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Valentin Jean Racataian

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VARIOUS STEINER SYSTEMS

A Project
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Mathematics

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by
Valentin Jean Racataian
March 2004

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Approved by:

[redacted]
Zahid Hasan, Committee Chair,

3/17/04
Date

[redacted]
John Sarli, Committee Member

[redacted]
Peter Williams, Committee Member

[redacted]
Peter Williams, Chair
Department of Mathematics

[redacted]
Charles Stanton,
Graduate Coordinator
Department of
Mathematics

ABSTRACT

The project deals with the automorphism group G of a Steiner system $S(3, 4, 10)$. S_{10}, the symmetrical group of degree 10, acts transitively on T, the set of all Steiner systems with parameters 3, 4, 10. The purpose of this project is to study the action of S_{10} on cosets of G. This will be achieved by means of a graph of S_{10} on $T\times T$. The orbits of S_{10} on $T\times T$ are in one-one correspondence with the orbits of G, the stabilizer of an $S\in T$, on T. The number of orbits of G on T is given by the length of the permutation character $1_G\uparrow S_{10}$, which turns out to be 10. Since $\{Sg \mid g\in GxG, S\in T\}$ is an orbit of G on T, and different double cosets give rise to different orbits of G on T, there is a natural one-one correspondence between the orbits of G on T and the double cosets GxG in S_{10}. Thus a part of the project deals with the double coset decomposition of S_{10} over G.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF GRAPHS

CHAPTER ONE

STEINER SYSTEMS

Introduction

Simple groups are groups whose normal subgroups have order one or the order of the original group, and they include alternating groups, cyclic groups of prime order, Lie-type Chevalley groups, and sporadic groups. Twenty-six finite simple groups comprise the sporadic groups. In 1861 Emile Mathieu (1835-1890) discovered the first simple sporadic groups: $M_{11}, M_{12}, M_{22}, M_{23}$, and M_{24}. Later on Frobenius proved that all Mathieu groups are subgroups of M_{24}. The Mathieu groups are defined as automorphism groups of Steiner systems. For example, M_{11} is the automorphism group of $S(4, 5, 11)$, M_{12} is the automorphism group of $S(5, 6, 12)$, M_{22} is the automorphism group of $S(3, 6, 22)$, M_{23} is the automorphism group of $S(4, 7, 23)$, and last M_{24} is the automorphism group of $S(5, 8, 24)$.

By definition, a Steiner system of type $S(t, k, v)$, with $1<t<k<v$ integers, is an ordered pair (X, B), where X is a set with v elements and B is a family of subsets of X called blocks, each having k elements, such that every t elements of X lie in a unique block. Having the parameters

$1<t<k<v$ it is unknown whether there exists a Steiner system of the type $S(t, k, v)$. The special case with $t=2$ and $k=3$ is called a Steiner triple system. It was shown by Kirkman that a triple system $S(v)=S(t=2,k=3,v)$ of order v exists iff $v \equiv 1, 3 (\mathrm{mod}\, 6)$. Moreover, the projective plane of order n is a Steiner system of type $S(2, n+1, n^2+n+1)$ where n is conjectured to be a prime power. Ryser proved that if Steiner triple systems S_1 and S_2 of orders v_1 and v_2 exist, then a Steiner system of order v_1v_2 also exists. Some examples of Steiner triples are S(2, 3, 7) = {{1, 2, 4}, {2, 3, 5}, {3, 4, 6}, {4, 5, 7}, {1, 5, 6}, {2, 6, 7}, {1, 3, 7}} , S(2, 3, 9) = {{1, 2, 3}, {4, 5, 6}, {7, 8, 9}, {1, 4, 7}, {2, 5, 8}, {3, 6, 9}, {1, 5, 9}, {2, 6, 7}, {3, 4, 8}, {1, 6, 8}, {2, 4, 9}, {3, 5, 7}}, and S(2, 3, 15) = {{1, 2, 3}, {1, 4, 5}, {1, 6, 7}, {1, 8, 9}, {4, 8, 12}, {2, 8, 10}, {2, 9, 11}, {2, 12, 14}, {5, 10, 15}, {3, 13, 14}, {3, 12, 15}, {3, 5, 6}, {6, 11, 13}, {6, 9, 15}, {4, 10, 14}, {4, 11, 15}, {4, 7, 9}, {7, 11, 12}, {5, 8, 13}, {7, 10, 13}, {1, 10, 11}, {1, 12, 13}, {1, 14, 15}, {2, 13, 15}, {2, 4, 6}, {2, 5, 7}, {3, 4, 7}, {3, 9, 10}, {3, 8, 11}, {5, 9, 12}, {5, 11, 14}, {4, 9, 13}, {6, 8, 14}, {7, 8, 11}, {6, 10, 12}}.

A Steiner system of the type $S(t=3, k=4, v)$ is called a Steiner quadruple system. Hanani proved in 1960 that a condition necessary and sufficient for the existence of such a system is that $v \equiv 2, 4 (\mathrm{mod} 6)$. In 1915 Fitting constructed the systems $S(3, 4, 26)$ and $S(3, 4, 34)$. In 1935 Bays and de Weck showed the existence of $S(3, 4, 14)$. Some examples of Steiner quadruple systems are $S(3, 4, 8)$ = {{1, 2, 4, 8}, {2, 3, 5, 8}, {3, 4, 6, 8}, {4, 5, 7, 8}, {1, 5, 6, 8}, {2, 6, 7, 8}, {1, 3, 7, 8}, {3, 5, 6, 7}, {1, 4, 6, 7}, {1, 2, 5, 7}, {1, 2, 3, 6}, {2, 3, 4, 7}, {1, 3, 4, 5}, {2, 4, 5, 6}}, and $S(3, 4, 10)$ which will be studied in detail later in this paper. Additional Steiner systems are exhibited in appendix A.

There is a one-to-one correspondence between the number of cosets of S_{10}/G and the elements of the set T containing all the different Steiner systems. Moreover, the orbits of S_n on $T \times T$ are in one-to-one correspondence with the orbits of G_s on T, and the double cosets $G_s \times G_s$ in S_n. Thus, it can be shown that different Steiner systems preserve Mathieu subgroups. Understanding the behavior of these subgroups would help in gaining a better understanding of the most complex Mathieu group, M_{24}.

Definitions and Results

- Group

A group $\langle G,*\rangle$ is a set G, closed under a binary operation $*$, such that the following axioms are satisfied:

1. For all $a,b,c\in G$, we have $(a*b)*c=a*(b*c)$. Associativity of $*$.

2. There is an element e in G such that for all $x\in G$, $e*x=x*e=x$. Identity e for $*$.

3. Corresponding to each $a\in G$, there is an element a' in G such that $a*a'=a'*a=e$. Inverse a' of a.

- $G-$set

Let X be a set and G a group. An action of G on X is a map $*:G\times X\to X$ such that

1. $ex=x$ for all $x\in X$,

2. $(g_1g_2)(x)=g_1(g_2x)$ for all $x\in X$ and all $g_1,g_2\in G$.

Under these conditions, X is a $G-$set.

- Transitive

A group G is transitive on a $G-$set X if and only if for each $x_1,x_2\in X$, there exists $g\in G$ such that $gx_1=x_2$.

- Orbit

If G acts on X and $x\in X$, the orbit of x is $\{gx:g\in G\}$.

- Stabilizer

If G acts on X and $x \in X$, then the stabilizer of x, denoted by G_x, is the subgroup $G_x = \{g \in G : gx = x\}$.

- k-transitive

Let G be a permutation group on Ω. G is said to be k-transitive on Ω if and only if for every pair of k-tuples having distinct entries in G, say $(x_1,\ldots,x_k)$ and $(y_1,\ldots,y_k)$, there exists a $\sigma \in G$ with $\sigma x_i = y_i$ for $i = 1,\ldots,k$.

- Action

A group G acts on a set X if there is a homomorphism $\varphi : G \to S_x$. If $a \in G$, write $\varphi(a) \in S_x$ and, if $x \in X$, write ax instead of $\varphi_a(x)$.

Useful results:

1. Let G act transitively on Ω, let $\alpha \in \Omega$ and let $H = G_\alpha$. Then $(1_H)^G$ is the permutation character of the action.
2. Let G act on Ω with permutation character χ. Suppose Ω decomposes into exactly k orbits under the action of G. Then $[\chi, 1_G] = k$.

3. Let G act transitively on Ω with permutation character χ. Suppose that $\alpha \in \Omega$ and that G_α has exactly r orbits on Ω. Then $[\chi, \chi] = r$.

   (Character Theory of Finite Groups by I. Martin Isaacs, Dover 1994. Page 68.)

4. (Definition) If X is a transitive G-set, then the rank of X is the number of G_x-orbits of X.
5. If X is a transitive G-set, then the rank of X is the number of $(G_x - G_x)$-double cosets in G.

   (An Introduction to the Theory of Groups, Fourth Edition, by Joseph J. Rotman, Springer-Verlag 1995. Page 249)

CHAPTER TWO

THE STEINER SYSTEM $S(3, 4, 10)$

There exists a Steiner system of the type $S(3, 4, 10)$ with $\frac{\binom{10}{3}}{\binom{4}{3}}=30$ elements. An example of such a Steiner system is exhibited below:

$S(3, 4, 10) = \{\{1, 2, 4, 5\}, \{1, 2, 3, 7\}, \{1, 3, 5, 8\}, \{2, 3, 5, 6\}, \{2, 3, 4, 8\}, \{2, 4, 6, 9\}, \{3, 4, 6, 7\}, \{3, 4, 5, 9\}, \{3, 5, 7, 10\}, \{4, 5, 7, 8\}, \{4, 5, 6, 10\}, \{1, 4, 6, 8\}, \{5, 6, 8, 9\}, \{1, 5, 6, 7\}, \{2, 5, 7, 9\}, \{6, 7, 9, 10\}, \{2, 6, 7, 8\}, \{3, 6, 8, 10\}, \{1, 7, 8, 10\}, \{3, 7, 8, 9\}, \{1, 4, 7, 9\}, \{1, 2, 8, 9\}, \{4, 8, 9, 10\}, \{2, 5, 8, 10\}, \{2, 3, 9, 10\}, \{1, 5, 9, 10\}, \{1, 3, 6, 9\}, \{1, 3, 4, 10\}, \{1, 2, 6, 10\}, \{2, 4, 7, 10\}\}$.

By definition, a Steiner system of type $S(t, k, v)$, with $1< t< k< v$ integers, is an ordered pair (X, B), where X is a set with v elements and B is a family of subsets of X (called blocks), each having k elements, such that every t elements of X lie in a unique block.

Observations About $S(3, 4, 10)$

There are $\frac{\binom{9}{2}}{\binom{3}{2}}=12$ four ads containing a given point. In addition, there are $\frac{\binom{8}{1}}{\binom{2}{1}}=4$ four ads containing a given two points. By the definition of a Steiner system there is only a four ad containing a given three points. It follows that there must be 30-12 = 18 four ads not containing a given point, 30-4 = 26 four ads not containing a given two points, and 30-1 = 29 four ads not containing a given three points. The following arrangement shows the connection between the entries of the different four ads. Moreover, the $j+1^{th}$ entry in the $i+1^{th}$ line is the number of four ads intersecting S_i in S_j.

30

18 12

10 8 4

5 5 3 1

3 2 3 0 1

Going from right to left the first entry in the $i+1^{th}$ line represents the number of four ads containing i points, the next entry the number of four ads containing $i-1$ points and not the other, the next one $i-2$ points and not the other, and so forth and so on. Finally, the last entry represents the number of four ads containing none of the i points.

Proposition 1

If $x_1,x_2 \in$ S(3, 4, 10) and $|x_1 \cap x_2|=1$, then $x_1+x_2 \notin S$(3, 4, 10).

Proof:

We add two four-ads by taking their symmetric differencing. Let $x_1=\{a_1,a_2,b_1,b_2\}$ and $x_2=\{a_1,a_3,b_3,b_4\}$ Then $x_1+x_2=(x_1 \cup x_2)-(x_1 \cap x_2)=\{a_2,a_3,b_1,b_2,b_3,b_4\} \notin S$(3, 4, 10).

Let $\Omega=\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define $G=\{\alpha \in S_{10} : S^{\alpha}=S\}$, where S is a Steiner system of the form S(3, 4, 10).

Proposition 2

The set of permutations $G \subseteq S_{10}$ that stabilize S form a group.

Proof:

a) The identity of S_{10}. stabilizes *S*.

b) Let $\alpha, \beta \in G$ s.t. $S^{\alpha} = S^{\beta} = S$. $S^{\beta^{-1}} = \left(S^{\beta}\right)^{\beta^{-1}} = S \Rightarrow \beta^{-1} \in G.$

c) Let $\alpha, \beta \in G$ s.t. $S^{\alpha} = S^{\beta} = S$. Then $S^{\alpha\beta} = (S^{\alpha})^{\beta} = S^{\beta} = S.$

In a similar manner it can be shown that G_{x_1}, G_{x_1,x_2} and G_{x_1,x_2,x_3} are also groups formed by the permutations stabilizing one point, two points, and three points, respectively.

