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An Exposition of the Curvature of Warped Product Manifolds

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An Exposition on the Curvature of Warped Product Manifolds

A Thesis

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In Partial Fulfillment

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by

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Abstract

The field of differential geometry is brimming with compelling objects, among which are warped products. These objects hold a prominent place in differential geometry and have been widely studied, as is evident in the literature. Warped products are topologically the same as the Cartesian product of two manifolds, but with distances in one of the factors in skewed. Our goal is to introduce warped product manifolds and to compute their curvature at any point. We follow recent literature and present a previously known result that classifies all flat warped products to find that there are flat examples of warped products which do not arise as the warped product of two flat manifolds. Lastly, using reasonable assumptions we will derive the metric of a spacetime containing one point mass representing the center of a black hole, where the point mass representing the singularity is not modeled. We identify this spacetime as a warped product and use a Weyl curvature invariant to show that the curvature of this spacetime is unbounded near this point mass. This will demonstrate that this model does not allow for such a point to be included in the spacetime, so a black hole really is a "hole" in spacetime. Similarly, but with an opposite conclusion, we also show that the curvature is bounded near the event horizon, suggesting that the event horizon still can be modeled.

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[1] "While it is always best to believe in one's self, a little help from others can be a great blessing."–A quote from a beloved character, Iroh.

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Chapter 1

Introduction

At its core, a warped product represents a generalization of the well-established concept of a product manifold, with a classical example being $\mathbb{R} \times \mathbb{R}$. In essence, warped products are topologically the same as the Cartesian product of two manifolds, but with distances in one of the factors is skewed. Our focus centers on examining manifolds that possess two crucial attributes: they are smooth and semi-Riemannian, as we will precisely define. In the context of this study, a warped product assumes the role of a specific smooth semi-Riemannian product manifold, furnished with a metric. This metric, in turn, becomes an invaluable tool in our journey, enabling the calculation of intriguing geometric properties, such as curvature.

To provide a clear road map for the forthcoming chapters, an outline is presented below. In section 1.1, notation is provided for the reader to bridge this thesis and other literature in differential geometry. A solid grasp of linear algebra, topology, and differential equations is advisable. We will presume familiarity with certain foundational definitions, given the assumption that the reader possesses proficiency in the aforementioned subjects. For those seeking specific resources for an introduction essential to comprehending the content, refer to "Introduction to Smooth Manifolds" [Lee12] and "Differential geometry of warped product manifolds and submanifolds" [Che17]. Another source would be "Riemannian Geometry A Beginner's Guide" [Mor98]. The reader could find that the visual representations in [Mor98] provide intuition for Riemannian geometries.

Chapter 2 will provide the necessary foundation of manifold theory and tangent spaces. The concepts addressed in the first few chapters will help prepare the reader for the formal exposition that unfolds.

It is our convention to regard elements within $\mathbb{R}^n$ as points expressed in coordinates. Still, vectors in $\mathbb{R}^n$ may not have the origin as their initial point and may emanate from anywhere. We will see in Chapters 2 and 3 that by taking directional derivatives, a "one to one correspondence between tangent vectors and linear maps" emerges. Such mappings are called derivations [Lee12].

In Chapter 3, we introduce one forms and the differential. A differential, in the form of a covector field, serves as a bridge for analyzing the behavior of covector fields under the influence of smooth mappings. As a result, we will establish that covector fields located within the codomain of a specified smooth map naturally undergo a process of pullback, giving rise to covector fields on the domain, much akin to the process of pushforward of tangent vectors on a manifold to tangent vectors on another manifold which we encounter in Chapter 2.

Our investigation in Chapter 3 has a focus on practical understanding of semi-Riemannian manifolds, and Chapter 4 will consist of explicitly computing the Christoffel symbols that define the Levi-Civita connection, a pivotal connection capturing the essence of curvature. Within Chapter 5 we will present a comprehensive exposition of curvature. Then we will illustrate the art of manipulating indices to create meaningful scalar quantities.

Building upon the foundational concepts introduced in Chapters 2 through 5, we will delve into the examination of intriguing scenarios involving warped products and their curvature. There is an obvious result of a flat manifold crossed with another that results in a flat manifold. However a compelling inquiry that arises is if given a flat warped product, does that imply the restraint that the warped product arose from two flat manifolds. Through our exploration, we aim to define a warped product's flatness. Following Akbar [Akb12], we will classify all flat warped products to find that there are flat warped products which do not arise as the warped product of two flat manifolds. We will prove the conditions for such results. In Chapter 6, we will provide an exact computation of curvature of warped product manifolds.

Lastly in Chapter 7, we will use reasonable assumptions to derive the metric of a spacetime that contains only a black hole. We identify this spacetime as a warped product and use a Weyl curvature invariant to show that the curvature of this spacetime is unbounded near the center of the black hole, but not near its event horizon.

It is pertinent to acknowledge that we utilize Maple software for assistance with some curvature calculations presented in this work.

1.1 Notation

This section outlines the specialized notation used throughout this thesis and commonly encountered in differential geometry texts. Refer to the accompanying list for a quick and easy guide to these unique mathematical symbols.

M	Manifold.
$\mathbb{R}^n_p$	All vectors in $\mathbb{R}$ originating at p.
$\mathfrak{F}(M)$	The set of all smooth real-valued functions on M.
$\mathfrak{X}(M)$	The set of all vector fields on M.
$\mathfrak{X}^*(M)$	The set of all smooth one forms on M.
$\mathfrak{T}^k_l$	The set of all mixed (k, l) tensor fields.
$\mathfrak{L}(M)$	The set of all lifts of vector fields on M.
T_pM	The set of all tangent vectors to M at $p \in M$.
$\mathscr{D}$	A derivation on $\mathfrak{F}(M)$.
∇	The Levi-Civita connection.
$\downarrow^a_b$	Lowering an index.
$\uparrow^a_b$	Raising an index.

In progressing through the more complicated content of this thesis, it might become apparent to readers that summation symbols are missing. Given the complex nature of tensors and the challenges associated with differentiating and computing curvature, many authors adopt the Einstein summation convention. Given the frequent appearance of summations represented by expressions such as $\sum_i x^i E_i$ in this study, we'll adopt a more concise notation.

$$E(x) = \sum_i x^i E_i \Rightarrow E(x) = x^i E_i$$

By this convention, if an index (for instance, i) is seen precisely two times in a monomial, once in the upper position and once in the lower, it implies a summation over all its potential values, typically ranging from 1 up to the dimension of the relevant space.

Chapter 2

Preliminary Content: Manifold Theory

In mathematics, manifolds consistently emerge as central objects of interest. Serving as multidimensional generalizations of curves and surfaces, they offer a comprehensive framework for grasping the concept of “space” in its myriad forms. Presently, manifold theory techniques have become integral to a plethora of subfields within pure mathematics. As such, we offer an informal notion to provide the reader an intuitive understanding of manifolds.

Suppose an observer is gazing at the full moon from Earth’s vantage point. If the observer zooms in on a very small portion of the moon’s surface and magnifies it, this fragment would resemble a flat piece paper. More precisely, this magnified image would resemble a patch of $\mathbb{R}^2$, the two-dimensional Euclidean space. This process of magnification and flattening can be repeated across the moon’s surface from different vantage points, creating a patchwork schematic of the moon, a concept generalized to the topological spaces we study. These local homeomorphisms provide coordinates near any point on our manifold. That is to say, we can create a map that transforms information such as points on the moon to coordinates that we can analyze. The moon’s surface is an example of a locally Euclidean topological space called a manifold.

