2002

The cyclic cutwidth of mesh cubes

Dwayne William Clarke

THE CYCLIC CUTWIDTH OF MESH CUBES

A Project

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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ABSTRACT

This project's purpose was to understand the workings of a new theorem introduced in a professional paper on the cutwidth of meshes and then use this knowledge to apply it to the search for the cyclic cutwidth of the n-cube. Before being able to use the theorem though, problems with the theorem needed to be worked out. In fact, the theorem was found to be stated wrong in the paper. Actual examples were found to contradict what the theorem stated.

It was then found that the theorem did not accurately represent the propositions, corollaries and a theorem that formed it. It was also found that another proposition was needed to complete the theorem, that is, to allow the theorem to accurately describe all cases under consideration. A proposition was then created and proved using techniques worked out in the proof of the other propositions and corollaries of the theorem.

After completing the theorem, product structures of meshes were looked at. These structures were called mesh cubes. Mesh cubes of the type $P_n \times P_n \times P_n$ are similar to the n-cube in many ways and so a theorem on the cyclic cutwidth of mesh cubes became the next step. Proving a

theorem on the upper bounds of the cyclic cutwidth of mesh cubes became the main goal of the application portion of our work in this project.

The Theorem states: If $n>2$

n is even $\qquad ccw\ (P_n \times P_n \times P_n) \leq n^2 - n + 1$

n is odd $\qquad ccw\ (P_n \times P_n \times P_n) \leq n^2 + 1$

In words, it says that the cyclic cutwidth of a three dimensional mesh cube of equal dimension is equal to the length of an edge (the number of vertices along an edge) squared, plus one , minus the length of the edge if the edge length is even; if the edge length is odd, then it is the square of an edge plus one.

The result of this work has been insightful and the result of our proof on the upper bounds is interesting. It indicates a, sort-of, "two dimensionality" of 3 dimensional objects when it comes to cyclic cutwidth. It may be this idea that helps reduce the complexity of the n-cube so that it is solvable.

ACKNOWLEDGEMENTS

I would like to take the time to acknowledge the help and contribution that several people have made in helping me attain the goal of this final project. First, I thank Dr. Chavez, for his patience and guidance, and his willingness to support me in my goals. Next, I would like to thank my other advisors for their contribution. Lastly, I thank several of my professors who have demanded high standards and have contributed greatly to my love and talent in mathematics. These Professors are Dr. Bob Stein, Dr. Griffing, and Dr. Okon.

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CHAPTER ONE

INTRODUCTION

Purpose of Project

We began this project with a common goal in mind; to contribute to the solution of the cyclic cutwidth of the n-cube. Dr. Chavez has been working toward this and I became interesting in it during Dr. Chavez's class on Graph Theory. During the class, Dr. Chavez distributed a work by Schroder et al., 5), which used a new technique to prove a theorem on meshes. However, when Dr. Chavez tested the theorem with examples he found flaws. He then encouraged students to consider the accuracy of the theorem.

I have chosen to explore the cyclic cutwidth of the n-cube as my graduate project. I will correct errors within Schroder's Theorem, master the techniques used in it's proof and finally apply these idea's to some aspect of the n-cube which may give us insight into the cyclic cutwidth solution.

Overview

The work in this project will focus on Schroder's Theorem on the cyclic cutwidth of meshes. The theorem as stated in 5) is:

Schroder's Theorem: For $m \geq n \geq 3$

$$ccw(P_m x P_n) = \begin{cases} n-1, & \text{if } m=n \text{ is even} \\ n, & \text{if } n \text{ is odd or } m=n+2 \text{ is even} \\ n+1, & \text{otherwise} \end{cases}$$

By looking at specific examples, problems with this theorem were encountered. As a result, the various propositions and corollaries that make up the lower bounds of Schroder's Theorem and Rolim's Theorem that makes up the upper bounds, were dissected. Subsequently, it was found that the theorem did not accurately reflect the theorem, propositions and corollaries that were used to construct it. Furthermore, a new proposition was needed to complete the lower bounds of the theorem. A proof was then constructed for the missing proposition:

Proposition 1: For $n \geq 3$,

$$ccw(P_{n+2} \times P_n) \geq n \text{ if } n \text{ is even}$$

Subsequently, a new theorem was constructed:

Theorem 1: For $m \geq n \geq 3$

$$ccw(P_m \times P_n) = \begin{cases} n - 1, & \text{if } m = n, \text{ and } n \text{ is even} \\ n, & \text{if } n + 2 > m \geq n, \text{ and } n \text{ odd or even} \\ n + 1, & \text{if } m \geq n + 2, \text{ and } n \text{ is odd or} \\ & \text{if } m > n + 2, \text{ and } n \text{ is even} \end{cases}$$

With a completed theorem on the cyclic cutwidth of meshes, the techniques and facts regarding these meshes were applied to a structure similar to the n-cube. Insight and conjectures on structures $P_l \times P_m \times P_n$ were found and a theorem for these structures was developed.

Technical Terms and Concepts

In order to ensure the reader's ability to understand the following discussions, a few important definitions and concepts are now presented. Non-technical definitions and illustrations are in the appendix.

Types of Graphs

Graph - A graph $G = (V, E, \partial)$, consists of a vertex-set V, edge-set E, and a boundary function ∂,

$$\partial : E \rightarrow \binom{V}{1} + \binom{V}{2}$$

which identifies the pair of vertices incident to each edge.

Tree - A graph that is connected and acyclic.

Mesh - A mesh, denoted $P_m \times P_n$, is the Cartesian product of two paths.

Path - A sequence of distinct vertices $(x_1, x_2, \ldots, x_n)$ such that x_i is adjacent to x_{i+1}.

H - Layout - An H-Layout of G is an ordered pair (π, P_π) consisting of:

(i) A one-to-one correspondence π between the vertices of G and those of H, and

(ii) A collection P_π of paths in H, one path joining $\pi(v)$ and $\pi(w)$ for each pair of adjacent vertices v and w in G.

Cutwidth - The cutwidth $c_\pi(G)$ of the layout (π, P_π) is the maximum number of times an edge e of H appears in the set of paths P_π. The cutwidth c(G) of G in H is the minimum of the cutwidths c_π taken over all layouts (π, P_π) of G in H.

Linear cutwidth - The linear cutwidth, denoted lcw(G), is the cutwidth of G embedded in a linear chassis layout.

Cyclic Cutwidth - The cyclic cutwidth, denoted ccw(G), is the cutwidth of G embedded in a cycle layout.

N-cube - The graph of the n-dimensional cube, Q_n, has vertex set $\{0,1\}^n$, the n-fold Cartesian product of $\{0,1\}$. Thus $|V| = 2^n$. Q_n has an edge between two vertices (n-tuples of 0's and 1's) if they differ in exactly one entry.

CHAPTER TWO

THE THEOREM AND ITS PROBLEMS

Confirming a Problem

The techniques in Schroder's Theorem, 5), may be of use in solving the cyclic cutwidth of the n-cube. Unfortunately, we have found several flaws in the theorem. Specifically, the theorem did not support certain examples we created to test it.

In this chapter we will dissect Schroder's Theorem and show that it does not accurately reflect the theorem, propositions, and corollaries that make it. We will construct a Theorem 1 using the theorem, propositions, and corollaries as the building blocks, and verify it's accuracy. Lastly, we will demonstrate that Theorem 1 will need an additional proposition in order to complete it. Hence the creation of Proposition 1.

Schroder's Theorem: For $m \geq n \geq 3$

$$ccw\ (P_m \times P_n) = \begin{cases} n - 1, & \text{if } m = n \text{ is even} \\ n, & \text{if } n \text{ is odd or } m = n + 2 \text{ is even} \\ n + 1, & \text{otherwise} \end{cases}$$

Schroder's Theorem can be broken down into four individual statements:

(1) if $m = n$ n is even, $ccw(P_n \times P_m) = n - 1$

(2) if $m \geq n$ n is odd, $ccw(P_m \times P_n) = n$

(3) if $n + 1 \leq m \leq n + 2$ n is even, $ccw(P_{n+2} \times P_n) = n$

(4) if $m > n + 2$ n is even, $ccw(P_{m>n+2} \times P_n) = n + 1$

By looking at easy to verify examples, a contradiction to part (2) of the theorem can be easily demonstrated. Following are a few examples.

Example 1. $ccw\ (P_7 \times P_3) = 4$, which is $n + 1$ and not $n = 3$. See Figure 1 below.

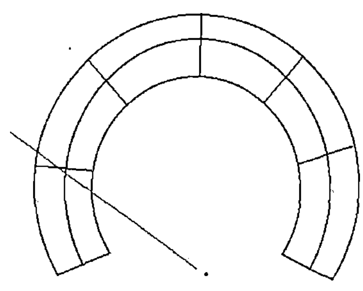


Figure 1. A 7 x 3
Mesh Has Cutwidth 4.

Example 2. $ccw\ (P_6 \times P_3) = 4$, which is $n + 1$ and not $n = 3$. See Figure 2 below. Notice the lower bound for this shape is found using the conventional method as opposed to example 1. The choice of method is determined by the shape of the structure.

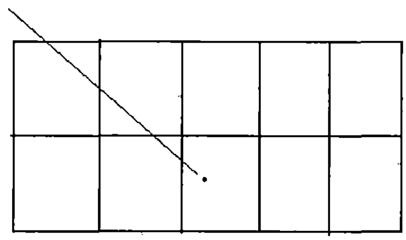


Figure 2. A 6 x 3 Mesh Has Cutwidth 4.

Example 3. ccw (P_5 x P_3) = 4, which is n + 1 and not n = 3. See Figure 3 below.

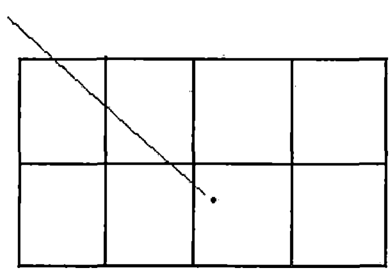


Figure 3. A 5 x 3 Mesh Has Cutwidth 4.

By noting these examples, the nature of the problem within the stated theorem is difficult to identify. It is then necessary to look at the individual components, the theorem, propositions, and corollaries of Schroder's theorem to see what parts of the theorem are implied.

A Look at the Components

The theorem as stated in 5) uses equalities, however it is actually composed of two separate parts; the lower bounds consisting of greater than or equal to symbols, and the upper bounds consisting of less than or equal to symbols. The proof in 5) is for the lower bounds of the theorem. The proof of the upper bounds is in 6).

We are particularly interested in the proof on the lower bounds because it gives us an idea of the minimum cutwidths of meshes. It is these concepts and techniques that may help us find a solution to the n-cube problem.

