Math, music, and membranes: A historical survey of the question "can one hear the shape of a drum"?

Tricia Dawn McCorkle

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MATH, MUSIC, AND MEMBRANES: A HISTORICAL SURVEY

OF THE QUESTION "CAN ONE HEAR THE SHAPE OF A DRUM"?

A Project
Presented to the
Faculty of
California State University,
San Bernardino

by
Tricia Dawn McCorkle

December 2005
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ABSTRACT

In 1966 Mark Kac posed an interesting question regarding vibrating membranes and the sounds they make. His article entitled "Can One Hear the Shape of a Drum?", which appeared in The American Mathematical Monthly, generated much interest and scholarly debate. The question of the article's title can be interpreted as follows: If two drums are different shapes, will they necessarily produce different sounds? Intuition says yes, since we know strings of different length or different material, when plucked, produce different pitches. In his article Kac expresses a hunch, however, that one could not hear the shape of a drum. Yet he points to the work of Herman Weyl, who shows that one can hear the area of a drum (that is, drums that sound the same have the same area). This result is called Weyl's theorem. Furthermore, decades after Kac's paper was written, Carolyn Gordon, David L. Webb, and Scott Wolpert proved that drums of different shapes can actually produce the same sound, by giving the first example of two drums of different shapes that sound the same. They did this by constructing a map between the eigenfunctions of the two domains. Their proof was later shortened considerably by Piére Bérard, who established a
simpler map from the eigenfunctions of the first drum to the eigenfunctions of the second drum. Bérard's method was simplified even further by S.J. Chapman, who in his article entitled "Drums that Sound the Same", reduced the method for constructing two isospectral, non-congruent drums to a process of folding and layering copies of the first drum to create the second.

The evolution of Kac's intriguing question will be the subject of this project. In Chapter One, I first solve the heat equation and the wave equation. In Chapter Two I solve the wave equation in two dimensions (and in particular, I relate it to a rectangular membrane), and then prove some prerequisite lemmas necessary for Weyl's theorem. The proof of Weyl's theorem follows, first for a special case, and finally for the general case. Chapter Three describes Chapman's explanation of his technique for constructing isospectral, non-congruent drums. Also in Chapter Three, I construct new examples based on the techniques given in Chapman's paper. Therefore, each chapter of this paper can be encapsulated in one word: Chapter One, Computation; Chapter Two, Proof; and Chapter Three, Examples.
ACKNOWLEDGEMENTS

Thank you to Belisario Ventura, for his guidance on this project. He has not only an encyclopedic knowledge of mathematics, but also possesses the ability to make it comprehensible! I have benefited from his academic expertise as well as his understanding and patience throughout this endeavor.

Thank you to my committee members, Dr. Charles Stanton and Gary Griffing, for the time they donated to the project and for their valuable input.
DEDICATION

With love and appreciation, this project is dedicated to my husband, Brock - my greatest supporter, not only in academic pursuits such as this, but in every arena of life.

Also to Dad, Mom, Rhonda, David, Nathan, WilliAnne, Brent, Kim, Nana, and Dad and Mom McCorkle - their encouragement throughout my graduate studies has been invaluable!
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CHAPTER ONE

THE HEAT AND WAVE EQUATIONS

Introduction: Kac’s Question

Famous mathematicians throughout the centuries, such as Pythagoras, Archytas and Kepler, had strong interest in studying music alongside mathematics. Followers of Pythagoras believed that music and math unlocked the secrets of the world. They discovered that simple ratios of frequencies, such as 2:1 and 3:2 for example, corresponded to certain combinations of pitches that they considered pleasing. Since then the concepts of acoustics, tuning, and pitch have occupied some of the brightest minds of science and mathematics.

The infinite connections between my two favored fields of study, math and music, are inherent and vast. Perhaps the most intriguing to me is the way musical instruments produce different pitches. Within any musical instrument (or voice) exists air, string, or membrane vibration. This vibration is governed by a partial differential equation called the wave equation. A change in length, tension, area, or frequency of the vibrating object causes differences in pitch.
In 1966 Mark Kac posed an interesting question regarding vibrating membranes and the sounds they make. (Throughout this paper the words drum, membrane, domain and region will be used interchangeably to refer to a subset of the Cartesian plane.) His article entitled "Can One Hear the Shape of a Drum?", which appeared in The American Mathematical Monthly, generated much interest and scholarly debate. The question of the article's title can be interpreted as follows: If two drums have different shapes, will they necessarily produce different sounds? Intuition says yes, since we know strings of different length or different material, when plucked, produce different pitches. In his article, Kac expresses a hunch, however, that one could not hear the shape of a drum. Yet he points to the work of Herman Weyl, who shows that one can hear the area of a drum (that is, drums that sound the same have the same area). This result is called Weyl's theorem.

Before I prove Weyl's theorem in Chapter Two, I will take the reader on a journey through the prerequisite material that I first explored in order to understand the theorem and its proof. Chapter One, the solution to the heat and wave equations, is to be seen only partly as a
means to an end. Some of the results in these computations will be necessary components of the proofs in Chapter Two, while others will not be used again. Those parts are included here simply to show the road I have taken during my studies of this subject.

The Heat Equation

The solution of the heat equation has applications in various branches of science such as diffusive processes, which behave much like acoustic wave problems. Hence this linear, second order, basic partial differential equation is a natural place to begin.

Consider a heat conduction problem for a straight bar of uniform cross-section and homogeneous material. Let the x-axis be chosen to lie along the axis of the bar, and let

\[ x=0 \]  \[ x=L \]

Figure 1. Heat Conduction Bar
x=0 and x=L denote the ends of the bar. Also assume that no heat passes through the sides of the bar due to perfect insulation, and that the cross-sectional dimensions are so small that the temperature $u$ can be considered constant on any given cross section. So $u$ is a function only of the coordinate $x$ along the axis and time $t$. The variation of temperature in the bar is ruled by a partial differential equation called the heat conduction equation, or heat equation for short, and has the following form:

$$\alpha^2 u_{xx} = u_t, \quad 0 < x < L, \quad t > 0 \quad (1.1)$$

In equation 1.1 above, $\alpha^2$ is a constant called the thermal diffusivity that depends on the properties of the bar's material. As for notation, $u_{xx}$ is the second partial derivative of $u$ with respect to $x$, the position along the bar (some readers may be accustomed to the notation $\frac{\partial^2 u}{\partial x^2}$ rather than $u_{xx}$) and $u_t$ is the first partial derivative of $u$ with respect to time $t$ (some readers may be used to the notation $\frac{\partial u}{\partial t}$ rather than $u_t$). Our present goal is to find such a $u(x,t)$ that satisfies the heat equation 1.1, given the initial condition.
\[ u(x,0) = f(x) \quad (1.2) \]

And the following boundary conditions:

\[ u(0,t) = 0 \quad (1.3) \]
\[ u(L,t) = 0 \quad (1.4) \]

Equation 1.2 above describes the initial temperature of the bar as a function of the position \( x \). Equations 1.3 and 1.4 above are the conditions requiring that at all times the temperature of the bar at both ends is zero.

To begin our solution we shall assume that \( u(x,t) \) can be separated into a function of \( x \) and a function of \( t \).

(This method of separation of variables is the oldest systematic method of solving partial differential equations, dating back to wave and vibration investigations by D’Alembert, Daniel Bernoulli, and Euler in about 1750). Separating the variables gives us

\[ u(x,t) = X(x)T(t). \quad (1.5) \]

We can substitute this \( u \) into the heat equation to obtain

\[ \alpha^2 \left( X(x)T(t) \right)_x = \left( X(x)T(t) \right)_t, \quad (1.6) \]

which can be differentiated appropriately to yield

\[ \alpha^2 X ''(x)T(t) = X (x)T '(t). \quad (1.7) \]
From equation 1.7 we can simply rearrange components of the terms using division to obtain

\[
\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2} \cdot \frac{T'(t)}{T(t)}.
\]  

(1.8)

Now, each of the sides of the equation 1.8 depend on one variable only (left side, x, and right side, t). The only way the two sides can be equal for all values of x and t, is that they both equal the same constant. Let's call that constant $-\lambda$. The constant is chosen to be negative due to physical considerations.

Note: From now on we shall drop the parenthesis notation and refer to $X''(x)$ as $X''$, $T'(t)$ as $T'$ etc. It will be up to the reader to remember that these are functions of position and time.