2.1 Preliminaries

A coordinate system or a chart is a mapping φ from an open subset U on a manifold M to Euclidean space $\mathbb{R}^n$. The coordinate system is often denoted (U, φ) where the coordinate functions $(x^1, \ldots, x^n)$ of φ defined by $\varphi(p) = (x^1(p), \ldots, x^n(p))$ are the local coordinates on U [Lee12].

Definition 2.1. *The coordinate chart $\varphi : U \to \mathbb{R}^n$ is a homeomorphism. That is, for $\varphi : U \to \varphi(U)$:*

1. *φ is a bijection.*
2. *φ is continuous.*
3. *φ^{-1} is continuous.*

Definition 2.2. *The topological space M is a manifold of dimension n if:*

1. *M is Hausdorff space: There exists open disjoint subsets $U, V \subseteq M$, for every pair of disjoint points $p, q \in M$, such that $p \in U$ and $q \in V$.*
2. *M is second-countable: There exists a countable basis for the topology of M. That is, a countable collection of open sets such that every open set is the union of some subcollection of a countable basis.*
3. *M is equipped with a complete atlas $\mathscr{A}$: For every point p on M there is an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$ that contains every possible chart on M smoothly overlapping with every chart in $\mathscr{A}$.*

Property 3 of the Definition 2.2 is what allows us to preform calculus on the manifold. By passing to the homeomorphic images, the homeomorphism between a neighborhood U on M and $\mathbb{R}^n$ provides us with coordinates near any point p on the manifold M.

Before discussing the intricacies of manifolds and their characteristics, we must build a comprehensive understanding of the mappings of vectors and tangent vectors between spaces.

It is frequent that one requires a change in coordinates when working with multiple charts. A familiar example is switching from spherical to Cartesian coordinates. This process is done by using transition functions.

Let there be n and m dimensional open subsets U and V that overlap smoothly around a point p on M. Where the point p is in the intersection of U and V. There exists smooth charts $\varphi_U : U \to \mathbb{R}^n$ and $\varphi_V : V \to \mathbb{R}^m$ where $m = n$. In order to translate coordinates of p we must find a mapping between $\varphi_U(p)$ and $\varphi_V(p)$. By appropriately restricting the domains of these charts we may view the composition $\varphi_V \circ \varphi_U^{-1}$ as a map from the open set $\varphi_U(U \cap V)$ to $\varphi_V(U \cap V)$.

We consider the case of real-valued functions f on a manifold M. If $\varphi : U \to \mathbb{R}^n$ is a coordinate system in M, then the composite function $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^1$ is called the coordinate expression for f in terms of φ [O'n83]. We sometimes identify f with $f \circ \varphi^{-1}$ for convenience.

2.2 Generalizing $\mathbb{R}^n$ to M

We must introduce a new condition in order to effectively manage derivatives of functions involving real values, curves, or mappings. Such manifolds are called smooth manifolds.

Definition 2.3. *[Lee12] Let U be a an open set on $\mathbb{R}^n$. We define a real-valued function $\varphi : U \to \mathbb{R}$. The function φ is smooth if in any coordinate system, all mixed partial derivatives exist and are continuous.*

We measure a function's smoothness by the number of continuous derivatives for each function. We define the set of all smooth real-valued functions on M as $\mathfrak{F}(M)$. Throughout this thesis, all manifolds will be considered infinitely differentiable and smooth.

Using information we have about smooth manifolds, we have the means to calculate directional derivatives of functions in Euclidean space.

We denote all vectors in $\mathbb{R}^n$ originating at p as $\mathbb{R}^n_p$. Any tangent vector v yields a map $D_v|_p : \mathfrak{F}(\mathbb{R}^n) \to \mathbb{R}$, which takes the directional derivative in the direction of v at p. When referring to vectors or tangent vectors, it is the common convention to denote them as lover case letter such as v. Whereas vector fields and tangent vector fields are denoted with uppercase letters such as X and V as we will see in the upcoming sections.

$$D_v|_p f = D_v f(p) = \frac{d}{dt}|_{t=0} f(p + tv).$$

It is easy to see this operation is linear over $\mathbb{R}$ and satisfies the product rule. If $v = v^i e_i|_p$ in terms of the standard basis e_i for $\mathbb{R}^n$, then by the chain rule the expression can be written as:

$$D_v|_p f = v^i \frac{\partial f}{\partial x^i}(p).$$

This motivates the following definition.

Definition 2.4. *[Lee12] If p is a point of $\mathbb{R}^n$, then a map $w : \mathfrak{F}(\mathbb{R}^n) \to \mathbb{R}$ is a derivation at a point p if it is linear and satisfies the following product rule:*

$$w(fg) = f(p)w(g) + g(p)w(f).$$

The set of all derivations of $\mathfrak{F}(\mathbb{R}^n)$ at point p is denoted $T_p\mathbb{R}^n$. The following proposition is useful.

Proposition 2.5. *The map $v \mapsto D_v|_p$ is an isomorphism from $\mathbb{R}^n_p$ onto $T_p\mathbb{R}^n$.*

The set $T_p\mathbb{R}^n$ is a vector space under the usual conditions:

$$(w_1 + w_2)f = w_1 f + w_2 f, \text{ and}$$

$$(cw)f = c(wf).$$

Therefore we identify the tangent space of $\mathbb{R}^n$ at p as a set of derivation. The subsequent propositions follow naturally [Lee12].

Proposition 2.6. *Let $p \in \mathbb{R}^n$, $w \in T_p\mathbb{R}^n$, and $f, g \in \mathfrak{F}(\mathbb{R}^n)$.*

(a) If f is a constant function, then $wf = 0$.

(b) If $f(p) = g(p) = 0$, then $w(fg) = 0$.

(c) For each tangent vector $v \in \mathbb{R}^n_p$, the map $D_v|_p f : \mathfrak{F}(\mathbb{R}^n) \to \mathbb{R}$ defined uniquely by (a) and (b) is a derivation at point p.

A pivotal advancement in extending calculus from the familiar Euclidean space $\mathbb{R}^n$ to a more general manifold lies in the definition below.

Definition 2.7. *[O'n83]. Let M be a smooth manifold and p be a point of M. A tangent vector v to M at p is a real-valued function $v : \mathfrak{F}(M) \to \mathbb{R}$ that is*

(a) $\mathbb{R}$-linear: $v(af + bg) = av(f) + bv(g)$, and

(b) Leibnizian: $v(fg) = v(f)g(p) + f(p)v(g)$.

for all $a, b \in \mathbb{R}$ *and* $f, g \in \mathfrak{F}(M)$*. Recall that* $\mathfrak{F}(M)$ *is the set of all smooth real-valued functions on* M*.*

The set of all tangent vectors to M at p is denoted $T_p(M)$, the tangent space to M at p. The tangent space is a vector space over the real numbers under the usual definitions of functional addition and scalar multiplication.

Here, we outline several properties of tangent vectors on manifolds we have adopted from Lee [Lee12].

