2007

Using the concrete-representation-abstract instruction to teach algebra to students with learning disabilities

Edward William Sung

Follow this and additional works at: https://scholarworks.lib.csusb.edu/etd-project

Part of the Science and Mathematics Education Commons

Recommended Citation
https://scholarworks.lib.csusb.edu/etd-project/3263

This Project is brought to you for free and open access by the John M. Pfau Library at CSUSB ScholarWorks. It has been accepted for inclusion in Theses Digitization Project by an authorized administrator of CSUSB ScholarWorks. For more information, please contact scholarworks@csusb.edu.
USING THE CONCRETE-REPRESENTATION-ABSTRACT INSTRUCTION TO TEACH ALGEBRA TO STUDENTS WITH LEARNING DISABILITIES

A Project
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Education:
Special Education

by
Edward William Sung
June 2007
USING THE CONCRETE-REPRESENTATION-ABSTRACT INSTRUCTION TO TEACH ALGEBRA TO STUDENTS WITH LEARNING DISABILITIES

A Project
Presented to the
Faculty of
California State University,
San Bernardino

by
Edward William Sung
June 2007

Approved by:

Gary Sherwin, Ph.D., First Reader

Thom Gehring, Ph.D., Second Reader
ABSTRACT

This project explored the Concrete to Representational to Abstract instruction (CRA instruction) as a strategy to teach abstract math concepts for secondary students with learning disabilities. Through the review of literature, multiple researchers suggested that students with learning disabilities need to be exposed to a variety of instructional strategies to develop problem-solving skills in algebra concepts.

The goal of the project was to emphasize the use of the CRA instruction method to deliver a better understanding of math concepts for students with learning disabilities. Furthermore, examples using CRA instruction were presented to offer comprehensible samples for special education teachers to utilize CRA instruction method to teach algebra concepts effectively to students with learning disabilities.
ACKNOWLEDGEMENTS

I would like to thank Dr. Sherwin and Dr. Gehring for their support and guidance in this project. The profound knowledge they gave me made it possible to complete this project.

Special thanks to my colleagues at Coachella Valley High School, Valerie Duffin, Sia Lux and Steve McCambly for reading this project and providing valuable advice.

Finally, I dedicate this project to my two daughters, Isabella and Victoria and my wife Martha for their support and endurance. Especially, my sincere appreciation goes to my wife Martha for her endless help these past few years.
# TABLE OF CONTENTS

**ABSTRACT** ................................................................. iii  
**ACKNOWLEDGEMENTS** ................................................. iv  

**CHAPTER ONE: BACKGROUND**

Introduction ................................................................. 1  
Significance of the Project ............................................. 3  
Definitions ................................................................. 4  
Assumptions ............................................................... 7  
Limitations ................................................................. 7  

**CHAPTER TWO: REVIEW OF RELATED LITERATURE**

Introduction ................................................................. 10  

What Cognitive Factors Influence Learning Math and How do these Factors Affect Learning Algebra Concepts? ........................................ 11  

What are the Characteristics of Students with Learning Disability? ................................. 15  

Why is Algebra Important for High School Students Including Students with Learning Disability? ............................................. 17  

What Instructional Strategies Can Be Implemented to Teach Algebra for Students with Learning Disability? ............................................. 19  

What does the Research Say about the Concrete-Representation-Abstract Instruction? ........ 24  

What are the Benefits in Using the Concrete-Representation-Abstract Instruction to Teach Algebra for Students with Learning Disability? ............................................. 26
CHAPTER THREE: CURRICULUM PROJECT

Introduction ........................................ 30

Discussion of Concrete-Representation-Abstract Examples ........................................ 32

  Adding and Subtracting Integers .............. 33
  Multiplying Integers ............................. 34
  Dividing Integers ................................. 35
  Multiplying Polynomials .......................... 36
  Factoring Trinomials .............................. 38

CHAPTER FOUR: CONCLUSION ........................................ 40

APPENDIX A: CONCRETE-REPRESENTATION-ABSTRACT EXAMPLES:
  ADDING INTEGERS ................................. 43

APPENDIX B: CONCRETE-REPRESENTATION-ABSTRACT EXAMPLES:
  SUBTRACTING INTEGERS ........................... 48

APPENDIX C: CONCRETE-REPRESENTATION-ABSTRACT EXAMPLES:
  MULTIPLYING INTEGERS ............................ 52

APPENDIX D: CONCRETE-REPRESENTATION-ABSTRACT EXAMPLES:
  DIVIDING INTEGERS ............................... 57

APPENDIX E: CONCRETE-REPRESENTATION-ABSTRACT EXAMPLES:
  MULTIPLYING POLYNOMIALS ....................... 61

APPENDIX F: CONCRETE-REPRESENTATION-ABSTRACT EXAMPLES:
  FACTORING TRINOMIALS ............................ 66

REFERENCES ........................................ 71
CHAPTER ONE

BACKGROUND

Introduction

High school students with learning disabilities (LD) have difficulties with algebra because they are required to have abstract thinking skills. Many students with LD are also overwhelmed with algebra concepts because they do not possess the basic math skills needed before they are exposed to algebra concepts. Bryant (2005) indicated that students with LD in mathematics have difficulties in learning new math skills and concepts.

According to Cawley and Miller’s 1989 study, junior and senior high school students with LD achieved between 5th and 6th grade level in math computation skills. This study showed that most LD students in this age group perform at about the 5th grade level. Balckorby and Wagner (1996) found that approximately one out of ten high school students with disabilities took algebra and other higher math courses according to the survey conducted by the National Longitudinal Transition Study of Special Education Students in 1990. In secondary level math, many
students with LD experience considerable difficulty with the subject.

Over the years, it has been important for high school students to complete algebra courses successfully. Kortering, deBettencourt and Braziel (2005) indicated that it is important for high school students including students with LD to possess the necessary knowledge of algebra to succeed during high school. This knowledge will also help students to further their education. Furthermore, in California, all high school students are required to pass the California High School Exit Exam to receive a high school diploma. The California High School Exit Exam includes knowledge in basic algebra (California Department of Education, 2006).