Therefore we have

\[
\frac{X''}{X} = -\lambda, \quad \text{that is } X'' + \lambda X = 0
\]  

(1.9)

and

\[
\frac{1}{\alpha^2} \cdot \frac{T'}{T} = -\lambda, \quad \text{that is } T' + \lambda \alpha^2 T = 0.
\]  

(1.10)
Our first step is to solve equation 1.9, \( X'' + \lambda X = 0 \). We know that the solution space is two dimensional. Let us look first for exponential solutions, that is, \( X = e^{rx} \) where \( r \) is a parameter to be determined. Then \( X' = re^{rx} \) and \( X'' = r^2 e^{rx} \), and we can substitute into our rewritten form of equation 1.9 to yield the following calculations:

\[
\begin{align*}
 r^2 e^{rx} + \lambda e^{rx} &= 0 \\
 (r^2 + \lambda) e^{rx} &= 0
\end{align*}
\]

Hence, since \( e^{rx} \neq 0 \) for all \( x \),

\[
(r^2 + \lambda) = 0.
\]

This last equation, \( (r^2 + \lambda) = 0 \), is called the characteristic equation for \( X'' + \lambda X = 0 \). Therefore we have \( r^2 = -\lambda \), and \( r = \pm \sqrt{\lambda} \), where these \( r \) values are the roots of the characteristic equation. Since we have two roots to the characteristic equation, we get two solutions, \( e^{\sqrt{\lambda} x} \) and \( e^{-\sqrt{\lambda} x} \), which, by the way, are linearly independent, and therefore form a basis for the solution space of \( X'' + \lambda X = 0 \). Thus the general solution of \( X'' + \lambda X = 0 \) (equation 1.9) is:
\[ X = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}. \] (1.11)

Now any linear combination like 1.11 will also be a solution, and since \( \sin x = \frac{e^{ix} - e^{-ix}}{2i} \) and \( \cos x = \frac{e^{ix} + e^{-ix}}{2} \), and since \( \sin x \) and \( \cos x \) are linearly independent, we have that \( \sin x \) and \( \cos x \) also form a basis for the solution space of equation 1.9. Thus, we can rewrite 1.11 as

\[ X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x. \] (1.12)

Note that the constants \( A \) and \( B \) from 1.11 are, in general, different than the constants from 1.12. The purpose of using 1.12 instead of 1.11 is to rid ourselves of imaginary numbers in our solution.

Now we must solve for \( A \) and \( B \) by using the boundary conditions discussed earlier. Boundary condition 1.3 requires that \( u(0,t) = 0 \). But since \( u(0,t) = X(0)T(t) \), we have that either \( X(0) = 0 \) or \( T(t) = 0 \). If the latter is true, then the temperature is zero at all times, since \( u(x,t) = X(x)T(t) \) becomes \( u(x,t) = X(x)(0) = 0 \). If this were the case, then we would have the trivial solution, which might not satisfy the initial condition 1.2. This implies that \( X(0) = 0 \).
Hence, \( A \cos \sqrt{\lambda}(0) + B \sin \sqrt{\lambda}(0) = 0 \), and since \( \sin 0 = 0 \), we have that \( A \cos 0 = 0 \). Furthermore, \( \cos 0 = 1 \), so \( A = 0 \), and the first term of equation 1.12 is eliminated. Boundary condition 1.4 implies that \( X(L) = 0 \). Hence, \( B \sin \sqrt{\lambda}L = 0 \), which implies that the argument of sine must be a multiple of \( \pi \). Therefore, \( \sqrt{\lambda}L = n\pi \), and \( \lambda = \left( \frac{n\pi}{L} \right)^2 \) where \( n \) is any integer. These values for \( \lambda \) are known as eigenvalues.

Now we can substitute \( \lambda = \left( \frac{n\pi}{L} \right)^2 \) into 1.12 and our final solution for 1.9 is (1.13), where each eigenvalue has a corresponding solution, called the eigenfunction.

\[
X = A \sin \frac{n\pi x}{L}.
\] (1.13)

We have now solved "half" of equation 1.5, namely the position function, \( X \). Now we turn our attention to the time function, \( T \). To solve 1.10 we simply repeat the process used to solve for \( X \). The equation 1.10 can be rewritten as \( T' + \alpha^2 \lambda T = 0 \), which clearly has a characteristic equation of \( r + \alpha^2 \lambda = 0 \), and a one-dimensional solution space.
Now since \( r = -\alpha^2 \lambda \) we have \( T = Ae^{-\alpha^2 \lambda t} \). Substituting \( \lambda = \left( \frac{n\pi}{L} \right)^2 \)

into this equation, we have the solution to 1.10, which is

\[
T = Ae^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}} \tag{1.14}
\]

Now remember that we were solving for \( X \) and \( T \) separately for the purpose of putting them together to as

\[ u(x,t) = X(x)T(t). \]

So we shall now substitute 1.13 and 1.14 into 1.5 to obtain

\[
u_n(x,t) = A \sin \frac{n\pi x}{L} e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}. \tag{1.15}
\]

This is one solution to the partial differential equation \( \alpha^2 u_{xx} = u_t \), namely the one corresponding to \( n \), denoted by \( u_n \). However, any linear combination of these solutions is also a solution. Hence we have

\[
u(x,t) = \sum_{n=0}^{\infty} c_n \sin \frac{n\pi x}{L} e^{-\frac{\alpha^2 n^2 \pi^2 t}{L^2}}. \tag{1.16}
\]

We now must find the collection of constant coefficients, the \( c_n \)'s (heretofore referred to as simply \( c_n \)). We will start by using the boundary condition 1.2, which states
that \( u(x,0) = f(x) \). When \( t=0 \), the exponential component of 1.16 is one. Therefore,

\[
 u(x,0) = f(x) = \sum_{n=0}^{\infty} c_n \sin \frac{n\pi x}{L}.
\]

(1.17)

Our next step is to take the inner product of both sides of the equation with \( \sin \left( \frac{m\pi x}{L} \right) \). The purpose is to uncover information about \( c_n \), which is clearly a component of \( f(x) \) itself. The justification for taking the inner product is neither immediate nor intuitive. Therefore, before actually doing so, let us take a short detour to uncover the motivation for this strategy.

Let us momentarily detour to the world of vectors in \( \mathbb{R}^n \), since it serves as an analogous situation to our problem. Consider the vector \( \vec{v} = (c_1, c_2, \ldots, c_n) \) and the standard basis \( \{ \hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n \} \) where

\[
\hat{e}_1 = (1,0,0\ldots0), \hat{e}_2 = (0,1,0\ldots0), \ldots, \hat{e}_n = (0,0\ldots1).
\]

That is, each basis vector \( e_i \) has a one in the \( i^{th} \) position and zeros elsewhere. Note that \( \vec{v} = c_1 \hat{e}_1 + c_2 \hat{e}_2 + \ldots + c_n \hat{e}_n \). What would happen if we took the usual inner product of \( \vec{v} \) and any basis vector \( \hat{e}_i \)? We
would get the $i^{th}$ component of $\vec{v}$ itself, as these computations show:

\[
\langle \vec{v}, \hat{e}_i \rangle = \langle c_1 \hat{e}_1 + c_2 \hat{e}_2 + \ldots + c_n \hat{e}_n, \hat{e}_i \rangle \\
= c_1 \langle \hat{e}_1, \hat{e}_i \rangle + c_2 \langle \hat{e}_2, \hat{e}_i \rangle + \ldots + c_i \langle \hat{e}_i, \hat{e}_i \rangle + \ldots + c_n \langle \hat{e}_n, \hat{e}_i \rangle \\
= 0 + 0 + \ldots + c_i(1) + \ldots + 0 \quad \text{since} \quad \langle \hat{e}_i, \hat{e}_j \rangle = \begin{cases} 0 & \text{when } i \neq j \\ 1 & \text{when } i = j \end{cases} \\
= c_i
\]

So we see that we were able to "pick out" the $i^{th}$ component of a vector $\vec{v}$, called $c_i$, by taking the inner product of the vector $\vec{v}$ with the $i^{th}$ basis vector, $\hat{e}_i$. In summary, $c_i = \langle \vec{v}, \hat{e}_i \rangle$

(this will be an important part of our analogy later when we need to pick out the $c_n$ component of $f(x)$). Also note that if we multiply that component $c_i$ by its corresponding basis vector, $\hat{e}_i$, we would get a vector with all zeros except for the $c_i$ in the $i^{th}$ spot. Do this $n$ times and add them all up, and you will have the vector $\vec{v}$ itself, as shown here: $\vec{v} = \sum_{i=1}^{n} \langle \vec{v}, \hat{e}_i \rangle \cdot \hat{e}_i$.

Continuing the "detour" of vector spaces, let us consider the case of functions defined on the interval $[-L,L]$ and develop an analogous construction to what we just
did with vectors. We consider a vector \( \tilde{x} = (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n \) as a function from \( \{1, 2, \ldots, n\} \) to \( \mathbb{R} \) such that each entry corresponds to its index. For example, \( f(1) = x_1 \). Thus to each value \( n \), we think of \( f(n) \) as the entry of the vector \( x \) corresponding to \( n \). Now instead of \( \mathbb{R}^n \), our vector space is \( L^2[-L, L] = \left\{ f: \int_{-L}^{L} |f(x)|^2 \, dx < \infty \right\} \), that is, the set of all functions with the property that the integral of their square is finite on the interval from \( -L \) to \( L \). Note: the main difference between our new vector space and \( \mathbb{R}^n \) is that it is infinite dimensional. Furthermore, instead of the usual inner product, let us use the following inner product:

\[
\langle f, g \rangle = \frac{1}{L} \int_{-L}^{L} f(x)g(x) \, dx.
\]

With this vector space and inner product, the set

\[
\left\{ \sin \frac{n\pi x}{L} : n = 1, 2, 3, \ldots \right\} \cup \left\{ \cos \frac{n\pi x}{L} : n = 0, 1, 2, \ldots \right\}
\]

(1.18)

is an orthogonal set. That is, the inner product of two different functions in the set is zero, similar to the usual inner product of any two basis vectors, \( \hat{e}_i \) and \( \hat{e}_j \) in
our analogous example above. The orthogonality of the set follows from the computation of the integrals below.

\[
\begin{align*}
& \frac{1}{L} \int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = 0 \text{ if } n \neq m \\
& \frac{1}{L} \int_{-L}^{L} \cos \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = 0 \text{ for all } n, m \\
& \frac{1}{L} \int_{-L}^{L} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = 0 \text{ if } n \neq m
\end{align*}
\]

(1.19)

Each of the integrals in 1.19 can be obtained by an application of integration by parts. The equations in 1.19 are analogous to \( \langle \hat{e}_i, \hat{e}_i \rangle = 0 \) when \( i = j \) in our previous example (see page 12).