Proposition 3

The Steiner system *S* is unique up to the relabelling of the points.

Proof:

All four-ads can be generated by the following arrangement:

1	3	5	7	9
2	4	6	8	10

X		X	O	O
X	X	O		O

O	X	X	O	
X		X	O	O

O	X		X	O
O	X	X	O	

	O	X	X	O
O	X		X	O

X	X	O	X	
X	O	O		O

O	X	O		O
X	X		X	O

	X	X	O	X
O	X	O	O	

O	X		X	X
O		O	X	O

X	X	X		O
O	O	O	X	

O	X	X	X	
	O	O	O	X

	X	O	O	O
O		X	X	X

X		O	X	X
O	X		O	O

O	O	X		X
	O	X	X	O

O		O	O	X
	X	O	X	X

X	X		O	X
O	O	X		O

The 2x5 array was divided into five bricks, where each brick represents a column. Since there are thirty four-ads there must be fifteen arrays containing all the Steiner systems. Since there are 12 four -ads containing a given point the first entry of each array was occupied in twelve of the fifteen arrays. For the second position there are four four-ads that contain two given points. Therefore, the second position is occupied in only four of the arrays. Since there could be only one four-ad containing a given three points the third position is occupied in only one of the four-ads. Following this reasoning all the arrays were constructed.

Proposition 4

The order of the stabilizer of three points of Ω is 2, or $|G_{x_1,x_2,x_3}| = 2.$

Proof:

Stabilizing three points is the same as stabilizing a four-ad since any three points determine a unique four-ad. Suppose we stabilize 1, 2, and 3. The unique four-ad containing these points is $\{1, 2, 3, 7\}$. Since these points are fixed so will be any other four-ad containing either of those points. There are three four-ads not containing 1, 2, 3, or 7: $\{4, 5, 6, 10\}$, $\{5, 6, 8, 9\}$, and $\{4, 8, 9, 10\}$. In looking for a permutation that will preserve all three four-ads it is clear that since two points, for example 4 and 10, appear in two of the four-ads, 4 must go to 10 and vice versa or (4, 10). Since 8 and 9 appear in two of the four-ads it must be the case that 8 goes to 9 and vice versa or (8, 9). Following this logic, it is clear to see that the only permutation stabilizing three points is (4, 10)(5, 6)(8, 9) and the identity. Under this permutation the first array becomes

X		X	O	O
X	X	O		O

→

X		O	O	
X	O	X	O	X

or namely the two four-ads {1, 2, 6, 10} and {4, 5, 7, 8}. Under the same permutations the second array becomes

O	X	X	O	
X		X	O	O

→

O	X	X	O	O
X	O	X		

or {2, 3, 5, 6} and {1, 4, 7, 9}. In a similar manner it can be checked that all the remaining arrays are preserved under this permutation.

Proposition 5

The Steiner system S is a G-set. $S-\{x_1\}$ is a G_{x_1} −set. $S-\{x_{1,}x_2\}$ is a G_{x_1,x_2} −set.

Proof:

Define the function $G\times S \to S$ denoted by $(g,\alpha)\to g^{\alpha}$.

1) Clearly the identity of G stabilizes S .

2) Let $\alpha,\beta \in G$ and $x \in S$ such that $x^{\alpha}=x^{\beta}=x$. Now, $(x^{\alpha})^{\beta}=x^{\beta}=x$, and $(x)^{\alpha\beta}=(x^{\alpha})^{\beta}=x^{\beta}=x$. Thus, S is a $G-$set.

Similarly it can be shown that $S-\{x_1\}$ is a G_{x_1}-set and $S-\{x_1, x_2\}$ is a G_{x_1,x_2}-set.

Proposition 6

The G-set Ω is transitive.

Proof:

In order to preserve the Steiner system under a permutation all four-ads have to be preserved by the permutation. We will show that the permutation (1, 4, 6, 5, 2, 9, 8, 10, 3, 7) is in G. This will clearly show that the G-set Ω is transitive.

Under this permutation the first array becomes

X		X	O	O
X	X	O		O

$$\longrightarrow$$

O	O	O		X
X	X	X	O	

or namely {2, 4, 6, 9} and {1, 3, 5, 8}. Under the same permutation the second array becomes

O	X	X	O	
X		X	O	O

$$\longrightarrow$$

O	O	X	X	X
X	O			O

or namely {2, 5, 7, 9} and {1, 3, 4, 10}. The rest of the arrays can be checked in a similar fashion. Hence, the permutation (1, 4, 6, 5, 2, 9, 8, 10, 3, 7) preserves S by sending elements of four-ads to other elements of four-ads without going outside the Steiner system.

Proposition 7

The G_{x_1}-set is transitive on $\Omega-\{x_1\}$.

Proof:

Consider the permutations (2, 4, 9, 6)(3, 10, 5, 7), (3, 9, 7, 8)(4, 10, 5, 6), (3, 10, 8, 4, 7, 6, 9, 5), and (4, 10)(5, 6)(8, 9) of G_1. These permutations preserving S show that the G_1-set is transitive on $\Omega-\{1\}$. Applying the first permutation the first array becomes

X		X	O	O
X	X	O		O

→

X	O	O	X	X
O	X	O		

or namely {1, 4, 7, 9} and {2, 3, 5, 6}. The second array becomes

O	X	X	O	
X		X	O	O

→

O	O	O	X	
X	X		O	X

or {2, 4, 7, 10} and {1, 3, 5, 8}. In a similar manner it can be checked that the rest of the arrays are preserved under these permutations that fix x_1.

Proposition 8

The G_{x_1,x_2} - set is transitive on $\Omega-\{x_1,x_2\}$.

Proof:

Consider the permutations (3, 9, 7, 8)(4, 10, 5, 6), (3, 10, 8, 4, 7, 6, 9, 5), and (4, 10)(5, 6)(8, 9) of $G_{1,2}$. These permutations preserving S show that the G_{x_1,x_2}-set is transitive on $\Omega-\{x_1,x_2\}$. Applying the first permutation the first array becomes

X		X	O	O
X	X	O		O

→

X		O	O	
X	O	X	O	X

or {1, 2, 6, 10} and {4, 5, 7, 8}. The second array becomes

O	X	X	O	
X		X	O	O

→

O	O	O		X
X	X	X	O	

or {1, 3, 5, 8} and {2, 4, 6, 9}. It can be checked in a similar fashion that the rest of the arrays are preserved under these permutations that preserve 1 and 2.

Proposition 9

The order of G_{x_1,x_2} =16. The order of G_{x_1} =144. Last, the order of G=1440.

Proof:

There is a known result that states that if X is a transitive $G-$set of degree n and $x \in X$, then $|G| = n|G_x|$. Using this result it is clear that $\left|G_{x_1,x_2}\right| = (n-2)\left|G_{x_1,x_2,x_3}\right| = 8 \cdot 2 = 16.$ Similarly, $\left|G_{x_1}\right| = (n-1)\left|G_{x_1,x_2}\right| = 9 \cdot 16 = 144.$ Last, $|G| = n \cdot \left|G_{x_1}\right| = 10 \cdot 144 = 1440.$

Proposition 10

The number of orbits of $S(3, 4, 10)$ is 10.

Proof:

There is a known result that states that the number of orbits could be calculated using the formula

$\Phi_\alpha^G = \frac{n}{h_\alpha} \sum_\omega \Phi(\omega)$, $(\omega \in C_\alpha \cap H)$. Since the number of cosets $n = [G:H] = \frac{10!}{1440} = 2520$, it follows that the number of orbits is 10.

Explanation

The number of orbits of $S(3,4,10)$ were calculated in the following manner.

$\Phi_n^G = \frac{[S_{10}:G]}{(\text{permutation length in class of } S_{10})} \bullet (\text{permutation length in class of } G)$. Thus,

$\Phi_1^G = 2520, \quad \Phi_2^G = \frac{2520}{3150} \cdot 30, \; \Phi_3^G = \frac{2520}{945} \cdot 36$, . . . , $\Phi_{13}^G = \frac{2520}{362880} \cdot 144$. The number of orbits is then obtained by calculating

$$\frac{\Phi_1^G \cdot \Phi_1^G \cdot (\text{number of elements in its class of } S_{10}) + \ldots + \Phi_{13}^G \cdot \Phi_{13}^G \cdot (\text{number of elements in its class of } S_{10})}{10!}$$

which is found to be 10.

The conjugacy classes of S_{10} and the conjugacy classes of group G are exhibited in appendix B.

Double Coset Decomposition of S_{10} Over G

As shown in proposition 10, there are ten distinct double cosets of S_{10} over G. A double coset of G determined by $x \in S_{10}$ is given by

$$GxG = \{Gxg \mid g \in G\} = \{Ggg^{-1}xg \mid g \in G\} = \{Gx^g \mid g \in G\}.$$

1. GeG where e is the identity of S_{10}.

2. GxG where x is a transposition. There is only one such double coset since G is doubly transitive over Ω. The orbit consists of $\frac{10\times 9}{2} = 45$ single cosets.

3. GxG where x is a 3-cycle. There is only one double coset of this type since G is triply transitive on Ω. The orbit consists of $\frac{10\times 9\times 8}{3} = 240$ single cosets.

4. GxG where x is the product of two disjoint transpositions. There are three orbits of this type, namely 45, 90, and 360. Each single coset of the 45 and 90 orbit contains two elements of this type, and each single coset of the 360 orbit contains one element of this type. This accounts for all 630 permutations of the class of two disjoint transpositions of S_{10}. Since there are 360 permutations in the 360 orbit there is exactly one permutation for each Steiner system and thus 360 single

cosets. The orbit consists of $\frac{1}{2}\left(\frac{10\times9}{2}\times\frac{8\times7}{2}\right)-90-180=360$ single cosets.

5. GxG where x is a 4-cycle. There are three orbits of this type, namely 90, 180, and 720. Each single coset of the 90 and 180 orbit contains two 4-cycles, and each single coset of the 720 orbit with coset representative (1, 2)(3, 4)(5, 6) contains one 4-cycle. As stated previously, each single coset of the 180 orbit contains two 4-cycles.

Therefore the 180 orbit consists of $\frac{1}{2}\left(\frac{10\times9\times8\times7}{4}-720-180\right)=180$ single cosets.

6. GxG where x is the product of three disjoint transpositions. There are eight orbits of this type, namely the identity, 20, 45, 90, 240, 360, and the two 720 orbits. Each single coset of the identity contains 30 such permutations, each single coset of the 20 orbit contains 12 such permutations, and each single coset of the 45 orbit contains 4 elements of this type. Each single coset of the 90 orbit contains 2 permutations, each single coset of the 240 orbit contains 3, and each single coset of the 360 orbit contains one such permutation. Each single coset of the 720 orbit with coset representative (1, 2)(3, 4)(5, 6)

contains one such permutation as does each single coset of the 720 orbit with coset representative (1, 2, 3, 4, 5). The 720 orbit with coset representative (1, 2)(3, 4)(5, 6), contains $\frac{1}{3!}\left(\frac{10\times 9}{2}\times\frac{8\times 7}{2}\times\frac{6\times 5}{2}\right)-30-240-180-180-720-360-720=720$ single cosets.

7. GxG where x is a 5-cycle. There are three orbits of this type: 144 and the two distinct 720 orbits. Each single coset of the 144 orbit contains two 5-cycles, each single coset of the 720 orbit with coset representative (1, 2)(3, 4)(5, 6) contains four 5-cycles, as does each single coset of the 720 orbit with coset representative (1, 2, 3, 4, 5). Hence, the 720 orbit with coset representative (1, 2, 3, 4, 5) contains $\frac{1}{4}\left(\frac{10\times 9\times 8\times 7\times 6}{5}-288-2880\right)=720$ single cosets.

8. GxG where x is the product of four disjoint transpositions. There are nine orbits of this type: identity, 45, 90, 144, 180, 240, 360, and the two distinct 720 orbits. Each single coset of the identity contains 45 such permutations, each single coset of the 45 orbit contains 8 permutations, each single coset of the 90 and 180 orbit contains 4, each single coset of the 144 orbit

contains 5, each single coset of the 240 orbit contains 3, each single coset of the 360 orbit contains one as do each of the two distinct 720 orbits. Thus, the 144 orbit contains

$$\frac{1}{5}\left[\frac{1}{4!}\left(\frac{10\times 9}{2}\times\frac{8\times 7}{2}\times\frac{6\times 5}{2}\times\frac{4\times 3}{2}\right)-45-360-360-720-720-360-720-720\right]=144$$

single cosets.

