Numbers of generators of ideals in local rings and a generalized Pascal's Triangle

Lucia Riderer

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NUMBERS OF GENERATORS OF IDEALS IN LOCAL RINGS
AND A GENERALIZED PASCAL'S TRIANGLE

A Project
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
Of the Requirements for the Degree
Masters of Arts
in
Mathematics

by
Lucia Riderer
September 2005
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Approved by:

James Okon, Committee Chair

Davida Fischman, Committee Member

Laura Wallace, Committee Member

Charles Stanton, Chair
Department of Mathematics

Terry Hallett
Graduate Coordinator
Department of Mathematics
ABSTRACT

In this paper, we will define generalized binomial coefficients. We will show that they can be used to generate generalized Pascal’s Triangles and have properties analogous to binomial coefficient. We will use our generalized binomial coefficients to compute the Dilworth number and the Sperner number of certain rings.
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CHAPTER ONE
INTRODUCTION

Pascal's triangle is one of the oldest and most important tools in mathematics. In the twelfth century, both Persian and Chinese mathematicians were working on a so-called arithmetic triangle that gives the coefficients of the expansion of the algebraic expression \((x + y)^n\) for different integer values of \(n\). Years after it first appeared, the triangle came to be known as Pascal's Triangle with Blaise Pascal's completion of Traité du triangle arithmétique in 1654. Making use of the array of binomial coefficients, the French mathematician developed many of the triangle's properties.

The binomial coefficients have been studied both in the context of binomial expansions and in combinatorics. They are called the binomial coefficients because they are the numbers that multiply terms of the expansion of the binomial \((x + y)^n\), which is a cornerstone of the study of algebra.

The binomial coefficient \(\binom{n}{k}\) (or, alternatively, the number of combinations of \(n\) items taken \(k\) at a time),
provides a well defined recursion derived from the
recurrence we get if we define the binomial coefficient in
terms of Pascal's Triangle: each binomial coefficient that
is not on the boundary of the triangle (that is, $n>k>0$) is
the sum of the two terms immediately above it in the
triangle. This recursion is illustrated in Figure 1.1.

\[
\begin{array}{cccc}
  n=0: & 1 \\
  n=1: & 1 & 1 \\
  n=2: & 1 & 2 & 1 \\
  n=3: & 1 & 3 & 3 & 1 \\
  n=4: & 1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

Figure 1.1. Pascal's Triangle

As mentioned above, the numbers in each row of
Pascal's triangle are precisely the same numbers that are
the coefficients of binomial expansions. For example, when
one expands the binomial, $(x + y)^3$, algebraically the
expression equals $1x^3+3x^2y+3xy^2+y^3$. The coefficients of
this binomial expansion, 1 3 3 1, correspond exactly to the
entries in the third row of Pascal's Triangle. In general,
starting from row 0, the $n^{th}$ row in Pascal’s Triangle gives the coefficients of $(x + y)^n$.

The entries of Pascal’s Triangle can also be used in probability to find out how many subsets of $r$ elements can be formed from a set with $n$ distinct elements. For example, to find out how many ways are there to choose 2 objects out of a collection of 5 distinct objects, one will have to calculate $\binom{5}{2} = 10$. So there are 10 ways. Consequently, if one wishes to find the number of subsets of 2 elements that can be formed from a set with 5 distinct elements, they would just have to look up the number in the second position of the fifth row in Pascal’s Triangle. Therefore, Pascal’s Triangle is a useful tool in finding, without tedious computations, the number of subsets of $r$ elements that can be formed from a set with $n$ distinct elements.

Among some of the most interesting natural numbers are the binomial coefficients. The binomial coefficients are classical combinatorial numbers with diverse interpretations that arise in various areas such as discrete mathematics, combinatorics, algebra, number theory, discrete probability, linear algebra, finite geometry, analysis of algorithms, and numerical analysis.
One area of algebra in which we will have a particular interest is determining the number of generators of ideals in a local ring.

A local ring $R$ is a commutative ring with a unique maximal ideal. If $I$ is an ideal of $R$, we denote by $\mu(I)$ the cardinality of a minimal generating set of $I$. The Dilworth number of $R$, $d(R)$ is $\sup\{\mu(I) | I \text{ is an ideal of } R\}$. If $M$ is the unique maximal ideal of $R$, then the Sperner number of $R$ is defined by $\text{sp}(R) = \sup\{\mu(M^i) | i \geq 0\}$.

In this thesis, we will define generalized binomial coefficients. We will show that they can be used to generate generalized Pascal's Triangles and have properties analogous to those involving binomial coefficients. We will also use generalized binomial coefficients to compute the Dilworth number and the Sperner number of certain rings.

We will give two ways of defining the binomial coefficients. One is the traditional definition given in college algebra and the other one is more combinatorial.

In Chapter 2 we will give some basic properties of the binomial coefficients and some generalizations of the binomial coefficients. We will introduce general binomial
coefficients, give their main properties and show how to construct the generalized version of Pascal’s Triangle.

In Chapter 3, we will introduce group rings, show how they can be represented as quotient rings of polynomial rings, and determine their ideal structure. We will also show how generalized binomial coefficients can be used to give recursive formulas for the minimal number of generators of an ideal for a class of rings.

The original motivation for determining the maximal number of generators of ideals in a ring goes back to linear algebra. Vector spaces and modules are studied in linear algebra. A module is an abelian group together with a scalar multiplication by elements of a ring. If the ring is a field, then the module is called a vector space. Vector spaces can be represented as a direct sum

\[ F \oplus F \oplus F \oplus \ldots \]

of copies of the field. Every module over a principal ideal domain \( \mu(D)=1 \) can be written as a direct sum of cyclic modules.

In 1963, H. Bass showed that for a class of rings \( R \) with \( \mu(R) \leq 2 \), finitely generated torsion free \( R \)-modules are isomorphic to direct sums of ideals. He began by trying to determine when \( \mu(R) \leq 2 \) for group rings over the integers.
Since that time, many papers have been written to determine the Dilworth number and Sperner number of group rings.
CHAPTER TWO

GENERALIZED BINOMIAL COEFFICIENTS

In this section, we begin with a combinatorial definition of the binomial coefficients. In Propositions 2.3 and 2.4 we prove some of their basic properties. We then prove in Lemma 2.5 that binomial coefficients correspond to cardinalities of certain sequences of natural numbers. We then give more general proofs of parts (1), (2), and (3) of Proposition 2.4 in Proposition 2.6.

Definition 2.7 is our first generalization of the binomial coefficients. The properties of these “trinomial coefficients” are given in Theorem 2.11.

Finally we introduce general binomial coefficients in Definition 2.12 and conclude with their basic properties in Theorem 2.15.

Definition 2.1. Let \( n \) be a positive integer and \( k \) a non-negative integer. Then \( \binom{k}{i} \) is the number of ways to place \( i \) indistinguishable objects into \( k \) distinct boxes.
with at most one object in each box. We define \( \binom{0}{i} = \binom{k}{i} = 0 \)

for \( i < 0 \) and \( \binom{0}{0} = 1 \).

The following example illustrates the above definition of the binomial coefficients.

Example 2.2. We illustrate the number of ways to put 2 objects into 4 boxes, with at most one object in each box. Each row represents the set of four boxes.

![Figure 2.1. Two Objects into Four Boxes](image)

It can be easily seen from figure 2.1 that there are six ways to put 2 objects into 4 boxes, with at most one object in each box.
So, \( \binom{4}{2} = 6 \).

Our next proposition shows that the binomial coefficients satisfy the recursion property

\[
\binom{k}{i} = \binom{k-1}{i} + \binom{k-1}{i-1}.
\]

This proposition is especially useful in construction of Pascal's Triangle.

Proposition 2.3. Let \( k \in \mathbb{N} \). Then:

1. \( \binom{0}{i} = \begin{cases} 1 & \text{if } i=0 \\ 0 & \text{otherwise} \end{cases} \)

   If \( k \geq 1 \), then:

2. \( \binom{k}{i} = \binom{k-1}{i} + \binom{k-1}{i-1} = \sum_{j=0}^{i} \binom{k-1}{i-j} \)

Proof.

1. This is immediate from the definition.

2. We proceed by induction on \( k \).

   If \( k=1 \) and \( i<0 \), then \( \binom{1}{i} = 0 \) and \( \binom{0}{i} + \binom{0}{i-1} = 0 \).

   If \( i>1 \) then \( \binom{1}{i} = 0 \) and \( \binom{0}{i} + \binom{0}{i-1} = 0 \).

   If \( i=0 \), then \( \binom{1}{0} = 1 \) and \( \binom{0}{0} + \binom{0}{-1} = 1 \).