Now, just as in the case of \( \mathbb{R}^n \), where \( \tilde{v} = \sum_{i=1}^{n} (\tilde{v}, \hat{e}_i) \hat{e}_i \), we have that a given function satisfies that:

\[
f(x) = \sum_{n=1}^{\infty} \left( f(x), \sin \frac{n\pi x}{L} \right) \sin \frac{n\pi x}{L} dx + \sum_{n=0}^{\infty} \left( f(x), \cos \frac{n\pi x}{L} \right) \cos \frac{n\pi x}{L} dx.
\]

Now recall the initial condition 1.17 of the heat equation, which was \( u(x,0) = f(x) = \sum_{n=0}^{\infty} c_n \sin \frac{n\pi x}{L} \) with \( f(x) \) defined on \([0,L] \). Since the right hand side of the equation 1.17 is an odd function on \([-L,L]\), and \( f(x) \) is defined only on \([0,L]\),
we can take the odd extension of \( f(x) \) on \([-L,L]\) to compute the inner products. Then

\[
\left\langle f(x), \cos \frac{n \pi x}{L} \right\rangle = \frac{1}{L} \int_{-L}^{L} \frac{f(x) \cos \frac{n \pi x}{L}}{\text{odd}} \, dx = 0 \quad \text{and} \quad \left\langle f(x), \sin \frac{n \pi x}{L} \right\rangle = \frac{1}{L} \int_{-L}^{L} \frac{f(x) \sin \frac{n \pi x}{L}}{\text{even}} \, dx = 0.
\]

\[c_n = \left\langle f(x), \sin \frac{n \pi x}{L} \right\rangle = \frac{1}{L} \int_{-L}^{L} \frac{f(x) \sin \frac{n \pi x}{L}}{\text{odd}} \, dx = \frac{2}{L} \int_{0}^{L} \frac{f(x) \sin \frac{n \pi x}{L}}{\text{odd}} \, dx = 0.
\]

Equation 1.20 is in analogy with \( c_i = \langle v, \hat{e}_i \rangle \) of page 12.

Let us show another way of calculating \( c_n \). Start with equation 1.17 and take the inner product of both sides, that is multiply both sides by \( \sin \frac{m \pi x}{L} \) and take the integral from zero to \( L \).

\[
f(x) = \sum_{n=0}^{\infty} c_n \sin \frac{n \pi x}{L},
\]

\[
\int_{0}^{L} f(x) \sin \frac{m \pi x}{L} \, dx = \sum_{n=0}^{\infty} c_n \sin \frac{n \pi x}{L} \int_{0}^{L} \sin \frac{m \pi x}{L} \, dx
\]

\[
= \sum_{n=0}^{\infty} c_n \int_{0}^{L} \sin \frac{n \pi x}{L} \sin \frac{m \pi x}{L} \, dx
\]

\[
= \sum_{n=0}^{\infty} c_n \left[ \frac{1}{2} \cos \frac{(n-m) \pi x}{L} - \frac{1}{2} \cos \frac{(n+m) \pi x}{L} \right] \, dx
\]
Computing the integral yields

\[ \sum_{n=0}^{\infty} c_n \left[ \frac{L}{(n-m)\pi} \frac{1}{2} \sin \frac{(n-m)\pi x}{L} - \frac{1}{(n+m)\pi} \frac{1}{2} \sin \frac{(n+m)\pi x}{L} \right]. \]

Notice that when \( m \neq n \), we are taking the sine of a multiple of \( \pi \), and we end up with zero. Therefore, the only term of this summation that survives is the term for which \( m = n \). So we return back to the integral \( \sum_{n=0}^{\infty} c_n \int_0^L \left[ \frac{1}{2} \cos \frac{(n-m)\pi x}{L} - \frac{1}{2} \cos \frac{(n+m)\pi x}{L} \right] dx \) to eliminate the summation and insert \( m \) for \( n \), resulting in the following calculation:

\[
\int_0^L f(x) \sin \frac{m\pi x}{L} \, dx = c_m \int_0^L \left[ \frac{1}{2} \cos \frac{(m-m)\pi x}{L} - \frac{1}{2} \cos \frac{(m+m)\pi x}{L} \right] dx \\
= c_m \left[ \int_0^L \frac{1}{2} \cos \frac{(m-m)\pi x}{L} \, dx - \int_0^L \frac{1}{2} \cos \frac{(m+m)\pi x}{L} \, dx \right] \\
= c_m \left[ \frac{1}{2} \int_0^L dx - \frac{1}{2} \cos \frac{2m\pi x}{L} \, dx \right] \\
\]

Now by evaluating the integral, we obtain the value of \( c_m \).

\[
c_m \left[ \frac{1}{2} x \bigg|_0^L + \left( \frac{1}{2} \frac{L}{2m\pi} \sin \frac{2m\pi x}{L} \right) \bigg|_0^L \right] \\
= c_m \left( \frac{1}{2} L - 0 \right) \\
= c_m \frac{L}{2} \\
\]
So we have $\int_0^L f(x) \sin \frac{m\pi x}{L} \, dx = c_m \frac{L}{2}$. Now we can substitute n for m, and solve for $c_n$. Therefore, $c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$, which is the same result we had obtained in 1.20.

We have now arrived at the conclusion to our present goal, the solution of the heat equation

$$\alpha^2 u_{xx} = u, \quad 0 < x < L, \quad t > 0$$

with the initial condition

$$u(x,0) = f(x) \quad (1.2)$$

and the boundary conditions:

$$u(0,t) = 0 \quad (1.3)$$

$$u(L,t) = 0. \quad (1.4)$$

$$u(x,t) = \sum_{n=0}^\infty c_n \sin \frac{n\pi x}{L} e^{-\alpha^2 n^2 \pi^2 t/L^2} \quad \text{where} \quad c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

Figure 2. Solution to the Heat Equation
The Wave Equation

We now turn our attention to the wave equation in one-dimension. This second order partial differential equation, or a generalization of it, arises in multiple applications, such as the studies of water waves, electromagnetic waves, seismic waves, and of course, acoustic waves. The equation is

\[ a^2 u_{xx} = u_t, \]  

(1.21)

where \( a^2 \) is a constant coefficient derived from the tension and mass of the vibrating string. Visualize an elastic string of length \( L \) that is tightly stretched between two fixed supports at the same horizontal level, with the \( x \)-axis lying along the string. The string is set in motion (by plucking, for example) which causes it to vibrate vertically. The function \( u(x,t) \) is the vertical
displacement experienced by the string at the point \( x \) and time \( t \). The wave equation therefore must have the following boundary conditions:

\[
\begin{align*}
  u(0,t) &= 0, \quad t \geq 0 \\
  u(L,t) &= 0, \quad t \geq 0
\end{align*}
\] (1.22) (1.23)

Equations 1.22 and 1.23 require that the ends of the string remain fixed at all times. These conditions are called the \textit{Dirichlet boundary conditions}. The wave equation also must have initial conditions - initial position of the string and initial velocity of the string - which are described by equations (1.24) and (1.25) respectively.

\[
\begin{align*}
  u(x,0) &= f(x) \\
  u_t(x,0) &= g(x)
\end{align*}
\] (1.24) (1.25)

We will assume that \( f(x) = 0 \). We further assume that \( u(x,t) \) can be separated into a function of \( x \) and a function of \( t \) (as we did with the heat equation). Separating the variables gives us

\[
u(x,t) = X(x)T(t)
\] (1.26)

and we can substitute this \( u \) into the wave equation to obtain

\[
ad^2(X(x)T(t))_x = (X(x)T(t))_t
\] (1.27)
which can be differentiated appropriately to yield

\[ a^2 X''(x)T'(t) = X(x)T''(t). \]  
(1.28)

Again, we will drop the parenthesis notation, use division, and arrive at this version of the equation:

\[ \frac{X''}{X} = \frac{T''}{a^2 T}. \]  
(1.29)

As in the heat equation, if each of the sides of the equation 1.25 depend on one variable (left side, \(x\), and right side, \(t\)), then each side is equal to the same constant, call it \(-\lambda\). Again, the constant is chosen to be negative due to physical considerations.

\[ \frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda \]

