2004

A partial differential equation to model the Tacoma Narrows Bridge failure

James Paul Swatzel

A PARTIAL DIFFERENTIAL EQUATION TO MODEL THE TACOMA NARROWS BRIDGE FAILURE

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

James Paul Swatzel

March 2004

A PARTIAL DIFFERENTIAL EQUATION TO MODEL THE TACOMA NARROWS BRIDGE FAILURE

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

---

by

James Paul Swatzel

March 2004

Approved by:

[redacted] 2/3/04

Shawnee McMurran, Committee Chair — Date

[redacted]

Nadejda Dyakevich, Committee Member

[redacted]

Chetan Prakash, Committee Member

[redacted]

Charles Stanton, Committee Member

[redacted]

Charles Stanton
Graduate Coordinator,
Department of
Mathematics

[redacted]

Peter Williams, Chair,
Department of Mathematics

ABSTRACT

Modern dynamical systems theory provides a means of describing how a system changes over time. On November 7, 1940, during a windstorm, the Tacoma Narrows Bridge broke apart and collapsed. This suspension bridge was more than a mile long and was known as "Galloping Gertie" because the roadbed oscillated with the wind. This thesis will examine the equation

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y + c(y) = -g$$

developed by Lazer and McKenna to model a suspension bridge in no wind. We will then examine a simple partial differential equation

$$\begin{aligned} &u_{tt} + u_{xxxx} + ku^{+} = W(x) + \varepsilon f(x,t), \\ &u(0,t) = u(L,t) = 0, \\ &u_{xx}(0,t) = u_{xx}(L,t) = 0, \end{aligned}$$

to model a suspension bridge.

ACKNOWLEDGMENTS

I would first like to thank Dr. Shawnee McMurran for her guidance on this thesis. Dr. McMurran as busy as she is, always found the time to assist me in any way that she could on this project. I am grateful to have had her for an advisor. I would also like to thank my project committee Dr. Nadejda Dyakevich, Dr. Chetan Prakash, and Dr. Charles Stanton for giving me valuable insight into this project. My thanks goes out to Dr. Terry Hallett our wonderful graduate coordinator here at CSUSB for advising me throughout the Masters degree program. I would also like to thank Dr. Janet Beery my undergraduate advisor at the University of Redlands. Without her support and encouragement I may not have majored in mathematics, hence not have come to the CSUSB math program, and thus not have written this thesis. I would also like to thank my fellow graduate students in mathematics. We spent countless hours together on Sundays in Jack Brown Hall. I would also like to thank my parents and grandparents for their support throughout my education. And certainly last but not least I want to acknowledge Kim for all of her support through this adventure.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER ONE

DYNAMICAL SYSTEMS

Modern dynamical systems theory provides a means of describing how a system changes over time. Continuous dynamical systems are described with differential equations. Equations that model "real world" applications are usually nonlinear and can therefore be complicated. The majority of nonlinear differential equations do not have a closed form solution, that is, we cannot come up with an explicit mathematical expression that solves the differential equation. However, we can analyze many of these systems using a geometric-analytic approach. This allows us to qualitatively view the behavior of the system and gain insight into how the system will behave when parameters are varied.

One technique for analyzing the behavior of a system is to look at its qualitative structure of flow. The qualitative structure of flow may change as parameters are varied. By varying the parameters of a system it is possible to create, destroy, or change the stability of fixed points. Fixed points correspond to steady states or equilibria of a system and occur where the time derivatives of the dependent variables are simultaneously zero. Qualitative changes in dynamics are called bifurcations, and the parameter values at which they occur are called bifurcation points.

Bifurcations are very important scientifically. For instance, a steel beam can support a small weight without deforming. However if we gradually increase the weight, the beam will eventually buckle due to the heavy load placed on it. This buckling point occurs at a bifurcation point.

Bifurcation points are classified by their effect on a system. The saddle-node bifurcation is a basic mechanism by which fixed points are created and destroyed. Consider the following system:

$$\begin{aligned} \dot{x} &= y - 2x \\ \dot{y} &= \mu + x^2 - y. \end{aligned} \tag{1.1}$$

The system's qualitative behavior will eventually change as the parameter μ is varied. To begin an analysis of such a system we find the fixed points for the system. Nullclines are valuable in the analysis of a system. Nullclines are defined as curves where $\dot{x}$ or $\dot{y}$ equal zero. The nullclines indicate where the flow is parallel to the coordinate axes.

Fixed points occur at the intersection of nullclines. For system (1.1) the only nullclines are the line $y = 2x$ and the parabola $y = x^2 + \mu$. The vector field is vertical along the line $y = 2x$ since $\dot{x} = 0$. The vector field is horizontal along the parabola $y = x^2 + \mu$ since $\dot{y} = 0$. Figure 1 shows the nullclines when $\mu = 0.5$.

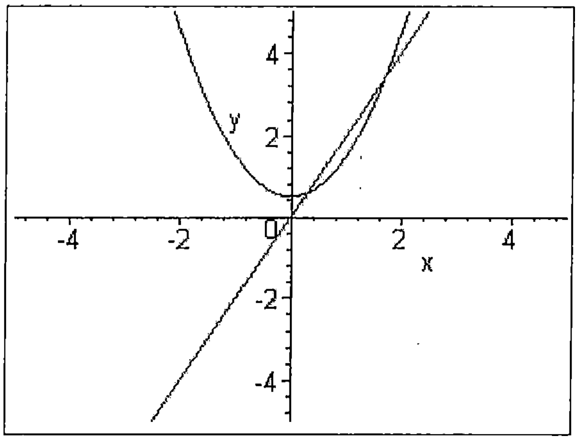

Figure 1. Intersecting Nullclines

We see that the nullclines intersect at two points. These are the fixed points for system (1.1) when $\mu = 0.5$. In general the fixed points of system (1.1) occur at

$x = 1 \pm \sqrt{1-\mu}$.

Thus, for $\mu = 0.5$, fixed points occur at $\left(1+\sqrt{0.5}, 2+2\sqrt{0.5}\right)$ and $\left(1-\sqrt{0.5}, 2-2\sqrt{0.5}\right)$. We will now attempt to classify the fixed points by examining the behavior of the system near the fixed points. In a small neighborhood around a fixed point, the solutions to a nonlinear system can often be approximated by solutions of a linear system. Frequently this will give us the information needed to determine long term behavior of the system near the fixed points.

Recall that a two-dimensional linear system is a system of the form $\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}$ where a,b,c,d are parameters.

This can be written in matrix form as $\dot{\vec{x}} = A\vec{x}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. One way to analyze a two-dimensional linear system is by looking at its phase portrait. The phase portrait gives a qualitative view of the trajectories for the system. Some systems may contain special straight-line solutions. To find these straight-line solutions we look for solutions of the form $\vec{x}(t) = e^{\lambda t}\vec{v}$, $\vec{v} \neq 0$. To find the conditions on $\vec{v}$ and λ we can substitute $\vec{x}(t) = e^{\lambda t}\vec{v}$ into $\dot{\vec{x}} = A\vec{x}$ obtaining $A\vec{v} = \lambda\vec{v}$. A scaler λ and a vector $\vec{v}$ which satisfy this equation are respectively called an eigenvalue and eigenvector of the system. Straight-line solutions exist when $\vec{v}$ is a real-valued eigenvector of A with corresponding real eigenvalue λ.[17] For a 2×2 matrix we can find the eigenvalues by solving the characteristic equation $\det(A-\lambda I)=\lambda^2-(trace(A))\lambda+\det A=0$ for λ, where $trace\,A$ is defined to be $a+d$ and $\det A=ad-bc$. Hence we find that the eigenvalues of the 2×2 system $A\bar{x}=\lambda\bar{x}$ are given by

$$\lambda=\frac{trace(A)\pm\sqrt{(trace(A))^2-4\det(A)}}{2}. \tag{1.2}$$

Thus the eigenvalues depend only on the trace and determinant of matrix A.

Let $\begin{matrix}\dot{x}=f(x,y)\\ \dot{y}=g(x,y)\end{matrix}$ be our given system with fixed point (x^*,y^*) such that $f(x^*,y^*)=0$ and $g(x^*,y^*)=0$. Let us invoke a small disturbance from the fixed point in our system. We need to see if this disturbance will grow or decay. If the disturbance grows we will call the fixed point a source. If the disturbance decays we will call the fixed point a sink.