Although algebra is an essential subject, there are many obstacles for students with LD to learn algebra concepts. In 2000, Maccini and Hughes pointed out that many high school students with LD do poorly in algebra, because it "requires knowledge of fundamental skills and terminology, problem presentation, problem solution, and self-monitoring" (p. 10).

This project examines how the Concrete to Representational to Abstract sequence of instruction (CRA
instruction) will help high school students with LD to conceptualize algebra concepts. CRA instruction is "an intervention for mathematics instruction that research suggests can enhance the mathematics performance of students with learning disabilities" (The Assess Center Improving Outcomes for All Students K-8, 2004, p. 1). I will discuss the definitions of related terms regarding learning disabilities and CRA instruction, point out the relationship between learning disabilities and learning abstract concepts, and reveal the benefits of applying CRA instruction to teach algebra concepts, and also explain the limitations of CRA instruction. I will also show examples of CRA instruction that teachers can use to deliver algebra concepts to students with LD.

Significance of the Project

Many high school students with LD experience difficulty with algebra concepts (Kortering, deBettencourt & Braziel, 2005). Over the past years, teachers attempted to use a variety of strategies to teach algebra concepts to students with LD. However, an extensive literature review reveals that no one has determined what teaching
strategy is most effective to deliver algebra concepts for students with LD.

According to various research studies, CRA instruction is effective in teaching algebra concepts to students with LD. This project will be informative for high school special education teachers who need more efficient instructional methods to teach algebra concepts and other related math concepts to students with LD. In addition, both high school special education teachers and general education teachers who teach students with LD will benefit by using CRA instruction. They will better understand CRA instruction as an effective instructional strategy to teach algebra concepts.

Definitions

For the purpose of this study the following definitions were used:

Specific Learning Disability (SLD) is defined as: A disorder in one or more of the basic psychological processes involved in understanding or in using language, spoken or written, which disorder may manifest itself in the imperfect ability to listen, think, speak,
read, write, spell, or do mathematical calculations (The Individuals with Disabilities Education Improvement Act (IDEA), 2004, STAT. 2657-2658, P. 118).

A math disability is defined as: ... an inability making sufficient school progress in mathematics similar to that of her peer group despite the implementation of effective teaching practices over time (Bryant, 2005, p. 1).

Algebra is defined as: a branch of mathematics that substitutes letters for numbers... Algebra can include real numbers, complex numbers, matrices, vectors etc. Moving from arithmetic to algebra will look something like this: arithmetic: $3 + 4 = 3 + 4$ in algebra it would look like: $x + y = y + x$ (Russell, 2006, p. 1).

Concrete-Representational-Abstract Sequence of Instruction (CRA instruction) is defined as: a three-part instructional strategy.

(a) Concrete: In the concrete stage, the teacher begins instruction by modeling each mathematical concept with concrete materials
(e.g., red and yellow chips, cubes, base-ten blocks, pattern blocks, fraction bars, and geometric figures).

(b) Representational: In this stage, the teacher transforms the concrete model into a representational (semiconcrete) level, which may involve drawing pictures; using circles, dots, and tallies; or using stamps to imprint pictures for counting.

(c) Abstract: At this stage, the teacher models the mathematics concept at a symbolic level, using only numbers, notation, and mathematical symbols to represent the number of circles or groups of circles. The teacher uses operation symbols (+, −, ×, ÷) to indicate addition, multiplication, or division (The Assess Center Improving Outcomes for All Students K-8, 2004, p. 1).
Assumptions

For the purpose of the project the following assumptions were made:

1. All students are able to learn regardless of their disabilities.
2. Students with LD have different learning styles and learn at different speeds (Gagnon & Maccini, 2001).
4. Special education mathematics teachers are able to help students with LD learn algebra concepts by using a variety of teaching strategies.

Limitations

One of the main limitations of using CRA instruction is teacher commitment and confidence. As a result of lack of training and practice, Hardy (2005) wrote that “teachers may not trust the usefulness or efficiency of manipulative objects for higher level algebra” (p. 20). One study (Howard, Perry & Tracey, 1997) indicated that the majority of elementary school teachers used
manipulatives weekly, but middle and high school teachers used them monthly or seldom.

The second limitation is time. There are three phases to complete CRA instruction: "1) concrete representations, 2) pictorial representations and 3) abstract reasoning" (Allsopp, 2006, p. 1). In order to apply CRA instruction effectively, teachers need to spend time to become familiar with using the materials.

Howard, Perry and Tracy (1997) found that today many high schools have neither flexible schedules nor enough instructional preparation time to use manipulatives. These high schools have also adopted certain math programs which make it difficult for teachers to use. Inflexible schedule and insufficient preparation time may discourage teachers from using manipulatives.

A planning period is also crucial. Some teachers are confident in using manipulatives; however, they do not have sufficient knowledge to use manipulatives successfully (Hardy, 2004). Teachers are required to spend ample planning time to be effectively implemented in CRA instruction.

The third limitation in CRA instruction may be financial constraints. A limited school budget may
inhibit the acquisition of required materials and objects to use in the first phase of concrete objects, and the second phase of pictorial representations.
CHAPTER TWO

REVIEW OF RELATED LITERATURE

Introduction

Many secondary math teachers believe that the use of manipulatives benefit students' understanding of algebra. But some educators often think that manipulatives are used only at the primary level of learning math (Howard, Perry & Tracey, 1997; Sobel & Maletsky, 1988). However, using manipulatives to teach algebra at the secondary level is also important to assist students with LD approach algebra concepts in a more meaningful way. Sobel and Maletsky (1988) wrote that algebra is an abstract subject. Therefore, when teachers introduce new algebra concepts for students with LD, they need to motivate these students by applying various manipulative teaching means such as visual aids and hands-on activities.

The purpose of this project is to devise a guide that will assist secondary algebra teachers who want to use CRA instruction to teach algebra concepts for students with LD. In order to understand the importance and benefit CRA instruction for students with LD, the following questions are addressed:
What Cognitive Factors Influence Learning Math and How do these Factors Affect Learning Algebra Concepts?

What are the Characteristics of Students with Learning Disability?

Why is Algebra Important for High School Students Including Students with Learning Disability?