9. GxG where x is the product of two 3-cycles and two 2-cycles. There are seven orbits of this type, namely the 90 orbit, 144, 180, 240, 360, and the two distinct 720 orbits. Each single coset of the 90 orbit contains 32 such permutations, each single coset of the 144 orbit contains 15 permutations, each single coset of the 180 orbit contains 16 permutations, each single coset of the 240 orbit contains 9 permutations, and each single coset of the 360 contains 10 permutations. Each single coset of the 720 orbit with coset representative (1, 2)(3, 4)(5, 6) contains 11 such permutations, and each single coset of the 720 orbit with coset representative (1, 2, 3, 4, 5) contains 5 permutations. Therefore, the 90 orbit contains

$$\frac{1}{32}\left[\frac{1}{2!}\left(\frac{10\times 9\times 8}{3}\times\frac{7\times 6\times 5}{3}\right)\times\frac{1}{2!}\left(\frac{4\times 3}{2}\times\frac{2\times 1}{2}\right)-2160-2880-2160-3600-7920-3600\right]=90$$

single cosets.

10. GxG where x is a 10-cycle. All ten orbits contain permutations of this type. Each single coset of the ten orbits contain 144 such permutations. Thus, the 20 orbit contains

$$\frac{1}{144}\left(\frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{10}-144-6480-20736-12960-25920-103680-103680-51840-34560\right)=$$

20 single cosets.

The following table shows every permutation of S_{10} over G. The orbit denoted by 720* has coset representative (1, 2, 3, 4, 5), and the orbit denoted by 720 has coset representative (1, 2)(3, 4)(5, 6).

Table 1. Double Coset Decomposition of S_{10} over G

S_{10} class	Orbit	Number of permutations in each orbit	Coset representative
$[2,1^8]$; 45	45	45	(4,9)
$[2^2,1^6]$; 630	45	90	(1,5)(2,4)
	90	180	(2,9)(5,10)
	360	360	(1,10)(5,7)
$[2^5]$; 945	1	36	(1,7)(2,9)(3,4)(5,8)(6,10)
	45	45	(1,10)(2,6)(3,4)(5,9)(7,8)
	90	360	(1,4)(2,5)(3,8)(6,7)(9,10)
	144	144	(1,4)(2,7)(3,8)(5,6)(9,10)
	360	360	(1,4)(2,10)(3,9)(5,8)(6,7)
$[2^3,1^4]$; 3150	1	30	(1,2)(4,5)(6,10)
	20	240	(1,10)(3,8)(4,6)
	45	180	(2,4)(3,8)(6,7)
	90	180	(1,8)(2,9)(5,7)
	240	720	(1,3)(4,8)(6,9)
	360	360	(1,5)(3,10)(4,7)
	720*	720	(1,2)(3,4)(5,6)
	720	720	(3,7)(5,9)(6,10)
$[2^4,1^2]$; 4725	1	45	(1,9)(2,8)(3,7)(4,6)
	45	360	(1,10)(2,6)(4,5)(7,8)
	90	360	(1,8)(2,9)(4,5)(6,10)
	144	720	(1,6)(2,4)(7,8)(9,10)
	180	720	(1,7)(2,3)(4,6)(5,8)
	240	720	(1,6)(2,9)(4,8)(5,10)
	360	360	(2,6)(3,8)(4,7)(9,10)
	720*	720	(2,7)(3,6)(4,9)(8,10)
	720	720	
$[3,1^7]$; 240	240	240	(2,9,3)

$[3^2,1^4]$;	20	240	(2,9,6) (5,7,8)
8400	90	720	(3,4,7) (6,8,10)
	144	720	(1,7,4) (2,9,6)
	180	720	(1,8,7) (3,10,5)
	240	240	(1,4,2) (3,10,7)
	360	2880	(1,2,9) (4,10,6)
	720*	1440	(1,3,7) (4,5,6)
	720	1440	
$[3^3,1]$;	1	80	(1,8,5) (2,10,9) (4,7,6)
22400	20	480	(1,4,5) (2,7,6) (8,10,9)
	45	720	(1,6,3) (4,8,10) (5,7,9)
	180	1440	(1,2,3) (4,10,8) (5,6,7)
	240	960	(1,2,9) (3,6,8) (4,5,7)
	360	1440	(1,10,3) (2,9,5) (6,8,7)
	720*	7200	(1,2,4) (5,6,10) (7,9,8)
	720	10080	
$[4,1^6]$;	90	180	(1,8,9,2)
1260	180	360	(1,10,2,5)
	720*	0	(4,10,6,9)
	720	720	
$[4,2^3]$;	20	360	(1,8,4,10) (2,5) (3,6) (7,9)
18900	45	1800	(1,4,10,5) (2,9) (3,6) (7,8)
	90	180	(1,6) (2,10) (3,9) (4,5,7,8)
	240	2160	(1,3) (2,8) (4,5,7,6) (9,10)
	360	2160	(1,4,2,6) (3,8) (5,10) (7,9)
	720*	8640	(1,7,2,10) (3,5) (4,8) (6,9)
	720	3600	
$[4,2,1^4]$	20	360	(1,5) (3,7,8,6)
18900	45	360	(1,5) (6,7,9,10)
	90	180	(2,5,6,3) (4,9)
	144	1440	(1,4) (2,6,3,9)
	180	2880	(3,8,9,10) (5,6)
	240	2160	(2,9,5,7) (3,6)
	360	2160	(1,10) (3,8,7,5)
	720*	5040	(1,7,4,2) (3,8)
	720	4320	

$[4^2,2]$; 56700	1	90	(1,6,9,3)(2,8,10,5)(4,7)
	45	990	(1,2,10,9)(3,5)(4,8,6,7)
	90	3780	(1,9,6,10)(2,7)(3,4,8,5)
	144	2160	(1,5,7,8)(2,3,9,4)(6,10)
	180	5760	(1,6,10,3)(2,5,8,7)(4,9)
	240	3600	(1,5)(2,3,6,4)(7,10,9,8)
	360	14400	(1,2,6,7)(3,8,5,9)(4,10)
	720*	12240	(1,5)(2,3,8,6)(4,10,7,9)
	720	13680	
$[4^2,1^2]$; 56700	1	270	(1,7,3,4)(2,6,10,9)
	45	810	(1,9,10,7)(2,3,6,8)
	90	3780	(1,7,6,2)(3,5,10,8)
	144	2160	(1,9,8,4)(2,7,3,6)
	180	4320	(2,9,7,10)(3,8,6,4)
	240	6480	(1,9,10,3)(2,6,4,5)
	360	12960	(1,6,4,2)(3,8,9,7)
	720*	10800	(3,9,5,10)(4,6,8,7)
	720	15120	
$[4,2^2,1^2]$ 56700	45	2160	(2,5,7,9)(3,10)(6,8)
	90	1620	(1,7)(2,3,5,6)(4,9)
	144	4320	(1,6,10,5)(3,8)(4,7)
	180	3240	(2,6)(3,7,8,10)(5,9)
	240	4320	(1,4)(2,9)(3,7,6,10)
	360	8640	(1,6)(3,7)(4,8,9,10)
	720*	15120	(2,8)(3,7)(4,5,9,10)
	720	17280	
$[5,1^5]$; 6048	144	288	(1,4,5,6,9)
	720*	2880	(1,9,4,10,5)
	720	2880	(1,2,3,4,5)
$[5^2]$; 72576	1	144	(1,4,7,6,2)(3,8,10,5,9)
	45	720	(1,5,3,6,7)(2,4,9,10,8)
	90	4320	(1,7,9,3,4)(2,6,8,5,10)
	144	6912	(1,8,4,7,10)(2,3,9,5,6)
	180	5760	(1,3,8,10,5)(2,9,6,4,7)
	240	5760	(1,3,9,7,5)(2,8,4,10,6)
	360	14400	(1,8,4,9,3)(2,6,5,7,10)
	720*	14400	(1,4,3,9,5)(2,10,7,6,8)
	720	20160	

$[3,2,1^5]$; 5040	240	720	(1,4,8)(2,9)
	360	1440	(1,10,8)(2,9)
	720*	2880	(4,5)(6,9,8)
	720	0	
$[6,1^4]$; 25200	90	720	(1,8,6,2,4,3)
	144	3600	(3,8,5,9,10,4)
	180	1440	(1,3,8,6,4,7)
	240	2160	(1,4,2,8,6,9)
	360	2880	(1,6,8,5,9,3)
	720*	5760	(1,4,6,9,3,10)
	720	8640	
$[3^2,2^2]$; 25200	90	2880	(1,3,6)(2,7,4)(5,9)(8,10)
	144	2160	(1,8,4)(2,3)(5,9,6)(7,10)
	180	2880	(1,9,8)(2,7,10)(3,4)(5,6)
	240	2160	(1,2)(3,7,8)(4,9)(5,6,10)
	360	3600	(1,9,5)(2,3)(4,10)(6,7,8)
	720*	3600	(1,10)(2,6,4)(3,5,9)(7,8)
	720	7920	
$[3,2^3,1]$; 25200	45	720	(1,10,6)(3,7)(4,5)(8,9)
	90	720	(1,5)(2,4)(3,9,8)(6,10)
	180	1440	(1,7,6)(3,5)(4,8)(9,10)
	240	3600	(1,5)(2,6)(3,10,8)(4,7)
	360	1440	(1,7)(2,3)(4,6,9)(8,10)
	720*	10080	(1,5)(2,3,9)(4,8)(6,10)
	720	7200	
$[3,2^2,1^3]$ 25200	45	720	(2,5)(3,6)(8,10,9)
	90	720	(2,5,6)(4,8)(9,10)
	180	1440	(1,4,5)(3,6)(7,10)
	240	2160	(1,9,10)(3,4)(6,7)
	360	5760	(1,6)(2,4)(5,10,9)
	720*	7200	(1,10,6)(2,8)(3,7)
	720	7200	

