2002

# 1p spaces

Anh Tuyet Tran

# $l^p$ SPACES

A Project

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Anh Tuyet Tran

March 2002

# $l^p$ SPACES

---

A Project

Presented to the

Faculty of

California State University,

San Bernardino

---

by

Anh Tuyet Tran

March 2002

Approved by:

[redacted] 02/21/02

Hajrudin Fejzic, Committee Chair — Date

[redacted]

Hasan Zahid, Committee Member

[redacted]

Prakash Chetan, Committee Member

[redacted]

Peter Williams, Chair
Department of Mathematics

[redacted]

Terry Hallett,
Graduate Coordinator
Department of Mathematics

ABSTRACT

In this paper we will study the $l^p$ spaces. We will begin with definitions and different examples of $l^p$ spaces. In particular, we will prove Holder's and Minkowski's inequalities for $l^p$sequence. We will illustrate the relationships between different spaces and conclude that $l^p$ spaces are Banach spapce. Moreover, we will prove that $l^2$ space is a Hilbert space.

## ACKNOWLEDGMENTS

I would first like to thank my mentor Dr. Hajrudin Fejzic who inspired me to teach my students with enthusiasm and understanding. His zeal for Mathematics in the classroom as well as in his guidance through my thesis has shown me the path of dedication I wish to take. This will help me mature into a Mathematician such as Dr. Fejzic. I now know that Mathematics is an art whose beauty is shown through the power of the idea and whose passion grows through the many failures, which lead to a great success. Thank you Dr. Fejzic.

As for the faculty at C.S.U.S.B., my experience here would never have been more rewarding if it were not for your overwhelming kindness and consideration. There have been many times when you have gone above and beyond the normal routine for me. I never felt like a student, but more like a family member.

To my family, never-ending gratitude for everything they have done for me. I would not be a fraction of the person I present to you today if it was not for the love and support from my family.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER ONE

$l^p$ SPACES

The set of real numbers, $\mathbb{R}$ together with the usual addition, and multiplication has many nice properties. Not only that $(\mathbb{R}, +)$ and $(\mathbb{R}\backslash\{0\},\cdot)$ are commutative groups, but the two operations also satisfy the distribution laws. Thus $(\mathbb{R},+,\cdot)$ is a commutative field. If one wants the same structure on the set $\mathbb{R} \times \mathbb{R}$, the multiplication has to be defined in a different way. Although the addition is defined in a natural way as $(a + c, b + d)$, for two points $(a, b)$, and $(c, d)$ in $\mathbb{R} \times \mathbb{R}$ we define the binary operation $\cdot \to \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, by $(a, b)\cdot(c, d) = (ac - bd, ad + bc)$. Only with this different multiplication $(\mathbb{R} \times \mathbb{R}, +, \cdot)$ becomes a commutative field. This multiplication can be better understood with the introduction of the imaginary unit, i, and the corresponding field is better known as the field of complex numbers. For $n > 2$, it is not possible to make the product of n-tuples, $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, into a field. Still one can make it not only into a vector space, but also into Banach and Hilbert spaces. For example it is still possible to measure a distance between two n-tuples, and moreover, that can be done in the most natural way as the Euclidean distance between two points. If one allows " infinite n-tuples", that is infinite sequence, "the distance" problem will have a solution only if we consider a smaller subset of all infinite sequences the so called $l^p$ spaces.

Definition 1.1: For $p \geq 0$ we define $l^p$ to be the set of all infinite sequence $(x_1, x_2, x_3, \cdots\cdots, x_n, \cdots)$ for which $\sum_{n=1}^{\infty} |x_n|^p$ is finite.

That set is indeed a proper subset of the set of all sequences. We provide the following example.

Infinite Sequence is Not in $l^1$

Example 1.1: Infinite sequence

$$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots, \frac{1}{n}, \cdots\right)$$

is not in $l^1$.

Solution:

To show that the sequence $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots, \frac{1}{n}, \cdots\right)$ is not in $l^1$, we have to verify that the series $\sum_{i=1}^{\infty} \frac{1}{n}$ is divergent. We consider the subsequence $\{s_{2^n}\}$ of the partial sums.

$$s_1 = 1 \qquad s_2 = 1+\frac{1}{2}$$

$$s_{2^2} = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right) > 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right) = 1+\frac{2}{2}$$

$$s_{2^3} = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)$$

$$> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)$$

$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 1+\frac{3}{2}$$

$$s_{2^4} = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}\right)$$

$$> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\cdots+\frac{1}{16}\right)$$

$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2} = 1+\frac{4}{2}$$

And so on, we get that for all n.

$$s_{2^n} > 1+\frac{n}{2}.$$

Thus $s_{2^n} \to \infty$ as $n \to \infty$ and so $\{s_n\}$ is divergent.

Thus, by definition 1.1 the sequence

$$\left(1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\cdots,\frac{1}{n},\cdots\right) \notin l^1 \text{ space}.$$

On the other hand the sequence $\left(1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\cdots,\frac{1}{n},\cdots\right)$ is in $l^p$ for $p > 1$.

