "\mu"'] & \\
(\alpha_1 1+\alpha_g g+\alpha_x x+\alpha_{gx} gx)(\beta\cdot 1) & (\alpha_1 1+\alpha_g g+\alpha_x x+\alpha_{gx} gx)\otimes\beta \arrow[ul, "id\otimes u"'] \arrow[dl] \\
\beta(\alpha_1 1+\alpha_g g+\alpha_x x+\alpha_{gx} gx) &
\end{tikzcd}$$

Again by definition of multiplication in H4, we have

$(\alpha_1 1+\alpha_g g+\alpha_x x+\alpha_{gx} gx)(\beta\cdot 1)$

$=\alpha_1\beta 1+\alpha_g\beta g+\alpha_x\beta x+\alpha_{gx}\beta gx$

$=\beta\alpha_1 1+\beta\alpha_g g+\beta\alpha_x x+\beta\alpha_{gx} gx$

$=\beta(\alpha_1 1+\alpha_g g+\alpha_x x+\alpha_{gx} gx)$

Thus, both sides of the second diagram commute. Therefore, (H_4,μ,u) is an algebra. H_4 is not commutative since $gx=-xg\neq xg$, unless char$k=2$.

2.2 Coalgebras

The next important step we take is to dualize the idea of an algebra, thus defining what is called a coalgebra. Here we will see the usefulness of the diagrams we used to define an algebra as we define this new structure. Also at this stage tensor products become vitally important, since comultiplication Δ"reverses" multiplication and sends a general element to a tensor product of elements from the same set.

Definition 2.9. A *coalgebra* is a triple (C, Δ, ε) with C a vector space and linear maps $\Delta : C \to C \otimes C$ (comultiplication) and $\varepsilon : C \to k$ (the counit) such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes id"] \\ C \otimes C \arrow[r, "id \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd} \quad . \qquad \begin{tikzcd} & C \otimes C \arrow[dl, "\varepsilon \oplus id"'] \arrow[dr, "id \oplus \varepsilon"] & \\ k \otimes C & & C \otimes k \\ & C \arrow[ul] \arrow[ur] \arrow[uu, "\Delta"] & \end{tikzcd}$$

These diagrams can be translated into formulas. The first diagram is coassociativity of Δ. In terms of functions we have that $\Delta \otimes id = id \otimes \Delta$. This gives us in terms of elements that if $\Delta(c) = \sum_{i=1}^{n} d_i \otimes e_j$, $\Delta(d_i) = \sum_{j=1}^{m} f_{ij} \otimes g_{ij}$, and $\Delta(e_i) = \sum_{k=1}^{l} h_{ik} \otimes p_{ik}$ then

$$\sum_{i,j} f_{ij} \otimes g_{ij} \otimes e_i = \sum_{i,k} d_i \otimes h_{ik} \otimes p_{ik}.$$

Unfortunately, this notation can become cumbersome when Δ is applied multiple times. Sweedler first introduced "sigma notation" for the application of Δ to address this problem [Swe69]. In this notation, we denote $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$, where $c_{(1)}$ and $c_{(2)}$ are symbolic, and do not represent specific elements in C. Since Δ is coassociative, when we apply Δ again, it does not matter which of the two elements we apply it to. Thus we get $\sum c_{(1)_{(1)}} \otimes c_{(1)_{(2)}} \otimes c_{(2)} = \sum c_{(1)} \otimes c_{(2)_{(1)}} \otimes c_{(2)_{(2)}}$, and so we may abbreviate and write $\sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$. When using this notation, we often even drop the parentheses and write that $\Delta(c) = \sum c_1 \otimes c_2$. This notation is useful when we are dealing with a general coalgebra. When dealing with a specific coalgebra for which Δ is specifically defined, we do not necessarily need this notation. This notation also allows us to write the formula for the second diagram as $\sum \varepsilon(c_1)c_2 = c = \sum \varepsilon(c_2)c_1$.

We also need to dualize the idea of an algebra map. This is done in the same manner as with a coalgebra, by reversing the arrows of the original definition.

Definition 2.10. Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras. $f : C \to D$ is a coalgebra map if the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f \otimes f"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\ & k & \end{tikzcd}$$

We can also translate these into formulas as we did with the algebra map. The first diagram gives that $\sum f(c_1) \otimes f(c_2) = \sum (f(c))_1 \otimes (f(c))_2$. The second diagram gives that $\varepsilon_C(c) = \varepsilon_D(f(c))$. Just as before, We will be examining our five example sets ($k[t]$, $M_n(k)$, $\mathcal{O}[M_n(k)]$, kG, and H_4) in the context of a coalgebra. Each of these will be shown to be a coalgebra with the appropriate maps.

Example 2.11. Consider $k[t]$ along with the maps Δ and ε defined on basis elements of the standard basis and extended linearly and multiplicatively as follows:

$$\Delta : k[t] \to k[t] \otimes k[t] \quad \Delta(t) = t \otimes 1 + 1 \otimes t$$
$$\varepsilon : k[t] \to k \quad \varepsilon(t) = 0$$

Since Δ is defined on t and extended linearly and multiplicatively, we only need to check coassociativity for the basis element t.

$$\begin{tikzcd}
t \arrow[r, "\Delta"] \arrow[dd, "\Delta"'] & t\otimes 1+1\otimes t \arrow[d, "\Delta\otimes id"] \\
& (t\otimes 1+1\otimes t)\otimes 1+1\otimes 1\otimes t \\
t\otimes 1+1\otimes t \arrow[r, "id\otimes\Delta"'] & t\otimes 1\otimes 1+1\otimes(t\otimes 1+1\otimes t)
\end{tikzcd}$$

We can see that if we expand each result we get $1\otimes 1\otimes t+t\otimes 1\otimes 1+1\otimes t\otimes 1$. Thus the first diagram commutes. Next, we chase a general element around the second diagram.

$$\begin{tikzcd}
& \sum \alpha_i \left(\binom{i}{k} t^{i-k}t^k\right) \arrow[dl, "\varepsilon\otimes id"'] \arrow[dr, "id\otimes\varepsilon"] & \\
\sum \alpha_i \left(\binom{i}{k}\varepsilon(t^{i-k})\otimes t^k\right) & & \sum \alpha_i \left(\binom{i}{k} t^{i-k}\otimes\varepsilon(t^k)\right) \\
1\otimes\sum \alpha_i t^i & & \sum \alpha_i t^i\otimes 1 \\
& \sum \alpha_i t^i \arrow[uuu, "\Delta"] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

On the left we use the scalar property of the tensor product and the fact that $\varepsilon(t)=0$ to reduce the sum. Also, since $\varepsilon(ti-k)=0$ except for when $i=k$, then only the case when $i=k$ is not zero. This reduces the sum to the following.

$$\sum \alpha_i \left(\binom{i}{k}\varepsilon(t^{i-k})\otimes t^k\right)=\sum\left(\varepsilon(t^{i-k})\otimes\alpha_i\binom{i}{k}t^k\right)=\sum 1\otimes\alpha_i t^i=1\otimes\sum\alpha_i t^i$$

Thus the left side is equivalent. We can show by a similar argument that the right side is also equivalent, and so the second diagram commutes. Therefore $(k[t],\Delta,\varepsilon)$ is a coalgebra. In general, if C is any coalgebra, and $c\in C$ such that $\Delta(c)=c\otimes 1+1\otimes c$ then c is called a primitive element. The set of primitive elements in C is denoted $P(C)$. More generally, we can replace each occurrence of 1 with any other group-like element as defined later in example 2.14. In this case $\Delta(c)=c\otimes g+h\otimes c$ for some $g,h\in G(C)$, and c is called quasi-primitive. We denote the set of all such c as $P_{g,h}(C)$.

Example 2.12. Consider $M_n(k)$ along with the maps Δ and ε which are defined as follows on the standard basis elements and extended linearly and multiplicatively:

$$\Delta : M_n(k) \to M_n(k) \otimes M_n(k) \quad \Delta(E_{ij}) = \sum_{k=1}^{j} E_{ik} \otimes E_{kj}$$
$$\varepsilon : M_n(k) \to k \quad \varepsilon(E_{ij}) = \delta_j^i$$

We know that $M_n(k)$ is a vector space, and since we have defined Δ and ε on basis elements extended linearly, they are linear on general elements. Thus, we only need to show coassociativity and the counit property, i.e. that the coalgebra diagrams commute. Let $A \in M_n(k)$. In terms of the standard basis $A = \sum_{i,j} \alpha_{ij} E_{ij}$. Chasing this element around the first diagram we have:

$$\begin{tikzcd}
\sum\limits_{i,j} \alpha_{ij} E_{ij} \arrow[rr, "\Delta"] \arrow[dd, "\Delta"'] & & \sum\limits_{i,j,k} \alpha_{ij} E_{ik} \otimes E_{kj} \arrow[d, "\Delta \otimes id"] \\
& & \sum\limits_{i,j,k,p} \alpha_{ij} E_{ip} \otimes E_{pk} \otimes E_{kj} \\
\sum\limits_{i,j,k} \alpha_{ij} E_{ik} \otimes E_{kj} \arrow[rr, "id \otimes \Delta"'] & & \sum\limits_{i,j,k,m} \alpha_{ij} E_{ik} \otimes E_{km} \otimes E_{mj}
\end{tikzcd}$$

Now, $\sum_{i,j,k,p} \alpha_{ij} E_{ip} \otimes E_{pk} \otimes E_{kj} = \sum_{i,j,k,m} \alpha_{ij} E_{ik} \otimes E_{km} \otimes E_{mj}$, since p and m run over the same values. So the first diagram commutes, and Mn(k) is cocommutative. We chase the same general element around the second diagram to verify the counit property.

$$\begin{tikzcd}
& \sum\limits_{i,j,k} \alpha_{ij} E_{ik} \otimes E_{kj} \arrow[dl, "\varepsilon \otimes id"'] \arrow[dr, "id \otimes \varepsilon"] & \\
\sum\limits_{i,j,k} \alpha_{ij} \varepsilon(E_{ik}) \otimes E_{kj} & & \sum\limits_{i,j,k} \alpha_{ij} E_{ik} \otimes \varepsilon(E_{kj}) \\
1 \otimes \sum\limits_{i,j} \alpha_{ij} E_{ij} & & \sum\limits_{i,j} \alpha_{ij} E_{ij} \otimes 1 \\
& \sum\limits_{i,j} \alpha_{ij} E_{ij} \arrow[ul, "\cong"] \arrow[ur, "\cong"'] \arrow[uuu, "\Delta"] &
\end{tikzcd}$$

On the left hand side we have $\sum_{i,j,k} \varepsilon(\alpha_{ij} E_{ik}) \otimes E_{kj}$. First, since ε is linear we can pull our α_{ij} out. This leaves us at $\sum_{i,j,k} \alpha_{ij} \varepsilon(E_{ik}) \otimes E_{kj}$. Next, by definition $\varepsilon(E_{ik}) = 0$

when $i \neq k$, and thus all these terms in the sum drop out. What is left is when $i = k$, and in this case $\varepsilon(E_{ik}) = 1$. This (along with the scalar property of a tensor product) reduces the sum to $1 \otimes \sum_{i,j} \alpha_{ij} E_{ij}$, and thus the left side of the diagram commutes. By a similar argument on the right hand side, only the case when $k = j$ survives, leaving $\sum_{i,j} \alpha_{ij} E_{ij} \otimes 1$, and thus the right hand side of the diagram commutes. Since both diagrams commute, $(M_n(k), \Delta, \varepsilon)$ is a coalgebra.

Example 2.13. When we proved $(\mathcal{O}[M_n(k)], \mu, u)$ was an algebra in example 2.6, we used the same maps as with $k[t]$. For its coalgebra structure we will use Δ and ε similar to $M_n(k)$. Specifically we define Δ and ε on the basis elements (extended linearly and multiplicatively) as follows:

$$\Delta : \mathcal{O}[M_n(k)] \to \mathcal{O}[M_n(k)] \otimes \mathcal{O}[M_n(k)] \quad \Delta(X_{ij}) = \sum_{k=1}^{j} X_{ik} \otimes X_{kj}$$
$$\varepsilon : \mathcal{O}[M_n(k)] \to k \quad \varepsilon(X_{ij}) = \delta_j^i$$

We start by chasing a base element around the first diagram to check coassociativity.

$$\begin{tikzcd}
X_{ij} \arrow[r, "\Delta"] \arrow[dd, "\Delta"'] & \sum_k X_{ik} \otimes X_{kj} \arrow[d, "\Delta \otimes id"] \\
 & \sum_k \left(\sum_p X_{ip} \otimes X_{pk} \right) \otimes X_{kj} \\
\sum_k X_{ik} \otimes X_{kj} \arrow[r, "id \otimes \Delta"'] & \sum_k X_{ik} \otimes \left(\sum_r X_{km} \otimes X_{mj} \right)
\end{tikzcd}$$

Checking the results we see that $\sum_k \left(\sum_p X_{ip} \otimes X_{pk} \right) \otimes X_{kj} = \sum_k X_{ik} \otimes (\sum_r X_{km} \otimes X_{mj})$, since p and r run over the same values. Thus $\mathcal{O}[M_n(k)]$ is coassociative. Chasing the same basis element around the second diagram we have:

$$\begin{tikzcd}
 & \sum_k X_{ik} \otimes X_{kj} \arrow[dl, "\varepsilon \oplus id"'] \arrow[dr, "id \oplus \varepsilon"] & \\
\sum_k \varepsilon(X_{ik}) \otimes X_{kj} & & \sum_k X_{ik} \otimes \varepsilon(X_{kj}) \\
1 \otimes X_{ij} & & X_{ij} \otimes 1 \\
 & X_{ij} \arrow[ul] \arrow[uuu, "\Delta"'] \arrow[ur] &
\end{tikzcd}$$

On the top left hand side we have $\sum_k \varepsilon(X_{ik}) \otimes X_{kj}$. By definition $\varepsilon(X_{ik}) = 0$ when $i \neq k$, and thus all these terms in the sum drop out. What is left is when $i = k$, and in this case $\varepsilon(X_{im}) = 1$. Because of this the sum reduces to $1 \otimes X_{ij}$, and thus the left side of the diagram commutes. By a similar argument on the right hand side, only the case when $k = j$ survives, leaving $X_{ij} \otimes 1$, and thus the right hand side of the diagram commutes. Since both diagrams commute, $(\mathcal{O}[M_n(k)], \Delta, \varepsilon)$ is a coalgebra.