Lemma 2.8. *Suppose* M *is a smooth manifold,* $p \in M$*,* $v \in T_pM$*, and* $f, g \in \mathfrak{F}(M)$*.*

(a) If f *is a constant function, then* $vf = 0$*.*

(b) If $f(p) = g(p) = 0$*, then* $v(fg) = 0$*.*

Definition 2.9. *The derivation on the set of all smooth functions on* M *follows the same* $\mathbb{R}$*-linearity, and Leibnizian property as in Definition 2.7. That is, a derivation on* $\mathfrak{F}(M)$ *is a map* $\mathscr{D} : \mathfrak{F}(M) \to \mathfrak{F}(M)$ *that is* $\mathbb{R}$*-linear,* $\mathscr{D}(af+bg) = a\mathscr{D}(f)+b\mathscr{D}(g)$ *and Leibnizian,* $\mathscr{D}(fg) = \mathscr{D}(f)g(p) + f(p)\mathscr{D}(g)$*.*

To establish partial differentiation on a manifold, the approach involves mapping the function f back to Euclidean space through a coordinate system (chart), and then taking the usual derivatives.

Definition 2.10. *[O'n83] Let* $\varphi = (x^1, \ldots, x^n)$ *be a chart in* M *at* p*. If* $f \in \mathfrak{F}(M)$*, let*

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\varphi(p) \quad (1 \leq i \leq n),$$

where $(u^1, \ldots, u^n)$ *are the natural coordinate functions of* $\mathbb{R}^n$*.*

A computation then shows that the function

$$\partial_i|_p = \frac{\partial}{\partial x^i}(p) : \mathfrak{F}(M) \to \mathbb{R}$$

that sends $f \in \mathfrak{F}(M) \to (\partial f/\partial x^i)(p)$ is tangent to M at p. We can picture $\partial_i|_p$ as an arrow at p tangent to the x^i-coordinate curve through p.

Definition 2.11. *Let* $F : M \to N$ *be a smooth map between manifolds* M *and* N*, and let* $p \in M$*. The pushforward* dF_p *is a linear map from* T_pM *to* $T_{F(p)}N$ *characterized by*

$$dF_p(X)(h) = X_p(h \circ F),$$

for $h \in \mathfrak{F}(M)$*, and* $X \in T_pM$*.*

The map dF_p is sometimes called the differential of F at p.

One of the most widely studied manifolds are product manifolds. We will regularly refer to the example of $\mathbb{R}^1 \times \mathbb{R}^1 = \mathbb{R}^2$. To contemplate calculus on not just one manifold but the calculus on a product manifold, we must consider the calculus on the manifolds separately. If we denote $\pi : M \times N \to N$ sending (p, q) to p and $\sigma : M \times N \to N$ sending (p, q) to q and assume $f : P \to M \times N$ is smooth, then the tangent spaces:

$$T_{(p,q)}M \cong T_{(p,q)}(M \times \{q\}) \quad \text{and} \quad T_{(p,q)}N \cong T_{(p,q)}(\{p\} \times N),$$

are subspaces of the tangent space to $M \times N$ at (p, q).

Lemma 2.12. *[O'n83] $T_{(p,q)}(M \times N)$ is the direct sum of its subspaces $T_{(p,q)}M$ and $T_{(p,q)}N$. That is, each element of $T_{(p,q)}(M \times N)$ has a unique expression as $x + v$, where $x \in T_{(p,q)}M$ and $v \in T_{(p,q)}N$.*

It might be helpful to demonstrate some of the concepts found in this chapter with an example.

Example 2.13. *Consider the manifold $\mathbb{R}^2$, with (global) coordinates (x, y). Consider the point $(1, 2) \in \mathbb{R}^2$, and denote $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ as the familiar partial derivatives from multivariable calculus. Let $v = (3, 2)$ be a vector in $\mathbb{R}^2$. We recall that if $h \in \mathfrak{F}(\mathbb{R}^2)$, that the derivative of h in direction v at $(1, 2)$ is computed as*

$$D_v|_{(1,2)}(h) = 3\frac{\partial h}{\partial x}(1, 2) + 2\frac{\partial h}{\partial y}(1, 2).$$

For instance, if $h(x, y) = x^2y$, then

$$D_v|_{(1,2)}(h) = 3 \cdot (2xy)|_{(1,2)} + 2(x^2)|_{(1,2)} = 14.$$

Now consider the function $F : \mathbb{R}^2 \to \mathbb{R}$, given by

$$F(x, y) = x.$$

As we will consider later in the thesis, this is called a projection. For convenience, and to connect this example to the notation used earlier, denote the derivation $D_v|_{(1,2)}$ above as w. We wish to provide an example of how to compute the pushforward $dF|_{(1,2)}(v)$ by considering its effect on a smooth function $h \in \mathfrak{F}(\mathbb{R}^2)$. By definition,

$$dF|_{(1,2)}(v)(h) = w(h \circ F),$$

and $(h \circ F)(x,y) = x$. *Thus,*

$$\frac{\partial(h \circ F)}{\partial x} = \frac{\partial h}{\partial x}, \text{ and } \frac{\partial(h \circ F)}{\partial y} = 0.$$

So,

$$w(h \circ F) = 3\frac{\partial h}{\partial x}(1,2).$$

So in this way, whereas

$$w = 3\frac{\partial}{\partial x}|_{(1,2)} + 2\frac{\partial}{\partial y}|_{(1,2)},$$

we have

$$dF|_{(1,2)}(w) = 3\frac{\partial}{\partial x}|_{(1,2)}$$

since their action on any smooth function h *is the same.*

Chapter 3

Fields, Spaces, and Forms

In the preceding chapter, we navigated through the foundational aspects of manifold theory, discussing the structures and mappings that characterize manifolds. As we transition into exploring the realm of fields, spaces, and forms, our focus shifts towards understanding manifolds through the lens of vector fields.

A vector field on a manifold M can be envisioned as attaching a tangent vector to each point in M, thereby creating a field of vectors that permeates through the manifold.

Definition 3.1. *[O'n83] A vector field V on a manifold M is a function that assigns to each point $p \in M$ a tangent vector to M at p. If V is a vector field on M and $f \in \mathfrak{F}(M)$, then Vf denotes the real-valued function on M given by*

$$(Vf)(p) = V(f), \quad \text{for } p \in M.$$

Then V is smooth provided Vf is smooth for all $f \in \mathfrak{F}(M)$.

Definition 3.2. *If V and W are vector fields on M, and $f, h \in \mathfrak{F}(M)$, then we define the vector fields $(V + W)(h) = V(h) + W(h)$ and $(fV)(h) = f \cdot V(h)$.*

It is easy to see that if V and W are smooth, then so are $V + W$ and fV. The set of all smooth vector fields M is denoted $\mathfrak{X}(M)$.

3.1 Coordinate Vector Fields

Definition 3.3. *[O'n83] If $\varphi = (x^1, \ldots, x^n)$ is a coordinate system on $U \subset M$, then for each i with $1 \leq i \leq n$, the vector field ∂_i on U sending each p to $\partial_i|_p$ is called the* *coordinate vector field of φ. These vector fields are smooth since $\partial_i|_p(f) = \frac{\partial f}{\partial x^i}$.*

The following theorem, called the Basis Theorem, is what O'Neil refers to as the "the fundamental link between coordinates and tangent vectors".

