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Geodesic on surfaces of constant Gaussian curvature

Veasna Chiek

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GEODESIC ON SURFACES OF CONSTANT GAUSSIAN CURVATURE

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Veasna Chiek

December 2006
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Approved by:

Wenxiang Wang, Committee Chair

John Sarli, Committee Member

Rolland Trapp, Committee Member

Peter Williams, Chair,
Department of Mathematics

Joseph Chavez,
Graduate Coordinator,
Department of Mathematics
ABSTRACT

This paper will cover the necessary definitions and theorems that are needed to study surfaces of constant Gaussian curvature. It will include a sufficient number of examples to clarify the definitions and theorems. The main goal of this paper is to study the properties of surfaces of constant Gaussian curvature. Thus, Minding's theorem which states that two surfaces of the same constant Gaussian curvature are locally isometric becomes very important in this paper, which then leads to the fact that surfaces of positive constant Gaussian curvature are locally isometric to a sphere of radius $a$, surfaces of zero Gaussian curvature are locally isometric to a plane and surfaces of negative constant Gaussian curvature are locally isometric to the pseudosphere. I will include examples of geodesics on these type of surfaces and discuss their properties. Moreover, this paper will incorporate MAPLE into some of its calculations and graphs. Finally, we conclude that the Gaussian curvature is a surface invariant and the geodesics of these surfaces will be the so-called best paths.
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Chapter 1

Introduction

The goal of my project is to study geodesics on surfaces of constant Gaussian curvature. This paper will be broken up into two chapters. The first chapter will consist of four sections. The first three sections will be dedicated to the definitions and theorems that are necessary to study surfaces of constant Gaussian curvature. The fourth section will contain examples of how these definitions and theorems are applied.

1.1 Regular Surfaces

We first begin by introducing the definition of a regular surface in $\mathbb{R}^3$ and some of its properties.

**Definition 1.1.** A surface in $\mathbb{R}^3$ is a subset $S \subset \mathbb{R}^3$ such that for each point $p \in S$ there are a neighborhood $V$ of $p$ in $\mathbb{R}^3$ and a mapping $\mathbf{x} : U \rightarrow V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ subject to the following conditions (Figure 1.1).

1. $\mathbf{x}$ is differentiable. This means that if we write

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where $(u, v) \in U$, and the functions $x(u, v), y(u, v), z(u, v)$ have continuous partial derivatives of all orders in $U$.

2. $\mathbf{x}$ is a homeomorphism. Since $\mathbf{x}$ is continuous by condition 1, this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : V \cap S \rightarrow U$ which is continuous, that is, $\mathbf{x}^{-1}$ is the restriction of a continuous map $F : W \subset \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined on an open set $W$ containing $V \cap S$. 

Figure 1.1: Regular Surface

3. \( \mathbf{x} \) is regular at each point \( q \in U \), meaning that for each \( q \in U \), the differential mapping \( d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3 \) is one-to-one or equivalently that the Jacobian Matrix \( J_x(q) \) of the mapping \( \mathbf{x} \) at each \( q \in U \) has rank 2. This implies that at each \( q \in U \) the vector product

\[
\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq 0,
\]

where \((u, v) \in U\). Thus \( \mathbf{x} \) is neither constant nor a function of \( u \) or \( v \) alone, so that the surface \( S \) is neither a point nor a curve. Furthermore, the vectors \( \frac{\partial \mathbf{x}}{\partial u} \) and \( \frac{\partial \mathbf{x}}{\partial v} \) are linearly independent at each \( q \).

The mapping \( \mathbf{x} \) is called a parametrization or a system of local coordinates in a neighborhood of \( p \). The neighborhood \( V \cap S \) of \( p \) in \( S \) is called a coordinate neighborhood. In addition, the mapping \( \mathbf{x} \) is also referred to as a local coordinate patch or simply a patch.

**Lemma 1.2.** If \( f : U \to \mathbb{R} \) is a differentiable function in an open set \( U \) of \( \mathbb{R}^2 \), then the graph of \( f \), that is, the subset of \( \mathbb{R}^3 \) given by \((x, y, f(x, y))\) for \((x, y) \in U\), is a regular surface.

This parametrization \( \mathbf{x} \) is called a Monge parametrization (or Monge patch), and the corresponding surface a simple surface, so that a general surface in \( \mathbb{R}^3 \) can be constructed by gluing together simple surfaces.

Moreover, condition 3 in the definition of a regular surface \( S \) guarantees that
for every point \( p \in S \) the set of tangent vectors to the parametrized curves of \( S \), passing through \( p \), constitutes a plane.

By a tangent vector to \( S \), at a point \( p \in S \), we mean the tangent vector \( \alpha'(0) \) of a differentiable parametrized curve \( \alpha: (-\epsilon, \epsilon) \rightarrow S \) with \( \alpha(0) = p \).

**Lemma 1.3.** Let \( x: U \subset \mathbb{R}^2 \rightarrow S \) be a parametrization of a regular surface \( S \) and let \( q \in U \). The vector subspace of dimension 2,

\[
dx_q(\mathbb{R}^2) \subset \mathbb{R}^3,
\]

coincides with the set of tangent vectors to \( S \) at \( x(q) \).

By the above proposition, the plane \( \dx_q(\mathbb{R}^2) \), which passes through \( x(q) = p \), does not depend on the parametrization \( x \). This plane will be called the tangent plane to \( S \) at \( p \) and will be denoted by \( T_p(S) \). The choice of parametrization \( x \) determines a basis \( \left\{ \frac{\partial x}{\partial u}(q), \frac{\partial x}{\partial v}(q) \right\} \) of \( T_p(S) \), called the basis associated to \( x \). Sometimes it is convenient to write \( \frac{\partial x}{\partial u} = x_u \) and \( \frac{\partial x}{\partial v} = x_v \).

The coordinates of a vector \( w \in T_p(S) \) in the basis associated to a parametrization \( x \) are determined as follows. \( w \) is the velocity vector \( \alpha'(0) \) of a curve \( \alpha = x \circ \beta \), where \( \beta: (-\epsilon, \epsilon) \rightarrow U \) is given by \( \beta(t) = (u(t), v(t)) \), with \( \beta(0) = q = x^{-1}(p) \). Thus,

\[
\begin{align*}
\alpha'(0) & = \frac{d}{dt}(x \circ \beta)(0) = \frac{d}{dt}(u(t), v(t))(0) \\
& = x_u(q)u'(0) + x_v(q)v'(0) = w.
\end{align*}
\]

Thus, in the basis \( \{x_u(q), x_v(q)\} \), \( w \) has coordinates \( (u'(0), v'(0)) \), where \( (u(t), v(t)) \) is the expression, in the parametrization \( x \), of a curve whose velocity vector at \( t = 0 \) is \( w \).

With the notion of a tangent plane, we can talk about the differential of a differentiable map between two surfaces. Let \( S_1 \) and \( S_2 \) be two regular surfaces and let \( \phi: V \subset S_1 \rightarrow S_2 \) be differentiable mapping of an open set \( V \) of \( S_1 \) into \( S_2 \). If \( p \in V \), we know that every tangent vector \( w \in T_p(S_1) \) is the velocity vector \( \alpha'(0) \) of a differentiable parametrized curve \( \alpha: (-\epsilon, \epsilon) \rightarrow V \) with \( \alpha(0) = p \). the curve \( \beta = \phi \circ \alpha \) is such that \( \beta(0) = \phi(0) \), and therefore \( \beta'(0) \) is a vector \( T_{\phi(p)}(S_2) \)

In the discussion above, given \( w \), the vector \( \beta'(0) \) does not depend on the choice of \( \alpha \). The map \( d\phi_p: T_p(S_1) \rightarrow T_{\phi(p)}(S_2) \) defined by \( d\phi_p(w) = \beta'(0) \) is linear. The linear
Figure 1.2: Tangent Planes

map $d\phi_p$ is called the differential of $\phi$ at $p \in S_1$. In a similar way we define the differential of a differentiable function $f : U \subset S \to R$ at $p \in U$ as a linear map $df_p : T_p(S) \to R$.

What we have been doing up to now is extending the notions of differential calculus in $R^2$ to regular surfaces. Since calculus is essentially a local theory, we defined the regular surface which locally was a plane, up to diffeomorphisms, and this extension then became natural. It might be expected therefore that the basic inverse function theorem extends to differentiable mappings between surfaces.

Understanding the properties of the tangent plane, allows us to speak of the angle of two intersection surfaces at a point of intersection.

Given a point $p$ on a regular surface $S$, there are two unit vectors of $R^3$ that are normal to the tangent plane $T_p(S)$; each of them is called a unit normal vector at $p$. The straight line that passes through $p$ and contains a unit normal vector at $p$ is called the normal line at $p$. The angle of two intersection surfaces at an intersection point $p$ is the angle of their tangent planes or normal lines at $p$ (Figure 1.2).

By fixing a parametrization $\mathbf{x} : U \subset R^2 \to S$ at $p \in S$, we can make a definite choice of a unit normal vector at each point $q \in x(U)$ by the rule

$$N(q) = \pm \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} (q) = \pm \frac{\mathbf{x}_u \times \mathbf{x}_v}{\sqrt{|\mathbf{x}_u \times \mathbf{x}_v|^2}} (q) = \pm \frac{\mathbf{x}_u \times \mathbf{x}_v}{\sqrt{\mathbf{x}_u^2 \mathbf{x}_v^2 - (\mathbf{x}_u \cdot \mathbf{x}_v)^2}} (q).$$

Thus, we obtain a differentiable map $N : \mathbf{x}(U) \to R^3$. 
Lemma 1.4. Let $p \in S$ be a point of a regular surface $S$ and let $x : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a map with $p \in x(U) \subset S$ such that conditions 1 and 3 of definition 1.1 hold. Assume that $x$ is 1-1. Then $x^{-1}$ is continuous.

1.2 The First and Second Fundamental Form

In the previous sections we have looked at surfaces from the point of view of differentiability. In this sections we shall begin the study of further geometric structures carried by the surface. The most important of these is perhaps the first fundamental form, which we shall now describe.

The natural inner product of $\mathbb{R}^3 \supset S$ induces on each tangent plane $T_p(S)$ of a regular surface $S$ an inner product, to be denoted by $\langle \ , \ \rangle_p :$ If $w_1, w_2 \in T_p(S) \subset \mathbb{R}^3,$ then $\langle w_1, w_2 \rangle_p$ is equal to the inner product of $w_1$ and $w_2$ as vectors in $\mathbb{R}^3.$ To this inner product, which is a symmetric bilinear form (i.e., $\langle w_1, w_2 \rangle = \langle w_2, w_1 \rangle$ and $\langle w_1, w_2 \rangle$ is linear in both $w_1$ and $w_2$), there corresponds a quadratic form $I_p : T_p(S) \rightarrow \mathbb{R}$ given by

$$I_p(w) = \langle w, w \rangle_p = |w|^2 \geq 0. \quad (1.1)$$

Definition 1.5. The quadratic form $I_p$ on $T_p(S),$ defined by (1.1), is called the first fundamental form of the regular surface $S \subset \mathbb{R}^3$ at $p \in S.$

Therefore, the first fundamental form is merely the expression of how the surface $S$ inherits the natural inner product of $\mathbb{R}^3.$ Geometrically, as we shall see in a while, the first fundamental form allows us to make measurements on the surface (lengths of curves, angles of tangent vectors, areas of regions) without referring back to the ambient space $\mathbb{R}^3$ where the surface lies.