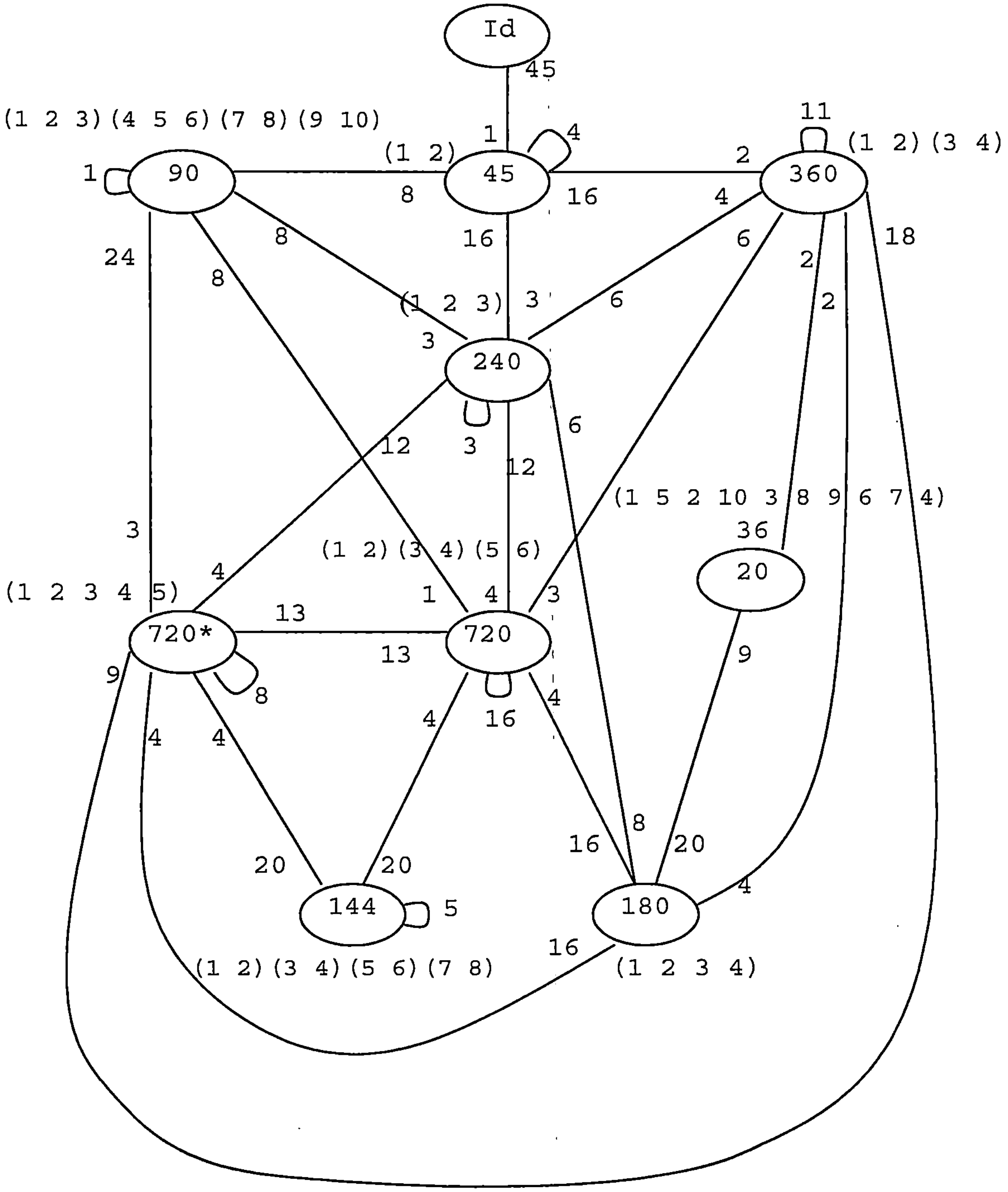
$[3^2,2,1^2]$	20	720	(1,4,7) (2,9,5) (3,8)
	45	720	(1,6,9) (2,8,4) (5,7)
50400	90	2160	(1,4) (2,10,5) (7,8,9)
	144	4320	(1,2) (3,9,10) (4,5,6)
	180	5040	(2,6) (3,5,8) (7,9,10)
	240	4320	(2,5,7) (3,9,6) (4,8)
	360	10080	(1,4) (3,5,9) (6,7,10)
	720*	12960	(1,4) (2,6,10) (3,8,9)
	720	10080	
$[6,2^2]$;	20	720	(1,2,3,6,9,10) (4,8) (5,7)
75600	45	1440	(1,2) (3,9,8,5,4,10) (6,7)
	90	2880	(1,5,9,6,2,10) (3,7) (4,8)
	144	5040	(1,10,9,8,6,5) (2,4) (3,7)
	180	6480	(1,8) (2,5,4,3,10,7) (6,9)
	240	10800	(1,10) (2,4,6,3,9,8) (5,7)
	360	13680	(1,9) (2,3) (4,7,6,8,10,5)
	720*	15120	(1,5) (2,9,7,8,4,6) (3,10)
	720	19440	
$[6,2,1^2]$	20	2160	(1,7,9,2,6,3) (4,5)
15120	45	2160	(1,3,6,5,2,4) (9,10)
0	90	5040	(2,3,7,8,4,6) (9,10)
	144	10080	(1,2) (4,8,9,7,10,6)
	180	6480	(1,10,2,6,3,4) (7,8)
	240	12960	(1,4,10,6,8,2) (3,5)
	360	14400	(1,6,4,2,8,3) (5,7)
	720*	51840	
	720	46080	
[6,3,1] ;	1	240	(1,3,9,10,4,5) (2,8,6)
20160	20	480	(1,7,9) (2,10,8,6,3,5)
0	90	8640	(1,2,3,4,6,5) (7,8,10)
	144	11520	(1,3,9,10,4,6) (2,7,5)
	180	12960	(1,2,7) (3,9,5,8,6,4)
	240	20160	(1,7,10,4,2,5) (6,8,9)
	360	36000	(1,5,9,2,7,10) (3,8,4)
	720*	44640	
	720	64800	

$[7,1^3]$; 86400	45	1440	(1,5,6,9,4,7,2)
	90	2880	(1,2,10,9,3,5,4)
	144	5760	(1,3,10,6,5,7,2)
	180	5760	(2,4,8,9,5,7,6)
	240	8640	(1,6,5,10,3,7,8)
	360	10080	(1,6,4,7,5,9,8)
	720*	24480	(2,8,6,7,9,4,10)
	720	27360	
[8,2]; 226800	1	180	(1,8,5,7,10,2,9,6)(3,4)
	20	2160	(1,5,8,2,6,3,10,7)(4,9)
	45	3060	(1,7)(2,6,3,5,8,9,10,4)
	90	5400	(1,5,2,7,6,4,10,8)(3,9)
	144	12960	(1,2,9,7,10,6,5,3)(4,8)
	180	18000	(1,8,2,6,10,5,4,9)(3,7)
	240	20160	(1,2,9,10,6,5,3,4)(7,8)
	360	35280	(1,5)(2,7,4,8,10,9,3,6)
	720*	65520	(1,4,8,3,9,10,6,7)(2,5)
	720	64080	
[8]; 226800	1	180	(2,6,8,10,4,7,3,9)
	20	720	(1,5,6,2,8,3,9,4)
	45	4500	(1,3,6,9,5,10,7,2)
	90	9720	(1,2,9,5,4,8,7,3)
	144	12960	(1,6,7,10,9,3,8,2)
	180	16560	(1,7,4,2,8,10,9,6)
	240	23040	(1,9,2,6,3,10,4,5)
	360	29520	(2,9,3,7,5,8,10,4)
	720*	68400	(1,9,8,10,2,3,5,4)
	720	61200	
[9]; 403200	20	2880	(1,9,10,3,4,5,6,8,2)
	45	8640	(1,3,7,6,2,4,5,8,9)
	90	14400	(1,3,6,8,9,10,2,5,7)
	144	23040	(1,5,6,8,9,4,10,2,7)
	180	31680	(1,9,3,5,7,2,4,10,6)
	240	34560	(1,9,4,5,6,10,7,3,2)
	360	57600	(1,8,10,7,2,4,6,5,9)
	720*	115200	(1,2,5,10,4,3,6,9,7)
	720	115200	

$[5,2,1^3]$	90	2880	(1,4,5,8,7)(3,6)
60480	144	2880	(1,9,4,7,3)(5,6)
	180	5760	(2,5,3,10,9)(6,8)
	240	5760	(1,3,9,4,2)(6,7)
	360	8640	(2,8)(3,5,4,9,10)
	720*	14400	(1,7,3,6,5)(8,9)
	720	20160	
$[5,2^2,1]$	90	4320	(1,4,3,8,7)(2,10)(5,9)
90720	144	4320	(1,9)(2,8,4,6,10)(5,7)
	180	8640	(1,9)(2,6,5,8,4)(3,10)
	240	8640	(1,10,4,7,5)(2,3)(6,9)
	360	12960	(1,7,9,2,6)(4,5)(8,10)
	720*	21600	(1,4,10,3,7)(2,5)(8,9)
	720	30240	
[10]	1	144	(1,5,2,10,3,8,9,6,7,4)
362880	20	2880	(1,6,10,8,7,9,4,5,2,3)
	45	6480	(1,8,6,4,2,10,7,5,9,3)
	90	12960	(1,9,2,3,5,8,7,6,10,4)
	144	20736	(1,9,7,2,5,6,8,10,3,4)
	180	25920	(1,2,7,10,8,9,6,3,4,5)
	240	34560	(1,7,8,4,5,9,6,2,3,10)
	360	51840	(1,10,3,2,5,8,6,4,7,9)
	720*	103680	
	720	103680	
$[4,3^2]$	45	720	(1,9,3,6)(2,4,8)(5,10,7)
50400	90	2880	(1,3,5)(2,10,8)(4,6,9,7)
	144	4320	(1,8,9)(2,5,6)(3,10,7,4)
	180	5760	(1,9,5,4)(2,10,3)(6,8,7)
	240	5760	(1,6,8)(2,7,9,10)(3,5,4)
	360	5040	(1,6,3,9)(2,8,10)(4,5,7)
	720*	12240	(1,9,6)(2,3,5,4)(7,10,8)
	720	13680	
$[4,3,1^3]$	20	1440	(1,10,3,9)(4,7,5)
50400	45	2160	(1,10,3)(4,7,6,5)
	90	720	(1,7,5)(2,10,3,9)
	180	2880	(1,6,5,10)(3,7,8)
	240	4320	(1,2,3,5)(4,10,8)
	360	7200	(4,6,8)(5,7,10,9)
	720*	17280	(2,7,8,9)(4,5,10)
	720	14400	

[6,4]	20	2880	(1,4,2,9)(3,10,7,5,6,8)
151200	45	3600	(1,3,6,10,8,4)(2,9,5,7)
	90	5760	(1,10,8,9)(2,6,7,5,3,4)
	144	4320	(1,3,9,10,2,7)(4,5,6,8)
	180	8640	(1,8,6,4,2,5)(3,9,7,10)
	240	17280	(1,6,4,10,3,8)(2,5,9,7)
	360	19440	(1,10,6,7,3,8)(2,9,5,4)
	720*	49680	(1,9,4,5)(2,3,8,10,7,6)
	720	39600	
[4,3,2,1]	20	1440	(1,6,9,3)(2,10,4)(5,7)
151200	45	2160	(1,10,5)(2,6,8,7)(3,4)
	90	3600	(1,7,9,5)(2,4)(6,10,8)
	144	11520	(1,5)(2,3,8,7)(4,6,10)
	180	8640	(1,7)(3,6,10,5)(4,9,8)
	240	15840	(1,2,5,8)(3,9)(4,10,7)
	360	24480	(1,2,4,9)(5,8)(6,10,7)
	720*	43200	(2,9)(3,7,6)(4,8,5,10)
	720	40320	
[7,2,1]	20	2880	(1,7,6,2,5,8,3)(4,10)
259200	45	1440	(1,9,7,3,6,10,5)(4,8)
	90	8640	(1,8,9,6,7,4,2)(3,5)
	144	17280	(1,8,5,4,9,10,3)(2,6)
	180	20160	(1,10,9,5,7,6,2)(4,8)
	240	20160	(1,2,7,9,4,5,6)(3,10)
	360	33120	(1,10,7,3,9,8,5)(2,6)
	720*	76320	(2,3,5,10,7,6,8)(4,9)
	720	79200	
$[5,3,1^2]$	20	1440	(1,7,5,6,2)(4,10,8)
120960	45	2880	(1,6,10,7,2)(3,9,4)
	90	4320	(2,9,10,7,8)(3,6,5)
	144	5760	(1,7,4,9,8)(3,5,10)
	180	10080	(1,10,7,2,9)(4,6,8)
	240	14400	(1,2,9)(3,8,10,4,6)
	360	18720	(3,6,4,9,8)(5,10,7)
	720*	33120	(1,9,5)(2,8,3,4,10)
	720	30240	

[5,4,1]	20	2880	(1,3,5,2,4)(6,10,7,9)
181440	45	5760	(1,6,7,8)(2,4,5,10,3)
	90	4320	(1,5,7,2,4)(3,8,9,10)
	144	8640	(1,4,9,3,7)(2,6,5,10)
	180	11520	(2,8,3,5)(4,10,6,7,9)
	240	20160	(1,10,9,7,2)(3,4,6,8)
	360	24480	(1,10,9,5)(2,3,4,8,7)
	720*	59040	(1,4,9,2,3)(5,8,10,7)
	720	44640	
[7,3]	45	2880	(1,3,10,6,9,2,5)(4,8,7)
172800	90	5760	(1,4,2,5,3,6,10)(7,9,8)
	144	11520	(1,9,8,7,10,5,2)(3,6,4)
	180	11520	(1,4,9,6,5,10,8)(2,7,3)
	240	17280	(1,9,4,6,7,2,8)(3,5,10)
	360	20160	(1,6,2)(3,4,10,5,7,8,9)
	720*	48960	(1,7,2,4,10,6,5)(3,8,9)
	720	54720	
[5,3,2]	20	1440	(1,4,6)(2,7)(3,10,5,8,9)
120960	45	2880	(1,5)(2,8,7,10,3)(4,6,9)
	90	1440	(1,5,2,6,4)(3,8,10)(7,9)
	144	5760	(1,3,6,5,8)(2,10)(4,9,7)
	180	4320	(1,8,9)(2,6,7,10,4)(3,5)
	240	8640	(1,9,7,10,3)(2,8)(4,5,6)
	360	10080	(1,5,2)(3,10,7,8,4)(6,9)
	720*	47520	(1,2)(3,10,9,4,6)(5,7,8)
	720	38880	



Graph 1. The Graph of S_{10} on *TxT*

Description of the Graph of S_{10} on *TxT*

The numbers shown in the ovals represent the numbers of Steiner systems in each orbit. The cosets next to each oval are the coset representatives for each orbit. The numbers near each oval represent the number of transpositions taking the Steiner systems of one orbit to another orbit.

Transpositions Moving Steiner Systems From
One Orbit to Another

In what follows, we will call the orbit containing N Steiner systems an N-orbit.

45 orbit

Transpositions taking the 45 orbit to the 360 orbit: {(4, 9),(5, 8),(4, 8),(3, 4),(4, 7),(5, 7),(9, 10),(7, 10),(6, 9),(6, 7),(3, 10),(3, 5),(8, 10),(6, 8),(3, 6),(5, 9)}.