(1.30)

\[ \frac{X''}{X} = -\lambda \]

(1.31)

Our first step is to solve equation 1.30, which can be rewritten as \(X'' + \lambda X = 0\), and is the exact same equation as the \(x\) component of the heat equation (see equation 1.9). We have already done this computation, and the answer is the same as equation 1.13. Each eigenvalue corresponds to an eigenfunction.
\[ X = \text{Asin} \frac{n\pi x}{L} \quad (1.32) \]

Our next step is to solve the equation 1.31, which can be rewritten as \( T'' + \lambda a^2 T = 0 \) and has characteristic equation \( r^2 + \lambda a^2 = 0 \). We shall substitute \( \lambda = \left( \frac{n\pi}{L} \right)^2 \) and solve for \( r \), resulting in the following roots of the characteristic equation: \( r = \pm \frac{n\pi a}{L} \). Now we can write a solution to 1.31, shown below:

\[ T = Ae^{\frac{n\pi a}{L} t} + Be^{-\frac{n\pi a}{L} t} \quad (1.33) \]

Changing the basis of the solution space of 1.33 to

\[ \sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}, \]

as we did in 1.11, we can rewrite the general solution as

\[ T = A\cos \frac{n\pi a}{L} t + B\sin \frac{n\pi a}{L} t. \quad (1.34) \]

Now our solution to (1.21) is

\[ u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( A_n \cos \frac{n\pi a}{L} t + B_n \sin \frac{n\pi a}{L} t \right) \quad (1.35) \]
Using the initial conditions, we can learn more about the constants of this equation. Since $u(0,t)=0$ (equation 1.22) and since $u(x,0)=X(x)T(0)$, either $X(x)$ or $T(0)$ must equal zero. But if $X(x)=0$, we would have the trivial solution, since $u(x,t)=X(x)T(t)$ would be $u(x,t)=(0)T(t)=0$. Therefore, $T(0)=0$ and referring to equation 1.34 we have

$$T(0)=A\cos\frac{n\pi a}{L}(0)+B\sin\frac{n\pi a}{L}(0)=0. \quad (1.36)$$

This implies that $A\cos\frac{n\pi a}{L}(0)=0$ and since $\cos 0=1, A=0$. This eliminates the first term of 1.36, and now we have

$$u(x,t)=\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \left( B_n \sin\frac{n\pi a}{L} t \right) \quad (1.37)$$

Now, one of the possible initial conditions, if in addition we assume that $f(x)=0$ in 1.24, is that the string is set into motion by an initial velocity (equation 1.25), $g(x)$. We can partially differentiate 1.37 with respect to $t$ and set it equal to $g(x)$ to obtain more information about our constant $B_n$. Hence,

$$u_t(x,t)=\sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} \cdot B_n \frac{n\pi a}{L} \cdot \cos\frac{n\pi a}{L} t,$$

and for $t=0$ we obtain
\[ u_0(x,0) = g(x) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \cdot B_n \cdot \frac{n\pi a}{L} \cdot \cos \frac{n\pi a}{L}(0) \]
\[ u_0(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi a}{L} \sin \frac{n\pi x}{L}. \]

Note that the coefficients \( B_n \cdot \frac{n\pi a}{L} \) are the coefficients in the Fourier sine series of period 2L for the odd extension of \( g(x) \). We could use this information to arrive at our final result for \( B_n \), but instead, we shall take the "scenic route" to the solution.

Let us again take the inner product, that is integrate against sine, to solve.

\[ \int_0^L g(x) \sin \frac{m\pi x}{L} \, dx = \int_0^L \left( \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi a}{L} \sin \frac{n\pi x}{L} \right) \sin \frac{m\pi x}{L} \, dx \]

\[ \int_0^L g(x) \sin \frac{m\pi x}{L} \, dx = \sum_{n=1}^{\infty} B_n \cdot \frac{n\pi a}{L} \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx \]

\[ = 0 \text{ if } n \neq m \text{ by (19)} \]
\[ = \frac{L}{2} \text{ if } n = m \]

Again, the only term of the summation that survives is when \( m=n \). And in that case, \( \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \frac{L}{2} \), resulting in a solution for \( B_m \).
\[
\int_0^L g(x) \sin \frac{m\pi x}{L} \, dx = B_m \frac{m\pi a}{L} \frac{L}{2}
\]

\[
B_m = \frac{2}{m\pi a} \int_0^L g(x) \sin \frac{m\pi x}{L} \, dx
\]

Substituting \( n \) for \( m \) yields 
\[
B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx.
\]

We have now arrived at the conclusion to our second goal, the solution of the wave equation \( a^2 u_{tt} = u_{xx} \) with conditions (restated):

\[
u(0,t) = 0, \quad t \geq 0 \quad (1.22)
\]

\[
u(L,t) = 0, \quad t \geq 0 \quad (1.23)
\]

\[
u(x,0) = f(x) = 0 \quad (1.24)
\]

\[
u_t(x,0) = g(x) \quad (1.25)
\]

\[
u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left( B_n \sin \frac{n\pi a}{L} t \right) 
\text{where } B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi x}{L} \, dx
\]

Figure 4. Solution to the Wave Equation
The Wave Equation in Two Dimensions

In the case of the vibrating string, there is one spatial variable, the position along the string between zero and L. But what about a vibrating drum, or membrane? In this case, there are two spatial variables, the horizontal and vertical position of a point. The drum can be pictured as a subset of the Cartesian plane. Each point of this drum has two components, the x and the y, and each point will vibrate up and down (along the z axis) according to the wave equation.

Hence we need to expand our wave equation to accommodate more than one dimension. If we have two or more spatial variables, our $X(x)$ in equation 1.5 becomes analogous to a new $U(x,y)$. Similarly, in three dimensions, our function would be $U(x,y,z)$ and so forth. This transforms our spatial component of the wave equation to be $a^2(U_{xx} + U_{yy} + U_{zz} + ...) + \lambda U = 0$ for many dimensions, or simply $a^2(U_{xx} + U_{yy}) + \lambda U = 0$ for two dimensions. Let us generalize for ANY number of dimensions, by using the Laplacian:
\[ U_{xx} + U_{yy} + U_{zz} + \ldots = \nabla^2 U. \]

Therefore, if we have two or more spatial variables, our spatial component of the wave equation becomes

\[
\begin{align*}
\alpha^2 \nabla^2 U + \lambda U &= 0 \text{ on } \Omega \\
\text{with } U &= 0 \text{ on } \Gamma
\end{align*}
\]

(2.1)

Note, in the above equation, for the case of dimension 2, \( \Gamma \) is the boundary of the membrane \( \Omega \). The boundary condition \( U = 0 \text{ on } \Gamma \) means that \( U(x,y) \to 0 \) as \( (x,y) \) approaches the boundary of \( \Omega \). This is the two-dimensional equivalent of the Dirichlet boundary conditions we used on page 19 for the one-dimensional case.

![Figure 5. Membrane \( \Omega \) With Boundary \( \Gamma \)](image)

Now, recall from Chapter 1 that the solution to the \( X \) part of the wave equation \( \alpha^2 X'' + \lambda X = 0, \ X(0) = 0 = X(L) \) gave us
eigenvalues \( \lambda_n = \left( \frac{n\pi}{L} \right)^2 \) for the one-dimensional wave equation, and eigenfunctions \( X_n(x) = \sin \left( \frac{n\pi x}{L} \right) \). We notice that for the one-dimensional case, the eigenvalues determine the length of the string in the sense that \( \lim_{n \to \infty} \frac{\sqrt{\lambda_n}}{n} = \frac{\pi}{L} \). Indeed,

\[
\frac{\sqrt{\lambda_n}}{n} = \sqrt{\left( \frac{n\pi}{L} \right)^2} = \frac{n\pi}{nL} = \frac{\pi}{L}.
\]

Similarly, we will have a sequence of eigenvalues for the equation 2.1. These eigenvalues represent the natural frequencies at which the membrane vibrates. For instance in two dimensions a different pitch is produced by the vibrating membrane for each eigenvalue. Our goal, therefore, is to show that the eigenvalues, \( \lambda_n \), are related to the area, \( |\Omega| \), of a drum, (just as the eigenvalues for a vibrating string are related to its length). To do this we will show that the area of the membrane can be expressed as a function of the eigenvalues. In colloquial terms, this can be expressed by saying that we can hear the area of a drum. Weyl expressed this as the following theorem:
Theorem 2.1 (Weyl's Theorem)

For a two-dimensional spatial component of the wave equation, $\nabla^2 U = -\lambda U$, in any plane domain $\Omega$ with $U=0$ on the boundary $\Gamma$ of $\Omega$, the eigenvalues satisfy the limit

$$\lim_{n \to \infty} \frac{\lambda_n}{n} = \frac{4\pi}{|\Omega|}$$

where $|\Omega|$ is the area of the domain $\Omega$.

