Aspects of the Jones polynomial

Alvin Mendoza Sacdalan

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ASPECTS OF THE JONES POLYNOMIAL

A Project
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Masters of Arts
in
Mathematics

by
Alvin Mendoza Sacdalan

June 2006
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A Project

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ABSTRACT

A knot invariant called the Jones polynomial is the focus of this paper. The Jones polynomial will be defined into two ways, as the Kauffman Bracket polynomial and the Tutte polynomial. Three properties of the Jones polynomial are discussed. Given a reduced alternating knot with n crossings, the span of its Jones polynomial is equal to n. The Jones polynomial of the mirror image $L^*$ of a link diagram $L$ is $V_{(L^*)}(t) = V_{(L)}(t)$. Given an oriented connected sum $L_1 \# L_2$, its Jones polynomial is $V_{L_1 \# L_2}(t) = V_{L_1}(t) \cdot V_{L_2}(t)$. Finally, we will see how mutant knots share the same Jones polynomial.
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CHAPTER ONE

INTRODUCTION

The studies in Knot Theory began with a "knotted" concept in chemistry. In the 1880's, in an attempt to explain all types of matter, many scientists thought that the universe was filled with a substance called ether, and all matter was knotted to this substance. A scientist named Lord Kelvin (William Thompson, 1824-1907) proposed that different knots in this substance would mean different elements. This idea, of course, was dismissed shortly after a more precise model of an atom was developed in the late nineteenth century. For almost one hundred years, physical scientists lost interest in Knot Theory.

In the 1980's, there was a renewed interest in Knot Theory. Biologists and chemists found that knotting occurs in DNA molecules. Experiments suggested that different types of knots in the DNA molecule could mean different properties of the molecule. Today, the study in Knot Theory is not limited to mathematicians; mainstream science has made it an integral part of their studies.

In this chapter we will learn some basic definitions in Knot Theory. This is essential when we discuss the Jones
polynomial in the later chapters. Basic examples of knots and links are also provided. The chapter also discusses the rules on deformations of knots and links through Reidemeister moves.

Knots and Links

What is Knot Theory? It is a branch of topology that deals with knots and links. This subfield of topology studies the geometric properties of knots and links.

Definition 1.1.1 A knot $K \subset \mathbb{R}^3$ is a closed loop that does not intersect itself.

Three examples of knots are shown in Figure 1.1. The first knot is a round circle called the trivial knot. The second knot is called the left-handed trefoil. It has a mirror image appropriately called the right-handed trefoil. The third knot is called the figure-eight knot. Its mirror image is itself. We will define mirror image in the later chapters. Two knots are equivalent if we can deform one knot to look like the other knot. Deformation is accomplished through a series of the so called Reidemeister moves described in the next section.
Definition 1.1.2 A link $L \subset \mathbb{R}^3$ is a union of knots. Each knot is called a component of the link.

Two examples of links are shown in Figure 1.2. The first link is called the Hopf link. It has two components. The second link is called the Whitehead link, and it also has two components. Two links are equivalent if we can deform one link to look like the other link.

![Figure 1.1 Examples of Knots](image-url)
Knots with simple diagrams are identified by their catalogue numbers. They are generally written as $N_m$ where $N$ is the order of the knot, and the subscript $m$ is the $m$th knot of order $N$. In the examples shown in Figure 1.1, knots $0_1$, $3_1$, and $4_1$ are shown. The order of the knot is also the crossing number of the knot.

Reidemeister Moves

We refer to the picture of a knot or a link as the projection of the knot or link. In this paper, the two will be used interchangeably. One of the difficult things in the study in Knot Theory is to distinguish between projections of knots. How do we know if two knots are the same or different?

We are allowed to deform our knots, but we must follow certain rules. How do we deform the projection of the knot? We accomplish this through ambient isotopy. The deformation
under ambient isotopy is described in three types of moves called the Reidemeister moves.

The first type can either add a twist or eliminate a twist from the knot (Figure 1.3). The second type can either add two crossings or eliminate two crossings (Figure 1.4). The third type allows us to slide a strand across a crossing. The strand can either be an overpass or an underpass (Figure 1.5).

![Figure 1.3 Type I Reidemeister Move](image1.jpg)

![Figure 1.4 Type II Reidemeister Move](image2.jpg)
We have seen through deformation, specifically Reidemeister move type I and II, that we can increase or decrease the number of crossings of a knot.

Definition 1.2.1 The minimal number of crossings of any projection of a knot is called the crossing number of the knot denoted $c(K)$.

The crossing numbers of the trivial knot, trefoil knot, and the figure-eight knot are 0, 3, and 4 respectively.

Another property of a knot is its writhe. First let us provide an orientation for our knot. We achieve this by choosing a direction to travel around the knot. Using the rules in Figure 1.6, we assign a $+1$ or $-1$ to each crossing of our knot.

- Figure 1.5 Type III Reidemeister Move
Definition 1.2.2 The sum of the positive and negative crossings is called the writhe of the oriented link.

Sometimes it’s quite difficult to tell if we have a positive crossing or a negative crossing. An easy method is given an oriented link, place your index finger directly above the direction of the overpass of the crossing, then your thumb should be pointing on the direction of the underpass of the crossing. If you use your right hand to achieve this, you have a positive crossing. If you use your left hand, you have a negative crossing. The writhes of the left-handed trefoil and figure-eight knot are $-3$ and $0$ respectively shown in Figure 1.7. Each crossing of the left-handed trefoil carries a $-1$ sign which adds up to $-3$. Two of the crossings of the figure-eight knot carry a $-1$ sign, while the other two of its crossings carry a $+1$ sign which adds up to zero.
We say that a knot property is an invariant of a knot if it is unchanged under all Reidemeister moves.

Lemma 1.2.1 The writhe of an oriented knot is an invariant under Reidemeister move Type II and Type III, but not under Reidemeister move Type I.

Proof

Under Reidemeister move Type II and Type III, the writhe of the oriented knot is preserved since it contributes a zero on the writhe as shown in Figure 1.8 and Figure 1.9 respectively. However, under Reidemeister move Type I, the twist contributes a +1 or -1 to the writhe as

Figure 1.7 Writhes of 3_1 and 4_1
shown in Figure 1.10. The result for other choices of orientations follows similarly.

\[
\begin{array}{c}
\uparrow \uparrow \\
\uparrow \uparrow +1 \\
\downarrow \downarrow -1 \\
\end{array}
\begin{array}{cc}
\text{or}
\begin{array}{c}
\uparrow \uparrow -1 \\
\downarrow \downarrow +1 \\
\end{array}
\end{array}
\]

Figure 1.8 Writhe is an Invariant Under Type II Move

\[
\begin{array}{c}
\uparrow \downarrow +1 \\
\downarrow \uparrow -1 \\
\end{array}
\begin{array}{cc}
\text{or}
\begin{array}{c}
\uparrow \downarrow -1 \\
\downarrow \uparrow +1 \\
\end{array}
\end{array}
\]

Figure 1.9 Writhe is an Invariant Under Type III Move
Figure 1.10 Writhe is not an Invariant Under
Type I Move

Hence, writhe is not a knot invariant. We now know sufficient definitions in Knot Theory to discuss a knot invariant called the Jones Polynomial.
CHAPTER TWO

THE JONES POLYNOMIAL

How can we tell knots apart? This chapter discusses a knot invariant called the Jones polynomial. We learn the two definitions of the Jones polynomial, which are the Kauffman Bracket polynomial and the Tutte polynomial. The Jones polynomial helps us distinguish between knots. Knots with nine or fewer crossings have their own distinct Jones polynomials. This does not hold true, however, for knots with ten or more crossings as we will see in Chapter Four.