Example 2.14. For kG we define the maps Δ and ε on the basis elements as follows and extend linearly and multiplicatively.

$$\Delta : kG \to kG \otimes kG \quad \Delta(g) = g \otimes g$$
$$\varepsilon : kG \to k \quad \varepsilon(g) = 1$$

Again, we already are given that kG is a vector space. We can then begin by checking that the first diagram commutes by chasing a basis element around the diagram.

$$\begin{tikzcd}
g \arrow[r, "\Delta"] \arrow[dd, "\Delta"'] & g \otimes g \arrow[d, "\Delta \otimes id"] \\
 & (g \otimes g) \otimes g \\
g \otimes g \arrow[r, "id \otimes \Delta"'] & g \otimes (g \otimes g)
\end{tikzcd}$$

It is easy to see that these two paths are equal (since all three elements of the tensor product are the same), and thus the first diagram commutes. We then chase a general element around the second diagram.

$$
\begin{tikzcd}
 & g \otimes g \arrow[dl, "\varepsilon \oplus id"'] \arrow[dr, "id \oplus \varepsilon"] & \\
\varepsilon(g) \otimes g & & g \otimes \varepsilon(g) \\
1 \otimes g & & g \otimes 1 \\
 & g \arrow[ul] \arrow[ur] \arrow[uuu, "\Delta"'] &
\end{tikzcd}
$$

On each side we need only the fact that $\varepsilon(g) = 1$ for all $g \in G$ to see that this diagram commutes. Therefore, $(kG, \Delta, \varepsilon)$ is a coalgebra. In general, if C is any coalgebra, then if $\Delta(c) = c \otimes c$ we call $c \in C$ a group-like element. The set of group-like elements in C is denoted $G(C)$. For kG the set of group-like elements is $G(C) = G$.

Example 2.15. For H_4 we define our maps as$\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes 1 + g \otimes x$, $\varepsilon(g) = 1$, and $\varepsilon(x) = 0$. Again we extend linearly and multiplicatively. Note that comultiplication of x generalizes comultiplication for a quasi-primitive element. Chasing a the two basis elements g and x around the first diagram we check that it commutes.

$$
\begin{tikzcd}
g \arrow[r, "\Delta"] \arrow[dd, "\Delta"'] & g \otimes g \arrow[d, "\Delta \otimes id"] \\
 & (g \otimes g) \otimes g \\
g \otimes g \arrow[r, "id \otimes \Delta"'] & g \otimes (g \otimes g)
\end{tikzcd}
$$

As in the previous example we need only the fact that $\varepsilon(g) = 1$ for all $g \in G$ to see that this diagram commutes. We next check coassociativity for x.

$$\begin{tikzcd}
x \arrow[rr, "\Delta"] \arrow[dd, "\Delta"'] & & x \otimes 1 + g \otimes x \arrow[d, "\Delta \otimes id"] \\
& & (x \otimes 1 + g \otimes x) \otimes 1 + (g \otimes g) \otimes x \\
x \otimes 1 + g \otimes x \arrow[rr, "id \otimes \Delta"'] & & x \otimes (1 \otimes 1) + g \otimes (x \otimes 1 + g \otimes x)
\end{tikzcd}$$

We see that if we expand each tensor product using the additive properties of a tensor product we get $x \otimes 1 \otimes 1 + g \otimes x \otimes 1 + g \otimes g \otimes x$ from both paths. Since Δ is linear and multiplicative, then coassociativity holds on a general element.

For the second diagram we only need to check the basis elements g, and x (since both maps are linear and multiplicative).

$$\begin{tikzcd}
& g \otimes g \arrow[dl, "\varepsilon \oplus id"'] \arrow[dr, "id \oplus \varepsilon"] & \\
\varepsilon(g) \otimes g & & g \otimes \varepsilon(g) \\
1 \otimes g & & g \otimes 1 \\
& g \arrow[ul] \arrow[ur] \arrow[uuu, "\Delta"'] &
\end{tikzcd}$$

For the above diagrams we have that it commutes since $\varepsilon(g) = 1$. We then check the second diagram for $x \in H_4$.

$$\begin{tikzcd}
 & x \otimes 1 + g \otimes x \arrow[dl, "\varepsilon \oplus id"'] \arrow[dr, "id \oplus \varepsilon"] & \\
\varepsilon(x) \otimes 1 + \varepsilon(g) \otimes x & & x \otimes \varepsilon(1) + g \otimes \varepsilon(x) \\
1 \otimes x & & x \otimes 1 \\
 & x \arrow[ul] \arrow[ur] \arrow[uuu, "\Delta"'] &
\end{tikzcd}$$

Using the fact that $\varepsilon(x) = 0$, $\varepsilon(1) = 1$, and $\varepsilon(g) = 1$ we see that this diagram commutes. Since both Δ and ε are both linear and multiplicative, we have that the second diagram commutes for a general element. Therefore $(H_4, \Delta, \varepsilon)$ is a coalgebra.

2.3 Bialgebras

We have seen so far that a lot of sets qualify as both an algebra and a coalgebra. The next step is to unify these two structures with a compatibility condition. This helps us gain an understanding of how the two previous structures work together, and provides a powerful mechanism for working with these sets.

Definition 2.16. Let B be a vector space over k such that (B, μ, u) is an algebra and (B, Δ, ε) is a coalgebra. $(B, \mu, u, \Delta, \varepsilon)$ is a *bialgebra* if either of the following (equivalent) conditions holds:

1. Δ and ε are algebra maps
2. μ and u are coalgebra maps.

It would be prudent for us to prove that these two conditions are in fact equivalent.

Proof. This proof could be worked through formulaically, but we would find this to be very tedious, and difficult to understand. Fortunately we have another tool, the commutative diagrams. By using the diagrams from the definition of an algebra map (definition 2.3) and the definition of a coalgebra map (definition 2.10) the proof becomes easier and clearer.

Let $a, b \in B$. We will be proving both directions of the equivalence of the statements simultaneously. We will start by looking at what the statement that Δ is an algebra map will give us. The diagram for the fact that Δ is multiplicative gives us the following.

$$\begin{tikzcd}
a \otimes b \arrow[r, "\delta \otimes \Delta"] \arrow[dd, "\mu"'] & \sum (a_1 \otimes a_2) \otimes (b_1 \otimes b_2) \arrow[d, "\mu_{B \otimes B}"] \\
 & \sum a_1 b_1 \otimes a_2 b_2 \\
ab \arrow[r, "\Delta"'] & \sum (ab)_1 \otimes (ab)_2
\end{tikzcd}$$

Thus we have that $\sum a_1 b_1 \otimes a_2 b_2 = \sum (ab)_1 \otimes (ab)_2$ whenever Δ is an algebra map. Now looking at the statement that μ is a coalgebra map, we get an interesting result.

$$\begin{tikzcd}
a \otimes b \arrow[rr, "\mu"] \arrow[dd, "\Delta_{B\otimes B}"'] & & ab \arrow[d, "\Delta"] \\
& & \sum (ab)_1 \otimes (ab)_2 \\
\sum (a_1 \otimes b_1) \otimes (a_2 \otimes b_2) \arrow[rr, "\mu \otimes \mu"'] & & \sum a_1 b_1 \otimes a_2 b_2
\end{tikzcd}$$

We then have here that $\sum a_1b_1 \otimes a_2b_2 = \sum (ab)_1 \otimes (ab)_2$ whenever μ is a coalgebra map. This is exactly what we got from the last diagram. This tells us the first diagram commutes if and only if the second diagram commutes. For each of the other statements above, as we check the diagrams we will find a similar if and only if condition, which will then give us that statement 1) and 2) are equivalent. $\square$

Before we move to our example sets we will prove one more proposition about the kernel of the counit. We will use this in two of our examples below.

Proposition 2.17. *Let (C, Δ, ε) be a coalgebra that is also a ring. If ε is an algebra map, then $ker(\varepsilon)$ is an ideal of C.*

Proof. Let (C, Δ, ε) be a coalgebra that is also a ring, and let ε be an algebra map. By definition of an algebra map, if $c, d \in C$ then $\varepsilon(cd) = \varepsilon(c)\varepsilon(d)$ (i.e. ε is multiplicative). Consider $ker(\varepsilon) = \{c \in C \mid \varepsilon(c) = 0\}$. We show that $ker(\varepsilon)$ is an ideal of C.
Let $x \in ker(\varepsilon)$ and $c \in C$. Then

$$\varepsilon(xc) = \varepsilon(x)\varepsilon(c) = 0\varepsilon(c) = 0$$

$$\varepsilon(cx) = \varepsilon(c)\varepsilon(x) = \varepsilon(c)0 = 0$$

Thus $xc, cx \in ker(\varepsilon)$, and so $ker(\varepsilon)$ is an ideal of C. $\square$

We now investigate our five examples with our new definition. At this point we will lose one of our sets.

Example 2.18. For $k[t]$ we use the way we defined Δ and ε to show it is a bialgebra. In example 2.11, we defined Δ and ε on basis elements and extended linearly and multiplicatively, Thus both Δ and ε are multiplicative. A consequence of these maps being multiplicative is that $\Delta(1) = 1$ and $\langle\varepsilon, 1\rangle = 1$ (since $\Delta(t) = \Delta(t \cdot 1) = \Delta(t)\Delta(1)$, and similarly for ε). These combined give us that Δ and ε are algebra maps, and therefore $k[t]$ is a bialgebra.

Example 2.19. With $M_n(k)$ we quickly run into a problem. The maps Δ and ε we defined in example 2.12 are not algebra maps, since neither of them preserves 1 (the $n \times n$ identity matrix):

$$\begin{aligned} \Delta(I) &= \Delta(\textstyle\sum_{i=1}^{n} E_{ii}) \\ &= \textstyle\sum_{i=1}^{n} \Delta(E_{ii}) \\ &= \textstyle\sum_{i=1}^{n} \sum_{k=1}^{n} (E_{ik} \otimes E_{ki}) \neq I \otimes I \end{aligned}$$

$$\begin{aligned} \varepsilon(I) &= \varepsilon(\textstyle\sum_{i=1}^{n} E_{ii}) \\ &= \textstyle\sum_{i=1}^{n} \varepsilon(E_{ii}) \\ &= \textstyle\sum_{i=1}^{n} 1 \\ &= n \neq 1 \end{aligned}$$

This might lead us to believe we have simply used the wrong maps when showing $M_n(k)$ is a coalgebra, but it turns out the problem is in the fact that $M_n(k)$ is a simple ring (a ring with only trivial ideals). Because of this $M_n(k)$ can never be a bialgebra. We prove this by contradiction.

Proof. Assume there exists $\varepsilon : M_n(k) \to k$ such that ε is an algebra map. Then by proposition 2.17 we have that $ker(\varepsilon)$ is an ideal of $M_n(k)$. We also know that $M_n(k)$ is a simple ring [FD93]. This tells us that $\{0\}$ and $M_n(k)$ are the only ideals of $M_n(k)$. We check whether $ker(\varepsilon)$ is equal to either of these ideals.

(1)Assume $ker(\varepsilon) = \{0\}$.

$0 \in ker(\varepsilon)$

$\Rightarrow E_{ii}E_{jj} = 0 \forall i \neq j$

$\Rightarrow \varepsilon(EiiEjj) = 0$

$\Rightarrow \varepsilon(Eii)\varepsilon(Ejj) = 0$ (since ε is multiplicative)

$\Rightarrow$ either $E_{ii} \in ker(\varepsilon)$ or $E_{jj} \in ker(\varepsilon)$ (a contradiction)

(2) $\varepsilon(1) = 1$ (Since ε is an algebra map)

$\Rightarrow 1 \notin ker(\varepsilon)$

$\Rightarrow ker(\varepsilon) \neq M_n(k)$

So by (1) and (2) $ker(\varepsilon)$ is not an ideal of $M_n(k)$ (a contradiction). This implies that $\varepsilon : M_n(k) \to k$ can never be an algebra map, and thus $M_n(k)$ can never be a bialgebra. $\square$

Example 2.20. Although the last example showed that $M_n(k)$ can never be a bialgebra, there is something different about the set $\mathcal{O}[M_n(k)]$ that will allow us to circumvent the problem. The differences we find are that $1_{M_n(k)} = I$ while $1_{\mathcal{O}[M_n(k)]} = 1_k$, and that $\mathcal{O}[M_n(k)]$ is not a simple ring. This allows us to prove that Δ and ε are algebra maps. First since we defined Δ and ε on basis elements and extended linearly and multiplicatively, then Δ and ε are multiplicative on a general element. From the fact that Δ is multiplicative, we know that

$$(X_{11}\otimes X_{11})(1\otimes 1) = (X_{11}\otimes X_{11}) = \Delta(X_{11}) = \Delta(X_{11}\cdot 1) = \Delta(X_{11})\Delta(1) = (X_{11}\otimes X_{11})\Delta(1)$$

which gives us that $\Delta(1) = 1 \otimes 1$. Similarly we know that $\langle \varepsilon, 1 \rangle = 1$. Thus Δ and ε are algebra maps, and $\mathcal{O}[M_n(k)]$ is a bialgebra.