In looking at the problems that are associated with Schroder's Theorem and the proofs of its components, it is important that we look at both the upper and lower bounds. This is what we will do in this section of the paper.

First, let us state the theorem and the component parts that make up the lower bounds and the upper bounds:

Schroder's Theorem: For $m \geq n \geq 3$

$$ccw\ (P_m \times P_n) = \begin{cases} n - 1, & \text{if } m = n \text{ is even} \\ n, & \text{if } n \text{ is odd or } m = n + 2 \text{ is even} \\ n + 1, & \text{otherwise} \end{cases}$$

The following propositions and corollaries give the lower bounds of Schroder's Theorem:

Schroder's Prop. 1: for $n \geq 3$

$$ccw(P_n \times P_n) \geq n - 1 \quad \text{if } n \text{ is even}$$

$$ccw(P_n \times P_n) \geq n \quad \text{if } n \text{ is odd}$$

Schroder's Prop. 2: for $n \geq 3$

$$ccw(P_{n+2} \times P_n) \geq n + 1 \quad \text{if } n \text{ is odd}$$

Schroder's Cor. 1: for $n \geq 3$

$$ccw(P_{n+1} \times P_n) \geq n \quad \text{for all } n$$

Schroder's Prop. 3: for $n \geq 4$

$$ccw(P_{n+3} \times P_n) \geq n+1 \quad \text{for all } n$$

The upper bounds of Schroder's Theorem are made up of Rolim's Theorem.

Rolim's Theorem For $m, n \geq 2$ it holds

(1) $ccw(P_2 \times P_n) = 2,$ if $n = 3, 4$

(2) $ccw(P_2 \times P_n) = 3,$ if $n \geq 5$

(3) $ccw(P_n \times P_n) \leq n - 1,$ if n is even

(4) $ccw(P_n \times P_i) \leq n,$ if $i = n, n + 1$

(5) $\min\{m+1, n+1\}/2 \leq ccw(P_m \times P_n) \leq \min\{m+1, n+1\}$

Let us attempt to form Schroder's Theorem by using the theorem, propositions and corollaries that make it. First, Schroder's Theorem broken into four statements looks like:

(1) if $m = n$, n is even ccw $(P_n \times P_m) \geq n - 1$

(2) if $m \geq n$, n is odd ccw $(P_m \times P_n) \geq n$

(3) if $m \leq n + 2$, n is even ccw $(P_m \times P_n) \geq n$

(4) if $m > n + 2$, n is even ccw $(P_{m>n+2} \times P_n) \geq n + 1$

Combining the various parts of the theorem, corollaries, and propositions of the lower and upper bounds, the following parts of the theorem are implied:

Schroder's Prop. 1 and Rolim's Thm. (3) $\Rightarrow$ part (1)

Schroder's Prop. 1 and Rolim's Thm. (4) $\Rightarrow$ part (2) conditionally

Schroder's Corr. 1 and Rolim's Thm. (5) $\Rightarrow$ part (3)

Schroder's Prop. 3 and Rolim's Thm. (5) $\Rightarrow$ part (4)

A problem is encountered when we look at Proposition 2 and part (2) of Schroder's Theorem. Proposition 2 states when m is two more than n and n is odd the cutwidth is n+1. For example for $P_5 \times P_3$, the cutwidth according to Proposition 2 is 4. By Schroder's Theorem it is 3. By our previous examples, a cutwidth of n + 1 or 4 is confirmed. In fact, below is Table 1 where the results of many meshes are listed. The starred items represent conflicts with the theorem and confirm Proposition 2.

Table 1. Samples of Mesh Cutwidths

ccw = n - 1	ccw = n	ccw = n + 1
$P_4 \times P_4$	$P_3 \times P_3$	$P_5 \times P_3$**
$P_6 \times P_6$	$P_5 \times P_4$	$P_6 \times P_3$**
	$P_4 \times P_3$	$P_7 \times P_3$**
	$P_6 \times P_4$	$P_7 \times P_4$
	$P_8 \times P_6$	

This indicates that the theorem as stated in the paper does not reflect the propositions, corollaries, and Theorem 6.1 which where used to make it. This requires a correction of the theorem.

A New Proposal

The best way to proceed at this point is to take a realistic look at what can be implied by the individual parts provided by the theorem, propositions and corollaries used to create Schroder's Theorem. Let us again look at the lower bounds (propositions and corollaries).

(1) for $n \geq 3$ $ccw(P_n \times P_n) \geq n - 1$ if n is even

(2) for $n \geq 3$ $ccw(P_n \times P_n) \geq n$ if n is odd

(3) for $n \geq 3$ $ccw(P_{n+1} \times P_n) \geq n$ for all n

(4) for $n \geq 3$ $ccw(P_{n+2} \times P_n) \geq n + 1$ if n is odd

(5) for $n \geq 4$ $ccw(P_{n+3} \times P_n) \geq n + 1$ for all n

Rolim's Theorem (upper bounds) in 6) implies:

(6) for $n \geq 3$ $ccw(P_n \times P_n) \leq n-1$ if n is even

(7) for $n \geq 3$ $ccw(P_i \times P_n) \leq n$ if n is even and $i = n$

(8) for $n \geq 3$ $ccw(P_i \times P_n) \leq n$ if n is even and $i = n+1$

(9) for $n \geq 3$ $ccw(P_i \times P_n) \leq n$ if n is odd and $i = n$

(10) for $n \geq 3$ $ccw(P_i \times P_n) \leq n$ if n is odd and $i = n+1$

(11) for $n \geq 3$ $ccw(P_m \times P_n) \leq n+1$ all values of $m \geq n$

If we combine these statements:

Statements (1) and (6) $\Rightarrow ccw(P_n \times P_n) = n-1$ if n is even

Statements (2) and (9) $\Rightarrow ccw(P_n \times P_n) = n$ if n is odd

Statements (3),(8) and (10) $\Rightarrow ccw(P_{n+1} \times P_n) = n$ for all n

Statements (4) and (11) $\Rightarrow ccw(P_{m>n+1} \times P_n) = n+1$ if n is odd

Statements (5) and (11) $\Rightarrow ccw(P_{m\geq n+3} \times P_n) = n+1$ for all n

With this new structure established, let us compare these statements with specific examples we have generated to see if we have full agreement. Table 2 illustrates a wide range of examples created for confirmation.

Table 2. A Table of Mesh Cutwidths

ccw = n - 1	ccw = n	ccw = n + 1
$P_4 \times P_4$	$P_3 \times P_3$	$P_5 \times P_3$
$P_6 \times P_6$	$P_5 \times P_4$	$P_6 \times P_3$
	$P_4 \times P_3$	$P_7 \times P_3$
	$P_6 \times P_4$***	$P_7 \times P_4$
	$P_8 \times P_6$***	

A quick review of these results indicates one problem with the above statements. The starred items in Table 2 are not addressed. These items represent the case:

$$ccw\ (P_{n+2} \times P_n) \qquad \text{if } n \text{ is even}$$

The question then is, "can the statement:

$$ccw\ (P_{n+2} \times P_n) \geq n \qquad \text{if } n \text{ is even}$$

be proved?"

In the following chapter, we review the methods of proof used for Schroder's Theorem. We then apply these methods to the above statement to get Proposition 1, which will then allow us to complete Schroder's Theorem to obtain Theorem 1.

CHAPTER THREE

A REVIEW OF METHODS

Introduction

So far we have shown flaws in Schroder's Theorem. We then attempted to reconstruct the theorem based on the theorem, propositions, and corollaries that were used to build it. At this point, we discovered that there was a case left out, therefore a new statement had to be constructed: $ccw(P_{n+2} \times P_n) \geq n$ if n is even. This statement will become our new proposition, Proposition 1. First, we will need to prove it. Before we prove Proposition 1, we will present the techniques used to prove the previous propositions and corollaries of Schroder's Theorem.

Proof Overview

The proof of Schroder's Theorem is made of several parts. For each corollary or proposition different techniques must be used. In this review of methods, we will review the techniques that apply generally to the theorem and specifically to the proof of Schroder's Proposition 1.

The format of the proof will be like this. First Schroder's Lemma is proven. This lemma applies to all cases of the proof. Next, the proof of Schroder's Proposition 1 will begin and the concept of embedding will be addressed. Finally, a claim will be made and proved. This claim will involve the two cases that arise when an even or odd number of vertices are used.

Schroder's Lemma

Schroder's Lemma is used as a tool in the following proof. It is developed to provide a means of separating vertices in a mesh into two separate and equal parts and colors on a cycle.

Schroder's Lemma: For k, $l \geq 1$, consider any $2(k + l)$ vertex cycle with vertices labeled consecutively by 0, 1, 2,…, $2(k + l) - 1$. Color arbitrary $2k$ vertices by black and the rest of the vertices by gray. Then there exist $k+ l$ consecutive vertices on the cycle containing exactly k black and l gray vertices.

What follows is a sketch of how the lemma works. Take any number of black and gray vertices, say $2k$ black and $2l$ gray. The total number of vertices is $2(k + l)$. Create a cycle of $2(k + l)$ vertices and color arbitrarily $2k$

vertices black and the rest gray. There is always a way of cutting the cycle into two parts such that we are guaranteed to have $k + l$ consecutive vertices in a part and where there is exactly k black and l gray vertices in it. Below in Figure 4 is an arbitrary arrangement of 14 vertices, 8 black and 6 gray:

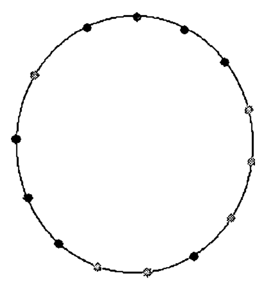


Figure 4. An Arbitrary Arrangement of 8 Black and 6 Gray Vertices.

In Figure 5 below, we attempt to cut the arrangement so

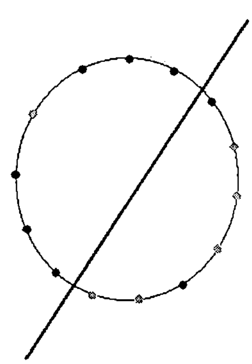


Figure 5. A First Cut.

that the vertices are divided equally, that is, 7 vertices in part I and 7 vertices in part II.

We see that in one part we have six black vertices and 1 gray and in the other part we have 2 black and 5 gray. By rotating the cut in a clockwise fashion by one vertex, nothing will change since each part will lose and gain a black vertex. The next rotation will exchange a gray for a black vertex. See Figure 6 below.

Figure 6. Successive Rotations of Cut.

After repeated rotations, the situation with the same number of black and gray vertices on one side can be achieved. This is shown below in Figure 7.