   Finally, if \( i=1 \), then \( \binom{1}{1} = 1 \) and \( \binom{0}{1} + \binom{0}{0} = 1 \).
Thus \( \binom{1}{i} = \binom{0}{i} + \binom{0}{i-1} \) for all \( i \).

Now assume \( k > 1 \) and the result holds for smaller values of \( k \). We may put either 0 objects in the \( k \)-th box and \( i \) objects in boxes 1, \( \ldots \), \( k-1 \) in \( \binom{k-1}{i} \) ways, or 1 object in the \( k \)-th box and \( i-1 \) in boxes 1, \( \ldots \), \( k-1 \), in \( \binom{k-1}{i-1} \) ways. So \( \binom{k}{i} = \binom{k-1}{i-1} + \binom{k-1}{i} \). Q.E.D

Some of the main properties of the binomial coefficients are introduced below. In the next proposition and throughout the paper, \([x]\) will denote the greatest integer \( n, n \leq x \). So \([x] = \sup\{n \in \mathbb{Z} | n \leq x\}\).

**Proposition 2.4.** Let \( i \) and \( k \) be two non-negative integers so that \( 0 \leq i \leq k \). Then:

1. \( \binom{k}{i} = \binom{k-1}{i-1} + \binom{k-1}{i} \)
2. \( \binom{k}{i} = \frac{k!}{i!(k-i)!} = \binom{k}{k-i} \)
3. \( \binom{k}{j} \leq \binom{k}{i} \) for \( j \leq i \leq \left[\frac{k}{2}\right] \)
4. \( \binom{k}{j} \geq \binom{k}{i} \) for \( \left[\frac{k}{2}\right] \leq j \leq i \)
(5) \[ \binom{k}{\frac{k+1}{2}} = \binom{k}{\frac{k}{2}} \]

(6) \[ \sum_{i=0}^{k} \binom{k}{i} = 2^k \]

Proof.

(1) Statement (1) is a direct result of Proposition 2.3.

(2) By (1) we have

\[ \binom{k}{i} = \binom{k-1}{i-1} + \binom{k-1}{i} = \frac{(k-1)!}{(i-1)!(k-i)!} + \frac{(k-1)!}{i!(k-1-i)!} \]

However, \( i! = i(i-1)! \) and \( (k-i)! = (k-i)(k-i-1)! \). So, the binomial coefficient becomes:

\[ \binom{k}{i} = \frac{i(k-1)!}{i!(k-i)(k-i-1)!} + \frac{(k-1)!}{i!(k-1-i)!} \]

\[ \binom{k}{i} = \frac{(k-1)!}{i!(k-i-1)!} \left( \frac{i}{k-i} + 1 \right) \]

\[ \binom{k}{i} = \frac{k(k-1)!}{(k-i)!((k-i-1)!} = \frac{k!}{i!(k-i)!} \]

Finally, \( \binom{k}{k-i} = \frac{k!}{(k-(k-i))!(k-i)!} \frac{k!}{(k-i)!} = \frac{k}{i} \).

(3) Using induction, it suffices to show: \( \binom{k}{j-1} \leq \binom{k}{j} \).
for \( j \leq \left\lfloor \frac{k}{2} \right\rfloor \). The case \( k=1 \) is clear.

Assume \( k>1 \) and that the inequality \( \binom{k-1}{j} \leq \binom{k}{i} \) holds for \( j\leq i \leq \left\lfloor \frac{k-1}{2} \right\rfloor \). Now, \( \binom{k}{j-1} \leq \binom{k}{j} \) if and only if

\[
\binom{k-1}{j-2} + \binom{k-1}{j-1} \leq \binom{k-1}{j} + \binom{k-1}{j} \quad \text{if and only if}
\]

\[
\binom{k-1}{j-2} \leq \binom{k-1}{j}.
\]

If \( j < \left\lfloor \frac{k}{2} \right\rfloor = \begin{cases} k/2 & \text{if } k \text{ even} \\ k-1/2 & \text{if } k \text{ odd} \end{cases} \), then \( j-1 < \left\lfloor \frac{k-1}{2} \right\rfloor = \begin{cases} k-2/2 & \text{if } k \text{ even} \\ k-1/2 & \text{if } k \text{ odd} \end{cases} \)

since \( j-1 < \frac{k-1}{2} = \frac{k-2}{2} \leq \left\lfloor \frac{k-1}{2} \right\rfloor \). Thus \( j-2 \leq j \leq \left\lfloor \frac{k-1}{2} \right\rfloor \) and

\[
\binom{k-1}{j-2} \leq \binom{k-1}{j} \quad \text{by the inductive hypothesis.}
\]

If \( j = \left\lfloor \frac{k}{2} \right\rfloor \), then

\[
\binom{k-1}{j} = \binom{k-1}{k-1-j} = \binom{k-1}{k-1 - \left\lfloor \frac{k}{2} \right\rfloor}.
\]

So it suffices to show that \( j-2 \leq k-1-j \text{ or } \left\lfloor \frac{k}{2} \right\rfloor - 2 \leq k - 1 - \left\lfloor \frac{k}{2} \right\rfloor \).
For this, \( \frac{k}{2} \geq k - 1 - \frac{k}{2} \) if and only if \( 2 \frac{k}{2} \leq k + 1 \)

if and only if \( \frac{k}{2} \leq \frac{k + 1}{2} \), which always holds.

(4) To prove that \( \binom{k}{j} \geq \binom{k}{i} \) for \( \frac{k}{2} \leq j \leq i \), we start with

the case of \( k \) even. Then \( k - i \leq k - j \leq \frac{k}{2} = \frac{k}{2} \), so

\[
\frac{k}{k-i} \leq \frac{k}{k-j} \quad \text{and} \quad \frac{k}{i} \leq \frac{k}{j},
\]

by (3) and (2).

If \( k \) is odd and \( j > \left\lfloor \frac{k}{2} \right\rfloor = \frac{k-1}{2} \), then \( j \geq \left\lfloor \frac{k+1}{2} \right\rfloor \).

So \( k - i \leq k - j \leq \left\lfloor \frac{k}{2} \right\rfloor - 1 = k - \frac{k-1}{2} - 1 = \frac{k-1}{2} = \left\lfloor \frac{k}{2} \right\rfloor \).

The result follows as in the even case.

The only case remaining is when \( k \) is odd, and

\( j = \left\lfloor \frac{k}{2} \right\rfloor = \frac{k-1}{2} \). If \( i = j \), we are done.

So assume \( i \geq \frac{k+1}{2} \). Then \( k - i \leq \frac{k-1}{2} = \left\lfloor \frac{k}{2} \right\rfloor \).

So, by (3), \( \binom{k}{k-i} \leq \binom{k}{k-1} \) and by (2) and (5),

\[
\frac{k}{i} \leq \frac{k}{k+1} = \frac{k}{k-1} = \frac{k}{j}.
\]
(5) First, assume \( k \) is even. Then \( \left\lfloor \frac{k+1}{2} \right\rfloor = \left\lfloor \frac{k}{2} \right\rfloor \) so the result is clear. Now, assume \( k \) is odd. Then \( k = 2n+1 \) for some \( n \in \mathbb{N} \). Then, we have

\[
\binom{k}{\left\lfloor \frac{k+1}{2} \right\rfloor} = \binom{2n+1}{n+1} = \binom{2n+1}{n} = \binom{2n+1}{\frac{2n+1}{2}} = \left\lfloor \frac{k}{2} \right\rfloor.
\]

(6) We proceed by induction on \( k \). The case \( k=0 \) is clear, so assume \( k>0 \), and \( \sum_{i=0}^{k-1} \binom{k-1}{i} = 2^{k-1} \). From Proposition 2.3(b), we know that \( \binom{k}{i} = \binom{k-1}{i-1} + \binom{k-1}{i} \).

Then

\[
\sum_{i=0}^{k} \binom{k}{i} = \sum_{i=0}^{k-1} \binom{k-1}{i-1} + \sum_{i=0}^{k-1} \binom{k-1}{i} = \binom{k-1}{0} + \binom{k-1}{1} + \binom{k-1}{2} + \ldots + \binom{k-1}{k-2} + \binom{k-1}{k-1} + \binom{k-1}{k-1} = 2^k. \quad \text{Q.E.D.}
\]

Some of the proofs in Proposition 2.4 involved the representation of \( \binom{k}{i} \) as a fraction, which will not be possible for generalized binomial coefficients. To get
around this problem, we will show that the number of ways of putting \( n \) indistinguishable balls into \( k \) boxes equals the number of sequences of integers \((m_1, m_2, ..., m_n)\), with \( 0 \leq m_i \leq 1 \).

Lemma 2.5. Let \( n \) be a positive integer, \( 0 \leq k \leq n \), and let \( S_{n,k} = \{(m_1, m_2, ..., m_n) | 0 \leq m_i \leq 1 \forall i, \sum m_i = k\} \). Then \( |S_{n,k}| = \binom{n}{k} \).