Example 2.21. Just as for $k[t]$, we use the way we defined Δ and ε to prove kG is a bialgebra. We defined both Δ and ε on basis elements and extended linearly and multiplicatively, thus they are both multiplicative. To show that the unit property holds for Δ and ε we use the fact that $1_{kG} = 1_G$. This gives us that $\Delta(1) = 1 \otimes 1$ and $\langle \varepsilon, 1 \rangle = 1$ since $1 \in G$. Therefore Δ and ε are algebra maps, and kG is a bialgebra.

Example 2.22. For H_4 we have a similar situation in that Δ and ε are defined on basis elements and extended linearly and multiplicatively. Thus Δ and ε are multiplicative. Also since $1 \in G(H_4)$, then $\Delta(1) = 1 \otimes 1$ and $\langle \varepsilon, 1 \rangle = 1$, so Δ and ε are algebra maps, and H_4 is a bialgebra.

2.4 Hopf Algebras

Our last step is to add an additional property to a bialgebra to make it a Hopf algebra. This additional property is one that creates an psuedo "inverse" for the set. This is important because often Hopf algebra elements do not have usual inverses.

Definition 2.23. Let $(H, \mu, u, \Delta, \varepsilon)$ be a bialgebra. if there exists $S \in Hom_k(H, H)$ such that $\sum_h S(h_1)h_2 = \varepsilon(h)1_H = \sum_h h_1 S(h_2)$ for all $h \in H$, then S is called the *antipode* of H, and H is called a *Hopf algebra.*

We now prove a few important properties of the antipode.

Theorem 2.24. *If H is a Hopf algebra with antipode S, then the following three properties hold.*

1. $S(hk) = S(k)S(h)$ *for all* $h, k \in H$ *and* $S(1) = 1$ *(i.e. S is antimultiplicative).*
2. $\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta$ *and* $\varepsilon \circ S = \varepsilon$ *(i.e. S is anticocommutative).*
3. $\sum h_2 S^{-1}(h_1) = \varepsilon(h) = \sum S^{-1}(h_2)h_1$.

Proof. We follow the original proof in [Swe69] which is updated in [Abe80] and [Kas95]. 1) Define elements $P, N \in Hom_k(H \otimes H, H)$ as follows. For $k, h \in H$ let $P(h \otimes k) = S(hk)$ and $N(h \otimes k) = S(k)S(h)$. If μ is multiplication, u the unit, and ε the counit in H, then if we can show $P * \mu = u\varepsilon = \mu * N$ then we have that $P = N$.

$$\begin{aligned}
(P * \mu)(h \otimes k) &= \sum P((h \otimes k)_1)\mu((h \otimes k)_2) \\
&= \sum P(h_1 \otimes k_1)\mu(h_2 \otimes k_2) \\
&= \sum S(h_1 k_1)h_2 k_2 \\
&= \sum S((hk)_1)(hk)_2 \\
&= \varepsilon(hk) \\
&= \varepsilon(h)\varepsilon(k)
\end{aligned}$$

$$\begin{aligned}
(\mu * N)(h \otimes k) &= \sum \mu((h \otimes k)_1)N((h \otimes k)_2) \\
&= \sum \mu(h_1 \otimes k_1)N(h_2 \otimes k_2) \\
&= \sum h_1 k_1 S(k_2)S(h_2) \\
&= \sum h_1 \varepsilon(k)S(h_2) \\
&= \varepsilon(h)\varepsilon(k)
\end{aligned}$$

Thus $S(hk) = S(k)S(h)$. Also since $\varepsilon(1) = 1$ and $\Delta(1) = 1 \otimes 1$ then:

$$1 = \varepsilon(1) = (id * S)(1) = S(1)$$

Thus $S(1) = 1$, and S is antimultiplicative.

2) Similar to above, we define elements $V, R \in Hom_k(H, H \otimes H)$ as $V = \tau \circ (S \otimes S) \circ \Delta$ and $R = \Delta \circ S$. Again if we can show $R \circ \Delta = ue = \Delta \circ V$ then we have that $R = V$.

$$\begin{aligned}
(R \circ \Delta)(h) &= \sum \Delta \circ S(h_1)\Delta(h_2) \\
&= \Delta(\sum S(h_1)h_2) \\
&= \Delta \circ (u \circ e(h)) \\
&= u_{H\otimes H} \circ \varepsilon(h)
\end{aligned}$$

$$\begin{aligned}
(\Delta \circ V)(h) &= \sum (h_1 \otimes h2)(S(h_4) \otimes S(h_3)) \\
&= \sum h_1 S(h_4) \otimes h_2 S(h_3) \\
&= \sum h_1 S(h_3) \otimes \varepsilon(h_2) \\
&= \sum h_1 S(h_3)\varepsilon(h_2) \otimes 1 \\
&= \sum h_1 S(h_2) \otimes 1 \\
&= \varepsilon(h) \otimes 1 \\
&= u_{H\otimes H} \otimes \varepsilon(h)
\end{aligned}$$

Thus $\tau \circ (S \otimes S) \circ \Delta = \Delta \circ S$. Also $\varepsilon \circ S = \varepsilon$ since

$$\varepsilon(h) = \varepsilon(h)\varepsilon(1) = \varepsilon \circ u \circ \varepsilon(h) = \sum \varepsilon(S(h_1)\varepsilon(h_2)) = \varepsilon \circ S(h)$$

Therefore S is anticocommutative.

3) Beginning with the antipode property of S we have that $\sum t_1 S(t_2) = \varepsilon(t) = \sum S(t_1)t_2$ for all $t \in H$. By theorem 4.6 we know S is bijective, and so S^{-1} is bijective. Thus there exist $h \in H$ such that $t = S^{-1}(h)$. Substituting this throughout the equation we have the following.

$$\begin{aligned}
\sum t_1 S(t_2) &= \varepsilon(t) &&= \sum S(t_1)t_2 \\
\sum (S^{-1}h)_1 S((S^{-1}h)_2) &= \varepsilon(S^{-1}h) &&= \sum S((S^{-1}h)_1)(S^{-1}h)_2 \\
\sum S^{-1}h_2 S(S^{-1}h_2) &= \varepsilon(h) &&= \sum S(S^{-1}h_2)S^{-1}h_1 \\
\sum S^{-1}(h_2)h_1 &= \varepsilon(h) &&= \sum h_2 S^{-1}(h_1)
\end{aligned}$$

Since t was arbitrary, this is true for all $h \in H$. $\square$

We are now ready to complete our remaining four example sets. For three of these we will indentify an antipode, and show that it is a Hopf algebra. For our fourth example, $\mathcal{O}[M_n(k)]$, we find that it does not have such an antipode, and so is not a Hopf algebra.

Example 2.25. Since example 2.18 tells us that $k[t]$ is a bialgebra, we only need to find an antipode. Consider S such that $S(t) = -t$ and extend linearly. We check the antipode property for a general element. Recall $\Delta(t) = t \otimes 1 + 1 \otimes t$ and Δ is multiplicative and linear so $\Delta(\sum \alpha_i t^i) = \sum \alpha_i \sum_{k=0}^{i} \binom{i}{k} t^{i-k} \otimes t^k$.

$$\sum \alpha_i \sum_{k=0}^{i} \binom{i}{k} S(t^{i-k}) t^k = \sum \alpha_i \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} t^1 = \alpha_0$$

$$\sum \alpha_i \sum_{k=0}^{i} \binom{i}{k} t^{i-k} S(t^k) = \sum \alpha_i \sum_{k=0}^{i} \binom{i}{k} (-1)^k t^1 = \alpha_0$$

The last step of each equality comes from the binomial theorem with $x = -1$. In this case $(1-1) = 0 = \sum_{k=0}^{i} (-1)^k \binom{i}{k}$, thus the only term that remains is when $i = 0$. We next note that by definition $\varepsilon(t) = 0$ so when we take ε of a general element from $k[t]$ the only term that survives is the constant. This gives us $\varepsilon(\sum \alpha_i t^i) = \alpha_0$. Therefore the antipode property holds, and $k[t]$ is a Hopf algebra.

In general for any Hopf algebra H, if $c \in P_{g,h}(H)$, then it must be true that $S(c) = -h^{-1} c g^{-1}$.

Example 2.26. with $\mathcal{O}[M_n(k)]$ we have an example of a bialgebra that can never be a Hopf algebra. This is proved by contradiction.

Proof. Assume there exists an antipode $S \in Hom_k(\mathcal{O}[M_n(k)], \mathcal{O}[M_n(k)])$ such that $\mathcal{O}[M_n(k)]$ is a Hopf algebra. Consider the element det$X$ =det$[X_{ij}]$. det$X$ is a a group-like element of $\mathcal{O}[M_n(k)]$. The calculation of this is extremely long, so we give it for the case of $n = 2$ to get the flavor. For $n = 2$, det$X = X_{11}X_{22} - X_{12}X_{21}$. We then show that $\Delta(\mathrm{detX}) = \mathrm{detX} \otimes \mathrm{detX}$.

$\Delta(\mathrm{detX}) = \Delta(\mathrm{X_{11}X_{22}} - \mathrm{X_{12}X_{21}})$

$= \Delta(X_{11}X_{22}) - \Delta(X_{12}X_{21})$

$= \Delta(X_{11})\Delta(X_{22}) - \Delta(X_{12})\Delta(X_{21})$
$= (X_{11} \otimes X_{11} + X_{12} \otimes X_{21})(X_{21} \otimes X_{12} + X_{22} \otimes X_{22}) - (X_{11} \otimes X_{12} + X_{12} \otimes X_{22})(X_{21} \otimes X_{11} + X_{22} \otimes X_{21})$
$= X_{11}X_{22} \otimes X_{11}X_{22} + X_{12}X_{21} \otimes X_{12}X_{21} - X_{11}X_{22} \otimes X_{12}X_{21} - X_{12}X_{21} \otimes X_{11}X_{22}$
$= (X_{11}X_{22} - X_{12}X_{21}) \otimes X_{11}X_{22} + (X_{11}X_{22} - X_{12}X_{21}) \otimes -X_{12}X_{21}$
$= (X_{11}X_{22} - X_{12}X_{21}) \otimes (X_{11}X_{22} - X_{12}X_{21})$

Also, since $\mathcal{O}[M_n(k)]$ is a Hopf algebra with antipode S, then $S(\mathrm{detX}) = (\mathrm{detX})^{-1}$ (since this is true for all group-like elements of any Hopf algebra). However, $(detX)^{-1}$ does not exists in $\mathcal{O}[M_n(k)]$ (since polynomials do not have polynomial multiplicative inverses), so S does not exist (a contradiction). Therefore, $\mathcal{O}[M_n(k)]$ can never be a Hopf algebra. □

Example 2.27. Again, since kG is a bialgebra by example 2.21 we just need to verify the antipode. Consider $S \in Hom_k(kG, kG)$ where $S(g) = g^{-1}$ for all $g \in G$ (extended linearly and multiplicatively). We check that S satisfies the antipode property, starting from the left.

$$\sum \alpha_i S(g_i) g_i = \sum \alpha_i g_i^{-1} g_i = \sum \alpha_i 1_{kG} = \varepsilon(\sum \alpha_i g_i) 1_k G$$
$$\sum \alpha_i g_i S(g_i) = \sum \alpha_i g_i g_i^{-1} = \sum \alpha_i 1_{kG} = \varepsilon(\sum \alpha_i g_i) 1_k G$$

Therefore kG is a Hopf algebra.

In general, for any Hopf algebra H, if $g \in G(H)$ then $S(g) = g^{-1}$.

Example 2.28. For the bialgebra $(H_4, \mu, u, \Delta, \varepsilon)$ we need to define the antipode S on each of the basis elements. In order for S to be an antipode then we must have $S(1) = 1$. Since $g \in G(H_4)$ than $S(g) = g^{-1}$ but g is its own inverse so $S(g) = g$. Also, since $x \in P_{1,g}(H_4)$ then $S(x) = -gx$. For gx we use the anti-algebra morphism property of S to get $S(gx) = S(x)S(g) = -gxg = ggx = x$. Since S and ε are linear and S is antimultiplicative, it is enough to check the antipode property for the basis elements.

$S(1)1 = 1 \cdot 1 = \varepsilon(1)1 = 1 \cdot 1 = 1S(1)$
$S(g)g = g^2 = 1 = \varepsilon(g)1 = 1 = g^2 = gS(g)$
$S(x)1 + S(g)x = -gx + gx = 0 = \varepsilon(x)1 = 0 = x - x = x - ggx = xS(1) + gS(x)$

Therefore H_4 is a Hopf algebra.

2.5 Duality

We will now end the chapter with a few notes about duality as it relates to the structures presented here. As mentioned before, we got our definition of a coalgebra by dualizing the definition of an algebra. This connection is so strong, that we can prove some theorems about the dual spaces of algebras and coalgebras with respect to their structure under all the important definitions of this chapter. We start with a formal definition of a dual space to a vector space.

Definition 2.29. Let V be a k-vector space. The dual space to V, denoted V^* is the set of all linear functionals on V. In set notation we have:

$$V^* = \{f \in End_k(V) \mid \alpha f(v) + \beta f(w) = f(\alpha v + \beta w) \forall \alpha, \beta \in k \text{ and } v, w \in V\}.$$

We can also dualize a map between vector spaces. Given two (finite dimensional) vector spaces U and V , and a mapping $f : U \to V$ we may dualize this concept by defining $f^* : V^* \to U^*$ as $f^*(v^*)(u) = v^*(f(u))$. In some circumstances this dualization also works in the infinite dimensional case, as we will see below. In particular, the dual of a coalgebra is an algebra.