The Jones polynomial of an oriented link denoted $V_L(t)$ follows two basic rules

Rule 1: $V_0(t) = 1$

Rule 2: $t[V_L(t)] - t^{-1}[V_L(t)] = (t^{1/2} - t^{-1/2})V_0(t)$

The first rule states that the Jones polynomial of the trivial knot is 1. The second rule deals with the skein relation of the knot projections as shown in Figure 2.1. The following two sections in this chapter define the Jones polynomial in alternative ways.
Kauffman Bracket Polynomial

In this definition of the Jones polynomial, we derive the Jones polynomial of a knot by following three steps. First, we compute the bracket polynomial of a knot using the method devised by Louis Kauffman [1]. Second, we use both the bracket polynomial and the writhe of the knot to compute the Kauffman polynomial of the knot. Lastly, we can obtain our desired Jones polynomial of a knot by a change of variable. We simply replace each $A$ in the Kauffman polynomial with $t^{-1/4}$.

Louis Kauffman developed rules for computing the bracket polynomial of a link projection. For example, we let the bracket polynomial of the trivial knot be 1, which is Rule 1 of the bracket polynomial.

Rule 1: $\langle [] \rangle = 1$
The second rule of our bracket polynomial deals with splitting the crossings of our knot. We utilize the following skein relation. Given a projection of a knot, we split its crossing vertically and horizontally thus obtaining a simpler projection of a knot with one less crossing. We now have a linear combination for our bracket polynomial. For now, let's assign A and B as coefficients of the linear combination. We repeat this process until all crossings are split. For example, given a projection of a trefoil knot, we apply the second rule of the bracket polynomial three times. Later, we will see some shortcuts to decrease the number of times we repeat this process. In the first equation, a vertical split is initially carried out followed by a horizontal split. In the second equation, we simply reversed the process.

Rule 2: \[
\begin{align*}
\langle \begin{array}{c} \\
\end{array} \rangle & = A \langle \begin{array}{c} \\
\end{array} \rangle + B \langle \begin{array}{c} \\
\end{array} \rangle \\
\langle \begin{array}{c} \\
\end{array} \rangle & = A \langle \begin{array}{c} \\
\end{array} \rangle + B \langle \begin{array}{c} \\
\end{array} \rangle
\end{align*}
\]

It may appear that the second rule has two different equations, but this is not the case. If we choose one of the equations and rotate the projection of its crossing by 90 degrees, we arrive at the other equation.
When we apply the second rule, we sometimes find that we split the knot or link diagram into two components. When this occurs, we multiply our polynomial with coefficient C. This is our third and final rule for our bracket polynomial.

\[ \langle L \cup \square \rangle = C\langle L \rangle \]

We now have three rules with three coefficients, A, B, and C. We want to simplify our rules such that we only deal with one coefficient. We stated that we want our polynomial to be a knot invariant. This means that our polynomial remains unchanged if we apply the Reidemeister moves. In Chapter One, we saw that writhe of a knot is not an invariant since it is affected by the Type I move. Let’s find out if our bracket polynomial is a knot invariant.

First, let’s apply the Type II Reidemeister move as shown in Figure 2.2.
If we factor out the vertical and horizontal split, we get the following:

\[
\langle \hat{L} \rangle = (A^2 + ABC + B^2)\langle \hat{I} \rangle + BA\langle \rangle\{\} = \langle \rangle\{\}
\]

We want to have \( \langle \hat{L} \rangle = \langle \rangle\{\} \). In order to accomplish this and show that our bracket polynomial is unaffected by type II, we let \( B = A^{-1} \). So from the vertical split, we have that \( BA = AA^{-1} = 1 \). We need to find a way to set the coefficient of the horizontal split equal to zero. For our horizontal split, we have \( A^2 + AA^{-1}C + A^{-2} = A^2 + C + A^{-2} \). Thus we make \( C = -A^2 - A^{-2} \). With these specializations the bracket polynomial is an invariant under Type II move. Now we only
have one variable to deal with, and our rules for bracket polynomial now become:

Rule 1: \( \langle \square \rangle = 1 \)

Rule 2: \( \langle \square \times \rangle = A \langle \rangle + A^{-1} \langle \square \rangle \)

Rule 3: \( \langle L \cup \square \rangle = (-A^2 - A^{-2}) \langle L \rangle \)

We saw that our bracket polynomial is unaffected by Type II Reidemeister move. Let's investigate Type III move. If we apply the fact that the bracket polynomial is unchanged under Type II move, Type III move is also unchanged as shown below.

\[
\langle \square \times \rangle = A \langle \square \rangle + A^{-1} \langle \square \rangle
\]

\[
= A \langle \square \rangle + A^{-1} \langle \square \rangle = \langle \square \times \rangle
\]

When we apply Type I Reidemeister move, we are not so lucky. Let's apply our rules of bracket polynomial to show that our polynomial is changed under Type I move as shown in Figure 2.3.
Lemma 2.1.1 The bracket polynomial of Figure 2.3a is
\[ A + A^{-1}(-A^2-A^2) = -A^3. \]

Lemma 2.1.2 The bracket polynomial of Figure 2.3b is
\[ A(-A^2-A^2) + A^{-1} = -A^3. \]

Type I move changes our polynomial, thus the bracket polynomial is not a knot invariant. This is fine since we can still show that the Jones polynomial is a knot invariant. The bracket polynomial and the writhe of the knot will be use when we compute the Jones polynomial.

Example 2.1.1 Figure 2.4 shows the decomposition of the Hopf link when we compute its bracket polynomial.
In the first step, we split one of the crossings (top) of the Hopf link. In the second step, we can use the result we found from Figure 2.3. We get

\[ A(-A^3) + A^{-1}(-A^{-3}) = -A^4 - A^{-4} \]

to be the bracket polynomial of the Hopf link.

Now that we know how to compute the bracket polynomial of a link projection, we can now compute the Kauffman polynomial. The Kauffman polynomial of an oriented link is defined to be

\[ X_{(L)} = (-A^3)^{-w(L)} \langle L \rangle \]
where \( w(L) \) is the writhe and \( \langle L \rangle \) is the bracket polynomial of the oriented link diagram.

Theorem 2.1.1 The Kauffman polynomial is a knot invariant.

Proof

We already have shown that both the writhe and bracket polynomial are unaffected by Type II and Type III Reidemeister moves, so the Kauffman polynomial is also unaffected by these two moves. We only need to show that the Kauffman polynomial is an invariant under the Type I move. Let us prove that the Kauffman polynomial is unchanged under Type I move. Suppose we have two strands \( L' \) and \( L \) as shown in Figure 2.5, then \( w(L') = w(L) + 1 \) and \( \langle L' \rangle = -A^3 \langle L \rangle \).

\[ X_{(L')} = (-A^3)^{-w(L')} \langle L' \rangle \]
\[ = (-A^3)^{-w(L)+1} \langle L' \rangle \]

Figure 2.5 Applying Type I Move on \( X_{(L)} \)
\[
\begin{align*}
&= (-A^3)^{-w(L)} \cdot (-A^3)^{-1}(-A^3) \langle L \rangle \\
&= (-A^3)^{-w(L)} \langle L \rangle \\
&= X_{(L)} \square
\end{align*}
\]

Definition 2.1.1 Given the Kauffman polynomial \( X_{(L)} \) of a knot, its Jones polynomial \( V_L(t) \) is obtained by letting \( A = t^{-1/4} \).

Example 2.1.2 Compute the Jones polynomial of the left-hand trefoil.