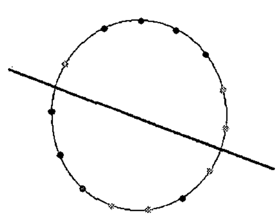


Figure 7. An Equal Number of Black and Gray.

This will always be the case, since by rotating the cut you can always separate an even number of vertices. Once this is done, you have separated both colors.

Proof of Schroder's Proposition 1

The techniques in the proof of Schroder's Proposition 1 highlight the essential strategies for all the proofs of the lower bounds that make up Schroder's Theorem. First, we will prove a technique called embedding. Then, a claim will be made and it will be proven. Its proof will involve the introduction of two routing algorithms. This will allow us to satisfy the requirements of Schroder's Proposition 1 when n is even. Next, a technique will be introduced to address when n is odd. It will make use of a dual graph to gain an extra cut.

Schroder's Proposition 1 : For any $n \geq 3$

$$ccw\ (P_n \times P_n) \geq \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

Proof:

<u>Embedding</u>

Take a $P_n \times P_n$ mesh and color only the outer vertices of the mesh. Color the bottom row and the last column black with the exception of the top vertex. Color the rest of the outer vertices gray. See Figure 8.

Take these colored vertices and embed them on a cycle. According to Schroder's Lemma, we can cut the

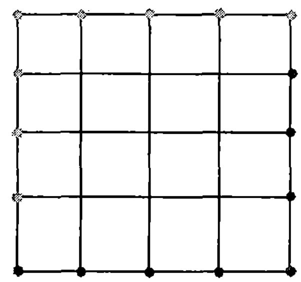


Figure 8. A 5 x 5 Mesh.

corresponding cycle such that each part contains n - 1 black and n - 1 gray vertices. This cut on the cycle induces a similar cut into two parts on the mesh, as shown below in Figure 9.

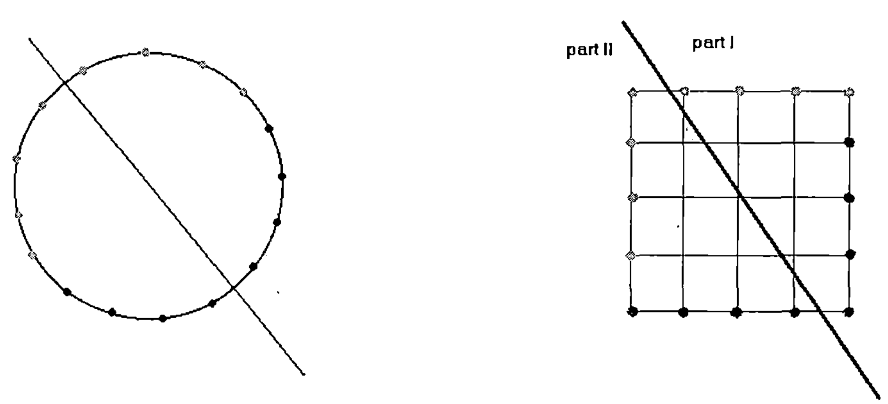


Figure 9. A Cut on Cycle that Induces a Similar Cut on the Mesh.

We now estimate the size of this cut. Ignoring any interior vertices, we can see this cut is at least as large as the number of edges it crosses, which is eight. These paths can be looked at as disjoint edge paths between each half of the cut mesh.

In the discussion below a few new terms will be used. Edge disjoint refers to the condition that the routing of edges between part I black vertices and part II black vertices must not share an edge. T_n is a subgraph of a P_n x P_n graph. It is the half of the graph that contains only one color of outer vertices using Schroder's Lemma.

Claim: There are n-1 edge disjoint paths between black vertices with one vertex in part I and the second in part II. Moreover, these paths can be routed in T_n only.

This claim refers to the graph in Figure 10 below.

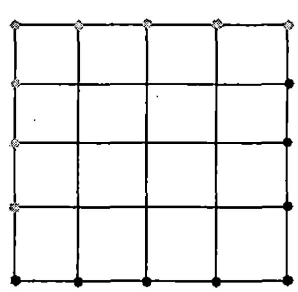


Figure 10. A 5 x 5 Mesh.

Proof of Claim: The proof uses induction on n. The assumption begins at $n \geq 5$ since the smaller cases are trivial. It is assumed that the claim holds for every subgraph T_{n-1}, T_{n-2}, and so on.

Consider a subgraph T_5 of P_5 x P_5 in Figure 11 below.

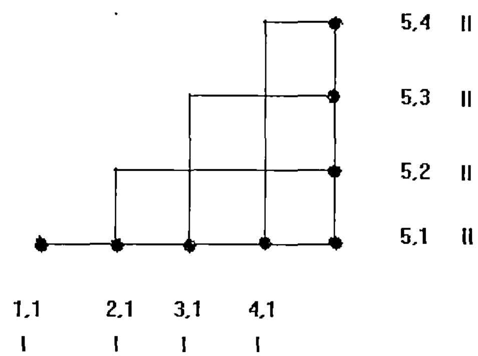


Figure 11. The Subgraph T_5.

Half of the black vertices are from part I of the cut, and the other half is from part II.

Schroder's Routing Algorithm 1: If two vertices are from different parts, that is one from Part I and the other from part II, and they are of the form (i,1), (n,i), where 1 < i < n, then we connect them by the following shortest path between them as in Figure 12. Now deleting the darkened edges creates a new T_{n-1} or T_4 subgraph as shown in Figure 13.

In figure 12 below, we see that vertex (2,1) is in part I and the corresponding vertex (5,2) is in part II. This allows Routing Algorithm 1 to be used.

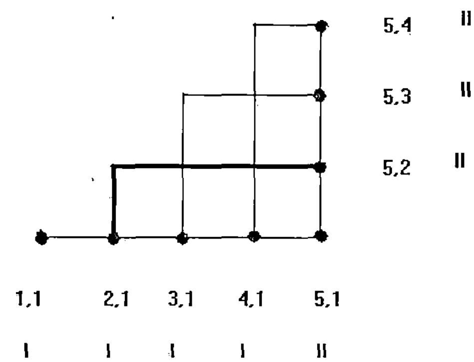


Figure 12. An Edge Using Algorithm 1.

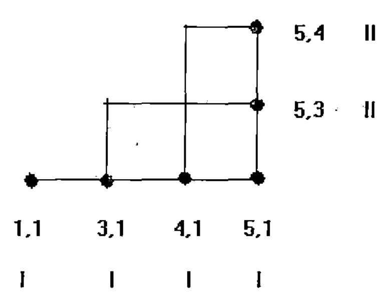


Figure 13. A T_4 Subgraph.

The claim is met by induction since we reduced the T_n graph to a T_{n-1} graph. This makes sense, intuitively, since we only need to do reductions n-2 times before we create a T_1 subgraph. A T_1 graph will give us only one edge and the claimed n-1 edge disjoint paths.

The T_n graph in Figure 11 could have been labeled differently such as below in Figure 14.

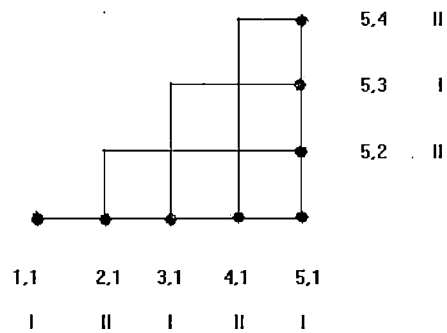


Figure 14. An Alternate Labeling
of Part I and II Vertices.

Here the outer vertices are mixed between the two parts the cycle was cut into. The algorithm above will not work on this graph because it requires us to find the corresponding vertex in the opposite half of the bisected cycle, but there is not one. For example, the vertex (3,1)'s corresponding vertex is (5,3), but in this case they are both in part I and cannot create an edge. A new algorithm is required.

Schroder's Routing Algorithm 2: Look at the corresponding vertices, for example, (2,1) and (5,2). When they are in the same half of the graph, in this case part II, look for a similar situation that occurs in part I, such as (3,1) and (5,3). This condition exists since we ruled out the corresponding vertices in algorithm 1. Next, route the edge disjoint path to the vertex of opposite part but on the same side of the mesh as below in Figure 15.

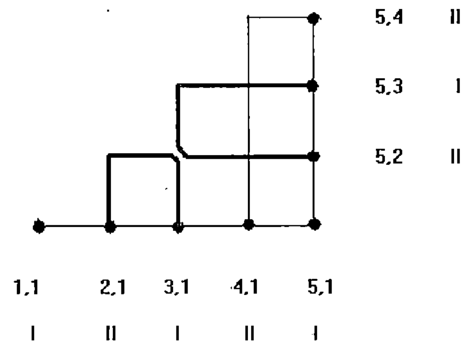


Figure 15. An Alternate Routing.

Now, delete these edges and reduce the graph to a T_{n-2} graph as below in Figure 16.

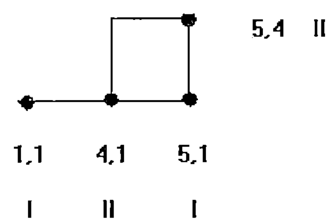


Figure 16. A T_{n-2}.

By induction, a T_{n-2} satisfies the claim. Hence, the claim is proven.

It must be noted that it depends on how the outer vertices are partitioned as to which algorithm must be used to achieve the induction process. A case in point is the graph below in Figure 17. The T_3 in this case yields 2 edge disjoint paths by the first algorithm.

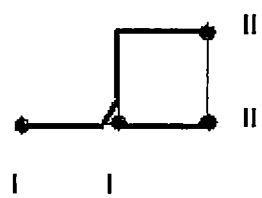


Figure 17.
Example 1 of T_3.

In Figure 18 below, we still have a T_3 graph but a different labeling. Using the second algorithm we get three disjoint paths.

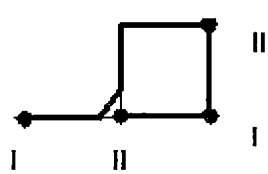


Figure 18. Example
2 of T_3.

The idea here is that we are only looking for the lower bounds and one or the other of the algorithms will yield this. Once we are able to reduce the graph, our goal is achieved.

It is also important to point out that we have only looked at edge disjoint paths of black vertices. We must do the same for the other half of the mesh, which is composed of only gray vertices. Since the structure of the mesh is symmetric, another exact copy of a T_n graph will be used with the same algorithms to get the same results; a cut of n-1. The overall graph then supplies n-1 + n-1 = 2n-2 disjoint paths which implies a cyclical cutwidth of at least (2n-2)/2 = n-1. Hence, the claim is proven.