Theorem 2.30. *If C is a coalgebra, then C^* is an algebra with multiplication $\mu = \Delta^*$ and unit $u = \varepsilon^*$.*

Proof. We first note that $(C \otimes C)^* \subseteq C^* \otimes C^*$ with equality holding in the finite dimensional case. This allows us to restrict Δ^* to simply $C^* \otimes C^*$ to get a map $\mu : C^* \otimes C^* \to C^*$. We now prove our statement by dualizing the diagrams. First we check that μ is associative. This is true since Δ is coassociative.

$$\begin{aligned}
\mu(\mu(f \otimes g) \otimes h)(c) &= \Delta^*(\Delta^*(f \otimes g) \otimes h)(c) \\
&= (\Delta^*(f \otimes g) \otimes h)(\Delta(c)) \\
&= \Delta^*\langle (f \otimes g), c_1 \rangle \langle h, c_2 \rangle \\
&= \langle f \otimes g, \Delta(c_1) \rangle \langle h, c_2 \rangle \\
&= \langle f, c_1 \rangle \langle g, c_2 \rangle \langle h, c_3 \rangle \\
&= \langle f, c_1 \rangle \langle g \otimes h, \Delta(c_2) \rangle \\
&= \langle f \otimes \Delta^*(g \otimes h), \Delta(c) \rangle \\
&= \Delta^*(f \otimes \Delta^*(g \otimes h))(c) \\
&= \mu(f \otimes \mu(g \otimes h))(c)
\end{aligned}$$

Thus μ ia associative, and so the first algebra diagram commutes. We now check the second diagram by formula.

$$\begin{aligned}(\varepsilon^*(\alpha)f)(c) &= (\alpha f)(\varepsilon(c))\\ &= \varepsilon(c)(\alpha f)(1)\\ &= \varepsilon(c)\varepsilon^*(\alpha f)\\ &= \varepsilon^*(\alpha f)(c)\end{aligned}$$

Thus the second diagram commutes, and so $(C^*, \Delta^*, \varepsilon^*)$ is an algebra. $\square$

It would be convenient if the converse of this theorem were also true, but unfortunately we run into a problem. All is well in the finite dimensional case, but in the general infinite dimensional case we have an issue. In the proof above we used the fact that $(C \otimes C)^* \subseteq C^* \otimes C^*$. When we have an algebra A, Then $A^* \otimes A^*$ is a proper subset of $(A \otimes A)^*$, so it is often the case that the image of our $\Delta = \mu^*$ will not lie in $A^* \otimes A^*$. To overcome this we will need a new definition.

Definition 2.31. Let A be a k-algebra. The *finite dual* of A is $A^\circ = \{f \in A^* \mid f(I) = 0,$ for some ideal I of A such that $dimA/I < 1\}$.

This new definition now allows us to prove a partial converse of Theorem 2.30.

Theorem 2.32. *If A is an algebra, then A° is a coalgebra, with comultiplication $\Delta = \mu^*$ and counit $\varepsilon = u^*$.*

Proof. We know from [Mon93] that if we restrict ourselves to A°, then $\mu^*(A^\circ) \subseteq A^* \otimes A^*$. We will need to show that the two coalgebra diagrams commute for our maps μ^* and u^*. Let $f \in A^\circ$ and $a, b, c \in A$. We will only check one side of the counit diagram for u^*.

$$\begin{aligned}(id \otimes u^*)(\mu^*)(f)(a \otimes b) &= id \otimes u^* f(\mu(a \otimes b))\\ &= id \otimes u^* f(ab)\\ &= f(au(b))\\ &= f(a)u(b)\\ &= (u \otimes f)(a \otimes b)\end{aligned}$$

Thus the counit diagram commutes for u^*. We next check coassociativity by using the fact that μ is associative.

$$\begin{aligned}(\mu^*)(id\otimes\mu^*)(f)(a\otimes(b\otimes c)) &= (\mu^*)f((id\otimes\mu)(a\otimes(b\otimes c)))\\ &= (\mu^*)f(a\otimes bc)\\ &= f(\mu(a\otimes bc))\\ &= f(a(bc))\\ &= f((ab)c)\\ &= f(\mu(ab\otimes c))\\ &= (\mu^*)f(ab\otimes c)\\ &= (\mu^*)f((\mu\otimes id)((a\otimes b)\otimes c))\\ &= (\mu^*)(\mu^*\otimes id)(f)((a\otimes b)\otimes c)\end{aligned}$$

Thus μ^* is coassociative, and so (A°, μ^*, u^*) is a coalgebra. □

With these two dualizations we will now be able to dualize our combined structures with the following theorem. Again our result is limited to the finite dual, but remember that if our vector space is finite dimensional, then $V^\circ = V^*$.

Theorem 2.33. *If $(B, \mu, u, \Delta, \varepsilon)$ is a bialgebra, then $(B^\circ, \Delta^*, \varepsilon^*, \mu^*, u^*)$ is a bialgebra. If $B = H$ is a Hopf algebra with antipode S, then H° is a Hopf algebra with antipode S^*.*

Proof. We know from the previous theorems that $(B^\circ, \Delta^*, \varepsilon^*)$ is an algebra and (B°, μ^*, u^*) is a coalgebra. We thus only need to check the compatibility conditions. We will check that $\Delta_{B^\circ} = \mu^*$ and $\varepsilon_{B^\circ} = u^*$ are algebra maps using the formula conditions. Let $a, b \in B$, $f \in B^\circ$, and $\alpha \in k$. First μ^* and u^* are multiplicative by definition. Also μ^* preserves 1 by the following.

$$\begin{aligned}\mu^*(id)(a\otimes b) &= id(\mu(a\otimes b))\\ &= id(ab)\\ &= ab\\ &= id(a)id(b)\\ &= (id\otimes id)(a\otimes b)\end{aligned}$$

Thus $\mu^*(id) = id\otimes id$. Also u^* preserves 1 since:

$$\begin{aligned}u^*(id)(\alpha) &= id(u(\alpha))\\ &= id(\alpha\cdot 1)\\ &= \alpha\cdot 1\end{aligned}$$

Thus $u^*(id) = 1$. This gives us that μ^* and u^* are algebra maps, and so B° is a bialgebra. We would now like to show that if $B = H$ is a Hopf algebra with antipode S, then H° is a Hopf algebra with antipode S^*. So for S^* to be an antipode, we need to show that S^* is the inverse of id_{H° under convolution, or $(S^* * id)(f)(h) = \varepsilon^*(f)u^*(h) = (id * S^*)(f)(h)$.

$$\begin{aligned}
(S^* * id)(f)(h) &= \Delta^*(S^* \otimes id)(\mu^* f)(h) \\
&= (S^* \otimes id)(\mu^* f)\Delta(h) \\
&= (S^* \otimes id)(\mu^* f)(\textstyle\sum h_1 \otimes h_2) \\
&= (S^* \otimes id)f(\mu(\textstyle\sum h_1 \otimes h_2)) \\
&= (S^* \otimes id)f(\textstyle\sum h_1 h_2)) \\
&= f(S(h1)h2) \\
&= f(\varepsilon(h)) \\
&= f(1)\varepsilon(h) \\
&= \varepsilon^*(f)u^*(h)
\end{aligned}$$

By similar calculation, we will get the other half of our equality, and so H° is a Hopf algebra with antipode S^*. □

We can extend each of the above theorems to the finite dimensional case with the following corollary. Each statement is proven simply by remembering that if V is a finite dimensional vector space then $V^\circ = V^*$.

Corollary 2.34. *Let H be a finite dimensional Hopf algebra. Then:*

1. *(H^*, μ^*, u^*) is a coalgebra.*
2. *$(H^*, \Delta^*, \varepsilon^*)$ is an algebra.*
3. *$(H^*, \Delta^*, \varepsilon^*, \mu^*, u^*)$ is a bialgebra.*
4. *H^* is a Hopf algebra with antipode S^*.*

Chapter 3

Modules to Hopf Modules

In this chapter we look at sets that act in various ways on the structures we have constructed. Beginning with algebras we have a module, which is similiar to a vector space. From there we will dualize to a comodule, and then combine these two into a Hopf module. Our ultimate goal is to then prove the Fundamental Theorem of Hopf Modules. This theorem tells us that all Hopf modules are essentially trivial. It is a key piece to proving the Nichols-Zeoller freeness theorem. Throughout this chapter H will represent a Hopf algebra.

3.1 Modules and Comodules

There are many ways to define a module. In general, a module is defined as an additive abelian group over a ring. For our purposes we will be defining a module as a vector space over an algebra. We can do this since a vector space is also an additive abelian group, and an algebra is also a ring. The difference we find with this definition is that we have an additional connection between the underlying field and the action of the module. This additional structure is represented in the second commutative diagram of the following definition.

Definition 3.1. Let A be an algebra. A vector space M is a (left) *A-module* along with the linear map $\gamma : A \otimes M \to M$ and scalar multiplication $\sigma : k \otimes M \to M$ if the following diagrams commute.

$$
\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "\mu \otimes id"] \arrow[d, "id \otimes \gamma"'] & A \otimes M \arrow[d, "\gamma"] \\
A \otimes M \arrow[r, "\gamma"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes M \arrow[r, "u \otimes id"] \arrow[dr, "\alpha"'] & A \otimes M \arrow[d, "\gamma"] \\
& M
\end{tikzcd}
$$

Just as with our algebra definition in the last chapter, this definition is a diagramatic version of the usual definition of a module. the first diagram gives us the associative property of the action, and the second diagram (as mentioned above) tells us that for each $\alpha \in k$ there is an element $a \in A$ that act on M in the same way as scalar multiplication of α on M. We can now dualize this definition to define a comodule.

Definition 3.2. Let C be a coalgebra. A vector space M along with the linear map $\rho : M \to M \otimes C$ is a (right) *C-comodule* if the following diagrams commute.

$$
\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[d, "id \otimes \Delta"] \\
M \otimes C \arrow[r, "\rho \otimes id"'] & M \otimes C \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "\rho"] \arrow[dr, "\oplus 1"'] & M \otimes C \arrow[d, "id \otimes \varepsilon"] \\
& M \otimes C
\end{tikzcd}
$$

The second diagram gives us a useful formula for working with comodules. It says that for $m \in M$ then $\sum \varepsilon(m_1)m_0 = m$. Next, we need some notation for the coaction ρ. As for Δ in the last chapter, we use summation notation for ρ. We write $\rho(m) = \sum m_{(0)} \otimes m_{(1)} = \sum m_0 \otimes m_1$ where $m_0 \in M$ and $m_1 \in C$. This preserves the notation in the last chapter, in that these are not specific elements, but symbolic representations of the elements. We can also apply ρ a second time to the left element, or we can apply Δ to the right element. Because the first diagram in the definition commutes, we have that $(\rho \otimes id) \circ \rho = (id \otimes \Delta) \circ \rho$, and so we may write $\sum (m_0)_0 \otimes (m_0)_1 \otimes m_2 = \sum m_0 \otimes (m_1)_1 \otimes (m_1)_2 = \sum m_0 \otimes m_1 \otimes m_2$ where $m_0 \in M$ and $m_1, m_2 \in C$.

Just like before there is a strong connection between modules and comodules with respect to dual spaces. We show this in the following two lemmas.

Lemma 3.3. *If M is a right C-comodule, then M is a left C^*-module.*

Proof. Let M be a right C-comodule with coaction $\rho : M \to M \otimes C$. Define $\gamma : C^* \otimes M \to M$ to be $\gamma(f \otimes m) = f \cdot m = \sum \langle f, m_1 \rangle m_0$. We show that for this map the two module diagrams commute, and so γ is an an action for C^* on M. Using the formula form, the first diagram checks for associativity of the action. Let $f, g \in C^*$.

$$\begin{aligned} f \cdot (g \cdot m) &= f \cdot (\textstyle\sum \langle g, m_1 \rangle m_0) \\ &= \textstyle\sum \langle g, m_1 \rangle \langle f \cdot m_0 \rangle \\ &= \textstyle\sum \langle g, m_2 \rangle \langle f, m_1 \rangle m_0 \\ &= \textstyle\sum \langle g \otimes f, \Delta(m) \rangle m_0 \\ &= \textstyle\sum \langle f * g, m_1 \rangle m_0 \\ &= (f * g) \cdot m \end{aligned}$$

Thus our action is associative. For the second diagram we need to show that $\varepsilon \cdot m = m$.

$$\begin{aligned} \varepsilon \cdot m &= \textstyle\sum \langle \varepsilon, m_1 \rangle m_0 \\ &= m \end{aligned}$$

Therefore M is a left C^*-module. □

The next question here becomes whether or not the converse of this lemma is true. Unfortunately a full converse is not possible, since Theorem 2.32 showed us that if A is an algebra, then only the finite dual A° is a coalgebra. So we restrict ourselves to the finite dual of A as follows to get a version of the converse.

Lemma 3.4. *Let M be a left A-module. Then M is a right A°-comodule $\Leftrightarrow$ $\{A \cdot m\}$ is finite dimensional for all $m \in M$.*

Proof. ($\Rightarrow$) We use the standard notation of the right A° coaction defined as $\rho(m) = \sum m_0 \otimes m_1$ where $m_1 \in A^\circ$. By the construction of Lemma 3.3 we use the action $a \cdot m = \sum \langle a, m_1 \rangle m_0$ for all $a \in A$. Thus $A \cdot m$ is spanned by the set $\{m_0\}$, and since this set is finite, $A \cdot m$ is finite diminsional.

($\Leftarrow$) Assume $A \cdot m$ is finite dimensional. For $m \in M$ denote by $\{m_1, ..., m_n\}$ a basis for $A \cdot m$. Thus for all $a \in A$, $a \cdot m = \sum f_i(a) m_i$, for some $fi \in A^\circ$. Now let $\varphi : A \to End_k(A \cdot m)$ be defined by $\varphi(a) \cdot (b \cdot m) = ab \cdot m$. Next consider the ideal $I = ker(\varphi)$. I is cofinite dimensional since $A \cdot M$ is finite dimensional by assumption.

This implies M/I is finite dimensional. Thus by Thereom 9.1.1 in [Mon93] each f_i vanishes on I, and so by Definition 2.31 $f_i \in A^\circ$. M then becomes an A°-comodule via $\rho : M \to M \otimes A^\circ$ given by $m \mapsto \sum_i m_i \otimes f_i$. □

For each of the two lemmas above we look at an example of how these are applied to general algebras and coalgebras.