Proof of Weyl's Theorem: Rectangular Membrane

Let us first prove this theorem for a simple example, the case of a rectangular drum. Position the drum with one vertex at the origin of the Cartesian plane, as shown:

Figure 6. Rectangular Membrane

The wave equation is

$$a^2 \nabla^2 u = u_{tt} \text{ on } R \quad \{ \begin{array}{l} \\
\text{with } u = 0 \text{ on the boundary of } R. \end{array} \}$$

(2.2)
We separate $T$ from the spatial variables first, so that

$$u(x,y,t) = U(x,y)T(t).$$

Then $a^2 \left( U_x + U_y \right) T = UT''$, that is

$$a^2 \left( \nabla^2 U \right) T = UT''$$

with $U = 0$ on the boundary of $R$. Thus

$$-\lambda = \frac{T''}{a^2 T} = \frac{\nabla^2 U}{U},$$

and we get the two equations $\nabla^2 U + U\lambda = 0$ with $U = 0$ on the boundary of $R$ and $T'' + a^2 \lambda T = 0$. The solution for the second equation is the same for the one-dimensional case, and will be heretofore ignored. In order to solve the first equation, we must separate the variables again, setting $U(x,y) = X(x)Y(y)$. Then we obtain $X''Y + XY'' = -\lambda XY$.

Thus,

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

with the boundary conditions

$$X(0) = X(a) = Y(0) = Y(b) = 0.$$  It follows that

$$\frac{X''}{X} = \left( -\lambda - \frac{Y''}{Y} \right) = -\mu,$$

where $-\mu$ is a constant chosen to be negative due to physical considerations. So we must solve $X'' + X\lambda = 0$ and $Y'' + (\lambda - \mu)Y = 0$.

The solution of $X'' + X\lambda = 0$ is like the wave equation for a string. We saw earlier that the eigenvalues for the solution of this equation are $\mu = \frac{l^2 \pi^2}{a^2}$ (see page 9 and substitute appropriate variables). Also we know by referring back to equation 1.13 that the eigenfunction is
\[ X_i(x) = \sin\left(\frac{l\pi x}{a}\right). \] (2.3)

(Note: Equation 1.13 shows a coefficient, \( A \), which when combined with the coefficient of the time component of the wave equation, results in a series of coefficients \( c_n \) in the final solution to the wave equation. Earlier we devoted much space to solving for the series of coefficients, \( c_n \). Heretofore, however, we will drop the coefficient that should be present in equation 2.3, since it has no effect on our final result.)

On to the second equation, \( y'' + (\lambda - \mu)y = 0 \). We can substitute \( \mu = \frac{l^2 \pi^2}{a^2} \) to obtain \( y'' + \left(\frac{\lambda - l^2 \pi^2}{a^2}\right)y = 0 \). As in the one-dimensional wave equation, we get

\[ Y(y) = A\cos\sqrt{\frac{\lambda - l^2 \pi^2}{a^2}y} + B\sin\sqrt{\frac{\lambda - l^2 \pi^2}{a^2}y}. \]

Requiring that \( Y(0) = 0 \) yields the following calculations:

\[
Y(0) = A\cos\sqrt{\frac{\lambda - l^2 \pi^2}{a^2}0} + B\sin\sqrt{\frac{\lambda - l^2 \pi^2}{a^2}0} = 0
\]

\[
Y(0) = A\cos 0 + B\sin 0 = 0
\]

\[
Y(0) = A(1) + B(0) = 0
\]

\[
Y(0) = A = 0
\]

\[
\Rightarrow Y(y) = B\sin\sqrt{\frac{\lambda - l^2 \pi^2}{a^2}y}
\]
Again, let us drop the coefficient, $B$. Now, requiring that $Y(b)=0$ yields further calculations:

$$Y(b) = \sin \sqrt{\lambda - \frac{l^2 \pi^2}{a^2}} b = 0$$

$$\Rightarrow \sqrt{\lambda - \frac{l^2 \pi^2}{a^2}} b = m \pi$$

$$\left( \lambda - \frac{l^2 \pi^2}{a^2} \right) b^2 = m^2 \pi^2$$

$$\lambda = \frac{l^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2}$$

Substituting for $\lambda$ in the equation for $Y(y)$, we get the following eigenfunction:

$$Y_m(y) = \sin \sqrt{\frac{l^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2} - \frac{l^2 \pi^2}{a^2}} y$$

$$Y_m(y) = \sin \left( \frac{m \pi y}{b} \right)$$

$$Y_m(y) = \sin \left( \frac{m \pi}{b} y \right)$$

Note that eigenfunctions 2.3 and 2.4 are analogous.

Now, to order the eigenvalues, one must evaluate every $\lambda_n$ and compare them. In the case of the rectangle, for instance, where the eigenvalues are $\lambda_{l,m} = \frac{l^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2}$, one must evaluate $\lambda_{l,m}$ for every $l, m$ with the given rectangle dimensions $a$ and $b$. Since the results are real-valued, one
can put them in order, $\lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots$, such that $\lambda_n$ is the $n^{th}$ eigenvalue.

Let us demonstrate Theorem 2.1 for the case of the rectangle $R = \{(x, y): 0 \leq x \leq a, 0 \leq y \leq b\}$ in the plane, with eigenvalues $\lambda_{l,m} = \frac{l^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2}$ and eigenfunctions $\sin\left(\frac{l\pi y}{a}\right)$ and $\sin\left(\frac{m\pi y}{b}\right)$. In order to see the result in this case, we introduce the enumeration function, $N(\lambda)$.

$N(\lambda)$ is the number of eigenvalues that do not exceed $\lambda$. In this case, $N(\lambda)$ is the number of points $(l,m)$ such that $\frac{l^2}{a^2} + \frac{m^2}{b^2} \leq \frac{\lambda}{\pi^2}$. Since both $l$ and $m$ are greater than zero, $N(\lambda)$ can be represented by the number of lattice points contained in the quarter ellipse $\frac{l^2}{a^2} + \frac{m^2}{b^2} \leq 1$ (see Figure 7, next page). Each point $(l,m)$ is the upper right corner of a unit square contained in the quarter ellipse, therefore $N(\lambda)$ is at most the area of the quarter ellipse, since there is a one-to-one correlation between the lattice points and the unit squares under the curve.
The area of an ellipse is $\pi a \beta$ where $\alpha$ and $\beta$ are the lengths of the semi-major and semi-minor axes of the ellipse. In our case, the semi-major and semi-minor axes are $\frac{a\sqrt{2}}{\pi}$ and $\frac{b\sqrt{2}}{\pi}$, so the area of the quarter ellipse is

$$\frac{1}{4} \frac{a\sqrt{\lambda}}{\pi} \frac{b\sqrt{\lambda}}{\pi} = \frac{\lambda ab}{4\pi}.$$ 

Hence we have $N(\lambda) \leq \frac{\lambda ab}{4\pi}$.

For very large $\lambda$, $N(\lambda)$ and the area of the quarter ellipse differ approximately by the length of the perimeter, which is proportional to $\sqrt{\lambda}$. Thus we obtain

$$\frac{\lambda ab}{4\pi} - k\sqrt{\lambda} \leq N(\lambda) \leq \frac{\lambda ab}{4\pi}.$$
Now since we ordered the eigenvalues so that
\[ \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n, \quad N(\lambda_n) = n, \] and equation 2.3 becomes
\[
\frac{\lambda_n ab}{4\pi} - k\sqrt{\lambda_n} \leq n \leq \frac{\lambda_n ab}{4\pi}. \]
Dividing by \( \lambda_n \) and taking the limit yields the following result:
\[
\frac{ab}{4\pi} \frac{k}{\sqrt{\lambda_n}} \leq n \leq \frac{ab}{4\pi} \frac{1}{\sqrt{\lambda_n}}
\]\
\[
\Rightarrow \lim_{n \to \infty} \frac{ab}{4\pi} \frac{k}{\sqrt{\lambda_n}} \leq n \leq \frac{ab}{4\pi} \frac{1}{\sqrt{\lambda_n}}
\]
\[
= \frac{ab}{4\pi} \lim_{n \to \infty} \frac{n}{\lambda_n} \leq \frac{ab}{4\pi}
\]
\[
\Rightarrow \lim_{n \to \infty} \frac{n}{\lambda_n} = \frac{ab}{4\pi}
\]
Reciprocate the result, and we have
\[
\lim_{n \to \infty} \frac{\lambda_n}{n} = \frac{4\pi}{ab}. \quad (2.5)
\]
Recall that Weyl's theorem asserts \( \lim_{n \to \infty} \frac{\lambda_n}{n} = \frac{4\pi}{|\Omega|} \). But our \( \Omega \) is a rectangle, hence \( |\Omega| = ab \). Therefore our equation 2.5 supports the theorem. Our eigenvalues, \( \lambda_n \), determine the area of the rectangular region R.
Minimum Principles for Eigenvalues

Before we extend the proof of Theorem 2.1 beyond rectangles, we will need some important properties of eigenvalues at our disposal. These properties are called the "Minimum Principle for the 1st Eigenvalue" and the "Minimum Principle for the n-th Eigenvalue". We will formally state and prove these theorems later. Before we do that, we also need some prerequisite lemmas!