Proof. For \( n = 1 \), \( S_{1,k} = \{(m_1) | m_1 = k, 0 \leq m_1 \leq 1\} \). Thus \( |S_{1,1}| = |S_{1,0}| = 1 \) and \( |S_{i,k}| = 0 \) for \( k \neq 0, 1 \). So, \( |S_{i,k}| = \binom{1}{k} \), for all \( k \). Assume \( n > 1 \).

By the definition of \( S_{n,k} \):

\[
S_{n,k} = \left\{ (m_1, m_2, ..., m_{n-1}, 0) \left| \sum_{i=1}^{n-1} m_i = k, 0 \leq m_i \leq 1, \forall i \right. \right\} \\
\cup \left\{ (m_1, m_2, ..., m_{n-1}, 1) \left| \sum_{i=1}^{n-1} m_i = k - 1, 0 \leq m_i \leq 1, \forall i \right. \right\}
\]

Thus \( |S_{n,k}| = |S_{n-1,k}| + |S_{n-1,k-1}| \). Therefore, \( |S_{n,k}| \) satisfies the same initial conditions and recurrence relation as \( \binom{n}{k} \). Hence they are equal for all \( n, k \). Q.E.D

Proposition 2.6. Let \( i \leq k \) be non-negative integers.
Then:

(1) \( \binom{k}{i} = \sum_{j=0}^{i} \binom{k-1}{i-1} \)

(2) \( \binom{k}{i} = \binom{k}{k-i} \)

(3) \( \sum_{i=0}^{k} \binom{k}{i} = 2^k \)

Proof.

(1) Statement (1) follows from the proof of Lemma 2.5.

(2) Recall that \( S_{k,l} = \{(m_1,\ldots,m_k) | 0 \leq m_i \leq 1, \sum m_i = i \leq k\} \) and let \( \phi: S_{k,l} \to S_{k,k-i} \) be the function defined by:

\[
\phi(m_1,\ldots,m_k) = (1-m_1,\ldots,1-m_k).
\]

If \( \bar{m} = (m_1,\ldots,m_k) \) and \( \bar{n} = (n_1,\ldots,n_k) \), then \( \phi(\bar{m}) = \phi(\bar{n}) \) if and only if \( 1-m_i = 1-n_i, \forall i \), if and only if \( m_i = n_i, \forall i \), if and only if \( \bar{m} = \bar{n} \). So \( \phi \) is one-to-one.

Also, if \( \bar{m} \in S_{k,k-i} \), then \( (1-m_1,\ldots,1-m_k) \in S_{k,l} \) since

\[
\sum_{i=1}^{k} (1-m_i) = k - (k-i) = i.
\]

Now, \( \phi(1-m_1,\ldots,1-m_k) = \bar{m} \), so \( \phi \) is onto. Therefore,

\[
|S_{k,l}| = |S_{k,k-i}|, \text{ so } \binom{k}{i} = \binom{k}{k-i} \text{ by Lemma 2.5.}
\]
\[
\sum_{i=0}^{k} \binom{k}{i} = \sum_{i=0}^{k} |S_{n,i}| = \sum_{i=0}^{k} \left| \left\{ (m_1, \ldots, m_k) \mid 0 \leq m_i \leq 1, \sum m_j = i \right\} \right| .
\]

The last sum is the number of sequences \((m_1, \ldots, m_k)\)
with \(0 \leq m_i \leq 1\) which is \(2^k\), by the Fundamental
Counting Theorem. Q.E.D.

The definition of a generalized binomial coefficient
\[
\binom{N_k}{i}
\]
using a \(k\)-tuple \(N_k = (3,3,\ldots,3)\) is now introduced together
with some examples to illustrate it.

Definition 2.7. Let \(i, j\) be two non-negative integers,
and let \(N_0 = 0\), and for \(k > 0\), \(N_k = (3,3,\ldots,3)\), a \(k\)-tuple. Then
\[
\binom{N_0}{0} = 1 \quad \text{and} \quad \binom{N_0}{i} = 0 \quad \text{for} \quad i \neq 0.
\]
For \(k > 0\), and \(i \geq 0\), \(\binom{N_k}{i}\) is the
number of ways to place \(i\) indistinguishable objects into \(k\)
distinct boxes with at most two objects in each box.

We define \(\binom{N_k}{i} = 0\) for \(i < 0\), and \(k > 0\).

Remark. These generalized binomial coefficients
satisfy a recurrence relation similar to binomial
coefficients. Consider \(\binom{N_k}{i}\), the number of ways to put \(i\)
objects into \(k\) boxes. We may put 0, 1 or 2 objects into the
k-th box leaving \(i, i-1, i-2\) (respectively) objects to place in first \(k-1\) boxes. Thus \(\begin{pmatrix} N_k \\ i \end{pmatrix} = \begin{pmatrix} N_{k-1} \\ i \end{pmatrix} + \begin{pmatrix} N_{k-1} \\ i-1 \end{pmatrix} + \begin{pmatrix} N_{k-1} \\ i-2 \end{pmatrix} \).

Example 2.8. Figure 2.2 illustrates the number of ways to put 4 objects into 3 boxes with at most two objects in each box.

![Figure 2.2. Four Objects into Three Boxes](image)

So \(\begin{pmatrix} (3,3,3) \\ 4 \end{pmatrix} = 6\)

Definition 2.9. Let \(N_k = (3,3,..3,..)\) be a \(k\)-tuple. Then, for \(0 \leq i \leq 2k\), let \(S_{N_{i,j}} = \{(m_1, ..., m_k) | 0 \leq m_i \leq 2, \forall i, \sum m_j = i\}\).
Lemma 2.10. Let $N_k = (3, 3, \ldots, 3, \ldots)$ be a $k$-tuple and for \(0 \leq i \leq 2k\), let \(S_{N_k,i} = \{(m_1, m_2, \ldots, m_k) | 0 \leq m_i \leq 2, \forall i, \sum m_j = i\}\). Then, \(|S_{N_k,i}| = \binom{N_k}{i}\).

Proof. For $k=1$, \(S_{N_1,i} = \{(m_1) | m_1 = i, 0 \leq m_1 \leq 2\}\). Thus \(|S_{N_1,2}| = |S_{N_1,1}| = |S_{N_1,0}| = 1\) and \(|S_{N_1,i}| = 0\) for \(i \neq 0, 1, 2\). So, \(|S_{N_1,i}| = \binom{N_1}{i}\), for all $i$. Assume $k>1$. By the definition of $S_{N_k,i}$:

\[
S_{N_k,i} = \{(m_1, m_2, \ldots, m_{k-1}, 0) \left| \sum_{j=1}^{k-1} m_j = i, 0 \leq m_i \leq 2, \forall i \right.\} \\
\cup \{(m_1, m_2, \ldots, m_{k-1}, 1) \left| \sum_{j=1}^{k-1} m_j = i-1, 0 \leq m_i \leq 2, \forall i \right.\} \\
\cup \{(m_1, m_2, \ldots, m_{k-1}, 2) \left| \sum_{j=1}^{k-1} m_j = i-2, 0 \leq m_i \leq 2, \forall i \right.\}.
\]

Thus \(|S_{N_k,i}| = |S_{N_1,i}^1| + |S_{N_1,i-1}| + |S_{N_1,i-2}|\). Therefore \(|S_{N_k,i}|\) satisfies the same initial conditions and recurrence relation as \(\binom{n}{k}\) (See the remark preceding Definition 2.9). Hence they are equal for all $N_k, i$. Q.E.D

The following theorem gives some important properties of the generalized binomial coefficients \(\binom{N_k}{i}\).

Theorem 2.11. Let $k>0$ and let $i \in Z$. Then:
(1) \[ \binom{N_k}{i} = \sum_{j=0}^{i} \binom{N_{k-1}}{i-j} \]

(2) \[ \binom{N_k}{i} = 0, \text{ for } i < 0 \text{ and } i > 2k \]

(3) \[ \binom{N_k}{i} = \binom{N_k}{2k-i} \]

(4) \[ \binom{N_k}{j} \leq \binom{N_k}{i}, \text{ for } j \leq i \leq k \]

(5) \[ \binom{N_k}{i} \geq \binom{N_k}{j}, \text{ for } k \leq i \leq j. \]

(6) \[ \sum_{i} \binom{N_k}{i} = 3^k \]

Proof.

(1) \( \binom{N_k}{i} \) is the number of ways to put \( i \) indistinguishable objects into \( k \) boxes with at most \( k-1 \) objects in one box. We may put 0, 1, 2 objects into the \( k^{th} \) box, leaving \( i, i-1, i-2 \) objects, respectively, to place in \( k-1 \) remaining boxes. Therefore, the equality \( \binom{N_k}{i} = \sum_{j=0}^{i} \binom{N_{k-1}}{i-j} \) holds.