## Hyperharmonic P-series

<u>Proposition 1.1</u>: Hyperharmonic P-series

$$\sum_{n=1}^{\infty}\frac{1}{n^p} = \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots+\frac{1}{n^p}+\cdots$$

converges if $p > 1$ and diverges if $0 < p \leq 1$.

Proof:

<u>Case I</u>: $p > 1$

Since $p > 0$, $f(x) = \frac{1}{x^p}$ is a positive decreasing function of x, and since for $p > 1$.

$$\int_1^\infty \frac{1}{x^p}dx = \int_1^\infty x^{-p}dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \frac{1}{1-p}\left(\lim_{b\to\infty} \frac{1}{b^{p-1}} - 1\right)$$

$$= \frac{1}{1-p}(0-1) = \frac{1}{p-1}$$

The series converges by the Integral Test.

<u>Case II</u>: $p < 1$

$$\int_1^\infty \frac{1}{x^p}dx = \frac{1}{1-p}\lim_{b\to\infty}\left(b^{1-p} - 1\right) = \infty$$

The series diverges by the Integral Test.

<u>Case III</u>: $p = 1$

If $p = 1$, we have the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

By example 1.1 the harmonic series diverges.

Thus, we have convergence for $p > 1$ but divergence for

$0 < p \leq 1$.

<u>Corollary</u>: The sequence $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots, \frac{1}{n}, \cdots\right)$ is in $l^p$ if and only if $p > 1$.

Notice that a sequence $\{a_n\}$ is in $l^p$ if the series made of p powers of absolute values of $a_n$ is convergent. So for example, although the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is convergent, the sequence $\left(1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\frac{1}{5},-\frac{1}{6},\cdots,(-1)^{n-1}\frac{1}{n},\cdots\right)$ is not in $l^1$.

The following two proposition show that the geometric Sequence $(a,ar,ar^2,\ldots)$ is in $l^p$ for all p > 0 if and only if $|r| < 1$.

## Geometric Series

Proposition 1.2: Geometric Series

$$a+ar+ar^2+ar^3+\cdots+ar^{n-1}+\cdots$$

Where a and r are real numbers with $a \neq 0$

i. Converges and has the sum $S=\frac{a}{1-r}$ if $|r|<1$

ii. Diverges if $|r| \geq 1$

Proof:

Case I: $r = 1$

If r =1, then $S_n=a+a+a+a+a+a+\cdots+a=na$ and the series diverges, since $\lim_{n\to\infty} S_n$ does not exist.

Case II: $r = -1$

If r = -1, then $S_k = a$ if k is odd and $S_k = 0$ if k is even. Since the sequence of partial sums oscillates between a and 0, the series diverges.

<u>Case III</u>: $r \neq 1$

If $r \neq 1$, then

$$S_n = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \cdots + ar^{n-1}$$

and

$$r \cdot S_n = ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6 + \cdots + ar^n$$

Subtracting corresponding sides of these equations, we obtain

$$(1 - r)S_n = a - ar^n$$

Dividing both sides by (1 - r) gives us

$$S_n = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

Consequently

$$\begin{aligned}\lim_{n\to\infty} S_n &= \lim_{n\to\infty}\left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right)\\ &= \lim_{n\to\infty}\frac{a}{1-r} - \lim_{n\to\infty}\frac{ar^n}{1-r}\\ &= \frac{a}{1-r} - \frac{a}{1-r}\lim_{n\to\infty} r^n\end{aligned}$$

<u>Case IV</u>: $|r| < 1$

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$

Hence, $\lim_{n\to\infty} S_n = \dfrac{a}{1 - r} = S$

Case V: $|r|>1$

If $|r|>1$, then $\lim_{n\to\infty} r^n$ does not exist. Hence $\lim_{n\to\infty} S_n$ does not exist and therefore the series diverges.

## Geometric Sequence

Proposition 1.3: The geometric sequence $(a, ar, ar^2, ar^3, \cdots, ar^n, \cdots)$ is in $l^p$ space if and only if $|r|<1$.

Proof:

$$\sum_{n=0}^{\infty} |ar^n|^p = a^p + a^p r^p + a^p r^{2p} + a^p r^{3p} + \cdots + a^p r^{np} + \cdots$$

The series $(a^p, a^p r^p, a^p r^{2p}, a^p r^{3p}, \cdots, a^p r^{np}, \cdots)$ is a geometric series. By Proposition 1.2, this series converges if and only if $|r^p| = |r|^p < 1$, which is true if and only if $|r|<1$.

By definition 1.1, the sequence $(a, ar, ar^2, ar^3, \cdots, ar^n) \in l^p$. Next we will discuss the relationships between different $l^p$ space. In particular, we have the following result.

Theorem 1.1: For p < q, $l^p \subsetneq l^q$.

Proof:

Let x = $(x_1, x_2, x_3, \cdots) \in l^p$. Then $\sum_{i=1}^{\infty} |x_i|^p$ is finite.