Let $u=x-x^*$ and $v=y-y^*$ denote a small disturbance from the fixed point. So we have:

$u=x-x^*\Rightarrow\frac{d}{dt}(u)=\frac{d}{dt}(x-x^*)\Rightarrow\dot{u}=\dot{x}$. By substitution we obtain $\dot{u}=f(x^*+u,y^*+v)$. Since we are working with a function of two variables and we want a linear approximation of that function near (x^*,y^*), we must construct the tangent plane at that point. The Taylor series expansion of $f(x^*+u, y^*+u)$ about (x^*,y^*) is given by

$f(x^*,y^*)+u\cdot\left[\frac{\partial f}{\partial x}(x^*,y^*)\right]+v\cdot\left[\frac{\partial f}{\partial y}(x^*,y^*)\right]+O(u^2,v^2,uv)$. Thus

$$\begin{aligned}\dot{u} &= f(x^*,y^*)+u\cdot\left[\frac{\partial f}{\partial x}(x^*,y^*)\right]+v\cdot\left[\frac{\partial f}{\partial y}(x^*,y^*)\right]+O(u^2,v^2,uv)\\ &= u\cdot\left[\frac{\partial f}{\partial x}(x^*,y^*)\right]+v\cdot\left[\frac{\partial f}{\partial y}(x^*,y^*)\right]+O(u^2,v^2,uv)\end{aligned}$$

since $f(x^*,y^*)=0$. Similarly

$$\begin{aligned}\dot{v} &= g(x^*,y^*)+u\cdot\left[\frac{\partial g}{\partial x}(x^*,y^*)\right]+v\cdot\left[\frac{\partial g}{\partial y}(x^*,y^*)\right]+O(u^2,v^2,uv)\\ &= u\cdot\left[\frac{\partial g}{\partial x}(x^*,y^*)\right]+v\cdot\left[\frac{\partial g}{\partial y}(x^*,y^*)\right]+O(u^2,v^2,uv)\end{aligned}$$

since $g(x^*,y^*)=0$. Thus $(\dot{u},\dot{v})$ can be expressed by the matrix equation

(1.3)
$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix}=\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}_{(x^*,y^*)}\begin{pmatrix}u\\ v\end{pmatrix}+O(u^2,v^2,uv)\ .$$

The matrix $J(x,y)=\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}$ is called the Jacobian matrix.

The Jacobian can be viewed as a multivariable analog of the derivative. When the quadratic terms $O(u^2,v^2,uv)$ are close to zero we may approximate (1.3) with the linearized system

(1.4) $$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The Jacobian for example (1.1) is given by $J = \begin{pmatrix} -2 & 1 \\ 2x & -1 \end{pmatrix}$. When we evaluate J at the fixed points of system (1.1) with $\mu = 0.5$ we obtain $J(1+\sqrt{0.5}, 2+2\sqrt{0.5}) = \begin{pmatrix} -2 & 1 \\ 2+2\sqrt{0.5} & -1 \end{pmatrix}$ and $J(1-\sqrt{0.5}, 2-2\sqrt{0.5}) = \begin{pmatrix} -2 & 1 \\ 2-2\sqrt{0.5} & -1 \end{pmatrix}$. Near $\left(1+\sqrt{0.5}, 2+2\sqrt{0.5}\right)$ our system should resemble the linearized system

(1.5) $$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2+2\sqrt{0.5} & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

and near $\left(1-\sqrt{0.5}, 2-2\sqrt{0.5}\right)$ our system should resemble the linearized system

(1.6) $$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2-2\sqrt{0.5} & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

We will use linear approximations to classify our fixed points. Before we do this, we will briefly review different types of fixed points. A sink is a fixed point where all solutions near (x^*,y^*) approach (x^*,y^*) as $t \to \infty$. If no straight line solutions exist, then the system has complex eigenvalues. Solutions will be of the form $\vec{x}(t) = e^{\lambda t}\vec{v}$, $\vec{v} \neq 0$, where $\lambda = \alpha \pm i\omega$. Thus we will have trajectories of the form $x(t) = c_1 e^{\alpha t}\cos\omega t + c_2 e^{\alpha t}\sin\omega t$. If $\alpha < 0$, the trajectories will approximate exponentially decaying oscillations, thus solutions will spiral toward the fixed point. We call such a fixed point a spiral sink. A sink is a stable fixed point. A source is a fixed point where all solutions with an initial condition near (x^*,y^*) move away from (x^*,y^*) as $t \to \infty$. If λ is complex with $\alpha > 0$ the trajectories will approximate exponentially growing oscillations, thus solutions will spiral out from the fixed point. We call such a fixed point a spiral source. A source is an unstable fixed point.

The classification of fixed points depends on the eigenvalues of the Jacobian matrix evaluated at the given fixed point. If both eigenvalues are negative, we have a sink. If the eigenvalues are complex with negative real parts, then we have a spiral sink. If both eigenvalues are positive, we have a source. If the eigenvalues are complex with positive real parts, then we have a spiral source. If the eigenvalues are purely imaginary the fixed point of the linear system is a center. Trajectories about a center are closed orbits that neither spiral into a sink nor spiral out like a source. In this situation, the classification of fixed points based on the linear system is inconclusive. The fixed point of the nonlinear system could be a center, spiral source, or spiral sink. Another possibility is a system for which one eigenvalue is positive and one eigenvalue is negative. When this is the case, the fixed point is called a saddle. A saddle is considered a semi - stable fixed point since we have one set of straight line solutions approaching it and one set of straight line solutions moving away from it. Systems for which one or both eigenvalues are zero require further analysis. Thus, in order to classify a fixed point for a system, it would be helpful to find the eigenvalues of the linearized system about the fixed point. The eigenvalues of the matrix

$A = \begin{pmatrix} -2 & 1 \\ 2+2\sqrt{0.5} & -1 \end{pmatrix}$ in system (1.5) are given by

$\lambda_A = \dfrac{-3 \pm \sqrt{9+8\sqrt{0.5}}}{2}$. Thus, this system has one positive eigenvalue which is approximately equal to 0.414, and one negative eigenvalue which is approximately equal to -3.414. Hence we have a saddle at $\left(1+\sqrt{0.5}, 2+2\sqrt{0.5}\right)$ as in figure 2.

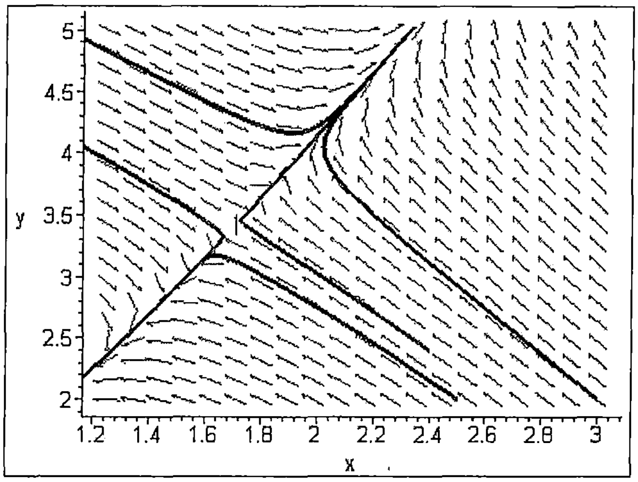

Figure 2. Phase Portrait, Near a Saddle

Similarly we find the eigenvalues of the matrix $B=\begin{pmatrix} -2 & 1 \\ 2-2\sqrt{0.5} & -1 \end{pmatrix}$ in system (1.6) are given by $\lambda_B = \dfrac{-3 \pm \sqrt{9-8\sqrt{0.5}}}{2}$. Since both eigenvalues are negative we have a sink at the fixed point $(1-\sqrt{0.5}, 2-2\sqrt{0.5})$.

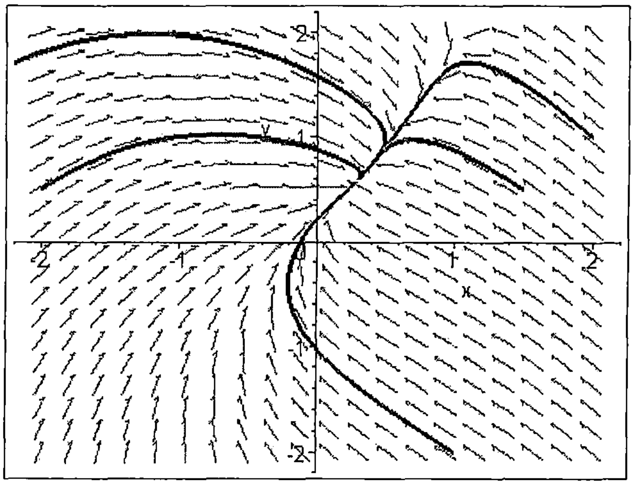

Figure 3. Phase Portrait, Near a Sink

CHAPTER TWO

THE TACOMA NARROWS BRIDGE

On July 1, 1940 the Tacoma Narrows Bridge was opened to traffic. Construction of the bridge took 19 months at a cost of 6.4 million dollars. The bridge was constructed at the beginning of World War II as a defense measure. Plans were to connect Seattle and Tacoma with the Puget Sound Navy Yard at Bremerton, Washington. This bridge was no ordinary suspension bridge. The bridge had been given the nickname "Galloping Gertie" even before being opened for traffic. The bridge received this nickname because the roadbed would oscillate dramatically in the wind.

This oscillation of the bridge was disconcerting to many of the consulting engineers called on to address the problem. Tacoma Narrows was no small bridge. In fact, at the time it was constructed, it was the third largest suspension span bridge in the world. The 2,800 foot center span stretched between two 425 foot towers, and the side spans were each 1,100 feet long. The design trend of the time was to streamline things as much as possible. This may have been one of the factors that led to the bridge's demise. The Tacoma Narrows Bridge had a very slender roadway which arched between the towers. The roadway was only 26 feet wide, and the actual deck of the bridge was 39 feet wide with the sidewalks and stiffening girders included. The problem with Tacoma Narrows occurs in the relation between the depth of the stiffening girders to the length of the center span.