Step 1: Compute the bracket polynomial of the left-handed trefoil as shown in Figure 2.6.
Figure 2.6 Bracket Polynomial of Left-Handed Trefoil

\[ \langle L \rangle = A((-A^3)(-A^3)) + A^{-1}(-A^4 - A^{-4}) \]
\[ = A^7 - A^3 - A^{-5} \]

Step 2: Compute the Kauffman polynomial of the left-handed trefoil. We computed the writhe to be -3 in the first chapter.

\[ X_{(L)} = (-A^3)^{-3} (A^7 - A^3 - A^{-5}) \]
\[ = -A^{16} + A^{12} + A^4 \]
Step 3: Compute the Jones polynomial by replacing each $A$ with $t^{-1/4}$.

$$V_L(t) = -(t^{-1/4})^{16} + (t^{-1/4})^{12} + (t^{-1/4})^{4}$$

$$= -t^{-4} + t^{-3} + t^{-1}$$

Example 2.1.3 Compute the Jones polynomial of the $5_2$ knot as shown in Figure 2.7.

![Figure 2.7](image.png)

Figure 2.7 $5_2$ Knot

Step 1: Compute the bracket polynomial of the $5_2$ knot. To simplify our bracket polynomial for the $5_2$ knot, let us first compute the bracket polynomial for the right-handed trefoil as shown in Figure 2.8. We will see that on the third step of the decomposition of the $5_2$ knot, we get the right-handed trefoil as noted by an asterisk.
\[ \langle L \rangle = A(-A^4 - A^{-4}) + A^{-2}((-A^{-3})(-A^{-3})) \]
\[ = -A^5 - A^{-3} + A^{-7} \]

Let us now compute the bracket polynomial of the 5_2 knot as shown in Figure 2.9.
Figure 2.9 Bracket Polynomial of $5_2$ Knot

Note: * = bracket polynomial for right-hand trefoil.

\[
\langle L \rangle = A((-A^3)(-A^5-A^{-3}+A^{-7})) + A^{-1}(A(-A^5-A^{-3}+A^{-7}) + A^{-1}(-A^{-9}))
\]
\[
= -A^{-11} + A^9 + A^{-7} - A^5 - 2A^{-3} + A
\]

Step 2: Calculate the Kauffman polynomial of the $5_2$ knot.

The writhe of the knot is shown in Figure 2.10.
Figure 2.10 Writhe of the $5_2$ Knot

\[ X(L) = (-A^3)^{-5} (-A^{-11} + A^9 + A^{-7} - A^5 - 2A^{-3} + A) \]

\[ = A^4 - A^{24} - A^8 + A^{20} + 2A^{12} - A^{16} \]

Step 3: Compute the Jones polynomial by replacing each $A$ with $t^{-1/4}$.

\[ V_L(t) = -t^{-6} + t^{-5} - t^{-4} + 2t^{-3} - t^{-2} + t^{-1} \]

**Tutte Polynomial**

In the preceding section, we computed the bracket polynomial of the link projection using the method created by Louis Kauffman. In this section, we will compute the bracket polynomial through a process found by Morwen B. Thistlethwaite [4]. We convert the projection of the knot into a graph and compute the Tutte polynomial of the graph. We will define the Tutte polynomial as the sum of the
weights of all its spanning trees given that we have a connected graph.

Let us now discuss some common definitions in graph theory that are necessary to compute the Tutte polynomial. Definition 2.2.1 A graph, $G = (V,E)$, consists of a finite set of vertices, $V = \{v_0,v_1,\ldots,v_n\}$, and a finite set of edges, $E = \{e_1,e_2,\ldots,e_k\}$, joining different pairs of distinct vertices. Example 2.2.1 Figure 2.11 shows a graph of $G$ with six vertices and eleven edges.

![Figure 2.11 Graph of G](image)

A spanning subgraph of graph $G$ is a subgraph which contains all vertices of $G$. Figure 2.12 shows several spanning subgraphs of graph $G$. 
A path in a graph is a sequence of distinct vertices \((v_1,v_2,\ldots,v_n)\) such that consecutive vertices are joined by an edge. In a path, the first vertex is called the start vertex and the last vertex is called the end vertex. A cycle in a graph is a path, such that the starting vertex is the same as the ending vertex. Unlike a path in a graph, any vertex in a cycle can be the start vertex. We call a graph acyclic if it contains no cycle. A graph is connected if given vertices \(v_1\) and \(v_n\), there is a path from \(v_1\) to \(v_n\). A spanning tree denoted \(T\) is a spanning subgraph of a graph \(G\) that is connected and acyclic.

Let \(T\) be a spanning tree of \(G\) and \(e\) be an edge of \(T\), then \(T-e\) is partitioned into two components. The cut\((T,e)\) of a graph are all edges in \(G-T\) that connect the two pieces of \(T-e\). Let us now define internally active. Arbitrarily label the edges of \(G\) \((e_1,e_2,\ldots,e_n)\). An edge \(e_j\) is said to be greater that \(e_i\) if \(i < j\). An edge \(e_i\) is internally active if

Figure 2.12 Spanning Subgraphs of \(G\)
for all $e_j \in \text{cut}(T,e_i)$, $e_j$ is greater than $e_i$. Otherwise, $e_i$ is internally inactive.

Let $T$ be a spanning tree of $G$ and $e$ be an edge of $G-T$, then $T \cup e$ contains a single cycle. The $\text{cyc}(T,e)$ of a graph is all edges in $T$ that forms a cycle with $e$. Let us now define externally active. An edge $e_i \in G-T$ is externally active if for all $e_j \in \text{cyc}(T,e_i)$, $e_j$ is greater that $e_i$. Otherwise, $e_i$ is externally inactive.

We now have sufficient knowledge in graph theory to compute the Tutte polynomial. In the beginning of the section, it was stated that we needed to convert the projection of the knot into a graph. We accomplish this by first shading the regions of the knot, but in such a way that no shaded regions are adjacent. This is called the checkerboard shading. We place a vertex inside all the shaded regions, and replace the crossings in our link diagram with edges. Figure 2.13 shows two different ways we can shade the left-handed trefoil, thus obtaining two graphs.

We also label the edges with $\pm 1$ using the rules shown in Figure 2.14.
Figure 2.13 Graphs of Left-Handed Trefoil

Figure 2.14 Rules for Signs for the Edges

The two graphs of the left-handed trefoil from Figure 2.13 with their associated signs are shown in Figure 2.15.
We have established that every edge $e_j$ in G can:

i. be active or inactive (internally or externally)

ii. be in the spanning tree of $T_i$ or not in $T_i$

iii. carry a sign of +1 or -1

Thus given a spanning tree $T_i$ of G, every edge $e_j$ of G has eight possible states. Thistlethwaite assigns a monomial, $u_{ij}$, for the eight possible states as shown in Table 2.1

Table 2.1 Value of $e_j$

<table>
<thead>
<tr>
<th>Label of $e_j$</th>
<th>Activity of $e_j$</th>
<th>Sign of $e_j$</th>
<th>$u_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>internally active</td>
<td>+1</td>
<td>$-A^{-3}$</td>
</tr>
<tr>
<td>D</td>
<td>internally inactive</td>
<td>+1</td>
<td>A</td>
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<tr>
<td>l</td>
<td>externally active</td>
<td>+1</td>
<td>$-A^3$</td>
</tr>
<tr>
<td>d</td>
<td>externally inactive</td>
<td>+1</td>
<td>$A^{-1}$</td>
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<tr>
<td>L</td>
<td>internally active</td>
<td>-1</td>
<td>$-A^3$</td>
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<td>D</td>
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<td>$A^{-1}$</td>
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<tr>
<td>l</td>
<td>externally active</td>
<td>-1</td>
<td>$-A^{-3}$</td>
</tr>
<tr>
<td>d</td>
<td>externally inactive</td>
<td>-1</td>
<td>A</td>
</tr>
</tbody>
</table>
Example 2.2.2 In this example we describe how to use the graph of the knot and Thistlethwaite’s monomial, $u_{ij}$, to find the bracket polynomial of the figure-eight knot. Its graph is shown in Figure 2.16.

![Graph of the Figure-Eight Knot](image)

Figure 2.16 Graph of the Figure-Eight Knot

Let us number the edges of the graph of the figure-eight knot as shown in Figure 2.17.