Example 3.5. Let C be a coalgebra. Then C is a right C-module using $\rho = \Delta$. From Lemma 3.3 we get a left action of C^* on C. Let $f \in C^*$ and $c \in C$. Explicitly the action is:

$$f \rightharpoonup c = \sum \langle f, c_2 \rangle c_1$$

However, we have also seen that C^* is an algebra by Theorem 2.32. We can then apply this to right multiplication in C^*. This yields us:

$$\langle g, f \rightharpoonup c \rangle = \langle g, \sum \langle f, c_2 \rangle c_1 \rangle = \sum \langle f, c_2 \rangle \langle g, c_1 \rangle = \langle gf, c \rangle$$

We denote this relationship by say that $\rightharpoonup$ is the transpose of right muliplication of C^* on itself.

Similarly, we can define a right action $\leftharpoonup$ which is the transpose of left multiplication of C^* on itself.

Example 3.6. If we start with an algebra A, then we can also define a left action on A^* by A. Given $a \in A$ and $f \in A^*$, then $a \rightharpoondown f$ is defined to be the element of A^* such that for all $b \in A$,

$$\langle a \rightharpoonup f, b \rangle = \langle f, ab \rangle.$$

We do not have an explicit formula for this action in general, because we saw in theorem 2.32 that A^* in general is not a coalgebra. Now, if $f \in A^\circ$, then it makes sense to talk about $\Delta(f)$, so we can then write:

$$a \rightharpoonup f = \sum \langle f_2, a \rangle f_1.$$

In this case we denote the relationship by saying that $\rightharpoonup$ is the transpose of right multiplication by a on A. Again, we can also define a right action $\leftharpoonup$ which is the transpose of left multiplication of a on A.

The next two definitions give us special subsets of both modules and comodules, which will be important to the Fundamental Theorem of Hopf Modules, and to the concept on an integral in the next chapter.

Definition 3.7. Let M be a left H-module. The invariants of H on M are the set $M^H = \{m \in M \mid h \cdot m = \varepsilon(h)m, \forall h \in H\}$.

Definition 3.8. Let M be a right H-comodule. The coinvariants of H in M are the set $M^{coH} = \{m \in M \mid \rho(m) = m \otimes 1\}$.

These sets are "special" in the sense that they provide us with elements in M for which the action or coaction of H on these elements are essentially trivial. This will be crucial in the understanding the structure of Hopf algebras, for example in the proof of Fundamental Theorem of Hopf Modules at the end of this chapter.

We now look at two lemmas that deal with the relationship between these sets with respect to dual spaces.

Lemma 3.9. *Let M be a right H-comodule that is also a left H^*-module. Then $M^{H^*} = M^{coH}$.*

Proof. Let M be a right H-comodule and a left H^*-module. We show by double inclusion that $M^{H^*} = M^{coH}$

1) $M^{H^*} \subseteq M^{coH}$

Let $m \in M^{H^*}$. Thus $m \cdot f = \varepsilon^*(f)m$ for all $f \in H^*$, and by definition $\varepsilon^*(f) = \langle f, 1\rangle$, so $m \cdot f = \langle f, 1\rangle m$. Now we show that $\rho(m) = m \otimes 1$.

Let $\rho(m) = \sum m_0 \otimes m_1 = \sum_{i=1}^{l} m_i \otimes h_i$. Assume without loss of generality that the m_i's are linearly independent and $m_1 = m$. This can be done since M is a vector space, so any set $\{m_i\}$ can be rewritten this way by simply multiplying its representation as basis elements by scalars. The scalars needed are taken from the other side of the tensor product. This is acceptable because it does not matter what the element on other side of the tensor product is. We now let $f \in H^*$, and apply f to m using this form of

$\rho(m)$.

$$\begin{aligned}\langle f,1\rangle m &= f\cdot m\\ &= \textstyle\sum_{i=1}^{l}\langle f,h_i\rangle m_i\\ &= \langle f,h_1\rangle m_1+\textstyle\sum_{i=2}^{l}\langle f,h_i\rangle m_i\\ &= \langle f,h_1\rangle m+\textstyle\sum_{i=2}^{l}\langle f,h_i\rangle m_i && \text{Since } m_1=m\\ &= \langle f,h_1\rangle m && \text{Since the set } \{m_i\}\text{is linearly independent}\end{aligned}$$

We get two important results from this. First we used the fact that $\langle f,h_i\rangle = 0$, for $i=2,\dots,l$ for all $f\in H^*$. This gives us that $h_i=0$, for $i=2,\dots,l$, since 0 is the only element of H that every element of H^* takes to 0. Second we have that $\langle f,1\rangle=\langle f,h_1\rangle$ which implies that $h_1=1$, again since 1 is the only element of H that every element of H^* takes to 1. These two combined give us $\rho(m)=\sum_{i=1}^{l}m_i\otimes h_i=m_1\otimes h_1=m\otimes 1$. Thus $m\in M^{coH}$.

2) $M^{H^*}\supseteq M^{coH}$.
Let $m\in M^{coH}$. This means $\rho(m)=m\otimes 1$ or that $\sum m_0\otimes m_1=m\otimes 1$ (by sigma notation of ρ). Now for any $f\in H^*$.

$$\begin{aligned}f\cdot m &= \textstyle\sum\langle f,m_1\rangle m_0 && \text{By the construction in the proof of Lemma 2.30}\\ &= \langle f,1\rangle m && \text{Since } \textstyle\sum m_0\otimes m_1=m\otimes 1\\ &= \varepsilon^*(f)m\end{aligned}$$

Thus $m\in M^{H^*}$. □

Lemma 3.10. *Let M be a left H-module that is also a right H°-comodule. Then $M^H=M^{coH^\circ}$.*

Proof. Let M be a left H-module and a right H°-comodule. We will show that $M^H=M^{coH^\circ}$ by double inclusion.

1) $M^H\subseteq M^{coH^\circ}$.
Let $m\in M^H$. This means that $h\cdot m=\varepsilon(h)m$ for all $h\in H$. As before, we can rearrange $\rho(m)=\sum m_0\otimes f_1$ as the finite sum $\sum m_i\otimes f_i$ where $m_1=m$. We also need

to remember that $\varepsilon = 1_{M^{coH^\circ}}$.

$$\begin{aligned}\langle \varepsilon, h\rangle m &= h \cdot m\\ &= \textstyle\sum_{i=1}^{n} \langle h, f_i\rangle m_i\\ &= \langle h, f_1\rangle m_1 + \textstyle\sum_{i=2}^{n} \langle h, f_i\langle m_i\\ &= \langle h, f_1\rangle m + \textstyle\sum_{i=2}^{n} \langle h, f_i\rangle m_i\\ &= \langle h, f_1\rangle m\end{aligned}$$

From this we get that $\langle \varepsilon, h\rangle = \langle f_1, h\rangle$ for all $h \in H$, thus $\varepsilon = f_1$. We also get that $\langle f_i, h_i\rangle = 0$ for $i = 2, \ldots, n$ and all $h \in H$, thus $f_i = 0$ for $i = 2, .., n$ since 0 is the only element of H^* that takes every element of H to 0. We now apply ρ.

$$\begin{aligned}\rho(m) &= \textstyle\sum m_i \otimes f_i & \\ &= m \otimes f_1 & f_i = 0 \text{ for } i = 2, \ldots, n\\ &= m \otimes \varepsilon & f_1 = \varepsilon\\ &= m \otimes 1_{M^{coH^\circ}} &\end{aligned}$$

Therefore $m \in M^{coH^\circ}$.

2)$M^H \supseteq M^{coH^\circ}$.

Let $m \in M^{coH^\circ}$. This means that $\rho(m) = \sum m_0 \otimes f_1 = m \otimes 1$. Let $h \in H$ and apply h to m.

$$\begin{aligned}h \cdot m &= \textstyle\sum \langle h, f_1\rangle m_0 & \\ &= \langle h, \varepsilon\rangle m & \text{Since } \textstyle\sum m_0 \otimes f_1 = m \otimes 1\\ &= \varepsilon(h) m &\end{aligned}$$

Therefore $m \in M^H$. □

We now define what it means to be free as a module. Our definition is a version of one that is stated (but not numbered) in [Lam76], but is specific for an algebra which, again, is a ring as one of its base structures.

Definition 3.11. Let A be an algebra and M be a left A-module. M is called *free* if there exists a set $\{m_i \mid i \in I, m_i \in M\}$ such that every element $m \in M$ can be written uniquely as $m = \sum_i a_i m_i$ where $a_i \in A$ and all but a finite number of a_i's are 0.

At face value this definition seems like a restatement of a basis of a vector space. It is true that, since a field is also a ring, a vector space also qualifies as a free module.

So this definition is, in fact, a generalization of a basis of a vector space. However for free modules that are not vector spaces, there is a fundamental difference. This difference is in the fact that the "scalars" come from a ring which is not a field. This difference causes us to lose some of the properties of a vector space. The most important of these has to do with zero. In a vector space if $\alpha v = 0$, then either $\alpha = 0$ or $v = 0$. In a free module it is possible that $a \cdot m = 0$ with $a \neq 0$ and $m \neq 0$, since a ring can have zero divisors.

Despite this difference, the most important property remains. Freeness gives us the ability to write a general element as a linear combination of basis elements. This ability is an invaluable tool in work with modules.

we also have the concept of a faithful module, which will be used in the proof of the Nichols-Zoeller Theorem. This definition also is a version of one found in [Lam76], but is specific to an algebra.

Definition 3.12. Let A be an algebra and M be a left A-module. M is called *faithful* if for all $a \in A$ such that $a \neq 0$ the set $\{a \cdot M\} \neq \{0\}$.

What this means is that if $a \neq 0$ then there exist at least one $m \in M$ such that $a \cdot m \neq 0$.

3.2 Hopf Modules

We begin this section with a definition of a module structure for the tensor product of two modules. We do this because it is necassary for the compatibility condition in the definition of a Hopf module. As before throughout this section H will represent a Hopf algebra.

Definition 3.13. Let V and W be left H-modules. Then $V \otimes W$ is also a left H-module, via

$$h \cdot (v \otimes w) = \sum (h_1 \cdot v) \otimes (h_2 \cdot w)$$

for all $h \in H$, $v \in V$, and $w \in W$.

The particular action in the above definition is interesting because it incorporates elements of the algebra and coalgebra structure of H. We use multiplication from the algebra structure, and comultiplication from the coalgebra structure. Because of this, H is required to be at least a bialgebra for this definition to hold. With this definition we can now move to the important structure of this section, the Hopf module, which uses this action in its definition. The crucial addition to the structure is a compatibility condition between the module and comodule structure on M.

Definition 3.14. Let M be a right H-module and a right H-comodule. Then M is a right *H-Hopf module* if $\rho : M \to M \otimes H$ from the comodule structure is a right H-module map. That is, $\sum (m \cdot h)_1 \otimes (m \cdot h)_2 = \sum m_0 \cdot h_1 \otimes m_1 h_2$ for all $m \in M$ and $h \in H$.

More generally, we can replace H in the module part with any subHopfalgebra K, and say that M is a right *(H,K)-Hopf module.* The next example demonstrates what it means for a Hopf module to be trivial.

Example 3.15. For a Hopf algebra H, H itself is an H-Hopf module by letting $\rho = \Delta$. This is due to the fact that since H is also a bialgebra, then Δ is an algebra map, thus making it also an H-module map.

Also, if we let W be any right H-module, Then $W \otimes H$ is a right H-Hopf module by setting $\rho = id \otimes \Delta$. As a specific case of this let W be the H-module defined by $w \cdot h = \varepsilon(h) w$ for all $w \in W$ and $h \in H$. Thus W is trivial in the sense that

$W = M^{coH}$. If we again consider the H-Hopf module $W \otimes H$ we now see that for all $w \in W$ and $k, h \in H$:

$$\begin{aligned}(w \otimes k) \cdot h &= \sum w \cdot h_1 \otimes kh_2 \\ &= \sum \varepsilon(h_1) w \otimes kh_2 \\ &= \sum w \otimes k\varepsilon(h_1)h_2 \\ &= w \otimes kh\end{aligned}$$

Thus $(w \otimes k) \cdot h = w \otimes kh$. Such an H-Hopf module is called a trivial Hopf module.

We end this chapter with the Fundamental Theorem of Hopf Modules. The proof of this theorem is based on an outline given in [Mon93], and also appears in [Swe69] and [Abe80]. In this theorem we see that all Hopf modules are essentially trivial. Hopf modules are nevertheless important, since the difficulty is in proving that something is in fact a Hopf module at all. In situations where it is possible to prove that a set is a Hopf module, the properties of that set are well understood.