(2) This follows from Definition 2.7 and the fact that the \( k \) boxes may hold at most 2 balls each.
if \( \binom{N_{k-1}}{i-3} \leq \binom{N_{k-1}}{i} \). If \( i < k \), then \( i-3 \leq i \leq k-1 \), so

\[ \binom{N_{k-1}}{i-3} \leq \binom{N_{k-1}}{i} \]

by the inductive hypothesis.

If \( i = k \), then by inductive hypothesis and part(3), we have \( \binom{N_{k-1}}{k-3} \leq \binom{N_{k-1}}{k-2} = \binom{N_{k-1}}{2(k-1)-(k-2)} = \binom{N_{k-1}}{k} \), since \( k-3 \leq k-2 \leq k-1 \), so we are done.

(5) The given condition \( k \leq i \leq j \) implies \( 2k-j \leq 2k-i \leq k \), so

\[ \binom{N_k}{2k-j} \leq \binom{N_k}{2k-i} \]

and so we have \( \binom{N_k}{j} \leq \binom{N_k}{i} \)

following from part(3).

(6) From Lemma 2.10,

\[ \sum_{i=0}^{k} \binom{N_k}{i} = \sum_{i=0}^{k} |S_{N_i,j}| = \sum_{i=0}^{k} \left| \left\{ (m_1,\ldots,m_k) \mid 0 \leq m_i \leq 2, \sum m_j = i \right\} \right| . \]

The last sum is the number of sequences \((m_1,\ldots,m_k)\) with \( 0 \leq m_i \leq 2 \), which is \( 3^k \) by the Fundamental Counting Theorem. Q.E.D.

Definition 2.12. Let \( N=(n_1,n_2,n_3,\ldots) \), where \( n_i \in \mathbb{Z}^+ \), let \( N_0=0 \), and for \( k>0 \), let \( N_k \) denote the k-tuple \((n_1,\ldots,n_k)\). We define \( \binom{N_0}{i} = \begin{cases} 0, & \text{if } i \neq 0 \\ 1, & \text{if } i = 0 \end{cases} \), and for \( k>0 \), we define \( N_i \) to be the
number of ways to put \( i \) indistinguishable objects into \( k \) boxes with at most \( n_i - 1 \) in each box.

Example 2.13. This example demonstrates the number of ways in which eight balls can be placed in three boxes, such that we have fewer than three balls in the first and third box, and fewer than five in the second box. The top row represents the number of objects, and the columns represent the number of boxes. Now, \( \binom{k}{i} \) match \( \binom{N_k}{i} \). This is equivalent to setting \( N_3 = (3,5,3) \).

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Figure 2.3. Illustration of \( N_3 = (3,5,3) \)

Now, we can construct the generalized Pascal’s triangle corresponding to the vector \( N_3 = (3,5,3) \). Note, for example,
that \( \binom{N_3}{3} \), the fifth number from left in the fifth row is formed by adding the third, fourth, and fifth numbers in the fourth row, which is equivalent to \( \binom{N_2}{3} + \binom{N_2}{2} + \binom{N_2}{1} \).

Figure 2.4 gives the generalized version of the triangle for this specific vector.

```
1
1 1 1
1 2 3 3 3 2 1
1 3 6 8 9 8 6 3 1
```

Figure 2.4. Generalized Pascal’s Triangle for \( N_3 = (3,5,3) \)

Example 2.14.

(a) We illustrate these general binomial coefficients with the vector \( N_3 = (3,4,5) \).

\[
\binom{N_3}{3} = \sum_{j=0}^{5-1} \binom{N_2}{3-j} = \binom{N_2}{3} + \binom{N_2}{2} + \binom{N_2}{1} + \binom{N_2}{0} = 9.
\]

(b) Figure 2.5 illustrates the ways in which 3 balls can be placed in 3 boxes. The first box will take a maximum of two balls, the second one a maximum
of three balls, and the third one a maximum of four balls.

![Table of Max2, Max3, and Max4 values]

Figure 2.5. Three Balls into Three Boxes

The following theorem will show that analogues of the properties of usual binomial coefficients hold for the generalized binomial coefficients. These properties will play an important role in the next section.
Theorem 2.15 ([5], Proposition 2.5). Let
\( N_k = (n_1+1, n_2+1, \ldots, n_k+1) \) be a sequence of integers with \( n_i \geq 1 \), \( 1 \leq i \leq k \), and let \( T_k = \sum_{i=1}^{k} n_i \). Then:

1. \( \binom{N_k}{i} = \sum_{j=0}^{n_k} \binom{N_{k-1}}{i-j} \)
2. \( \binom{N_k}{i} = 0 \), for \( i < 0 \) and \( i > T_k \).
3. \( \binom{N_k}{i} = \binom{N_k}{T_k-i} \)
4. \( \binom{N_k}{j} \leq \binom{N_k}{i} \), for \( j \leq i \leq \left\lfloor \frac{T_k}{2} \right\rfloor \)
5. \( \binom{N_k}{j} \geq \binom{N_k}{i} \), for \( \left\lfloor \frac{T_k}{2} \right\rfloor \leq j \leq i \)
6. \( \sum_{i=0}^{n_k} \binom{N_k}{i} = (n_1+1)(n_2+1)\ldots(n_k+1) \), for \( k \geq 1 \)

Proof.

(1) \( \binom{N_k}{i} \) is the number of ways to put \( i \) indistinguishable objects into \( k \) boxes with at most \( n_i \) objects in one box. We may put

0, 1, 2, ..., \( n_k \) objects into the \( k^{th} \) box, leaving
\(i, i-1, i-2, \ldots, i-n_k\) objects, respectively, to place in \(k-1\) remaining boxes. Therefore, the equality
\[
\binom{N_k}{i} = \sum_{j=0}^{n_k} \binom{N_{k-1}}{i-j}
\]
holds.

(2) This follows from Definition 2.12 and the fact that the \(i\)-th box holds at most \(n_i\) objects.

(3) Define the function
\[
\phi: S_{N_k,i} \rightarrow S_{N_k,k-1} \text{ by } \phi(m_1, \ldots, m_k) = (n_i - m_i, \ldots, n_k - m_k).
\]
We will show that this function is one-to-one and onto.

If \(\overline{m} = (m_1, \ldots, m_k)\) and \(\overline{p} = (p_1, \ldots, p_k)\), then \(\phi(\overline{m}) = \phi(\overline{p})\), if and only if \(n_i - m_i = n_i - p_i\), \(\forall i\), if and only if \(m_i = p_i\), \(\forall i\), if and only if \(\overline{m} = \overline{p}\). So \(\phi\) is one-to-one.

Also, if \(\overline{m} \in S_{k, T_{k-i}}\), then \((n_i - m_i, \ldots, n_k - m_k) \in S_{k,i}\), since
\[
\sum_{i=1}^{k} (n_i - m_i) = \sum_{i=1}^{k} n_i - (T_k - i) = i.
\]
Now, \(\phi(n_i - m_i, \ldots, n_k - m_k) = \overline{m}\), so \(\phi\) is onto. Therefore,
\[
\left| S_{N_k,i} \right| = \left| S_{N_k,k-1} \right|, \text{ so } \binom{N_k}{i} = \binom{N_k}{T_k-i}.
\]
(4) Induction on \( k \) will be used. The case \( k=1 \) is trivial. Assume \( k \geq 2 \) and \( \binom{N_{k-1}}{j} \leq \binom{N_{k-1}}{i} \) for \( j \leq i \leq \left\lfloor \frac{T_{k-1}}{2} \right\rfloor \).

Now assume \( j \leq i \leq \left\lfloor \frac{T_{k}}{2} \right\rfloor \) and note that it suffices to prove the result for \( i=j+1 \). Then \( \binom{N_k}{j} \leq \binom{N_k}{j+1} \) if and only if

\[
\sum_{s=0}^{n_k} \binom{N_{k-1}}{j-s} \leq \sum_{s=0}^{n_k} \binom{N_{k-1}}{j+1-s}
\]

since all terms cancel except for 0 on the right hand side, and \( n_k \) on the left hand side. This inequality holds by the induction hypothesis if \( j+1 \leq \left\lfloor \frac{T_{k-1}}{2} \right\rfloor \). So we will consider the case \( \left\lfloor \frac{T_{k-1}}{2} \right\rfloor < j+1 \leq \left\lfloor \frac{T_{k}}{2} \right\rfloor \). But \( \left\lfloor \frac{T_{k-1}}{2} \right\rfloor < j+1 \leq \left\lfloor \frac{T_{k}}{2} \right\rfloor \), which implies that \( T_{k-1} \leq 2j+2 \leq T_k = T_{k-1} + n_k \). This gives us

\( -n_k - T_{k-1} \leq -2j-2 \leq T_{k-1} \), which means that \( -n_k \leq T_{k-1} - 2j - 2 \)

and \( 2T_{k-1} - 2j - 2 \leq T_{k-1} \). From here, we have that

\( j - n_k \leq T_{k-1} - j - 2 \) and \( T_{k-1} - j - 1 \leq T_{k-1} - j - 1 \).