What Instructional Strategies Can Be Implemented to Teach Algebra for Students with Learning Disability?

What does the Research Say about the Concrete-Representation-Abstract Instruction?

What are the Benefits in Using the Concrete-Representation-Abstract Instruction to Teach Algebra for Students with Learning Disability?

What Cognitive Factors Influence Learning Math and How do these Factors Affect Learning Algebra Concepts?

In 2005, the U.S. Department of Education reported that of the thirteen categories of special education services, the largest group of students qualifies under the category of specific learning disabilities. Lerner (1985) found that specific learning disabilities at primary schools were revealed differently from these at the secondary school level.
Many primary grade students with LD have difficulty focusing on academic work. In many cases this factor inhibits students with LD from learning basic math concepts and skills. When the inability to focus continues to increase throughout the elementary school years, students with LD become discouraged and do not perform well (McDevitt & Ormrod, 2002).

In 2003, Blackorby, Chrost, Garza and Guzman found that the average high school student with LD reads at least three grades below or more compared to general education high school students. Heward (1996) also wrote that at the secondary level students depend highly on reading and writing for academic subjects.

According to Reisman and Kauffman (1980), in order to teach math concepts to students with LD, a teacher does not need to know how a student with LD is severely disabled. This fact does not clarify whether students with LD can learn math or not; moreover, it does not explain how teachers should teach math concepts for students with LD.

It is, however, helpful to know what kind of cognitive factors inhibit the learning of mathematics.
The following are several cognitive considerations that influence learning mathematics (Reisman & Kauffman, 1980):

[a]. Rate and amount of learning compared to age peers.
[b]. Speed of learning related to specific content.
[c]. Ability to retain information.
[d]. Need for repetition.
[e]. Verbal skills.
[f]. Ability to learn symbol system and arbitrary associations.
[g]. Size of vocabulary compared with peers.
[h]. Ability to form relationships, concepts, and generalizations.
[i]. Ability to attend to salient aspects of a situation.
[j]. Use of problem-solving strategies.
[k]. Ability to make decisions and judgments.
[l]. Ability to draw inferences and conclusions and to hypothesize.
[m]. Ability in general to abstract and to cope with complexity. (p. 3).

Among these factors we might tie one characteristic which is a student’s “ability to learn symbol systems and
arbitrary associations" (Reisman & Kauffman, 1980, p. 3). It will be beneficial for teachers to understand this critical cognitive ability to teach algebra concepts to students with LD.

In mathematics, a variety of symbols are used to communicate mathematics and solve math problems. "In order to communicate thoughts... There must be a conventional system of signs or systems which, when used by some persons, are understood by other persons receiving them" (Gelb, 1963, p. 1). The understanding of a conventional system of mathematics symbols is essential for students with LD to understand and solve math problems, including algebra and higher levels of math.

Reisman and Kauffman (1980) found that many students with LD have difficulty understanding algebra concepts if they do not have the knowledge of math terms and language. These students are often confused and do not understand symbolic function.

Arieti (1976) stated that "the general process of cognition... is based on two fundamental characteristics: progressive abstraction and progressive symbolization... and that abstraction leads to symbolization" (p. 54). As a result, this makes it easier for students with LD to
understand abstract concepts when teachers use concrete demonstrations of numbers. This abstraction is also related to symbols.

The other factor we need to consider is the "ability to think and cope with abstract and complex problems" (Reisman & Kauffman, 1980, p. 3). Students with LD need thinking abilities to solve abstract and complex problems. These abilities facilitate them to "find logical relationships or analogies, perform simple operations of logical deductions, use similes and metaphors, and solve constructive tasks" (Reisman & Kauffman, 1980, p. 7-8).

In 2001, Gagnon and Maccini mentioned that students with LD have difficulty with higher level mathematics which require abstract and complex problem solving skills. Therefore, new math strategies should be presented with concrete representations for those students with LD.

What are the Characteristics of Students with Learning Disability?

Many students with LD face difficulties with a various area of mathematics because of their learning disabilities. These disabilities often hinder their academic performance (Maccini & Gagnon, 2001). Maccini
and Gagnon (2001) characterize students with learning disabilities related to mathematics as having:

[a]. Difficulty processing information, which results in problems learning to read and problem-solve.
[b]. Difficulty with distinguishing the relevant information in story problems.
[c]. Low motivation, self-esteem, or self-efficacy to learn because of repeated academic failure.
[d]. Problems with higher-level mathematics that require reasoning and problem-solving skills.
[e]. Reluctance to try new academic tasks or to sustain attention to task.
[g]. Difficulty with arithmetic, computational deficits. (p. 8).

Among these characteristics, reasoning and problem-solving skills can be particularly difficult for students with LD because these skills necessitate the combination of algebra and problem solving skills. Those who have difficulties with algebra concepts also have difficulties with arithmetic skills. The California Department of Education, Mathematics Framework for California Public
Schools (2000) stated that many students have trouble transferring from arithmetic skills to algebra concepts because the lack of understanding in symbolic system of algebra.

In many cases students with LD have difficulty finding information that is related to solving word problems. It is difficult for them to comprehend math questions. To solve word problems, students with LD need good math computation skills and an understanding of related math terms and language.

Why is Algebra Important for High School Students Including Students with Learning Disability?

To understand algebra, students with LD must understand the math language by which it is communicated (National Council of Teachers of Mathematics, 2000). Educators believe "algebra is considered a gateway to abstract thought" (Witzel, Mercer & Miller, 2003, p. 121). Algebra is a key to open doors to access enormous ideas (Witzel, Mercer & Miller, 2003).

Algebra is a fundamental course to higher level math classes and sciences. Algebra is also essential for students who pursue a postsecondary education. High school students, who successfully take and pass algebra
courses, are enabled to pursue higher math and education courses. Without a strong background in algebra skills, it would be difficult for students to be successful in further postgraduate study.

Moreover, many states require high school students to pass a high school graduation exam that consists of the knowledge in algebraic understanding. The State of California requires all high school students, including English language learners and students with learning disabilities, to pass the California high school exit exam in order to receive a high school diploma.