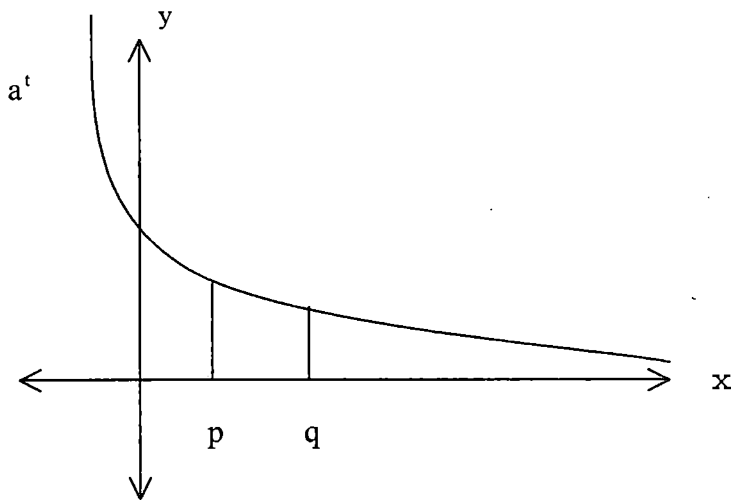



Figure 1.1.

Exponential Function of $a^t$

Consider the exponential function $a^t$, for $0 < a < 1$. The function is decreasing, thus $a^p > a^q$. (*)

Since, $\sum_{i=1}^{\infty} |x_i|^p$ is finite $\lim_{a\to\infty} |x_i| = 0$. Therefore, for all but finitely many i's $|x_i| < 1$. For such i's by (*), with $a = |x_i|$ we have $|x_i|^p > |x_i|^q$. By the comparison test $\sum_{i=1}^{\infty} |x_i|^q$ is also finite, and hence $x \in l^q$. Thus $l^p \subseteq l^q$. To show that $l^p$ is a proper subset of $l^q$, notice that the sequence

$x = \left(\frac{1}{1}, \frac{1}{2^{\frac{1}{p}}}, \frac{1}{3^{\frac{1}{p}}}, \ldots, \frac{1}{n^{\frac{1}{p}}}, \ldots\right)$ is in $l^q$. This is because $\sum_{n=1}^{\infty} \left(\frac{1}{n^{\frac{1}{p}}}\right)^q = \sum_{n=1}^{\infty} \left(\frac{1}{n^{\frac{q}{p}}}\right)$

is convergent since it is a hyperharmonic q/p series, and because $q > p$, it is convergent. On the other hand $x \notin l^p$, since $\sum_{n=1}^{\infty}\left(\frac{1}{n^{\frac{1}{p}}}\right)^p = \sum \frac{1}{n}$ which is divergent.

# CHAPTER TWO

# VECTOR SPACE

Definition 2.1: Vector Space

Let $\mathbb{R}$ be a field. A vector space over $\mathbb{R}$ consists of an abelian group (V,+) under addition together with operation of scalar multiplication of each element of V by each element of $\mathbb{R}$ on the left, such that for all a, b $\in \mathbb{R}$ and $\alpha$, $\beta \in V$ the following conditions are satisfied:

i. $a\alpha \in V$

ii. $a(b\alpha) = (ab)\alpha$

iii. $(a + b)\alpha = (a\alpha) + (b\alpha)$

iv. $a(\alpha + \beta) = (a\alpha) + (a\beta)$

$1\alpha = \alpha$

To make $l^p$ into a vector space we have to define addition and scalar multiplication on $l^p$.

For $\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n, \ldots) \in l^P$

$\beta = (\beta_1, \beta_2, \beta_3, \cdots, \beta_n, \ldots) \in l^P$ and $c \in \mathbb{R}$

We define,

1. $\alpha + \beta = (\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n, \cdot\cdot) + (\beta_1, \beta_2, \beta_3, \cdots, \beta_n, \cdots)$
$= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3, + \cdots +, \alpha_n + \beta_n, + \cdots)$

2. $c\alpha = c(\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n, \cdot\cdot)$

$$=(c\alpha_1, c\alpha_2, c\alpha_3, \cdots, c\alpha_n, \cdot\cdot)$$

It is not a simple fact to show that $l^p$ is closed under this addition, that is if x, and y are two sequences in $l^p$, then x + y is in $l^p$. That this is indeed the case it will follow from the so called Minkowski's Inequality. In order to prove Minkowski's Inequality, first we need the following result.