Finally, this implies that \( j - n_k \leq T_{k-1} - j - 2 \leq T_{k-1} - j - 1 \leq \left\lfloor \frac{T_{k-1}}{2} \right\rfloor \).
By the inductive hypothesis, we have

\[
\left( \frac{N_{k-1}}{j-n_k} \right) \leq \left( \frac{N_{k-1}}{T_{k-1}-(j+1)} \right).
\]

Thus \( \left( \frac{N_{k-1}}{j-n_k} \right) \leq \left( \frac{N_{k-1}}{j+1} \right) \), by part (3).

(5) We need to show that \( \left( \frac{N_k}{j} \right) \geq \left( \frac{N_k}{i} \right) \) for \( \left\lfloor \frac{T_k}{2} \right\rfloor \leq j \leq i \). We start with the case \( T_k \) is even. Then

\[
T_k-i \leq T_k-j \leq T_k \frac{T_k}{2} = \frac{T_k}{2}.
\]

So \( \left( \frac{N_k}{T_k-i} \right) \leq \left( \frac{N_k}{T_k-j} \right) \) by part (3).

Thus \( \left( \frac{N_k}{j} \right) \leq \left( \frac{N_k}{i} \right) \) by part (3). If \( T_k \) is odd and \( j > \left\lfloor \frac{T_k}{2} \right\rfloor \), then the result follows as in the even case. Finally, we consider \( i > j = \left\lfloor \frac{T_k}{2} \right\rfloor \), with \( T_k \) odd.

We note that \( T_k \frac{T_k}{2} = \frac{T_k+1}{2} \), so \( \left( \frac{N_k}{T_k-1} \right) = \left( \frac{N_k}{T_k+1} \right) \) by part (3). Now, \( i \geq \frac{T_k+1}{2} > \left\lfloor \frac{T_k}{2} \right\rfloor \). This case was considered above, so \( \left( \frac{N_k}{i} \right) \leq \left( \frac{N_k}{T_k+1} \right) = \left( \frac{N_k}{T_k-1} \right) = \left( \frac{N_k}{j} \right) \), and we are done.
(6) Since $\binom{N_k}{i} \neq 0$, $\sum_{i=0}^{n_i} \binom{N_k}{i}$ will be the number of ways of putting any number of indistinguishable objects into $k$ different boxes with the $i$-th box containing at most $n_i$ objects. This equals the number of sequences of integers $(m_1, \ldots, m_k)$ with $0 \leq m_i \leq n_i$, which is $(n_1 + 1)(n_2 + 1)\ldots(n_k + 1)$ by the Fundamental Counting Theorem, Q.E.D.

The general binomial coefficients have applications in various areas of mathematics, such as discrete mathematics, combinatorics, algebra, number theory, discrete probability, linear algebra, finite geometry, analysis of algorithms, and numerical analysis. In the next chapter we will introduce some of the applications of the generalized binomial coefficients in algebra, specifically in ring theory.
CHAPTER THREE
APPLICATIONS OF THE GENERALIZED BINOMIAL
COEFFICIENTS TO ALGEBRA

In this section, we will begin with the definition of a group ring, followed by examples in which we give the sum and product of two elements of specific group rings, such as \( \mathbb{Q}[Z] \) and \( \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}_2] \). Throughout this section we will assume that all rings are commutative with unity.

Proposition 3.5 shows that for divisors of polynomials, which are monic polynomials, a division algorithm holds for rings of polynomials.

In Proposition 3.8 and Corollary 3.9 we show that for finite abelian groups, group rings are homomorphic images of polynomial rings.

In Examples 3.14-3.16 we are then able to describe group rings in some detail. In particular, we classify elements as units or zero divisors. This enables us to determine the maximal ideals of these group rings. The main problem is that in this characterization \( x \), which is an element of the group ring, is a unit since \( x^n = 1, \exists n \).

However, if the ring has characteristic \( p \), we are able to
give a characterization of group rings as factor rings of polynomials where the indeterminates are non-units.

In Theorem 3.29, we will prove that the ring $R = F[Z_{p^r}]$ has a unique maximal ideal generated by these indeterminates. These indeterminates will serve as the boxes into which we place objects. Then, we will give the definitions for the Dilworth number and the Sperner number.

Finally, we will conclude with Theorem 3.33, which will prove that the numbers of generators of powers of a maximal ideal of a group ring is determined by an appropriate generalized binomial coefficient.

Definition 3.1. Let $R$ be a ring and $G$ be an abelian group. Then the group ring $R[G]$ is the set

$$\left\{ \sum_{g \in G} a_g x^g \mid a_g \in R, g \in G \right\}$$

with multiplication $a_g x^g a_h x^h = a_g a_h x^{g+h}$ and addition $a_g x^g + b_g x^g = (a_g + b_g) x^g$.

Proposition 3.2. If $R$ is a ring and $G$ is an abelian group, then $R[G]$ is a ring.

We will not give a proof for this proposition because the ring properties of $R[G]$ are similar to the ones for polynomial rings. We will only give its multiplicative and additive identity, and its inverse. The multiplicative
identity is 1, the multiplicative identity of \( R \): \( 1x^0 = 1 \in R[G] \), and the additive identity is 0, the additive identity of \( R \): \( 0x^0 = 0 \in R[G] \). The inverse of \( \sum a_x x^x \) is \( \sum (-a_x) x^x \in R[G] \).

The following two examples will illustrate multiplication and addition in two common group rings, \( Q[Z] \) and \( Z[Z \oplus Z_2] \).

Example 3.3. Let \( Q[Z] \) be the group ring of \( Z \) with coefficients from \( Q \). Let \( f = 2x^3 + x^2 + x^6 \) and \( g = 3x^3 - x^6 \). Then, \( f + g = 5x^3 + x^2 \), and \( f \cdot g = 6x^6 + 3x^1 + x^3 - x^8 - x^{12} \).

Example 3.4. Let \( Z[Z \oplus Z_2] \) be a group ring. Let \( f = 2x^{(-3,1)} + x^{(-3,0)} - 3x^{(1,1)} \) and \( g = x^{(-3,1)} - x^{(1,1)} \). Then, \( f + g = 3x^{(-3,1)} + x^{(-3,0)} - 4x^{(1,1)} \) and \( f \cdot g = 2x^{(-6,0)} + x^{(-6,1)} - x^{(-2,0)} - x^{(-2,1)} + 3x^{(2,0)} \)

Next, we give the Division Algorithm for rings of polynomials over a ring \( R \). First, we must explain two of the terms involved: a monic polynomial and the degree of a polynomial.

An element of \( R[X] \) is monic if and only if the leading coefficient is the multiplicative identity of \( R \).
The degree of a polynomial \( f(x) = a_0 + a_1 x + \ldots + a_m x^m \) is the maximal nonnegative integer \( m \), such that \( a_m \neq 0 \). The standard notation for the degree of a polynomial \( f \) is \( \deg f \).

In the next proposition, we will modify the usual division algorithm to accommodate the lack of multiplicative inverses in \( R \).

Proposition 3.5. Let \( R \) be a ring and let \( f, g \in R[X] \), with \( g \) monic. Then \( \exists q, r \in R[X] \) such that \( f = q \cdot g + r \) and \( r = 0 \) or \( \deg r < \deg g \).

Proof. Let \( f(x) = a_0 + a_1 x + \ldots + a_m x^m \) and \( g(x) = b_0 + b_1 x + \ldots + x^n \), with \( a_i, b_i \in R \). Since \( g(x) \neq 0 \), we can assume that \( \deg g(x) = n \).

The proof is trivial for \( f(x) = 0 \). So, we also assume that \( a_m \neq 0 \) so that \( \deg f(x) = m \neq 0 \). The existence of \( q \) and \( r \) will be proved by induction on \( m \). If \( m < n \), then \( q = 0 \), and \( r = f \) satisfy the requirements. Assume that \( m \geq n \). If \( m = 0 \), then \( q = a_0 \) and \( r = 0 \). Now, we must prove that if \( \deg f = m > 0 \), \( \exists q, r \in R[X] \) with \( f = q \cdot g + r \) and \( r = 0 \) or \( \deg r < \deg g \). Let \( f_1(x) = f(x) - a_m x^m \cdot g(x) \). Then \( \deg f_1(x) < \deg f(x) \). So, by the inductive hypothesis, there exists \( q \) and \( r \) such that \( f_1 = q \cdot g + r \) with \( r = 0 \) or \( \deg r < \deg g \).
This implies that \( f(x) = g(x)\left[a_m x^{m-n} + q(x)\right] + \eta_1(x) \). Thus \( q = a_m x^{m-n} + q_1 \) and \( r = \eta_1 \). Q.E.D

We will now give an example to show that if \( g \) is not monic, then no such \( q \) and \( r \) exist.