Many suspension bridges failed in the eighteenth and nineteenth centuries. These failures usually exhibited some kind of twisting or torsional movement before collapse. The bridges that failed were very narrow in relation to their width. The Golden Gate Bridge is considered to be a thin bridge and has a width to length ratio of 1 to 47, whereas Tacoma Narrows had a width to length ratio of 1 to 72. This "thinness" makes the bridge very weak torsionally, meaning it is quite susceptible to twisting motions, especially when there is not sufficient stiffening built into the bridge. This caused the bridge to oscillate even during light winds. Engineers tried several methods to stabilize the bridge and control the oscillations. One method was attaching 1.5 inch steel tie-down cables close to each end of the bridge. These cables were then anchored to 50-ton concrete blocks as a means to dampen the oscillations. This attempt to curb the oscillations failed. The steel cables snapped during the first windstorm following their installation. Engineers tried other methods, all to no avail. The bridge failed on November 7, 1940.

To begin our analysis of the bridge failure we will look at a one-dimensional model developed by A.C. Lazer and P.J. McKenna. Lazer and McKenna developed the following equation to model a suspension bridge in no wind:

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y + c(y) = -g \tag{2.1}$$

This equation is based on the following assumption. We consider the bridge to exhibit an up and down vertical oscillation. This vertical position of the bridge will be denoted as $y(t)$, with $y=0$ corresponding to the rest position of the bridge, $y<0$ corresponding to stretched cables, and $y>0$ corresponding to slack cables. See figure 4.

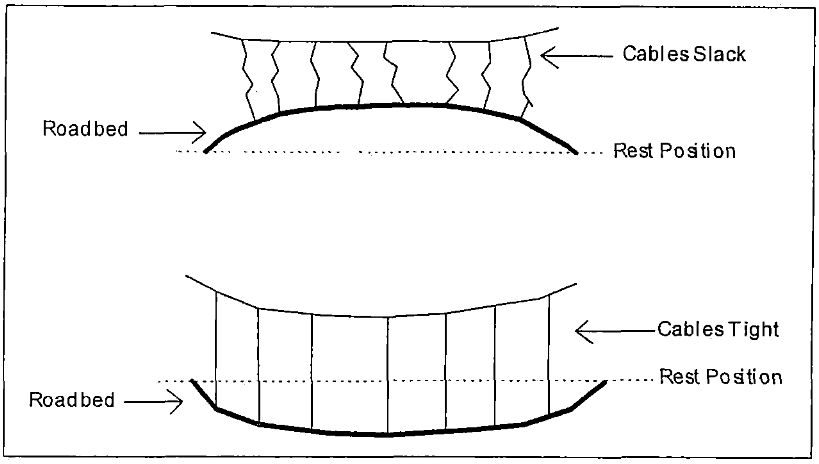

Figure 4. Bridge, Cables Tight and Slack

In developing the model, many forces that act on the bridge must be considered. Gravity provides a constant downward force. Gravity is acting in the negative y direction. If the cables are stretched, that is $y<0$, then the cables will provide a force that will pull the bridge back towards its rest position. When $y>0$ the cables are slack so they provide no force. The roadbed is a semi-rigid object due to its composition, thus when $y \neq 0$ a restoring force pulls the roadbed back to its rest position, $y=0$. We should also include a damping term since the roadbed is being pulled back to its original position when we have an oscillation. We shall assume the damping term to be proportional to $\frac{dy}{dt}$. In order to simplify the model, units are chosen so that the mass of the bridge is 1. We now examine equation (2.1). Since this equation models the vertical position $y(t)$, the first term $\frac{d^2y}{dt^2}$ represents vertical acceleration. The second term $\alpha\frac{dy}{dt}$ represents damping. The third term βy represents the restoring force, the force exerted by the material of the bridge to pull it back into its resting position $y=0$. The behavior of the stretched cable is similar to the behavior of a spring. Hence the piecewise function $c(y)=\begin{cases}\gamma y, & \text{if } y<0\\ 0, & \text{if } y\geq 0\end{cases}$ represents the force exerted by the cable when $y<0$ where γ can be thought of as the spring constant. The constant g represents the force due to gravity. We can now convert this second order equation into a first order system of equations. Let $\frac{dy}{dt}=v$. Then $\frac{d^2y}{dt^2}=\frac{dv}{dt}$ and we obtain the first order system of equations:

(2.2)
$$\begin{aligned}\frac{dy}{dt}&=v\\ \frac{dv}{dt}&=-\beta y-c(y)-\alpha v-g.\end{aligned}$$

We may simplify our system by combining the $-\beta y$ term with the terms in $c(y)$ to obtain the function $h(y)$ where

$$h(y) = \begin{cases} (\beta + \gamma)y & \text{if } y < 0; \\ \beta y & \text{if } y \geq 0. \end{cases}$$

Thus we obtain the first order system of equations

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -h(y) - \alpha v - g. \end{aligned} \tag{2.3}$$

To get an idea of the behavior of this system we look at its corresponding phase plane. In order to do this we must choose particular values for the parameters. We shall follow the example of Lazer and McKenna and choose $\beta = 13$, $\gamma = 4$, and $\alpha = 0.01$. In our system g represents acceleration due to gravity, hence we let $g = 9.8$. Thus our system (2.3) becomes

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -h(y) - 0.01v - 9.8 \end{aligned} \tag{2.4}$$

where

$$h(y) = \begin{cases} (13 + 4)y, & \text{if } y < 0; \\ 13y, & \text{if } y \geq 0. \end{cases}$$

When $y<0$, system (2.4) has an equilibrium point at $(y,v)=\left(-\frac{49}{85},0\right)$. Evaluating the Jacobian at this fixed point, we find $J\left(\frac{-49}{85},0\right)=\begin{pmatrix} 0 & 1 \\ -17 & -0.01 \end{pmatrix}$. We shall now attempt to determine the stability of the equilibrium point. With $trace(J)=-0.01$ and $\det(J)=17$, (1.2) gives eigenvalues $\lambda \approx \frac{-0.01 \pm \sqrt{-67.999}}{2}$. Thus we have a stable spiral sink at $\left(\frac{-49}{85},0\right)$. The corresponding phase portrait for the system is shown below.

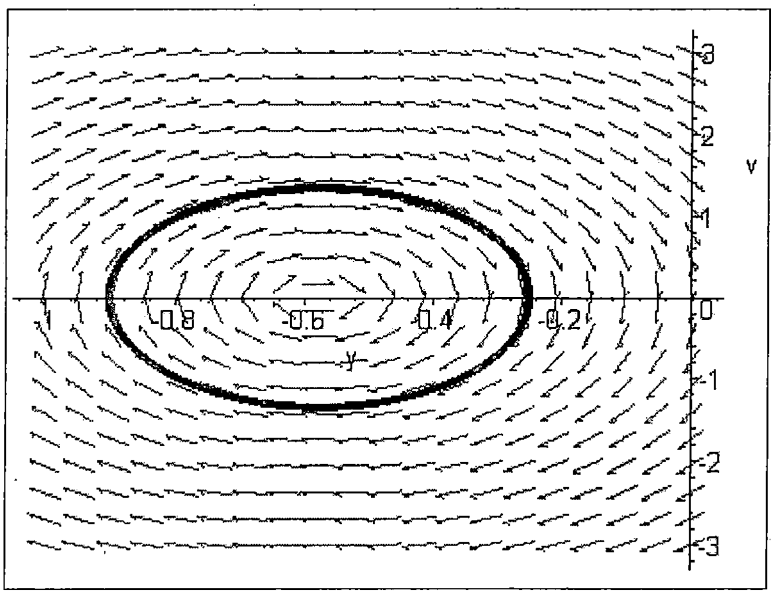

Figure 5. Phase Portrait, Spiral Sink

The phase portrait confirms our previous analysis that the solution trajectory spirals in toward the equilibrium point, that is, that the bridge tends toward its rest position. However, we note that the portrait indicates that the amplitude of the oscillations decreases slowly.