If n is even, the above technique will meet the n-1 cutwidth requirement of Schroder's Proposition 1 stated below.

$$\text{ccw } (P_n \times P_n) \geq n-1 \quad \text{if n is even.}$$

If n is odd, as in $P_5 \times P_5$, an additional disjoint edge path must be found to satisfy the cutwidth requirement of n for odd n as stated in Schroder's Proposition 1 below.

$$\text{ccw } (P_n \times P_n) \geq n \quad \text{if n is odd.}$$

A new technique must be introduced.

Dual Graph Technique: Take the $P_5 \times P_5$ mesh and create its dual graph. A dual graph is created by placing a vertex in every region of the original graph, including the exterior region, and if any two regions share a face, an edge is placed there. The new vertices are given the coordinates of the original vertex in the lower left corner of the region where it was created. The exterior vertex is labeled v. The dual for $P_5 \times P_5$ is below in Figure 19.

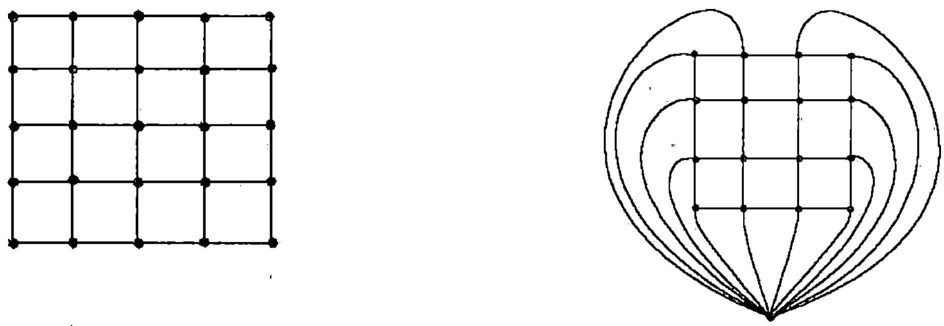


Figure 19. A Dual Graph Created from $P_5 \times P_5$.

A new mesh-like graph is created similar to a $P_4 \times P_4$.

Let us look at the properties between the original mesh, which we will call G, and the dual, which we will call G′. In Figure 20 below is Graph G with all edge paths created by the routing algorithms.

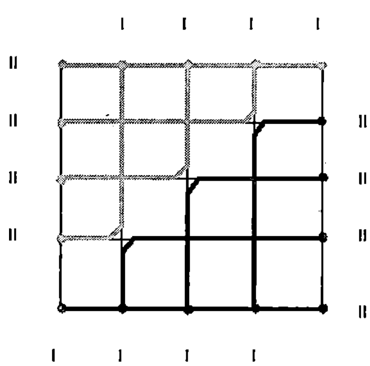


Figure 20. Graph G.

The existing algorithms gave us 2n-2 edge disjoint paths as a lower bound. Let us now create an edge cut.

The edge cut is created by a path, which cuts each disjoint edge path above. Such a path is below in Figure 21.

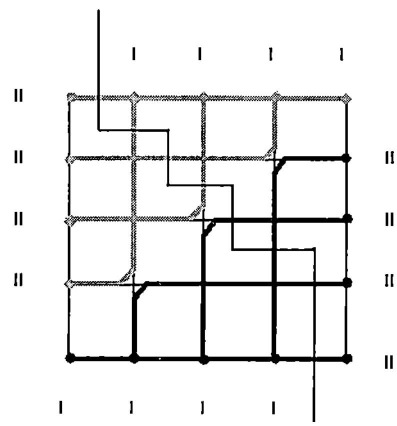


Figure 21. An Edge Cut Path.

The path cuts all 8 disjoint edge paths. This path creates a corresponding cycle C in the dual G′ in Figure 22.

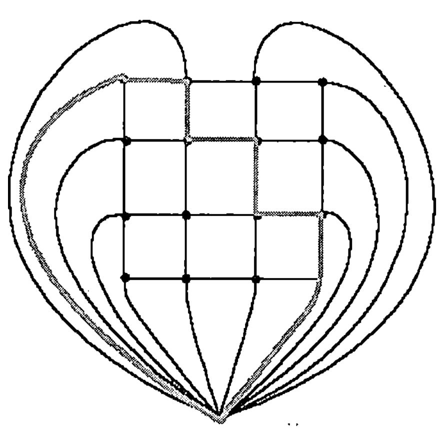


Figure 22. A Cycle in G′.

The cycle, which is 8 edges in length, is shown in gray in Figure 22.

Now let us create a new P_5 x P_5 mesh with black vertices placed across the top outer vertices (except the far left) and down vertically along the far right column. This corresponds to a new orientation of the original mesh. Creating its dual and cycle would create the cycle C′ in the dual below in Figure 23.

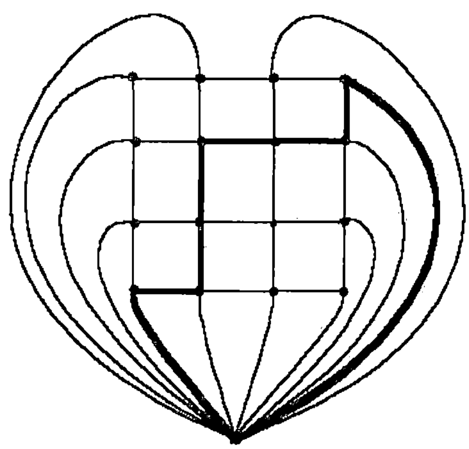


Figure 23. The Cycle C′.

Combining these two orientations in one graph we have Figure 24 below.

Now let us consider the paths created by C ∪ C′. We see that these cycles must always intersect in two places by nature of the fact that cycle C must begin at one corner vertex and finish at the opposite corner, while the cycle C′ must do the same with the other two corner vertices. Without loss of generality, we can make the requirement that these two graphs must intersect at a vertex (k, l), where $l \geq (n+1)/2$. This condition simply

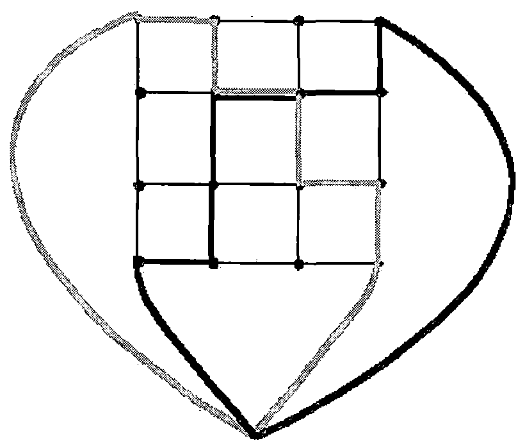


Figure 24. Cycle C and C′ Together.

requires the newly created path, $C \cup C'$, to go at least halfway up the mesh. Without this, a lower bound would not be achieved.

The shortest path formed by $C \cup C'$ and contains in general v, (1, 1), (k, l), (n-1, 1), and looks like the example in Figure 25 below.

The number of edges used moving vertically is 2[(n+1)/2 -1]. The number of edges used horizontally is n-2. An expression for the total paths are 2[(n+1)/2] + (n-2) + 2 or 2n-1. Therefore, the edge length of this path is then 2n-1 or with n = 5 this becomes 9. Notice in Figure 26 below, that this edge cut also separates vertices

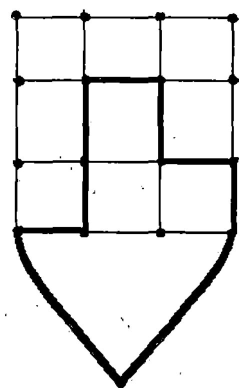


Figure 25. The Shortest Path in $C \cup C'$.

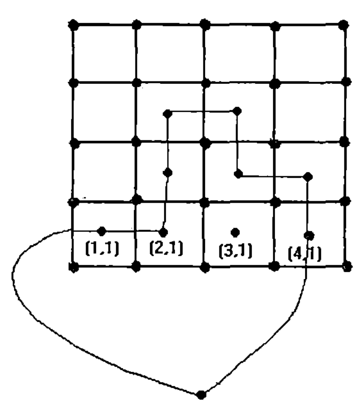


Figure 26. The Separation of Vertices (i,1) Through (n-1,1).

(i, 1) where i = 1,2,3, …, n-1, from the rest of the boundary vertices in P_n x P_n. Hence, this edge cut corresponds to a cut on a cycle where in part I is the vertices (i,1), where i = 1,2,3, …, n-1, and in part II is the rest of the vertices. This edge cut then corresponds to a cyclical cutwidth of $P_n \times P_n \geq [(2n-1)/2]$ and hence $ccw(P_n \times P_n) = n$, if n is odd. This then satisfies the cutwidth of n required by Schroder's Proposition 1 for odd n. End of proof.

In conclusion, we have introduced the proof of Proposition 1 of Schroder's Theorem. In the next chapter, we will use these techniques to prove our new proposition, which will then complete Schroder's Theorem and become our Theorem 1.

CHAPTER FOUR

PROOF OF NEW PROPOSITION

Overview

In chapter 2, we learned that Schroder's Theorem was incomplete and that a new statement was necessary to finish it. In chapter 3, we reviewed the techniques of the proof that are necessary for creating a proof of our new statement, for $n \geq 3$ $ccw(P_{n+2} \times P_n) \geq n$ if n is even. In this chapter, we prove the above statement and so it becomes Proposition 1.

As aid to understanding the proof, we will create an example to work with. This example must meet the requirements of a $P_{n+2} \times P_n$ mesh. We will use a $P_6 \times P_4$. Next, we will color the outer vertices black and gray, as required by 5). Then we partition these vertices to part I and part II. We will use a labeling that we know will give us the ideal lower bound. Because our mesh is asymmetric, we create T_{n+1} subgraphs, instead of T_n subgraphs. Next, we apply our routing algorithms on it and reduce it to a T_n. At this point we achieve a edge cut of n by induction. Finally, we double it for both subgraphs, the black and gray, then divide by two to get our cut of n.

The Proof of Proposition 1

Proposition 1: for $n \geq 3$ $ccw(P_{n+2} \times P_n) \geq n$ if n is even

Proof:

First, we label the outer vertices of the mesh. Because our mesh is rectangular, we must adjust the coloring and labeling of the outer vertices to fit our graph. Label the vertices black if (i,1), where i = 2,3,4,…,n+2 and (n+2,j), where j = 2,3,4,…,n. Label the rest of the outer vertices gray. See Figure 27 below.

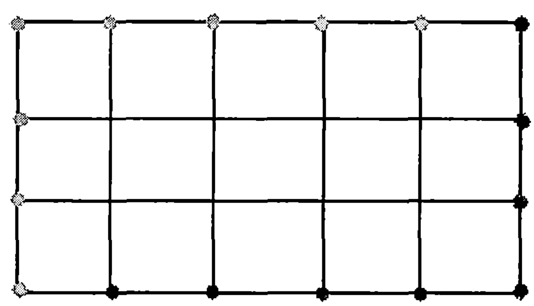


Figure 27. A $P_6 \times P_4$ with Outer Vertices Colored.