Theorem 3.16. The Fundamental Theorem of Hopf Modules. *Let M be a right H-Hopf module. Then $M \cong M^{coH} \otimes H$ as right H-Hopf modules, where $M^{coH} \otimes H$ is a trivial Hopf module. In particular, M is a free right H-module of rank $= dim_k(M^{coH})$.*

Proof. We begin by defining maps $\alpha : M^{coH} \otimes H \to M$ by $\alpha(m \otimes h) = m \cdot h$ and $\beta : M \to M \otimes H$ by $\beta(m) = \sum m_0 \cdot (Sm_1) \otimes m_2$. First we show that $\beta(M) \subseteq M^{coH} \otimes H$ by showing that $\sum m_0 \cdot (Sm_1) \in M^{coH}$ for all $m \in M$. Let $m \in M$.

$$\begin{aligned}\rho(\sum m_0 \cdot (Sm_1)) &= \sum (m_0 \cdot (Sm_1))_0 \otimes (m_0 \cdot (Sm_1))_1 \\ &= \sum m_0 \cdot (Sm_2)_1 \otimes m_1 (Sm_2)_2 \\ &= \sum m_0 \cdot S(m_3) \otimes m_1 S(m_2) \\ &= \sum m_0 \cdot S(m_2) \otimes \varepsilon(m_1) \\ &= \sum m_0 \cdot \varepsilon(m_1) S(m_2) \otimes 1 \\ &= \sum m_0 \cdot S(m_1) \otimes 1\end{aligned}$$

Thus $\beta(M) \subseteq M^{coH} \otimes H$. Next we need to show that $\alpha\beta = id$ and $\beta\alpha = id$.

$$\begin{aligned}\alpha\beta(m) &= \alpha(\sum m_0 \cdot (Sm_1) \otimes m_2) \\ &= \sum (m_0 \cdot (Sm_1)) \cdot m_2 \\ &= \sum m_0 \cdot ((Sm_1)m_2) \\ &= \sum m_0 \cdot \varepsilon(m_1) \\ &= m\end{aligned}$$

Thus $\alpha\beta = id$. Now we show the other direction. Recall that since $m \in M^{coH}$ then $\rho(m) = m \otimes 1$ or $\sum m_0 \otimes m_1 = m \otimes 1$.

$$\begin{aligned}
\beta\alpha(m \otimes h) &= \beta(m \cdot h) \\
&= \textstyle\sum (m \cdot h)_0 \cdot S(m \cdot h)_1 \otimes (m \cdot h)_2 \\
&= \textstyle\sum m_0 \cdot h_1 S(m_1 \cdot h_2) \otimes m_2 h_3 \\
&= \textstyle\sum m \cdot h_1 S(1_H \cdot h_2) \otimes h_3 \\
&= \textstyle\sum m \cdot h_1 S(h_2) \otimes h_3 \\
&= \textstyle\sum m \cdot \varepsilon(h_1) \otimes h_2 \\
&= \textstyle\sum m \otimes \varepsilon(h_1) h_2 \\
&= m \otimes h
\end{aligned}$$

Thus $\beta\alpha = id$. We now show that α and β are both H-module maps.

$$\begin{aligned}
\alpha(m \otimes hg) &= m \cdot (hg) \\
&= (m \cdot h)g \\
&= \alpha(m \otimes h)g
\end{aligned}$$

Thus α is an H-module map. Now applying β to $m \cdot h$ we show that β is an H-module map.

$$\begin{aligned}
\beta(m \cdot h) &= \textstyle\sum (m \cdot h)_0 \cdot S(m \cdot h)_1 \otimes (m \cdot h)_2 \\
&= \textstyle\sum m_0 \cdot h_1 S(m_1 \cdot h_2) \otimes m_2 h_3 \\
&= \textstyle\sum m_0 \cdot h_1 S(h_2) S(m_1) \otimes m_2 h_3 \\
&= \textstyle\sum m_0 \cdot \varepsilon(h_1) S(m_1) \otimes m_2 h_2 \\
&= \textstyle\sum m_0 \cdot S(m_1) \otimes m_2 \varepsilon(h_1) h_2 \\
&= \textstyle\sum m_0 \cdot S(m_1) \otimes m_2 h \\
&= \beta(m)h
\end{aligned}$$

This tells us that α and β are both H-module maps, and since α and β are inverses, then α is an isomorphism of H-modules. We now check that α and β are H-comodule maps.

$$\begin{aligned}
(\alpha \otimes id)(\rho_{M \otimes H})(m \otimes h) &= (\alpha \otimes id)(\textstyle\sum m_0 \otimes h_1 \otimes m_1 h_2) \\
&= \textstyle\sum m_0 \cdot h_1 \otimes m_1 h_2 \\
&= \textstyle\sum m \cdot h_1 \otimes h_2 \\
&= \rho(m)h \\
&= \rho(m \cdot h) \\
&= \rho(\alpha(m \otimes h))
\end{aligned}$$

Thus α is an H-comodule map. Checking β we get the following.

$$\begin{aligned}
(\beta \otimes id)\rho(m) &= (\beta \otimes id)(\sum m_0 \otimes m_1) \\
&= (m_0)_0 \cdot S(m_0)_1 \otimes (m_0)_2 \otimes m_1 \\
&= \sum m_0 S(m_1) \otimes m_2 \otimes m_3 \\
&= (id \otimes \Delta)\beta(m)
\end{aligned}$$

So α and β are H-comodule maps, and since α and β are inverses, then α is an isomorphism of H-comodules. Also since α is also an isomorphism of H-modules, then α is an isomorphism of Hopf modules. $\square$

Chapter 4

Freeness of Hopf Algebras

In this final chapter we will take a deeper look at the concept of a free module, and then prove the important result of the paper. The concept of a free module is very important to the study of Hopf algebras. In linear algebra, when we work with vector spaces one of the most common techniques used is to use the basis of the vector space by writing general elements in terms of the basis. This gives us the ability to prove results for just the basis elements, and then easily extend those result to general elements. The same idea works for modules. If we can show a module is free, then we get a set of elements for the module that is similar to a basis. The difficulty is that although a vector space always has a basis, modules are not always free.

Mathematicians ran into this problem when researching Hopf algebras. It was known that a Hopf algebra was a module over any subHopfalgebra, but determining whether it was a free module was not easy. Kaplansky conjectured in 1975 that it was always true that a Hopf algebra was free over any subHopfalgebra [Kap75]. His original conjecture did not specify the dimension of the Hopf algebra. Immediately after this, it was shown that freeness does not hold in the general, infinite dimensional case [OS74]. It took fourteen years, but Warren Nichols (a former student of Kaplansky) and M. Bettina Zoeller (a student of Nichols) settled the question for the finite dimensional case in 1989. We will give a proof of this result here, as well as one important generalization they proved three years later. We will end by stating a few other important generalizations that have been proven since then, as well as some open questions on the subject.

4.1 Integrals

In this section we lay out two important concepts. First we discuss integrals which are special subsets of elements from a Hopf algebra. We then take a brief look at direct sums and semisimplicity, which are important to the main result of this chapter as well as the first generalization. We will begin with a definition of an integral, which comes from the idea of invariants introduced in the last chapter.

Definition 4.1. A *left integral* in H is an element $t \in H$ such that $ht = \varepsilon(h)t$ for all $h \in H$; a *right integral* in H is an element $t' \in H$ such that $t'h = \varepsilon(h)t'$ for all $h \in H$. The sets of left and right integrals are denoted by $\int_H^l$ and $\int_H^r$, respectively. H is called *unimodular* if $\int_H^l = \int_H^r$.

It is interesting to note that when we consider H as a vector space, then $\int_H^l$ and $\int_H^r$ are each subspaces of H. We prove this with the next proposition.

Proposition 4.2. *$\int_H^l$ and $\int_H^r$ are each subspaces of H.*

Proof. We will only prove in detail that $\int_H^l$ is a subspace, since the calculations will be similar for $\int_H^r$. We first show that $0 \in \int_H^l$. This is true since for any $h \in H$, $h0 = 0 = \varepsilon(h)0$. Next we show that $\int_H^l$ is closed under addition and scalar multiplication. Let $t_1, t_2 \in \int_H^l$ and $\alpha \in k$.

$$\begin{aligned} h(t_1 + t_2) &= ht_1 + ht_2 \\ &= \varepsilon(h)t_1 + \varepsilon(h)t_2 \\ &= \varepsilon(h)(t_1 + t_2) \end{aligned}$$

$$\begin{aligned} h(\alpha t_1) &= (h\alpha)t_1 \\ &= \alpha h t_1 \\ &= \alpha\varepsilon(h)t_1 \\ &= \varepsilon(h)\alpha t_1 \end{aligned}$$

Thus $t_1 + t_2 \in \int_H^l$ and $\alpha t_1 \in \int_H^l$. Therefore $\int_H^l$ is a subspace of H. □

The issue of unimodularity becomes trivial when the Hopf algebra is commutative. This is because $ht = th$ so if $t \in \int_H^l$ then $t \in \int_H^r$ and visa versa. In the following examples we will consider the integrals of two non-commutative Hopf algebras.

Example 4.3. The Hopf algebra H_4 has left and right integrals but is not unimodular. This is due mostly to the fact that H_4 is not commutative. We begin by considering a general element H_4 and assume that it is a right integral. In doing this we will discover what form this element must have in order to be a right integral. Let $t = \alpha_1 + \alpha_g g + \alpha_x x + \alpha_{gx} gx \in H_4$ be a right integral. Thus $ht = \varepsilon(h)t$ for all $h \in H_4$. Take the specific case of when $h = g$. since $\varepsilon(g) = 1$, then in order for t to be an integral we must have $gt = t$.

$$\begin{aligned} ht &= g(\alpha_1 + \alpha_g g + \alpha_x x + \alpha_{gx} gx) \\ &= \alpha_1 g + \alpha_g g^2 + \alpha_x gx + \alpha_{gx} g^2 x \\ &= \alpha_g + \alpha_1 g + \alpha_{gx} x + \alpha_{gx} x \end{aligned}$$

So in order for t to be a right integral, then $\alpha_1 = \alpha_g$ and $\alpha_x = \alpha_{gx}$. Next consider the case when $h = x$. Since $\varepsilon(x) = 0$ then in order for t to be a right integral we must get $xt = 0$.

$$\begin{aligned} ht &= x(\alpha_1 + \alpha_1 g + \alpha_x x + \alpha_x gx) \\ &= \alpha_1 x + \alpha_1 xg + \alpha_x x^2 + \alpha_x xgx \\ &= \alpha_1 x - \alpha_1 gx \end{aligned}$$

this time we have the requirement that $\alpha_1 = 0$ and $\alpha_g = 0$ for t to be a right integral. Combining this with what we learned previously, we know that all right integral will have the form $\alpha x + \alpha gx$. Thus the space of right integrals is generated by the element $x + gx$, or $\int_{H_4}^{r} = \{\alpha(x + gx) \mid \alpha \in k\}$. By a similar process we arrive at the fact that all left integrals will have the form $\alpha x - \alpha gx$, and so $\int_{H_4}^{l} = \{\alpha(x - gx) \mid \alpha \in k\}$. Therefore H_4 is not unimodular.

Example 4.4. Consider the group algebra kG. As noted in Example 2.7, kG is commutative iff G is an abelian group. We have already seen that if kG is commutative, then kG is unimodular. We will consider the case where G is not necessarily abelian and show that kG is unimodular even then. Let $\sum \alpha_g g \in kG$ represent a general element of kG. we know that $\varepsilon(g) = 1$ for all $g \in G$, so $\varepsilon(\sum \alpha_g g) = \sum \alpha_g$. This means for a right integral, we need to find an element $t \in kG$ such that $t(\sum \alpha_g g) = (\sum \alpha_g)t$. Consider the element $t = \sum_{h \in G} h$. Multiplying this by a general element from the right we get the following:

$$\left(\sum_{h \in G} h\right)\left(\sum_{g \in G} \alpha_g g\right) = \sum_{h,g \in G} \alpha_g hg = \left(\sum_{g \in G} \alpha_g\right)\left(\sum_{g \in G} g\right)$$

Thus $\sum_{h \in G} h$ is a right integral. By similar calculations $\sum_{h \in G} h$ is also a left integral. This is the only element (up to multiplication by a scalar) that is either type of integral, so kG is unimodular.

It is interesting to note that all the spaces of left and right integrals in these examples are one dimensional. This will turn out to be true in general. Before we can prove this though, we need the following lemma about the dual space H^* being an H-Hopf module when H is finite dimensional.

Lemma 4.5. *Let H be a finite dimensional Hopf algebra. Then H^* is a right H-Hopf module.*

Proof. First, we show that H^* is a right H-comodule. We know that H^* is a left H^*-module via left multiplication. So if $\{f_1, \dots, f_n\}$ is a basis for H^* and $f \in H^*$, then there exist $h_1, \dots, h_n \in H$ such that for any $g \in H$ then $gf = \sum \langle g, h_i \rangle g_i$. Corollary 2.34 gives us that H^* is a right $(H^*)^*$-comodule, thus H^* is a right H-comodule. The comodule map $\rho : H^* \to H^* \otimes H$ is given by $\rho(f) = \sum g_i \otimes h_i$.

Also, H^* is a right H-module as follows. Let $\leftharpoondown$ be the action of H on H^* such that if $f \in H^*$ and $l, h \in H$, then $f \leftharpoondown h = Sh \rightharpoonup f$ and $\langle f \leftharpoondown h, l \rangle = \langle f, lS(h) \rangle$. We next show that ρ is an H-module map. We start with a technical fact.

$$
\begin{aligned}
\sum((h_2 \rightharpoonup g)f) \leftharpoondown h_1 &= \sum(\langle g_2, h_2 \rangle g_1 f) \leftharpoondown h1 \\
&= Sh_1 \rightharpoonup (\langle g_2, h_2 \rangle g_1 f) \\
&= \langle g_2, h_2 \rangle Sh_1 \rightharpoonup (g_1 f) \\
&= \langle g_2, h_3 \rangle (Sh_2 \rightharpoonup g_1)(Sh_1 \rightharpoonup f) \\
&= (\langle g_3, h_3 \rangle \langle g_2, Sh_2 \rangle g_1)(Sh_1 \rightharpoonup f) \\
&= (\langle g_2, h_3 Sh_2 \rangle g_1)(Sh_1 \rightharpoonup f) \\
&= \langle g_2, \varepsilon(h_2) 1_H \rangle g_1 (Sh_1 \rightharpoonup f) \\
&= \varepsilon(h_2)\varepsilon(g_2) g_1 (Sh_1 \rightharpoonup f) \\
&= \varepsilon(g_2) g_1 (\varepsilon(h_2) Sh_1 \rightharpoonup f) \\
&= g(S_h \rightharpoonup f) \\
&= g(f \leftharpoondown h)
\end{aligned}
$$

Now we need to show that $\rho(f \leftharpoondown h) = \sum(f_0 \leftharpoondown h_1) \otimes f_1 h_2 = \rho(f) \cdot h$. This is equivalent to showing that $g(f \leftharpoondown h) = \sum \langle g, f_1 h_2 \rangle (f_0 \leftharpoondown h_1)$. Using the equality above we can do this as follows.

$$\begin{aligned} g(f \leftharpoondown h) &= \sum((h_2 \rightharpoonup g)f) \leftharpoondown h_1 \\ &= \sum(\langle h_2 \rightharpoonup g, f_1\rangle f_0) \leftharpoondown h_1 \\ &= \sum\langle h_2 \rightharpoonup g, f_1\rangle (f_0 \leftharpoondown h_1) \\ &= \sum\langle g, f_1 h_2\rangle (f_0 \leftharpoondown h_1) \end{aligned}$$

Therefore ρ is an H-module map, and so by Definition 3.14 H^* is a right H-Hopf module.