![Numbered graph of the Figure-Eight Knot](image)

Figure 2.17 Numbered graph of the Figure-Eight Knot
We have five spanning trees \( T_i, 1 \leq i \leq 5 \). Given a spanning tree \( T_i \), we compute its activities. We then use Table 2.1 to find the corresponding monomial, \( u_{ij} \), for each activity. Let the product of \( u_{ij} \) be the weight of \( T_i \). We repeat this process for all the spanning trees. The sum of all the weights of \( T_i \) is the bracket polynomial of our knot.

\( T_1 \)

\[
\begin{align*}
\text{cut}(T_1, 1) &= \{2\} \quad \text{internally active (L)} \\
\text{cut}(T_1, 3) &= \{2, 4\} \quad \text{internally inactive (D)} \\
\text{cyc}(T_1, 2) &= \{1, 3\} \quad \text{externally inactive (d)} \\
\text{cyc}(T_1, 4) &= \{3\} \quad \text{externally inactive (d)} \\
LDdd &= (-A^{-3})(A)(A^{-1})(A^{-1}) = -A^{-4}
\end{align*}
\]

\( T_2 \)

\[
\begin{align*}
\text{cut}(T_2, 1) &= \{2\} \quad \text{internally active (L)} \\
\text{cut}(T_2, 4) &= \{2, 3\} \quad \text{internally inactive (D)}
\end{align*}
\]
cyc(T_2, 2) = \{1, 4\} \quad \text{externally inactive (d)}
cyc(T_2, 3) = \{4\} \quad \text{externally active (l)}
LDdl = (-A^{-3}) (A)(A^{-1})(-A^2) = 1

T_3

\begin{align*}
\text{cut}(T_3, 2) &= \{1\} \quad \text{internally inactive (D)} \\
\text{cut}(T_3, 3) &= \{1, 4\} \quad \text{internally inactive (D)} \\
cyc(T_3, 1) &= \{2, 3\} \quad \text{externally active (l)} \\
cyc(T_3, 4) &= \{3\} \quad \text{externally inactive (d)} \\
\text{DDld} &= (A)(A) (-A^{-3})(A^{-1}) = -A^4
\end{align*}

T_4

\begin{align*}
\text{cut}(T_4, 2) &= \{1, 3\} \quad \text{internally inactive (D)} \\
\text{cut}(T_4, 4) &= \{1, 3\} \quad \text{internally inactive (D)} \\
cyc(T_4, 1) &= \{2, 4\} \quad \text{externally active (l)} \\
cyc(T_4, 3) &= \{4\} \quad \text{externally active (l)} \\
\text{DDll} &= (A)(A)(-A^3)(-A^3) = A^8
\end{align*}
\( T_5 \)

\[
\begin{array}{c}
1 \\
\downarrow \\
2
\end{array}
\]

\( \text{cut}(T_5,1) = \{3,4\} \quad \text{internally active (L)} \)

\( \text{cut}(T_5,2) = \{3,4\} \quad \text{internally active (L)} \)

\( \text{cyc}(T_5,3) = \{1,2\} \quad \text{externally inactive (d)} \)

\( \text{cyc}(T_5,4) = \{1,2\} \quad \text{externally inactive (d)} \)

\( \text{LLdd} = (-A^{-3})(-A^{-3})(A^{-1})(A^{-1}) = A^{-8} \)

Theorem 2.2.1 (Thistlethwaite [4]) Given a knot with its corresponding graph \( G \), the bracket polynomial is the sum of the weights of all its spanning trees, \( \sum_{T \in \mathcal{G}} (\prod_{e \in G} \mu_e) = \langle L \rangle \).

Thus, the bracket polynomial of the figure-eight knot is \( \langle L \rangle = \sum_{T \in \mathcal{G}} (\prod_{e \in G} \mu_e) = -A^{-4} - A^4 + 1 + A^8 + A^{-8} \).

Now that we have our bracket polynomial, we follow the same steps from the preceding section to obtain the Kauffman polynomial and the Jones polynomial. The writhe of the figure-eight knot is zero.

\( X_{(L)} = (-A^3)^0 (-A^{-4} - A^4 + 1 + A^8 + A^{-8}) \)

\( = -A^{-4} - A^4 + 1 + A^8 + A^{-8} \)

\( V_{(L)}(t) = -t^1 - t^{-1} + 1 + t^{-2} + t^2 \)
Notice that the Jones polynomial of the figure-eight knot is palindromic or symmetric. We will take a closer look at this when we discuss mirror image of knots in the next chapter.
CHAPTER THREE

PROPERTIES OF THE JONES POLYNOMIAL

Span of the Jones Polynomial

Definition 3.1.1 The span of the polynomial is the difference between the highest power of the polynomial and the lowest power of the polynomial.

Example 3.1.1 The Jones polynomial of the left-handed trefoil is

\[ V(t) = -t^{-4} + t^{-3} + t^{-1} \]

Thus the span of the left-handed trefoil is:

\[ \text{span} = -1 - (-4) = 3 \]

An alternating projection is an oriented projection of a knot in which the overpass and underpass alternate between crossings. A projection of a knot is said to be reduced if there exists no nugatory crossings. Thus, there exists neither Type I nor Type II move to reduce the number of crossings.

Theorem 3.1.1 Given a reduced alternating projection of a knot of \( n \) crossings, the span of the Jones polynomial is \( n \).

We will need another alternative method on how to compute the bracket polynomial of a link projection as well.
as several lemmas to prove Theorem 3.1.1. We borrow ideas used by Kauffman for our new method.

Given any crossings of a link projection, we can split the crossings into two ways. We call these splits the A-split and the B-split as shown in Figure 3.1.

![Figure 3.1 A-split and B-split](image_url)

Given a link projection $L$ of $n$ crossings, by Rule 3 of the Kauffman bracket polynomial, we can compute the bracket polynomial of $L$ using the bracket polynomials for two links $L_1$ and $L_2$. We now have a pair of links with one fewer crossing than $L$. We perform the similar steps to compute the bracket polynomials for $L_1$ and $L_2$. Thus the bracket polynomial of $L$ now depends on four links with two fewer crossings than $L$. We continue this process until we have links with no crossings. If we have a link projection $L$ of $n$ crossings, how many total links should we have at the end? Since we have two choices for every crossing, A-split or B-split, we will have a total of $2^n$ links.
Let us call a choice of splitting all the n crossings of L a state. Suppose L' is a particular state of L. L' has no crossings, thus it's made up of nonoverlapping unknotted loops. Let $|S|$ be the number of loops of L', then by Rule 3 of the Kauffman bracket polynomial, the bracket polynomial of L' is $(-A^2-A'^{-2})^{[S]}$.

We still need to consider what factor $(-A^2-A'^{-2})^{[S]}$ is multiplied by when we add it into the bracket polynomial of L. Let's recall that at each crossing we have a choice of either an A-split or B-split. The A-split contributes a factor of $A$, while the B-split contributes a factor of $A'^{-1}$ to the bracket polynomial L. Hence, $(-A^2-A'^{-2})^{[S]}$ is multiplied by $A^{a(S)}A^{-b(S)}$, where $a(S)$ and $b(S)$ are the number of A-splits and B-splits respectively. Our new method for computing the bracket polynomial is the sum over all states of these contributions written

$$\langle L \rangle = \sum_S A^{a(S)}A^{-b(S)}(-A^2-A'^{-2})^{[S]}$$

Example: Two states of the left-handed trefoil are shown in Figure 3.2. (a) has an all A-split, while (b) has an all B-split.
Figure 3.2 Two States of the Left-Handed Trefoil

Note: Given two states of a link projection with \( n \) crossings, one state having an all A-splits, and the other state having an all B-splits, the sum of the loops \( |S| \) of these two states is two more than the crossing. Let us prove this using the fact that the Euler characteristic of a disk is 1. Our Euler characteristic is defined as

\[
v - e + r = 1
\]

where \( v \) is the number vertices, \( e \) is the number of edges, and \( r \) is the number of regions. We project our knot onto an \( \mathbb{R}^2 \) plane, thus the crossings become the vertices, \( v \). The number of vertices \( v \) is equal to the number of crossings \( n \) of our knot. We need to find out what \( e \) is equal to. Let us first define the degree of a vertex. The degree of a vertex is the number of edges incident to the vertex. Let us use a well known fact in Graph Theory to find \( e \).
Lemma 3.1.1 In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

Our knot graph has sum of degree 4n, thus by Lemma 3.1.1

\[ 4n = 2e \]

\[ 2n = e \]

So far, we know that the number of vertices is n, and the number of edges is 2n. We need to find how many regions we will have. Since we are projecting our knot graph onto a plane, we have one less region. We have \( r-1 \) regions. Thus our Euler characteristic is

\[ v - e + r = 1 \]

\[ n - 2n + (r-1) = 1 \]

\[ r = n + 2 \]

We will use this fact on Lemma 3.1.2.