Transpositions taking the 45 orbit to the 240 orbit: {(1, 7),(2, 7),(1, 4),(2, 4),(2, 5),(1, 5),(2, 6),(2, 3),(1, 10),(2, 10),(1, 3),(2, 8),(2, 9),(1, 6),(1, 8),(1, 9)}.

Transpositions taking the 45 orbit to the 90 orbit: {(5, 6),(7, 8),(5, 10),(4, 10),(3, 9),(3, 8),(4, 6),(7, 9)}.

Transpositions taking the 45 orbit into itself: {(4, 5),(6, 10),(3, 7),(8, 9)}.

Transpositions taking the 45 orbit to identity: {(1, 2)}.

90 orbit

Transpositions taking the 90 orbit to the 720* orbit: {(4, 10),(5, 8),(1, 5),(1, 8),(3, 8),(7, 10),(6, 8),(1, 2),(6, 10),(4, 8),(3, 9),(6, 9),(2, 10),(8, 9),(7, 8),(4, 6),(3, 4),(5, 10),(2, 8), (2, 7),(3, 10),(9, 10),(5, 7),(1, 10)}.

Transpositions taking the 90 orbit to the 720 orbit: {(4, 5),(2, 4),(1, 3),(2, 9),(3, 7),(6, 7),(1, 6),(5, 9)}.

Transpositions taking the 90 orbit to the 45 orbit: {(2, 5),(3, 6),(1, 7),(4, 9)}.

Transpositions taking the 90 orbit to the 240 orbit: {(5, 6),(1, 4),(2, 3),(2, 6),(3, 5),(1, 9),(7, 9),(4, 7)}.

Transpositions taking the 90 orbit into itself: {(8, 10)}.

240 orbit

Transpositions taking the 240 orbit to the 45 orbit: {(2, 3),(1, 3),(1, 2)}.

Transpositions taking the 240 orbit to the 360 orbit: {(4, 5),(6, 10),(4, 8),(5, 8),(9, 10),(6, 9)}.

Transpositions taking the 240 orbit to the 180 orbit: {(1, 4),(1, 10),(3, 5),(3, 6),(2, 8),(2, 9)}.

Transpositions taking the 240 orbit to the 720 orbit: {(3, 9),(3, 8),(3, 4),(2, 4),(2, 5),(1, 5),(3, 10),(2, 6),(2, 10),(1, 6),(1, 8),(1, 9)}.

Transpositions taking the 240 orbit to the 720* orbit: {(5, 7),(7, 8),(6, 8),(5, 10),(7, 10),(8, 10),(4, 9),(4, 6),(5, 9),(7, 9),(4, 7),(6, 7)}.

Transpositions taking the 240 orbit to the 90 orbit: {(2, 7),(1, 7),(3, 7)}.

Transpositions taking the 240 orbit into itself: {(5, 6),(4, 10),(8, 9)}.

360 orbit

Transpositions taking the 360 orbit to the 45 orbit: {(3, 4),(1, 2)}.

Transpositions taking the 360 orbit to the 240 orbit: {(6, 10),(6, 7),(8, 9),(5, 9)}.

Transpositions taking the 360 orbit to the 720 orbit: {(5, 6),(2, 3),(7, 9),(6, 8),(9, 10),(1, 4)}.

Transpositions taking the 360 orbit to the 20 orbit: {(7, 10),(5, 8)}.

Transpositions taking the 360 orbit to the 180 orbit: {(2, 4),(1, 3)}.

Transpositions taking the 360 orbit to the 720* orbit: {(1, 7),(4, 9),(4, 6),(3, 9),(4, 8),(3, 8),(2, 7),(4, 10),(2, 5),(1, 5),(5, 10),(3, 10),(2, 6),(3, 6),(7, 8),(2, 9),(1, 6),(1, 9)}.

Transpositions taking the 360 orbit into itself: {(5, 7),(1, 8),(3, 7),(8, 10),(1, 10),(4, 5),(3, 5),(2, 10),(6, 9),(2, 8),(4, 7)}.

20 orbit

Transpositions taking the 20 orbit to the 360 orbit: {(8, 9),(6, 7),(9, 10),(6, 8),(4, 7),(2, 4),(5, 7),(4, 10),(6, 9),(1, 4),(1, 8),(2, 5),(5, 8),(1, 5),(4, 8),(1, 7),(1, 9),(2, 9),(8, 10), (2, 6),(1, 10),(7, 8),(6, 10),(4, 6),(5, 6),(2, 8),(2, 10),(7, 10),(5, 10),(7, 9),(4, 9),(1, 6),(1, 2),(5, 9),(2, 7),(4, 5)}.

Transpositions taking the 20 orbit to the 180 orbit: {(2, 3),(3, 4),(1, 3),(3, 7),(3, 9),(3, 8),(3, 5),(3, 10),(3, 6)}.

180 orbit

Transpositions taking the 180 orbit to the 360 orbit: {(2, 4),(7, 10),(1, 3),(5, 8)}.

Transposition taking the 180 orbit to the 20 orbit: {(6, 9)}.

Transpositions taking the 180 orbit to the 240 orbit: {(5, 7),(2, 3),(5, 10),(8, 10),(7, 8),(3, 4),(1, 2),(1, 4)}.

Transpositions taking the 180 orbit to the 720 orbit: {(6, 10),(4, 9),(4, 6),(3, 9),(9, 10),(7, 9),(6, 7),(2, 6),(8, 9),(6, 8),(3, 6),(5, 9),(5, 6),(2, 9),(1, 6),(1, 9)}.

Transpositions taking the 180 orbit to the 720* orbit: {(1, 7),(3, 7),(4, 5),(4, 8),(3, 8),(2, 7),(4, 7),(4, 10),(2, 5),(1, 5),(3, 10),(3, 5),(1, 10),(2, 10),(2, 8),(1, 8)}.

720 orbit

Transpositions taking the 720 orbit to the 240 orbit: {(4, 7),(4, 10),(3, 10),(3, 7)}.

Transpositions taking the 720 orbit to the 360 orbit: {(5, 6),(7, 10),(1, 2)}.

Transpositions taking the 720 orbit to the 180 orbit: {(2, 9),(5, 8),(1, 9),(6, 8)}.

Transpositions taking the 720 orbit to the 144 orbit: {(2, 5),(7, 8),(1, 6),(9, 10)}.

Transpositions taking the 720 orbit to the 720* orbit: {(4, 5),(5, 7),(1, 4),(6, 9),(6, 7),(8, 9),(2, 3),(1, 10),(2, 10),(3, 6),(5, 9),(2, 8),(1, 8)}.

Transpositions taking the 720 orbit to the 90 orbit: {(3, 4)}.

Transpositions taking the 720 orbit into itself: {(6, 10),(1, 7),(4, 9),(4, 6),(3, 9),(4, 8),(3, 8),(2, 7),(7, 9),(2, 4),(1, 5),(5, 10),(2, 6),(3, 5),(8, 10),(1, 3)}.

<u>144 orbit</u>

Transpositions taking the 144 orbit to the 720 orbit: {(4, 10),(1, 6),(2, 4),(4, 5),(3, 8),(6, 8),(1, 2),(1, 3),(3, 6),(2, 5),(6, 9),(2, 10),(8, 9),(7, 8),(4, 7),(7, 9),(3, 10),(5, 7),(1, 10),(5, 9)}.

Transpositions taking the 144 orbit to the 720* orbit: {(2, 6),(2, 3),(5, 8),(1, 5),(2, 9),(1, 8),(7, 10),(6, 10),(4, 8),(5, 6),(3, 9),(4, 9),(3, 7),(3, 4),(1, 9),(5, 10),(6, 7),(2, 7),(8, 10),(1, 4)}.

Transpositions taking the 144 orbit into itself: {(3, 5),(1, 7),(4, 6),(9, 10),(2, 8)}.

720* orbit

Transpositions taking the 720* orbit to the 90 orbit: {(5, 10),(2, 6),(3, 4)}.

Transpositions taking the 720* orbit to the 240 orbit: {(2, 5),(6, 10),(3, 7),(4, 9)}.

Transpositions taking the 720* orbit to the 720 orbit: {(1, 7),(4, 5),(2, 7),(9, 10),(4, 10),(7, 9),(6, 7),(8, 9),(2, 3),(3, 6),(5, 9),(7, 8),(1, 9)}.

Transpositions taking the 720* orbit to the 144 orbit: {(5, 6),(4, 7),(3, 9),(2, 10)}.

Transpositions taking the 720* orbit to the 180 orbit: {(8, 10),(1, 5),(6, 8),(1, 2)}.

Transpositions taking the 720* orbit to the 360 orbit: {(5, 7),(1, 4),(1, 3),(1, 8),(7, 10),(4, 8),(3, 8),(6, 9),(2, 9)}.

Transpositions taking the 720* orbit into itself: {(2, 4),(5, 8),(1, 10),(1, 6),(3, 5),(4, 6),(3, 10),(2, 8)}.

CHAPTER THREE

THE STEINER SYSTEM $S(2, 4, 16)$

By the definition of a Steiner system, each element of $S(2, 4, 16)$ should contain four points out of sixteen with less than two repeating. Clearly, the system exhibited below follows the requirement.

There exists a Steiner system of the form $S(2, 4, 16)$ with $\binom{16}{2}/\binom{4}{2} = 20$ elements. Here is an example of such Steiner system:

$S(2, 4, 16) = \{\{1, 2, 3, 4\}, \{1, 5, 9, 13\}, \{1, 6, 11, 16\}, \{1, 7, 12, 14\}, \{1, 8, 10, 15\}, \{5, 6, 7, 8\}, \{2, 6, 10, 14\}, \{4, 7, 10, 13\}, \{2, 7, 9, 16\}, \{2, 8, 11, 13\}, \{9, 10, 11, 12\}, \{3, 7, 11, 15\}, \{3, 8, 9, 14\}, \{3, 5, 10, 16\}, \{3, 6, 12, 13\}, \{13, 14, 15, 16\}, \{4, 8, 12, 16\}, \{2, 5, 12, 15\}, \{4, 5, 11, 14\}, \{4, 6, 9, 15\}\}$.

Observations about $S(2, 4, 16)$

There are $\binom{15}{1}/\binom{3}{1} = 5$ four-ads containing a given point. By the definition of $S(2,4,16)$ there is only one four-ad containing a given two points. There must be 20-5=15 four-ads not containing a given point and 20-1 not containing a given two points. The following arrangement shows the connection between the entries of the different four-ads. Moreover, the $j+1^{th}$ entry in the $i+1^{th}$ line is the number of four-ads intersecting S_i in S_j.

				20				
			15		5			
		11		4		1		
	7		4		0		1	
3		4		0		0		1

(Explanation. Follow row four for example. Going from right to left the first entry represents the number of four-ads in $S(2,4,16)$ containing three given points. Next entry to the left shows how many of the remaining four-ads contain two of the given points, next how many contain one of the given three points, and finally the last one how many contain none of the given three points.

Proposition 1

If $x_1, x_2 \in S(2,4,16)$ and $|x_1 \cap x_2| = 1$, then $x_1 + x_2 \notin S(2, 4, 16)$.
Proof:

We add two four-ads by taking their symmetric differencing. Let $x_1 = \{a_1, a_2, b_1, b_2\}$ and $x_2 = \{a_1, a_3, b_3, b_4\}$. The four-ad containing a_2, a_3 must contain two other points that cannot be from x_1 or x_2, $\therefore x_1 + x_2 \notin S(2, 4, 16)$. $x_1 + x_2 = (x_1 \cup x_2) - (x_1 \cap x_2) = \{a_2, a_3, b_1, b_2, b_3, b_4\} \notin S(2, 4, 16)$.

Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$. Define $G = \{\alpha \in S_{16} : S^{\alpha} = S\}$, where S is a Steiner system of the form $s(2, 4, 16)$.

Proposition 2

The set of permutations $G \subseteq S_{16}$ that stabilize S form a group.

Proof: a) The identity of S_{16} stabilizes S.

b) Let $\alpha, \beta \in G$ s.t. $S^{\alpha} = S^{\beta} = S$. $S^{\beta^{-1}} = \left(S^{\beta}\right)^{\beta^{-1}} = S \Rightarrow \beta^{-1} \in G$.

c) Let $\alpha,\beta \in G$ s.t. $S^{\alpha}=S^{\beta}=S.$ Then $S^{\alpha\beta}=(S^{\alpha})^{\beta}=S^{\beta}=S.$

Following the same steps it can be shown that G_{x_1} and G_{x_1,x_2} are also groups formed by the permutations stabilizing one point and two points respectively.

Proposition 3

The Steiner system *S* is unique up to the relabelling of the points.