Holder's Inequality for Infinite Sum

Theorem 2.1: Holder's Inequality for Infinite Sum:

Let p and q be such that $1 < p,\ q < \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Let

$x \in l^p$ and $y \in l^q$, then

$$\sum_{i=1}^{\infty}|x_i y_i| \le \left(\sum_{i=1}^{\infty}|x_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{\infty}|y_i|^q\right)^{\frac{1}{q}}$$

Proof:

Given $\frac{1}{p}+\frac{1}{q}=1$ for $p > 1$

$$\frac{1}{p}=1-\frac{1}{q}$$

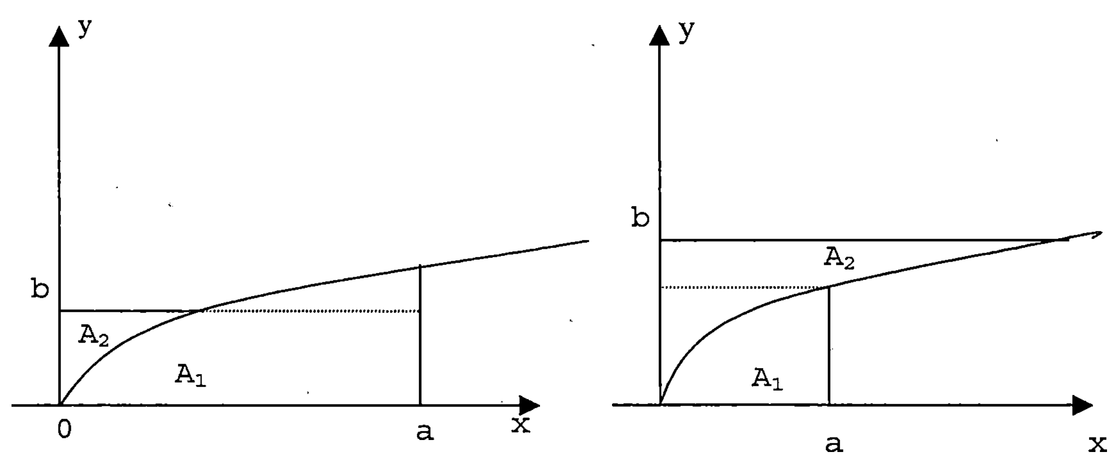
$$p = \frac{q}{q-1}$$

Let $y = x^{p-1}$. Then

$x = y^{\frac{1}{p-1}}$ Substitute $p = \frac{q}{q-1}$, to get

$$x = y^{\frac{1}{\frac{q}{q-1}-1}}$$

$x = y^{\frac{1}{\frac{q-q+1}{q-1}}} = y^{q-1}$. Since $y' = (p-1)x^{p-2} > 0$, y is increasing on $[0,+\infty)$. Let a, b >0 and let $A_1$ denotes the area between the x-axis, the curve $y = x^{p-1}$, the line x = a, and let $A_2$ denotes the area between the lines y = b, x = 0 and the curve $y = x^{p-1}$. See the figure below.




Case 1: $a \geq b$ Case 2: $a < b$

Figure 2.1.

Square Root Function

From the figures above in both cases, we see that,

$ab \le A_1 + A_2$ and equality holds if and only if a = b

$$A_1 + A_2 = \int_0^a x^{p-1}dx + \int_0^b y^{q-1}dy$$

$$= \frac{a^p}{p} + \frac{b^q}{q}. \qquad \text{So } ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Let $a_i = \dfrac{|x_i|}{\left(\sum |x_i|^p\right)^{\frac{1}{p}}}$ and $b_i = \dfrac{|y_i|}{\left(\sum |y_i|^q\right)^{\frac{1}{q}}}$

Then, $a_i b_i = \dfrac{|x_i|}{\left(\sum |x_i|^p\right)^{\frac{1}{p}}} \dfrac{|y_i|}{\left(\sum |y_i|^q\right)^{\frac{1}{q}}} \le \dfrac{|x_i|^p}{\sum |x_i|^p} \cdot \dfrac{1}{p} + \dfrac{|y_i|^q}{\sum |y_i|^q} \cdot \dfrac{1}{q}$. Since $x \in l^p, y \in l^q$

the series $\sum a_i b_i$ is convergent,

and $\sum a_i b_i \le \dfrac{\sum |x_i|^p}{\sum |x_i|^p} \cdot \dfrac{1}{p} + \dfrac{\sum |y_i|^q}{\sum |y_i|^q} \cdot \dfrac{1}{q} = \dfrac{1}{p} + \dfrac{1}{q} = 1$.

Therefore, $\sum |x_i||y_i| \le \left(\sum |x_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum |y_i|^q\right)^{\frac{1}{q}}$.