We shall now express the first fundamental form in the basis $x_u, x_v$ associated to a parametrization $x(u,v)$ at $p.$ Since a tangent vector $w \in T_p(S)$ is the tangent vector to a parametrized curve $\alpha(t) = x(u(t), v(t)), t \in (-\epsilon, \epsilon),$ with $p = \alpha(0) = x(u_0, v_0),$ we
obtain
\[ I_p(\alpha'(0)) = \langle \alpha'(0), \alpha'(0) \rangle_p \]
\[ = \langle x_u u' + x_v v', x_u u' + x_v v' \rangle_p \]
\[ = \langle x_u, x_u \rangle_p (u')^2 + 2 \langle x_u, x_v \rangle_p u' v' + \langle x_v, x_v \rangle_p (v')^2 \]
\[ = E(u')^2 + 2Fuv' + G(v')^2 \]
\[ = Edu^2 + 2Fduv + Gdv^2 \]

where the values of the function involved are computed for \( t = 0 \), and

\[ E(u_0, v_0) = \langle x_u, x_u \rangle_p \]
\[ = x_u \cdot x_u \]
\[ F(u_0, v_0) = \langle x_u, x_v \rangle_p \]
\[ = x_u \cdot x_u \]
\[ G(u_0, v_0) = \langle x_v, x_v \rangle_p \]
\[ = x_u \cdot x_u \]

are the coefficients of the first fundamental form in the basis \( \{x_u, x_v\} \) of \( T_p(S) \). By letting \( p \) run in the coordinate neighborhood corresponding to \( x(u, v) \) we obtain functions \( E(u, v), F(u, v), G(u, v) \) which are differentiable in that neighborhood.

From now on we shall drop the subscript \( p \) in the indication of the inner product \( \langle \ , \ \rangle_p \) or the quadratic form \( I_p \) when it is clear from the context which point we are referring to. It will also be convenient to denot the natural inner product of \( R^3 \) by the same symbol \( \langle \ , \ \rangle \) rather than the previous dot.

We will now introduce the Gauss map and its properties before introducing the second fundamental form. A regular surface is orientable if it admits a differentiable field of unit normal vectors defined on the whole surface; the choice of such a field \( N \) is called an orientation of \( S \). Every surface covered by a singe coordinate system is trivially orientable. Thus, every surface is locally orientable, and orientation is definitely a global property in the sense that it involves the whole surface. An orientation \( N \) on \( S \) induces an orientation on each tangent space \( T_p(S), \ p \in S \), as follows. Define a basis \( \{v, w\} \in T_p(S) \) to be positive if \( \langle v \times w, N \rangle \) is positive. Therefore, \( S \) will denote a regular
orientable surface in with an orientation (i.e., differentiable field of unit normal vectors $N$) has been chosen; this will be simply called a surface $S$ with an orientation $N$.

**Definition 1.6.** Let $S \subset \mathbb{R}^3$ be a surface with an orientation $N$. The map $N : S \to \mathbb{R}^3$ takes its values in the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$$

The map $N : S \to S^2$, thus defined, is called the Gauss Map of $S$ (Figure 1.3).

It is straightforward to verify that the Gauss map is differentiable. The differential $dN_p$ of $N$ at $p \in S$ is a linear map from $T_p(S)$ to $T_{N(p)}(S^2)$. Since $T_p(S)$ and $T_{N(p)}(S^2)$ are parallel planes, $dN_p$ can be looked upon as a linear map on $T_p(S)$. The linear map $dN_p : T_p(S) \to T_p(S)$ operates as follows. For each parametrized curve $\alpha(t)$ in $S$ with $\alpha(0) = p$, we consider the parametrized curve $N \circ \alpha(t) = N(t)$ in the sphere $S^2$; this amounts to restricting the normal vector $N$ to the curve $\alpha(t)$. The tangent vector $N'(0) = dN_p(\alpha'(0))$ is a vector in $T_p(S)$ (Figure 1.2). It measures the rate of change of the normal vector $N$, restricted to the curve $\alpha(t)$, at $t = 0$. Thus, $dN_p$ measures how $N$ pulls away from $N(p)$ in a neighborhood of $p$. In the case of curves, this measure is given by a number, the curvature. In the case of surfaces, this measure is characterized by a linear map.
Definition 1.7. The quadratic form $II_p$, defined in $T_p(S)$ given by

$$II_p(v) = -\langle dN_p(v), v \rangle$$

(1.2)

is called the second fundamental form of $S$ at $p$.

To given an interpretation of the second fundamental form $II_p$, consider a regular curve $C \subset S$ parametrized by $\alpha(s)$, where $s$ is the arc length of $C$, and with $\alpha(0) = p$. Fe we denote $N(s)$ the restriction of the normal vector $N$ to the curve $\alpha(s)$, we have $\langle N(s), \alpha'(s) \rangle = 0$. Hence,

$$\langle N(s), \alpha''(s) \rangle = -\langle N'(s), \alpha'(s) \rangle.$$

Therefore,

$$II_p(\alpha'(0)) = -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle$$

$$= -\langle N'(0), \alpha'(0) \rangle$$

$$= -\langle N(0), \alpha''(0) \rangle$$

$$= \langle N, kn(p) \rangle$$

$$= k_n(p)$$

Definition 1.8. Let $C$ be a regular curve in $S$ passing through $p \in S$, $k$ the curvature of $C$ at $p$, and $\cos \theta = \langle n, N \rangle$, where $n$ is the normal vector to $C$ and $N$ is the normal vector to $S$ at $p$. The number $k_n = k \cos \theta$ is then called the normal curvature of $C \subset S$ at $p$. 
In other words, \( k_n \) is the length of the projection of the vector \( k\mathbf{n} \) over the normal to the surface at \( p \), with a sign given by the orientation \( N \) of \( S \) at \( p \). Also note, the normal curvature of \( C \) does not depend on the orientation of \( C \) but changes sign with a change of orientation for the surface.

**Definition 1.9.** The maximum normal curvature \( k_1 \) and the minimum normal curvature \( k_2 \) are called the principle curvatures at \( p \); the corresponding directions, that is, the direction given by the eigenvectors \( e_1, e_2 \), are called principal directions at \( p \).

**Definition 1.10.** Let \( p \in S \) and let \( dN_p : T_p(S) \to T_p(S) \) be the differential of the Gauss map. The determinant of \( dN_p \) is the Gaussian curvature \( K \) of \( S \) at \( p \). The trace of \( dN_p \) is called the mean curvature \( H \) of \( S \) at \( p \).

In terms of the principle curvatures, we can write mean curvature \( H \) and Gaussian curvature \( K \) as follows:

\[
H = \frac{1}{2}(k_1 + k_2),
\]

\[
K = k_1 k_2,
\]

and, the two principle curvatures, \( k_1, k_2 \) can be expressed in terms of \( H, K \) as follows:

\[
k_1, k_2 = H \pm \sqrt{H^2 - K}.
\]

Moreover, we will now introduce some concepts related to the local behavior of the Gauss map. More importantly, we will obtain an expression of the second fundamental form.

Let \( \mathbf{x}(u, v) \) be a parameterizations at a point \( p \in S \), and let \( \alpha(t) = \mathbf{x}(u(t), v(t)) \) be a parametrized curve on \( S \), with \( \alpha(0) = p \). The tangent vector to \( \alpha(t) \) at \( p \) is

\[
\alpha' = x_u u' + x_v v'
\]

and

\[
dN(\alpha') = N'(u(t), v(t)) = N_u u' + N_v v'.
\]

Since \( N_u \) and \( N_v \) belong to \( T_p(S) \), we may write

\[
N_u = a_{11} x_u + a_{21} x_v,
\]

(1.7)
\[ N_v = a_{12}x_u + a_{22}x_v, \]  

and therefore,  
\[ dN(\alpha') = (a_{11}u' + a_{12}v')x_u + (a_{21}u' + a_{22}v')x_v; \]

hence,  
\[ dN \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}. \]

This shows that the basis \( \{x_u, x_v\} \), \( dN \) is given by the matrix \( (a_{ij}) \), \( i, j = 1, 2 \). The matrix are not necessarily symmetric, unless \( \{x_u, x_v\} \) is an orthonormal basis.

The expression of the second fundamental form in the basis \( \{x_u, x_v\} \) is given by

\[ \II_p(\alpha') = -\langle dN(\alpha'), \alpha' \rangle \]
\[ = -\langle N_u'u' + N_v'v', x_u'u', x_v'v' \rangle \]

hence,  
\[ \II_p(\alpha') = L(u')^2 + 2M u'v' + N(v')^2, \]  

where, since \( \langle N, x_u \rangle = \langle N, x_v \rangle = 0 \) yields,

\[ L = -\langle N_u, x_u \rangle = \langle N, x_{uu} \rangle = x_{uu} \cdot N \]
\[ M = -\langle N_v, x_u \rangle = \langle N, x_{uv} \rangle = \langle N, x_{uv} \rangle = -\langle N_u, v \rangle = x_{uv} \cdot N \]
\[ N = -\langle N_v, x_v \rangle = \langle N, x_{vv} \rangle = x_{vv} \cdot N \]

We will now obtain the values of \( a_{ij} \) in terms of the coefficients \( L, M, N \). From (1.3) and (1.4) we have

\[ -L = \langle N_u, x_u \rangle = a_{11}E + a_{21}F \]
\[ -M = \langle N_u, x_v \rangle = a_{11}F + a_{21}G \]
\[ -M = \langle N_v, x_u \rangle = a_{12}E + a_{22}F \]
\[ -N = \langle N_v, x_v \rangle = a_{12}F + a_{22}G \]
where $E, F,$ and $G$ are the coefficients of the first fundamental form in the basis $\{x_u, x_v\}$. We can express the above relation in matrix form by

$$
\begin{pmatrix}
L & M \\
M & N
\end{pmatrix}
\begin{pmatrix}
a_{11} & a_{21} \\
a_{12} & a_{22}
\end{pmatrix}
\begin{pmatrix}
E & F \\
F & G
\end{pmatrix}
$$

hence,

$$
\begin{pmatrix}
a_{11} & a_{21} \\
a_{12} & a_{22}
\end{pmatrix}
= -
\begin{pmatrix}
L & M \\
M & N
\end{pmatrix}
\begin{pmatrix}
E & F \\
F & G
\end{pmatrix}^{-1}
$$

then

$$
\begin{align*}
a_{11} &= \frac{MF - LG}{EG - F^2} \\
a_{12} &= \frac{NF - MG}{EG - F^2} \\
a_{21} &= \frac{LF - ME}{EG - F^2} \\
a_{22} &= \frac{MF - NE}{EG - F^2}
\end{align*}
$$

Note that relations (1.3) and (1.4), along with the above coefficients are known as the equations of Weingarten.

From (1.10) we obtain

$$
K = \text{det}(a_{ij}) = \frac{LN - M^2}{EG - F^2}
$$

To compute the mean curvature, we recall that $-k_1, -k_2$ are eigenvalues of $dN$. Therefore, $k_1$ and $k_2$ satisfy the equation

$$
dN(v) = -k v = k I v \quad \text{for some } v \in T_p(S), \ v \neq 0
$$

where $I$ is the identity map. Then the linear map $dN + k I$ is not invertible; hence, its discriminant is zero. Thus,

$$
\text{det}
\begin{pmatrix}
a_{11} + k & a_{12} \\
a_{21} & a_{22} + k
\end{pmatrix}
= 0
$$
or

\[ k^2 + k(a_{11} + a_{22}) + a_{11}a_{22} - a_{21}a_{12} = 0. \]

Since \( k_1 \) and \( k_2 \) are the roots of the above quadratic equation, we conclude that

\[ H = \frac{1}{2}(k_1 + k_2) = -\frac{1}{2}(a_{11} + a_{22}) = \frac{1}{2}\frac{LN - 2MF + NE}{EG - F^2}; \quad (1.12) \]

hence,

\[ k^2 - 2Hk + K = 0, \]

and therefore,

\[ k = H \pm \sqrt{H^2 - K}. \quad (1.13) \]

With equations (1.3) and (1.4) in terms of the first and second fundamental coefficients yields

\[ H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}\frac{EN - 2FM + GL}{EG - F^2}; \quad (1.14) \]

\[ K = k_1k_2 = \frac{LN - M^2}{EG - F^2}; \quad (1.15) \]

and, these properties allows us to define points on a surface as follows:

1. If \( \det(dN_p) = LN - M^2 > 0 \), then a point on the surface \( S \) is called elliptical. In this case the Gaussian curvature is positive. Both principle curvature have the same sign, and therefore all curves passing through this point have their normal vectors pointing toward the same side of the tangent plane (Figure 1.5(a)).