□

Theorem 4.6. *Let H be a finite dimensional Hopf algebra. Then*

1. *$\int_H^l$ and $\int_H^r$ are each one dimensional vector spaces.*
2. *The antipode S of H is bijective, and $S(\int_H^l) = \int_H^r$.*

Proof.

1) We begin with Theorem 4.5 that states H^* is a right H-Hopf module. This also means H is a right H^*-Hopf module, since it is also true that H is the dual space of H^*. Thus by the Fundamental Theorem of Hopf Modules, Theorem 3.16, we see that $H \otimes H^{coH^*} \cong H^*$. Also $\dim H^* = \dim H$ (since this is true for any finite dimensional vector space [Lam76]), then $\dim(H^{coH^*}) = 1$. But by Lemma 3.10, we have that:

$$H^{coH^*} = H^H = \{h \in H \mid ht = \varepsilon(h)t, \forall t \in H\} = \int_H^l$$

where ht represents left multiplication in H. Since $\dim(H^{coH^*}) = 1$ and $H^{coH^*} = \int_H^l$, then $\dim \int_H^l = 1$. Before we show that $\int_H^r$ is one dimensional, we will need to prove part 2) of the theorem.

2) To show that S is bijective, let $f \in \int_{H^*}^l$ and $f \neq 0$. If $h \in kerS$, and $\alpha : H^* \otimes H \to H$ is the map from the proof of Theorem 3.16, then:

$$\alpha(f \otimes h) = Sh \rightharpoonup f = 0$$

Since α is an isomorphism, and thus injective, then $f \otimes h = 0$. Thus since $f \neq 0$ and H^* has no zero divisors, then $h = 0$. So since $kerS = 0$ then S is injective. Also, since H is finite dimensional, then S is bijective.

We can now show that $S(\int_H^l) = \int_H^r$. Let $tin \int_H^l$ and let $h \in H$. We need to show that $S(t) \in \int_H^r$. This one direction is sufficient since S is bijective. First, since S is bijective then for all $h \in H$ there exist $b \in H$ such that $S(b) = h$.

$$\begin{aligned} hS(t) &= S(b)S(t) \\ &= S(tb) \\ &= S(\varepsilon(b)t) \\ &= \varepsilon(b)S(t) \\ &= \varepsilon(S^{-1}h)S(t) \\ &= \varepsilon(h)S(t) \end{aligned}$$

Thus $S(\int_H^l) = \int_H^r$, and $\int_H^r$ is one dimensional since $\int_H^l$ is one dimensional. □

For completeness, we will include the following definitions and result that will be necessary for the last section. Some of the results list are include without proof, but a reference as to where the proof can be found is provided.

Definition 4.7. Let A be an algebra and let $\{M_i\}_{i \in I}$ be a family of left A-modules.

1. Let $m_i \in M_i$ for all $i \in I$. The *direct sum* of $\{M_i\}_{i \in I}$ (denoted $\oplus_{i \in I} M_i$) is the set of all finite formal sums $\sum_i m_i$. For a finite family we write $M_1 \oplus \cdots \oplus M_n$. Also by $M_i^{(r)}$ we mean the direct sum of r copies of M_i
2. A left module M is called *indecomposable* if it is not the direct sum of non-zero submodules.
3. The *principal indecomposable modules* of A (denoted P_i) are all the indecomposable A-modules which are also a submodule of A.

Definition 4.8. Let A be an algebra and M be a left A-module. M is called *simple* if the only subsets of M that are also left A-modules are $\{0\}$ and M. M is called *semisimple* if it is the direct sum of simple modules

Theorem 4.9. *Let A be an algebra and M be a right A-module. M is a free right A-module $\Leftrightarrow M \cong A^{(s)}$ for some $s > 0$.*

Proof. This proof simply shows how the definition of the first statement is equivalent to the definition of the second statement.

M is a free right A-module

$\Leftrightarrow$ There exist a set $\{m_i\}$ such that if $m \in M$ then $m = \sum a_i m_i$ for a finite number of $a_i \in A$

$\Leftrightarrow m = \sum \alpha_i a_i$ for $\alpha_i \in k$ (by the second diagram of the definition of a module)

$\Leftrightarrow M \cong A^{(s)}$ for some $s > 0$. $\square$

Theorem 4.10. **(Classic Krull-Schmidt Theorem [Lam76])** *If A is an algebra and M is a right A-module, then every indecomposable summand of M is a principal indecomposable right A-module.*

Theorem 4.11. **(Theorem 59.3 in [CR62])** *Let A be an algebra, P_i be the principle indecomposable modules of A, and let M be any right A-module. M is faithful $\Leftrightarrow$ each P_i is isomorphic to a summand of M.*

4.2 The Nichols-Zoeller Theorem

The next series of lemmas and theorems follows the original sequence presented in [NR92]. This sequence then ends with the original Nichols-Zoeller theorem. With the exception of Lemma 4.12 (which is proposition 1 from [Rad78]), an outline of each proof was provided in the original paper. This lemma will be stated without proof. Note that for the first lemma H need only be a bialgebra which is not necessarily a Hopf algebra. This is fine since the definition of a Hopf module does not use the antipode S in its definition, so we can still have a Hopf module over a bialgebra.

Lemma 4.12. *Let H be a bialgebra, and let K be a finite dimensional subbialgebra of H. If every finite dimensional left (H, K)-Hopf module M is free over K, then every left (H, K)- Hopf module M is free over K.*

Lemma 4.13. *Let H be a Hopf algebra with bijective antipode, let K be a subHopfalgebra, and let M be a right (H,K)-Hopf module. Then $M \otimes H \cong M^{(dimH)}$ as right K-modules.*

Proof. Let H_0 be H as a vector space with the action $h \cdot k = \varepsilon(k)h$ for all $h \in H$ and $k \in K$. H_0 is thus a right K-module that is trivial in the sense that $H_0 = H_0^K$ as K-modules. We then know from Definition 3.13 that $M \otimes H_0$ is a right K-module. Since $M \otimes H_0 \cong M^{(dimH)}$, we need only to show that $M \otimes H_0 \cong M \otimes H$ as right K-modules.

Let $\Phi : M \otimes H_0 \to M \otimes H$ with $\Phi(m \otimes h) = (1 \otimes h)\rho(m) = \sum m_0 \otimes hm_1$. We show that Φ is an isomorphism of vector spaces. First, Using the fact that ρ is a right K-module map (since M is a right (H, K)-Hopf module), we show that Φ is a right K-module map.

$$\begin{aligned}
\Phi(m \otimes h) \cdot k &= (1 \otimes h)\rho(m)k \\
&= (1 \otimes h)\rho(mk) \\
&= \Phi(mk \otimes h) \\
&= \Phi((m \otimes h) \cdot k)
\end{aligned}$$

Next, define $\Psi : M \otimes H \to M \otimes H_0$ as $\Psi(m \otimes h) = \sum m_0 \otimes hS^{-1}(m_1)$. We now show that Φ and Ψ are in fact inverses.

$$\begin{aligned}
\Phi\Psi(m \otimes h) &= \Phi(\sum m_0 \otimes hS^{-1}(m_1)) \\
&= \sum (m_0)_0 \otimes h(S^{-1}m_1)(m_0)_1 \\
&= \sum m_0 \otimes h(S^{-1}m_2)m_1 \\
&= \sum m_0 \otimes h\varepsilon(m_1) \\
&= \sum \varepsilon(m_1)m_0 \otimes h \\
&= m \otimes h
\end{aligned}$$

Thus $\Phi\Psi = id$. Checking the other direction we get:

$$\begin{aligned}
\Psi\Phi(m \otimes h) &= \Psi(\sum m_0 \otimes hm_1) \\
&= \sum m_0 \otimes hm_2S^{-1}(m_1) \\
&= \sum m_0 \otimes h\varepsilon(m_1) \\
&= \sum \varepsilon(m_1)m_0 \otimes h \\
&= m \otimes h
\end{aligned}$$

So we also have that $\Psi\Phi = id$. Thus Φ is indeed an isomorphism of K-modules. $\square$

Lemma 4.14. *Let K be a Hopf algebra, and let W be a right K-module. Then $W \otimes K$ is free over K. If the antipode is bijective, then $K \otimes W$ is free over K.*

Proof. From [Swe69] $W \otimes K$ has a right K-comodule structure via $\rho = id \otimes \Delta : W \otimes K \to (W \otimes K) \otimes K$ defined as $w \otimes h \mapsto (w \otimes h_1) \otimes h_2$. We also have a right K action defined as $m \otimes h) \cdot k = (m \otimes h)(1 \otimes k) = m \otimes hk$. We show that ρ is a K-module map. Let $h, k \in K$ and $w \in W$.

$$\begin{aligned}
\rho(w \otimes h) \cdot k &= (w \otimes h_1 \otimes h_2) \cdot k \\
&= (w \otimes h_1 \otimes h_2)(1 \otimes k_1 \otimes k_2) \\
&= w \otimes h_1k_1 \otimes h_2k_2 \\
&= w \otimes (hk)_1 \otimes (hk)_2 \\
&= \rho(w \otimes hk) \\
&= \rho((w \otimes h)(1 \otimes k)) \\
&= \rho((w \otimes h) \cdot k)
\end{aligned}$$

Thus $W \otimes K$ is a right K-Hopf module, so by the Fundamental Theorem of Hopf Modules, Theorem 3.16, $W \otimes K$ is free over K. $\square$

Lemma 4.15. *Let A be a finite-dimensional algebra, and let W be a finitely-generated left A-module. Then there exist a positive integer r such that $W^{(r)} \cong F \oplus E$ as A-modules, where F is free and E is not faithful.*

Proof. Let W be any finitely generated left A-module. Consider the two cases of when W is not faithful and when W is faithful. For the case when W is not faithful, then the conclusion holds for $r = 1$, $F = (0)$, and $W = E$.

Consider the case when W is faithful. Let $P_1, \dots, P_t$ be the principal indecomposable left A-modules, or $A \cong P_1^{(n_1)} \oplus \cdots \oplus P_t^{(n_t)}$ as A-modules. Let r be the least common multiple of $n_1, \dots, n_t$. Note that by Theorem 59.3 of [CR62], W is faithful if and only if each P_i is isomorphic to a summand of W. This allows us to write $W \cong P_1^{(w_1)} \oplus \cdots \oplus P_t^{(w_t)} \oplus Q$ where no summand of Q is a principal indecomposable module. Then we have $W^{(r)} \cong P_1^{(rw_1)} \oplus \cdots \oplus P_t^{(rw_t)} \oplus Q^{(r)}$. Now let s be the minimum of $\frac{rw_1}{n_1}, \dots, \frac{rw_t}{n_t}$, thus $s = \frac{rw_j}{n_j}$ for some j. We can regroup our P_i's so that we get $W^{(r)} \cong A^{(s)} \oplus E$, where E is not isomorphic to any summand of P_j because all copies of P_j have been placed in $A^{(s)}$. We know that $A^{(s)}$ if always free as a left A-modules, so we may set $F = A^{(s)}$. It follows from theorem 59.3 of [CR62] that since there is one principal indecomposable module P_j that is not isomorphic to a summand of E, that E is not faithful. thus $W^{(r)} \cong F \oplus E$ where where F is free and E is not faithful. □

Lemma 4.16. *Let K be a finite-dimensional Hopf algebra and W a finitely generated left K-module. If there exists an integer $r > 0$ such that $W^{(r)}$ is free over K, then W is free over K.*

Proof. We begin by writing $K = K_1 \oplus \cdots \oplus K_N$ as a direct sum of principal indecomposable left K-modules. Let λ be a nonzero left integral of K. We can write $\lambda = \lambda_1 + \cdot + \lambda_n$ where $\lambda_i \in K_i$ for $i = 1, \dots, n$. Since $\int_K^l$ is a vector space, and thus closed under addition, then each λ_i is also left integral. Also since by Theorem 4.6 $\int_K^l$ is one dimensional we must have that $\lambda = \lambda_i$ for some i. Without loss of generality, we can say that $\lambda = \lambda_1$, so $\lambda_i = 0$ for $i > 1$. This tells us for $i > 1$, K_i cannot contain any nonzero left integral of K, so K_i is not isomorphic to K_1.

Next, let $P_1, P_2, \dots, P_t$ be the principal indecomposable left K-modules, with $P_1 = K_1$ from above. We write $K \cong K_1^{(n_1)} \oplus P_2^{(n_2)} \oplus \cdots \oplus P_t^{(n_t)}$ as left K-modules. Since we have that K_1 contains all the left integrals of K, then $n_1 = 1$. Now, let W

be a finitely generated left K-module such that $W^{(r)}$ is free as a left K-module for some integer $r > 0$. We say that $W^{(r)} \cong K^{(s)}$ for some integer $s > 0$ since this is true for any free module. By the classic Krull-Schmidt theorem, every indecomposable summand of W is a principal indecomposable left K-module [Lam76]. Thus we can write $W \cong K_1^{(w_1)} \oplus P_2^{(w_2)} \oplus \cdots \oplus P_t^{(w_t)}$, and so $W^{(r)} \cong K_1^{(rw_1)} \oplus P_2^{(rw_2)} \oplus \cdots \oplus P_t^{(rw_t)}$. Since $W^{(r)} \cong K^{(s)}$ then $rw_i = sn_i$ for $i = 1, ..., t$. We already know that $n_1 = 1$ so $rw_1 = s$. This gives us that $rw_i = rw_1 n_i$. Thus $w_i = w_1 n_i$ for all i, and so $W \cong K_1^{(w_1 n_1)} \oplus P_2^{(w_1 n-2)} \oplus \cdots \oplus P_t^{(w_1 n_t)} \cong K^{(w_1)}$ or $W \cong K^{(w_1)}$. Therefore W is a free left K-module. □

The two theorems that end this section are the main results of the original paper. The first one is a partial converse of Lemma 4.14, and the second is the Nichols-Zoeller Theorem.