Proof: All the four ads can be generated by the following arrangement:

Brick 1 Brick 2

1	2	9	10
3	4	11	12
5	6	13	14
7	8	15	16

Brick 3 Brick 4

Array 1

X	X	O	O
X	X	O	O
.	.	I	I
.	.	I	I

Array 2

X	.	X	.
O	I	O	I
X	.	X	.
O	I	O	I

Array 3

X	O	I	.
I	.	X	O
O	X	.	I
.	I	O	X

X	O	I	.
.	I	O	X
.	I	O	X
X	O	I	.

Array 4

O	X	X	O
I	.	.	I
.	I	I	.
X	O	O	X

Array 5

Any four-ad of *S* will intersect the rows, columns or diagonals in two, one or zero places. The 4x4 array was divided into four bricks. Since there are twenty four-ads there must be five 4x4 arrays containing all the four-ads.

The first array contains bricks that each have a four-ad. Because the Steiner system $S(2, 4, 16)$ cannot have two repeating points in any of its four-ads the remaining 4x4 arrays will have to share a point from each brick. As a first choice the first entry in each brick was selected. The next 4x4 array could not have the first entry in the second brick as in the previous array or else there will be two common points for a four-ad. Rather, the first entry in the second brick of array 3 was moved to another position. Following this pattern all the 4x4 arrays were created.

Proposition 4

The order of the stabilizer of two points of Ω is 12, or $|G_{x_1,x_2}| = 12$.

Proof: Stabilizing two points is the same as stabilizing a four-ad since any two points determine a unique four-ad. Suppose we stabilize the first brick in all five 4x4 arrays. The only permutations stabilizing the first brick in arrays 2, 3, 4, and 5 are (X, X, X)(O, O, O)(I, I, I)(., ., .). These permutations in array two are (5, 9, 13)(6, 10, 14)(7, 11, 15)(8, 12, 16) and (5, 13, 9)(6, 14, 10)(7, 15, 11)(8, 16, 12). Similarly, another such permutation in the third array is (5, 12, 15)(6, 11, 16)(7, 10, 13)(8, 9, 14). Since there are two permutations of the same type for arrays 2 through 5 there are a total of eight permutations for these arrays. The first array yields a set of four permutations including the identity that fix the first brick. Within the second brick there are three two-cycles that start with 0 going to 0 horizontally, vertically, and diagonally. These are placed together with the following two-cycles of brick three: vertical, diagonal, and horizontal. Finally, the last brick completes the three, six two-cycles: diagonal, horizontal, and vertical. This shows that there are 12 permutations preserving two points of a four-ad.

It was verified using Magma that the 12 permutations listed above form a group.

Proposition 5

The Steiner system S is a G-set. $S-\{x_1\}$ is a G_{x_1}-set.

Proof: Define the function $G\times S\rightarrow S$ denoted by $(g,\alpha)\rightarrow g^{\alpha}$. Clearly the identity of G stabilizes S.

Let $\alpha, \beta \in G$ and $x \in S$ such that $x^{\alpha} = x^{\beta} = x$. Now, $(x^{\alpha})^{\beta} = x^{\beta} = x$, and $(x)^{\alpha\beta} = (x^{\alpha})^{\beta} = x^{\beta} = x$. Thus, S is a G-set. Similarly it can be shown that $S - \{x_1\}$ is a G_{x_1}-set.

<u>Proposition 6</u>

The group G is transitive on Ω.

Proof: In order to preserve the Steiner system elements of four-ads have to be preserved. The following permutations: (1, 4, 11, 6, 15, 7, 8, 10, 13, 2, 14, 16, 9, 3, 5) and (1, 11, 13, 15, 5, 3)(2, 16)(4, 6, 8, 14, 12, 10)(7, 9) preserve S as follows.

Under the first permutation the first array becomes

$$\begin{array}{|cc|cc|}\hline X & X & O & O \\ X & X & O & O \\ \hline . & . & I & I \\ . & . & I & I \\ \hline \end{array} \longrightarrow \begin{array}{|cc|cc|}\hline . & I & I & . \\ O & X & X & O \\ \hline X & O & O & X \\ I & . & . & I \\ \hline \end{array}$$

Clearly this is one of the initial arrays. The rest of the arrays can be checked in a similar fashion. Thus S is preserved by sending elements of four-ads to other elements of four-ads without going outside the Steiner system. Thus G is transitive on Ω.

Proposition 7

The G_{x_1}-set is transitive on $\Omega-\{x_1\}$.

Proof: Consider the permutations: (2, 13, 15, 11, 14, 3, 5, 8, 16, 7, 4, 9, 10, 6, 12), (2, 11)(3, 16)(4, 6)(5, 10)(8, 13)(9, 15), and (2, 16, 9)(3, 6, 13)(4, 11, 5)(8, 10, 15) of G_1. These permutations fix 1. Under the first permutation the first array becomes

$$\begin{array}{|cc|cc|}\hline X & X & O & O \\ X & X & O & O \\ \hline . & . & I & I \\ . & . & I & I \\ \hline \end{array} \longrightarrow \begin{array}{|cc|cc|}\hline X & O & X & O \\ I & . & I & . \\ \hline X & O & X & O \\ I & . & I & . \\ \hline \end{array}$$

Under the same permutation the second array becomes

$$\begin{array}{|cc|cc|}\hline X & . & X & . \\ O & I & O & I \\ \hline X & . & X & . \\ O & I & O & I \\ \hline \end{array} \longrightarrow \begin{array}{|cc|cc|}\hline X & I & I & X \\ . & O & O & . \\ \hline O & . & . & O \\ I & X & X & I \\ \hline \end{array}$$

The permutation exhibited above preserves the point in each array and moves array one to array two and array two to array five. In a similar manner it can be checked that the rest of the arrays are preserved under this permutation, thus S is preserved. This implies that the G_{x_1}-set is transitive on $\Omega-\{x_1\}$.

In a similar manner it can be shown that the G_{x_1,x_2}-set is transitive on $\Omega-\{x_1,x_2\}$. The permutations showing this are: (5, 9, 13)(6, 10, 14)(7, 11, 15)(8, 12, 16), and (5, 11, 14)(6, 12, 13)(7, 9, 16)(8, 10, 15).

Proposition 8

The order of G_{x_1} is 180.

Proof: There is a known result that states: if X is a transitive G-set of degree n and $x \in X$, then $|G| = n|G_x|$. It follows that if $S-\{x_1\}$ is a transitive G_{x_1}-set of degree $(n-1)$ and $x_1 \in S$, then $|G_{x_1}| = (n-1)|G_{x_1,x_2}|$. It was shown in Proposition 4 that $|G_{x_1,x_2}| = 12$. This implies that $|G_{x_1}| = 15 \cdot 12 = 180$.

The Order of G and Number of Orbits

Proposition 9

The order of G is 2,880.

Proof: Using the result quoted above it follows that $|G| = n|G_{x_1}| = 16 \cdot 180 = 2880$.

Proposition 10

The number of orbits of $S(2, 4, 16)$ is 2,523,920.

Proof: There is a known result that states that the number of orbits could be calculated using the formula $\Phi_\alpha^G = \frac{n}{h_\alpha}\sum_\omega \Phi(\omega)$, $(\omega \in C_\alpha \cap H)$. Since $n = [G:H] = \frac{16!}{2880} = 7{,}264{,}857{,}600$, it follows that the number of orbits is 2,523,920.

APPENDIX A

THE STEINER SYSTEMS $S(2, 3, 13)$ AND $S(2, 5, 25)$

S(2, 3, 13) = {{1, 2, 3}, {1, 4, 5}, {1, 6, 7}, {1, 8, 9}, {1, 10, 11}, {1, 12, 13}, {2, 4, 6}, {2, 5, 7}, {2, 8, 10}, {2, 9, 12}, {2, 11, 13}, {3, 4, 8}, {3, 5, 9}, {3, 6, 10}, {3, 7, 13}, {3, 11, 12}, {4, 7, 11}, {4, 9, 13}, {4, 10, 12}, {5, 6, 12}, {5, 8, 11}, {5, 10, 13}, {6, 8, 13}, {6, 9, 11}, {7, 8, 12}, {7, 9, 10}}.

S(2, 5, 25) = {{1, 2, 3, 4, 5}, {6, 7, 8, 9, 10}, {11, 12, 13, 14, 15}, {16, 17, 18, 19, 20}, {21, 22, 23, 24, 25}, {1, 6, 11, 16, 21}, {2, 7, 12, 17, 22}, {3, 8, 13, 18, 23}, {4, 9, 14, 19, 24}, {5, 10, 15, 20, 25}, {5, 9, 13, 17, 21}, {1, 7, 13, 19, 25}, {2, 8, 14, 20, 21}, {1, 10, 14, 18, 22}, {4, 8, 12, 16, 25}, {5, 6, 12, 18, 24}, {4, 10, 11, 17, 23}, {1, 8, 15, 17, 24}, {2, 9, 11, 18, 25}, {4, 6, 13, 20, 22}, {5, 7, 14, 16, 23}, {2, 6, 15, 19, 23}, {1, 9, 12, 20, 23}, {3, 10, 12, 19, 21}, {3, 7, 11, 20, 24}, {5, 8, 11, 19, 22}, {3, 9, 15, 16, 22}, {2, 10, 13, 16, 24}, {3, 6, 14, 17, 25}, {4, 7, 15, 18, 21}}.

APPENDIX B

CONJUGACY CLASSES OF S_{10} AND OF GROUP G OF AUTOMORPHISMS OF $S(3,4,10)$

Conjugacy Classes of S_{10}

```
[1]      Order 1       Length 1
          Rep Id(S10)

[2]      Order 2       Length 45
          Rep (1, 2)

[3]      Order 2       Length 630
          Rep (1, 2)(3, 4)

[4]      Order 2       Length 945
          Rep (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)

[5]      Order 2       Length 3150
          Rep (1, 2)(3, 4)(5, 6)

[6]      Order 2       Length 4725
          Rep (1, 2)(3, 4)(5, 6)(7, 8)

[7]      Order 3       Length 240
          Rep (1, 2, 3)

[8]      Order 3       Length 8400
          Rep (1, 2, 3)(4, 5, 6)

[9]      Order 3       Length 22400
          Rep (1, 2, 3)(4, 5, 6)(7, 8, 9)

[10]     Order 4       Length 1260
          Rep (1, 2, 3, 4)

[11]     Order 4       Length 18900
          Rep (1, 2, 3, 4)(5, 6)(7, 8)(9, 10)

[12]     Order 4       Length 18900
          Rep (1, 2, 3, 4)(5, 6)

[13]     Order 4       Length 56700
          Rep (1, 2, 3, 4)(5, 6, 7, 8)(9, 10)
[14]     Order 4       Length 56700
          Rep (1, 2, 3, 4)(5, 6, 7, 8)

[15]     Order 4       Length 56700
          Rep (1, 2, 3, 4)(5, 6)(7, 8)
```

[16] Order 5 Length 6048
Rep (1, 2, 3, 4, 5)

[17] Order 5 Length 72576
Rep (1, 2, 3, 4, 5)(6, 7, 8, 9, 10)

[18] Order 6 Length 5040
Rep (1, 2, 3)(4, 5)

[19] Order 6 Length 25200
Rep (1, 2, 3, 4, 5, 6)

[20] Order 6 Length 25200
Rep (1, 2, 3)(4, 5, 6)(7, 8)(9, 10)

[21] Order 6 Length 25200
Rep (1, 2, 3)(4, 5)(6, 7)(8, 9)

[22] Order 6 Length 25200
Rep (1, 2, 3)(4, 5)(6, 7)

[23] Order 6 Length 50400
Rep (1, 2, 3)(4, 5, 6)(7, 8)

[24] Order 6 Length 75600
Rep (1, 2, 3, 4, 5, 6)(7, 8)(9, 10)

[25] Order 6 Length 151200
Rep (1, 2, 3, 4, 5, 6)(7, 8)

[26] Order 6 Length 201600
Rep (1, 2, 3, 4, 5, 6)(7, 8, 9)

[27] Order 7 Length 86400
Rep (1, 2, 3, 4, 5, 6, 7)

[28] Order 8 Length 226800
Rep (1, 2, 3, 4, 5, 6, 7, 8)(9, 10)
[29] Order 8 Length 226800
Rep (1, 2, 3, 4, 5, 6, 7, 8)

[30] Order 9 Length 403200
Rep (1, 2, 3, 4, 5, 6, 7, 8, 9)
[31] Order 10 Length 60480

Rep (1, 2, 3, 4, 5)(6, 7)

[32] Order 10 Length 90720
Rep (1, 2, 3, 4, 5)(6, 7)(8, 9)

[33] Order 10 Length 362880
Rep (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)

[34] Order 12 Length 50400
Rep (1, 2, 3, 4)(5, 6, 7)(8, 9, 10)

[35] Order 12 Length 50400
Rep (1, 2, 3, 4)(5, 6, 7)

[36] Order 12 Length 151200
Rep (1, 2, 3, 4, 5, 6)(7, 8, 9, 10)