We add the effect of wind to the model by including a nonlinear forcing term. Since wind speed and direction are not constant, modeling the wind can be very complicated. To simplify the model we will assume the forcing term $A\sin\mu t$ with a range of values for A and μ to be determined. Thus our modified system becomes

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -h(y) - 0.01v - 9.8 + A\sin\mu t. \end{aligned} \tag{2.5}$$

We now analyze our modified system in order to evaluate the effect of the forcing term on its behavior. Following the example of Glover, Lazer, and McKenna [7], let A be very small $(A < 0.05)$ so that we have small forcing in our system such as a light wind. Let's look at y versus t for the bridge if we let $A = 0.04$ and $\mu = 4$. This gives us a forcing component of relatively small magnitude. The time series graph for position y with initial conditions $y(0) = \frac{-10}{17}$ and $v(0) = 1$ is shown in figure 6:

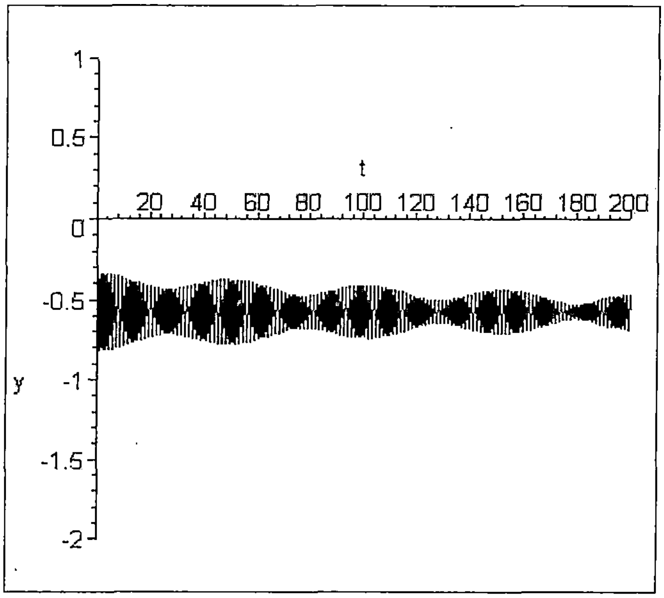

Figure 6. Time Series

Here we have an oscillation of a small amplitude. Notice $y<0$ for all time. This is due to gravity constantly pulling downward on the bridge. Thus in a light wind the cables will remain stretched ($y<0$), acting as linear springs. We conclude that with very small forcing term the solution never reaches $y=0$. Interpreting this result with regards to the bridge indicates that when there is light wind the bridge will oscillate with small amplitude.

The system behaves quite differently when $v(0)$ is increased. Suppose $v(0)=5$ which can be thought of as a large gust of wind. The time series graph of the vertical displacement for the bridge is illustrated in figure 7:

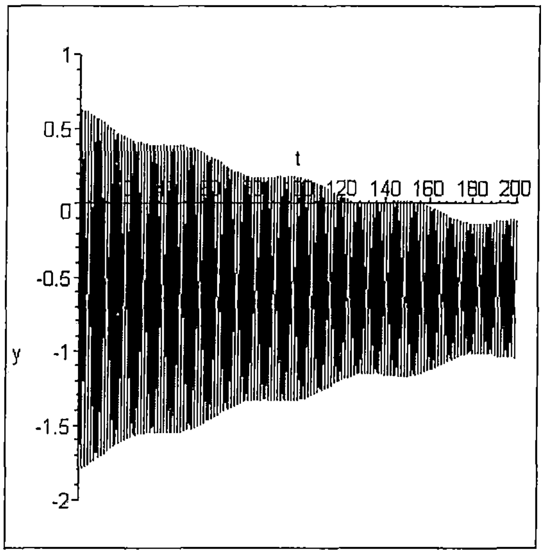

Figure 7. Time Series

Notice the bridge rises above $y=0$. This in turn makes our linear model inaccurate. When the bridge rises above $y=0$ the cables that were acting as linear springs suddenly go slack. This brings non-linearity into the system. Hence we need a better model.

CHAPTER THREE

THE PARTIAL DIFFERENTIAL EQUATION

The one-dimensional model is useful for giving us some insight into the bridge behavior, but we need a better model if we are to study the effects of outside forces, such as wind, on the bridge. We will now examine a simple partial differential equation model developed by Lazer and McKenna. In this model we make the following assumptions. We will treat the cables as non-linear springs under tension where, if the cables are stretched, then the restoring force is proportional to the distance the cables were stretched. If the cables are slack there will be no restoring force. We will also consider the roadbed as a one-dimensional vibrating beam of length L with hinged ends. The movement of the cables will be ignored. The cables will be used only to transmit force to the roadbed. According to Lazer and McKenna we have the partial differential equation model

$$\begin{aligned} &u_{tt} + u_{xxxx} + ku^{+} = W(x) + \varepsilon f(x,t), \\ &u(0,t) = u(L,t) = 0, \\ &u_{xx}(0,t) = u_{xx}(L,t) = 0, \end{aligned} \tag{3.1}$$

where $u^+ = \max\{u,0\}$. Why did Lazer and McKenna choose this partial differential equation model? This was my question and the main driving force behind this paper.

Recall the bridge is being modeled as a vibrating beam, which is hinged at both endpoints. Cables that stretch, but do not compress are supporting the bridge. The downward deflection of the beam is given by $u(x,t)$ where the restoring force of the cables is equal to ku if u is positive (downward), and zero if u is negative (upward). The weight per unit length of the beam is given by $W(x)$, where we will assume the beam is length L and constant density with $\varepsilon f(x,t)$ acting as an external forcing term.

To analyze the bending of the beam consider the following figure:

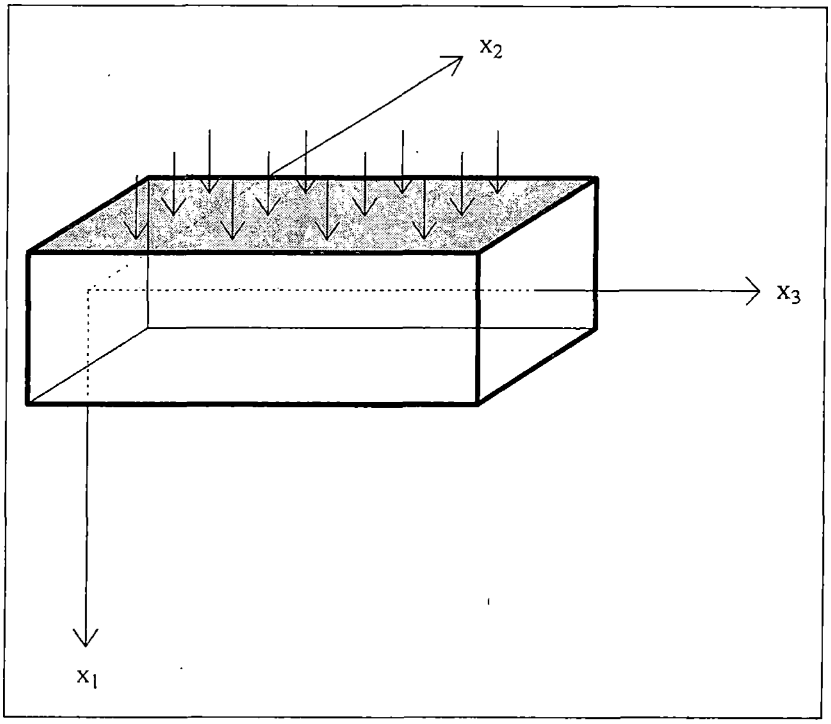

Figure 8. Beam Diagram

Here we will assume the top surface is horizontal where A represents the cross-section area, σ represents the mass per unit volume, and $\rho = \sigma A$ represents the mass per unit length. Let $\mathbf{f}(\mathbf{x},t) = f(x_3,t)\mathbf{i}$ be a force given in terms of its distribution per unit length that acts in the vertical direction such that it varies only along the length of the beam. The following figure shows a portion of the beam taken between two arbitrary points a and b:

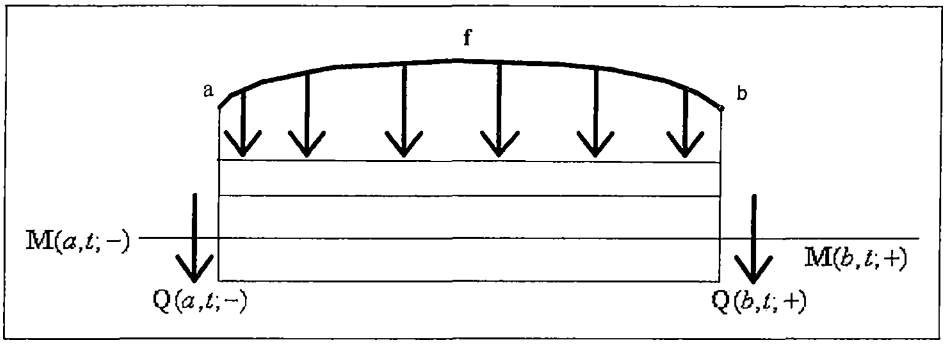

Figure 9. Forces Acting On Bridge

We also have forces that are acting on the ends of the beam. These forces can be resolved into vertical and horizontal components.

In figure 9, the shear force $\mathbf{Q}$ given by $\mathbf{Q}(b,t;+)$ represents the total vertical force exerted by the material on the side $x_3 > b$ on the material on the side $x_3 < b$ across the cross section $x_3 = b$. $\mathbf{Q}(a,t;-)$ represents the total vertical force exerted by the material on the side $x_3 < a$ on the material on the side $x_3 > a$ across the cross section $x_3 = a$. Since the shear force acts vertically in the x_1-direction we can represent the shear force as

(3.2) $$\mathbf{Q}(b,t;+) = Q(b,t)\mathbf{i} \text{ and } \mathbf{Q}(b,t;-) = -Q(b,t)\mathbf{i}.$$

In equation (3.2),

$$Q(b,t) = \iint_A T_{31}(x_1, x_2, b, t)\,dx_1 dx_2$$

represents the magnitude of the net vertical force due to the stress. T_{31} represents the 1st component of the stress tensor whose exterior normal points in the direction of the 3rd Cartesian base vector $\vec{k}$.