Lemma 3.1.2 [1] If K has a reduced alternating projection of n crossings, then the span of the bracket polynomial of K is 4n.

Proof

Given a reduced alternating projection, we know that the span is the difference between the highest degree and the lowest degree of the polynomial. If we expand our new bracket polynomial of \( A^{(s)} a(-A^2) |s|^{-1} \), the highest degree of A occurs in \( A^{(s)} a(-A^2) |s|^{-1} \). We want to find the
highest value for $a(S) - b(S) + 2(|S|-1)$. To find the maximum value, we want the least amount of B-split. We let all the splits of the link projection of $n$ crossings be an A-split, so $a(S) = n$, and $b(S) = 0$. Let us make the number of loops $|S|$ be equal to $J$ in this state. So our highest degree is

$$a(S) - b(S) + 2(|S|-1) = n + 2(J - 1)$$

Similarly, to find the lowest degree of our polynomial, we want to find the lowest value for $a(S) - b(S) - 2(|S|-1)$. We let all the splits of the link projection of $n$ crossings be a B-split, so $a(S) = 0$ and $b(S) = n$. Let us make the number of loops $|S|$ be equal to $K$ in this state. So our lowest degree is

$$a(S) - b(S) - 2(|S|-1) = -n - 2(K - 1)$$

The span of the bracket polynomial is

$$n + 2(J - 1) - (-n - 2(K - 1))$$

$$= n + 2(J - 1) + n + 2(K - 1)$$

$$= 2n + 2(J + K - 2)$$

$$= 2n + 2((n + 2) - 2)$$

$$= 4n$$

We replaced $J + K$ with $n + 2$ as we earlier proved that the sum of the loops of the two states of an all A-splits and B-splits is two more than the number of crossings.
Lemma 3.1.3 If K has a reduced alternating projection of n crossings, then the span of the Kauffman polynomial is equal to 4n.

Proof

We want to find the highest and lowest power in the Kauffman polynomial. Recall that in Kauffman polynomial, we merely adds \(-3w(L)\) to each exponents of the bracket polynomial. Since we are adding \(-3w(L)\) to each exponents of the bracket polynomial, the highest and lowest degree in the bracket polynomial remain the highest and lowest degree in the Kauffman polynomial with \(-3w(L)\) added to both. Thus the span of the Kauffman polynomial is

\[
-3w(L) + n + 2(J - 1) - (-3w(L) - n - 2(K-1))
= -3w(L) + n + 2(J - 1) + 3w(L) + n + 2(K - 1)
= 2n + 2(J + K - 2)
= 2n + 2((n + 2) - 2)
= 4n
\]

We can now easily prove Theorem 3.1.1.

Proof (Theorem 3.1.1)

We want to find the highest and lowest power in our Jones polynomial. Each exponents of the Kauffman polynomial is multiplied by \(-1/4\). Since we are multiplying by a negative, the highest and lowest power in the Kauffman polynomial is
polynomial interchange, then we divide by 4. Thus the span of the Jones polynomial is

\[-(-3w(L) - n - 2(K - 1))/4 - (-(-3w(L) + n + 2(J - 1)))/4\]

\[= 3w(L) + n + 2(K-1)/4 - 3w(L) + n + 2(J-1)/4\]

\[= (2n + 2(n + 2 - 2))/4\]

\[= 4n/4\]

\[= n \square\]

Mirror Image

If we reflect the projection of link L onto a mirror, we obtain its mirror image L*. Another way to think of the mirror image of link L is to change all its crossings. If we reverse all the crossings of the left-handed trefoil, we obtain its mirror image of the right-handed trefoil shown in Figure 3.3.

![Mirror Image of Left-Handed Trefoil](image)

Figure 3.3 Mirror Image of Left-Handed Trefoil
Let us compute the bracket polynomials of the left-handed trefoil and the right-handed trefoil using the method of Tutte polynomial. The two graphs for the left-handed trefoil and right-handed trefoil are shown in Figure 3.4.

![Figure 3.4 Graphs of the Left-Handed Trefoil and the Right-Handed Trefoil](image)

Notice that the graphs for both trefoils are the same, therefore, the number of spanning trees and their activities will be the same for both graphs. The signs of the edges, however, differ. We obtain a $-$ sign for all the edges of the left-handed trefoil, while we obtain a $+$ sign for all the edges of the right-handed trefoil from the graph of Figure 3.4. Figure 3.5 illustrates this. We observe that if we change each crossing, it results to also changing the signs of the edges.
Figure 3.5 Signs of the Edges of the Left-Handed Trefoil and the Right-Handed Trefoil

When we compute the bracket polynomial of the right-handed trefoil, we get the following states for its spanning trees

$$LLd + LdD + Ldd$$

From Table 2.1, these equals to

$$(-A^{-3})(-A^{-3})(A^{-1}) + (-A^{-3})(A^{-1})(A) + (-A^{3})(A)(A)$$

$$= A^{-7} - A^{-3} - A^{5}$$

Since the edges of the left-handed trefoil have - signs, we get the following states for its spanning trees

$$LLd + LdD + Ldd$$

From our table, these equals to

$$(-A^{3})(-A^{3})(A) + (-A^{3})(A)(A^{-1}) + (-A^{-3})(A^{-1})(A^{-1})$$

$$= A^{7} - A^{3} - A^{-5}$$

We make the following observations that the monomial, $u_{ij}$, of the edges with the same activity of opposite signs is
\[ L = L(A^{-1}), \cdot = D(A^{-1}), \perp = l(A^{-1}) \] and \[ d = d(A^{-1}), \] thus we obtain the following lemma.

Lemma 3.2.1 The bracket polynomial of the mirror image \( L^* \) of a link diagram \( L \) is

\[ \langle L^* \rangle (A) = \langle L \rangle (A^{-1}) \]

Proof

Let \( L^* \) be the mirror image of a link diagram \( L \). The graphs for \( L \) and \( L^* \) are the same, thus the number of spanning trees, and the activities for each spanning trees are the same for \( L \) and \( L^* \). Because of the change in crossings, the corresponding edges of \( L \) and \( L^* \) have opposite signs. Therefore,

\[ \langle L^* \rangle (A) = \langle L \rangle (A^{-1}) \]

Lemma 3.2.2 The Kauffman polynomial of the mirror image \( L^* \) of a link diagram \( L \) is

\[ X_{(L^*)}[A] = X_{(L)}[A^{-1}] \]

Proof

Let \( L^* \) be the mirror image of \( L \). Let's consider the writhe of \( L \) and \( L^* \). Let \( w(L) \) be the writhe of \( L \), then \( w(L^*) = -w(L) \). We obtain the following Kauffman polynomial for \( L^* \).