We have n black vertices across the bottom and n black vertices along the vertical column to the far right.

Next, we partition the outer vertices. The partition in Figure 28 below usually produces the lower bound case.

The reason for this is due to the fact that it allows us to use Routing Algorithm 1.

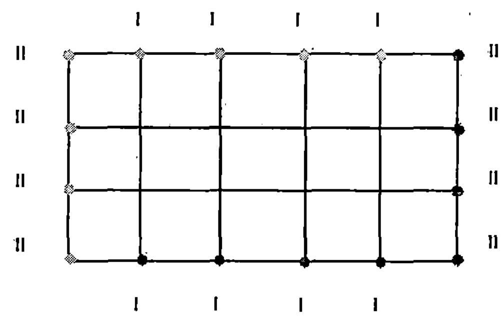


Figure 28. A Lower Bound Partition.

In this case, instead of creating two T_n subgraphs, we create two T_{n+1} subgraphs. One T_{n+1} subgraph is shown below in Figure 29.

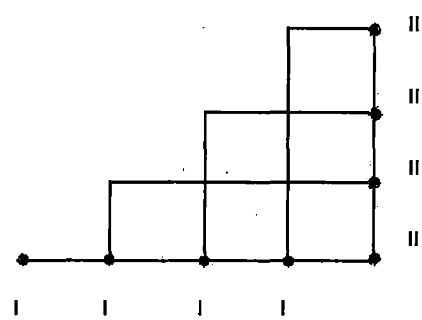


Figure 29. A T_{n+1} Subgraph.

Beginning with T_{n+1}, and using routing algorithm 1, we get a series of subgraphs formed by creating an edge and then eliminating it and reducing the graph. The first reduction is illustrated in Figure 30 below. Once this reduction is is performed we have a T_n subgraph, and an edge cut of n by induction.

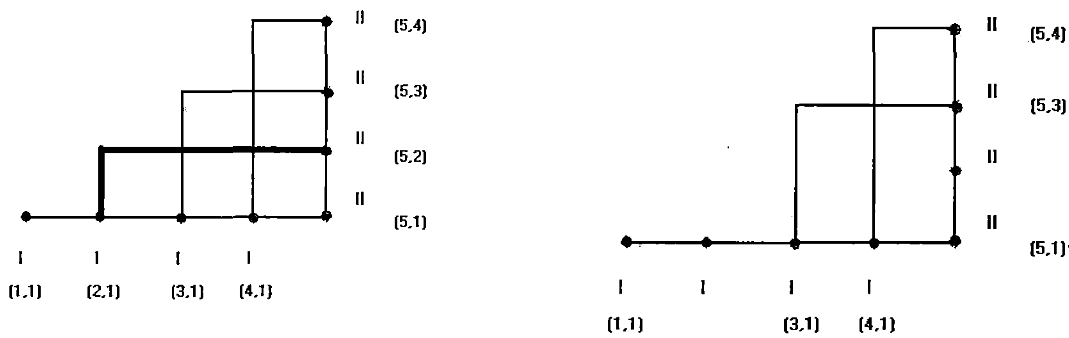


Figure 30. The First Edge Reduction.

Since there are actually two T_{n+1} subgraphs that are reduced to T_n, we then have a cutwidth of $2(4)/2 = 4$ or, in general, $2n/2 = n$.

End of proof.

CHAPTER FIVE

CYCLIC CUTWIDTH OF MESH CUBES

Introduction

In the preceding chapters we introduced a theorem on cyclic cutwidths of meshes and illustrated techniques used in the proof of such a theorem. We also corrected problems with the stated theorem and introduced Theorem 1 and Proposition 1 in support of that process. We now come to the main purpose of the paper.

The purpose of the paper was to use the preceding work to gain insight into the cyclic cutwidth of the n-cube. The next step in this process is to look at structures that are similar to the n-cube, yet not as complex. A structure that could provide the necessary insight might be what we will call mesh cubes. We define a mesh cube as the product of a mesh (P_m x P_n) and P_l. These structures are three-dimensional and can be square or rectangular. Since n-cubes can be represented as square in shape, square mesh cubes may offer the needed insight. Thus the final topic of this paper will be exploring the cyclic cutwidth of square mesh cubes, graphs formed by the product of a mesh of form P_n x P_n and P_n or P_n x P_n x P_n.

Overview

In this chapter we will present a new theorem on the upper bounds of square mesh cubes. Next, we will prove the theorem using examples for verification.

Terminology

To aid in understanding, a few new terms must be created. Below are terms that will apply in this chapter.

Mesh cube - The product graph $P_l \times P_m \times P_n$.

Square mesh cube - The product graph $P_n \times P_n \times P_n$.

Rectangular mesh cube - The product graph $P_l \times P_m \times P_n$ where l, m, and n cannot all be equal.

Vertical connecting edge - A vertical edge that connects a mesh to a copy of that mesh to form a mesh cube.

Augmented graph - A mesh with the additional edges that connect it to another mesh.

Augmenting edges - edges that form an augmented graph.

Augmenting vertex - a vertex incident to an augmenting edge.

The New Theorem

Based on explorations with mesh cubes, the following theorem is on the upper bounds of mesh cubes.

Theorem 2: If $n>2$

$$ccw(P_n \times P_n \times P_n) \leq n^2 - n + 1 \qquad \text{if } n \text{ is even}$$

$$ccw(P_n \times P_n \times P_n) \leq n^2 + 1 \qquad \text{if } n \text{ is odd}$$

Proof:

Case 1 (even n)

We will use $P_4 \times P_4 \times P_4$ as in Figure 31 below to illustrate the proof.

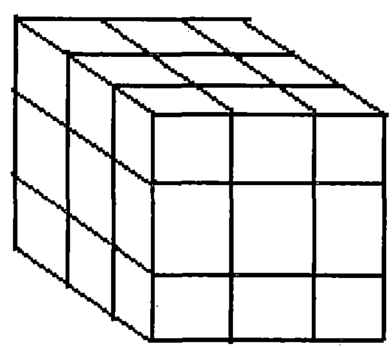


Figure 31. $P_4 \times P_4 \times P_4$.

Laying the cube flat on one face we can see that $P_4 \times P_4 \times P_4$ can be thought of as four copies of $P_4 \times P_4$ with corresponding vertices connected as below in Figure 32. In this proof, we will orient this graph vertically so that the edges which connect each $P_4 \times P_4$ are vertical and will be referred to as vertical connecting edges.

The four copies will then be looked at separately and a relationship between a copy and linear cutwidth will be developed.

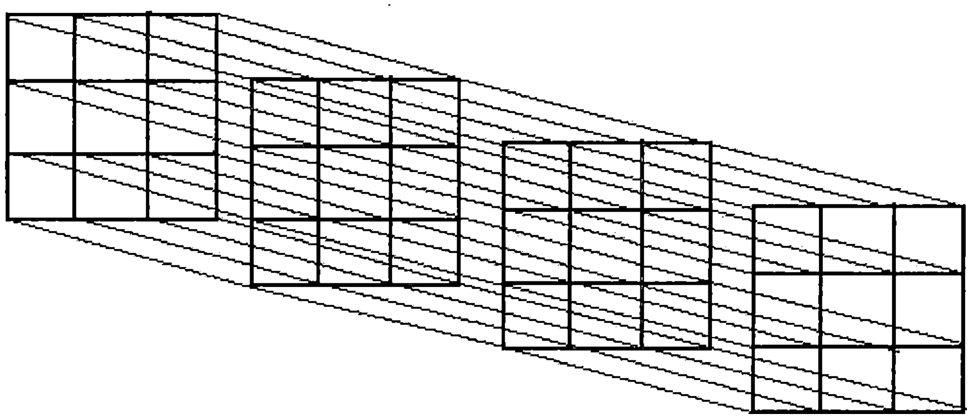


Figure 32. Four Copies of $P_4 \times P_4$. This graph is oriented horizontally for purposes of illustration.

Looking at one copy of $P_4 \times P_4$, we can visualize how it may be embedded on a cyclic chassis as in Figure 33 below.

Figure 33. An Example of $P_4 \times P_4$ Embedded on a Cycle.

Since $P_4 \times P_4$ is symmetric about a point in it's center, we can make a cut from the center perpendicular to the outside edge and consider only half of P_4 x P_4 and it's augmenting edges as below in Figure 34.

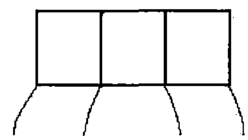


Figure 34. P_4 x P_4
Consists of Two
P_4 x P_2's
With Augmenting Edges.

This half is P_4 x P_2 with augmenting edges or P_n x $P_{n/2}$ with n augmenting edges in general. We will use the notation AUG( )when speaking of augmented graphs.

It is easy to see, the upper bound of ccw(P_4 x P_4)must be at least lcw(P_4 x P_2)as in Figure 35 below. In general, this is lcw(P_n x $P_{n/2}$).

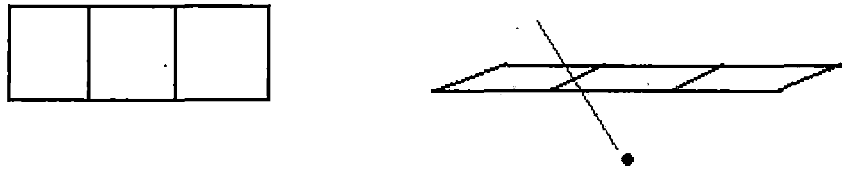


Figure 35. A Mesh Represented in a Linear Embedding.

Lemma 1: For even n, lcw (P_n x $P_{n/2}$) $\leq$ (n/2)+1

Proof of Lemma 1:

Looking at P_4 x P_2 we can see that, as a minimum, the linear cutwidth will be the number of horizontal paths shown in black below in Figure 36.

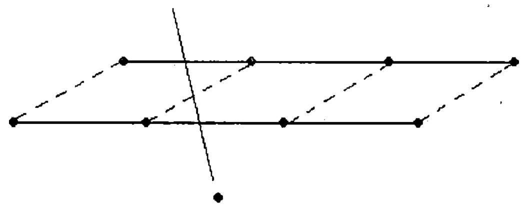


Figure 36. Horizontal Paths.

Since the other edges are vertical, we must consider them. In the best case, we will consider that a vertical path does not overlap or double up on itself as shown below in Figures 37 and 38.