The bending moment $\mathbf{M}$ given by $\mathbf{M}(b,t;+)$ represents the total moment exerted by the material on the side $x_3 > b$ on the material on the side $x_3 < b$ across the cross section $x_3 = b$. $\mathbf{M}(a,t;-)$ represents the total moment exerted by the material on the side $x_3 < a$ on the material on the side $x_3 > a$ across the cross section $x_3 = a.$ Thus we can represent the bending moment as

(3.3) $$\mathbf{M}(b,t;+) = M(b,t)\mathbf{j} \text{ and } \mathbf{M}(b,t;-) = -M(b,t)\mathbf{j}.$$

In order to derive our model we assume the center line of the beam moves only vertically with displacement $y(x_3,t)$. All vertical displacements are equal to the vertical displacement of the center line, there is no displacement in the x_2-direction, and plane sections that are normal to the center line before deformation will remain planar and normal to the deformed center line.

Let $\mathbf{u}$ be a displacement vector with components u_1, u_2, and u_3. In order to analyze the displacement $\mathbf{u}(Q)$ of a point Q on the beam it is helpful to look at the following figure which represents deformation of a small segment of the beam.

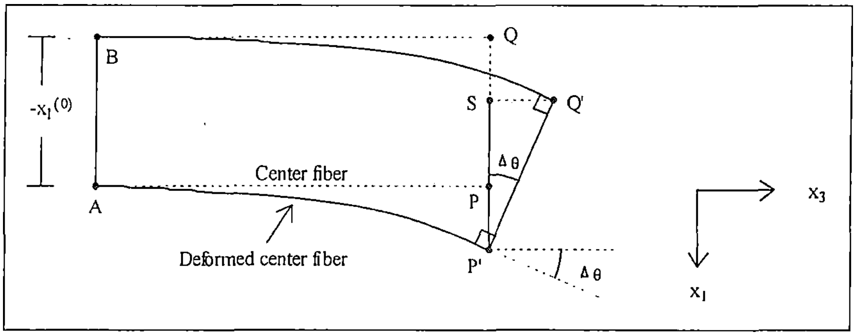

Figure 10. Beam Deformation

Let u_1 be displacement in the x_1 direction. Thus $u_1 = y(x_3, t)$ since the center line of the beam moves only vertically with the displacement $y(x_3,t)$, and all vertical displacements are equal to the vertical displacement of the center line. Let u_2 be displacement in the x_2-direction, thus $u_2 = 0$ since there is no displacement in the x_2-direction. Let Q be a point on the beam before deformation occurs. Now we can describe the displacement of u_3 by considering point Q. Point Q undergoes both horizontal and vertical displacement. Since all vertical displacements are equal to the vertical displacement of the center line, the vertical displacement, QS, is equal in magnitude to that of PP'. From figure 10 we see that $SP' = SP + PP' = SP + QS = QP$. In order to compute the horizontal displacement given by SQ' we consider triangle $SP'Q'$. Let the "displacement angle", $\angle SP'Q'$, be denoted by $\Delta\theta$. Then $\tan(\Delta\theta) = \dfrac{SQ'}{SP'}$. Hence $SQ' = SP' \tan(\Delta\theta)$. Thus $u_3(x_1, x_3, t) = -x_1 \tan(\Delta\theta)$. Since plane sections that are normal to the center line before deformation will remain planar and normal to the deformed center line, $\angle AP'Q'$ is a right angle. Hence $\tan(\Delta\theta) = \dfrac{\partial y(x_3,t)}{\partial x_3}$. Thus $u_3 = -x_1 \dfrac{\partial y(x_3,t)}{\partial x_3}$.

Now that we have described the components u_1, u_2, and u_3 of our displacement vector $\mathbf{u}$, we can write down the equations of motion governing a portion of the beam. Given an arbitrary portion of the beam between $x_3 = a$ and $x_3 = b$ as in figure 10 we have

(3.4) $$\mathbf{Q}(a,t;-) = -\mathbf{i}\left[\iint_A T_{31}(x_1, x_2, a, t)\,dx_1 dx_2\right]$$

and

(3.5)
$$\mathbf{Q}(b,t;+) = \mathbf{i}\left[\iint_A T_{31}(x_1,x_2,b,t)dx_1dx_2\right].$$

$\mathbf{Q}(a,t;-)$ and $\mathbf{Q}(b,t;+)$ can be added to get

(3.6)
$$\mathbf{Q}(a,t;-)+\mathbf{Q}(b,t;+) = \mathbf{i}\left[\iint_A \left[T_{31}(x_1,x_2,b,t)-T_{31}(x_1,x_2,a,t)\right]dx_1dx_2\right].$$

From the Fundamental Theorem of Calculus we have

(3.7)
$$T_{31}(x_1,x_2,b,t)-T_{31}(x_1,x_2,a,t) = \int_a^b \frac{\partial}{\partial x_3}\left(T_{31}(x_1,x_2,x_3,t)\right)dx_3,$$

where we assume T_{31} is continuous with respect to x_3. By substitution we obtain

(3.8)
$$\mathbf{Q}(a,t;-)+\mathbf{Q}(b,t;+) = \mathbf{i}\left[\iint_A \int_a^b \frac{\partial}{\partial x_3}\left(T_{31}(x_1,x_2,x_3,t)\right)dx_3dx_1dx_2\right].$$

Applying Fubini's Theorem to (3.8) we obtain

(3.9)
$$\mathbf{Q}(a,t;-)+\mathbf{Q}(b,t;+) = \mathbf{i}\left[\int_a^b \iint_A \frac{\partial}{\partial x_3}\left(T_{31}(x_1,x_2,x_3,t)\right)dx_1dx_2dx_3\right].$$

The equation

(3.10)
$$\sigma\frac{Dv_i}{Dt} = \sigma f_i + \frac{\partial T_{ji}}{\partial x_j}$$

is a standard equation from continuum mechanics which represents the balance of linear momentum. Here f_i and σ

represent force per unit volume and mass per unit length respectively. Since f_i represents the force per unit volume and $\mathbf{f}(\mathbf{x},t) = f(x_3,t)\mathbf{i}$ represents the force per unit length as previously defined, we have $f(x_3,t) = \iint_A f_i$. Note that the velocity is given by

$$v_i = \frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + \sum_{j=1}^{3} u_j \frac{\partial u_i}{\partial x_j} \approx \frac{\partial u_i}{\partial t}. \tag{3.11}$$

We shall assume the displacements we are encountering are small, hence we approximate v_i by

$$v_i \approx \frac{\partial u_i}{\partial t} . \tag{3.12}$$

In the same manner we obtain

$$\frac{Dv_i}{Dt} \approx \frac{\partial^2 u_i}{\partial t^2}. \tag{3.13}$$

Substitution of the approximation (3.13) into the balance of linear momentum equation (3.10) gives us

$$\sigma \frac{\partial^2 u_i}{\partial t^2} = \sigma f_i + \frac{\partial T_{ji}}{\partial x_j}. \tag{3.14}$$

In particular

(3.15)
$$\sigma\frac{\partial^2 u_1}{\partial t^2} = \sigma f_i + \frac{\partial\left(T_{31}(x_1, x_2, x_3, t)\right)}{\partial x_3}.$$
$$\Rightarrow\ \frac{\partial\left(T_{31}(x_1, x_2, x_3, t)\right)}{\partial x_3} = \sigma\frac{\partial^2 u_1}{\partial t^2} - \sigma f_i.$$

Substituting (3.15) into (3.9) we obtain

(3.16)
$$\mathbf{Q}(a,t;-) + \mathbf{Q}(b,t;+) = \mathbf{i}\left[\int_a^b \iint_A \left[\sigma\frac{\partial^2 u_1}{\partial t^2} - \sigma f_i\right] dx_1 dx_2 dx_3\right].$$

We claim that

(3.17)
$$\mathbf{i}\int_a^b \iint_A \frac{\partial^2 u_1}{\partial t^2} = \int_a^b \iint_A \frac{\partial^2 \mathbf{u}}{\partial t^2}.$$

To establish this result, writing $\mathbf{u}$ in terms of its components gives us

(3.18)
$$\begin{aligned}\int_a^b \iint_A \left[\frac{\partial^2 \mathbf{u}}{\partial t^2}\right] dx_1 dx_2 dx_3 &= \mathbf{i}\int_a^b \iint_A \frac{\partial^2 u_1}{\partial t^2} dx_1 dx_2 dx_3 \\ &+ \mathbf{j}\int_a^b \iint_A \frac{\partial^2 u_2}{\partial t^2} dx_1 dx_2 dx_3 \\ &+ \mathbf{k}\int_a^b \iint_A \frac{\partial^2 u_3}{\partial t^2} dx_1 dx_2 dx_3.\end{aligned}$$