Figure 1.5: Points on a Surface
2. If $\det(dN_p) = LN - M^2 < 0$, then a point on the surface $S$ is called hyperbolic. In this case the Gaussian curvature is negative. The principle curvatures have opposite signs, and therefore there are curves through $p$ whose normal vectors at $p$ point toward any of the sides of the tangent plane at $p$ (Figure 1.5(b)).

3. If $\det(dN_p) = LN - M^2 = 0$ with $dN_p = L^2 + M^2 + N^2 \neq 0$, then a point on the surface $S$ is called parabolic. In this case the Gaussian curvature is zero, but one of the principle curvatures is not zero (Figure 1.5(c)).

4. If $\det(dN_p) = L = M = N = 0$, then a point on the surface $S$ is called planar. In this case all principle curvatures are zero. The points of a plane trivially satisfy this condition.

We will later observe how the first three cases relates to surfaces of constant Gaussian curvature.

Since, $N$ now represents a coefficient from the second fundamental form we will then let $U$ represent the unit normal vector and $O$ be an open set.

1.3 Geodesics

This section will focus on the important concepts associated to a regular surface depend only on the first fundamental form and see how they will relate to Geodesics on surfaces of constant Gaussian curvature. Therefore, it is convenient that we formulate in a precise way what is meant by two regular surfaces having equal first fundamental forms.

**Definition 1.11.** An isometry $\phi : S \to \tilde{S}$ of two regular surfaces $S, \tilde{S}$ in $\mathbb{R}^3$ is bijective differentiable mapping that preserves the first fundamental form; when this is the case, the two surfaces $S, \tilde{S}$ are said to be isometric.

**Definition 1.12.** A mapping $\phi : V \to \tilde{S}$ of a neighborhood $V$ of a point $p$ on a surface $S$ into a surface $\tilde{S}$ is local isometry at $p$ if there exists a neighborhood $\tilde{V}$ of $\phi(p) \in \tilde{S}$ such that $\phi : V \to \tilde{V}$ is an isometry. If there exists a local isometry of $S$ on $\tilde{S}$ at every point $p \in S$, the surface $S$ is said to be locally isometric to $\tilde{S}$. Surfaces $S$ and $\tilde{S}$ are locally isometric if $S$ is locally isometric to $\tilde{S}$ is locally isometric to $S$. 
To get a better understanding of the above definitions consider a one-sheeted cone minus the vertex and a generator and show that it is locally isometric to a plane.

This can be seen geometrically by cutting the cone along the generator and unrolling it onto a piece of plane. In this case the unrolling is a bending of the surface without stretching or shrinking; therefore it is an isometry. This is also clear, perhaps even clearer, from the inverse process of the unrolling a piece of paper in the form of a circular sector can be bent around a cone; the radius of the sector becomes the length of the generator (Figure 1.6).

**Theorem 1.13.** Assume the existence of parametrization $x : O \rightarrow S$ and $\bar{x} : O \rightarrow \bar{S}$ such that $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$ in $O$. Then the map $\phi : \bar{x} \circ x^{-1} : x(O) \rightarrow \bar{S}$ is a local isometry.

**Proof.** Let $p \in x(O)$ and $w \in T_p(S)$. Then $W$ is tangent to a curve $x(\alpha(t))$ at $t = 0$, where $\alpha(t) = (u(t), v(t))$ is a curve in $U$; thus, $w$ may be written

$$w = x_u u' + x_v v'.$$

By definition, the vector $d\phi_p(w)$ is the tangent vector to the curve $\bar{x} \circ x^{-1} \circ x(\alpha(t))$, i.e., to the curve $\bar{x}(\alpha(t))$ at $t = 0$. Thus,

$$d\phi_p(w) = \bar{x}_u u' + \bar{x}_v v'.$$

Since

$$I_p(w) = E(u')^2 + 2Fu'v' + G(v')^2, $$
$$I_{\phi(p)}(d\phi_p(w)) = \bar{E}(u')^2 + 2\bar{F}u'v' + \bar{G}(v')^2,$$
we conclude that $I_p(w) = I_{\phi(p)}(d\phi_p(w))$ for all $p \in x(O)$ and all $w \in T_p(S)$; hence, $\phi$ is a local isometry. \hfill \Box

Now, I will turn my focus to studying the analogue of Frenet’s trihedron and the derivatives of its vector. The purpose of this is to show that the Gaussian curvature can be written in terms of the first fundamental coefficients and therefore, locally isometric. $S$ will denote, as usual, a regular, orientable, and oriented surface. Let $x : O \to S$ be a parametrization in the orientation of $S$. It is possible to assign to each point of $x(O)$ a natural trihedron given by the vectors $x_u, x_v, u$.

By expressing the derivatives of the vectors $x_u, x_v, U$ in terms of its basis $\{x_u, x_v, U\}$, yields

\begin{align*}
x_{uu} &= \Gamma_1^1 x_u + \Gamma_1^2 x_v + L_1 U, \\
x_{uv} &= \Gamma_2^1 x_u + \Gamma_2^2 x_v + L_2 U, \\
x_{vu} &= \Gamma_2^1 x_u + \Gamma_2^2 x_v + L_2 U, \\
x_{vv} &= \Gamma_2^1 x_u + \Gamma_2^2 x_v + L_3 U, \\
U_u &= a_{11} x_u + a_{21} x_v, \\
U_v &= a_{12} x_u + a_{22} x_v,
\end{align*}

where the $a_{ij}, i, j = 1, 2$, were obtain chapter 2.6 and the other coefficients are left to be determined. The coefficients $\Gamma_i^k, i, j, k = 1, 2$, are called the Christoffel symbols of $S$ in the parametrization $x$. Since $x_{uv} = x_{vu}$, we conclude that $\Gamma_1^1 = \Gamma_2^1$ and $\Gamma_1^2 = \Gamma_2^2$.

By taking the inner product of $x_{uu}, x_{uv}, x_{vu}, x_{vv}$ with $N$, we obtain $L_1 = L, L_2 = \tilde{L}_2 = M, L_3 = N$, where $L, N, M$ are the coefficients of the second fundamental form of $S$. Moreover, to determine the Christoffel symbols, we take the inner
product of \( x_{uu}, x_{uv}, x_{vu}, x_{vv} \) with \( x_u \) and \( x_v \)

\[
\begin{align*}
\Gamma^{1}_{1} E + \Gamma^{2}_{1} F &= \langle x_{uu}, x_u \rangle = \frac{1}{2} E_u, \\
\Gamma^{1}_{1} F + \Gamma^{2}_{1} G &= \langle x_{uu}, x_v \rangle = F_u - \frac{1}{2} E_v, \\
\Gamma^{1}_{2} E + \Gamma^{2}_{1} F &= \langle x_{uv}, x_u \rangle = \frac{1}{2} E_v, \\
\Gamma^{1}_{2} F + \Gamma^{2}_{1} G &= \langle x_{uv}, x_v \rangle = \frac{1}{2} G_u, \\
\Gamma^{1}_{2} E + \Gamma^{2}_{2} F &= \langle x_{vv}, x_u \rangle = F_v - \frac{1}{2} G_u, \\
\Gamma^{1}_{2} F + \Gamma^{2}_{2} G &= \langle x_{vv}, x_v \rangle = \frac{1}{2} G_v.
\end{align*}
\]

Note that all geometric concepts and properties expressed in terms of the Christoffel symbols are invariant under isometries.

The Gaussian curvature can tell us a lot about a surface. We compute \( K \) using the unit normal \( U \), so that it would seem reasonable to think that the way in which we embed the surface in three space would affect the value of \( K \) while leaving the geometry of \( S \) unchanged. This would mean that the Gaussian curvature would not be a geometric invariant and, therefore, would not be as helpful in studying surfaces. Our goal is to find a formula for \( K \) that does not depend on \( U \). We will give a formula for \( K \) that only depends on \( E, F, \) and \( G \). These three quantities \( \{E, F, G\} \) are called the metric of a surface. We will state the general formula later but for now we will look at the case when \( F = x_u \cdot x_v = 0 \).

**Theorem 1.14.** The Gaussian curvature of a surface depends only on the metric \( E, F, \) and \( G, \)

\[
K = -\frac{1}{\sqrt{EG}} \left( \left( \frac{E_u}{\sqrt{EG}} \right)_v + \left( \frac{G_u}{\sqrt{EG}} \right)_u \right)
\]

where

\[
E_v = (x_u \cdot x_u)_v \quad \text{and} \quad G_u = (x_v \cdot x_v)_u
\]

**Proof.** Since, \( S \) is a regular surface and \( x(u,v) \) is a parametrization of \( S \) implies that \( \{x_u, x_v, N\} \) form a basis for \( \mathbb{R}^3 \). Recall, \( L = x_{uu} \cdot U, M = x_{uv} \cdot U \) and \( N = x_{vv} \cdot U \).
Expanding $x_{uu}$, $x_{uv}$, and $x_{vv}$ in terms of its basis yields,

\[
x_{uu} = \Gamma_1^{11}x_u + \Gamma_1^{11}x_v + LU,
\]
\[
x_{uv} = \Gamma_1^{12}x_u + \Gamma_1^{12}x_v + MU,
\]
\[
x_{vv} = \Gamma_2^{22}x_u + \Gamma_2^{22}x_v + NU,
\]

and

\[
x_{uu} \cdot x_u = \Gamma_1^{11}x_u \cdot x_u + 0 + 0 = \Gamma_1^{11}E.
\]

If we compute $x_{uu} \cdot x_u$ then we will know $\Gamma_1^{11}$. Since,

\[
E = x_u \cdot x_u, \quad \text{so} \quad E_u = x_{uu} \cdot x_u + x_u \cdot x_{uu} = 2x_{uu} \cdot x_u.
\]

Thus,

\[
x_{uu} \cdot x_u = \frac{E_u}{2} \quad \text{and} \quad \Gamma_1^{11} = \frac{E_u}{2E}.
\]

Further, $x_u \cdot x_v = 0$ so taking the partial with respect to $u$ gives

\[
0 = x_{uu} \cdot x_v + x_u \cdot x_{uv} \quad \text{or} \quad x_{uu} \cdot x_v = -x_u \cdot x_{uv}.
\]

Also, $E = x_u \cdot x_u$, so taking the partial with respect to $v$ gives $E_v = 2x_u \cdot x_{uv}$ and, consequently, $E_v/2 = x_u \cdot x_{uv} = -x_{uu} \cdot x_v$. Moreover,

\[
\Gamma_1^{11} = \frac{(x_{uu} \cdot x_v)}{G} = -\frac{E_v}{2G}
\]

and

\[
\Gamma_1^{12} = \frac{(x_{uv} \cdot x_u)}{E} = \frac{E_v}{2E}.
\]

Continuing on, $G = x_u \cdot x_v$, so $G_u/2 = x_{uu} \cdot x_u$. Then, since $0 = x_v \cdot x_u$, we have

\[-x_v \cdot x_{uv} = x_{vv} \cdot x_u
\]

with

\[
\Gamma_2^{12} = \frac{x_{uv} \cdot x_v}{G} = \frac{G_u}{2G} \quad \text{and} \quad \Gamma_2^{22} = \frac{x_{vv} \cdot x_u}{E} = -\frac{G_u}{2E}.
\]
therefore, the Cristoffel symbols are
\[\Gamma^1_{11} = \frac{E_u}{2E}, \quad \Gamma^2_{22} = \frac{G_v}{2G},\]
\[\Gamma^1_{12} = \frac{E_v}{2E}, \quad \Gamma^2_{12} = \frac{G_u}{2G},\]
\[\Gamma^1_{22} = \frac{G_u}{2G}, \quad \Gamma^2_{11} = \frac{G_v}{2G}.\]

Finally, \(x_v \cdot x_v = G\), so \(x_{uv} \cdot x_v = G_v/2\) and \(\Gamma^0_{22} = x_{uv} \cdot x_u/G = G_u/2G\) and we end up with the following formulas.