Theorem 3.4. *(The Basis Theorem) [O'n83] If $\varphi = (x^1, \ldots, x^n)$ is a chart in M at p, then its coordinate vectors $\partial_1|_p, \ldots, \partial_n|_p$ form a basis for the tangent space T_pM, and*

$$v = \sum_{i=1}^{n} v(x^i)\partial_i|_p \quad \textit{for all } v \in T_pM.$$

In various works of literature, authors commonly represent this linear combination using the following notion in the equation below. For each index j, the components of v are given by the expression $v^j = v(x^j)$.

$$v = v^i \frac{\partial}{\partial x^i}|_p$$

The ordered basis denoted by $\left(\frac{\partial}{\partial x^i}|_p\right)$ serves as the coordinate basis for the tangent space T_pM. Simultaneously, we use $(v^1, \ldots, v^n)$ to represent the vector v in terms of its components with respect to this coordinate basis. Given our knowledge of the vector v, its components can be deduced through its interactions with the coordinate functions. Consider x^j as a smooth real-valued function defined on an open subset of M, then we characterize our equation by [Lee12]:

$$v(x^j) = \left(v^i \frac{\partial}{\partial x^i}\bigg|_p \right)(x^i) = v^i \frac{\partial x^j}{\partial x^i}(p) = v^j.$$

We further generalize the Basis Theorem to vector fields.

Corollary 3.5. *[O'n83] Using the same notation as in Theorem 3.4, for any vector field V on U:*

$$V = \sum V(x^i)\partial_i.$$

3.2 Navigating Interactions on Manifolds

Given a smooth map $F : M \to N$ between two smooth manifolds M and N, the pushforward of a vector field V on M is a new vector field F_*V on N defined by the action of V on functions from N to $\mathbb{R}$ as follows: For any smooth function $f : N \to \mathbb{R}$, the pushforward F_*V acts on f by $(F_*V)(f) = V(f \circ F)$. This definition ensures that the pushforward F_*V encapsulates the directional derivative information of V and transports it to the manifold N through the map F. The notation F_*V is sometimes denoted $dF(V)$ or $F_*(V)$. The pushforward $F_* : TM \to TN$ is a map that takes a tangent vector from the tangent space of M to a tangent vector in the tangent space of N.

Recall in Chapter 2 the discussion of tangent spaces on product manifolds. We can relate calculus of $M \times N$ to that of its factors by a notion of lifting.

Definition 3.6. *[O'n83] Let $\pi : M \times N \to M$ be the projection on the first factor described above: $\pi(p, q) = p$.*

1. *If $A \in \mathfrak{X}(M)$ then the lift of A to $M \times N$ is the vector field B whose values at each (p, q) is the lift of A_p to (p, q). We call the set of all such horizontal lifts B denoted $\mathfrak{L}(M)$.*

2. *If $f \in \mathfrak{F}(M)$, then the lift of f to $M \times N$ is $h = f \circ \pi \in \mathfrak{F}(M \times N)$.*

Similarly, we define vertical vector fields in the same way but using the projection $\sigma : M \times N \to N$ projection on xthe second factor. The vertical lifts are denoted $\mathfrak{L}(N)$. Both $\mathfrak{L}(M)$ and $\mathfrak{L}(N)$ are vector subspaces of $\mathfrak{X}(M \times N)$. As seen in the example at the end of Chapter 2, we sometimes do not distinguish between a vector field and its lift unless there is cause for confusion.

Having delved into the intricacies of pushforwards and the concept of lifting in the realm of smooth manifolds and tangent spaces, we now transition to another fundamental operation in differential geometry, the bracket operation. As we proceed, we will explore how the bracket operation contributes to defining curvature, as we will see in Chapter 5.

Definition 3.7. *[O'n83] Let $X, Y \in \mathfrak{X}(M)$. Then the differential operator called the bracket operation of X and Y assigns a smooth vector field on M by the smooth function $f : M \to \mathbb{R}$:*

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

A straightforward computations provides the following properties.

Lemma 3.8. *[O'n83] The bracket operation on $\mathfrak{X}(M)$ has the following properties:*

1. $\mathbb{R}$-*bilinearity:* $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ *and* $[Z, aX + bY] = a[Z, X] + b[Z, Y]$ *for* $a, b \in \mathbb{R}$.

2. *Skew-symmetry:* $[Y, X] = -[X, Y]$.

3. *Jacobi identity:* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

The bracket operation on $\mathfrak{X}(M)$, though $\mathbb{R}$-bilinear, is not $\mathfrak{F}(M)$-bilinear. In fact, it is easy to demonstrate that

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

It is important to note that there are special cases when the bracket is always zero [O'n83].

Corollary 3.9. *If* $X \in \mathfrak{L}(M)$, *then*

1. $[X, X] = 0$ *by the skew symmetry property.*

2. *For any two coordinate vector fields of the same coordinate system* $[\partial_i, \partial_j] = 0$, *since* $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$ *for smooth functions* f.

To provide a more concrete understanding see the proceeding example below.

Example 3.10. *Let* (x, y) *be the natural coordinates of the manifold* $M = \mathbb{R}^2$. *Consider the coordinate vector fields* $X = x\partial_y$ *and* $Y = y\partial_y$ *on* $\mathbb{R}^2$ *and a smooth function* h. *Then*

$$\begin{aligned}
[X, Y](h) &= X(Y(h)) - Y(X(h)) \\
&= x\partial_y(y\partial_y(h)) - y\partial_y(x\partial_y(h)) \\
&= x\partial_y\left(y\frac{\partial h}{\partial_y}\right) - y\partial_y\left(x\frac{\partial h}{\partial_y}\right) \\
&= xy\frac{\partial^2 h}{\partial_y^2} - x\frac{\partial h}{\partial_y} - yx\frac{\partial^2 h}{\partial_y^2} \\
&= -x\partial_y h \\
&= -X(h)
\end{aligned}$$

Thus $[X, Y] = -X$.

A third important instance of when the bracket of two vector fields is 0, is given in the corollary below. Recall the notion of lifting from Definition 3.6.

Corollary 3.11. *(Extension of Corollary 3.9)[O'n83]*

3. If $X \in \mathfrak{L}(M)$ and $V \in \mathfrak{L}(N)$ on the product manifold $M \times N$, then $[X, V] = 0$.

Proof. With a slight abuse of notation, we will prove $[X, V] = 0$. Given a product manifold $M \times N$, where $X \in \mathfrak{L}(M)$ and $V \in \mathfrak{L}(N)$, there is a function $h : M \times N \to \mathbb{R}$. Suppose (x_i) be coordinates in M and (v_j) be coordinates in N. Then by definition,

$$X = \sum a_i(x)\partial_{x_i} \quad \text{and,}$$
$$V = \sum b_i(v)\partial_{v_j}.$$

Then $X(V(h))$ can be expressed as:

$$\begin{aligned}
X(V(h)) &= X\left(\sum b_i(v) \cdot \partial_{v_j}(h)\right) \\
&= \sum X(b_i(v) \cdot \partial_{v_j}(h)) \\
&= \sum X(b_i(v))\partial_{v_j}(h) + b_i(v) \cdot X(\partial_{v_j}(h)) \\
&= \sum b_i(v) \cdot X(\partial_{v_j}(h)) \\
&= \sum b_i(v)a_i(x)\partial_{x_i}\partial_{v_j}(h).
\end{aligned}$$

In summary, we have expressed $X(V(h))$ through a sequence of steps, starting with the substitution of $V(h)$ and ending with the simplified form of the expression after applying the product rule. Lastly, since we are differentiating with respect to the direction of X, the term $X(b_i(v))\partial_{v_j}(h)$ reduces to zero.