\[ X_{(L^*)}(A) = (-A^3)^{-w(L^*)}\langle L^* \rangle \]
Since \( w(L^*) = -w(L) \) and from Lemma 3.2.1

\[
\langle L^* \rangle (A) = \langle L \rangle (A^{-1})
\]

\[
X_{(L^*)} (A) = (-A^3)^{-w(L^*)} \langle L^* \rangle
\]

\[
= (-A^3)^{-(-w(L))} \langle L \rangle (A^{-1})
\]

\[
= (-A^3)^{w(L)} \langle L \rangle (A^{-1})
\]

\[
= -(A^{-1})^3 \cdot w(L) \langle L \rangle (A^{-1})
\]

\[
= \{ (-A^3)^{-w(L)} \langle L \rangle \} [A^{-1}]
\]

\[
= X_{(L)} [A^{-1}] \quad \square
\]

Theorem 3.2.1 The Jones polynomial of the mirror image \( L^* \) of a link diagram \( L \) is

\[
V_{(L^*)} (t) = V_{(L)} (t^{-1})
\]

Proof

Let \( L^* \) be the mirror image of \( L \). From Lemma 3.2.2, we showed that \( X_{(L^*)} [A] = X_{(L)} [A^{-1}] \). By definition of the Jones polynomial, we replace each \( A \) in the Kauffman polynomial with \( t^{-1/4} \). Therefore, it immediately follows

\[
V_{(L^*)} (t) = V_{(L)} (t^{-1}) \quad \square
\]

Composition of Knots

Definition 3.3.1 Given two knots \( L_1 \) and \( L_2 \), we can obtain a new knot by removing an arc on both knots and connecting
their four endpoints. The new knot is called the composition of knots \( L_1 \) and \( L_2 \).

We also refer to composition of knots \( L_1 \) and \( L_2 \) as connected sum of knots \( L_1 \) and \( L_2 \) denoted \( L_1 \# L_2 \). \( L_1 \) and \( L_2 \) are called the factor knots of the connected sum. We remove two arcs on the outside of both \( L_1 \) and \( L_2 \) since we do not want any unwanted crossings. If \( L_1 \) and \( L_2 \) have \( n \) and \( m \) crossings respectively, then \( L_1 \# L_2 \) will have \( n+m \) crossings.

Note: If we compose any knots with the unknot (trivial knot), it yields the same knot.

Figure 3.6 shows a composition of two left-handed trefoils. The knot \( 3_1 \# 3_1 \) is also called the granny knot.

![Figure 3.6 3_1 \# 3_1 or the Granny Knot](image)

Lemma 3.3.1 Given a connected sum \( L_1 \# L_2 \), its bracket polynomial is \( \langle L_1 \# L_2 \rangle = \langle L_1 \rangle \langle L_2 \rangle \).
Proof

Let \( L_1 \) and \( L_2 \) be the factor knots of the oriented connected sum, and let

\[
\langle L_1 \rangle = \sum_{s_1} A^{a(s_1)} b(s_1) (-A^2 - A^{-2}) |s_1|^{-1}
\]

and

\[
\langle L_2 \rangle = \sum_{s_2} A^{a(s_2)} b(s_2) (-A^2 - A^{-2}) |s_2|^{-1}
\]

be the bracket polynomials of \( L_1 \) and \( L_2 \). There exists a 1-1 correspondence between states in \( L_1 \# L_2 \) and separate states of \( L_1 \) and \( L_2 \). Thus the bracket polynomial of \( L_1 \# L_2 \), are all the states of \( L_1 \) multiplied with all the states of \( L_2 \).

\[
\langle L_1 \# L_2 \rangle = \sum_{s_1} \left( \sum_{s_2} A^{a(s_1)} A^{a(s_2)} A^{b(s_1)} b(s_1) (-A^2 - A^{-2}) |s_1|^{-1} \right)
\]

\[
(-A^2 - A^{-2}) |s_2|^{-1}
\]

\[
= \sum_{s_1} \left( \sum_{s_2} A^{a(s_1)+a(s_2)} A^{b(s_1)+b(s_2)} (-A^2 - A^{-2}) |s_1| |s_2|^{-2} \right)
\]

\[
= \sum_{s_1} \left( A^{a(s_1)+b(s_1)} (-A^2 - A^{-2}) |s_1|^{-1} \sum_{s_2} A^{a(s_2)+b(s_2)} \right)
\]

\[
(-A^2 - A^{-2}) |s_2|^{-1}
\]

\[
= \sum_{s_1} A^{a(s_1)+b(s_1)} (-A^2 - A^{-2}) |s_1|^{-1} \| \langle L_2 \rangle
\]

\[
= \langle L_2 \rangle \sum_{s_1} A^{a(s_1)+b(s_1)} (-A^2 - A^{-2}) |s_1|^{-1}
\]

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Lemma 3.3.2 Given an oriented connected sum $L_1 \# L_2$, its Kauffman polynomial is $X_{L_1 \# L_2} = X_{L_1} X_{L_2}$.

Proof

Let $L_1$ and $L_2$ be the factor knots of the oriented connected sum, and let $w(L_1)$ and $w(L_2)$ be their corresponding writhes, then the writhe of the oriented connected sum is $w(L_1) + w(L_2)$. Applying the results from Lemma 3.3.1, we have

$$X_{L_1 \# L_2} = (-A^3)^{-w(L_1) + w(L_2)} \langle L_1 \rangle \langle L_2 \rangle$$

$$= (-A^3)^{-w(L_1)} \langle L_1 \rangle (-A^3)^{-w(L_2)} \langle L_2 \rangle$$

$$= X_{L_1} X_{L_2} \square$$

Theorem 3.3.1 Given an oriented connected sum $L_1 \# L_2$, its Jones polynomial is $V_{L_1 \# L_2}(t) = V_{L_1}(t) V_{L_2}(t)$.

Proof

It immediately follows from Lemma 3.3.1 and Lemma 3.3.2, and the definition of the Jones polynomial that

$$V_{L_1 \# L_2}(t) = V_{L_1}(t) V_{L_2}(t) \square$$
In this chapter, we will discuss mutant knots. These are knots obtained by cutting a 2-strand tangle from the knot projection $K_1$, rotating that tangle 180 degrees, and fusing the tangle back to the knot projection, thus obtaining a new knot $K_2$. We will show how these two knots have the same Jones polynomial.

**Tangles**

John H. Conway used tangles as the building blocks of knots. He developed what is called the Conway notation to describe tangles.

**Definition 4.1.1** A tangle is a circular region in a knot or link projection in which the knot or link projection crosses the circle exactly four times.

An example of a tangle is shown in Figure 4.1 with the ends coming out of the circle labeled a, b, c, and d.

Let's call these ends a, b, c, and d as NW, NE, SW, and SE respectively. The two basic tangles are the $\infty$-tangle and the $0$-tangle. The $\infty$-tangle is made up of two vertical
strands and the 0 tangle is made up of two horizontal strands as shown in Figure 4.2.

![Tangle Diagram]

Figure 4.1 Example of a Tangle

![Tangle Diagrams]

Figure 4.2 a) \(\infty\) Tangle b) 0 Tangle

Two tangles are said to be equivalent if there exist an ambient isotopy (Reidemeister moves) from one to the
other. Under the ambient isotopy the four ends must remain fixed. Let's construct a tangle. Let's start with the $\infty$ tangle and wind it with 3 negative twists. The slope of the overpass is negative under negative twists. Conversely, the slope of the overpass is positive under positive twists. Next, choose the NE and SE ends to construct 3 positive twists. Lastly, choose the SW and SE ends to construct 3 negative twists. Figure 4.3 illustrates these steps.

Figure 4.3 Constructing a Tangle
The tangle constructed in Figure 4.3 is denoted \( -3 \ 3 \ -3 \). If we trace back our steps, we can unwind our tangle back to the \( \infty \) tangle. Given a 0 tangle or a \( \infty \) tangle, we choose two endpoints and twist them, and then we choose another pair of endpoints and twist them, and so on for a finite number of times. A tangle that is constructed in this manner is called a rational tangle. If we close the ends of the rational tangle in Figure 4.3, it becomes a rational link. When we talk about mutant knots in the next section, we can not use rational tangle to form mutant knots by Cromwell’s Theorem [3].