Moreover, using a similar argument we can show that the general case \((F \neq 0)\) yields
\[K = \frac{1}{(EG - F^2)^2} \begin{pmatrix} -\frac{E_u}{2} + f_{uv} - \frac{G_{uv}}{2} & \frac{E_u}{2} & F_u - \frac{E_v}{2} \\ F_v - \frac{G_{uv}}{2} & E & F \\ \frac{G_{uv}}{2} & F & G \end{pmatrix} - \begin{pmatrix} 0 & \frac{E_u}{2} & \frac{G_{uv}}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_{uv}}{2} & F & G \end{pmatrix}\]

Therefore, the Gaussian curvature \(K\) of a surface \(S\) is invariant by local isometries.

We begin by defining the covariant derivative of vector field, which is the analogue for surfaces of the usual differentiation of vectors in the plane. Recall the tangent vector field in an open set \(O \subset S\) is a correspondence \(w\) that assigns to each \(p \in O\) a vector \(w(p) \in T_p(S)\). The vector field \(w\) is differentiable at \(p\) if, for some parametrization \(x(u, v)\) in \(p\), the components \(a\) and \(b\) of \(w = ax_u + bx_v\) in the basis \(\{x_u, x_v\}\) are differentiable function at \(p\) and \(w\).

**Definition 1.15.** Let \(w\) be a differential a vector field in an open set \(O \subset S\) and \(p \in O\). Let \(y \in T_p(S)\) and consider a parametrized curve
\[\alpha : (-\epsilon, \epsilon) \to O,\]
with \(\alpha(0) = p\) and \(\alpha'(0) = y\), and let \(w(t)\), \(t \in (-\epsilon, \epsilon)\), be the restriction of the vector field \(w\) to the curve \(\alpha\). The vector obtain by the normal projection of \((dw/dt)(0)\) onto the plane \(T_p(S)\) is called the covariant derivative at \(p\) of the vector field \(w\) relative to the vector \(y\). This covariant derivative is denoted by \((Dw/dt)(0)\) (Figure 1.3).
Figure 1.7: The Covariant Derivative

dw

geometry and that it does not depend on the choice of the curve \( \alpha \), we will obtain its expression in terms of a parametrization \( \mathbf{x}(u, v) \) of \( S \) in \( p \).

Let \( \mathbf{x}(u(t), v(t)) = \alpha(t) \) be the expression of the curve \( \alpha \) and let

\[
\mathbf{w}(t) = a(u(t), v(t))\mathbf{x}_u + b(u(t), v(t))\mathbf{x}_v \\
= a(t)\mathbf{x}_u + b(t)\mathbf{x}_v,
\]

be the expression of \( \mathbf{w}(t) \) in the parametrization \( \mathbf{x}(u, v) \). Then

\[
\frac{d\mathbf{w}}{dt} = a(\mathbf{x}_{uu}u' + \mathbf{x}_{uv}v') + b(\mathbf{x}_{vu}u' + \mathbf{x}_{vv}v') + a'\mathbf{x}_u + b'\mathbf{x}_v,
\]

where prime denotes the derivative with respect to \( t \).

Since \( \frac{D\mathbf{w}}{dt} \) is the component of \( d\mathbf{w}/dt \) in the tangent plane, we use the expression found in sections 2.7 for \( \mathbf{x}_{uu}, \mathbf{x}_{uv}, \) and \( \mathbf{x}_{vv} \) and, by dropping the normal component yields

\[
\frac{D\mathbf{w}}{dt} = (a' + \Gamma^1_{11}au' + \Gamma^1_{12}bu')\mathbf{x}_u \\
+ (b' + \Gamma^2_{11}au' + \Gamma^2_{12}bu')\mathbf{x}_v,
\]

and substituting the values for the Christoffel symbols yields,

\[
\frac{D\mathbf{w}}{dt} = \left( a' + \frac{E_u}{2E}au' + \frac{E_v}{2E}av' + \frac{E_u}{2E}bu' + \frac{G_v}{2G}bv' \right)\mathbf{x}_u \\
+ \left( b' + \frac{G_v}{2G}au' + \frac{G_u}{2G}av' + \frac{G_u}{2G}bu' + \frac{G_v}{2G}bv' \right)\mathbf{x}_v.
\]
Hence, the covariant derivative $Dw/dt$ of a tangent vector $w$ of a surface $S$ at a point $p$ depends only on the first fundamental form of the surface $S$ and is a generalization of the usual derivative of vectors in the plane.

**Definition 1.16.** A tangent vector field $w$ of a surface is parallel along a parametrized curve $C$ on $S$ if $Dw/dt = 0$ along $C$.

**Theorem 1.17.** On a surface $S$ there exists a unique parallel tangent vector field $w(t)$ such that $v(t)$ is a given tangent vector; in other words, every tangent vector $v(t_0)$ of the surface $S$ at the point $x(t_0)$ can be parallely transported along a curve $x(t)$ on $S$, and the vector $v(t_1)$ is called the parallel transport of $v(t_0)$ along the curve $x(t)$ at the point $x(t_1)$.

**Lemma 1.18.** The parallelism on a surface $S$ preserves the inner product of two tangent vector fields $v$ and $w$ on $S$ and, in particular, the angle $\theta$ between $v$ and $w$ is constant if they are constant lengths.

**Proof.** Since $v \cdot U = w \cdot U = 0$, we have

$$d(v \cdot w) = dv \cdot w + v \cdot dw = \frac{Dv}{dt} \cdot w + v \cdot \frac{Dw}{dt}.$$  

If $v$ and $v$ are parallel along any curve $C$ on $S$, then $Dv/dt = Dw/dt = 0$ which implies that $d(v \cdot w) = 0$. Therefore, $v \cdot w$ is constant. \qed

Consider a family of unit tangent vectors $y(s)$ of a surface $S$, which are parallel along a curve $C$ on $S$ where $s$ is the arc length of $C$. Let $t$ be the angle between $y(s)$ and the unit tangent vector $x'(s)$ of $C$, where the prime denotes the derivative with respect to $s$.

**Definition 1.19.** The rate of change of $t$ with respect to $s$

$$k_g = \frac{dt}{ds}$$  \hspace{1cm} (1.16)

is called the geodesic curvature at a point $p$ of $C$ on $S$, and a curve on a surface with zero geodesic curvature everywhere is a geodesic of the surface.
Let $\Delta \theta$ be the angle between the tangent vector $x'(s + \Delta s)$ and the parallel transport of the vector $x'(s)$ along $C$ at $x(s + \Delta s)$. Since the angle between two tangent vectors of constant lengths remain unchanged under a parallel transport, it follows that

$$k_g = \lim_{\Delta s \to 0} \frac{\Delta \theta}{\Delta s},$$

which shows that $k_g$ is independent of the choice of $y(s)$.

Geometrically, the tangent vectors of a geodesic are parallel to themselves along the geodesic. When the enveloping developable surface of a geodesic is unrolled onto a plane, the geodesic goes to a straight line.

We shall give an alternate definition of geodesic curvature and then state the properties of geodesics curvature and geodesics.

The acceleration along a curve $\alpha$ along a surface $S$ can be broken up into the tangential and normal components $\alpha'' = v'T + kv^2N$, but we can drop the normal component because we are on the tangent plane.

Now, suppose that $\alpha$ has unit speed. Then we have two perpendicular unit vectors $T = \alpha'$ and $U$, the unit normal of $S$. We get a third unit vector orthogonal to $T$ and $U$ by taking $T \times U$. These three vectors have to form a basis for $R^3$. Thus, $\alpha'' = aT + bT \times U + cU$ where

$$a = \alpha'' \cdot T, \quad b = \alpha'' \times U, \quad c = \alpha'' \cdot U.$$

Rewriting $\alpha$ yields

$$\alpha'' = (\alpha'' \cdot T)T + (\alpha'' \times U)T \times U + (\alpha'' \cdot U)U.$$

Since $\alpha' \cdot \alpha' = 1$ so differentiating we find that $\alpha \cdot \alpha'' = 0$ or $\alpha'' \cdot T = \alpha'' \cdot \alpha' = 0$. Thus, there is no T-component for $\alpha''$,

$$\alpha'' \cdot (T \times U) = -\alpha'' \cdot (U \times T)$$

$$= -(U \cdot (\alpha'' \times T))$$

$$= U \cdot (\alpha'' \times \alpha').$$

Our first condition above ($U \cdot (T \times U) = 0$) tells us that $T \times U$ lies in $T_p(S)$ for all $p \in S$. 
The second condition tells us

\[ \alpha'' \cdot (T \times U) = U \cdot (\alpha'' \times \alpha') \]

\[ = \|U\| \|\alpha'' \times \alpha'\| \cos \theta \]

\[ = \|\alpha'' \times \alpha'\| \cos \theta \]

\[ = k_\alpha \cos \theta \]

where \( k_\alpha \) is the curvature of \( \alpha \) and \( \theta \) is the angle between \( \alpha'' \times \alpha' \) and \( U \). This quantity is called the geodesic curvature of \( \alpha \) and is denoted

\[ k_g = k_\alpha \cos \theta. \]

We shall define a system of parameters bearing a simple relationship to geodesics. We take a curve \( C_0 \) on a surface \( S \) give by \( x(u, v) \), and construct geodesics of \( S \) through the points of \( C_0 \) and orthogonal to \( C_0 \). Take these geodesics as \( u \)-curves and their orthogonal trajectories as \( v \)-curves. We restrict our discussion to a neighborhood in which this coordinate system is valid. Since the parametric curves are orthogonal we have \( F = 0 \). Denoting the arc length on \( u \)-curves by \( s \) we find, along \( u \)-curves,

\[ x_s = x_u \frac{du}{ds} = \frac{x_u}{\sqrt{E}} \quad \text{and} \quad x_{ss} = x_{su} \frac{x_u}{\sqrt{E}} = \frac{x_{uu}}{E} - \frac{E_u x_u}{2E^2}, \]

therefore

\[ k_g = -\frac{E_v}{2GE^{3/2}} |x_u, x_v, U| = -\frac{E_v}{2E\sqrt{G}} = -\frac{(\sqrt{E})_v}{\sqrt{EG}} = 0, \]

which implies that \( E \) is a function of \( u \) alone. If we introduce a new parameter \( \int \sqrt{E} du \) and denote it again by \( u \), the first fundamental form of the surface \( S \) becomes

\[ ds^2 = du^2 + G(u, v)dv^2. \quad (1.17) \]

For convenience of studying intrinsic geometry we can define geodesic in terms of polar coordinates and equation 1.17 can be written as

\[ ds^2 = dr^2 + G(r, \phi)d\phi^2, \quad (1.18) \]

where \( r \) is the arc length along the geodesics through some point \( P \), and \( \phi \) can be chosen to be the angle the geodesic makes with a fixed directions at \( P \).
Moreover, we will introduce a way to handle the calculation of geodesics in a more general way. In the following we only consider orthogonal parameterizations \( x(u, v) \) (i.e. \( F = x_u \cdot x_u = 0 \)). Let \( \alpha \) be a geodesic in \( x(u, v) \). Then \( \alpha = x(u(t), v(t)) \) and \( \alpha' = x_u u' + x_v v' \) with

\[
\alpha'' = x_{uu}(u')^2 + x_{uv} v'u' + x_u u'' + x_{vu} u'v' + x_{vv}(v')^2 + x_v v''.
\]

Using the formulas for \( x_{uu}, x_{uv} \) and \( x_{vv} \) from section 2.7 yields

\[
\alpha'' = x_u \left( u'' + \frac{E_u}{2E}(u')^2 + \frac{E_v}{E} u'v' - \frac{G_u}{2E}(v')^2 \right)
+ x_v \left( v'' - \frac{E_v}{2G}(u')^2 + \frac{G_u}{G} u'v' + \frac{G_v}{2G}(v')^2 \right)
+ U \left( L(u')^2 + 2M u' v' + N(v')^2 \right)
\]

where the first two terms give the tangential part of \( \alpha'' \). For \( \alpha \) to be a geodesic then, it is both necessary and sufficient that the following geodesic equations

\[
u'' + \frac{E_u}{2E}(u')^2 + \frac{E_v}{E} u'v' - \frac{G_u}{2E}(v')^2 = 0
\]

(1.19)

\[
v'' - \frac{E_v}{2G}(u')^2 + \frac{G_u}{G} u'v' + \frac{G_v}{2G}(v')^2 = 0
\]

(1.20)

are satisfied.