We will do the same for $V(X(h))$ expressed in terms of coordinates:

$$\begin{aligned}
V(X(h)) &= V\left(\sum a_i(x) \cdot \partial_{x_i}(h)\right) \\
&= \sum V(a_i(x) \cdot \partial_{x_i}(h)) \\
&= \sum V(a_i(x))\partial_{x_i}(h) + a_i(x) \cdot V(\partial_{x_i}(h)) \\
&= \sum a_i(x) \cdot V(\partial_{x_i}(h)) \\
&= \sum a_i(x)b_i(v)\partial_{v_j}\partial_{x_i}(h).
\end{aligned}$$

By definition of the bracket operation,

$$\begin{aligned}[X,V] &= X(V(h)) - V(X(h)) \\ &= \sum b_i(v)a_i(x)\partial_{x_i}\partial_{v_j}(h) - \sum a_i(x)b_i(v)\partial_{v_j}\partial_{x_i}(h) \\ &= \sum b_i(v)a_i(x)\partial_{x_i}\partial_{v_j} - a_i(x)b_i(v)\partial_{v_j}\partial_{x_i}(h) \\ &= 0.\end{aligned}$$

□

3.3 The Dual Space

Tangent vectors provide a coordinate-independent approach to comprehending derivatives of objects on manifolds, tangent covectors correspond to the derivatives of real-valued functions defined over a manifold. Meaning, tangent covectors are linear functions on the tangent space at a point on a manifold. The space of all covectors at a point is a vector space, namely the cotangent space, commonly known as the a dual space to T_pM and is denoted T_p^*M. Typically we denote objects in a dual with Greek letters and objects in the vector space with lowercase Latin / Roman letters.

Let V be a real finite-dimensional vector space. We define a covector on V to be a real-valued linear functional on V, that is, a linear map $\varphi : V \to \mathbb{R}$. The space of all covectors on V itself is a real vector space under the operations of function addition and real scalar multiplication. The Kronecker delta symbol, denoted δ^i_j, is defined by

$$\delta^i_j = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

Proposition 3.12. *[Lee12] Let V be a finite-dimensional vector space. Given any basis $(E_1, \dots, E_n)$ for V, let $(\epsilon^1, \dots, \epsilon^n) \in V^*$ be the covectors defined by*

$$\epsilon^i(E_j) = \delta^i_j.$$

Then $\{\epsilon^1, \dots, \epsilon^n\}$ forms a basis for V^*. Therefore, dim$V$=dimV^*.

By the preceding proposition, we can express any arbitrary covectors in terms of a dual basis. That is, $\varphi \in V^*$ in terms of a dual basis is $\varphi = \varphi_i\epsilon^i$ where the components are determined by $\varphi_i = \varphi(E_i)$. The action of φ on a vector $v = v^j E_j$ is $\varphi(v) = \varphi_i v^i$. The convention is to write basis covectors with upper indices and components of covectors with lower indices [Lee12].

3.4 One Forms

Let M be a smooth manifold. For each $p \in M$, the cotangent space at p as T_p^*M is a dual space to T_pM as we saw above. Elements of T_p^*M are tangent covectors, sometimes just referred to as covectors or one forms at p, which are linear maps from T_pM to $\mathbb{R}$.

Definition 3.13. *[O'n83] A one form θ on a manifold M is a function that assigns to each point p an element θ_p of the cotangent space T_p^*M.*

A one form θ is smooth if θX is smooth for all $X \in \mathfrak{X}(M)$. The set of all smooth one forms on M is denoted $\mathfrak{X}^*(M)$. By Proposition 3.12, the one forms dx^i dual to ∂_i form a basis for T_p^*M.

3.5 Differentials

An operation exists, called the differential, which transforms functions into one forms, as mentioned in Chapter 2.

Definition 3.14. *[O'n83] The differential $d : \mathfrak{F}(M) \to T^*M$ of $f \in \mathfrak{F}(M)$ is the one form df such that $(df)(v) = v(f)$ for every tangent vector v to M.*

Lemma 3.15. *[O'n83] The differential has the following properties:*

1. *$d : \mathfrak{F}(M) \to \mathfrak{X}^*(M)$ is $\mathbb{R}$-linear.*

2. *Product rule: If $f, g \in \mathfrak{F}(M)$, then $d(fg) = gdf + fdg$.*

3. *If $f \in \mathfrak{F}(M)$ and $h \in \mathfrak{F}(\mathbb{R}^1)$, then $d(h(f)) = h'(f)df$.*

As observed previously, a smooth mapping induces a linear transformation on tangent vectors. Dualizing this concept leads to a linear map on convectors functioning in the reverse direction.

Consider a smooth mapping $F : M \to N$ between two manifolds, and let $p \in M$ be an arbitrary point. The differential $dF_p : T_pM \to T_{F(p)}N$, which gives rise to a corresponding dual linear map [Lee12]:

$$dF_p^* : T_{F(p)}^*N \to T_p^*M.$$

This is referred to as the pullback by F at p, and dF^*p is defined by the relationship

$$dF_p^*(\omega)(v) = \omega(dF_p(v)),$$

where $\omega \in T_{F(p)}^*N, v \in T_pM$.

Interestingly, covectors, which are one forms by Definition 3.13, exhibit a different behavior; namely, that covector fields pull back to covector fields, which are maps which produce a covector at each point on your manifold.

Given a smooth mapping $F : M \to N$ and a covector field ω defined on N, we define a covector field $F^*\omega$ as the pullback of ω by F, with its value at a point p determined by the following for $v \in \mathfrak{X}(M)$:

$$(F^*\omega)_p(v) = \omega_{F(p)}(dF_p(v)).$$

Hence, the value of the pullback covector field $F^*\omega$ at point p corresponds to the pullback of ω at the location $F(p)$ [Lee12].

3.6 Tensors

A generalization of real-valued functions, vector fields, and one forms are what are known as tensor fields on a manifold. Tensors and tensor fields provide us the means for describing objects on a manifold.

Lee introduces two distinct yet crucial definitions of tensors on a vector space. Firstly, tensors can be regarded as elements within tensor products of a dual space with itself. Essentially, a tensor product can be conceptualized as a mapping that assigns real values to a specified set of vectors or covectors. Secondly, they take the form of real-valued multilinear functions acting on multiple vectors or covectors. Both of these definitions hold significance and will have a pivotal role in our study.

Tensors manifest in various forms, yet their defining feature remains consistent: multilinearity. Our adopted definition stresses this fundamental property and seamlessly translates into the conventional coordinate representation of a tensor. Recall that a map is multilinear if it is linear as a function of each variable separately when the others are held fixed. As a reminder to the reader, a multilinear function of two variables is called bilinear. There are several familiar examples of multilinear functions, such as the dot product, cross product, and more recently discussed, the bracket operation.

Let us begin with a basic example. Let V be a real vector space, $v_1, v_2 \in V$, and $\omega, \mu \in V^*$. We define the tensor product of ω and μ as the function $\omega \otimes \mu : V \times V \to \mathbb{R}$ given by

$$\omega \otimes \mu(v_1, v_2) = \omega(v_1) \cdot \mu(v_2).$$

Since ω and μ are each linear, then $\omega \otimes \mu$ is multilinear, more precisely a bilinear function of v_1, v_2 [Lee12].

Example 3.16. *If* (e^1, e^2) *denotes the standard dual basis for* $(\mathbb{R}^2)^*$, *then* $e^1 \otimes e^2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ *is the bilinear function*

$$e^1 \otimes e^2((u, x), (y, z)) = uz.$$

We can generalize this to any arbitrary multilinear function. Consider the multilinear mapping of a covariant k-tensor on V:

$$F : \underbrace{V \times \ldots \times V}_{k \text{ copies (vectors)}} \to \mathbb{R}.$$

Similarly, a multilinear contravariant l-tensor on V^*,

$$F : \underbrace{V^* \times \ldots \times V^*}_{l \text{ copies (covectors)}} \to \mathbb{R}.$$

A multilinear map of mixed tensor type $\binom{k}{l}$ is defined as

$$F : \underbrace{V^* \times \ldots \times V^*}_{l \text{ copies}} \times \underbrace{V \times \ldots \times V}_{k \text{ copies}} \to \mathbb{R}.$$

Where the type of a tensor is the number of arguments it takes. For the above mixed type of $(l + k)$. The space of all covariant k, contravariant l, and mixed $\binom{k}{l}$, or commonly written as (k, l), tensor fields are denoted respectfully as $\mathfrak{T}^k(V), \mathfrak{T}_l(V), \mathfrak{T}^k_l(V)$.