However, many high school special education math teachers have difficulties teaching algebra concepts to students with LD. One main reason is that students with LD have difficulties in processing abstract thinking problems. Another reason is that some high school special education teachers feel they are not trained adequately to teach algebra concepts.

According to the Center for the Future of Teaching and Learning (2004), from the 2003 to 2004 school year survey in California, nearly 30% of special education teachers were not fully credentialed. In these schools the student population consisted of over 90% minorities.
What Instructional Strategies Can Be Implemented to Teach Algebra for Students with Learning Disability?

The U.S. Department of Education, National Center for Education Statistics (2006) reported that during the 2003 and 2004 school year, nearly 6% of K-12 public school students were receiving special education services. This group had a specific learning disability under the Individual with Disabilities Education Act (IDEA). Students' learning disabilities prevent them from learning algebra concepts in a traditional classroom environment. They cannot benefit fully from a conventional education approach.

In traditional math classes, many secondary students with LD often do not see algebra concepts as integrated with other math concepts. For example, students with LD need to learn necessary previous skills such as integer rules before solving equation and inequality problems. However, students with LD who have used manipulatives understand the process of solving equations and inequalities as integrated steps (Insights into algebra 1, 2007).

A mathematics curriculum is a key for successful teaching and learning algebra concepts to deliver lessons
effectively to students with LD. A math curriculum should consist of simple steps and explanations to provide a better understanding of the math concepts and terms (Stacey & MacGregor, 1997).

According to Gagnon and Maccini (2001), teachers can increase students’ understanding levels by integrating visual and hand-on activities such as objects and manipulatives. When students with LD use objects and manipulatives to learn algebra concepts, they understand better regardless of their grade levels. Maccini and Gagnon (2000) suggested the following strategies when teachers apply manipulatives with students with LD:

[a]. Select manipulatives that are connected to the concept and to students’ developmental levels.

[b]. Incorporate a variety of manipulatives for concept exploration and attainment.

[c]. Provide verbal explanations and questions with demonstrations.

[d]. Provide opportunities for student interaction and explanation.

[e]. Encourage the use of manipulatives and strategies across settings.
Program for transition from concrete to symbolic representation. (p. 11).

According to Smith and Geller (2004), several procedures are needed to deliver effective math instruction to students with LD. To help students with LD learn algebra concepts, teachers should "organize content information in these higher-level math classes into concepts" (Smith & Geller, 2004, p. 23). Teachers should provide algebra lessons that are meaningful to students with LD (Smith & Geller, 2004).

Smith and Geller (2004) suggested one teaching strategy, which is the use of cognitive strategies. This offers a technique to solve math problems. According to Smith and Geller (2004), the following procedures and cognitive strategies were used in many studies that have shown the effectiveness in math instruction for students with LD:

[a]. Teacher modeling: Presentation of concept and steps to solve problem.

[b]. Self-Questioning: Student talks himself or herself through the task in a series of small steps.
[c]. Guided practice: Teacher assists students with procedural steps.
[d]. Cueing prior knowledge: Ensure prerequisite information is evident.
[e]. Feedback: Systematic and detailed corrective feedback given by teacher to students.
[f]. CRA sequence of instruction: Concrete, representational, and abstract examples.
[g]. Hands-on experience: Allows understanding of how abstract operates at a concrete level.
[h]. Review opportunities: Given over time, sufficient for obtaining mastery/fluency, cumulative, and varied for generalization.
[i]. Mediated scaffolding: Gradually reducing support as the student becomes more proficient at the skill. (p. 23)

By reviewing multiple studies, Hardy (2004) found that "components of effective instruction for algebra are teacher-based activities, computer assisted instruction and strategy instruction" (p. 3). Teacher-based activities provide "direct and explicit instruction" to students with LD (Hardy, 2004, p. 3). These activities introduce "prerequisite skills" before teachers teach
actual abstract algebra skills (Hardy, 2004, p. 3). They also use "CRA instruction with manipulatives" (Hardy, 2004, p. 3).

An instructional strategy includes the use of a "structured worksheet and graphic organizers" that help students organize information they learn (Hardy, 2004, p. 3). These worksheets and organizers also help them retain information.

Studies show that many students attempt to solve problems without connecting to their prior knowledge. Hardy (2004) found that before teachers teach problem solving skills, they can teach how to use "metacognitive and self-monitoring strategies" to guide students in the right direction (Hardy, 2004, p. 4). Metacognitive strategies direct students how to "re-read information for clarification, use a diagram, and write algebraic equations to solve the problems" (Hardy, 2004, p. 4). Self-monitoring strategies provide students with solutions on how to use "question and answer skills and organize problem-solving activities by using structured worksheets" (Hardy, 2004, p. 4).
What does the Research Say about the Concrete-Representation-Abstract Instruction?

Witzel (2005) studied the effectiveness of CRA instruction by making comparisons between CRA instruction and an explicit instruction model. In Witzel’s research (2005), 231 students from four middle schools in the southeastern United States were divided into a treatment group and a comparison group (p. 49).

Witzel, Mercer and Miller (2003) also conducted a research study about the effectiveness of CRA instruction. In Witzel, Mercer and Miller’s research (2003), 358 students at sixth and seventh grade participated in this study (p. 122). Among 358 students, 34 students were identified as learning disabled or at risk students for algebra difficulty in the treatment group. In the comparison group 34 similar characteristic students across the same teacher’s classes were involved.

For both studies students in the treatment group were taught using CRA instruction. These students had minimal previous experience with algebra. They were introduced to algebraic thinking through CRA instruction. Each lesson consisted of four steps: "(a) introduce the lesson, (b) model the new procedure, (c) guide students through
procedures, and (d) begin students working at the independent level” (Witzel, 2005, p. 53; Witzel, Mercer & Miller, 2003, p. 126).

In the comparison group, the instructional techniques included “introducing a lesson, modeling a procedure, working with students through guided and independent practice, and assessment of knowledge” (Witzel, 2005, p. 53; Witzel, Mercer & Miller, 2003, p. 126). In the comparison group the abstract approach of instruction was introduced to students to think and solve algebraic concepts.