[37] Order 12 Length 151200
Rep (1, 2, 3, 4)(5, 6, 7)(8, 9)

[38] Order 14 Length 259200
Rep (1, 2, 3, 4, 5, 6, 7)(8, 9)

[39] Order 15 Length 120960
Rep (1, 2, 3, 4, 5)(6, 7, 8)

[40] Order 20 Length 181440
Rep (1, 2, 3, 4, 5)(6, 7, 8, 9)

[41] Order 21 Length 172800
Rep (1, 2, 3, 4, 5, 6, 7)(8, 9, 10)

[42] Order 30 Length 120960
Rep (1, 2, 3, 4, 5)(6, 7, 8)(9, 10)

Conjugacy Classes of Group G of Automorphisms of $S(3, 4, 10)$

[1] Order 1 Length 1
Rep Id(G)

[2] Order 2 Length 30
Rep (1, 3)(5, 8)(6, 9)

[3] Order 2 Length 36
Rep (1, 9)(2, 7)(3, 5)(4, 8)(6, 10)

[4] Order 2 Length 45
Rep (2, 3)(4, 9)(5, 6)(8, 10)

[5] Order 3 Length 80
Rep (1, 10, 4)(2, 6, 5)(7, 8, 9)

[6] Order 4 Length 90
Rep (2, 5, 3, 6)(4, 8, 9, 10)

[7] Order 4 Length 90
Rep (1, 10, 4, 3)(2, 9, 7, 5)(6, 8)

[8] Order 4 Length 180
Rep (1, 8, 4, 10)(3, 7, 6, 9)

[9] Order 5 Length 144
Rep (1, 6, 2, 8, 3)(4, 5, 9, 10, 7)

[10] Order 6 Length 240
Rep (1, 9, 10, 7, 4, 8)(2, 5, 6)

[11] Order 8 Length 180
Rep (1, 7)(2, 9, 5, 10, 3, 4, 6, 8)

[12] Order 8 Length 180
Rep (1, 8, 4, 2, 10, 6, 3, 7)

[13] Order 10 Length 144
Rep (1, 4, 6, 5, 2, 9, 8, 10, 3, 7)

APPENDIX C

CONJUGACY CLASSES OF GROUP *G* OF AUTOMORPHISMS OF *S*(2, 4, 16)

```
[1]      Order 1       Length 1
          Rep Id(g)

[2]      Order 2       Length 15
          Rep (1, 15)(2, 16)(3, 13)(4, 14)(5, 11)(6, 12)(7,
9)(8, 10)

[3]      Order 2       Length 60
          Rep (2, 9)(3, 13)(4, 5)(6, 12)(7, 16)(11, 14)

[4]      Order 3       Length 16
          Rep (1, 13, 9)(2, 15, 12)(3, 16, 10)(4, 14, 11)(6,
7, 8)

[5]      Order 3       Length 16
          Rep (1, 9, 13)(2, 12, 15)(3, 10, 16)(4, 11, 14)(6,
8, 7)

[6]      Order 3       Length 80
          Rep (1, 13, 5)(3, 11, 15)(4, 8, 12)(6, 14, 10)

[7]      Order 3       Length 80
          Rep (1, 5, 13)(3, 15, 11)(4, 12, 8)(6, 10, 14)

[8]      Order 3       Length 320
          Rep (1, 16, 6)(2, 9, 4)(3, 7, 15)(5, 8, 10)(12,
14, 13)

[9]      Order 4       Length 180
          Rep (1, 8, 2, 7)(3, 6, 4, 5)(9, 14, 10, 13)(11,
16, 12, 15)

[10]     Order 5       Length 192
          Rep (1, 11, 8, 12, 3)(2, 16, 13, 4, 6)(7, 15, 10,
9, 14)

[11]     Order 5       Length 192
          Rep (1, 8, 3, 11, 12)(2, 13, 6, 16, 4)(7, 10, 14,
15, 9)

[12]     Order 6       Length 240
          Rep (1, 15, 8)(2, 11, 6, 9, 14, 12)(3, 7, 5, 13,
16, 4)

[13]     Order 6       Length 240
```

Rep (1, 11, 13, 15, 5, 3)(2, 16)(4, 6, 8, 14, 12, 10)(7, 9)

[14] Order 6 Length 240
Rep (1, 3, 5, 15, 13, 11)(2, 16)(4, 10, 12, 14, 8, 6)(7, 9)

[15] Order 6 Length 240
Rep (1, 8, 15)(2, 12, 14, 9, 6, 11)(3, 4, 16, 13, 5, 7)

[16] Order 15 Length 192
Rep (1, 16, 15, 8, 4, 9, 3, 2, 7, 11, 13, 10, 12, 6, 14)

[17] Order 15 Length 192
Rep (1, 7, 16, 11, 15, 13, 8, 10, 4, 12, 9, 6, 3, 14, 2)

[18] Order 15 Length 192
Rep (1, 6, 10, 11, 2, 9, 8, 16, 14, 12, 13, 7, 3, 4, 15)

[19] Order 15 Length 192
Rep (1, 10, 2, 8, 14, 13, 3, 15, 6, 11, 9, 16, 12, 7, 4)

APPENDIX D

COMPUTER PROGRAMS

Program for Calculating the Orbits of *S*(3,4,10)

```
S10:=SymmetricGroup(10);
S3410:={{1,2,4,5},{1,2,3,7},{1,3,5,8},{2,3,5,6},{2,3,4,8},{
2,4,6,9},{3,4,6,7},{3,4,5,9},{3,5,7,10},{4,5,7,8},{4,5,6,10
},{1,4,6,8},{5,6,8,9},{1,5,6,7},{2,5,7,9},{6,7,9,10},{2,6,7
,8},{3,6,8,10},{1,7,8,10},{3,7,8,9},{1,4,7,9},{1,2,8,9},{4,
8,9,10},{2,5,8,10},{2,3,9,10},{1,5,9,10},{1,3,6,9},{1,3,4,1
0},{1,2,6,10},{2,4,7,10}};
x:=S10!(1,3)(5,8)(6,9);
y:=S10!(1,4,6,5,2,9,8,10,3,7);
G:=sub<S10|x,y>;
T:=Transversal(S10,G);
ST:=Seqset([2..#T]);
orbits:=[{1}: i in [1..10]];
for k:=1 to 9 do
a:=Rep(ST); tt:={};
for i:=2 to #T do
for x in G do
if T[a]^x*T[i]^(-1) in G then
tt:=tt join {i};
end if;
end for;
end for;
orbits[k]:=tt;
ST:=ST diff tt;
end for;
sum:=0;
for jj:=1 to 9 do sum:=sum + #orbits[jj]; end for;
print sum;
print #T;
for jj:=1 to 9 do print jj; print #orbits[jj]; print
orbits[jj]; end for;
for jj:=1 to 9 do print T[Rep(orbits[jj])]; end for;
sum:=0;
for jj:=1 to 9 do sum:= sum + #orbits[jj]; end for;
print sum;
2519
print #T;
2520
for jj:=1 to 9 do print jj; print #orbits[jj]; print
orbits[jj]; end for;
```

Program for Finding the Decomposition of S_{10} Over G

```
S10:=SymmetricGroup(10);
x:=S10!(1,4,6,5,2,9,8,10,3,7);
y:=S10!(1,3)(5,8)(6,9);
G:=sub<S10|x,y>;
C:=Classes(S10);
C28:=Class(S10,C[28][3]);
S:={20,45,90,144,180,240,360,720};
for i in S do
mm:=0;
for x in C28 do
if Index(G,G^x meet G) eq i then mm:=mm+1; yy:= x; end if;
end for; PrintFileMagma("temporary",i);
PrintFileMagma("temporary",mm);
PrintFileMagma("temporary",yy);
end for;
mm:=0;
for x in C28 do
if x in G then mm:=mm+1; end if; end for;
PrintFileMagma("temporary",mm);
C29:=Class(S10,C[29][3]);
S:={20,45,90,144,180,240,360,720};
for i in S do
mm:=0;
for x in C29 do
if Index(G,G^x meet G) eq i then mm:=mm+1; yy:= x; end if;
end for; PrintFileMagma("temporary",i);
PrintFileMagma("temporary",mm);
PrintFileMagma("temporary",yy); end for;
mm:=0;
for x in C29 do
if x in G then mm:=mm+1; end if; end for;
PrintFileMagma("temporary",mm);
C30:=Class(S10,C[30][3]);
S:={20,45,90,144,180,240,360,720};
for i in S do
mm:=0;
for x in C30 do
if Index(G,G^x meet G) eq i then mm:=mm+1; yy:= x; end if;
end for; PrintFileMagma("temporary",i);
PrintFileMagma("temporary",mm);
PrintFileMagma("temporary",yy); end for;
mm:=0;
```

```
for x in C30 do
if x in G then mm:=mm+1; end if; end for;
PrintFileMagma("temporary",mm);
C36:=Class(S10,C[36][3]);
S:={20,45,90,144,180,240,360,720};
for i in S do
mm:=0;
for x in C36 do
if Index(G,G^x meet G) eq i then mm:=mm+1; yy:= x; end if;
end for; PrintFileMagma("temporary",i);
PrintFileMagma("temporary",mm);
PrintFileMagma("temporary",yy); end for;
mm:=0;
for x in C36 do
if x in G then mm:=mm+1; end if; end for;
PrintFileMagma("temporary",mm);
C37:=Class(S10,C[37][3]);
S:={20,45,90,144,180,240,360,720};
for i in S do
mm:=0;
for x in C37 do
if Index(G,G^x meet G) eq i then mm:=mm+1; yy:= x; end if;
end for; PrintFileMagma("temporary",i);
PrintFileMagma("temporary",mm);
PrintFileMagma("temporary",yy); end for;
mm:=0;
for x in C37 do
if x in G then mm:=mm+1; end if; end for;
PrintFileMagma("temporary",mm);
C38:=Class(S10,C[38][3]);
S:={20,45,90,144,180,240,360,720};
for i in S do
mm:=0;
for x in C38 do
if Index(G,G^x meet G) eq i then mm:=mm+1; yy:= x; end if;
end for; PrintFileMagma("temporary",i);
PrintFileMagma("temporary",mm);
PrintFileMagma("temporary",yy); end for;
mm:=0;
for x in C38 do
if x in G then mm:=mm+1; end if; end for;
PrintFileMagma("temporary",mm);
C39:=Class(S10,C[39][3]);
S:={20,45,90,144,180,240,360,720};
for i in S do
```

```
mm:=0;
for x in C39 do
if Index(G,G^x meet G) eq i then mm:=mm+1; yy:= x; end if;
end for; PrintFileMagma("temporary",i);
PrintFileMagma("temporary",mm);
PrintFileMagma("temporary",yy); end for;
mm:=0;
for x in C39 do
if x in G then mm:=mm+1; end if; end for;
PrintFileMagma("temporary",mm);
C40:=Class(S10,C[40][3]);
S:={20,45,90,144,180,240,360,720};
for i in S do
mm:=0;
for x in C40 do
if Index(G,G^x meet G) eq i then mm:=mm+1; yy:= x; end if;
end for; PrintFileMagma("temporary",i);
PrintFileMagma("temporary",mm);
PrintFileMagma("temporary",yy); end for;
mm:=0;
for x in C40 do
if x in G then mm:=mm+1; end if; end for;
PrintFileMagma("temporary",mm);
C41:=Class(S10,C[41][3]);
S:={20,45,90,144,180,240,360,720};
for i in S do
mm:=0;
for x in C41 do
if Index(G,G^x meet G) eq i then mm:=mm+1; yy:= x; end if;
end for; PrintFileMagma("temporary",i);
PrintFileMagma("temporary",mm);
PrintFileMagma("temporary",yy); end for;
mm:=0;
for x in C41 do
if x in G then mm:=mm+1; end if; end for;
PrintFileMagma("temporary",mm);
C42:=Class(S10,C[42][3]);
S:={20,45,90,144,180,240,360,720};
for i in S do
mm:=0;
for x in C42 do
if Index(G,G^x meet G) eq i then mm:=mm+1; yy:= x; end if;
end for; PrintFileMagma("temporary",i);
PrintFileMagma("temporary",mm);
PrintFileMagma("temporary",yy);end for;
```

```
mm:=0;
for x in C42 do
if x in G then mm:=mm+1; end if; end for;
PrintFileMagma("temporary",mm);
 > s:={{1,3,2,4}, {1,3,5,7}, {1,3,6,8},{2,4,5,7},
{2,4,6,8}, {5,7,6,8},
 > {1,2,5,6}, {1,2,7,8}, {3,4,5,6}, {3,4,7,8}, {1,4,5,8},
{1,4,7,6},
{3,2,5,8},\
  {3,2,6,7}};
 > S8:=SymmetricGroup(8);
 > G:=sub<S8|Id(S8)>;
 > for x in S8 do if s^x eq s then G:=sub<S8|x,G>; end if;
end for;
 > print Order(G);
1344
```