Lemmas, Theorems, and Proofs

**Lemma 2.2 (Green's First Identity).** If \(u, v\), are \(C^2\) functions on a neighborhood of \(\bar{\Omega}\) (that is, functions whose second derivative exists and is continuous), then

\[
\int_{\Gamma} \frac{\partial u}{\partial n} dl = \iint_{\Omega} \nabla v \cdot \nabla u\, dx + \iint_{\Omega} \nabla^2 u\, dx.
\]

Moreover, if \(u\) and \(v\) vanish on the boundary \(\Gamma\) of \(\Omega\), then

\[
0 = \iint_{\Omega} \nabla v \cdot \nabla u\, dx + \iint_{\Omega} \nabla^2 u\, dx
\]

*Proof.* Consider \((v_{x} x) = v_{x} u_{x} + v_{xx}\) and \((v_{y} y) = v_{y} u_{y} + v_{yy}\), which follow from the product rule. Then, adding the two equations, we obtain \(\nabla \cdot (v \nabla u) = \nabla u \cdot \nabla v + \nabla^2 u\). Integrating over \(\Omega\) yields

\[
\iint_{\Omega} \nabla \cdot (v \nabla u)\, dx = \iint_{\Omega} \nabla u \cdot \nabla v\, dx + \iint_{\Omega} \nabla^2 u\, dx.
\]

Applying Gauss' divergence theorem on the left hand side, we obtain:
\[
\int_{\Gamma} \frac{\partial v}{\partial n} \, dl = \iint_{\Omega} \nabla u \cdot \nabla v \, d\tilde{x} + \iint_{\Omega} v \nabla^2 u \, d\tilde{x} \quad \text{which is Green's First Identity.}
\]

If \( v \) vanishes on \( \Gamma \), the integral on the left hand side becomes zero, so
\[
0 = \iint_{\Omega} \nabla u \cdot \nabla v \, d\tilde{x} + \iint_{\Omega} v \nabla^2 u \, d\tilde{x}.
\]

**Lemma 2.3 (Green's Second Identity).** If \( u, v \), are \( C^r \) functions on a neighborhood of \( \tilde{\Omega} \), then
\[
\iint_{\Omega} (u \nabla^2 v - v \nabla^2 u) \, d\tilde{x} = \iint_{\Gamma} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, dl.
\]
Moreover if \( u \) and \( v \) vanish on \( \Gamma \), we get:
\[
\iint_{\Omega} u \nabla^2 v \, d\tilde{x} = \iint_{\Omega} v \nabla^2 u \, d\tilde{x} \quad (2.7)
\]

**Proof.** By Greens First Identity, we have
\[
\iint_{\Gamma} \frac{\partial v}{\partial n} \, dl = \iint_{\Omega} \nabla u \cdot \nabla v \, d\tilde{x} + \iint_{\Omega} u \nabla^2 v \, d\tilde{x} \quad \text{and} \quad \iint_{\Gamma} \frac{\partial u}{\partial n} \, dl = \iint_{\Omega} \nabla v \cdot \nabla u \, d\tilde{x} + \iint_{\Omega} v \nabla^2 u \, d\tilde{x}.
\]

Subtracting the second equation from the first, we obtain
\[
\iint_{\Gamma} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, dl = \iint_{\Omega} (u \nabla^2 v - v \nabla^2 u) \, d\tilde{x} \quad \text{which is Green's Second Identity.}
\]

If \( u \) and \( v \) vanish on \( \Gamma \), the left hand side in the previous equation is zero, so
\[
\iint_{\Omega} u \nabla^2 v \, d\tilde{x} = \iint_{\Omega} v \nabla^2 u \, d\tilde{x}.
\]

Recall that we are considering the basic eigenvalue problem with the following Dirichlet boundary conditions:
\[
\nabla^2 u + \lambda u = 0, \quad u = 0 \quad \text{on the boundary} \ \Gamma \ \text{of} \ \Omega, \quad (2.8)
\]
where Ω is an open set in \( \mathbb{R}^2 \) with piecewise smooth boundary.

Let's denote the eigenvalues by

\[
0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \leq \ldots \tag{2.9}
\]

Repeating each value according to its multiplicity. The inner product of two functions is \( \langle f, g \rangle = \iint_{\Omega} f(\mathbf{x}) g(\mathbf{x}) \, d\mathbf{x} \) and the norm of a function \( f \) is \( \|f\| = \langle f, f \rangle^{\frac{1}{2}} = \left( \iint_{\Omega} |f(\mathbf{x})|^2 \, d\mathbf{x} \right)^{\frac{1}{2}} \). We will say that \( f \) is orthogonal to \( g \), that is \( f \perp g \) if \( \langle f, g \rangle = 0 \). It turns out, as we will prove in Theorem 2.4, that the first eigenvalue, \( \lambda_1 \), of the problem is equal to:

\[
m = \min \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} : w = 0 \text{ on } \Gamma, w \text{ being a } C^2 \text{ non-zero function on } \Omega \right\} \tag{2.10}
\]

We define a trial function to be a \( C^2 \) function \( w \) such that \( w = 0 \) on \( \Gamma \) and \( w \neq 0 \) on \( \Omega \). With this definition, 2.10 can be rewritten as: \( m = \min \left\{ \frac{\|\nabla w\|^2}{\|w\|^2} : w \text{ is a trial function} \right\} \). The quantity \( \frac{\|\nabla w\|^2}{\|w\|^2} \) is called the Rayleigh quotient.
Theorem 2.4 (Minimum Principle for the 1st Eigenvalue).

If $v$ satisfies that $\frac{\|\nabla v\|^2}{\|v\|^2} = m$ as in 2.10, then $m = \lambda_1 \nabla^2 v = -\lambda v$, that is, $v$ is an eigenfunction for $\lambda$.

Proof. Note that, from 2.10, we get $m \geq 0$. If the minimum is attained at a function $u$, then

$$m = \frac{\|\nabla u\|^2}{\|u\|^2} = \frac{\iint |\nabla^2 u| \, dx}{\iint |u|^2 \, dx} \leq \frac{\iint |\nabla^2 w| \, dx}{\iint |w|^2 \, dx}$$

for any trial function $w$. If we take another trial function $v$ and set $\bar{w}(x) = u(x) + tv(x)$, and

$$f(t) = \frac{\iint (u^2 + 2tv + t^2 |v|^2) \, dx}{\iint (u^2 + 2tu + t^2 |v|^2) \, dx}$$

then we get that $f$ has a minimum at $t = 0$. Thus $f'(0) = 0$.

Now since $f(t) = \frac{\iint (|\nabla u|^2 + 2t(\nabla u \cdot \nabla v + t^2 |\nabla v|^2) \, dx}{\iint (u^2 + 2tu + t^2 |v|^2) \, dx}$ we get

$$0 = f'(0) = \left( \frac{2 \iint \nabla u \cdot \nabla v \, dx}{\iint u^2 \, dx} \right) \left( \frac{\iint u^2 \, dx}{\iint \nabla u^2 \, dx} \right) - \left( \frac{2 \iint uv \, dx}{\iint u^2 \, dx} \right) \left( \frac{\iint \nabla u^2 \, dx}{\iint \nabla u^2 \, dx} \right).$$

Thus,

$$\left( \iint \nabla u \cdot \nabla v \, dx \right) \left( \iint u^2 \, dx \right) = \left( \iint uv \, dx \right) \left( \iint \nabla u^2 \, dx \right),$$

and therefore
\[
\int_{\Omega} \nabla u \cdot \nabla v d\bar{x} = \frac{\int_{\Omega} \nabla u d\bar{x} \left( \int_{\Omega} |\nabla v|^2 d\bar{x} \right)}{\int_{\Omega} |\nabla u|^2 d\bar{x}} = \int_{\Omega} \nabla u d\bar{x} \cdot \int_{\Omega} \nabla v d\bar{x} = m \int_{\Omega} \nabla v d\bar{x}.
\]

Since \( v \) is an arbitrary trial function, we conclude that 
\[ \nabla^2 u + mu = 0 \text{ on } \Omega, \] so \( m \) is an eigenvalue with eigenfunction \( u \).

If \( \mu \) is another eigenvalue with eigenfunction \( w \), we
\[
\int_{\Omega} \nabla |w|^2 d\bar{x} = \int_{\Omega} (-\nabla^2 w) d\bar{x} = \int_{\Omega} (\mu w) d\bar{x}
\]
get that 
\[
m \leq \frac{\int_{\Omega} |w|^2 d\bar{x}}{\int_{\Omega} |\nabla w|^2 d\bar{x}} = \frac{\int_{\Omega} |\nabla w|^2 d\bar{x}}{\int_{\Omega} |\nabla w|^2 d\bar{x}} = \mu,
\]
where the first inequality follows from Green's First Identity as 
\[
\int_{\Omega} |\nabla w|^2 d\bar{x} = \int_{\Omega} \nabla w \cdot \nabla w d\bar{x} = \int_{\Omega} (\nabla w)^2 d\bar{x}. \] Thus \( m \leq \mu \) so \( \mu \) is the smallest eigenvalue \( \lambda_1 \).

The corresponding result for the \( n^{th} \) eigenvalue is the following.