Example 3.6. Consider the group ring \( \mathbb{Z}[X] \) and let \( f = x^3 \) and \( g = 2x+1 \). We will show that \( x^3 \neq q(2x+1) + r \), with \( q, r \in \mathbb{Z}[X] \), as in the division algorithm.

If \( q = ax + b \) and \( r = cx + d \), and \( x^3 = q(2x+1) + r = (ax + b)(2x+1) + cx + d = 2ax^2 + x(2b + a + c) + (b + d) \). Thus \( 2a = 1 \), which is not possible for \( a \in \mathbb{Z} \). So there are no \( q, r \in \mathbb{Z}[X] \), such that \( x^3 = q(2x+1) + r \).

Before we introduce the next proposition and prove it, we want to remind the reader of the definition of the kernel of a homomorphism.

Definition 3.7. Let \( \phi: R \to S \) be a ring homomorphism. Then the set \( \{ a \in R | \phi(a) = 0 \} \) is the kernel of \( \phi \).

We use this definition to give a characterization of group rings \( R[G] \), with \( G \) cyclic. We will represent the elements of \( \mathbb{Z}_n \) as \( \bar{k} \), or if their meaning is clear from the context, simply \( k \).

Proposition 3.8. If \( R \) is a ring, then \( R[\mathbb{Z}_n] \cong R[x]/(1-x^n) \).
Proof. Let \( f \in R[x] \) and \( \phi: R[x] \rightarrow R[\mathbb{Z}_n] \) be defined by
\[
\phi\left(\sum a_k x^k\right) = \sum a_k x^k.
\]
Since \( x^n - 1 \) is monic, there exist \( q, r \in R[x] \) such that \( f = q(1 - x^n) + r \), where \( \deg r < \deg f \) or \( r = 0 \) by Proposition 3.6. Then \( \phi(f) = \phi(q)(1 - x^n) + \phi(r) = \phi(r) \). So \( \phi(f) = 0 \) if and only if \( \phi(r) = 0 \) if and only if \( r = 0 \). Thus \( f \in (1 - x^n) \) if and only if \( f \in \ker \phi \). Thus, \( \ker \phi = (1 - x^n) \), and by the Fundamental Isomorphism Theorem, stated later in this section, we have
\[
R[\mathbb{Z}_n] \cong R[x]/(1 - x^n).
\]
Q.E.D.

The next corollary extends Proposition 3.8 to a finite sum of cyclic groups. We do not give a proof, but refer the reader to [7], Okon, J.S, Vicknair J.P., "Numbers of Generators of ideals in a group ring of an Elementary Abelian p-group." Journal of Algebra. 2000, Vol.224, (pg. 1-22)

Corollary 3.9. Let \( R \) be a ring. Then
\[
R[\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_k}] \cong R[x_1, x_2, \ldots, x_k]/(1 - x_{n_1}^n, 1 - x_{n_2}^n, \ldots, 1 - x_{n_k}^n)
\]

Next, we will give a few examples of common group rings with their multiplication tables. The element 0 is omitted from these tables since multiplication by zero is
trivial. Also, we will give the units and the zero divisors of each group ring, along with their maximal ideals. Definitions 3.10-3.14 precede these examples.

Definition 3.10. An element \( u \) in a ring \( R \) such that \( ua=au=1 \) for some \( a \in R \) is a unit element for \( R \).

Definition 3.11. An element \( a \in R, a \neq 0 \), is a zero divisor if and only if there is an element \( b \in R, b \neq 0 \), such that \( ab=ba=0 \).

Definition 3.12. A nonempty subset \( I \) of a ring \( R \) is an ideal in \( R \) if and only if for all \( a,b \in I \), \( a-b \in I \); and for each \( x \in R \) and for each \( a \in I \), \( xa \in I \).

Definition 3.13. Let \( M \) be an ideal in a ring \( R \). \( M \) is a maximal ideal in \( R \) if and only if \( M \neq R \), and if \( I \) is an ideal such that \( M \subseteq I \subseteq R \), then \( I=M \) or \( I=R \).

Definition 3.14. Let \( R \) be a ring. If there is a positive integer \( n \) such that \( na=0 \) for each \( a \in R \), then the least such integer is called the characteristic of \( R \). If there is no such integer, then \( R \) has characteristic 0.

Example 3.15. Consider the group ring \( \mathbb{Z}_2[\mathbb{Z}_2]=\{0,1,x,1+x\} \). Figure 3.1 gives the multiplication table for this group ring.
Figure 3.1. Multiplication Table for $\mathbb{Z}_2[\mathbb{Z}_2]$

From the multiplication table above we see that the set of unit elements of this group ring is \{1, x\}, and the only zero divisor element is 1+x. This ring has three ideals 0 \subset (1+x) \subset (1).

Note that $\mathbb{Z}_3[\mathbb{Z}_2]$ has a unique maximal ideal. Example 3.16 will show that this need not be true if the ring and group have different characteristic.

Example 3.16. Consider the group ring $\mathbb{Z}_3[\mathbb{Z}_2]=\{0,1,2,x,2x,1+x,2+x,1+2x,2+2x\}$. Figure 3.2 gives the multiplication table for this group ring.
Figure 3.2. Multiplication Table for $\mathbb{Z}_3[\mathbb{Z}_2]$

From the multiplication table above we see that the set of unit elements of this group ring is $\{1, 2, x, 2x\}$ and the zero divisors are $\{1+x, 2+x, 1+2x, 2+2x\}$.

The maximal ideals of this group ring are:

$M_1 = (1+x) = \{0, 1+x, 2+2x\}$ and $M_2 = (2+x) = \{0, 2+x, 1+2x\}$. This is illustrated in Figure 3.3.
Example 3.17. Consider the group ring $\mathbb{Z}_2[\mathbb{Z}_3] = \{0, 1, x, 1+x, x^2, 1+x^2, x+x^2, 1+x+x^2\}$. Figure 3.4 gives the multiplication table for this group ring.

\[
\begin{array}{cccccccc}
. & 1 & x & 1+x & x^2 & 1+x^2 & x+x^2 & 1+x+x^2 \\
1 & 1 & x & 1+x & x^2 & 1+x^2 & x+x^2 & 1+x+x^2 \\
x & x & x^2 & x+x^2 & 1 & 1+x & 1+x^2 & 1+x+x^2 \\
1+x & 1+x & x+x^2 & 1+x^2 & x+x^2 & 1+x & 0 & \\
x^2 & x^2 & 1 & 1+x^2 & x & x+x^2 & 1+x & 1+x+x^2 \\
1+x^2 & 1+x^2 & 1+x & x+x^2 & x+x^2 & 1+x & 1+x^2 & 0 \\
x+x^2 & x+x^2 & 1+x^2 & 1+x & 1+x & 1+x^2 & x+x^2 & 0 \\
1+x+x^2 & 1+x+x^2 & 1+x+x^2 & 0 & 1+x+x^2 & 0 & 0 & 1+x+x^2 \\
\end{array}
\]
From the multiplication table above we see that the units are: \( \{1, x, x^2\} \) and the zero divisors are: \( \{1+x, 1+x^2, x+x^2, 1+x+x^2\} \).

The maximal ideals are \( M_1 = (1+x) = \{0, 1+x, 1+x^2, x+x^2\} \) and
\( M_2 = (1+x+x^2) \).

For the group ring in the next example, we do not give the multiplication table. We list the elements and we classify them as zero divisors or units. We also give the maximal ideal of this group ring. We will revisit this example later in the chapter.

Example 3.18. Consider the group ring \( \mathbb{Z}_3[\mathbb{Z}_3] \). This group ring has the following 27 elements:

\[
0, 1, 2, x, 2x, x^2, 2x^2, 1+x, 2+x, 1+2x, 2+2x, 1+x^2, 2+x^2, 2+2x^2, x+x^2, 2x+x^2, 2+x+2x^2, 2x+2x^2, 1+2x+2x^2, 2+2x+2x^2, 2+2x^2, 1+2x+2x^2, 2+2x+2x^2, 2+2x^2.
\]

The units of this group are:

\[
1, 2, x, 2x, x^2, 2x^2, 1+x, 2+2x, 1+x^2, 1+2x+x^2, 2+x+2x^2, 2+2x+2x^2, 1+2x^2, x+x^2, 1+x+2x^2, 2+2x^2, 2+x+2x^2, 2x+2x^2.
\]

The zero-divisors of this group are:

\[
0, 2+x, 1+2x, 1+x+x^2, 1+2x^2, 2+x^2, 2x+x^2, 2+2x+2x^2.
\]

The maximal ideal of this group ring is \( M = (1+2x) \).
Lemma 3.19. If $p$ is a prime and $n \geq 1$, then $p$ divides $\binom{p^n}{i}$ for $0 < i < p^n$.