Witzel (2005) indicated that according to results of the posttest and follow-up test, both groups of students improved their test scores. But the students who were taught lessons through CRA instruction scored above average on state achievement tests.

Witzel, Mercer and Miller (2003) also indicated that according to the pretest and follow-up test, both groups of students improved considerably on answering single variable algebraic equation questions. However, Witzel (2005), and Mercer, Witzel and Mercel (2003) found that students who were taught algebra concepts with CRA
instruction scored higher on pre-and-post tests than those students who received traditional math instruction.

What are the Benefits in Using the Concrete-Representation-Abstract Instruction to Teach Algebra for Students with Learning Disability?

Many math educators agree that math concepts should be taught by making them meaningful to students with LD. Some teachers, however, may have difficulty in doing so who are not trained in using these teaching methods.

To create more meaningful math instruction, teachers can provide students with hands-on activities and explicit instruction. When students use hands-on activities to solve algebra questions, they understand better how numbers and abstract equations can work together (Witzel, Smith & Brownell, 2001; Maccini & Gagnon, 2000). A variety of instruction methods to use hands-on activities are accessible. A part of CRA instruction is the use of visual and hands-on activities.

There are several benefits when teachers use CRA instruction appropriately. The first benefit of CRA instruction is to "teach conceptual understanding by connecting concrete understanding to abstract math processes" (Allsopp, 2006, p. 2). CRA instruction consists of three phases to teach abstract concepts.
A lesson begins with concrete objects to teach math concepts, and then abstract concepts are gradually introduced after practicing representational level (Allsopp, 2006). In 2001, Gagnon and Maccini mentioned that "CRA instruction supports students' understanding of underlying math concepts before learning rules" (p. 12).

The second benefit is to provide students with LD with a better understanding of math concepts by making the lessons meaningful (Allsopp, 2006). According to Miller and Mercer (1993), students who learned basic math concepts through CRA instruction demonstrate a better understanding of new math concepts and also maintain them longer.

The third benefit of using CRA instruction is to merge conceptual and procedural understanding together and apply it to solve abstract math problems.

This process makes it possible for students to learn both the 'How' and the 'Why' to the problem solving procedures they learn to do; and, they learn the 'What,' that is, they develop a conceptual understanding of the mathematics concept that underlies the problem solving process (Allsopp, 2006, p. 3).
According to Witzel (2005), the use of CRA instruction is beneficial for students who have difficulties in math. However, it also benefits students who do not have difficulties in math.

CRA instruction can assist students to understand algebra concepts better and expand their understanding beyond their expectations. The statistical results from Witzel's research (2005) indicated that the group of students, including students with LD who learned through CRA instruction, improved over the comparison group who learned through traditional explicit instructional method.

Through an extensive literature review, a variety of instructional strategies to teach algebra concepts to students with LD were found. Among them, CRA instruction was an effective way to teach algebra concepts.

CRA instruction helps students with LD develop knowledge of math concepts by connecting concrete and representational relationships. Eventually, they will extend this understanding to solve algebra problems using the process of abstract thinking.

The research base, however, is narrow because CRA instruction is still a relatively new teaching strategy.
No research was found that contradicted the effectiveness of CRA instruction versus traditional algebra instruction.
CHAPTER THREE
CURRICULUM PROJECT

Introduction

The purpose of this chapter is to provide classroom examples using CRA instruction to teach algebra concepts for students with LD. The CRA examples in the appendices I created by studying a variety of resources and adapting some resources.

The resources included professional research papers and articles. I also referred to different levels of high school curriculum textbooks (Glencoe/McGraw-Hill, 2002; Larson, R., Boswell, L., Kanold, T., & Stiff, L., 2001) and supplemental material (ETA/Cuisenaire, 2003).

The curriculum textbooks and supplemental materials provided guidelines on how to use manipulative objects. These textbooks and materials demonstrated how the transfer process works from the concrete representation level to pictorial representation level, and pictorial representation level to abstract approach level. Research papers and articles offered ideas on the steps to be taken in systematically using CRA instruction.
The goal of CRA instruction is to teach math concepts effectively so students with LD understand not only math skills and concepts but that those skills and concepts are related (Allsopp, 2006). When teachers implement CRA instruction, they need to consider the materials and elements for each phase.

First, at the concrete representation level, teachers use three-dimensional objects to teach particular math concepts or skills. These objects can be small sticks, pencils, chips, blocks, cups, rulers, markers, toy bear counters, coins, beans, balances, Algeblocks or Algebra tiles.

Second, at the pictorial representation level, teachers use two-dimensional objects to represent problems. In this phase, students are taught how to represent problems in a drawing form. Picture drawing forms look similar to concrete objects.

Third, at the abstract approach level, teachers teach students how to retain math skills they mastered at the pictorial representation stage level. Teachers also help students make a connection between math concepts and the abstract thinking process. After this stage, teachers use
explicit instruction methods to teach how to solve abstract math problems (Allsopp, 2006; Paulsen, 2003).

In the appendices I explained how the examples for each algebra question work when teachers use CRA instruction to teach algebraic concepts for high school students with LD. It can be used by inexperienced teachers to implement CRA instruction to teach algebra concepts to students with LD.

Discussion of Concrete-Representation-Abstract Examples

Fourteen high school students with LD in the special education algebra class were taught how to solve algebra problems by using CRA instruction in order to evaluate the effectiveness of CRA instruction. For this group of students with LD, each lesson included four steps. These were to introduce the lesson, model the new procedure, guide students with practice and facilitate students working at the independent level.

Students spent approximately four weeks to complete six lessons. Each lesson consisted between two and three hours of class time.
In the six appendices CRA examples are illustrated to solve algebra problems. These examples include the procedures to solve the problems using CRA instruction for students with LD.

In this chapter I will explain how the CRA procedures were taught to my students with LD to solve algebra problems. I will also discuss my students' response to the procedures of CRA instruction. The following CRA examples will be discussed:

- Adding and Subtracting Integers
- Multiplying Integers
- Dividing Integers
- Multiplying Polynomials
- Factoring Trinomials

Adding and Subtracting Integers

Appendices A and B illustrate the CRA examples of adding and subtracting integers. Approximately four hours were needed to complete these two lessons. Before the lessons, the materials of Algebblocks green unit blocks and a basic mat were introduced.