**Theorem 2.5 (Minimum Principle for the \( n^{th} \) Eigenvalue).**

Suppose \( \lambda_1, ..., \lambda_{n-1} \) are already known, with eigenfunctions \( v_1(\bar{x}), v_2(\bar{x}), ..., v_{n-1}(\bar{x}) \) respectively. Then
\[
\lambda_n = \min \left\{ \frac{\int_{\Omega} |\nabla w|^2 d\bar{x}}{\int_{\Omega} |w|^2 d\bar{x}} : w \text{ trial function, } w \perp v_1, ..., v_{n-1} \right\}
\] (2.13)
Assuming that the minimum exists. Furthermore, the minimizing function is the \( n \)th eigenfunction \( \psi_n(x) \).

**Proof.** Let \( u(x) \) be the minimizing function for 2.13. This function exists by assumption. Let \( m^* \) be the value of the right hand side of 2.13. Thus \( m^* = \frac{\| \nabla u \|^2}{\| u \|^2} \), and \( u \) is zero on \( \Gamma \), and orthogonal to \( \psi_1, \psi_2, \ldots, \psi_{n-1} \). Consider \( w = u + tv \), where \( v \) is a trial function orthogonal to \( \psi_1, \psi_2, \ldots, \psi_{n-1} \). By a similar argument to the one used in Theorem 2.13, we arrive at the version of 2.12 for \( m^* \), namely

\[
\int_{\Omega} (m^* u + \nabla^2 u) v \, d\bar{x} = 0 \tag{2.12a}
\]

This equation holds for all trial functions \( v \) that are orthogonal to \( \psi_1, \psi_2, \ldots, \psi_{n-1} \).

For \( \psi_1, \psi_2, \ldots, \psi_{n-1} \), with the use of Green's Second Identity, 2.2.3, we obtain:

\[
\int_{\Omega} (m^* u + \nabla^2 u) v_j \, d\bar{x} = \int_{\Omega} u(\nabla^2 v_j + m^* v_j) \, d\bar{x} = \int_{\Omega} u(- \lambda_j v_j + m^* v_j) \, d\bar{x} = (m^* - \lambda_j) \int_{\Omega} uv_j \, d\bar{x} = 0 \tag{2.14}
\]

**Note:** The last equality holds because \( u \) is orthogonal to \( \psi_1, \psi_2, \ldots, \psi_{n-1} \).
Combining 2.12a and 2.13, we will obtain 2.12a for arbitrary trial functions. In fact, for an arbitrary trial function \( f \), let \( v(x) = f(x) - \sum c_k v_k(x) \) where \( c_k = \frac{\langle f, v_k \rangle}{\langle v_k, v_k \rangle} \), that is \( v(x) \) is the component of \( f \) orthogonal to the span of \( v_1, v_2, \ldots, v_{n-1} \). Then \( \langle v, v_j \rangle = 0 \) for \( j = 1, 2, \ldots, n-1 \). Hence \( v \) satisfies the constraints in 2.13, and therefore \( v \) also satisfies 2.3.7a, that is \( \iint_{\Omega} (m^* u + \nabla^2 u) \left( f - \sum_{k=1}^{n-1} c_k v_k \right) \, d\bar{x} = 0 \). Hence,
\[
\iint_{\Omega} (\nabla^2 u + m^* u) \, f d\bar{x} - \sum_{k=1}^{n-1} c_k (m - \lambda_k) \iint_{\Omega} u v_k \, d\bar{x} = 0.
\]
Since \( \langle u, v_k \rangle = 0 \) for \( k = 1, \ldots, n-1 \), the previous equation becomes \( \iint_{\Omega} (\nabla^2 u + m^* u) \, f d\bar{x} = 0 \).

And since \( f \) is an arbitrary trial function, we conclude that \( \nabla^2 u + m^* u = 0 \) on \( \Omega \), so \( u \) is an eigenfunction and \( m^* \) is an eigenvalue. For \( i = 1, \ldots, n-1 \), \( \lambda_i \) satisfies less constraints than \( m^* \), so \( \lambda_i \leq m^* \). Thus \( m^* = \lambda_n \).

**Theorem 2.6 (Maximum Principle).** Let \( n \geq 2 \) be a fixed integer, and \( y_1(x), y_2(x), \ldots, y_{n-1}(x) \) be given trial functions.

Let \( \lambda_n = \min \left\{ \frac{\| \nabla w \|^2}{\| w \|^2} : w \text{ trial function}, w \perp y_1, y_2, \ldots, y_{n-1} \right\} \) (2.15)
Then $\lambda_n = \max \{\lambda_i\}$ where $\lambda_i$ corresponds to a choice of trial functions $y_1, y_2, ..., y_{n-1}$ by 2.15. That is, $\lambda_n$ is the greatest possible value over all possible choices of trial functions $y_1, y_2, ..., y_{n-1}$.

Proof. Choose $y_1, y_2, ..., y_{n-1}$ as $v_1, v_2, ..., v_{n-1}$ respectively, the first n-1 eigenfunctions. By Theorem 2.5, $\lambda_n = \lambda_*$ for this choice of trial functions. Thus $\lambda_n \leq \max \lambda_n$.

For the reverse inequality, take and arbitrary choice of trial functions $y_1, y_2, ..., y_{n-1}$, and let $v_1, v_2, ..., v_n$ be the first n eigenfunctions. Assume $\|v_j\|=1$. Let $w(\tilde{x})=\sum_{j=1}^{n} c_i v_i(\tilde{x})$ such that $w \perp y_1, y_2, ..., y_{n-1}$. Note that such w exists because $c_1, c_2, ..., c_n$ satisfies $\left(\sum_{i=1}^{n} c_i v_i, y_j\right) = \sum_{i=1}^{n} \left(v_i, y_j\right) c_i = 0$ for $j=1,2,...,n-1$ which is a system of n-1 equations in n unknowns. Such system has a non-trivial solution, which we can take as $c_1, c_2, ..., c_n$.

Thus, for this choice of $y_1, y_2, ..., y_{n-1}$,

$$\lambda_n \leq \frac{\|\nabla w\|^2}{\|w\|^2} = \sum_{i,j} c_i c_j \left(-\nabla^2 v_i, v_j\right) = \sum_{i=1}^{n} \lambda_i c_i^2 \leq \sum_{i=1}^{n} \lambda_n c_i^2 = \lambda_n. \quad \text{(Note: we have used } \|v_j\|=1 \text{ in the denominator)}$$
Now that we have shown both $\lambda_n \leq \max \lambda$, and $\lambda_n' \leq \lambda$, we have proven that $\lambda_n = \max \{\lambda_n'\}$.

It follows from Theorem 2.6 that any orthogonality constraint other than those of Theorems 2.4 and 2.5 leads to smaller values of the Rayleigh quotient.

**Theorem 2.7.** Let $\Omega \in \Omega'$ be two domains for the problem $\nabla^2 u = -\lambda u$, $u = 0$ on $\Gamma$ the boundary of $\Omega$ and $\Gamma'$ the boundary of $\Omega'$. Let $\lambda_n$ be eigenvalues for the problem on $\Omega$ and $\lambda_n'$ be eigenvalues for the problem on $\Omega'$. Then $\lambda_n' \leq \lambda_n$. That is, if the domain is enlarged, each eigenvalue is decreased.

Figure 8. Domains for Theorem 2.7
Summary of the Proof. The full proof is not included here due to time and space. But the outline is as follows: If \( w(x) \) is a trial function on \( \Omega \), we extend to \( \Omega' \) by defining
\[
    w'(x) = \begin{cases} 
    w(x) & x \in \Omega \\
    0 & x \in \Omega' \setminus \Omega
    \end{cases}
\]
Thus the trial functions in \( \Omega \) are trial function in \( \Omega' \) with the extra constraint that the functions vanish outside of \( \Omega \). Thus, \( \lambda_\ast \geq \lambda'_\ast \), by Theorem 2.6, as the maximum value for \( \Omega \) is larger than the maximum value for \( \Omega' \). (Notice that \( w'(x) \) may not be a trial function, because extending \( w(x) \) by zero as we did may cause \( w'(x) \) to loose the property of being a \( c^2 \) function.)

Proof of Weyl's Theorem: Square Domains

Let \( \Omega \) be a square domain, that is, \( \Omega \) is the finite union of \( p \) squares, \( s_1, s_2, \ldots, s_p \), each with side \( a \). Clearly, the area of \( \Omega \) is \( pa^2 \). Let \( N(\lambda) \) be the number of eigenvalues less than or equal to \( \lambda \) for the boundary condition \( u=0 \) on \( \Gamma \), and let \( N_q(\lambda) \) be the number of eigenvalues less than or equal to \( \lambda \) for the boundary condition \( u=0 \) on the
boundary of \( s_i \). The example of a rectangular domain gave us

\[
\frac{\lambda a b}{4\pi} - k\sqrt{\lambda} \leq N(\lambda) \leq \frac{\lambda a b}{4\pi} \quad \text{(pg. 33)},
\]

where the length of the sides of the rectangle were a and b. Now the sides are both of length a, hence we have

\[
\frac{\lambda a^2}{4\pi} - k\sqrt{\lambda} \leq N(\lambda) \leq \frac{\lambda a^2}{4\pi}.
\]

So we see that \( N(\lambda) \), since it is a continuous expression, takes every value between \( \frac{\lambda a^2}{4\pi} - k\sqrt{\lambda} \) and \( \frac{\lambda a^2}{4\pi} \), hence

\[
N_i(\lambda) = \frac{a^2}{4\pi} \lambda + k\sqrt{\lambda}
\]

(2.16)

for some \( k \), where \( k \) is a constant independent of \( i \), a, and \( \lambda \). Note, this \( k \) may be a different constant than the one used in the line above it.