Theorem 4.1.1 Rational tangles cannot be used to form mutants.

Proof

The tangle is described to be in pillowcase form when the arcs of a rational tangle are made to lie on its boundary. Under the pillowcase form, the rational tangle exhibits a lot of symmetry. Thus, a rotation by 180 degrees would produce the same tangle. Therefore, it results to the same link. \( \Box \)
Mutant Knots

In this section, we will show how different knots (mutant knots) possess the same Jones polynomial.

Definition 4.2.1 Let $K_1$ be an oriented knot diagram with tangle $t$. Cut out tangle $t$, rotate it 180 degrees, and glue it back to form the $K_2$. If the resulting knot $K_2$ is different from $K_1$, then they are called mutants of each other.

Figure 4.4 Tangle of a Link Projection

After cutting out the tangle (Figure 4.4) from the oriented knot diagram, we can rotate the tangle in three different ways obtain the mutant knot. One way is to flip the tangle from north to south. Another way is to flip the tangle from east to west. Finally, we can rotate the tangle 180 degrees about z-axis. All three are shown in Figure 4.5.
Figure 4.5 Rotation of the Tangle About the Three Principal Axes

Example 4.2.1 One of the more well-known pairs of mutant knots are the Kinoshita-Terasaka (K-T) knot and the Conway knot with the marked tangle shown in Figure 4.6. A marked tangle is the tangle utilized to form mutant knots. One knot can be obtained from the other by rotating its marked tangle by 180 degrees about z-axis. Note that both knots share the same Jones polynomial.

\[ V_L(t) = -t^{-4} + 2t^{-3} - 2t^{-2} + 2t^{-1} + t^2 - 2t^3 + 2t^4 - 2t^5 + t^6 \]
Figure 4.6 a) Conway Knot b) K-T Knot

Theorem 4.2.1 Given oriented mutant knots $K_1$ and $K_2$, their Jones polynomial are equal.

We will prove the theorem through two lemmas. The first lemma proving that the bracket polynomials of $K_1$ and $K_2$ are equal. The second lemma proving that their Kauffman polynomials are equal.

Let’s examine the bracket polynomial of both the Conway and Kinoshita-Terasaka knot (K-T). The knots are the same everywhere but the marked tangle. Let’s show that the marked tangle for both knots have the same bracket polynomial. Each tangle has five crossings; therefore, we will have $2^5 = 32$ states. Let’s examine some states. Let’s split the crossings of the Conway’s tangle and call this
state $S_1$. Using the same splits from $S_1$, split the corresponding crossings of the K-T's tangle and call this $S_1'$. Following the same process, let's construct $S_2$ and $S_2'$. $S_1$ and $S_1'$ are shown in Figure 4.7, and $S_2$ and $S_2'$ are shown in Figure 4.8. Notice that if we rotate $S_1$ by 180 degrees about the z-axis, we obtain $S_1'$. The endpoints remain fixed under the rotation, and $S_1$ and $S_1'$ contain the same number of components. Therefore, $S_1$ and $S_1'$ contribute the same amount to the bracket polynomial. The same holds true for states $S_2$ and $S_2'$. If we construct the rest of the states, we would see that the bracket polynomials of the mutants are the same. In general, any state of the tangle results in some number of components together with either $\infty$ tangle or 0 tangle. Under mutation, $\infty$ tangle and 0 tangle are invariant under mutation. We formulize this observation with a lemma.
Figure 4.7 States $S_1$ and $S_1'$
Lemma 4.2.1 Given mutant knots $K_1$ and $K_2$, their bracket polynomials are equal.

Proof

We only need to show that the bracket polynomials of their marked tangles are equal and connected endpoints are the same. The states of the tangles of $K_1$ and $K_2$ can be in the following two forms accompanied by its corresponding number of components (Figure 4.9). The tangles of $K_1$ and $K_2$
have the same number of states. Given a state for the
tangle of $K_1$, the corresponding state for the tangle of $K_2$
can be obtained by rotating it about the appropriate axis
with the $\infty$ tangle and 0 tangle invariant under the
rotation. Thus, each state from the tangle of $K_1$ has a
matching state from the tangle of $K_2$ which contributes the
same amount on the bracket polynomial. Therefore, the
marked tangles of mutant knots $K_1$ and $K_2$ have equal bracket
polynomials. Since they are the same everywhere else, the
bracket polynomials of mutant knots $K_1$ and $K_2$ are equal. □

![Figure 4.9 Two Possible States](image)

Lemma 4.2.2 Given oriented mutant knots $K_1$ and $K_2$, their
Kauffman polynomials are equal.

Proof

Lemma 4.2.1 showed that mutant knots have equal
bracket polynomials. We only need to show that the writhe
is preserved under rotation by preserving its orientation.
Consider an oriented tangle shown in Figure 4.10a. One of its strands comes in the tangle at point A and exits at point B. The other strand comes in the tangle at point C and exits at point D. We have three principal axes of rotation. We need to show how to preserve the orientation for all three rotations. If we flipped the tangle from North to South, the orientation is preserved (Figure 4.10b). If we flipped the tangle from East to West, the order of entry and exit needs to be reverse to preserve orientation. One of the strands now enters at point D and exits at point C. The other strand enters at point B and exits at point A (Figure 4.10c). The change in orientation does not change the sign of the crossings in the tangle. Similarly, if we rotate the tangle by 180 degrees about the z-axis, we need to reverse the order of entry and exit to preserve orientation. One of the strands enters at point B and exits at point A. The other strand enters at point D and exits at point C (Figure 4.10d). Again, the change in orientation does not change the sign of the crossings in the tangle. Now, orientations are preserved under all three principal axes of rotation, thus preserving the writhe.
We can now easily prove the Theorem 4.2.1.

By Lemma 4.2.1 and 4.2.2, and by the definition of the Jones polynomial, it immediately follows that mutant knots $K_1$ and $K_2$ have equal Jones polynomial.
Summary

In the first chapter, we discussed some common definitions in Knot Theory. We discussed the way to deform our knot using three types of Reidemeister moves. In the second chapter, we defined the Jones polynomial into two ways, as the Kauffman Bracket polynomial and the Tutte polynomial.

In the third chapter, we discussed some properties of the Jones polynomial. We saw that given an alternating knot with n crossings, the span of the knot’s Jones polynomial is n. We also saw the relation between the Jones polynomial of the knot and its mirror image. Finally, we saw that the Jones polynomial of connected sums is the product of the polynomials of its factors.

In the fourth chapter, we saw that mutant knots share the same Jones polynomial.