However

(3.19)
$$\mathbf{j}\int_a^b \iint_A \frac{\partial^2 u_2}{\partial t^2} dx_1 dx_2 dx_3 = 0$$

since $u_2 = 0$. Also

$$\mathbf{k}\left[\int_a^b\iint_A \frac{\partial^2 u_3}{\partial t^2}\,dx_1dx_2dx_3\right] = \mathbf{k}\left[\int_a^b\iint_A \frac{\partial^2}{\partial t^2}\left[-x_1\,\frac{\partial y(x_3,t)}{\partial x_3}\right]dx_1dx_2dx_3\right]$$
$$= \mathbf{k}\left[\int_a^b\iint_A -x_1\,\frac{\partial^2}{\partial t^2}\left[\frac{\partial y(x_3,t)}{\partial x_3}\right]dx_1dx_2dx_3\right]$$
$$= \mathbf{k}\left[\left[\int_a^b \frac{\partial^2}{\partial t^2}\left[\frac{\partial y(x_3,t)}{\partial x_3}\right]dx_3\right]\cdot\left[\iint_A -x_1\,dx_1dx_2\right]\right].$$

Since x_1, x_2 pass through the centroid of the beam as in figure 8, we have $\iint_A -x_1dx_1dx_2 = 0$. Hence

(3.20)
$$\mathbf{k}\left[\int_a^b\iint_A \frac{\partial^2 u_3}{\partial t^2}\,dx_1dx_2dx_3\right] = 0\,.$$

Substituting (3.19) and (3.20) into (3.18) we see that (3.17) holds. So (3.16) becomes

(3.21)
$$\mathbf{Q}(a,t;-)+\mathbf{Q}(b,t;+) = \sigma\left[\int_a^b\iint_A \frac{\partial^2 \mathbf{u}}{\partial t^2}\,dx_1dx_2dx_3\right] - \sigma\mathbf{i}\left[\int_a^b\iint_A f_i\,dx_1dx_2dx_3\right]$$

which implies

(3.22)
$$\mathbf{Q}(a,t;-)+\mathbf{Q}(b,t;+) + \sigma\mathbf{i}\int_a^b\iint_A f_i\,dx_1dx_2dx_3 = \sigma\left[\int_a^b\iint_A \frac{\partial^2 \mathbf{u}}{\partial t^2}\,dx_1dx_2dx_3\right].$$

We recall that $f(x_3,t) = \iint_A f_i$. We substitute this into (3.22) to obtain

(3.23) $$\mathbf{Q}(a,t;-)+\mathbf{Q}(b,t;+)+\sigma\mathbf{i}\int_a^b f(x_3,t)dx_3=\sigma\left[\int_a^b\iint_A\frac{\partial^2\mathbf{u}}{\partial t^2}dx_1dx_2dx_3\right].$$

Note that $\iint_A\frac{\partial^2\mathbf{u}}{\partial t^2}dx_1dx_2=\mathbf{i}\iint_A\frac{\partial^2 y(x_3,t)}{\partial t^2}dx_1dx_2$ since $u_2=0$. Therefore we obtain

(3.24) $$\iint_A\frac{\partial^2\mathbf{u}}{\partial t^2}dx_1dx_2=\mathbf{i}\iint_A\frac{\partial^2 y(x_3,t)}{\partial t^2}dx_1dx_2=A\frac{\partial^2 y(x_3,t)}{\partial t^2}\mathbf{i}.$$

Substitution of (3.24) and (3.2) into (3.23) yields

(3.25) $$\sigma\int_a^b f(x_3,t)dx_3+Q(b,t)-Q(a,t)=\sigma A\int_a^b\frac{\partial^2 y(x_3,t)}{\partial t^2}dx_3\ .$$

From the Fundamental Theorem of Calculus we can write $Q(b,t)-Q(a,t)=\int_a^b\frac{\partial Q(x_3,t)}{\partial x_3}dx_3$ where we assume Q is continuous with respect to x_3. Thus we obtain

(3.26) $$\int_a^b\left[f(x_3,t)+\frac{\partial Q(x_3,t)}{\partial x_3}-\rho\frac{\partial^2 y(x_3,t)}{\partial t^2}\right]dx_3=0\,,$$

where we recall that $\rho=\sigma A$. We claim that the integrand of (3.26) is identically equal to zero.

Lemma 1: Suppose $\int_a^b f(x)dx=0$ for every interval $[a,b]$ contained in an interval I. If $f(x)$ is continuous on I, then $f(x)\equiv 0$ on I.

Proof: Suppose $f(x_1)\neq 0$ for some $x_1\in I$ where x_1 is an interior point of I. $f(x_1)$ can either be positive or negative. Without loss of generality we let $f(x_1)>0$. Since f is continuous there exists $\delta>0$ such that $f(x)>0$ for $x\in[x_1-\delta,x_1+\delta]$. Then, by the Mean Value Theorem, $\int_{x_1-\delta}^{x_1+\delta} f(x)dx = f(x^*)\cdot[(x_1+\delta)-(x_1-\delta)]$ for some $x^*\in(x_1+\delta,x_1-\delta)$. This implies $\int_{x_1-\delta}^{x_1+\delta} f(x)dx = f(x^*)\cdot 2\delta>0$. We have a contradiction. The argument is similar if x_1 is an endpoint if we replace $[x_1-\delta,x_1+\delta]$ with $(x_1-\delta,x_1]$ or $[x_1,x_1+\delta)$. □

Recall that the interval $[a,b]$ was an arbitrary interval contained in $[0,L]$, where L is the length of the beam. Hence, it follows from (3.26) and Lemma 1 that

$$\frac{\partial Q(x_3,t)}{\partial x_3}+f(x_3,t)=\rho\frac{\partial^2 y(x_3,t)}{\partial t^2}. \tag{3.27}$$

This is the partial differential equation describing the balance of linear momentum.

Now we must describe the balance of angular momentum where the moments are taken about the point $x_1=x_2=0,\ x_3=a$. Let h represent the width in the x_2 direction of the top of the beam where the external load $\mathbf{f}(x_3,t)$ is applied. We define torque, τ, as a vector quantity relative to a fixed point given by $\tau = \mathbf{x} \times \mathbf{F}$ where $\mathbf{F}$ is a force applied to a particle and $\mathbf{x}$ is a position vector locating the particle relative to the fixed point.[8] The angular momentum $\mathbf{l}$ of a particle with linear momentum $\mathbf{p}$, mass m, and linear velocity $\mathbf{v}$ is a vector quantity defined relative to a fixed point as $\mathbf{l} = m(\mathbf{r} \times \mathbf{v})$. We can take the sum of the angular momenta of the individual particles and obtain $\mathbf{L}$, the angular momentum of a system of particles. Thus $\mathbf{L} = \mathbf{l}_1 + \mathbf{l}_2 + \cdots + \mathbf{l}_n = \sum_{i=1}^{n} \mathbf{l}_i$. [8] Newton's second law for a particle can be written in angular form as $\sum \tau = \frac{d\mathbf{l}}{dt}$, where $\sum \tau$ is the net torque acting on the particle, and $\mathbf{l}$ is the angular momentum of the particle. Hence the time rate of change of angular momentum for a system of particles is equal to the sum of the external torques on the system. Therefore $\sum \tau = \frac{d\mathbf{L}}{dt}$.

$\mathbf{M}(a,t;-) + \mathbf{M}(b,t;+)$ represents the torques exerted by the material where $x_3 < a$ and the material where $x_3 > b$. The integral $\int_a^b \int_{-h/2}^{h/2} (\mathbf{x} - a\mathbf{k}) \wedge \left(\frac{\mathbf{f}}{h}\right) dx_2 dx_3$ represents the torque acting on the beam between a and b, and $(b\mathbf{k} - a\mathbf{k}) \wedge \mathbf{Q}(b,t;+)$ represents the shearing force acting on the beam. The operation defined by $\wedge$ acts as a cross or vector product for our situation.