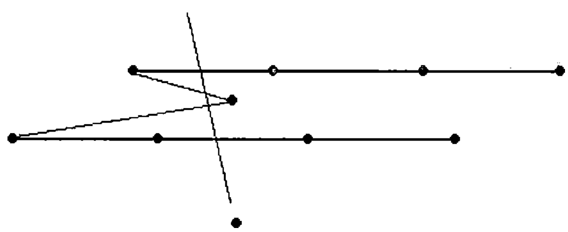


Figure 37. Double Up.

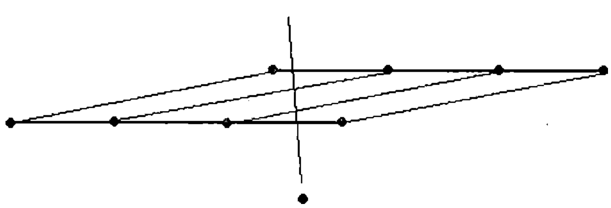


Figure 38. Overlap.

As a result, the vertical edges contribute at least one edge, but one is certainly achievable. Generalizing, an expression for the number of horizontal edges will be n/2, and hence, an expression for the upper bounds would be (n/2) + 1. End of proof of Lemma 1.

Next, we consider the augmenting edges between one $P_n \times P_{n/2}$ and the other as shown below in Fugre 39 as thin curved lines.

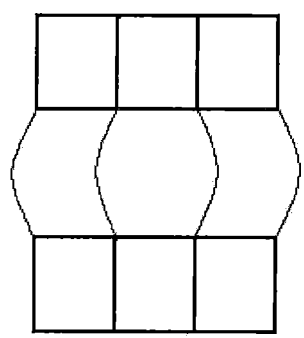


Figure 39. Augmenting Edges.

An augmented $P_4 \times P_2$ or AUG($P_4 \times P_2$)is shown in Figure 40.

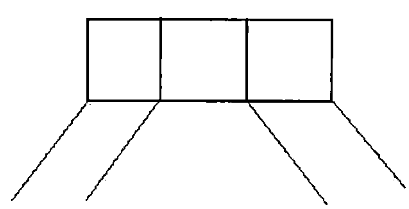


Figure 40. An Augmented $P_4 \times P_2$.

From this point on, we will use AUG when we speak of an augmented mesh as shown in Figure 40.

Lemma 2: For even n, $lcw[AUG(P_n \times P_{n/2})] \leq lcw(P_n \times P_{n/2}) + n/2 - 2 = n - 1$.

Proof of Lemma 2:

We will use $P_8 \times P_4$ to illustrate the proof. An augmented $P_8 \times P_4$ is shown below in Figure 41.

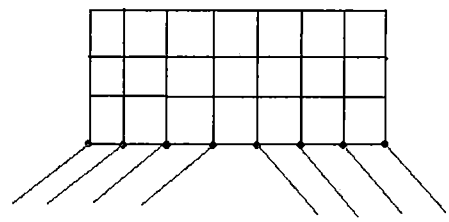


Figure 41. An Augmented $P_8 \times P_4$.

By Lemma 1, $lcw(P_8 \times P_4) \leq 5$. Next, we consider the augmenting edges and what they contribute. Let us consider each half of the graph and its augmenting edges as shown below in Figure 42.

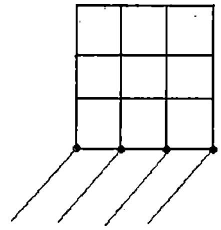


Figure 42. Half of an Augmented $P_8 \times P_4$.

Embedding each half in a linear chassis and exploring cutwidth contributions, we can determine the maximum cutwidth values and see how the augmenting edges influence the value of the cutwidth.

First, $P_8 \times P_4$ has a linear cutwidth of 5. For the embedding in Figure 43, we see that we reach a maximum of 7 then the cutwidth drops again. This layout was achieved by letting the vertical edges simply fall right to the Horizontal, we will call this the "Falling Fence" method.

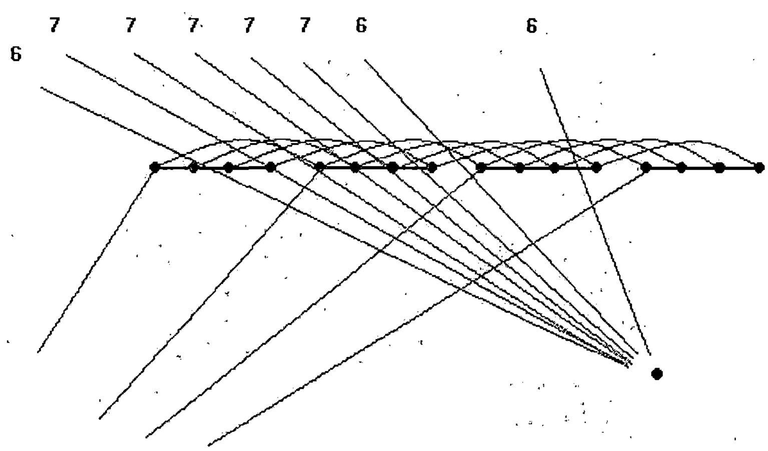


Figure 43. A Series of Cutwidths for a Left Linear Embedding.

Exploring the other half of the graph, using the "Falling Fence" method, we see that the cutwidth values are different than in Figure 43. We see in Figure 44 that three of the four augmenting edges add to the cut of $P_8 \times P_4$ in this half and may establish our upper bound.

Yet, upon exploring our layouts, we see that the right half could have been laid out the same as the left half.

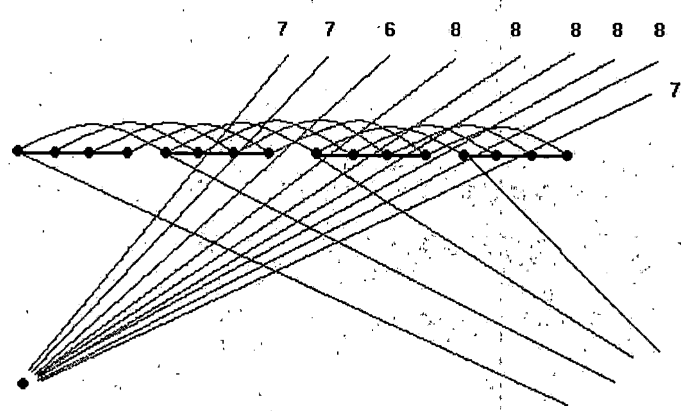


Figure 44. A Series of Cutwidths for a Right Linear Embedding.

Figure 45 below shows the two layouts.

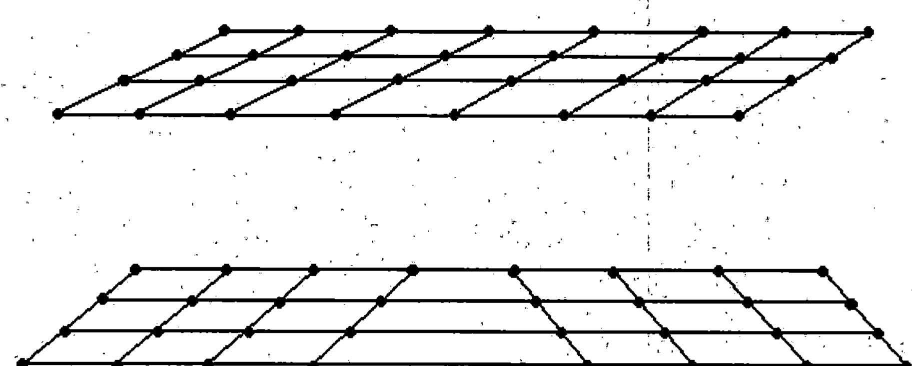


Figure 45. The Standard "Falling Fence" Method Above and the "Falling Inwards" Method Below.

The cutwidth of the non-augmented graph in either method is the same because the right half becomes the left half with augmenting edges in a similar orientation.

In general, we can gain an upper bound by assuming all augmenting edges except the outside two will contribute to the cut. Since in each half, there are n/2 augmenting edges and the outside 2 do not contribute, we gain (n/2) - 2 edges. Thus, $lcw\ [AUG(P_n \times P_{n/2})] \leq lcw(P_n \times P_{n/2}) + n/2 - 2 = n/2 + 1 + n/2 - 2 = n - 1$. End of proof of Lemma 2.

Next, let us consider the effect that multiple copies of $P_n \times P_{n/2}$ have on the linear cutwidth. We consider n copies of $P_n \times P_{n/2}$ and the vertical connecting edges.

Lemma 3: For even n, $lcw[AUG(P_n \times P_{n/2} \times P_n)] \leq n\{lcw[AUG(P_n \times P_{n/2})]\}+1 = n[lcw(P_n \times P_{n/2})+(n/2)-2] + 1 = n^2 - n + 1$.

Proof of Lemma 3:

We will use 4 copies of $P_4 \times P_2$ to illustrate the proof. Below in Figure 46 is a linear layout of $P_4 \times P_2$.

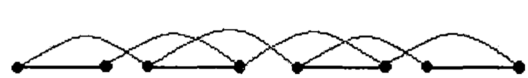


Figure 46. A Linear Layout of $P_4 \times P_2$.

Taking the layout of one copy, then creating n copies and connecting their vertical edges, we have Figure 47 below.

Without considering the effect of the gray vertical connecting edges, we see, in Figure 47, that the upper

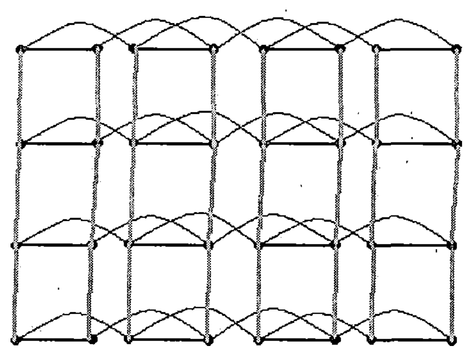


Figure 47. Four Copies of $P_4 \times P_2$ with Connecting Vertical Edges. The vertical edges are thick gray.

bound is achieved by multiplying lcw ($P_4 \times P_2$) by 4, yielding 4•3 = 12. Hence, for n copies of $P_n \times P_{n/2}$, the expression would be $n[lcw(P_n \times P_{n/2})+(n/2)-2]$.

In addition, we must consider the vertical connecting edges in Figure 47 above, shown in gray. To explore this, we must create a linear layout of Figure 46. We create the layout by representing the gray vertical connecting edges in Figure 47 as horizontal lines between the n-1 copies of the original as shown below in Figure 48.