Now we are ready to begin our study of surfaces of constant Gaussian curvature. Since in a geodesic polar system \( E = 1 \) and \( F = 0 \), the Gaussian curvature \( K \) can be written

\[
K = \frac{-(\sqrt{G})_{rr}}{\sqrt{G}}.
\]

This expression may be considered as the differential equation which \( \sqrt{G}(r, \phi) \) should satisfy if we want the surface to have curvature \( K(r, \phi) \). If \( K \) is constant, then their expression can be written

\[
(\sqrt{G})_{rr} + K \sqrt{G} = 0,
\]

(1.21)

a linear differential equation of second order with constant coefficients.

### 1.4 Examples

Before studying surfaces of constant Gaussian curvature, we will present some examples that show the use of the definitions and theorems.
Example 1.20. Show that the sphere

\[ \{ s^2 = (x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = a^2 \}, \]

which consists of all points at distance \( a \) from the origin \( O(0,0,0) \), is a surface.

We first need to verify that the map \( \mathbf{x}_1 : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) given by

\[ \mathbf{x}_1(x, y) = \left( x, y, +\sqrt{a^2 - (x^2 + y^2)} \right), \quad (x, y) \in U, \]

where \( U = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < a^2 \} \) and \( U = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < a^2 \} \), is a parametrization of \( S^2 \). Observe that \( \mathbf{x}_1(U) \) is the open part of \( S^2 \) above the xy plane.

Since \( x^2 + y^2 < a^2 \), the function \( +\sqrt{a^2 - (x^2 + y^2)} \) has continuous partial derivatives of all order. Thus, \( \mathbf{x}_1 \) is differentiable and condition 1 holds.

Next, we observe that \( \mathbf{x}_1 \) is one-to-one and that \( \mathbf{x}_1^{-1} \) is the restriction of the continuous projection \( \pi(x, y, z) = (x, y) \) to the set \( \mathbf{x}_1(U) \). Thus, \( \mathbf{x}_1^{-1} \) is continuous in \( \mathbf{x}_1(U) \) and satisfies condition 2.

To check condition 3, we can compute its Jacobian Matrix

\[
\begin{pmatrix}
\frac{\partial x}{\partial x} & \frac{\partial y}{\partial x} & \frac{\partial f}{\partial x} \\
\frac{\partial x}{\partial y} & \frac{\partial y}{\partial y} & \frac{\partial f}{\partial y} \\
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & \frac{\partial f}{\partial x} \\
0 & 1 & \frac{\partial f}{\partial y} \\
\end{pmatrix}
\]

whose rank is obviously 2 at each point \((x, y, z) \in S^2_+\). Hence \( \mathbf{x}_1 \) is a parametrization of \( S^2_+ \).

Next we cover the whole sphere with a similar parametrization as follows. Define \( \mathbf{x}_2 : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) by

\[ \mathbf{x}_2(x, y) = \left( x, y, -\sqrt{a^2 - (x^2 + y^2)} \right). \]

Moreover, \( \mathbf{x}_1(U) \cup \mathbf{x}_2(U) \) covers \( S^2 \) minus the equator

\[ \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 = a^2, z = 0 \}. \]

Then, using the xz and yz plane, we define the parametrization

\[ \mathbf{x}_3(x, z) = (x, +\sqrt{a^2 - (x^2 + z^2)}, z), \]
\[ \mathbf{x}_4(x, z) = (x, -\sqrt{a^2 - (x^2 + z^2)}, z), \]
\[ \mathbf{x}_5(y, z) = (+\sqrt{a^2 - (y^2 + z^2)}, y, z), \]
\[ \mathbf{x}_6(y, z) = (-\sqrt{a^2 - (y^2 + z^2)}, y, z), \]
which, together with $x_1$ and $x_2$, cover $S^2$ completely (Figure 1.8) and show that $S^2$ is a regular surface.

For most applications, it is convenient to relate parametrization to the geographical coordinates on $S^2$. Let $V = \{(\theta, \phi); 0 < \theta < \pi, 0 < \phi < 2\pi\}$ and let $x : V \to \mathbb{R}^3$ be given by

$$x(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

Clearly, $x(V) \subseteq S^2$. We shall prove that $x$ is a parametrization of $S^2$. $\theta$ is usually called the colatitude and $\phi$ the longitude (Figure 1.9).

It is clear that the function $\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta$ have continuous partial derivative of all order; hence $x$ is differentiable. Moreover, in order that the Jacobian determinants

$$\frac{\partial (x, y)}{\partial (\theta, \phi)} = a^2 \cos \theta \sin \theta,$$

$$\frac{\partial (y, z)}{\partial (\theta, \phi)} = a^2 \sin^2 \theta \cos \phi,$$

$$\frac{\partial (x, z)}{\partial (\theta, \phi)} = a^2 \sin^2 \theta \sin \phi,$$
vanish simultaneously, it is necessary that

\[ a^4(\sin^2 \theta \cos^2 \theta + \sin^4 \theta \cos^2 \phi + \sin^4 \theta \sin^2 \phi) = a^4 \sin^2 \theta = 0. \]

This does not happen in \( V \), and so conditions 1 and 3 of definition 1.1 are satisfied.

Next, we observe that given \((x, y, z) \in S^2 - C\), where \( C \) is the semicircle

\[ C = \{(x, y, z) \in S^2; y = 0, x \geq 0\}, \]

\( \theta \) is uniquely determine by \( \theta = \cos^{-1} z \), since \( 0 < \theta < \pi \). By knowing \( \theta \), we find \( \sin \phi \) and \( \cos \phi \) from \( x = \sin \theta \cos \phi \), \( y = \sin \theta \sin \phi \), and this determines \( \phi \) uniquely \((0 < \phi < 2\pi)\).

It follows that \( x \) has an inverse \( x^{-1} \). By lemma 1.4 \( x^{-1} \) is continuous.

We remark that \( x(V) = S^2 - \text{semicircle} \) through the two poles \((0, 0, a)\) and \((0, 0, -a)\) and that we can cover \( S^2 \) by the coordinate neighborhoods of two parameterizations of this type.

**Example 1.21.** Calculate the Mean and Gaussian curvature of the Enneper’s surface parametrized by

\[ x(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3}, u^2, u^2 - v^2) \]
Then, $\mathbf{x}_u = (1 - u^2 + v^2, 2uv, 2u)$, $\mathbf{x}_v = (2uv, 1 - v^2 + u^2, -2v)$ and we compute $E$, $F$ and $G$

$$E = \mathbf{x}_u \cdot \mathbf{x}_u$$
$$= 1 + 2u^2 + 2v^2 + u^4 + 2u^2v^2 + v^4$$
$$= (1 + u^2 + v^2)^2,$$

$$F = \mathbf{x}_u \cdot \mathbf{x}_v$$
$$= 2uv - 2u^3v + 2uv^3 + 2uv - 2uv^3 + 2u^3v - 4uv$$
$$= 0,$$

$$G = \mathbf{x}_v \cdot \mathbf{x}_v$$
$$= 4u^2v^2 + 1 - v^2 + u^2 - v^2 + v^4 - v^2u^2 + u^2 - u^2v^2 + u^4 + 4v^2$$
$$= 1 + 2u^2 + 2v^2 + u^4 + 2u^2v^2 + v^4$$
$$= (1 + u^2 + v^2)^2.$$
The unit normal

\[
U = \frac{x_u \times x_v}{|x_u \times x_v|}
\]

\[
= \left( \frac{-2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{1 - (u^2 + v^2)^2}{u^2 + v^2 + 1} \right).
\]

The second partials \(x_{uu} = (-2u, 2v, 2), x_{uv} = (2v, 2u, 0), x_{vv} = (2u, -2v, -2)\) yields

\[
L = x_{uu} \cdot U = \frac{2(u^2 + v^2 + 1)^2}{(u^2 + v^2 + 1)^2} = 2
\]

\[
M = x_{uv} \cdot U = 0
\]

\[
N = x_{vv} \cdot U = \frac{-2(u^2 + v^2 + 1)^2}{(u^2 + v^2 + 1)^2} = -2
\]

Then the Gaussian curvature of the Enneper's surface is

\[
K = \frac{LN - M^2}{EG - F^2}
\]

\[
= \frac{-4}{(u^2 + v^2 + 1)^4}
\]

and the mean curvature is

\[
H = \frac{GL + EN - 2FM}{2(EG - F^2)}
\]

\[
= \frac{(u^2 + v^2 + 1)^2(2) + (u^2 + v^2 + 1)^2(-2) - 0}{2(u^2 + v^2 + 1)^4}
\]

\[
= 0.
\]
Moreover, since the mean curvature is zero implies that the Enneper’s surface is a minimal surface.

Also note that calculating the mean and Gaussian curvature can be very time consuming and from now on MAPLE will be used to aid us in our studies (see Appendix A).

**Example 1.22.** Prove that the one-sheeted cone (minus the vertex)

\[ z = \pm k \sqrt{x^2 + y^2}, \quad (x, y) \neq (0, 0), \]

is locally isometric to a plane.

**Proof.** Let \( U \subset \mathbb{R}^2 \) be the open set given in polar coordinates \((\rho, \theta)\) by

\[ 0 < \rho < \infty, \quad 0 < \theta < 2\pi \sin \alpha, \]

where \( 2\alpha (0 < 2\alpha < \pi) \) is the angle at the vertex of the cone (i.e., where \( \cot \alpha = k \)), and let \( F: U \to \mathbb{R}^3 \) be the map (Figure 2-17)

\[ F(\rho, \theta) = \left( \rho \sin \alpha \cos \left( \frac{\theta}{\sin \alpha} \right), \rho \sin \alpha \sin \left( \frac{\theta}{\sin \alpha} \right), \rho \cos \alpha \right). \]

\( F(U) \) is contained in the cone because

\[ k \sqrt{x^2 + y^2} = \cot \alpha \sqrt{\rho^2 \sin^2(\alpha)} = \rho \cos \alpha = z. \]

Furthermore, when \( \theta \) describes the interval \((0, 2\pi \sin \alpha)\), \( \theta/\sin \alpha \) describes the interval \((0, 2\pi)\). Thus, all points of the cone except the generator \( \theta = 0 \) are covered by \( F(U) \).

Since \( F \) and \( dF \) are one-to-one in \( U \) implies that \( F \) is a diffeomorphism of \( U \) onto the cone minus a generator.

To show that \( F \) is an isometry, one can think of \( U \) as a regular surface parametrized by

\[ \bar{\mathbf{x}}(\rho, \theta) = (\rho \cos \theta, \rho \sin \theta, 0), \quad 0 < \rho < \infty, \quad 0 < \theta < 2\pi \sin \alpha. \]

The coefficients of the first fundamental form of \( U \) in this parametrization are

\[ \bar{E} = 1, \quad \bar{F} = 0, \quad \bar{G} = \rho^2. \]
On the other hand, the coefficients of the first fundamental form of the cone in the parametrization $F \circ \bar{x}$ are

\[ E = 1, \quad F = 0, \quad G = \rho^2. \]

Therefore, $F$ is a local isometry. \qed
Chapter 2

Surfaces of Constant Gaussian Curvature

2.1 Introduction

A geometric property of a surface is called intrinsic if it depends only on the first fundamental form of the surface. Geometrically, intrinsic properties of a surface are invariant under a bending of the surface without stretching or shrinking. We have shown that the Gaussian curvature and the covariant derivatives are intrinsic. The result that covariant derivatives are intrinsic gives the important fact that any two surfaces with the same first fundamental form have the same parallelism. For example, since the cone and plane have the same first fundamental form, we can obtain the parallelism on the cone by unrolling it onto the plane, transporting parallelly in the Euclidean sense and then rolling the plane back to the cone. Therefore, I will focus the remainder of my studies toward geodesics on surfaces of constant Gaussian curvature.