Students were also allowed to have about 10 minutes to manipulate the Algebblocks materials. Students worked in groups to become familiar with materials such as the
green block as a unit which represents the number 1. After practicing with the manipulative objects, the lesson went through three levels.

At the concrete representational level Algeblocks basic mat and green unit blocks were utilized to represent integers. Students were guided to place green blocks on the Algeblocks basic mat and remove them from the Algeblocks basic mat to calculate addition and subtraction problems.

At the pictorial representational level, students were taught how to put the expression of the integer question in the form of a pictorial representation. They simply used paper and pencil to represent the question.

At the abstract approach level, students were already familiar with positive and negative signs, and gradually facilitated to write the question in Arabic symbols. They were also given about 20 minutes to practice solving problems on their own after guided practice of approximately 15 minutes.

**Multiplying Integers**

Appendix C shows the CRA examples of multiplying integers. Students spent about 15 minutes with a partner. They practiced how to place green unit blocks on the
factor track and build a rectangle area in a quadrant that is bounded by the units in the Factor Track.

Second, students were guided in the drawing of positive or negative circles that represented the question. After that, students drew three negative signs inside a circle.

Third, students were taught how to write the question in Arabic symbols. The rules of multiplying integers were also introduced to them.

The product of any pair of negative numbers is positive. An even number of negative factors can be paired perfectly to produce a positive result. An odd number of negative factors cannot be paired exactly, so a negative result is produced. About 20 minutes was provided for students to practice various integer multiplication questions on independent work.

Dividing Integers

Appendix D illustrates the CRA examples of dividing integers. Students spent about 10 minutes with a partner. They practiced how to place a divisor number by using green unit blocks on the factor track and then build a rectangle of units in the quadrant bounded by the divisor.
Students were taught how to complete the other dimension of the rectangle.

Second, students drew positive or negative circles that represented the question. They drew twelve negative circles and made four sets of three negative circles.

Third, they were introduced how to write the question in Arabic symbols and the rules of dividing integers which were similar to multiplying integers. About 20 minutes were given to students to practice various integer division questions. The examples in appendices C and D can be expanded to estimate the square root.

Multipling Polynomials

Appendix E shows the CRA examples of multiplying polynomials. Students spent three hours finishing the lesson. Students also needed about 10 minutes to become familiar with the Algebblocks quadrant mat, factor track, square unit blocks, x unit blocks and unit blocks.

After becoming familiar with materials, students were guided in the placement of various units in the positive or negative parts of horizontal and vertical axis. They were also shown how to build a rectangle area in a quadrant that is bounded by the units in the factor track.
Second, the FOIL pattern which represents First, Outer, Inner and Last pattern was introduced. Students were guided on how to solve the problem by using the FOIL pattern.

Third, after learning the FOIL pattern, students were taught how to use the distributive property rule to find the product. They still used the mixed methods of the FOIL pattern and the distributive property.

After this procedure, students spent about 30 more minutes practicing examples with the teacher. They also had an opportunity to try to solve some problems.

This algebraic concept can be more complicated than the concepts that were in the appendices A through D because students needed to understand multiple steps to solve the problems. Most of the students expressed that if they practiced enough, they could solve problems by multiplying polynomials.

Teachers may need to re-teach the FOIL pattern and distributive rules for multiplying polynomials to students with LD, so the students will understand how to properly apply those rules to solve polynomial problems. Students with LD will also need more guided practice to master this skill before they actually work independently.
Factoring Trinomials

Appendix F illustrates the CRA examples of factoring trinomials. Students spent four hours to complete the lesson. The materials of Algebra Tiles containing x-square tiles, x-tiles and 1-tiles were presented. After becoming familiar with the materials, students were guided on how to place x-square tiles, x-tiles and 1-tiles on a product mat and complete the rectangle.

Second, in this pictorial representational stage, students used the traditional paper and pencil method to solve problems. They were taught how to use a guess-and-check strategy to find the correct numbers.

Third, students used the guess-and-check strategy and were led to utilize the distributive property to factor trinomials. Students were given about 30 minutes to work with a partner to solve problems.

Factoring a trinomial can be one of the most difficult algebraic concepts for students with LD. Most of the students expressed that there was a high level of difficulty on this lesson.

Five out of 14 students almost gave up because of the complicated and long process they had to follow. Nine students tried very hard to understand and solve problems
but they also had a difficult time. The examples of factoring a trinomial can also be expanded to factor binomials.
Prior to my first lesson using CRA instruction, my students with LD had great difficulties understanding abstract thinking concepts. They were not exposed to a variety of learning strategies such as hand-on activities and the CRA instruction method. Lacking the basic math skills, my students with LD were frustrated and poorly motivated to learn. Motivating my students with LD and using CRA instruction were keys to understanding the abstract thinking process which led them to learn algebra concepts.

After teaching six lessons, most students stated they felt more confident solving algebraic problems. They expressed more interest in studying algebra. They also felt it was easier to understand algebraic concepts because they were able to visualize the questions before they solved the problems. Overall, my students were satisfied to learn six examples of algebraic concepts through CRA instruction.

The lessons revealed that CRA instruction could be used to teach a variety of algebraic concepts.
Inexperienced teachers would benefit from training on how to utilize CRA instruction; however, they may need to spend a certain amount of time to practice how to deliver algebra lessons effectively.

Students should have the opportunity to practice extensively after each lesson to master algebraic concepts. However, further research is needed to determine the overall and long-term effectiveness of CRA instruction compared to traditional instruction.

The purpose of the project was to explore how CRA instruction would help learning disability high school students understand algebra concepts. This project also showed examples how to use CRA instruction to teach algebra concepts to students with LD.

To be successful in teaching algebra concepts, teachers need to consider other strategies to reinforce students' learning. One of the strategies that students can use is with extensive examples and practice. In 2004, Rivera stated that students with LD in math learn math skills better when they repeatedly practice the same skills and processes.