**Proof.** From $\binom{p^n}{i} = \frac{p^n!}{(p^n-i)!i!}$, we have

$$\frac{(p^n-(i-1))(p^n-(i-2))...(p^n-1)p^n}{i(i-1)(i-2)...(1)}.$$ For $1 \leq s \leq i-1$, $p'|s$ if and only if $p'|p^n-s$. So all powers of $p$ dividing $(i-1)!$ are cancelled in $(p^n-1)-(p^n-i+1)$. Now $i < p^n$, so $p^n$ does not divide $i$, and thus $p|\binom{p^n}{i}$ for $0 < i < p^n$. Q.E.D.

Before we introduce the next proposition and we give its proof, we want to remind the reader of the Fundamental Homomorphism Theorem for rings. We state this theorem below, without giving a proof for it.

**Theorem 3.20** (Fundamental Homomorphism Theorem)
Let $\phi: R \to S$ be a ring homomorphism. Then $\phi(R)$ is a subring of $S$, $R/\ker(\phi)$ is a ring, and $\phi(R) \cong R/\ker(\phi)$.

**Proposition 3.21.** If $R$ is a ring of characteristic $p$, then $R[\mathbb{Z}_p]\cong R[x]/(x^p)$. 

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Proof. Let $\phi : R[X] \to R \left[\mathbb{Z}_{p^n}\right]$ be defined by

$$
\phi \left( \sum a_k x^k \right) = \sum a_k \left(1-x^k\right). \quad \text{We claim that ker}\phi = (x^{p^n}). \quad \text{Let } f \in R[x].
$$

Since $x^{p^n}$ is monic, by Proposition 3.5, we have $f = qx^{p^n} + r$. Then

$$
\phi(f) = \phi(q) \cdot \phi(x^{p^n}) + \phi(r). \quad \text{We claim that } \phi(x^{p^n}) = 0. \quad \text{Now}
$$

$$
\phi \left( x^{p^n} \right) = (1-x)^{p^n} = \sum_{i=0}^{p^n} (-1)^i \binom{p^n}{i} x^i. \quad \text{However, if } i \neq 0 \text{ or } i \neq p^n, \text{ we have}
$$

$$
\binom{p^n}{i} = 0, \text{ by Lemma 3.19. So } \phi \left( x^{p^n} \right) = (1-x)^{p^n} = 1^{p^n} - x^{p^n} = 1-1 = 0.
$$

Also, since $f = qx^{p^n} + r$ by Proposition 3.5, we have

$$
\phi(f) = \phi(q) \phi(x^{p^n}) + \phi(r) = \left( \phi(x) \right)^{p^n} = (1-x)^{p^n} = \sum_{i=0}^{p^n} (-1)^i \binom{p^n}{i} x^i = 1^{p^n} - x^{p^n} = 1-1 = 0.
$$

Thus $\phi(f) = \phi(r)$. So $\phi(f) = 0$ if and only if $\phi(r) = 0$ if and only if $r = 0$. Thus $f \in \ker \phi$ if and only if $f \in (x^{p^n})$, which means that $\ker \phi = (x^{p^n})$. Now, by the Fundamental Homomorphism Theorem, $R \left[\mathbb{Z}_{p^n}\right] \cong R[x]/(x^{p^n})$. Q.E.D

The next corollary extends Proposition 3.21 to a finite sum of cyclic groups. We do not give a proof, but refer the reader to [6].
Corollary 3.22. Let \( R \) be a ring of characteristic \( p \).

Then \( R\left[\mathbb{Z}_{p^n} \oplus \ldots \oplus \mathbb{Z}_{p^n}\right] \cong R[x_1, \ldots, x_n]/(x_1^{p^n}, \ldots, x_n^{p^n}) \).

Example 3.23. Consider the group ring
\( \mathbb{Z}_2[\mathbb{Z}_2 \oplus \mathbb{Z}_2] \cong \mathbb{Z}_2[x, y]/(x^2, y^4) \). The set of elements of \( \mathbb{Z}_2[\mathbb{Z}_2 \oplus \mathbb{Z}_2] \) is 
\{0, 1, x, y, 1+x, 1+y, xy, 1+xy\}. The set of units of this group ring will be \{1, 1+x, 1+y, 1+xy\}. Now, the group ring \( \mathbb{Z}_2[\mathbb{Z}_2 \oplus \mathbb{Z}_2] \) has a maximal ideal, \( M=(x, y) \). Note that \( M^2=(xy) \), and \( M^3=0 \). If we denote the number of generators of \( M^n \) by \( \mu(M^n) \), then \( \mu(M)=1 \), \( \mu(M^2)=2 \), and \( \mu(M^3)=1 \). This corresponds to the third row of Pascal's triangle, illustrated in Figure 3.5.

\[
\begin{array}{ccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
\end{array}
\]

Figure 3.5. Generalized Pascal's Triangle for \( \mathbb{Z}_2[\mathbb{Z}_2 \oplus \mathbb{Z}_2] \)

The proof of Proposition 3.25 is based on one of the main isomorphism theorems. We will give the statement and proof for this theorem next.
Theorem 3.24. Let $R$ be a ring and $I \subseteq J$ be ideals of $R$. Then $(R/I)/J/I \cong R/J$.

Proof. Define a map $\phi: R/I \to R/J$ by $a + I \mapsto a + J$ for any coset $a + I$ of $I$ in $R$. This is clearly onto. An element $a + I$ is in the kernel of $\phi$ if $a + J = J$. This will only happen if $a \in J$, since $I \subseteq J$. Thus $\ker(\phi) = J/I \subseteq R/I$ and the result follows from the First Isomorphism Theorem, which states that, given a ring homomorphism $\phi: R \to S$, then $R/\ker\phi \cong \text{im}\phi$. Q.E.D.

Proposition 3.25. Let $F$ be a field of characteristic $p$. Then $R = F\left[Z_p[x]\right]$ has a unique maximal ideal.

Proof. We have $R \cong F[x]/(x^p)$ and $F[x]/(x^p)/(x)/(x^p) \cong F[x]/(x)$ by Theorem 3.24. Next we show that $F[x]/(x) \cong F$. We define $F[x] \xrightarrow{\phi} F$ by $\phi\left(\sum a_kx^k\right) = a_0$. The kernel of $\phi$ is $\ker\phi = (x)$. Thus $F[x]/(x) \cong F$ and so $(x)$ is a maximal ideal of $R$. To show it is the only maximal ideal, we must show that if $a_0 \neq 0$, then $a_0 + a_1x + \ldots + a_nx^n$ is a unit. To do this, we will first factor out $x$, so we now we have $a_0 + a_1x + \ldots + a_nx^n = a_0 + x(a_1 + \ldots + a_nx^{n-1}) = a_0 + x_f$. 45
But since \( x^p = 0 \), \( (a_0 + x_f)^p = a_0^p + x_f^p = a_0^p \), which is a unit, so we are done. Q.E.D

In the following two lemmas we will show some properties of nilpotent elements. We recall that an element \( r \) of a ring \( R \) is nilpotent if \( r^n = 0 \), for some positive integer \( n \).

Lemma 3.26. The sum of two nilpotent elements in a ring \( R \) is nilpotent.

Proof. Let \( r, s \) be nilpotent elements of \( R \) with
\[
r^n = s^m = 0, \quad n, m \in \mathbb{N}.
\]
Then \( (r+s)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} r^k s^{m+n-k} \). If \( k < n \) and \( m+n-k < m \), then \( k+m+n-k < m+n \). So \( k \geq n \) or \( m+n-k \geq m \). Then \( r^k = 0 \) or \( s^{m+n-k} = 0 \). Therefore \( (r+s)^{m+n} = 0 \) and \( r+s \) is nilpotent. Q.E.D.

Lemma 3.27. Let \( R \) be a ring and let \( u, r \in R \) with \( r \) nilpotent and \( u \) a unit. Then \( u+r \) and \( u-r \) are units.

Proof. The polynomial \( x^n - r^n \) has \( r \) as a root. Thus \( x^n - r^n = (x-r)q(x) \), where \( q(x) \) is a monic polynomial in \( R[x] \).