2.2 Geodesics on Surfaces of Constant Gaussian Curvature

We begin by stating Minding’s theorem.

Theorem 2.1. Two surfaces of the same constant Gaussian curvature are locally isometric.
Proof. Let $S$ and $\bar{S}$ be two surfaces of the same constant Gaussian curvature $K$, and let $f$ be a mapping between two neighborhoods of two arbitrary points $P, \bar{P}$ on $S, \bar{S}$ such that each pair of corresponding points has the same geodesic polar coordinates $(r, \phi)$ with respect to two arbitrary tangent directions of $S, \bar{S}$ at $P, \bar{P}$, respectively. Then by equation 1.18, the first fundamental forms of $S, \bar{S}$ at $P, \bar{P}$ are, respectively,

$$ds^2 = dr^2 + Gd\phi^2, \quad ds^2 = d\bar{r}^2 + \bar{G}d\phi^2,$$

(2.1)

where

$$G(P, \phi) = \bar{G}(P, \phi) = 0, \quad G(P, \phi)_r = \bar{G}(P, \phi)_r = 1.$$

(2.2)

Furthermore, we have

$$K = -\frac{1}{\sqrt{G}}(\sqrt{G})_{rr}, \quad K = -\frac{1}{\sqrt{\bar{G}}}(\sqrt{\bar{G}})_{rr}.$$

(2.3)

Thus $\sqrt{G}$ and $\sqrt{\bar{G}}$ are solutions to the differential equation

$$z_{rr} = -Kz$$

(2.4)

with a constant coefficient $K$.

Since $K$ does not depend on $r$, equation 2.4 can be treated as an ordinary differential equation, and a solution of the equation is uniquely determined by the initial values of $z$ and $z_r$, which might also depend on $\phi$ as a parameter. But by equation 2.2 but $\sqrt{G}$ and $\sqrt{\bar{G}}$ satisfy the same initial conditions. Thus, $\sqrt{G} = \sqrt{\bar{G}}$ everywhere, so that $ds^2 = d\bar{s}^2$. Hence, $f$ is a local isometry of $S, \bar{S}$. \hfill \Box

Before we can state the following corollary we need to show that the Gaussian curvature for the plane is zero, sphere is positive everywhere and pseudosphere is negative everywhere. Recall that the Gaussian curvature is $K = \frac{LN - M^2}{EG - F^2}$ and consider a plane with parametrization

$$x(u, v) = (u, v, u - v).$$

Then

$$x_u = (1, 0, -1), \quad (2.5)$$

$$x_v = (0, 1, -1). \quad (2.6)$$
Since, $x_{uv} = x_{vu} = x_{vv} = 0$ the Gaussian curvature $K = 0$ for a plane.

Now, consider a sphere with radius $a$ with parametrization

$$x(u, v) = (a \sin u \cos v, a \sin u \sin v, a \cos u).$$

we have $x_u = (a \cos u \cos v, a \cos u \sin v, -a \sin u)$ and $x_v = (-a \sin u \sin v, a \sin u \cos v, 0)$.

Compute the coefficients for the first fundamental form and the unit normal vector

$$E = a^2 \cos^2 u \cos^2 v + a^2 \cos^2 u \sin^2 v + a^2 \sin^2 u$$
$$= a^2 \cos^2 u (\cos^2 v + \sin^2 v) + a^2 \sin^2 u$$
$$= a^2 (\cos^2 u + \sin^2 u)$$
$$= a^2,$$

$$F = -a^2 \cos u \sin u \cos v \sin v + a^2 \cos u \sin u \cos v \sin v$$
$$= 0,$$

$$G = a^2 \sin^2 u \sin^2 v + a^2 \sin^2 u \cos^2 v$$
$$= a^2 \sin^2 u (\sin^2 v + \cos^2 v)$$
$$= a^2 \sin^2 u,$$

$$U = \frac{a^2 \sin^2 u \cos v, a^2 \sin^2 u \sin v, a^2 \cos u \sin u}{\sqrt{a^4 \sin^4 u \cos^2 v + a^4 \sin^4 u \sin^2 v + a^4 \cos^2 u \sin^2 u}}$$
$$= (\sin u \cos v, \sin u \sin v, \cos u).$$
Now we need to compute the coefficients of the second fundamental form

\[
L = (-a \sin u \cos v, -a \sin u \sin v, -a \cos u) \cdot (\sin u \cos v, \sin u \sin v, \cos u)
\]

\[
= -a \sin^2 u \cos^2 v - a \sin^2 u \sin^2 v - a \cos^2 u
\]

\[
= -a \sin^2 u (\cos^2 v + \sin^2 v) - a \cos^2 u
\]

\[
= -a (\sin^2 u + \cos^2 u)
\]

\[
= -a,
\]

\[
M = (-a \cos u \sin v, a \cos u \cos v, 0) \cdot (\sin u \cos v, \sin u \sin v, \cos u)
\]

\[
= -a \cos u \sin u \cos v \sin v + a \cos u \sin u \cos v \sin v
\]

\[
= 0,
\]

\[
N = (-a \sin u \cos v, -a \sin u \sin v, 0) \cdot (\sin u \cos v, \sin u \sin v, \cos u)
\]

\[
= -a \sin^2 u \cos^2 v - a \sin^2 u \sin^2 v
\]

\[
= -a \sin^2 u (\cos^2 v + \sin^2 v)
\]

\[
= -a \sin^2 u.
\]

Finally, we have

\[
K = \frac{LN - M^2}{EG - F^2}
\]

\[
= \frac{a^2 \sin^2 u}{a^4 \sin^2 u}
\]

\[
= \frac{1}{a^2}.
\]

We now consider the pseudosphere, which is a surface of revolution obtained by revolving a tractrix about its asymptote (Figure 2.2). A tractrix is a plane curve with the following property: the segment of its tangent between the point of contact and some fixed straight line in the plane (the asymptote of the tractrix) is of constant length $a$. Thus, the pseudosphere with respect to the tractrix is given by the following parametrization

\[
x(u, v) = \left( a \sin u \cos v, a \sin u \sin v, a \left( \cos u + \ln \tan \frac{u}{2} \right) \right).
\]
Then

\[ \mathbf{x}_u = \left( a \cos u \cos v, a \cos u \sin v, \frac{\cos^2 u}{\sin u} \right), \]
\[ \mathbf{x}_v = (-a \sin u \sin v, a \sin u \cos v, 0), \]
\[ \mathbf{x}_{uu} = \left( -a \sin u \cos v, -a \sin u \sin v, -\frac{\cos u (2 \sin^2 u + \cos^2 u)}{\sin^2 u} \right), \]
\[ \mathbf{x}_{uv} = (-a \sin u \cos v, -a \cos u \sin v, 0), \]
\[ \mathbf{x}_{vv} = (-a \sin u \cos v, -a \sin u \sin v, 0), \]
\[ \mathbf{x}_v \times \mathbf{x}_u = \left( -a^2 \cos^2 u \cos v, -a^2 \cos^2 u \sin v, a^2 \cos u \sin u \cos^2 v + a^2 \cos u \sin^2 u \right) \]
\[ = \left( -a^2 \cos^2 u \cos v, -a^2 \cos^2 u \sin v, a^2 \cos u \sin u \right), \]
\[ |\mathbf{x}_u \times \mathbf{x}_v| = \sqrt{a^4 \cos^4 u \cos^2 v + a^4 \cos^4 u \sin^2 v + a^4 \cos^2 u \sin^2 u} \]
\[ = \sqrt{a^4 \cos^4 u + a^4 \cos^2 u \sin^2 u} \]
\[ = \sqrt{a^4 \cos^2 u \left( \cos^2 u + \sin^2 u \right)} \]
\[ = a^2 \cos u, \]
\[ U = (- \cos u \cos v, - \cos u \sin v, \sin u). \]
We now then can compute the coefficients for the first and second fundamental form

\[
E = a^2 \cos^2 u \cos^2 v + a^2 \cos^2 u \sin^2 v + \frac{a^2 \cos^4 u}{\sin^2 u}
\]

\[
= \frac{a^2 \cos^2 u \sin^2 u + a^2 \cos^4 u}{\sin^2 u}
\]

\[
= \frac{a^2 \cos^2 (\sin^2 u + \cos^2 u)}{\sin^2 u}
\]

\[
= \frac{a^2 \cos^2 u}{\sin^2 u},
\]

\[
F = -a \cos u \sin u \cos v \sin v + a \cos u \sin u \cos v \sin v
\]

\[
= 0,
\]

\[
G = a^2 \sin^2 u \sin^2 v + a^2 \sin^2 u \cos^2 v
\]

\[
= a^2 \sin^2 u,
\]

\[
L = a \cos u \sin u \cos^2 v + a \cos u \sin u \sin^2 v - \frac{-a \cos u (2 \sin^2 u + \cos^2 u)}{\sin u}
\]

\[
= \frac{a \cos u \sin^2 u - 2a \cos u \sin^2 u - a \cos^2 u}{\sin u}
\]

\[
= \frac{a \cos u \sin^2 u + a \cos^3 u}{\sin u}
\]

\[
= \frac{a \cos u (\sin^2 u + \cos^2 u)}{\sin u}
\]

\[
= \frac{a \cos u}{\sin u},
\]

\[
M = a \cos^2 u \cos v \sin v - a \cos^2 u \cos v \sin v
\]

\[
= 0,
\]

\[
N = a \cos u \sin u \cos^2 v + a \cos u \sin u \sin^2 v
\]

\[
= a \cos u \sin u.
\]

Finally, we have the Gaussian curvature for a pseudosphere

\[
K = \frac{LN - M^2}{EG - F^2}
\]

\[
= \frac{LN}{EG}
\]

\[
= -\frac{a^2 \cos^2 u}{a^4 \cos^2 u}
\]

\[
= -\frac{1}{a^2}.
\]

Since, we have shown the constant Gaussian curvature for the plane, sphere,
and pseudosphere to be $K = 0, K = a^{-2}, K = -a^{-2}$, respectively. Then it is now possible to state the following corollary.

**Corollary 2.2.** A surface of zero constant Gaussian curvature $K = 0$ is locally isometric to a plane. A surface of positive constant Gaussian curvature $K = a^{-2}$ is locally isometric to a sphere of radius $a$. A surface of negative constant Gaussian curvature $K = -a^{-2}$ is locally isometric to the pseudosphere.

The first fundamental coefficients vary with different types of surfaces of constant Gaussian curvature. To get a better idea of what these coefficients look like, consider equation (2.3)

If $K = 0$ then $(\sqrt{G})_{rr} = 0$. Thus, $(\sqrt{G})_r = g(\phi)$, where $g(\phi)$ is a function of $\phi$. Since

$$\lim_{r \to 0} (\sqrt{G})_r = 1,$$

we conclude that $(\sqrt{G})_r \equiv 1$. Therefore, $\sqrt{G} = r + f(\phi)$, where $f(\phi)$ is a function of $\phi$. Since

$$f(\phi) = \lim_{r \to 0} \sqrt{G} = 0,$$

we have in this case,

$$E = 1, \quad F = 0, \quad \text{and} \quad G(r, \phi) = r^2.$$

If $K > 0$, the general solution of equation (2.3) is given by

$$\sqrt{G} = A(\phi) \cos(\sqrt{K}r) + B(\phi) \sin(\sqrt{K}r),$$

where $A(\phi)$ and $B(\phi)$ are functions of $\phi$, which can be verified by differentiation.