Now, if we take all the eigenvalues for each of the squares in our region and order them, the \( n^{\text{th}} \) eigenvalue of that union will be smaller than the \( n^{\text{th}} \) eigenvalue of the region \( \Omega \). This is because our union of squares is contained in \( \Omega \). Hence the number of eigenvalues for the union of squares not exceeding \( \lambda \) is smaller than the corresponding number for the region \( \Omega \). Now the number of those eigenvalues for the union of squares is just the sum of the corresponding eigenvalues (that is, those not
exceeding \( \lambda \) for each square. Hence, by Theorem 2.6, we have:

\[
N_{s_1}(\lambda) + N_{s_2}(\lambda) + \ldots + N_{s_p}(\lambda) \leq N(\lambda).
\]

Hence, 2.16 applied to the above inequality allows us to obtain

\[
N(\lambda) = \frac{p a^2 \lambda}{4\pi} + k \sqrt{\lambda}
\]

(2.17)

Now if \( \lambda_n \) is the \( n^{\text{th}} \) eigenvalue, \( N(\lambda_n) = n \), so 2.18 becomes

\[
n = \frac{|\Omega|}{4\pi} \lambda_n + k \sqrt{\lambda_n}.
\]

Divide by \( \lambda_n \) to obtain

\[
\frac{n}{\lambda_n} = \frac{|\Omega|}{4\pi} \frac{k}{\lambda_n} + \frac{1}{\sqrt{\lambda_n}}.
\]

As \( \lambda_n \) increases, \( \frac{k}{\sqrt{\lambda_n}} \) approaches zero. Therefore,

\[
\lim_{n \to \infty} \frac{n}{\lambda_n} = \frac{|\Omega|}{4\pi}, \quad \text{and the reciprocal takes the form} \quad \lim_{n \to \infty} \frac{\lambda_n}{n} = \frac{4\pi}{|\Omega|}.
\]

Proof of Weyl's Theorem: General Case

We have finally arrived to the proof of Weyl's theorem for the general case. That is, we need to prove that the eigenvalues are proportional to the area of a drum regardless of the shape of the drum.
Proof. Let $\Omega$ be a plane domain, and let $s_1, s_2, \ldots, s_p$ be squares whose union is a square domain approximating $\Omega$.

Again, $N(\lambda)$ is the number of eigenvalues less than $\lambda$, for the boundary problem for $\Omega$, and $N_q(\lambda)$ the corresponding number for $s_i$. Thus, equation 2.18 becomes

$$N(\lambda) = \frac{pa^2}{4\pi} + k\sqrt{\lambda} = \lambda \left( \frac{pa^2}{4\pi} + \frac{k}{\sqrt{\lambda}} \right).$$

Divide by $\lambda$ to obtain

$$\frac{N(\lambda)}{\lambda} = \frac{pa^2}{4\pi} + \frac{k}{\sqrt{\lambda}}.$$
Now, in Figure 9, what would happen if we used smaller squares? First, more of them would fit inside the domain $\Omega$, and second, there would be less space not covered by squares, resulting in a better approximation of $\Omega$. Using more squares is the same thing as letting $\lambda$ go to infinity.

Hence, $\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} = \frac{1}{4\pi} \lim_{pa^2 \to |\Omega|} (pa^2) = \frac{|\Omega|}{4\pi}$ as $pa^2 \to |\Omega|$. Reciprocate, let $N(\lambda_n) = n$, and we have arrived at our conclusion: $\lim_{n \to \infty} \frac{\lambda_n}{n} = \frac{4\pi}{|\Omega|}$. 
Summary of Findings by Gordon, Webb, Wolpert, and Bérard

Kac popularized the question "Can you hear the shape of a drum", the question of whether or not the eigenvalues for the two-dimensional wave equation of Chapter Two determine the region \( \Omega \) up to congruence. This problem remained unsolved until 1992, when Carolyn Gordon, David Webb, and Scott Wolpert were able to construct two plane domains that are isospectral (that is, the wave equation has the same set of eigenvalues in the two domains) and not congruent.

The drums in Figure 10 were first presented by Gordon, Webb, and Wolpert as an example of isospectral, non-congruent domains. Their proof was simplified considerably by Piérre Bérard who reduced it to the process of constructing a map that takes eigenfunctions for the first domain onto eigenfunctions of the second domain, with the same eigenvalue. Bérard proved that the eigenvalues of \( D \) and \( D^* \) in Figure 10 are identical by constructing the map shown in the figure. The domain \( D \) is divided into
congruent sections, each labeled with a different letter, A, B, C,...etc. The notation \( \overline{A}, \overline{B}, \overline{C}, \ldots \) etc. represents the reflection of a particular region. Note: Figures 10-13 were taken from S. J. Chapman's article. The captions have been changed to fit this context.

![Diagram](image)

**Figure 10.** Gordon, Webb, and Wolpert's Drums with Bérard's Transpositions

**Chapman's Method of Constructing Isospectral Non-Congruent Drums**

S. J. Chapman sought to explain Bérard's transformations in a concrete and comprehensible way. He arrived at the conclusion that layering folded versions of one drum could create an isospectral, non-congruent domain
for the second drum. Consider a given domain, D. Each folded version of a copy of D is a transposition on the domain, denoted by \( D_1, D_2, \ldots \) etc. Attaching these domains together is symbolic to adding the transpositions. Call this sum \( D^* \). Chapman states that if we follow certain folding and gluing guidelines, we have created an eigenfunction on the new domain \( D^* \) that is the same as the eigenfunction of the original domain D, meaning the drums would sound the same. Yet the domains will be different shapes!

Chapman asserts the following guidelines for the folding and gluing of drums in pursuit of two isospectral, non-congruent membranes. When attaching folded versions together, one must make sure that 1) all folds are placed along an outside edge of the figure, and 2) each edge that lies on the interior of the final shape must be adjacent to its reflection on another copy of the original shape. Chapman proposes that these qualifications for folding and gluing are necessary to ensure that the first derivative of the transposition is continuous.

Figure 11 shows the three different domains, \( D_1, D_2, \) and \( D_3 \), which were layered to create domain \( D^* \) in Figure 10.
Note that the dotted outline of figure D' helps the reader place each domain in context.

Figure 11. Chapman's Folding of Gordon, Webb, and Wolpert's Drum

Chapman's Examples

The congruent regions, A, B, C, ..., G in the previous examples were isosceles right triangles. Chapman claims
that many shapes can be used as the congruent regions, provided that copies of the region can be connected and folded as in Figure 11. He provides the following examples of domains made from different shaped congruent regions and then folded along the same boundaries as the example in Figure 11. For each example, the figure on the left is the first domain, and the figure on the right is the one that has been constructed by layering folded versions of the first according to the rules.

Figure 12. Chapman's Scalene Triangle Example and Square Example
Not only can one make drums by using different polygons for the congruent regions, but any cutout of any polygonal region that cooperates (that is, one that folds as in Figure 11) will also work. For instance, substituting the region \( \triangle \) for each region \( \triangle \) in the example from Figure 10 will result in the following isospectral non-congruent drums:

![Diagram showing the substitution process](image-url)

Figure 13. Chapman’s Cutout Example
Original Examples:

My first attempt at finding a cooperating polygon for the congruent regions A, B, C, ... G was a regular octagon. Consider the following drums, formed by connected octagons. The drums are isospectral and non-congruent, since the second drum was formed by layering folded versions of the first according to Chapman's rules.

Figure 14. Original Example #1
After choosing the octagons, constructing the domain, cutting the copies of the domain, and folding the copies of the domain to make the second domain, I realized that my octagon is a subset (or "cutout") of the isosceles right triangle region of Figure 10, therefore, my drums are subsets of the drums in Figure 10, as shown in Figure 15!

Figure 15. Original Example #1 as a Subset of Gordon’s Drums
The octagon incident led me to pursue a shape that is not a subset of the isosceles right triangle. I have yet to achieve success with the various parallelograms and non-regular pentagons that I attempted. I did, however, construct two isospectral, non-congruent drums from regions of right triangles that folded appropriately (shown in Figure 16) as well as a pair of drums made up of dodecadon regions (shown in Figure 17), which is a subset of Chapman's square example in Figure 12.

Figure 16. Original Example #2
Note that in Figure 17 below, the dashed lines represent a place where there is a disconnect in the membrane.

Figure 17. Original Example #3

For each of the original examples, I cut out and folded the domains to be sure that they would behave
properly. It would be interesting to actually build these membranes as real musical instruments and hear the sounds they make. For now, we shall take the mathematical evidence as sufficient proof that these drums would sound the same!
REFERENCES