Setting \( x = u \) we get \( u^n - r^n = (u-r)q(u) \). So \( 1 = (u-r) q(u) \cdot u^{-n} \), and \( u-r \) is a unit. If \( r \) is nilpotent, then \( -r \) is also nilpotent, so \( u-(-r) = u+r \) is nilpotent. Q.E.D.
The following Lemma, called Zorn's Lemma, is needed for Theorem 3.29. We state it without giving any proof.

Lemma 3.28 (Zorn's Lemma). If $S$ is any nonempty partially ordered set in which every chain has an upper bound, then $S$ has a maximal element.

The next theorem shows that every non-unit of a ring $R$ is contained in a maximal ideal.

Theorem 3.29. Let $R$ be a ring. Then every non-unit of $R$ is an element of the set $N = \bigcup \{ M_a \mid M_a$ is a maximal ideal of $R \}$.

Proof. Let $a$ be a non-unit and let $A=\{ I \mid I$ is a proper ideal of $R$ and $a \in I \}$. We know that $A \neq \emptyset$, since $aR \in A$. If $I_1 < I_2 < I_3 < \ldots$ is a chain of elements of $A$, then $I = \bigcup_{i=1}^{\infty} I_i \in A$. To show that $I$ is an ideal containing $a$, let $r \in R$ and $x \in \bigcup I_i$. Since $x \in I_j$ for some $j \in \mathbb{N}$, and $I$ is an ideal, we must have $rx \in I_j$, so $rx \in \bigcup I_i$ as well. For closure under addition, let $x, y \in \bigcup I_i$. Then $x \in I_a \exists a$, $y \in I_\beta \exists \beta$. Assume $a \leq \beta$. Then $x, y \in I_\beta$ and $x + y \in I_\beta \subseteq I$. Thus $\bigcup I$ is an ideal of $R$. By Zorn's Lemma, $A$ must have a maximal element. Let $M$ be a maximal element of $A$. We claim that $M$ is a maximal ideal contained in $R$.  

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Now, \( a \in M \) since \( M \in A \), and if \( M \) is not a maximal, then there exists a proper ideal \( N \) containing \( M \). Let \( a \in N - M \). Then \( M + aR \subset N \) so \( M + aR \in A \). This contradicts our choice of \( M \). Therefore, \( M \) is a maximal ideal. Q.E.D

The next theorem will show that rings of the form
\[ F[X_1, \ldots, X_k]/(X_1^n, \ldots, X_k^n) \]
have a unique maximal ideal. By Corollary 3.22 this class of rings includes group rings of the form \( F[Z_p \oplus Z_p \oplus \cdots \oplus Z_p] \) with characteristic \( \text{char}(F) = p \).

Theorem 3.30. Let \( F \) be a field, and for each \( i \), let \( x_i \) denote the image of \( X_i \) in \( F[X_1, \ldots, X_k]/(X_1^n, \ldots, X_k^n) \). Then \( (x_1, \ldots, x_k) \) is the unique maximal ideal of \( R \).

Proof. By Proposition 3.25,
\[ R[X_1, \ldots, X_k]/(X_1^n, \ldots, X_k^n)/(X_1^n, \ldots, X_k^n) \equiv F[X_1, \ldots, X_k]/(X_1, \ldots, X_k) \equiv F. \]
Thus \( m = (x_1, \ldots, x_k) \) is a maximal ideal. Since \( x_i^n = 0 \), each \( x_i \) is nilpotent. Therefore a polynomial \( f = a_0 + \sum a_{i_1 \cdots i_k} x_1^{i_1} \cdots x_k^{i_k} \), where
\( i_n > 0 \) is a unit if and only if \( a_0 \neq 0 \) by Lemma 3.27. Thus \( f \notin m \) if and only if \( f \) is a unit. Therefore \( m \) is the unique maximal ideal of \( R \). Q.E.D

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Next we give the definitions of the Dilworth number $D(R)$ and the Sperner number $S_p(R)$. We then state and prove Theorem 3.33 that relates the Dilworth number and the Sperner number to the general binomial coefficients. We will conclude with an example that illustrates this theorem.

Definition 3.31. Let $(R,M)$ be a local ring and let $\mu(I)$ be the number of elements in a minimal generating set for $I$. Then the Sperner number is $S_p(R) = \sup \{ \mu(M^n) | n \geq 0 \}$ and the Dilworth number is $D(R) = \sup \{ \mu(I) | I \leq R \}$.

The following example adds another cyclic group to the group ring $\mathbb{Z}_3[\mathbb{Z}_3]$ described earlier in the chapter.

Example 3.32. We represent the group ring $\mathbb{Z}_3[\mathbb{Z}_3 \otimes \mathbb{Z}_3]$ as $\mathbb{Z}_3[x,y]/(x^3,y^3)$. The group has a unique maximal ideal, $M = (x,y)$. Now, $M^2 = (x^2,xy,y^2)$, $M^3 = (x^3, y^3, x)$, $M^4 = (x^2, y^2)$, and $M^5 = 0$, so we can see that the numbers in the third row of Pascal's Triangle coincide with $\mu(M^j)$, for each of the nonzero powers $M^j$, $j = 0, \ldots, 4 (= (3-1) + (3-1))$, as illustrated in Figure 3.6, below.
In [5] it is shown that for rings of the form
\[ R = F[x_1, ..., x_k]/(x_1^{a_1}, ..., x_k^{a_k}) \]
the Dilworth and the Sperner numbers are identical. The next theorem can be found in [5].

**Theorem 3.33.** ([5], Theorem 3.1) Let \( F \) be a field, let
\[ R = F[x_1, ..., x_k]/(x_1^{a_1}, ..., x_k^{a_k}) \]
and let \( N_k = (n_1, ..., n_k) \) be a sequence of positive integers. Then
\[ S_p(R) - D(R) = \binom{N_k}{T_k} \binom{2}{2} \]

**Proof.** By Theorem 3.30, the unique maximal ideal of \( R \)
is \( M = (x_1, ..., x_k) \). Thus \( M^n = (x_1, ..., x_k)^n \) which is generated by the non-zero elements in the set \( B_n = \{ x_1^{m_1}, ..., x_k^{m_k} \mid 0 \leq m_i \leq n_i \} \) and
\[ \sum m_i = n \}. \] By Theorem 2.15, the largest value of \( \binom{N_k}{n} \) is
\[ \binom{N_k}{T_k} \binom{2}{2} \] where \( T_k = \sum n_i \). So \( |B_n| = |S_{N_k,n}| = \binom{N_k}{n} \), where
Let $N_k = (n_1 + 1, \ldots, n_k + 1)$. These are $\mu(M^n) = \binom{N_k}{n}$ and

$$\sup \left\{ \binom{N_k}{n} \mid n \geq 0 \right\} = \binom{T_k}{2}.$$ 

Thus $S_p(R) = \binom{T_k}{2}$, which by the remark preceding Theorem 3.33 is also $D(R)$. Q.E.D

The following example is an application of Theorem 3.33 to the ring $R = F[x,y,z]/(x^3,y^5,z^3)$.

Example 3.34. Let $R = F[x,y,z]/(x^3,y^5,z^3)$ be a ring with maximal ideal $M = (x,y,z)$. We will calculate the nonzero powers $M^j, j=1,\ldots, 8 (= (3-1)+(5-1)+(3-1))$, of the maximal ideal in the above ring: $M^0 = 1$, $M = (x,y,z)$, $M^2 = (x^2,xy,xz,yz,y^2,z^2)$, $M^3 = (x^2y,x^2z,y^2x,y^2z,z^2x,z^2y,xyz,y^3)$, $M^4 = (x^2y^2,x^2yz,x^2z^2,y^2z^2,y^2xz,xyz^2,xyz,xy^3,y^4)$, $M^5 = (x^2y^2z,x^2yz^2,x^2y^2z^2,y^2z^2y,y^2xz^2,xyz^3,xyz,y^3z^3,y^4z^3)$, $M^6 = (x^2y^2z^2,y^3xz^2,y^3xz^2,y^4x^2,y^4z^2,y^4xz)$, $M^7 = (x^2y^3z^2,xy^4z^2,x^2y^4z)$, and $M^8 = (x^2y^4z^2)$. It is easy to see that the entries in the third row of the generalized Pascal's Triangle in Example 2.13, Figure 2.4, from Chapter 2 of this paper coincide with $\mu(M^j)$ for each of the nonzero powers $M^j$. Also, we can see that the maximum occurs at $\frac{T_k}{2} = \frac{2+4+2}{2} = 4$. So, for the ring...
\[ R = F[x, y, z]/(x^3, y^5, z^3) \] we have \( \mu(M^4) = D(R) = S_p(R) = 9 \), by our last theorem, Theorem 3.33.
REFERENCES