## Minkowski's Inequality for Infinite Sum

Theorem 2.2: Minkowski's Inequality for Infinite Sum: Let

$x, y \in l^p$. Then

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty} |y_i|^p\right)^{\frac{1}{p}}$$

Proof:

The case when p =1 is clear. So assume that p > 1.

$$\left(|x_i| + |y_i|\right)^p = \left(|x_i| + |y_i|\right)^{p-1}\left(|x_i| + |y_i|\right)$$

$$= \left(|x_i| + |y_i|\right)^{p-1} |x_i| + \left(|x_i| + |y_i|\right)^{p-1} |y_i|$$

Hence,

$$\sum_{i=1}^{\infty} |x_i + y_i|^p \le \sum_{i=1}^{\infty} \left(|x_i| + |y_i|\right)^p$$

$$= \sum_{i=1}^{\infty}(|x_i|+|y_i|)^{p-1}\ |x_i| + \sum_{i=1}^{\infty}(|x_i|+|y_i|)^{p-1}|y_i|$$

Now let $z_i \equiv (|x_i|+|y_i|)^{p-1}$

Substitute z into the above inequality, we have

$$\sum(|x_i|+|y_i|)^{p-1}\ |x_i| = \sum_{i=1}^{\infty}|z_i||x_i|$$

and $\sum_{i=1}^{\infty}(|x_i|+|y_i|)^{p-1}|y_i| = \sum_{i=1}^{\infty}|z_i||y_i|$

Let $q = \dfrac{p}{p-1}$. Then $z = (z_1, z_2, z_3, \cdots)$ is in $l^q$, and $\dfrac{1}{p}+\dfrac{1}{q}=1$. So by Holder's inequality:

We have,

$$\sum_{i=1}^{\infty}|x_i||y_i| \le \left(\sum_{i=1}^{\infty}|z_i|^q\right)^{\frac{1}{q}} \cdot \left(\sum_{i=1}^{\infty}|x_i|^p\right)^{\frac{1}{p}}$$

and

$$\sum_{i=1}^{\infty}|z_i||y_i| \le \left(\sum_{i=1}^{\infty}|z_i|^q\right)^{\frac{1}{q}} \cdot \left(\sum_{i=1}^{\infty}|y_i|^p\right)^{\frac{1}{p}}$$

Thus, $\sum_{i=1}^{\infty}(|x_i|+|y_i|)^p \le \left\{\sum_{i=1}^{\infty}(|x_i|+|y_i|)^{(p-1)\cdot q}\right\}^{\frac{1}{q}}\left\{\left(\sum_{j=1}^{\infty}|x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{\infty}|y_j|^p\right)^{\frac{1}{p}}\right\}$

Since $(p - 1)q = p$.

$$\sum_{i=1}^{\infty}(|x_i|+|y_i|)^p \le \left[\sum_{i=1}^{\infty}(|x_i|+|y_i|)^p\right]^{\frac{1}{q}} \cdot \left\{\left(\sum_{j=1}^{\infty}|x_i|^p\right)^{\frac{1}{p}}\right\} + \left(\sum_{j=1}^{\infty}|y_i|^p\right)^{\frac{1}{p}}$$

Dividing both side by $\left\{\sum_{i=1}^{\infty}(|x_i|+|y_i|)^p\right\}^{\frac{1}{q}}$, we have:

$$\left(\sum_{i=1}^{\infty}(|x_i|+|y_i|)^p\right)^{1-\frac{1}{p}} \le \left(\sum_{i=1}^{\infty}|x_i|^p\right)^{\frac{1}{p}} + \left(\left(\sum_{i=1}^{\infty}|y_i|^p\right)^{\frac{1}{p}}\right)$$

Finally, since $|x_i + y_i|^p \le (|x_i|+|y_i|)^p$.

We have, $\sum_{i=1}^{\infty}\left(\left|x_i+y_i\right|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{\infty}\left|x_i\right|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty}\left|y_i\right|^p\right)^{\frac{1}{p}}$.

By simplifying the left-hand side we get.

$$\left[\sum_{i=1}^{\infty}\left(\left|x_i\right|+\left|y_i\right|\right)^p\right]^{\frac{1}{p}} \le \left(\sum_{i=1}^{\infty}\left|x_i\right|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty}\left|y_i\right|^p\right)^{\frac{1}{p}}$$

# CHAPTER THREE

# METRIC SPACE

Definition 3.1: Metric Space

Let X be a nonempty set. A function $d: X \times X \to \mathbb{R}$, taking pairs of elements of X into real numbers, is called a *metric* on the set X if the following conditions hold:

(i) $d(x,y) \geq 0$ for every $x,y \in X$

(ii) $d(x,y) = 0$ if and only if $x = y$

(iii) $d(x,y) = d(y,x)$ for every $x,y \in X$

(iv) $d(x,z) \leq d(x,y) + d(y,z)$ for every $x,y,z \in X$

The pair $(X,d)$ is called a metric space.

## The Pair $(l^p, d_p)$ is a Metric Space

Theorem 3.1: The pair $(l^p, d_p)$ where $d_p(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{\frac{1}{p}}$ is a metric space.