Theorem 4.17. *Let K be a finite-dimensional Hopf Algebra and W a finitely-generated right K-module. Suppose there exists a finitely-generated faithful right K-module L such that $W \otimes L \cong W^{(dimL)}$ as right K-modules. Then W is free over K.*

Proof. First by Lemma 4.16 it suffices to show that for some integer $r > 0$ $W^{(r)}$ is a free left K-module. By Lemma 4.15 we know that there exists $r > 0$ such that $W^{(r)} \cong F \oplus E$ where F is free and E is not faithful.

Now suppose that there exist a finitely generated faithful left K-module L such that $L \otimes W \cong W(dimL)$. By the additive property of a tensor we have that $L \otimes W^{(r)} \cong (L \otimes W)^{(r)}$, and then by the above we would have $(L \otimes W)^{(r)} \cong W^{(rdimL)} = (W^{(r)})^{(dimL)}$. Since we only need to show that the statement is true for $W^{(r)}$, then without loss of generality we may replace $W^{(r)}$ with W, which tells us $W \cong F \oplus E$. Similarly we may replace L with $L^{(r')}$ for some positive integer r', thus by Lemma 4.15 we may assume $L \cong F' \oplus E'$, where F' is free and E' is not faithful. Also, since L is faithful then L is not isomorphic to E', and so F' is a nonzero free left K-module.

Let $t = dimL$. We now get the following isomorphisms:

$$\begin{aligned}
F^{(t)} \oplus E^{(t)} &\cong (F \oplus E)^{(t)} && \text{By the definition of a direct sum} \\
&\cong W^{(t)} && \text{Since } F \oplus E \cong W \\
&\cong L \otimes W && \text{Since } L \otimes W \cong W^{(dimL)} \\
&\cong L \otimes (F \oplus E) && \\
&\cong (L \otimes F) \oplus (L \otimes E) && \text{By the additive property of a tensor product}
\end{aligned}$$

Now Lemma 4.14 tells us $F^{(t)} \cong L \otimes F$ so $E^{(t)} \cong L \otimes E$. This then gives us the following isomorphisms.

$$\begin{aligned}
E^{(t)} &\cong L \otimes E && \\
&\cong (F' \oplus E') \otimes E && \text{Since } L \cong F' \oplus E' \\
&\cong (F' \otimes E) \oplus (E' \otimes E) && \text{By the additive property of a tensor product}
\end{aligned}$$

If $E \neq 0$, then by Lemma 4.14, $F' \otimes E$ is a nonzero free left K-module, but this is not possible since since $E^{(t)}$ is not faithful. So $E = 0$ and thus $W = F$, and so W is free. $\square$

Theorem 4.18. Nichols-Zoeller Theorem. *Let H be a finite dimensional Hopf algebra and let K be a subHopfalgebra. Then every right (H,K)-Hopf module is free as a right K-module. In particular H is free as a right K-module.*

Proof. By Lemma 4.12 it is sufficient to show that every finite dimensional right (H, K)-Hopf algebra is free as a right K-module. By Lemma 4.13 we know that $M \otimes H \cong M^{(dimH)}$ as right K-modules. Since H itself is faithful as a right K-module and finite dimensional (thus finitely generated), then we can apply Theorem 4.17 to see that M is free as a right K-module. $\square$

4.3 Generalizations of The Nichols-Zoeller Theorem

After the publication of Theorem 4.18, the question of "How far can we generalize?" became the focus of researchers. Since then many important generalizations have been proven. The first important generalization was made three years after the original theorem by Nichols and Richmond (previously Zoeller). This generalization specified that if the Hopf algebra was not finite dimensional, then it would still be free if the subHopfalgebra was finite dimensional and semisimple. We will prove this generalization following the original paper [NR92]. After this we will state, without proof, a few other generalizations of the Nichols-Zoeller Theorem. We begin with a lemma which is a generalization of Lemma 4.13 in the previous section.

Lemma 4.19. *Let H be a Hopf algebra with antipode S, let K be a subHopfalgebra of H, and let M be a (H,K)-Hopf module. Consider $(H \otimes M, *)$ to be a left K-module via the action $k * (h \otimes m) = \sum hS(k_1) \otimes k_2 m$ for all $k \in K$, $h \in H$ and $m \in M$. Then $(H \otimes M, *) \cong M^{(dimH)}$ as left K-modules.*

Proof. Let H_0 be the vector space as in the proof of Lemma 4.13. From that proof H_0 is a left K-module via the action $k \cdot h = \varepsilon(k)h$ for all $kinK$ and $h \in H$. and so we had that $H_0 \otimes M \cong M^{(dimH)}$ as K-modules. Define $F : H_0 \otimes M \to (H \otimes M, *)$ as $F(h \otimes m) = sumhS(m_1) \otimes m_2$ for all $h \in H$ and $m \in M$. Also define $T : (H \otimes M, *) \to H_0 \otimes M$ as $T(h \otimes m) = \sum hm_1 \otimes m_2$ for all $h \in H$ and $m \in M$. We begin by showing that F is a K-module map.

$$\begin{aligned} k * F(h \otimes m) &= k * (\textstyle\sum hS(m_1)m_2) \\ &= \textstyle\sum hS(m_1)S(k_1) \otimes k_2 m_2 \\ &= \textstyle\sum hS(k_1 m_1) \otimes k_2 m_2 \\ &= \textstyle\sum hS((km)_1) \otimes (km)_2 \\ &= F(h \otimes km) \end{aligned}$$

Thus F is a K-module map. We next need to show that $T = F^{-1}$.

$$\begin{aligned}
FT(h \otimes m) &= F(\textstyle\sum hm_1 \otimes m_2) \\
&= \textstyle\sum hm_1 S((m_2)_1) \otimes (m_2)_2 \\
&= \textstyle\sum hm_1 S(m_2) \otimes m_3 \\
&= \textstyle\sum h\varepsilon(m_1) \otimes m_2 \\
&= \textstyle\sum h \otimes \varepsilon(m_1)m_2 \\
&= h \otimes m
\end{aligned}$$

$$\begin{aligned}
TF(h \otimes m) &= T(\textstyle\sum hS(m_1) \otimes m_2) \\
&= \textstyle\sum hS(m_1)(m_2)_1 \otimes (m_2)_2 \\
&= \textstyle\sum h(Sm_1)m_2 \otimes m_3 \\
&= \textstyle\sum h\varepsilon(m_1) \otimes m_2 \\
&= \textstyle\sum h \otimes \varepsilon(m_1)m_2 \\
&= h \otimes m
\end{aligned}$$

Thus $T = F^{-1}$, and so F is an isomorphism of K-modules. □

Lemma 4.20. *Let H be a Hopf algebra, and let K be a subHopfalgebra whose antipode S is injective. Then every nonzero (H,K)-Hopf module M is faithful as a left K-module.*

Proof. Consider the K-module $(K \otimes M, *)$ as defined in the last proof, the K-module homomorphism $S \otimes id : K \otimes M \to (K \otimes M, *)$, and the inclusion map $i : (K \otimes M, *) \to (H \otimes M, *)$. Since S is injective, then $i \circ (S \otimes id)$ embeds $K \otimes M$ into $(H \otimes M, *)$ as a K-module. By Lemma 4.19 it follows that $M^{(dimH)}$ contains a K-submodule isomorphic to $K \otimes M$. By Lemma 4.14 of the last section, $K \otimes M$ is a free K-module, thus $M^{(dimH)}$ is faithful, and so M is faithful. □

We are now ready to state and prove our generalization. This is proved by contradiction, using the lemma we just proved above.

Theorem 4.21. **(Generalized Nichols-Zoeller Theorem 1)** *Let H be a Hopf algebra, and let K be a finite dimensional, semisimple subHopfalgebra of H. Then H is free as a K-module. More generally, every infinite dimensional (H,K)-Hopf module is free as a left K-module.*

Proof. Let M be an infinite dimensional (H, K)-Hopf module. Assume M is not a free K-module. This tells us there exists some simple K-module L, for which the sum W of submodules of M which are isomorphic L has dimension less than the dimension of M. In other words $\dim W < \dim M$. Let $N = W \leftharpoonup H^*$ be the (H, K)-Hopf module generated by W. Since $\dim N \leq \dim W$, then it is also true that $\dim N < \dim M$. So the quotient M/N is a nonzero (H, K)-Hopf module, but contains no copy of L, since we have removed all copies of L with quotient. Thus M/N is not faithful, a contradiction to Lemma 4.20. Therefore M is a free K-module. $\square$

Nichols and Zoeller also showed in a paper previous to this that the theorem could not be strengthened to the assertion that if H is infinite dimensional, then the theorem holds for every (H, K)-Hopf module (i.e. the finite ones also) [NZ89a]. They also conjectured (but did not provide a counterexample) that it was not possible to remove the condition that K be semisimple. That same year Schneider found that instead of completely removing the condition that the subHopfalgebra K be semisimple, it could be replaced with the property of being normal [Sch93]. The following theorem was the result of this replacement.

Theorem 4.22. **(Generalized Nichols-Zoeller Theorem 2)** *Let H be a Hopf algebra, and let K be a finite dimensional, normal subHopfalgebra of H. Then H is free as a K-module. More generally, every infinite dimensional (H,K)-Hopf module is free as a left K-module.*

More recently, Schauenburg extended the Nichols-Zoeller theorem to another version of a Hopf algebra called a quasi-Hopf algebra [Sch04]. Without giving a formal definition, a quasi-Hopf algebra is basically a Hopf algebra that lacks coassociativity. The result of his paper was the following theorem.

Theorem 4.23. **(Generalized Nichols-Zoeller Theorem 3)** *Let H be a finite dimensional quasi-Hopf algebra, and let $K \subset H$ be a subquasibialgebra which has a quasi-antipode. Then every (H, K)-Hopf module that is finitely generated as a K-module, is free as a left K-module.*

Several other generalizations have been proven, but these replace K in the theorem with something other than a subHopfalgebra. These generalizations are not apparently helpful in settling the question of freeness of Hopf algebras over subHopfalgebras. One question that has not been settled is "Over what type of infinite dimensional subHopfalgebras are Hopf algebras free?" There is also an open question as to whether a weaker version of freeness called *faithful flatness* is true in the general infinite dimensional case. Schauenburg has found several possible counterexamples that depend on properties that are not proven yet, but give an indication that the answer will be negative [Sch00].

The study of Hopf algebras has benefited greatly from the work done on the concept of freeness. This can be seen by the frequency with which the original Nichols-Zoeller theorem is referenced in current research articles. The ability to consider a Hopf algebra as a free module has allowed researchers to expand what is known about Hopf algebras, and will continue to open the doors to other researchers who look to advance knowledge on this subject.

Bibliography

[Abe80] Eiichi Abe. *Hopf algebras*, volume 74 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1980.

[Bor53] Armand Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)*, 57:115–207, 1953.

[Car57] P. Cartier. *Séminaire "Sophus Lie" de la Faculté des Sciences de Paris, 1955-56. Hyperalgèbres et groupes de Lie formels.* Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1957.

[CR62] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras.* Pure and Applied Mathematics, Vol. XI. Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[FD93] Benson Farb and R. Keith Dennis. *Noncommutative algebra*, volume 144 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1993.

[Hop41] Heinz Hopf. Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen. *Ann. of Math. (2)*, 42:22–52, 1941.

[Kap75] Irving Kaplansky. *Bialgebras.* Department of Mathematics, University of Chicago, Chicago, Ill., 1975.

[Kas95] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[Lam76] Joachim Lambek. *Lectures on rings and modules.* Chelsea Publishing Co., New York, 1976.

[MM65] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.

[Mon93] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.

[NR92] Warren D. Nichols and M. Bettina Richmond. Freeness of infinite-dimensional Hopf algebras. *Comm. Algebra*, 20(5):1489–1492, 1992.

[NZ89a] Warren D. Nichols and M. Bettina Zoeller. Freeness of infinite-dimensional Hopf algebras over grouplike subalgebras. *Comm. Algebra*, 17(2):413–424, 1989.

[NZ89b] Warren D. Nichols and M. Bettina Zoeller. A Hopf algebra freeness theorem. *Amer. J. Math.*, 111(2):381–385, 1989.

[OS74] Ulrich Oberst and Hans-Jürgen Schneider. Untergruppen formeller Gruppen von endlichem Index. *J. Algebra*, 31:10–44, 1974.

[Rad78] David E. Radford. Freeness (projectivity) criteria for Hopf algebras over Hopf subalgebras. *J. Pure Appl. Algebra*, 11(1-3):15–28, 1977/78.

[Sch93] Hans-Jürgen Schneider. Some remarks on exact sequences of quantum groups. *Comm. Algebra*, 21(9):3337–3357, 1993.

[Sch00] Peter Schauenburg. Faithful flatness over Hopf subalgebras: counterexamples. In *Interactions between ring theory and representations of algebras (Murcia)*, volume 210 of *Lecture Notes in Pure and Appl. Math.*, pages 331–344. Dekker, New York, 2000.

[Sch04] Peter Schauenburg. A quasi-Hopf algebra freeness theorem. *Proc. Amer. Math. Soc.*, 132(4):965–972 (electronic), 2004.

[Swe69] Moss E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.