Knot Theory is fairly new relative to other subjects in mathematics. There are many things yet to be discovered in this area of mathematics. In this paper we explored a knot invariant called the Jones polynomial. One of the
central goals in Knot Theory is to continue to find other knot invariants. Knot Theory challenges us to discover them.
APPENDIX:

JONES POLYNOMIAL
\begin{align*}
3_1 & \quad -t^4 + t^3 + t^{-1} \\
4_1 & \quad t^{-2} - t^{-1} + 1 - t + t^2 \\
5_1 & \quad -t^{-7} + t^{-6} - t^{-5} + t^{-4} + t^{-2} \\
5_2 & \quad -t^{-6} + t^{-5} - t^{-4} + 2t^{-3} - t^{-2} + t - 1 \\
6_1 & \quad t^{-4} - t^{-3} + t^{-2} - 2t^{-1} + 2 - t + t^2 \\
6_2 & \quad t^{-5} - 2t^{-4} + 2t^{-3} - 2t^{-2} + 2t^{-1} - 1 + t \\
6_3 & \quad -t^{-3} + 2t^{-2} - 2t^{-1} + 3 - 2t + 2t^2 - t^3 \\
7_1 & \quad -t^{-10} + t^{-9} - t^{-8} + t^{-7} - t^{-6} + t^{-5} + t^{-3} \\
7_2 & \quad -t^{-8} + t^{-7} - t^{-6} + 2t^{-5} - 2t^{-4} + 2t^{-3} - t^{-2} + t^{-1} \\
7_3 & \quad t^2 - t^3 + 2t^4 - 2t^5 + 3t^6 - 2t^7 + t^8 - t^9 \\
7_4 & \quad t - 2t^2 + 3t^3 - 2t^4 + 3t^5 - 2t^6 + t^7 - t^8 \\
7_5 & \quad -t^{-9} + 2t^{-8} - 3t^{-7} + 3t^{-6} - 3t^{-5} + 3t^{-4} - t^{-3} + t^{-2} \\
7_6 & \quad -t^{-6} + 2t^{-5} - 3t^{-4} + 3t^{-3} - 3t^{-2} + 3t^{-1} - 2 + t \\
7_7 & \quad -t^{-3} + 3t^{-2} - 3t^{-1} + 4 - 4t + 3t^2 - 2t^3 + t^4 \\
8_1 & \quad t^6 - t^{-5} + t^{-4} - 2t^{-3} + 2t^{-2} - 2t^{-1} + 2 - t + t^2 \\
8_2 & \quad t^{-8} - 2t^{-7} + 2t^{-6} - 3t^{-5} + 3t^{-4} - 2t^{-3} + 2t^{-2} - t^{-1} + 1 \\
8_3 & \quad t^{-4} - t^{-3} + 2t^{-2} - 3t^{-1} + 3 - 3t + 2t^2 - t^3 + t^4 \\
8_4 & \quad t^{-3} - t^{-2} + 2t^{-1} - 3 + 3t - 3t^2 + 3t^3 - 2t^4 + t^5 \\
8_5 & \quad 1 - t + 3t^2 - 3t^3 + 3t^4 - 4t^5 + 3t^6 - 2t^7 + t^8 \\
8_6 & \quad t^{-7} - 2t^{-6} + 3t^{-5} - 4t^{-4} + 4t^{-3} - 4t^{-2} + 3t^{-1} - 1 + t \\
8_7 & \quad -t^{-2} + 2t^{-1} - 2 + 4t - 4t^2 + 4t^3 - 3t^4 + 2t^5 - t^6 \\
8_8 & \quad -t^{-3} + 2t^{-2} - 3t^{-1} + 5 - 4t + 4t^2 - 3t^3 + 2t^4 - t^5 \\
8_9 & \quad t^{-4} - 2t^{-3} + 3t^{-2} - 4t^{-1} + 5 - 4t + 3t^2 - 2t^3 + t^4 \\
\end{align*}
\begin{align*}
8_{10} & \quad -t^{-2}+2t^{-1}-3+5t-4t^2+5t^3-4t^4+2t^5-t^6 \\
8_{11} & \quad t^{-7}-2t^{-6}+3t^{-5}+5t^{-4}+5t^{-3}-4t^{-2}+4t^{-1}-2t \\
8_{12} & \quad t^{-4}-2t^{-3}+4t^{-2}-5t^{-1}+5-5t+4t^2-2t^3+t^4 \\
8_{13} & \quad t^{-3}+3t^{-2}-4t^{-1}+5-5t+5t^2-3t^3+2t^4-t^5 \\
8_{14} & \quad t^{-7}-3t^{-6}+4t^{-5}-5t^{-4}+6t^{-3}-5t^{-2}+4t^{-1}-2t \\
8_{15} & \quad t^{-10}-3t^{-9}+4t^{-8}-6t^{-7}+6t^{-6}-5t^{-5}+5t^{-4}-2t^{-3}+t^{-2} \\
9_{1} & \quad t^4+t^6-t^7+t^8-t^9+t^{10}-t^{11}+t_{12}-t^{13} \\
9_{2} & \quad t^4t^2+2t^3-2t^4+2t^5-2t^6+2t^7-t^8+t^9-t^{10} \\
9_{3} & \quad t^3-t^4+2t^5-2t^6+3t^7-3t^8-2t^9+t^{10}-t^{11} \\
9_{4} & \quad t^2-t^3+2t^4-3t^5+4t^6-3t^7+3t^8-2t^9+t^{10}-t^{11} \\
9_{5} & \quad t^2-t^3+3t^4-4t^5+4t^6-3t^7-3t^8+2t^9-t^{10} \\
9_{6} & \quad t^3-t^4+3t^5-4t^6+4t^7-5t^8+4t^9-3t^{10}+2t^{11}-t^{12} \\
9_{7} & \quad t^2-t^3+3t^4-4t^5+5t^6-5t^7+4t^8-3t^9+2t^{10}-t^{11} \\
9_{8} & \quad t^{-3}-2t^{-2}+3t^{-1}-4+5t-5t^2+5t^3-3t^4+2t^5-t^6 \\
9_{9} & \quad t^3-t^4+3t^5-4t^6+5t^7-5t^8+5t^9-4t^{10}+2t^{11}-t^{12} \\
9_{10} & \quad t^2-2t^3+4t^4-5t^5+6t^6-5t^7+5t^8-3t^9+t^{10}-t^{11} \\
9_{11} & \quad 1-2t+3t^2-4t^3+6t^4-5t^5+5t^6-4t^7+2t^8-t^9 \\
9_{12} & \quad t^{-3}+2t^2-5t^3+6t^4+5t^5-3t^6+2t^7-t^8 \\
9_{13} & \quad t^2-2t^3+4t^4-5t^5+7t^6-6t^7+5t^8-4t^9+2t^{10}-t^{11} \\
9_{14} & \quad t^{-3}+3t^{-2}-4t^{-1}+6-6t+6t^2-5t^3+3t^4-2t^5+t^6 \\
9_{15} & \quad t^{-1}-2+4t-6t^2+7t^3-6t^4+6t^5-4t^6+2t^7-t^8 \\
9_{16} & \quad t^3-t^4+4t^5-5t^6+6t^7-7t^8+6t^9-5t^{10}+3t^{11}-t^{12} \\
9_{17} & \quad t^{-3}-2t^{-2}+4t^{-1}+5+6t-7t^2+6t^3-4t^4+3t^5-t^6
\end{align*}
\[ t^2 - 2t^3 + 5t^4 - 6t^5 + 7t^6 - 7t^7 + 6t^8 - 4t^9 + 2t^{10} - t^{11} \]
\[ t^{-4} - 2t^{-3} + 4t^{-2} - 6t^{-1} + 7 - 7t + 6t^2 - 4t^3 + 3t^4 - t^5 \]
\[ 1 - 2t + 4t^2 - 5t^3 + 7t^4 - 7t^5 + 6t^6 - 5t^7 + 3t^8 - t^9 \]
\[ t^{-1} - 3 + 5t - 6t^2 + 8t^3 - 7t^4 + 6t^5 - 4t^6 + 2t^7 - t^8 \]
\[ t^{-3} - 2t^{-2} + 4t^{-1} - 6 + 7t - 7t^2 + 7t^3 - 5t^4 + 3t^5 - t^6 \]
\[ t^2 - 2t^3 + 5t^4 - 6t^5 + 8t^6 - 8t^7 + 6t^8 - 5t^9 + 3t^{10} - t^{11} \]
\[ t^{-4} - 3t^{-3} + 5t^{-2} - 7t^{-1} + 8 - 7t + 7t^2 - 4t^3 + 2t^4 - t^5 \]
\[ t^{-1} - 2 + 5t - 7t^2 + 8t^3 - 8t^4 + 7t^5 - 5t^6 + 3t^7 - t^8 \]
REFERENCES