The linear cutwidth of the layout in Figure 48 is 12 as shown below in Figure 49. From Figure 49 the linear cutwidth is achieved without crossing a vertical connecting edge. Hence the vertical connecting edges do

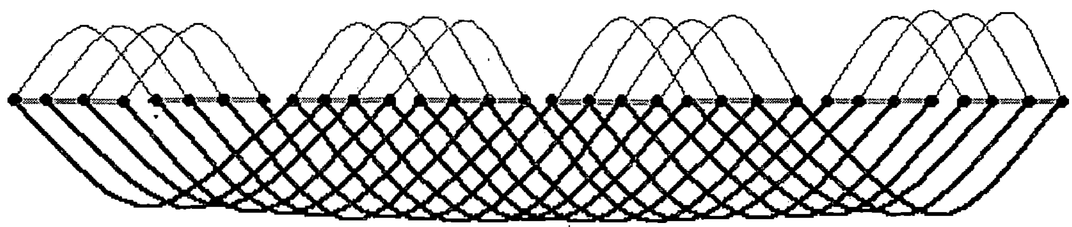


Figure 48. A Linear Layout of N Copies. The vertical connecting edges are the horizontal gray edges.

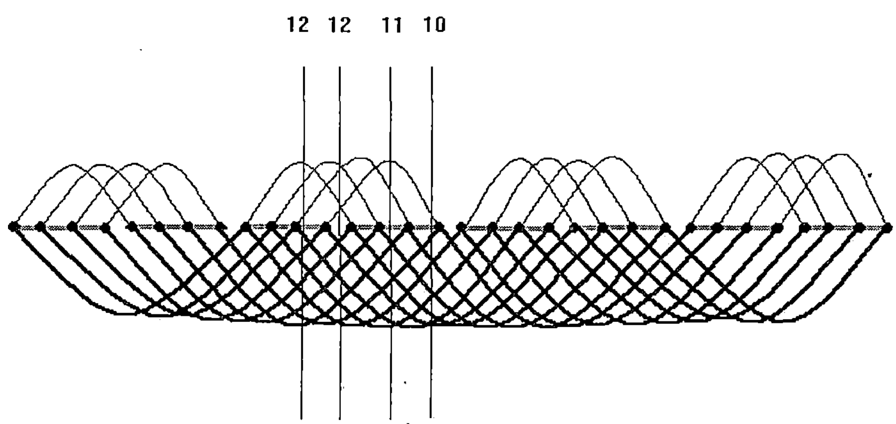


Figure 49. The Linear Cutwidth is 12.

not contribute to the cut. Thus, the linear cutwidth of a vertically connected P_4 x P_2 or (P_4 x P_2) x P_4 is less than or equal to 4[lcw (P_4 x P_2)] = 4(3) = 12.

Finally, we must look at the contributions of augmenting (P_4 x P_2)x P_4. Below in Figure 50 is the left half of Figure 48 with augmenting edges.

We can approach Figure 50, by comparing it to Figure 49. In Figure 50 we see that an additional edge was found that increased the cutwidth to 13. This edge is due to

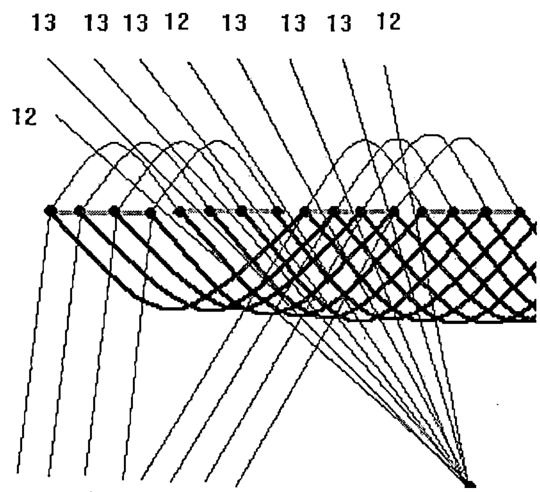


Figure 50. Cutwidths of an Augmented (P_4 x P_2) x P_4.

the fact that in Figure 50 a vertical edge must be cut across to achieve the maximum cutwidth. Recall in Lemma 2, we determined that the two outside augmenting edges do not need to be considered. Thus in Figure 50, an upper bound of 13 can be achieved by adding one edge cut for the connecting vertical edge not counting the two outside augmenting edges and their 3 copies.

In conclusion, the expression n-1 for the initial one copy, as in Figure 46, is then multiplied by n for the n copies of that graph, as in Figure 49, to get n^2 -n, and then adding one for the vertical connecting edge to get $n^2 - n + 1$. The resulting expression is then developed: $lcw[AUG(P_n \times P_{n/2} \times P_n)] \leq n[lcw\ (P_n \times P_{n/2}) + (n/2) - 2] + 1 = n^2 - n + 1$. End of proof of Lemma 3.

Earlier, we showed that a mesh could be embedded in a cycle as in Figure 51 below.

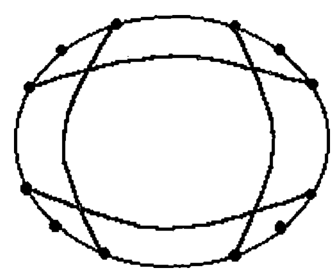


Figure 51. A Mesh Represented as a Cycle.

Next we considered only half of the mesh and found it's linear cutwidth. As a last step, we must show how this translates to cyclic cutwidth.

Any linear embedding can be represented as a cycle by bending the linear chassis to form a cycle or an arc of a cycle. Since we only need to be concerned with half of the mesh, our linear embedding dealt with only half the mesh and half the cycle as shown in Figure 52 and 53.

Since the linear cutwidth will then represent cyclic cutwidth, our formula becomes $ccw(P_n \times P_n \times P_n) = n[lcw\ (P_n \times P_{n/2}) + (n/2) - 2] + 1$.

In conclusion, we began with a single copy of $P_n \times P_{n/2}$. We found its linear cutwidth as

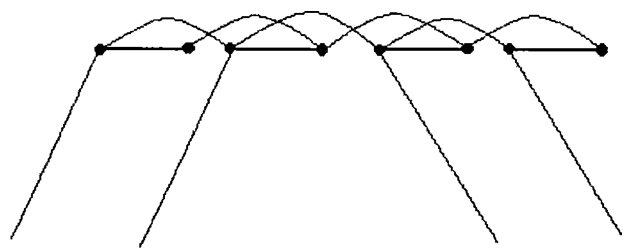


Figure 52. A Linear Embedding.

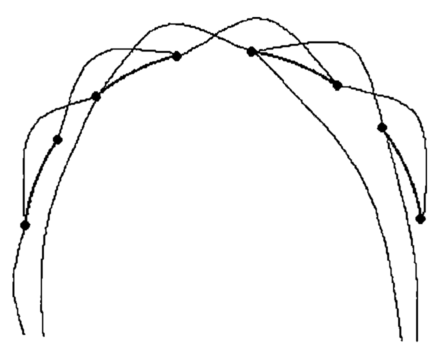


Figure 53. A Cyclic Representation of a Linear Embedding.

$lcw(P_n \times P_{n/2}) = n/2 + 1$. We considered the connecting edges so we have $n/2 + 1 + n/2 - 2 = n - 1$. Since, we have n copies, we have $n(n - 1) = n^2 - n$. Next, the n copies contribute an additional vertical edge and we get $n^2 - n + 1$. And finally, since $ccw(P_n \times P_n \times P_n) \le n[lcw\ (P_n \times P_{n/2}) +$

$(n/2)-2] + 1$, we get $ccw(P_n \times P_n \times P_n) \leq n^2 - n + 1$, if n is even.

End of proof of case 1.

<u>Case 2: n is odd</u>

We will use a $P_5 \times P_5 \times P_5$ as in Figure 54 below to illustrate the proof.

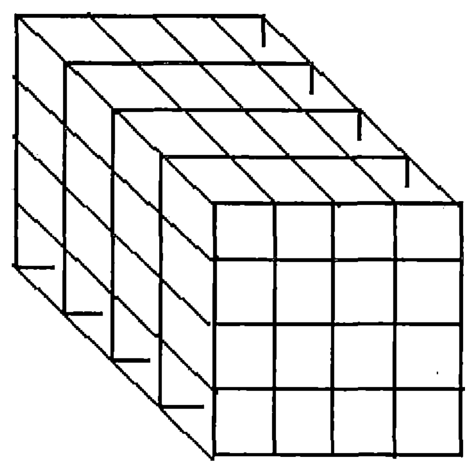


Figure 54. $P_5 \times P_5 \times P_5$.

Much of the proof will be the same as the n is even case yet there are significant differences. Let us first look at how 1 of the five copies of the $P_5 \times P_5$ mesh will be embedded on a circular chassis as in Figure 55.

We see that we lose the symmetry of the even mesh. Because of this, we must cut the mesh in a way such that one part is larger than the other is. Since we are interested in an upper bound, we must consider the larger half, which is the left portion of the mesh shown below in Figure 56.

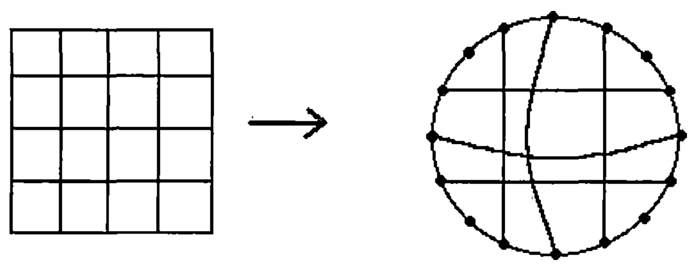


Figure 55. Embedding a P_5 x P_5 in a Cycle.

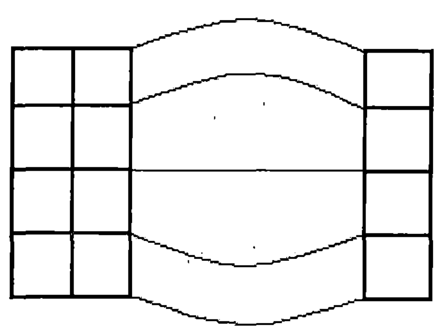


Figure 56. A Separated P_5 x P_5.

We see that the larger half is P_5 x $P_{(5+1)/2} = P_5$ x P_3 and in general this will be P_n x $P_{(n+1)/2}$. As a result, the upper bound of the cyclic cutwidth of P_5 x P_5 will be found by looking at the 5 copies of P_5 x P_3 and its augmenting edges.

Lemma 4: For odd n, lcw $(P_n \times P_{(n+1)/2}) \leq (n+1)/2 + 1$

The proof of this lemma is similar to Lemma 1.

Lemma 5: For odd n, lcw $[AUG(P_n \times P_{(n+1)/2})] \leq$ lcw $(P_n \times P_{(n+1)/2}) + (n+1)/2 - 2$.