Proof:

i) $d(x,y) \geq 0$ for every $x,y \in l^p$

$$d(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{\frac{1}{p}} \geq 0$$

ii) $d(x,y) = 0$ If and only if $x = y$

Proof:

d(x,y)= 0 if and only if $\sum_{i=1}^{\infty}|x_i - y_i|^p = 0$.

if and only if $|x_i - y_i|^p = 0$

for all i, if and only if $x_i = y_i$ for all i.

if and only if x = y

iii) $d(x,y) = d(y,x)$ for every $x, y \in X$

$$d(x,y) = \left(\sum_{i=1}^{\infty}|x_i - y_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty}|y_i - x_i|^p\right)^{\frac{1}{p}} = d(y,x)$$

iv) $d(x,z) \le d(x,y) + d(y,z)$ for every $x, y, z \in X$

$$d(x,z) = \left(\sum_{i=1}^{\infty}|x_i - z_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty}|x_i - y_i + y_i - z_i|^p\right)^{\frac{1}{p}}$$
$$= \left(\sum_{i=1}^{\infty}|(x_i - y_i) + (y_i - z_i)|^p\right)^{\frac{1}{p}}$$

By Minkowski's inequality, we have:

$$\le \left(\sum_{i=1}^{\infty}|x_i - y_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{\infty}|y_i - z_i|^p\right)^{\frac{1}{p}}$$
$$= d(x,y) + d(y,z)$$

Thus, $d(x,z) \le d(x,y) + d(y,z)$.

## $l^p$ is a Complete Metric Space

Theorem 3.2: $l^p$ is a complete metric space.

Proof:

Let $\{x_n\} \subseteq l^p$ be a Cauchy sequence. This means that for every ε > 0, ∃ an N such that for m, n ≥ N, $d(x_m, x_n) < \varepsilon$. We have to show that $x_n$ is convergent. We use the notation $x_n^k$ to denote the kth term of the sequence $x_n$, that is, $x_n = (x_n^1, x_n^2, x_n^3, \ldots, x_n^k, \ldots)$. For k ≥ 1, we have

$$\left|x_n^k - y_m^k\right| \leq \sum_{i=1}^{\infty}\left|x_n^i - y_m^i\right|^p \text{ so}$$
$$\left|x_n^k - x_m^k\right| \leq d(x_n, x_m) < \varepsilon$$

Here the last inequality is true for all n, m ≥ N. This implies that for each fixed k, the sequence $\{x_n^k\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. Since every sequence in $\mathbb{R}$ is convergent if and only if it is Cauchy, we see that for each k the sequence $\{x_n^k\}_{n=1}^{\infty}$ is convergent. Let $x_0^k = \lim_{n\to\infty} x_n^k$. It remains to show the sequence $x_0 = (x_0^1, x_0^2, x_0^3, \ldots, x_0^k, \ldots)$ is

1. The limit of the sequence $\{x_n\}$.

2. $x_0 \in l^p$.

Proof of 1:

Let ε > 0, we have to show that there is an N such that for n ≥ N, $d(x_n, x_0) < \varepsilon$. Since $\{x_n\}$ is a Cauchy sequence, there exists an N such that for all m, n $\geq$ N, $d(x_n, x_m) < \varepsilon$. That is

$$\sum_{i=1}^{\infty} \left| x_n^i - x_m^i \right|^p < \varepsilon^p .$$

Let $s_r = \sum_{i=1}^{r} \left| x_n^i - x_0^i \right|^p$

Then, $s_r = \lim_{m \to \infty} \sum_{i=1}^{r} \left| x_n^i - x_m^i \right|^p$

Since, $\sum_{i=1}^{r} \left| x_n^i - x_m^i \right|^p \leq \sum_{i=1}^{\infty} \left| x_n^i - x_m^i \right|^p < \varepsilon^p$

for m $\geq$ N, we have

$s_r \leq \varepsilon^p$, since this is true for all r we have:

$\sum_{i=1}^{\infty} \left| x_n^i - x_0^i \right|^p \leq \varepsilon^p$, and hence

$d(x_n, x_0) < \varepsilon$.

Proof 2: $X_o \in l^p$

To show that $X_o \in l^p$, by Minkowski's inequality

$$\left( \sum_{i=1}^{r} \left| x_0^i \right|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{r} \left| x_0^i - x_n^i \right|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{r} \left| x_n^i \right|^p \right)^{\frac{1}{p}}$$

$$\leq \left( \sum_{i=1}^{\infty} \left| x_0^i - x_n^i \right|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{\infty} \left| x_n^i \right|^p \right)^{\frac{1}{p}}$$

$$= d(x_n, x_0) + \left( \sum_{i=1}^{\infty} \left| x_n^i \right|^p \right)^{\frac{1}{p}}$$

$$\leq \varepsilon + \left(\sum_{i=1}^{\infty}\left|x_n^i\right|^p\right)^{\frac{1}{p}}$$

Where the last inequality is true for sufficiently large n.

Since $x_n \in l^p$, we have $\left(\sum_{i=1}^{\infty}\left|x_n^i\right|^p\right)^{\frac{1}{p}} \leq M$ for some constant M. Hence

$\left(\sum_{i=1}^{r}\left|x_0^i\right|^p\right)^{\frac{1}{p}} \leq \varepsilon + M$. Since this is true for all r, we have

$$\left(\sum_{i=1}^{\infty}\left|x_n^i\right|^p\right)^{\frac{1}{p}} \leq \varepsilon + M .$$

Thus, $x_0 \in l^p$.