Definition 3.17. *For a tensor $F \in \mathfrak{T}^k_l(V)$ and $G \in \mathfrak{T}^p_q(V)$ the tensor product of F and G denoted $F \otimes G$ of type $(k+p, l+q)$, we define as:*

$$F \otimes G \in \mathfrak{T}^{k+p}_{l+q}(V).$$

by

$$\begin{aligned}&(F \otimes G)(\omega^1, \ldots, \omega^{k+p}, v_1, \ldots, v_{l+q}) = \\ &\Rightarrow F(\omega^1, \ldots, \omega^k, v_1, \ldots, v_l) \cdot G(\omega^{k+1}, \ldots, \omega^{k+p}, v_{l+1}, \ldots, v_{l+q}).\end{aligned}$$

It is clear that the tensor product operation is associative but it is not commutative. However functions commute with tensors of any type. To be more precise, observe that, given a vector space $V : \omega \in \mathbb{R}$, then for the tensor fields F, G:

$$\omega F \otimes G = F \otimes \omega G.$$

This can be generalized for a smooth mapping $h : M \to N$ with tensor fields F, G

$$h(F \otimes G) = hF \otimes G = F \otimes hG.$$

Much like the approach used for covector fields, one can extend the idea of pulling back covariant tensor fields through a smooth mapping. This technique enables us to generate tensor fields within the domain of the map[1].

Definition 3.18. *Consider a smooth mapping $F : M \to N$. For any point $p \in M$ and any k-tensor $\omega \in \mathfrak{T}^k(T^*_{F(p)}N)$, we introduce a tensor denoted[2] as $dF^*(\omega) \in \mathfrak{T}^k(T^*_p M)$, which is referred to as the pullback of ω by F at p. This operation is defined by the relation:*

$$dF^*_p(\omega)(v_1, \ldots, v_k) = \omega(dF_p(v_1), \ldots, dF_p(v_k)),$$

where $(v_1, \ldots, v_k) \in T_p M$.

Definition 3.19. *In the scenario where A represents a covariant k-tensor field defined on N, we establish a corresponding k-tensor field F^*A on M, known as the pullback of A by F. This new field is constructed as [Lee12]:*

$$(F^*A)_p = dF^*(A_{F(p)}).$$

[1] It is important to note that this particular construction for pulling back tensor fields is applicable solely to covariant tensor fields. The distinctions between various types of tensor fields are not a focal point within this thesis but are explored in detail in [Lee12]. In particular, there is a natural way to transfer from tensors of any type from $M \to N$. When given an appropriate function $f : M \to N$.

[2] The pull back $dF^*(\omega)$ is sometimes just called F^*.

For any vectors $v_1, \ldots, v_k \in T_pM$, the action of this tensor field is defined by:

$$(F^*A)_p(v_1, \ldots, v_k) = A_{F(p)}(dF_p(v_1), \ldots, dF_p(v_k)).$$

There is a remarkable operation called contraction that shrinks (r, s) tensors to $(r-1, s-1)$ tensors. The general definition derives from the following special case.

Lemma 3.20. *[O'n83] There is a unique $\mathfrak{F}(M)$-linear function $C : \mathfrak{T}^1_1(M) \to \mathfrak{F}(M)$ called $(1,1)$ contraction, such that $C(X \otimes \theta) = \theta(X)$ for all $X \in \mathfrak{X}(M)$ and $\theta \in \mathfrak{X}^*(M)$.*

It is necessary to extend $(1,1)$ contraction C to tensors of higher type. The scheme is to specify one covariant slot and one contravariant slot, and apply C to these [O'n83]. This will be expanded upon in Chapter 5 when discussing curvature.

Chapter 4

Semi-Riemannian Manifolds

In our study of Semi-Riemannian geometry, we delve into the intricacies of a specific $(0,2)$ tensor defined on tangent spaces. We introduce a real finite dimensional vector space denoted V, and a real valued function on $V \times V$, denoted $b : V \times V \to \mathbb{R}$ defined below. Our focus lies solely on the symmetric case, where $b(v,w) = b(w,v)$ holds true for all v and w. Let us first acquaint ourselves with several fundamental statements introduced by O'Neil [O'n83].

Definition 4.1. *A symmetric bilinear form b on V is,*

1. *positive definite provided $v \neq 0$ implies $b(v,v) > 0$,*
2. *non-degenerate provided $b(v,w) = 0$, for all $w \in V$, implies $v = 0$.*

If $\{e_1, \ldots, e_n\}$ is a basis for V, then we may discuss the matrix relative to a bilinear form. If b is of such form, then we call its associated matrix $[b_{ij}]$, where $b_{ij} = b(e_i, e_j)$. Then, transitioning from the above definition, there is a pertinent lemma that clarifies the nature of non-degenerate symmetric bilinear forms and their matrices.

Lemma 4.2. *A symmetric bilinear form is non-degenerate if and only if its matrix relative to one (hence every) basis is invertible.*

For example, an inner product is a symmetric bilinear form for which, on an orthonormal basis, its associated matrix is diagonal and nonsingular. The literature sometimes refers to inner products and scalar products interchangeably, and we do so here as well.

We may extend the notion of an inner product on a vector space to an inner product on each tangent space of a manifold.

Lemma 4.3. *Let M be a manifold, and let $\langle \cdot, \cdot \rangle_p$ be an inner product on T_pM for each $p \in M$. If $\langle a, b \rangle_p = 0$ for all tangent vectors b on M, then $a = 0$.*

Corollary 4.4. *Given three tangent vectors a, b, and c on a manifold M, and a non-degenerate inner product at each point $p \in M$ $\langle \cdot, \cdot \rangle_p$, if $\langle b, a \rangle_p = \langle c, a \rangle_p$ for all vector fields a on M, then b must be equal to c.*

This corollary extends the preceding lemma, showcasing that if two vector fields exhibit identical inner products with all vector fields under a non-degenerate inner product, they must be the same vector field.

The index of a non-degenerate bilinear form is the dimension of a maximal subspace on which $b(v, v) < 0$ for all $v \neq 0$. This number is well defined by Sylvester's Law of Inertia.

Theorem 4.5. *(Sylvester's Law of Inertia) [FIS03] Let b be a symmetric bilinear form on a finite dimensional real vector space V. Then the number of positive diagonal entries and the number of negative diagonal entries in any associated matrix for b which is diagonal is independent of the basis chosen.*

4.1 Metrics

A metric on a manifold in essence, serves as a tool to quantify geometric properties. It provides a way of measuring distances as we will see in Definition 4.6 below. A metric g determines an inner product on each tangent space $T_p(M)$.