To teach algebra concepts to students with LD, the teachers' role is important. Teachers should establish a
classroom atmosphere where students can learn new concepts by interacting and reinforcing prior knowledge. Teachers also should continue to explore hands-on activities and representational approaches to present abstract concepts and procedures, which require producing efficient math skills connected to conceptual understanding.
APPENDIX A

CONCRETE-REPRESENTATION-ABSTRACT EXAMPLES:

ADDING INTEGERS
Question 1: Find the sum.

\[ 3 + (-6). \]

Step One: Concrete Representation

Before students begin to solve the problem, they are introduced to the materials: Algeblox Green Unit Blocks and Basic Mat. Students practice understanding what each material represents and how it works; for example, a green block stands for the number 1. During the open discussion period, students manipulate Basic Mat and Unit Blocks and this enables them to be familiar with the procedure.

To add two integers 3 and -6, students lay the Algeblox Basic Mat on their desks. They place three Green Unit Blocks on the positive side of the Basic Mat and six Green Unit Blocks on the negative side of the Basic Mat as below.

![Algeblox Basic Mat Diagram]
Students count the unit blocks from each side. They take three Green Unit Blocks off both sides of the mat shown as follows.

After removing three block units from each side of the mat, students count the remaining units on the negative side as below. The mat shows that the sum of 3 and -6 is -3.
Step Two: Pictorial Representation

At this phase, the question $3 + (-6)$ is expressed with a pictorial representation form. Students draw the question $3 + (-6)$ as follows.

\[\begin{array}{ccc}
+ & - & - \\
+ & - & - \\
+ & - & - \\
\end{array}\]

Students cross out the first column of three lines with the positive signs on the left side. They cross out the third column of three lines with the negative signs on the right side as follows.

\[\begin{array}{ccc}
+ & - & - \\
+ & - & - \\
+ & - & - \\
\end{array}\]

Students have the three positive lines left as follows. The answer remains as $-3$.

\[\begin{array}{ccc}
- & - & - \\
- & - \\
\end{array}\]

Or $-3$. 


Step Three: Abstract Approach

At the abstract approach phase, students write the question in Arabic symbols. Students also have an opportunity to learn the rules of integers. When adding negative and positive integers, students find the difference between two integers by subtracting two numbers. After that, they determine which sign should be used for the answer. They use the sign of the greater addend. The following is the procedure.

\[ 3 + (-6) = 3 - 6 = -3 \]

Adapted from

ETA/Cuisenaire (2003, p. 6-10);
Larson, R., Boswell, L., Kanold, T., and Stiff, L (2001, p. 78-80);
Witzel, B. (2005, p. 54);
APPENDIX B

CONCRETE-REPRESENTATION-ABSTRACT EXAMPLES:

SUBTRACTING INTEGERS
Question 2: Find the difference.

9 - 3.

Step One: Concrete Representation

To solve this integer question, the process is similar and uses the same materials as in the appendix A. Students place nine Green Unit Blocks on the positive side of the Basic Mat as follows.

![Concrete Representation Diagram]

Students read the question again and make sure how many Unit Blocks they need to take off. They take three Green Unit Blocks off the mat as follows.

![Concrete Representation Diagram]
Students count the remaining Green Unit Blocks. The mat shows the difference of nine and three. The answer remains as follows. \[ +9 - 3 = 6 \]

Step Two: Pictorial Representation

Students draw the question \(9 - 3\) in the form of a pictorial representation. They draw nine lines, a minus sign and three lines on the paper as follows.

\[
\begin{array}{c}
\text{+} \\
\text{} \\
\text{--} \\
\end{array}
\]

Students cross out the second column of three lines with the positive signs on the left side. They cross out the third column of three lines with the negative signs on the right side as follows.

\[
\begin{array}{c}
\text{} \\
\text{--} \\
\text{} \\
\end{array}
\]
Students count the following six lines. The answer remains 6.

\[ \begin{align*}
&\_\_\_\_ \\
&\_\_ \\
&\_ \\
\end{align*}\]

Step three: Abstract Approach

At this phase, students write the question in Arabic symbols. Students apply the rules of integers. When subtracting integers, the answer should follow the sign of the greater addend. The answer is \(9 - 3 = 6\).

Adapted from

ETA/Cuisenaire (2003, p. 6-10);

Larson, R., Boswell, L., Kanold, T., and Stiff, L (2001, p. 86-88);

Witzel, B (2005, p. 54);

APPENDIX C

CONCRETE-REPRESENTATION-ABSTRACT EXAMPLES:

MULTIPLYING INTEGERS
Question 3: Find the product.

$5 \times -3$.

Step One: Concrete Representation

The materials, Algeblocks Quadrant Mat, Factor Track, Unit Blocks are introduced. Before students use Algeblocks to find the product of two numbers, they practice to understand how to use Quadrant Mat, Factor Track and Unit Blocks. Students manipulate Unit Blocks in different quadrant with the appropriate signs. Students place five green unit blocks on the Factor Track on the positive side of the vertical axis and three green unit blocks on the Factor track on the negative side of the horizontal axis as below.

Algeblocks Quadrant Mat

![Quadrant Mat Diagram]
After students place eight green unit blocks on the both sides of the Factor Tracks, they start building a rectangle area in quadrant II by using green unit blocks. Quadrant II is bounded by the units in the Factor Track. Students count green unit blocks in quadrant II to find the answer. The answer carries + or - sign of the quadrant. The product of 5 and -3 is -15 unit blocks as shown below.

Algebblocks Quadrant Mat

Step Two: Pictorial Representation

At this phase, the question of how to find product of 5 times -3 is drawn with a pictorial representation form. Students put three negative circles together as a group and draw five sets as follows.
Students put three negative signs inside a circle as follows. Since there are 15 negative circles, the product is -15. Thus, the answer remains as $5 \times -3 = -15$.

Step Three: Abstract Approach

After students understand how to group numbers together and find the product by using the pictorial representational forms, they solve the question by using a traditional process that is written in Arabic symbols. Students have an opportunity to practice the rules of integers.
The product of two numbers having the same sign is positive, and the product of two numbers having the different signs is negative. The answer remains as $5 \times -3 = -15$.