Recall that $\mathbf{l} = m(\mathbf{r} \times \mathbf{v})$ represents the angular momentum for a particle. Thus $\left[(\mathbf{x} - a\mathbf{k}) \wedge \sigma \frac{\partial \mathbf{u}}{\partial t}\right]$ represents the angular momentum for a particle of the beam relative to point a. Hence the angular momentum for the beam relative to point a is given by $\int_a^b \iint_A \left[(\mathbf{x} - a\mathbf{k}) \wedge \sigma \frac{\partial \mathbf{u}}{\partial t}\right] dx_1 dx_2 dx_3$. Therefore we can equate the torques acting on the beam to the rate of change of angular momentum obtaining

$$\text{(3.28)} \qquad \mathbf{M}(a,t;-) + \mathbf{M}(b,t;+) + \int_a^b \int_{-h/2}^{h/2} (\mathbf{x} - a\mathbf{k}) \wedge \left(\frac{\mathbf{f}}{h}\right) dx_2 dx_3 + (b\mathbf{k} - a\mathbf{k}) \wedge \mathbf{Q}(b,t;+)$$

$$= \frac{\partial}{\partial t} \int_a^b \iint_A \left[(\mathbf{x} - a\mathbf{k}) \wedge \sigma \frac{\partial \mathbf{u}}{\partial t}\right] dx_1 dx_2 dx_3 .$$

We calculate the vector products of the right hand side of (3.28) to obtain

$$(\mathbf{x} - a\mathbf{k}) \wedge \frac{\partial \mathbf{u}}{\partial t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 - a \\ \frac{\partial y(x_3,t)}{\partial t} & 0 & -\frac{\partial x_1 \partial y(x_3,t)}{\partial t \partial x_3} \end{vmatrix} .$$

It follows that

$$(\mathbf{x}-a\mathbf{k})\wedge\frac{\partial\mathbf{u}}{\partial t}=\mathbf{i}\begin{vmatrix} x_2 & x_3-a \\ 0 & -\dfrac{x_1\partial^2 y(x_3,t)}{\partial t\partial x_3}\end{vmatrix}-\mathbf{j}\begin{vmatrix} x_1 & x_3-a \\ \dfrac{\partial y(x_3,t)}{\partial t} & -\dfrac{x_1\partial^2 y(x_3,t)}{\partial t\partial x_3}\end{vmatrix}+\mathbf{k}\begin{vmatrix} x_1 & x_2 \\ \dfrac{\partial y(x_3,t)}{\partial t} & 0\end{vmatrix}.$$

Hence

$$(\mathbf{x}-a\mathbf{k})\wedge\frac{\partial\mathbf{u}}{\partial t}=-\mathbf{i}\left[x_1x_2\frac{\partial^2 y(x_3,t)}{\partial t\partial x_3}\right]+\mathbf{j}\left[x_1{}^2\frac{\partial^2 y(x_3,t)}{\partial t\partial x_3}+(x_3-a)\frac{\partial y(x_3,t)}{\partial t}\right]-\mathbf{k}\left[x_2\frac{\partial y(x_3,t)}{\partial t}\right].$$

To simplify notation we will denote $y(x_3,t)$ as y. Now taking the derivative of $(\mathbf{x}-a\mathbf{k})\wedge\frac{\partial\mathbf{u}}{\partial t}$ with respect to t we obtain

$$\text{(3.29)}\quad \frac{\partial}{\partial t}\left[(\mathbf{x}-a\mathbf{k})\wedge\frac{\partial\mathbf{u}}{\partial t}\right]=-\mathbf{i}\left[x_1x_2\frac{\partial^3 y}{\partial t^2\partial x_3}\right]+\mathbf{j}\left[x_1{}^2\frac{\partial^3 y}{\partial t^2\partial x_3}+(x_3-a)\frac{\partial^2 y}{\partial t^2}\right]-\mathbf{k}\left[x_2\frac{\partial^2 y}{\partial t^2}\right].$$

By substituting (3.29) into (3.28) and applying Leibnitz's Rule we obtain

$$\text{(3.30)}\quad \mathbf{M}(a,t;-)+\mathbf{M}(b,t;+)+\int_a^b\int_{-h/2}^{h/2}(\mathbf{x}-a\mathbf{k})\wedge\left(\frac{\mathbf{f}}{h}\right)dx_3dx_2+(b\mathbf{k}-a\mathbf{k})\wedge\mathbf{Q}(b,t;+)$$

$$=\sigma\int_a^b\iint_A\left[-\mathbf{i}\left[x_1x_2\frac{\partial^3 y}{\partial t^2\partial x_3}\right]+\mathbf{j}\left[x_1{}^2\frac{\partial^3 y}{\partial t^2\partial x_3}+(x_3-a)\frac{\partial^2 y}{\partial t^2}\right]-\mathbf{k}\left[x_2\frac{\partial^2 y}{\partial t^2}\right]\right]dx_1dx_2dx_3$$

$$=\sigma\int_a^b\left[\left[-\mathbf{i}\frac{\partial^3 y}{\partial t^2\partial x_3}\iint_A x_1x_2dx_1dx_2\right]+\left[\mathbf{j}\frac{\partial^3 y}{\partial t^2\partial x_3}\iint_A x_1{}^2dx_1dx_2+\iint_A(x_3-a)\frac{\partial^2 y}{\partial t^2}dx_1dx_2\right]\right]dx_3$$

$$-\int_a^b\left[\mathbf{k}\frac{\partial^2 y}{\partial t^2}\iint_A x_2dx_1dx_2\right]dx_3.$$

Since the x_1 and x_2 axes pass through the centroid of the beam and are directed along the principal axes of inertia, we have $\iint_A x_1x_2dx_1dx_2=0$, $\iint_A x_2dx_1dx_2=0$, and $\iint_A x_1{}^2dx_1dx_2=I$ where I

represents the moment of inertia. Hence we are left with only the $\mathbf{j}$ component of the right hand side of(3.30).

Since

$$\sigma\int_a^b\left[\frac{\partial^3 y}{\partial t^2\partial x_3}\iint_A x_1{}^2 dx_1 dx_2+\iint_A (x_3-a)\frac{\partial^2 y}{\partial t^2}dx_1 dx_2\right]dx_3 = \tag{3.31}$$

$$\sigma\int_a^b\left[\frac{\partial^3 y}{\partial t^2\partial x_3}I+A(x_3-a)\frac{\partial^2 y}{\partial t^2}\right]dx_3,$$

(3.30) becomes

$$\begin{aligned}&-\mathbf{M}(a,t)+\mathbf{M}(b,t)+\int_a^b\int_{-h/2}^{h/2}(\mathbf{x}-a\mathbf{k})\wedge\left(\frac{\mathbf{f}}{h}\right)dx_2 dx_3+(b\mathbf{k}-a\mathbf{k})\wedge\mathbf{Q}(b,t;+)\\&=\sigma\mathbf{j}\int_a^b\left[\frac{\partial^3 y}{\partial t^2\partial x_3}I+A(x_3-a)\frac{\partial^2 y}{\partial t^2}\right]dx_3.\end{aligned}\tag{3.32}$$

We calculate the vector products of the left hand side of (3.32) to obtain

$$(\mathbf{x}-a\mathbf{k})\wedge\frac{\mathbf{f}}{h}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\x_1&x_2&x_3-a\\\dfrac{f(x_3,t)}{h}&0&0\end{vmatrix}.$$

It follows that

$$(\mathbf{x}-a\mathbf{k})\wedge\frac{\mathbf{f}}{h}=\mathbf{i}\begin{vmatrix}x_2&x_3-a\\0&0\end{vmatrix}-\mathbf{j}\begin{vmatrix}x_1&x_3-a\\\dfrac{f(x_3,t)}{h}&0\end{vmatrix}+\mathbf{k}\begin{vmatrix}x_1&x_2\\\dfrac{f(x_3,t)}{h}&0\end{vmatrix}.$$

Hence

$$(\mathbf{x}-a\mathbf{k})\wedge\frac{\mathbf{f}}{h}=\mathbf{j}\left[(x_3-a)\frac{f(x_3,t)}{h}\right]-\mathbf{k}\left[x_2\frac{f(x_3,t)}{h}\right].$$

Calculating $(b\mathbf{k}-a\mathbf{k})\wedge\mathbf{Q}(b,t;+)$ we obtain

$$(b\mathbf{k}-a\mathbf{k})\wedge\mathbf{Q}(b,t;+)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 0 & b-a\\ Q & 0 & 0\end{vmatrix}.$$

It follows that

$$(b\mathbf{k}-a\mathbf{k})\wedge\mathbf{Q}(b,t;+)=-\mathbf{j}\begin{vmatrix}0 & b-a\\ Q & 0\end{vmatrix}.$$

Hence

$$(b\mathbf{k}-a\mathbf{k})\wedge\mathbf{Q}(b,t;+)=\mathbf{j}Q(b-a).$$

Thus (3.32) becomes

(3.33)
$$\begin{aligned}&-\mathbf{M}(a,t)+\mathbf{M}(b,t)+\int_a^b\int_{-h/2}^{h/2}\mathbf{j}\left[(x_3-a)\frac{f(x_3,t)}{h}\right]-\mathbf{k}\left[x_2\frac{f(x_3,t)}{h}\right]dx_2dx_3+\mathbf{j}Q(b-a)\\&=\sigma\mathbf{j}\int_a^b\left[\frac{\partial^3 y}{\partial t^2\partial x_3}I+A(x_3-a)\frac{\partial^2 y}{\partial t^2}\right]dx_3.\end{aligned}$$

Simplifying the integrals on the left hand side of (3.33) we obtain

(3.34)

$$-M(a,t)+M(b,t)+\int_a^b(x_3-a)f(x_3,t)dx_3+(b-a)Q(b,t)-\sigma\int_a^b\left[\frac{\partial^3 y}{\partial t^2\partial x_3}I+A(x_3-a)\frac{\partial^2 y}{\partial t^2}\right]dx_3=0.$$

Recall from the Fundamental Theorem of Calculus that

$(b-a)Q(b,t)=\int_a^b\left[\frac{\partial(x_3-a)Q(x_3,t)}{\partial x_3}\right]dx_3$. Therefore it follows from the