Definition 4.6. *[O'n83] A metric tensor g on a smooth manifold M is a symmetric non-degenerate $(0, 2)$ tensor field on M of constant index.*

Depending on the context we may refer to the inner product of vectors V and W as either $g(V, W)$ or $\langle V, W \rangle$. We denote the metric in terms of a coordinate basis on a manifold M as $g = \sum g_{ij} dx^i \otimes dx^j$ or more simply written, $g = g_{ij} dx^i \otimes dx^j$, where the components g_{ij} are smooth functions on M and $dx^i dx^j = dx^j dx^i$ is understood to be the symmetric tensor product of dx^i and dx^j. For example, the manifold $\mathbb{R}^n$ with global coordinates $(x_1, \ldots, x_n)$ can be endowed with the standard metric $g = dx^i dx^i$ [Lee06]. When a metric is expressed in the form $g = \sum g_{ij} dx^i dx^j$, sometimes authors refer to this as a "line element" instead of a metric. Thus, the terms line element and metric can be used interchangeably.

In other words $g \in \mathfrak{T}_2^0(M)$ smoothly assigns to each point p of M a scalar product g_p on a tangent space T_pM, and the index of g_p is the same for all p. Earlier we referred to the scalar product as a bilinear form b.

Proposition 4.7. *[O'n83] Let M be a manifold with metric g, let and $V \in \mathfrak{X}(M)$. Let V^* be the one form on M such that*

$$V^*(X) = g(V, X) \quad \textit{for all} \quad X \in \mathfrak{X}(M).$$

Then the function $V \to V^*$ is an $\mathfrak{F}(M)$-linear isomorphism for $\mathfrak{X}(M)$-linear isomorphism from $\mathfrak{X}(M)$ to $\mathfrak{X}^*(M)$.

Thus in the presence of a metric we can freely transform a vector field into a unique one-form and vice versa. Corresponding pairs $V \leftrightarrow \theta$ contain exactly the same information and are said to be metrically equivalent [O'n83].

Definition 4.8. *[O'n83] A semi-Riemannian manifold is a smooth manifold M furnished with a metric tensor g. The common value v of index g_p on a semi-Riemannian manifold M is called the index of $M : 0 \leq \nu \leq n = \dim M$.*

From this definition, two special cases emerge.

1. If $\nu = 0$, then M is a Riemannian manifold; each g_p is then a positive definite inner product on T_pM.

2. If $\nu = 1$, then M is a Lorentzian manifold. Lorentzian manifold holds significance for us in Chapter 7.

A pseudo-Riemannian manifold might refer to a manifold with metric tensor of any index. A semi-Riemannian metric g on a smooth manifold is characterized by its symmetric 2-tensor field g, which is non-degenerate at each point $p \in M$. In terms of a local frame, if $g = g_{ij}\varphi^i\varphi^j$, then non-degeneracy translates to the invertibility of the matrix g_{ij}, as demonstrated in Lemma 4.2.

It is worth noting that if a metric g is Riemannian, non-degeneracy naturally follows from its positive-definiteness. Therefore, every Riemannian metric also qualifies as a semi-Riemannian metric. However, not all semi-Riemannian metrics are Riemannian in nature. For instance, let us consider the metric $g = dx^2 - dy^2$ on $\mathbb{R}^2$ with global coordinates (x, y). This is is a semi-Riemannian manifold that does not meet the criteria of a Riemannian metric in this context, since $g(\partial_y, \partial_y) = -1 < 0$.

We can apply metrics to varying types of manifolds including product manifolds. This construction is called the product metric. Take for instance the Euclidean $n-$space, $\mathbb{R}^n$. This is the product manifold $\underbrace{\mathbb{R}^1 \times \ldots \times \mathbb{R}^1}_{n \text{ - factors}}$. To be more precise:

Lemma 4.9. *[O'n83] If M and N are manifolds, then the set of all product coordinate systems in $M \times N$ is an atlas on $M \times N$ making it the product manifold of M and N.*

The calculus on a product manifold $M \times N$ is a direct result of the individual calculus on M and N, as discussed in Chapter 2. This relationship is closely analogous to the calculus of the plane, $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$, which is fundamentally built from two copies of the real line.

Definition 4.10. *[O'n83] Let M and N be semi-Riemannian manifolds with metrics g_M and g_N. If π and σ are the projections of $M \times N$ onto M and N, respectively, then*

$$g(v, w) = g_M(d\pi(v), d\pi(w)) + g_N(d\sigma(v), d\sigma(w)).$$

We call g the product metric on $M \times N$.

4.2 Connections

Consider vector fields V and W defined on a semi-Riemannian manifold M. The goal is to demonstrate the procedure for constructing a new vector field, denoted $D_V W$ on M. At each point p, the vector field $D_V W$ characterizes the derivative of the vector field W along the direction defined by the vector field V. A natural method for achieving this objective is readily available within $\mathbb{R}^n$.

Definition 4.11. *[O'n83] Let $(x^1, \ldots, x^n)$ be the natural coordinates on $\mathbb{R}^n$ which we recall from Definition 2.10, and V and $W = W^i \partial_i$ are vector fields on $\mathbb{R}^n$. The vector* *field,*

$$D_V W = V(W^i)\partial_i,$$

is called the natural covariant derivative of W with respect to V.

Given that this formulation relies on unique coordinates within $\mathbb{R}^n$, its extension to an arbitrary semi-Riemannian manifold is not immediately apparent. Therefore we axiomize the key properties of the covariant derivative.

Definition 4.12. *(Properties of a connection D)[O'n83] A connection D on a smooth manifold M is a function $D : \mathfrak{X}(M) \times \mathfrak{X}(M)$ such that,*

1. *$D_V W$ is $\mathfrak{F}(M)$-linear in V,*
2. *$D_V W$ is $\mathbb{R}$-linear in W, and*
3. *$D_V(fW) = (Vf)W + fD_V W$ for $f \in \mathfrak{F}(M)$.*

$D_V W$ is called the covariant derivative of W with respect to V for the connection D.

Notice, Definition 4.11 satisfies Property 3 if $D_{\partial_i}\partial_j = 0$ for all i, j.

Theorem 4.13. *(Extension of Definition 4.12)[O'n83] On a smooth semi-Riemannian manifold M, there is a unique connection ∇ that is,*

4. *torsion free: $[V, W] = \nabla_V W - \nabla_W V$, and*
5. *metric compatible: $X\langle V, W\rangle = \langle \nabla_X V, W\rangle + \langle V, \nabla_X W\rangle$,*

for all $X, V, W \in \mathfrak{X}(M)$, called the Levi-Civita connection of M.

The Levi-Civita connection[1] is characterized by the Koszul formula:

$$\begin{aligned} 2\langle \nabla_V W, X\rangle =& V\langle W, X\rangle + W\langle X, V\rangle - X\langle V, W\rangle \\ & - \langle V, [W, X]\rangle + \langle W, [X, V]\rangle + \langle X, [V, W]\rangle. \end{aligned}$$

When $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$ the Koszul formula is written in the following way:

$$2\langle \nabla_{\partial_i}\partial_j, \partial_k\rangle = \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}.$$

[1]Some authors do not distinguish ∇ differently than D and often times use D for both connections and Levi-Civita connections.

Given a Levi-Civita connection expressed in a coordinate basis, the coefficients are given by Christoffel symbols of the first kind. In order to compute covariant derivatives for specific coordinate system we adopt the following definition.

Definition 4.14. *[O'n83] Let* $(x^1, \ldots, x^n)$ *be a coordinate system on a neighborhood* U *in a semi-Riemannian manifold* M *and* ∇ *be a connection on* U*. The Christoffel symbols of the first kind for this coordinate system are real-valued functions* ${\Gamma_{ij}}^k$ *on* U *such that*

$$\nabla_{\partial_i}\partial_j = \sum {\Gamma_{ij}}^k \partial_k \quad (1 \leq i, k \leq n).$$

By definition, the Levi-Civita connection is torsion free. Hence, $\nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$ and it follows that ${\Gamma_{ij}}^k = {\Gamma_{ji}}^k$.