Since $\lim_{r \to 0} \sqrt{G} = 0$, we obtain $A(\phi) = 0$. Thus,

$$(\sqrt{G})_r = B(\phi) \sqrt{K} \cos(\sqrt{K}r),$$

and since $\lim_{r \to 0} (\sqrt{G})_r = 1$, we can conclude that

$$B(\phi) = \frac{1}{\sqrt{K}}.$$

There, in this case,

$$E = 1, \quad F = 0, \quad \text{and} \quad G = \frac{1}{K} \sin^2(\sqrt{K}r).$$
If $K < 0$, the general solution of equation (2.3) is

$$\sqrt{G} = A(\phi) \cosh(\sqrt{-K}r) + B(\phi) \sinh(\sqrt{-K}r).$$

By using the initial condition, we verify that in this case

$$E = 1, \quad F = 0, \quad \text{and} \quad G = \frac{1}{-K} \sinh^2(\sqrt{-K}r).$$

Now, we will going to investigate the geodesics of these models of constant Gaussian curvature case by case. For case 1, $K = 0$, we will show that the geodesics in the plane are straight lines. We begin by parameterizing the plane so that $x(u, v) = (u, v, 0)$ and then calculating the partial derivatives $x_u = (1, 0, 0)$ and $x_v = (0, 1, 0)$. Then, the coefficients of the first fundamental form are: $E = 1, F = 0,$ and $G = 1$. The geodesics equations (1.19) and (1.20) yield,

$$u'' = v'' = 0.$$

Hence, $u(t) = at + b$ and $v(t) = ct + d$ where $a, b, c,$ and $d$ are constants. Then $x(t) = (at + b, ct + d, 0)$ is linear and only straight lines on the plane take this form and are geodesics on the plane.

Another example of a surface with constant Gaussian curvature that is isometric to the plane is the circular cylinder. Consider the cylinder $x^2 + y^2 = 1$ parametrized by $x(u, v) = (\cos u, \sin u, v)$. First we need to compute $x_u$ and $x_v$

$$x_u = (-\sin u, \cos u, 0) \quad \text{and} \quad x_v = (0, 0, 1),$$

and, then

$$E = \sin^2 u + \cos^2 u = 1$$

$$F = 0$$

$$G = 1.$$ 

By the geodesic equations (1.19) and (1.20),

$$u''(s) = a \quad \text{and} \quad v''(s) = b,$$

where $s$ is the arc length of $C$. Then we can easily see that $u'(s) = a$ and then $u(s) = as + b$ where $a$ and $b$ are constants. Likewise, $v(s) = cs + d$ where $c$ and $d$ are constant.
If $u(s)$ is constant, then the geodesics will be straight lines. If $v(s)$ is constant, then the geodesics will be circles. If neither equation is constant, then the geodesics of a cylinder is in the form

$$(\cos(as + b), \sin(cs + d), cs + d)$$

and, thus the geodesics is the helix.

Furthermore, we know that the plane and cylinder both have zero Gaussian curvature and mapping geodesics on a right circular cylinder to a plane by local isometry will yield straight lines (Figure 2.2).

Now, I will on to the next case when $K > 0$. Consider a surfaces of positive constant Gaussian curvature such at the unit sphere. A know fact is that the geodesics of a sphere are the great circle, curves that are contained in the intersection of a plane through the origin. Recall, the parametrization of a unit sphere to be

$$x(u, v) = (\cos u \sin v, \sin u \cos v, \sin v).$$

Then

$$x_u = (-\sin u \cos v, \cos u \cos v, 0),$$
$$x_v = (-\cos u \sin v, -\sin u \sin v, \cos v),$$

and the coefficients

$$E = \sin^2 u \cos^2 v + \cos^2 u \cos^2 v = \cos^2 v,$$
$$F = \sin u \cos u \cos v \sin v - \sin u \cos u \sin v \sin v + 0 = 0,$$
$$G = \cos^2 u \sin^2 v + \sin^2 u \sin^2 v + \cos^2 v = 1.$$
The geodesic equations (1.19) and (1.20) yields

\[ u'' - 2 \tan vu' v' = 0 \quad \text{and} \quad v'' + \sin v \cos v(u')^2, \]

and we also know that \( 1 = E(u')^2 + G(v')^2 \), which then implies that \( 1 = \cos^2 v(u')^2 + (v')^2 \).

We first solve the first geodesic equation as follows:

\[
\begin{align*}
  u'' &= 2 \tan vu' v', \\
  \frac{u'}{u'} &= 2 \tan vu', \\
  \int \frac{u''}{u'} &= \int 2 \tan vu', \\
  \ln u' &= -2 \ln \cos v + c, \\
  e^{\ln u'} &= e^{-2 \ln \cos v + c}, \\
  u' &= c \cos^{-2} v, \\
  u' &= \frac{c}{\cos^2 v}.
\end{align*}
\]

Substituting \( u' \) into \( 1 = \cos^2 v(u')^2 + (v')^2 \) yields,

\[
\begin{align*}
  1 &= \cos^2 \left( \frac{c}{\cos^2 v} \right) + (v')^2, \\
  (v')^2 &= 1 - \frac{c^2}{\cos^2 v}, \\
  (v')^2 &= \frac{\sqrt{\cos^2 v - c^2}}{\cos v}, \\
  v' &= \frac{\sqrt{\cos^2 v - c^2}}{\cos v},
\end{align*}
\]

and dividing \( u' \) by \( v' \) produces the separable differential equation

\[
\frac{du}{dv} = \frac{c}{\cos v \sqrt{\cos^2 v - c^2}}
\]

and, then

\[
du = \frac{c}{\cos v \sqrt{\cos^2 v - c^2}} dv.
\]

Integrating both sides

\[
\begin{align*}
u &= \int \frac{c}{\cos v \sqrt{\cos^2 v - c^2}} dv \\
&= \int \frac{c \sec^2 v}{\sqrt{1 - c^2 \sec^2 v}} dv \\
&= \int \frac{c \sec^2 v}{\sqrt{1 - c^2 - c^2 \tan^2 v}} dv.
\end{align*}
\]
Let $w = \frac{e}{\sqrt{1-c^2}}\tan v$, $dw = \frac{e}{\sqrt{1-c^2}} \sec^2 v dv$ and then substituting it into the above equation yields,

$$u = \int \frac{1-c^2}{\sqrt{1-c^2}\sqrt{1-c^2-c^2\tan^2 v}} dw$$

$$= \int \frac{1-c^2}{\sqrt{1-2c^2+c^4-c^2\tan^2 v+c^4\tan^2 v}} dw$$

$$= \int \frac{1}{(1-c^2)^2-c^2\tan^2 v(1-c^2)} dw$$

$$= \int \frac{1}{1-c^2} dw$$

$$= \int \frac{dw}{1-w^2}$$

and now let $w = \sin \theta$

$$u = \int d\theta$$

$$= \arcsin \left( \frac{c\tan v}{\sqrt{1-c^2}} \right) + d.$$ 

Now, we let $\lambda = \frac{c}{1-c^2}$ and then we have

$$u = \arcsin (\lambda \tan v) + d,$$

which can be written as,

$$\sin(u - d) = \lambda \tan v$$

and expanding the left side

$$\sin u \cos d - \sin d \cos u = \frac{\lambda \sin v}{\cos v}$$

and, then

$$\frac{\sin u \cos v \cos d - \sin d \cos u \cos v - \lambda \sin v}{\cos v} = 0$$

Finally, letting $x = \cos u \cos v$, $y = \sin u \cos v$, and $z = \sin v$ and only considering the numerator, we get

$$y \cos d - x \sin d - \lambda z = 0.$$
Hence, the geodesic equations imply that the geodesics lie on a plane \( ax + by + cz = 0 \) through the origin and therefore, the great circle. Moreover, surfaces with constant Gaussian curvature are locally isometric to \( S^2 \).

MAPLE was used to aid these calculations and, more importantly, to graph the geodesics on various types of surfaces with constant Gaussian curvature. Thus, we have included MAPLE procedures that we have used to graph the surfaces of constant Gaussian curvature (see APPENDIX A).

Figure 2.3 is a geodesic going through \((1,0,1)\) and \((0,1,1)\) created by using the MAPLE procedures that can be found in APPENDIX A.

We would like to introduce rotation surfaces of constant curvature, before introducing surfaces with negative constant Gaussian curvature. The best known types of surfaces of constant curvature \( K \neq 0 \) are the rotation surfaces of constant curvature. Such surfaces can be given by the parametric representation

\[
\mathbf{x}(r, \phi) = (r \cos \phi, r \sin \phi, f(r)),
\]

with corresponding first fundamental form

\[
ds^2 = (1 + (f')^2)dr^2 + r^2 d\phi^2.
\]

Letting \( du = \sqrt{1 + (f')}dr \) yields

\[
ds^2 = du^2 + Gd\phi^2, \quad G = G(u) = r^2
\]
from equation (2.8). The function $G$ has the special property that depends only on $u$, and this property is a characteristic of all surfaces isometric with rotation surfaces. In fact, if a surface of revolution having parametrization (2.7) then any meridian is a geodesic.

Now, we will introduce the case of negative constant Gaussian curvature, $K = -a^{-2}$. The solution for $K = -a^{-2}$ is given by

$$\sqrt{G} = c_1 e^{u/a} + c_2 e^{-u/b},$$

where $c_1$ and $c_2$ are constants. From equations (2.7), (2.8), (2.9), and (2.10) we can write

$$r = \sqrt{G}, \quad z = f(r) = \int \sqrt{1 - \left(\frac{dr}{du}\right)^2} \, du,$$

we find the profile of the rotation surfaces of constant Gaussian curvature, $K = -a^{-2}$ to be

$$r = c_1 e^{(u/a)} + c_2 e^{(-u/a)},$$

$$z = \int \sqrt{1 - a^{-2} \left(c_1 e^{(u/a)} - c_2 e^{(-u/a)}\right)^2} \, du,$$

where the choice of $c_1$ and $c_2$ may result in different types of surfaces.

Now, we will examine the case when $K < 0$ of the pseudosphere. We have shown earlier that $(K = -a^{-2})$ and we will consider the case when $c_1 = a$ and $c_2 = 0$. Then

$$ds^2 = du^2 + a^2 e^{(2u/a)} dv^2,$$

and

$$r = ae^{(u/a)}, \quad z = \int \sqrt{1 - e^{(2u/a)}} du.$$

Instead of evaluating the integral, we observe that

$$\frac{dr}{dz} = \frac{r}{\sqrt{a^2 - r}},$$

which shows that the curve $r = f(z)$ has the property that the segment of the tangent between the point of tangency and the point of intersection with the $z$-axis is constant and is equal to $a$. The curve described here is called the tractix, and the corresponding rotation surface is called the pseudosphere, with pseudoradius $a$. This is why every
surface with constant negative Gaussian curvature can be isometrically mapped on a pseudosphere of pseudoradius \( b \) (Figure 2.1).

Furthermore, we will examine the pseudosphere by using the geodesic equations (1.19) and (1.20). Hence,

\[
E = a^2 \frac{\cos^2 u}{\sin^2 u}, \quad F = 0, \quad G = a^2 \sin^2 u,
\]

and

\[
E_u = 2a^2 \frac{\cos u}{\sin u} \frac{1}{\sin^2 u} = -2a^2 \frac{\cos u}{\sin^3 u},
\]

\[
G_u = 2a^2 \sin u \cos u.
\]

Then the geodesic equations have the form

\[
u'' - \frac{1}{\sin u \cos u} (u')^2 - \frac{\sin^2 u (v')^2}{\cos u} = 0,
\]

\[
u'' + 2 \frac{\cos u}{\sin u} u' v' = 0,
\]

and letting \( v' = 0 \) yields

\[
u'' - \frac{1}{\sin u \cos u} (u')^2 = 0,
\]

which then implies that there is \( u = u(t) \) that makes the above equation true. Moreover, \( u(t) \) are the geodesics and are the meridians of the pseudosphere.