# CHAPTER FOUR

## NORMED VECTOR SPACE

The definition of the vector space was derived from properties of vectors in the plane and in the space. Originally vectors were directed arrows. We measure the size of these arrows by measuring their length. In an arbitrary vector space, a vector size may not be defined. Take for example the vector space, C(0,1) of all continuous functions over the interval (0,1). What is a size of the vector $\frac{1}{x}$ from this space? On the other hand if we look at the vector space $\mathbb{R}^n$, we know that the size of an arbitrary vector $x = (x_1, x_2, x_3, \cdots, x_n)$ is $\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. A vector space, in which the size of vectors is defined as Normed Vector Space. More precisely if V is a vector space and the function $\|\cdot\|: V \to \mathbb{R}^+$ satisfies the following properties.

1. $\|x\| \geq 0$ for all $x \in V$ with $\|x\| = 0$ if and only if $x = 0$.
2. $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in \mathbb{R}$, $x \in V$.
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Then we say that V is a normed vector space.

In Chapter 1, we showed that $l^p$, is a vector space. For $p \geq 1$, $l^p$ is also a normed vector space were the norm is defined by $\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$. Clearly the function $\|\cdot\|$ satisfies the first two properties. The third property is equivalent to Minkowski's inequality.

## CHAPTER FIVE

## BANACH SPACE

We have seen that $l^p$ is both a Normed Vector Space and a Metric Space. In fact every Normed Vector Space is automatically a Metric Space, since it is easy to check that the function d: $V \times V \to \mathbb{R}^+$, defined by $d(x,y)=\|x-y\|$ is a metric on V. This metric is called the induced metric, since it is induced by the norm. Recall our definition of the metric on $l^p : d(x,y)=\left(\sum_{n=1}^{\infty}|x_n-y_n|^p\right)^{\frac{1}{p}}$. It is clear that this metric satisfies $d(x,y)=\|x-y\|$, thus it is the induced metric on $l^p$. A Normed Vector Space, which is complete with respect to the induced metric is called a Banach Space. In Chapter 2, we showed that $l^p$ with the metric defined by $d(x,y)=\left(\sum_{n=1}^{\infty}|x_n-y_n|^p\right)^{\frac{1}{p}}$ is a complete metric space. Since this is also the induced metric on $l^p$, we know that $l^p$ is a Banach space.

# CHAPTER SIX

# HILBERT SPACE

One of the important problems in plane geometry is how to measure the angle between two lines. Vectors give us an elegant solution to this problem. Namely if u and v are two nonzero vectors parallel to two lines that intersect at an angle $\alpha$, then we have the following formula $\cos\alpha = \frac{u \cdot v}{\|u\|\|v\|}$.

The product in the numerator is a special product called the dot product. If $u = (u_1, u_2)$ and $v = (v_1, v_2)$, then $u \cdot v = u_1 v_1 + u_2 v_2$. That the definition of the dot product deserves to be called product, follows from its distribution property. Namely, the dot product satisfies the following distributive property;

$u \cdot (v + w) = u \cdot v + u \cdot w$ and $(u + v) \cdot w = u \cdot w + v \cdot w$.

The dot product has been proved to be much more useful than just to help us measure angles between lines. Its definition to $\mathbb{R}^n$ extends with the following formula

$u \cdot v = \sum_{i=1}^{n} u_i v_i$. In general vector space, the dot product is commonly called the Inner Product.

## Inner Product

Definition 6.1: Inner Product

Let X be a vector space. An inner product ( , ): X × X → C, is a bilinear function that satisfies the following properties.

i) (x,y) = (y,x) is symmetric

ii) $(\lambda x+\mu y,z)=\lambda(x,z)+\mu(y,z)$ $\forall\lambda,\mu$ $\mathbb{R}$ and x, y, z ∈ X.

iii) (x,x) ≥ 0 and, (x,x) = 0 if and only if x = 0.

Definition 6.2: A vector space V together with the inner product is called an inner product space.

Theorem 6.1: Every inner product space is a normed space. In fact, if x is an element in an inner product space V, the mapping

$$x\to\|x\|=[(x,x)]^{\frac{1}{2}}$$

defines a norm on V.

Proof:

Define the norm on V by $\|x\|=[(x,x)]^{\frac{1}{2}}$. To check that this is indeed a norm on V we have to check the following.

1°. $\|x\|\ge 0$ for all x ∈ V with $\|x\|=0$ if and only if x = 0.

2°. $\|\alpha x\|=|\alpha|\|x\|$ for all α ∈ $\mathbb{R}$, x ∈ V.

3°. $\|x+y\| = \|x\| + \|y\|$ for all x, y $\in$ V.

Proof:

1° follows from definition 6.1 iii.