Proof of Lemma 5: We will use $P_5 \times P_5$ to illustrate the lemma. In the proof of the related lemma in the even case, we saw that either half of the mesh provided the upper bound. In the odd case, we see that, similar to the splitting of the $P_5 \times P_5$ mesh in Figure 56 above, we must split the $P_5 \times P_3$ or $P_n \times P_{(n+1)/2}$ in general, and consider the routing of the augmenting edges. By the location of the augmenting vertices, colored gray in Figure 57 below,

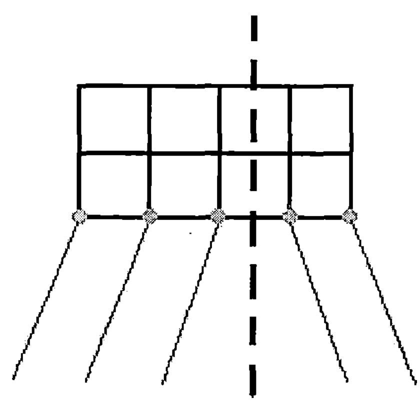


Figure 57. Splitting a $P_5 \times P_3$.

we see that we must route the additional augmenting edge on the left side in the linear layout as in Figure 58.

In fact, this splitting could now be looked at as increasing the graph to a half a P_6 x P_3.

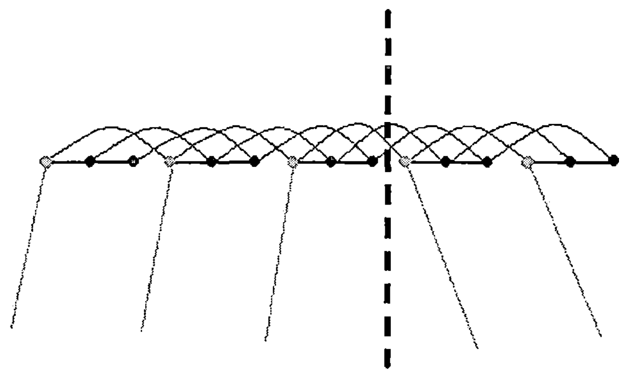


Figure 58. Splitting a Linear Layout of P_5 x P_3.

The left side routing will then determine the upper bound as in Figure 59.

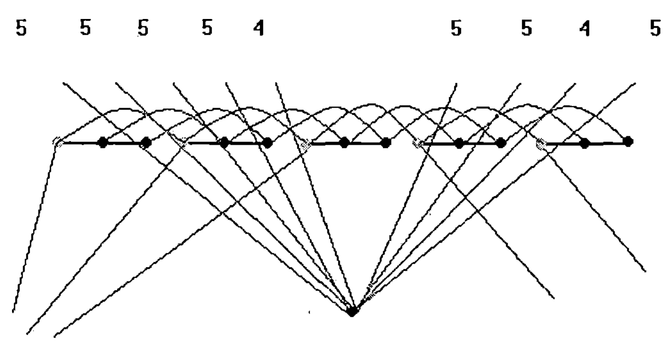


Figure 59. Cutwidths From a Left-side Routing.

Without the augmenting edges, the cutwidth of the linear layout of P_5 x P_3 is 4. As a result of the left-side routing, only 1 of the three edges, or $(5+1)/2 - 2 = 1$, contributes an edge to the cut. In general, this will be $(n+1)/2 - 2$. End of proof.

Lemma 3, which deals with the contributions of the vertical edges holds for odd n. Hence, n copies of a $P_n \times P_{(n+1)/2}$, will contribute a cutwidth of $n[lcw(P_n \times P_{(n+1)/2}) + (n+1)/2 - 2] + 1$.

The rest of the odd n proof is analogus to the even case. Simplifying $n[lcw(P_n \times P_{(n+1)/2}) + (n+1)/2 - 2] + 1$, we substitute $lcw\ (P_n \times P_{(n+1)/2}) = (n+1)/2 + 1$, so we get $n[(n+1)/2 + 1 + (n+1)/2 -2] + 1$. Combing like terms inside the brackets, we get $n[n+1 -1] + 1 = n^2 + 1$. Finally, the $ccw(P_n \times P_n \times P_n) \leq n^2 + 1$, if n is odd.

End of proof. □

CHAPTER SIX

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

In chapter 5 we proved that the cyclic cutwidth of square mesh cubes, mesh cubes of the form $P_n \times P_n \times P_n$, has an upper bound of $n^2 - n + 1$, if n is even, and $n^2 + 1$, if n is odd. In this chapter we turn our attention to the purpose of that endeavor, to gain insight and direction into the cyclic cutwidth of the n-cube. In the following discussion we list possible insights gained from this project that may help in the solution of the n-cube.

1. In determining an upper bound we learned that vertical edges contribute one additional edge to the cutwidth. This may also be true of the n-cube.
2. The symmetry of the mesh cube allowed us to reduce the complexity of the problem and consider only half of the original cube. N-cubes are also symmetric and may be approached in a similar manner.
3. Mesh cubes involved multiple copies of less complex structures. We were able to find the essential structure, the mesh, and then use it and its multiple copies to find the upper bounds of a more complex

structure. The n-cube is a structure built from multiple copies of other structures; hence, a similar technique may be useful.

4. Embedding the structure on a linear chassis provided the tool for determining the upper bounds. Linear embedding may be a useful tool in looking at the n-cube upper bounds.
5. The center of a cube is the center of the cycle.
6. A square orientation in which the cube stands on a face which is the base of the structure and the cyclic distribution of the mesh cubes vertices within that plane seems to be the optimum orientation for determining mesh cube cyclic cutwidths.
7. The cutwidths of square mesh cubes can be found by taking the cutwidth the outside mesh (for example n-1, if n even) and multiplying it by the width of the mesh (n) and then adding one for the vertical edges.

Recommendations

As a result of our work, we see several directions in which future research could go. Below is a list of possible research areas in the future.

1. Determine the upper bounds on asymmetrical mesh

cubes.

2. Determine lower bounds of mesh cubes.

3. Utilize the edge-counting techniques introduced in Chapter 3 on structures that have diagonal edges such as Q_3.

4. Develop an edge counting technique for mesh cubes such as that used for meshes.

5. Determine if the cyclic cutwidth of all mesh cubes can be found by taking the product of the outside mesh and multiplying by the width and then adding one for the vertical edges.

APPENDIX

TECHNICAL TERMINOLOGY

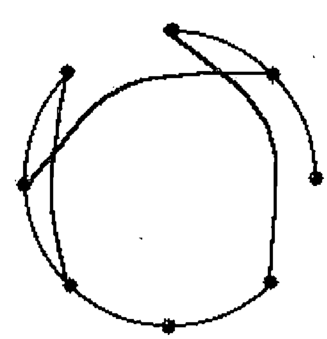
APPENDIX

TECHNICAL TERMINOLOGY

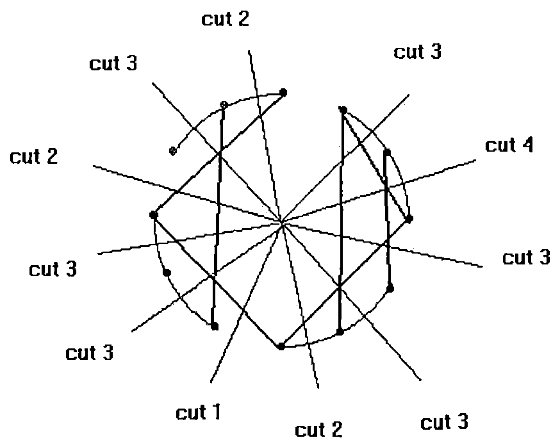
Cut - The maximum number of edges between adjacent vertices that are cut across on a specific graph.

Cutwidth - The cutwidth of a graph is the minimum cut achieved through all possible orientations of the graph. The cutwidth from one type of graph can be compared by translating its vertices to another type of graph and finding its cutwidth, usually the translation is to a linear graph or a cyclic (circular) graph.

Cycle Graph - A graph where the vertices are arranged in a circular fashion.

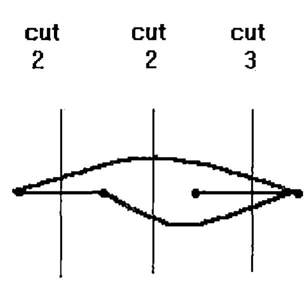


Cyclic Cutwidth - The minimum of the cut values of various arrangements of a set of vertices and its corresponding edges. Within one arrangement, the ccw is the maximum cut between any set of vertices, as shown below.



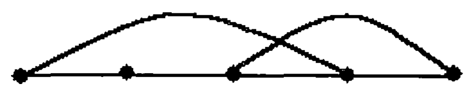
cyclic cutwidth is 4

Linear Cutwidth - The minimum of the cutwidth values of various linear arrangements of a set of vertices and its connecting edges. Within one arrangement, the lcw is the maximum cut between any set of vertices.

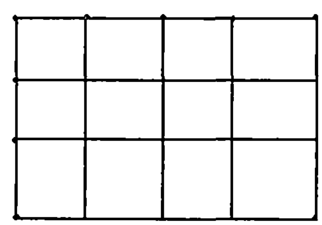


linear cutwidth is 3

Linear Graph - A graph where the vertices are arranged in a linear fashion.

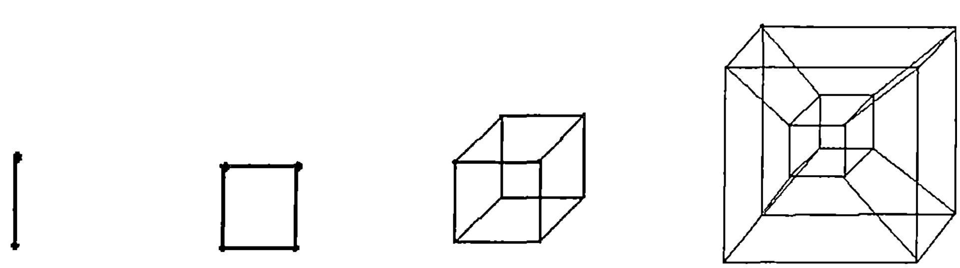


Mesh - A P_m x P_n mesh is a graph that is constructed with m columns and n rows of vertices.

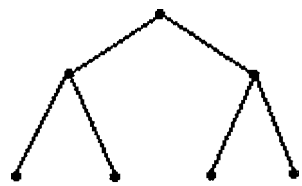


N-cube - An n-cube indicated by Q_n ,is built by beginning with a Q_1 (one cube) and continually duplicating the previous graph and then connecting the corresponding vertices.

Q_1 Q_2 Q_3 Q_4



Tree - A graph that consists of branches and paths with no complete cycles.



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