Adapted from

ETA/Cuisenaire (2003, p. 10-13);

APPENDIX D

CONCRETE-REPRESENTATION-ABSTRACT EXAMPLES:

DIVIDING INTEGERS
Question 4: Find the quotient.

\[-12 \div 3.\]

Step One: Concrete Representation

The materials used in the appendix C are applied to solve this question. Students place the divisor 3 in the positive part of horizontal axis and then build a rectangle of 12 units in the quadrant IV bounded by the divisor as below.

Students put four green unit blocks on the Factor Track on the negative side of the vertical axis to complete the other dimension of the rectangle as following. The answer remains as \(-12 \div 3 = -4\).
Step Two: Pictorial Representation

At this phase the question, \(-12 \div 3\) is expressed with a pictorial representation form. Students draw 12 negative circles as below.

\[
\begin{array}{c}
\circ \circ \\
\circ \circ \\
\circ \circ \\
\circ \circ \\
\end{array}
\]

Students then draw four sets of three negative circles. Since there are 12 negative circles, the quotient is \(-4\). So, the answer remains as \(-12 \div 3 = -4\).
Step Three: Abstract approach

After students understand how to group numbers together and find the quotient of the product by using the pictorial representational forms, they solve the question with Arabic symbols. The quotient of two numbers with the same sign is positive, and the product of two numbers with the different signs is negative. The answer remains as \(-12 \div 3 = -4\)

Adapted from

ETA/Cuisenaire (2003, p. 10-13);

APPENDIX E

CONCRETE-REPRESENTATION-ABSTRACT EXAMPLES:

MULTIPLYING POLYNOMIALS
Question 5: Find the product.

\((x + 2)(x + 1)\).

Step One: Concrete Representation

The following materials were introduced to students:

Algebblocks Quadrant Mat; Factor Track; Blocks (Square unit:)

Before students solve the problem, they practice how to identify and use Algebblocks. They place x and 1 unit in the positive part of a horizontal axis an x and 2 units in the positive part of vertical axis as below.
Students start building a rectangle area in quadrant I that is bounded by the units in the Factor Track as following. The answer is \((x + 2)(x + 1) = x^2 + 3x + 2\).

Step Two: Pictorial Representation

Students use paper and pencils to multiply two binomials based on what students learned from the concrete presentation level. The teacher teaches the definition of the FOIL (First, Outer, Inner, and Last terms) pattern as below.

\[(x + 2)(x + 1)\]

First terms

Inner terms

Outer terms

Last terms
At this phase, students develop the knowledge how to connect between the concrete representational objects with the pictorial representational forms. The solution is written on the board as shown below.

\[(x + 2)(x + 1)\]

\[= x^2 + 1x + 2x + 2\]

\[= x^2 + 3x + 2\] Combine like terms.

Step Three: Abstract Approach

After mastering the skills at the concrete representational level, students multiply two binomials without having to rely on the Algeblocks. Although students still rely on the FOIL pattern to obtain products, they should understand the process of finding products of binomials as a result of the distributive property. They find the product of \((x + 2) (x + 1)\) by using the distributive property rule as follows.
Solution

\[(x+2)(x+1)\]

\[= x(x+1)+2(x+1)\]

Distribute \((x + 1)\) to each term of \((x + 2)\).

\[= x(x)+x(1)+2(x)+2(1)\]

Distribute \(x\) and 2 to each term of \((x + 1)\).

\[= x^2+1x+2x+2\]

Multiply.

\[= x^2+3x+2\]

Combine like terms.

Adapted from

ETA/Cuisenaire (2003, p. 11-13);

APPENDIX F

CONCRETE-REPRESENTATION-ABSTRACT EXAMPLES:

FACTORIZING TRINOMIALS
Question 6: Factor the trinomial.

\[ x^2 + 5x + 6. \]

Step One: Concrete Representation

The materials: Algebra Tiles (one \( x^2 \)-tile, five \( x \)-tiles and six 1-tiles) were introduced. Students manipulate various tiles to become familiar with the concept. The teacher shows how to put the polynomial \( x^2 + 5x + 6 \) as below.

After understanding the materials, students place the \( x^2 \) tile at the corner of the product mat, and display the three 1-tiles each in two rows as below.
Students complete the rectangle by placing three $x$ tiles on the right side of the $x^2$ tile and two $x$ tiles below the $x^2$ tile as follows. The rectangle has a length of $x + 3$ and a width of $x + 2$. Thus, the answer remains as $x^2 + 5x + 6 = (x + 2)(x + 3)$.

Step Two: Pictorial Representation

Students use paper and pencils to factor trinomials. For the question of $x^2 + 5x + 6$, the product of one and six is six that is written as $x^2 + (\_ + \_)(x + 6)$, or the product of two and three is six that is written as $x^2 + (\_ + \_)(x + 6)$. To find two integers whose product is six and the sum is five, students use guess-and-check strategy to find these numbers.
Students make a table to guess and find the factor of six and sum of factors as follows.

<table>
<thead>
<tr>
<th>Factor of 6</th>
<th>Sum of Factors</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 6</td>
<td>1 + 6 = 7</td>
<td>No</td>
</tr>
<tr>
<td>2, 3</td>
<td>2 + 3 = 5</td>
<td>yes</td>
</tr>
</tbody>
</table>

Step 3: Abstract Approach

At this stage, students still rely on a guess-and-check strategy from the previous pictorial representation to solve the problem. Students solve the question to factor trinomial by grouping terms, factoring and using the distributive property as below.

\[ x^2 + 5x + 6 \]

= \[ x^2 + (2+3)x + 6 \] Select the factors of 1 and 3.

= \[ x^2 + 2x + 3x + 6 \] Distributive property.

= \[ (x^2 + 2x) + (3x + 6) \] Group terms.

= \[ x(x+2) + 3(x+2) \] Factor.

= \[ (x+2)(x+3) \] Distributive property.
The answer remains as \( x^2 + 5x + 6 = (x + 2)(x + 3) \).

Adapted from

Glencoe/McGraw-Hill (2002, p. 572-575);

REFERENCES


Rivera, F. D. (2004). Teaching techniques for 5th to 9th grade Students with learning disabilities in algebra and geometry. Alternative Education. San Jose State University.