Another surface with negative constant Gaussian curvature is the Poincare Plane. However, the Poincare plane does not exists in \( \mathbb{R}^3 \). So, we will take the same approach as if it were in \( \mathbb{R}^3 \) and modify the dot product. Thus, I define \( P \) to be the upper half-plane \( P = \{(x, y) \in \mathbb{R}^2; y > 0\} \) with parametrization \( x(u, v) = (u, v) \) and with the metric,

\[
w_1 \circ w_2 = \frac{w_1 \cdot w_2}{v^2},
\]

where \( w_1, w_2 \in T_p(P) \) and \( p = (u, v) \). This definition of metric means that the usual dot product is scaled down by the height of \( p \), in our case \( v \), about the x-axis. Computing the coefficients of the first fundamental form

\[
x = (1, 0), \quad v = (0, 1),
\]
\[ E = x_u \cdot x_u \frac{1}{v^2}, \quad F = x_u \cdot x_v = 0, \quad \text{and} \quad G = x_v \cdot x_v = \frac{1}{v^2}. \]

Note that \( G_u = 0 \) and \( E_v = -\frac{2}{v^3} \) and recall that the Gaussian curvature is defined to be

\[ K = -\frac{1}{\sqrt{EG}} \left( \left( \frac{E_v}{\sqrt{EG}} \right)_v + \left( \frac{G_u}{\sqrt{EG}} \right)_u \right) \]

hence,

\[
K = -\frac{1}{1/v^4} \left( -2/v^3 \sqrt{1/v^4} \right)_v \\
= -\frac{v^2}{2} \left( -\frac{2}{v} \right)_v \\
= \frac{v^2}{2} \frac{2}{v^2} \\
= -1.
\]

Thus, \( P \) has constant curvature equal to \(-1\) at each point. \( P \) is then a negative curvature analogue of the unit sphere in \( R^3 \).

I will now examine the geodesics on the Poincare plane \( P \). Let the parametrization \( x(u,v) = (u,v) \) and with the metric,

\[
w_1 \circ w_2 = \frac{w_1 \cdot w_2}{v^2},
\]

where \( w_1, w_2 \in T_p(P) \) and \( p = (u,v) \) define the Poincare plane \( P \). The geodesic equation yields,

\[ u'' - \frac{2}{v} u' v' = 0 \quad \text{and} \quad v'' + \frac{1}{v} (u')^2 - \frac{1}{v} (v')^2 = 0. \]

Then

\[
\frac{u''}{u'} = \frac{2}{v} v' \\
\ln(u') = 2 \ln(v) + c \\
u' = cv^2.
\]
Given the fact that $1 = (u')^2(1/v^2) + (v')^2(1/u^2)$ yields $v' = v\sqrt{1 - c^2v^2}$. Then

$$\int du = \int \frac{cv}{1 - c^2v^2} dv$$

$$u - d = \frac{1}{c} \int \sin w dw, \quad \text{where } v = \frac{1}{c} \sin w$$

$$= -\frac{1}{c} \cos w$$

$$= -\frac{1}{c} \sqrt{1 - c^2v^2}$$

$$c^2(u - d)^2 = 1 - c^2v^2$$

$$(u - d)^2 + v^2 = \frac{1}{c^2}.$$ 

This is the equation of a circle centered on the $u$-axis. Since $G_u = 0$, vertical lines are also geodesics. Therefore, the Poincare plane $P$ has as its geodesics arcs of circles centered on the $u$-axis and vertical lines.

Therefore, geodesics on a surface of constant Gaussian curvature are obtained by finding the coefficients of the first fundamental form and substituting them into the geodesics equations. Solutions of the geodesic equation must generally be obtained numerically and then, since the solutions are in terms of the parameters $u$ and $v$, plugged into the surface parametrization. This task is now simplified with the aid of MAPLE and it allows us to have a better visualization of geodesics on surfaces. Moreover, the idea of geodesics in $R^3$ holds true in higher dimension such as $R^4$ by fixing the dot product in $R^3$. In addition, when we calculate the Christoffel symbols for $x_{uu}$, $x_{uv}$ and $x_{vv}$ we never used $U$. Hence, the same formulas holds for surfaces not in $R^3$. In particular, the calculation of $\alpha''$ remain the same, but without the final term involving $U$. 
Chapter 3

Conclusion

The idea that the shortest distance between two points is a line only holds true in $R^2$. What happens when the surfaces is in $R^3$? This was the questions that I had before I started my studies on geodesics on surfaces of constant Gaussian curvature. Now the answer is clearer. It was important for me to understand that the Gaussian curvature was intrinsic, meaning depending only on the first fundamental form. From here I was able to conclude that two surfaces of the same constant Gaussian curvature are locally isometric. Thus, the Gaussian curvature became a surface invariant. Then I started to study certain characteristics on surfaces of constant Gaussian curvature. For example, the case when the Gaussian curvature was everywhere zero, the case when the Gaussian curvature was everywhere positive and the case when it was every negative. These conditions played a critical role on the local isometry of the Gaussian curvature. In addition, I investigated geodesics of higher dimensions such as the Poincare plane. The procedure was almost the same as if it was in $R^3$ but the dot product was defined differently. From here, I see that it is possible to study geodesics on surfaces of higher dimension. It was also imperative that I also introduce MAPLE into my studies. It helped me visualize and calculate geodesics equations. In conclusion, geodesics on surfaces of constant Gaussian curvatures may have many forms, but they are all locally isometric to the plane, sphere or pseudosphere. Therefore, understanding the properties of the plane, sphere and pseudosphere was the ideal way for me to understand the concept of geodesics on surfaces of constant Gaussian curvature.
Appendix A

MAPLE procedures used to calculate the Gaussian and Mean Curvature. For the geodesic equations we need only the coefficients of the first fundamental form.

We then define the dot product and cross product
> dp := proc(X,Y)
> end:
>
> nrm := proc(X)
> sqrt(dp(X,X));
> end:
>
> xp := proc(X,Y)
> local a,b,c;
> a := X[2]*Y[3]-X[3]*Y[2];
> b := X[3]*Y[1]-X[1]*Y[3];
> c := X[1]*Y[2]-X[2]*Y[1];
> [a,b,c];
> end:
>
We then need the Jacobian matrix whose columns comprise the tangent vectors to the parameter curves
> Jacf := proc(X)
> local Xu, Xv;
> Xu := [diff(X[1],u), diff(X[2],u), diff(X[3],u)];
> Xv := [diff(X[1],v), diff(X[2],v), diff(X[3],v)];
> simplify([Xu,Xv]);
> end:
>
We then need the unit normal vector and coefficients for the first and second fundamental form
> EFG := proc(X)
local E,F,G,Y;
> Y := Jacf(X);
> E := dp(Y[1],Y[1]);
> F := dp(Y[1],Y[2]);
> G := dp(Y[2],Y[2]);
> simplify([E,F,G]);
> end:
>
> UN := proc(X)
> local Y,Z,s;
> Y := Jacf(X);
> Z := xp(Y[1],Y[2]);
> s := nrm(Z);
> simplify([Z[1]/s, Z[2]/s, Z[3]/s]);
> end:
>
> LMN := proc(X)
> local Xu, Xv, Xuu,Xuv, Xvv, U, L, M, N;
> Xu := [diff(X[1],u), diff(X[2],u), diff(X[3],u)];
> Xv := [diff(X[1],v), diff(X[2],v), diff(X[3],v)];
> Xuu := [diff(Xu[1],u), diff(Xu[2],u), diff(Xu[3],u)];
> Xuv := [diff(Xu[1],v), diff(Xu[2],v), diff(Xu[3],v)];
> Xvv := [diff(Xv[1],v), diff(Xv[2],v), diff(Xv[3],v)];
> U := UN(X);
> L := dp(U,Xuu);
> M := dp(U,Xuv);
> N := dp(U,Xvv);
> simplify([L,M,N]);
> end:
>
Putting everything together yields the procedurea
to calculate the Gaussian and mean curvature
> GK := proc(X)
> local E,F,G,L,M,N,S,T;
> S := EFG(X);
> T := LMN(X);
> E := S[1];
> F := S[2];
> G := S[3];
> L := T[1];
> M := T[2];
> N := T[3];
> simplify((L*N-M^2)/(E*G-F^2));
\begin{center}
\textsc{Maple} procedures used to calculate the geodesics on a surface
\end{center}

For Plotting Surfaces we should always begin with
\begin{verbatim}
> with(plots):
\end{verbatim}

We then define the dot product and cross product
\begin{verbatim}
> dp := proc(X,Y)
> end:
>
> nrm := proc(X)
> sqrt(dp(X,X));
> end:
>
> xp := proc(X,Y)
> local a,b,c;
> a := X[2]*Y[3]-X[3]*Y[2];
> b := X[3]*Y[1]-X[1]*Y[3];
> c := X[1]*Y[2]-X[2]*Y[1];
> [a,b,c];
> end:
\end{verbatim}

We then need the Jacobian matrix whose columns
comprise the tangent vectors to the parameter curves
\begin{verbatim}
> Jacf := proc(X)
> local Xu, Xv;
\end{verbatim}
\[X_u := \{\text{diff}(X[1],u), \text{diff}(X[2],u), \text{diff}(X[3],u)\}\;\]
\[X_v := \{\text{diff}(X[1],v), \text{diff}(X[2],v), \text{diff}(X[3],v)\}\;\]
\[\text{simplify}([X_u,X_v]);\]

We then need the unit normal vector and coefficients for the first and second fundamental form

\[EFG := \text{proc}(X)\]
\[\text{local } E,F,G,Y;\]
\[Y := \text{Jacf}(X);\]
\[E := \text{dp}(Y[1],Y[1]);\]
\[F := \text{dp}(Y[1],Y[2]);\]
\[G := \text{dp}(Y[2],Y[2])\;\]
\[\text{simplify}([E,F,G]);\]

The geodesic equations

\[\text{geoeq}:=\text{proc}(X)\]
\[\text{local } M,\text{eq1,eq2; }\]
\[M := \text{EFG}(X);\]
\[\text{eq1} := \text{diff}(u(t),t2)+\text{subs}\{u=u(t),v=v(t)\}, \text{diff}(M[1],u)/(2*M[1]))*\text{diff}(u(t),t)^2\]
\[+\text{subs}\{u=u(t),v=v(t)\}, \text{diff}(M[1],v)/(M[1]))*\text{diff}(u(t),t)*\text{diff}(v(t),t)\]
\[-\text{subs}\{u=u(t),v=v(t)\}, \text{diff}(M[3],u)/(2*M[1]))*\text{diff}(v(t),t)^2=0;\]
\[\text{eq2} := \text{diff}(v(t),t2)-\text{subs}\{u=u(t),v=v(t)\}, \text{diff}(M[1],v)/(2*M[3]))*\text{diff}(u(t),t)^2\]
\[+\text{subs}\{u=u(t),v=v(t)\}, \text{diff}(M[3],u)/(M[3]))*\text{diff}(u(t),t)*\text{diff}(v(t),t)\]
\[+\text{subs}\{u=u(t),v=v(t)\}, \text{diff}(M[3],v)/(2*M[3]))*\text{diff}(v(t),t)^2=0;\]
\[\text{eq1,eq2; }\]
\[\text{end;}\]

Graphing the geodesics on a surface

\[\text{plotgeo} := \text{proc}(X,u\text{start},u\text{end},v\text{start},v\text{end}, u0,v0,Du0,Dv0,T,N,\text{gr},\text{theta},\text{phi})\]
\[\text{local sys,desys,u1,v1,listp,geo,plotX; }\]
\[\text{sys} := \text{geoeq}(X);\]
\[\text{desys} := \text{dsolve}\{\text{sys},u(0)=u0,v(0)=v0, D(u)(0)=Du0,D(v)(0)=Dv0\},\{u(t),v(t)\},\]

\[>\]
> type = numeric, output = listprocedure;
> ul := subs(desys, u(t)); v1 := subs(desys, v(t));
> geo := spacecurve
  (subs(u = 'u1'(t), v = 'v1'(t), X), t = 0 .. T,
   color = black, thickness = 2, numpoints = N):
> plotX := plot3d(X, u = ustart .. uend, v = vstart .. vend,
   grid = [gr1, gr2], shading = XY):
> display({geo, plotX}, style = wireframe,
   scaling = constrained, orientation
   = [theta, phi], shading = xy, lightmodel = light2);
> end:
Bibliography