Fundamental Theorem of Calculus and the properties of integrals that

(3.35)

$$\int_a^b \left(\frac{\partial M}{\partial x_3} + (x_3 - a) f(x_3,t) + \frac{\partial\left[(x_3 - a)Q(x_3,t)\right]}{\partial x_3} - \sigma\left[\frac{\partial^3 y}{\partial t^2 \partial x_3} I + A(x_3 - a)\frac{\partial^2 y}{\partial t^2}\right]\right)\partial x_3 = 0.$$

Applying Lemma 1 we obtain

$$(3.36)\quad \frac{\partial M}{\partial x_3} + (x_3 - a) f(x_3,t) + \frac{\partial\left[(x_3 - a)Q(x_3,t)\right]}{\partial x_3} - \sigma\left[\frac{\partial^3 y}{\partial t^2 \partial x_3} I + A(x_3 - a)\frac{\partial^2 y}{\partial t^2}\right] = 0.$$

Hence we obtain

$$(3.37)\quad \frac{\partial M}{\partial x_3} + (x_3 - a) f(x_3,t) + (x_3 - a)\frac{\partial Q(x_3,t)}{\partial x_3} + Q(x_3,t) = \sigma I \frac{\partial^3 y}{\partial t^2 \partial x_3} + \rho(x_3 - a)\frac{\partial^2 y}{\partial t^2}.$$

If we multiply equation (3.27) through by ($x_3 - a$) we have

$$(3.38)\qquad (x_3 - a)\frac{\partial(x_3,t)}{\partial x_3} + (x_3 - a) f(x_3,t) = \rho(x_3 - a)\frac{\partial^2 y(x_3,t)}{\partial t^2}.$$

Subtracting (3.38) from (3.37) we obtain the result

$$(3.39)\qquad \frac{\partial M}{\partial x_3} + Q(x_3,t) = \sigma I \frac{\partial^3 y(x_3,t)}{\partial t^2 \partial x_3}.$$

Since $\rho = \sigma A$, (3.39) becomes

$$(3.40)\qquad \frac{\partial M}{\partial x_3} + Q(x_3,t) = \frac{\rho I \partial^3 y(x_3,t)}{A \partial t^2 \partial x_3}.$$

This shows that we can find the shear force $Q(x_3,t)$ from the bending moment M and the vertical movement $y(x_3,t)$. To eliminate the shear completely we can differentiate (3.40) with respect to x_3 and obtain

$$\frac{\partial^2 M}{\partial x_3{}^2}+\frac{\partial Q(x_3,t)}{\partial x_3}=\frac{\rho I\partial^4 y(x_3,t)}{A\partial t^2\partial x_3{}^2} \quad . \tag{3.41}$$

Subtracting (3.27) from (3.41) we have

$$\frac{\partial^2 M}{\partial x_3{}^2}+\frac{\rho\partial^2 y(x_3,t)}{\partial t^2}-\frac{\rho I\partial^4 y(x_3,t)}{A\partial t^2\partial x_3{}^2}=f(x_3,t)\ , \tag{3.42}$$

which is a single partial differential equation in terms of the bending moment and vertical movement of the beam.

We need to introduce the Euler-Bernoulli rule for the bending of beams. The Euler-Bernoulli rule states that the applied moment M is directly proportional to the curvature R^{-1}, with proportionality constant the flexural rigidity EI. [12] That is, $M=EIR^{-1}$. Here Young's modulus E represents stiffness, and rigidity increases with stiffness. From the Euler-Bernoulli rule we have $\frac{M}{EI}=\frac{1}{R}$. [15] Since we have assumed small displacement, we can impose the linearized Euler-Bernoulli rule developed for the problem of pure bending [15] obtaining

$$\frac{\partial^2 y(x_3,t)}{\partial x_3^2}=\frac{M}{EI}. \tag{3.43}$$

Hence $M = EI\frac{\partial^2 y(x_3,t)}{\partial x_3^2}$. By substitution of (3.43) into (3.42) we obtain

$$EI\frac{\partial^4 y(x_3,t)}{\partial x_3^4} + \frac{\rho\partial^2 y(x_3,t)}{\partial t^2} - \frac{\rho I\partial^4 y(x_3,t)}{A\partial t^2 \partial x_3{}^2} = f(x_3,t)\ . \tag{3.44}$$

The term $\frac{\rho I\partial^4 y(x_3,t)}{A\partial t^2 \partial x_3{}^2}$ represents the effect of rotary inertia. In many applications, its effect is small and therefore negligible.[15] Thus (3.44) becomes

$$EI\frac{\partial^4 y(x_3,t)}{\partial x_3^4} + \frac{\rho\partial^2 y(x_3,t)}{\partial t^2} = f(x_3,t)\ . \tag{3.45}$$

Equation (3.45) provides a model for the bending of beams that retains many of the essential components for modeling an oscillating bridge.

CHAPTER 4

CONCLUSION

What really happened on November 7, 1940? How did the Tacoma Narrows bridge collapse? There are many theories on why the bridge collapsed. One theory focused on the idea of forced resonance. Mechanical structures have at least one natural frequency. The natural frequency represents the frequency the mechanical structure would oscillate with if disturbed and then left to oscillate freely. This idea can be visualized by striking a tuning fork with a mallet. The tuning fork vibrates with a certain frequency depending upon the rigidity of the fork. The general idea of the collapse is as follows: The bridge being a mechanical structure had at least one natural frequency. The wind started to drive the bridge at its natural frequency causing the oscillations of the bridge to increase without bound. Being a structure the bridge had mechanical limits, when these limits were exceeded the bridge broke apart. For this phenomena to occur we would need the wind to force the system with some well defined periodicity. Gusts of wind would not be represented well with defined periodicity, but instead with erratic forcing behavior that would actually cause the oscillations to decrease in amplitude.

Research done by Glover, Lazer, and McKenna points to another theory that is gaining popularity in both the mathematics and engineering communities. This research shows that there exists two stable solutions, one of small amplitude and one of large amplitude when (2.5) is solved numerically. These solutions occur when $.06 \leq A \leq .60$ and $\mu = 4$. As A is increased from .06 to .60 both solutions exist. At $A = .61$ the smaller solution disappears and the solution exhibits hysterisis or a jump to the large amplitude periodic solution. This behavior is suggestive of a Hopf bifurcation. Thus even if A was decreased the bridge would be forcing itself and oscillate at the large amplitude solution. This large amplitude oscillation would then destroy the bridge. According to Glover, Lazer, and McKenna more research on this phenomena is needed.

REFERENCES

[1] Ammann, O. , von Karman, T., Woodruff, G., The Failure of the Tacoma Narrows Bridge, A report to the Honorable John M. Carmody Administrator, Federal Works Agency, Washington D.C. March 28, 1941.

[2] Yusuf Billah, K., Scanlan, Robert H., Resonance, Tacoma Narrows bridge failure, and undergraduate physics textbooks, American Journal of Physics, 59, 1991, 118-124.

[3] Blanchard, P., Devaney, R.L., and Hall, G., Differential Equations, Brooks/Cole Publishing, Pacific Grove, 1998.

[4] Braun, M., Differential Equations and Their Applications, Springer-Verlag, New York, 1983.

[5] Finney, R., Weir, M., Giordano, F., Thomas' Calculus, Addison Wesley, Boston, 2001.

[6] Freiman, F.L., Schlager, N., Failed Technology Vol. 2, UXL, New York, 1995.

[7] Glover, J., Lazer, A.C., McKenna, P.J., Existence and stability of large scale nonlinear oscillations in suspension bridges, Journal of Applied Mathematics and Physics, Vol. 40, 1989, 171-200.

[8] Halliday, D., Resnick, R., Walker, J., Fundamentals of Physics, Wiley, New York, 1993.

[9] Kaplan, W., Advanced Calculus, Addison Wesley, Reading, 1991.

[10] Lazer, A. C., McKenna, P.J., Large-Amplitude Periodic Oscillations in Suspension Bridges: Some New Connections with Nonlinear Analysis, SIAM Review Vol.32, No.4, 1990, 537-578.

[11] Lin, C.C., Segel, L.A., Mathematics Applied to Deterministic Problems in the Natural Sciences, Macmillan Publishing, New York, 1974.

[12] McKenna, P.J., Moore, K.S., Mathematics arising from suspension bridge dynamics: recent developments, Jber. D. Dt. Math.-Verein., 101, 1999, 178-195.

[13] McKenna, P.J., Walter, W., Travelling Waves in a suspension bridge, SIAM J. APPL. MATH., Vol 50, No.3, 1990, 703-715.

[14] Ross, S., Construction Disasters: design failures, causes, and prevention, McGraw-Hill, New York, 1984.

[15] Segel, L.A., Mathematics Applied to Continuum Mechanics, Dover Publications, Inc., New York, 1987.

[16] Spivak, M., Calculus on Manifolds, Perseus, Cambridge, 1965.

[17] Strogatz, S.H., Nonlinear Dynamics and Chaos, Perseus Publishing, Cambridge, 1994.