Program for Graph

```
S10:=SymmetricGroup(10);
x:=S10!(1,4,6,5,2,9,8,10,3,7);
y:=S10!(1,3)(5,8)(6,9);
G:=sub<S10|x,y>;
C:=Classes(S10);
o45:=S10!(1,2);
o240:=S10!(1,2,3);
o360:=S10!(1,2)(3,4);
o180:=S10!(1,2,3,4);
o7201:=S10!(1,2,3,4,5);
o7202:=S10!(1,2)(3,4)(5,6);
o144:=S10!(1,2)(3,4)(5,6)(7,8);
o90:=S10!(1,2,3)(4,5,6)(7,8)(9,10);
o20:=S10!(1,5,2,10,3,8,9,6,7,4);
C:=Class(S10,S10!(1,2));
temp:=[[0: j in [1..10]]: i in [1..10]];
SS:=[[{Id(S10)} : j in [1..10]]: i in [1..10]];
S:={o45, o240, o360, o180, o7201, o7202,o144,o90,o20};
i:=0;
for s in S do
i:=i+1;
for x in C do
if Index(G, G meet (G^s)^x) eq 360 then
temp[i][1]:=temp[i][1]+1; SS[i][1]:=SS[i][1] join {x};
end if;
end for;
for x in C do
if Index(G, G meet (G^s)^x) eq 240 then
temp[i][2]:=temp[i][2]+1; SS[i][2]:=SS[i][2] join {x};
end if;
end for;
for x in C do
if Index(G, G meet (G^s)^x) eq 20 then
temp[i][3]:=temp[i][3]+1; SS[i][3]:=SS[i][3] join {x};
end if;
end for;
for x in C do
if Index(G, G meet (G^s)^x) eq 1 then
temp[i][4]:=temp[i][4]+1; SS[i][4]:=SS[i][4] join {x};
end if;
end for;
for x in C do
if Index(G, G meet (G^s)^x) eq 144 then
```

```
temp[i][5]:=temp[i][5]+1; SS[i][5]:=SS[i][5] join {x};
end if;
end for;
for x in C do
if Index(G, G meet (G^s)^x) eq 180 then
temp[i][6]:=temp[i][6]+1; SS[i][6]:=SS[i][6] join {x};
end if;
end for;
for x in C do
if Index(G, G meet (G^s)^x) eq 90 then
temp[i][7]:=temp[i][7]+1; SS[i][7]:=SS[i][7] join {x};
end if;
end for;
for x in C do
for g in G do
if G*(s*x) eq G*S10!(1,2,3,4,5)^g then
temp[i][8]:=temp[i][8]+1; SS[i][8]:=SS[i][8] join {x};
break;
end if;
end for; end for;
for x in C do
for g in G do
if G*(s*x) eq G*S10!(1,2)(3,4)(5,6)^g then
temp[i][9]:=temp[i][9]+1; SS[i][9]:=SS[i][9] join {x};
break;
end if; end for;
end for;
for x in C do
if Index(G, G meet (G^s)^x) eq 45 then
temp[i][10]:=temp[i][10]+1; SS[i][10]:=SS[i][10] join {x};
end if;
end for;
print s;
end for;
```

```
(1, 2, 3, 4, 5)
(1, 2)(3, 4)(5, 6)
(1, 2)(3, 4)
(1, 2, 3, 4)
(1, 2, 3)
(1, 2)
(1, 2)(3, 4)(5, 6)(7, 8)
(1, 2, 3)(4, 5, 6)(7, 8)(9, 10)

(1, 2, 3, 4, 5)
```

```
[ 9, 4, 0, 0, 4, 4, 3, 8, 13, 0 ]
{(5, 7),(1, 4),(1, 3),(1, 8),(7, 10),(4, 8),(3, 8),(6,
9),(2, 9)},
{(2, 5),(6, 10),(3, 7),(4, 9)},
{Id(S10)},
{Id(S10)},
{(5, 6),(4, 7),(3, 9),(2, 10)},
{(8, 10),(1, 5),(6, 8),(1, 2)},
{(5, 10),(2, 6),(3, 4)},
{(2, 4),(5, 8),(1, 10),(1, 6),(3, 5),(4, 6),(3, 10),(2,
8)},
{(1, 7),(4, 5),(2, 7),(9, 10),(4, 10),(7, 9),(6, 7),(8,
9),(2, 3),(3,
6),(5, 9),(7, 8),(1, 9)},
{Id(S10)}

(1, 2)(3, 4)(5, 6)
[ 3, 4, 0, 0, 4, 4, 1, 13, 16, 0 ]
{(5, 6),(7, 10),(1, 2)},
{(4, 7),(4, 10),(3, 10),(3, 7)},
{Id(S10)},
{Id(S10)},
{(2, 5),(7, 8),(1, 6),(9, 10)},
{(2, 9),(5, 8),(1, 9),(6, 8)},
{(3, 4)},
{(4, 5),(5, 7),(1, 4),(6, 9),(6, 7),(8, 9),(2, 3),(1,
10),(2, 10),(3,
6),(5, 9),(2, 8),(1, 8)},
{(6, 10),(1, 7),(4, 9),(4, 6),(3, 9),(4, 8),(3, 8),(2,
7),(7, 9),(2, 4),(1,
5),(5, 10),(2, 6),(3, 5),(8, 10),(1, 3)},
{Id(S10)}

(1, 2)(3, 4)
[ 11, 4, 2, 0, 0, 2, 0, 18, 6, 2 ]
{(5, 7),(1, 8),(3, 7),(8, 10),(1, 10),(4, 5),(3, 5),(2,
10),(6, 9),(2,
8),(4, 7)},
{(6, 10),(6, 7),(8, 9),(5, 9)},
{(7, 10),(5, 8)},
{Id(S10)},
{Id(S10)},
{(2, 4),(1, 3)},
{Id(S10)},
{(1, 7),(4, 9),(4, 6),(3, 9),(4, 8),(3, 8),(2, 7),(4,
```

```
10),(2, 5),(1, 5),(5,
10),(3, 10),(2, 6),(3, 6),(7, 8),(2, 9),
(1, 6),(1, 9)},
{(5, 6),(2, 3),(7, 9),(6, 8),(9, 10),(1, 4)},
{(3, 4),(1, 2)}

(1, 2, 3, 4)
[ 4, 8, 1, 0, 0, 0, 0, 16, 16, 0 ]
{(2, 4),(7, 10),(1, 3),(5, 8)},
{(5, 7),(2, 3),(5, 10),(8, 10),(7, 8),(3, 4),(1, 2),(1,
4)},
{(6, 9)},
{Id(S10)},
{Id(S10)},
{Id(S10)},
{Id(S10)},
{(1, 7),(3, 7),(4, 5),(4, 8),(3, 8),(2, 7),(4, 7),(4,
10),(2, 5),(1, 5),(3,
10),(3, 5),(1, 10),(2, 10),(2, 8),(1, 8)},
{(6, 10),(4, 9),(4, 6),(3, 9),(9, 10),(7, 9),(6, 7),(2,
6),(8, 9),(6,
8),(3, 6),(5, 9),(5, 6),(2, 9),(1, 6),(1, 9)},
{Id(S10)}

(1, 2, 3)
[ 6, 3, 0, 0, 0, 6, 3, 12, 12, 3 ]
{(4, 5),(6, 10),(4, 8),(5, 8),(9, 10),(6, 9)},
{(5, 6),(4, 10),(8, 9)},
{Id(S10)},
{Id(S10)},
{Id(S10)},
{(1, 4),(1, 10),(3, 5),(3, 6),(2, 8),(2, 9)},
{(2, 7),(1, 7),(3, 7)},
{(5, 7),(7, 8),(6, 8),(5, 10),(7, 10),(8, 10),(4, 9),(4,
6),(5, 9),(7,
9),(4, 7),(6, 7)},
{(3, 9),(3, 8),(3, 4),(2, 4),(2, 5),(1, 5),(3, 10),(2,
6),(2, 10),(1,
6),(1, 8),(1, 9)},
{(2, 3),(1, 3),(1, 2)}

(1, 2)
[ 16, 16, 0, 1, 0, 0, 8, 0, 0, 4 ]
{(4, 9),(5, 8),(4, 8),(3, 4),(4, 7),(5, 7),(9, 10),(7,
10),(6, 9),(6,
```

```
7),(3, 10),(3, 5),(8, 10),(6, 8),(3, 6),(5, 9)},
{(1, 7),(2, 7),(1, 4),(2, 4),(2, 5),(1, 5),(2, 6),(2,
3),(1, 10),(2,
10),(1, 3),(2, 8),(2, 9),(1, 6),(1, 8),(1, 9)},
{Id(S10)},
{(1, 2)},
{Id(S10)},
{Id(S10)},
{(5, 6),(7, 8),(5, 10),(4, 10),(3, 9),(3, 8),(4, 6),(7,
9)},
{Id(S10)},
{Id(S10)},
{(4, 5),(6, 10),(3, 7),(8, 9)}

(1, 2)(3, 4)(5, 6)(7, 8)
[ 0, 0, 0, 0, 5, 0, 0, 20, 20, 0 ]
{Id(S10)},
{Id(S10)},
{Id(S10)},
{Id(S10)},
{(3, 5),(1, 7),(4, 6),(9, 10),(2, 8)},
{Id(S10)},
{Id(S10)},
{(2, 6),(2, 3),(5, 8),(1, 5),(2, 9),(1, 8),(7, 10),(6,
10),(4, 8),(5,
6),(3, 9),(4, 9),(3, 7),(3, 4),(1, 9),(5, 10),
(6, 7),(2, 7),(8, 10),(1, 4)},
{(4, 10),(1, 6),(2, 4),(4, 5),(3, 8),(6, 8),(1, 2),(1,
3),(3, 6),(2, 5),(6,
9),(2, 10),(8, 9),(7, 8),(4, 7),(7, 9),
(3, 10),(5, 7),(1, 10),(5, 9)},
{Id(S10)}

(1, 2, 3)(4, 5, 6)(7, 8)(9, 10)
[ 0, 8, 0, 0, 0, 0, 1, 24, 8, 4 ]
{Id(S10)},
{(5, 6),(1, 4),(2, 3),(2, 6),(3, 5),(1, 9),(7, 9),(4, 7)},
{Id(S10)},
{Id(S10)},
{Id(S10)},
{Id(S10)},
{(8, 10)},
{(4, 10),(5, 8),(1, 5),(1, 8),(3, 8),(7, 10),(6, 8),(1,
2),(6, 10),(4,
8),(3, 9),(6, 9),(2, 10),(8, 9),(7, 8),
```

```
(4, 6),(3, 4),(5, 10),(2, 8),(2, 7),(3, 10),(9, 10),(5,
7),(1, 10)},
{(4, 5),(2, 4),(1, 3),(2, 9),(3, 7),(6, 7),(1, 6),(5, 9)},
{(2, 5),(3, 6),(1, 7),(4, 9)}

(1, 5, 2, 10, 3, 8, 9, 6, 7, 4)
[ 36, 0, 0, 0, 0, 9, 0, 0, 0, 0 ]
{(8, 9),(6, 7),(9, 10),(6, 8),(4, 7),(2, 4),(5, 7),(4,
10),(6, 9),(1,
4),(1, 8),(2, 5),(5, 8),
(1, 5),(4, 8),(1, 7),(1, 9),(2, 9),(8, 10),(2, 6),(1,
10),(7, 8),(6,
10),(4, 6),(5, 6),(2, 8),
(2, 10),(7, 10),(5, 10),(7, 9),(4, 9),(1, 6),(1, 2),(5,
9),(2, 7),(4, 5)},
{Id(S10)},
{Id(S10)},
{Id(S10)},
{Id(S10)},
{(2, 3),(3, 4),(1, 3),(3, 7),(3, 9),(3, 8),(3, 5),(3,
10),(3, 6)},
{Id(S10)},
{Id(S10)},
{Id(S10)},
{Id(S10)}


s3410:={{1,2,4,5},{1,2,3,7},{1,3,5,8},{2,3,5,6},{2,3,4,8},{
2,4,6,9},{3,4,6,7
},{3,4,
5,9},{3,5,7,10},{4,5,7,8},{4,5,6,10},{1,4,6,8},{5,6,8,9},{1
,5,6,7},{2,5,7,9}
,{6,7,9,10},{2,6,7,8},{3,6,8,10},{1,7,8,10},{3,7,8,9},{1,4,
7,9},{1,2,8,9},{4
,8,9,10},{2,5,8,10},{2,3,9,10},{1,5,9,10},{1,3,6,9},{1,3,4,
10},{1,2,6,10},{2
,4,7,10}};
```

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