To check 2°, notice that $\|\alpha x\| = [(\alpha x, \alpha x)]^{\frac{1}{2}} = [\alpha(x, \alpha x)]^{\frac{1}{2}}$ by definition 6.1 ii. By definition 6.1 i. this last equality is

$[\alpha(x, \alpha x)]^{\frac{1}{2}} = [\alpha \cdot \alpha(x, x)]^{\frac{1}{2}}$ again by definition 6.1 ii., and finally this equality is $|\alpha|(x,x) = |\alpha|\|x\|$.

To check 3°, we have

$$\|x+y\|^2 = (x+y, x+y) = (x, x+y) + (y, x+y) = (x+y, x) + (x+y, y)$$

$$= (x,x) + (y,x) + (x,y) + (y,y)$$

$$= \|x\|^2 + 2(x,y) + \|y\|^2$$

$$\leq \|x\|^2 + 2|(x,y)| + \|y\|^2 \qquad **$$

Now we show that

$$|(x,y)|^2 \leq (x,x) \cdot (y,y)$$

Consider the quadratic polynomial in λ.

$$|(x,y)|^2 (y,y)\lambda^2 + 2|(x,y)|^2 \lambda + (x,x) \qquad *$$

This polynomial is equal to

$(x + \lambda(x,y)y,\ x + \lambda(x,y)y)$ since

$$(x + \lambda(x,y)y,\ x + \lambda(x,y)y) = (x,x) + \lambda(x,y)(y,x) + \lambda(x,y)(x,y) + \lambda^2 (x,y)^2 (y,y)$$

$$= |(x,y)|^2(y,y)\lambda^2 + 2|(x,y)|^2\lambda + (x,x)$$

But $(x + \lambda(x,y)y, x + \lambda(x,y)y) \geq 0$ by definition 6.1 i., so the polynomial in * is a nonnegative polynomial in λ.

Since it is a quadratic polynomial its discriminant has to be less than or equal to zero. That is

$$4|(x,y)|^4 - 4(x,x)\cdot|(x,y)^2|(y,y) \leq 0$$

Simplifying the inequality above we get.

$$|(x,y)|^2 \leq (x,x)(y,y).$$

Now going back to **, we have

$$\|x\|^2 + 2|(x,y)| + \|y\|^2 \leq \|x\|^2 + 2\sqrt{(x,x)}\cdot\sqrt{(y,y)} + \|y\|^2 = \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$$

Therefore, $\|x+y\| \leq \|x\| + \|y\|$ and $3^\circ$ is proved.

<u>Definition 6.3:</u> Let V be an inner product space, if V with the norm defined by $\|x\| = [(x,x)]^{\frac{1}{2}}$ is a Banach space, then we say that V is a Hilbert space.

## $l^2$ is a Hilbert Space

<u>Theorem 6.2:</u> $l^2$ is a Hilbert space.

Proof:

Define the inner product on $l^2$ by $(x,y) = \sum_{i=1}^{\infty} x_i y_i$ where

$x = (x_1, x_2, x_3, \ldots)$ and $y = (y_1, y_2, y_3, \ldots)$ are in $l^2$. We check that

< , > is an inner product.

i. $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i = \sum_{i=1}^{\infty} y_i x_i = \langle y, x\rangle$

ii. Let $\lambda, \mu \in \mathbb{R}$ and $x, y, z \in l^2$.

$$\langle \lambda x + \mu y, z\rangle = \sum_{i=1}^{\infty} (\lambda x_i + \mu y_i) z_i$$

$$= \sum_{i=1}^{\infty} \lambda x_i z_i + \mu y_i z_i$$

$$= \lambda \sum_{i=1}^{\infty} x_i z_i + \mu \sum_{i=1}^{\infty} y_i z_i$$

$$= \lambda(x, z) + \mu(y, z)$$

iii. $(x, x) = \sum_{i=1}^{\infty} x_i x_i = \sum_{i=1}^{\infty} x_i^2 \geq 0$ and $(x, x) = 0$ if and only if

$x_i = 0$ for all i, that is $x = 0$.

The norm is defined by $\|x\| = [(x,x)]^{\frac{1}{2}}$, that is, $\|x\| = \left(\sum_{i=1}^{\infty} x_i^2\right)^{\frac{1}{2}}$. This norm agrees with our previous definition of the norm on $l^2$, and since we have shown that $l^2$, with this norm, is a Banach space, we have that $l^2$ is a Hilbert space.

## REFERENCES


[1] Day, Mahlon; Normed Linear Spaces. Third Edition, 1973. Springer-Verlag New York.

[2] Lindenstrauss, Joram; Classical Banach Spaces I. 1997 Springer-Verlag Berlin.

[3] Maddox, I.J; Element of Functional Analysis. 1970 Cambridge University Press.

[4] Oden, Tinsley; Applied Functional Analysis. 1979 Prentice-Hall, Inc.

[5] Packel, Edward; Functional Analysis. 1974 Intext, Inc.

[6] Vulikh, B.Z; Functional Analysis for Scientists and Technologists. 1963 Addison-Wesley Publishing Company, Inc.

[7] Young, Nicholas; An Introduction to Hilbert Space. 1988 Cambridge University Press.