It is important to note that by Property 3 in Definition 4.12, the connection ∇ is not a tensor, so the Christoffel symbols do not obey the usual tensor transformation rule under change of coordinates.

Proposition 4.15. *[O'n83] For a coordinate system* $(x^1, \ldots, x^n)$ *on* U *on a manifold* M*, a metric* g*, and a Levi-Civita Connection* ∇*,*

$$\nabla\partial_i\left(\sum W^j\partial_j\right) = \sum_k \left\{\frac{\partial W^k}{\partial x^i} + \sum_j {\Gamma_{ij}}^k W^j\right\}\partial_k.$$

Hence, the Christoffel symbols are given by,

$${\Gamma_{ij}}^k = \frac{1}{2}\sum_m g^{km}\left\{\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m}\right\}.$$

There is another way of expressing Christoffel symbols. We refer to this alternate form as the second kind[2].

Definition 4.16. *Christoffel symbols of the second kind are given by*

$$\Gamma_{ijk} = \frac{1}{2}\left(\partial_i g(\partial_j, \partial_k) + \partial_j g(\partial_i, \partial_k) - \partial_k g(\partial_i, \partial_j)\right) = g(\nabla_{\partial_i}\partial_j, \partial_k).$$

We will simplify this expression in the following way for ease of notation. We define the one jets $g_{ij/k}$.

$$\Gamma_{ijk} = \frac{1}{2}(g_{jk/i} + g_{ik/j} - g_{ij/k}).$$

[2]There are some disagreements on which Christoffel symbols are referred to as of the first or second kind. The reader may find in other literature the reverse.

One might use both Christoffel symbols of the first and second kind to calculate the covariant derivatives given a metric. To illustrate this procedure, let us construct a specific example. This example will serve as a recurring reference in the subsequent sections of this thesis. To demonstrate how to put these concepts in practice.

In our approach, we first compute the Christoffel symbols of the second kind and from these, ascertain the symbols of the first kind. Then we utilize the first kind Christoffel symbols to derive the covariant derivatives. It will suffice to explicitly calculate two of covariant derivatives along with their corresponding Christoffel symbols, while the others, as a similar computation, will be provided without computation.

Let M be a product of the two manifolds $\mathbb{R}^+$ and S^2. The manifold $\mathbb{R}^+$ is the set of all positive real numbers and the manifold S^2 is the 2-sphere. We express this product in the following way, $M = \mathbb{R}^+ \times S^2$. We define a semi-Riemannian metric g on M with the following nonzero entries:

$$\begin{aligned} g(\partial_r, \partial_r) &= 1, \\ g(\partial_\theta, \partial_\theta) &= r^2 \sin^2(\varphi), \\ g(\partial_\varphi, \partial_\varphi) &= r^2. \end{aligned}$$

Remark. *Often times authors will denote this metric in the following way:*

$$g = dr \otimes dr + r^2 \sin^2(\varphi) d\theta \otimes d\theta + r^2 d\varphi \otimes d\varphi.$$

We will derive both $\nabla_{\partial_r}\partial_\theta$ and $\nabla_{\partial_r}\partial_\varphi$. The scheme of this computation will unfold in four segments per covariant derivative.

Example 4.17. *1. Solving for* $\nabla_{\partial_r}\partial_\theta = \nabla_{\partial_\theta}\partial_r$.

By definition:

$$\nabla_{\partial_r}\partial_\theta = \nabla_{\partial_\theta}\partial_r = {\Gamma_{\theta r}}^{r}\partial_r + {\Gamma_{\theta r}}^{\theta}\partial_\theta + {\Gamma_{\theta r}}^{\varphi}\partial_\varphi.$$

(a) Christoffel symbols of the first kind:

$$\begin{aligned} g(\nabla_{\partial_\theta}\partial_r, \partial_r) &= {\Gamma_{\theta r}}^{r} g(\partial_r, \partial_r) + {\Gamma_{\theta r}}^{\theta} g(\partial_\theta, \partial_r) + {\Gamma_{\theta r}}^{\varphi} g(\partial_\varphi, \partial_r), \\ g(\nabla_{\partial_\theta}\partial_r, \partial_r) &= {\Gamma_{\theta r}}^{r} \cdot (1). \end{aligned}$$

Given the specified metric, the other terms involving the Christoffel symbols *are identically zero.*

$$g(\nabla_{\partial_\theta}\partial_r, \partial_\theta) = {\Gamma_{\theta r}}^{r} g(\partial_r, \partial_\theta) + {\Gamma_{\theta r}}^{\theta} g(\partial_\theta, \partial_\theta) + {\Gamma_{\theta r}}^{\varphi} g(\partial_\varphi, \partial_\theta),$$
$$g(\nabla_{\partial_\theta}\partial_r, \partial_\theta) = {\Gamma_{\theta r}}^{\theta} \cdot (r^2 \sin^2(\varphi)).$$

Similar to above, the other two terms involving the Christoffel symbols vanish, since by the metric, they are equal to 0.

$$g(\nabla_{\partial_\theta}\partial_r, \partial_\varphi) = {\Gamma_{\theta r}}^{r} g(\partial_r, \partial_\varphi) + {\Gamma_{\theta r}}^{\theta} g(\partial_\theta, \partial_\varphi) + {\Gamma_{\theta r}}^{\varphi} g(\partial_\varphi, \partial_\varphi),$$
$$g(\nabla_{\partial_\theta}\partial_r, \partial_\varphi) = {\Gamma_{\theta r}}^{\varphi} \cdot (r^2).$$

(b) *Christoffel symbol of the second kind:*

$$\Gamma_{\theta rr} = \frac{1}{2}(g_{rr/\theta} + g_{\theta r/r} - g_{\theta r/r}),$$
$$\Gamma_{\theta rr} = \frac{1}{2}(g_{rr/\theta}),$$
$$\Gamma_{\theta rr} = \frac{1}{2}\partial_\theta \cdot (1),$$
$$\Gamma_{\theta rr} = 0.$$

By the metric, the only one jet that does not vanish is $g_{rr/\theta}$.

$$\Gamma_{\theta r\theta} = \frac{1}{2}(g_{r\theta/\theta} + g_{\theta\theta/r} - g_{\theta r/\theta}),$$
$$\Gamma_{\theta r\theta} = \frac{1}{2}(g_{\theta\theta/r}),$$
$$\Gamma_{\theta r\theta} = \frac{1}{2}(r^2 \sin^2(\varphi))\partial_r,$$
$$\Gamma_{\theta r\theta} = r \sin^2(\varphi).$$

By the metric, the only one jet that does not vanish is $g_{\theta\theta/r}$.

$$\Gamma_{\theta r\varphi} = \frac{1}{2}(g_{r\varphi/\theta} + g_{\theta\varphi/r} - g_{\theta r/\varphi}),$$
$$\Gamma_{\theta r\varphi} = \frac{1}{2} \cdot (0),$$
$$\Gamma_{\theta r\varphi} = 0.$$

(c) *Now we solve for the Christoffel symbols of the first kind using the second kind:*

$$\begin{aligned}
{\Gamma_{\theta r}}^{r} \cdot (1) &= \Gamma_{\theta r r} = 0, \\
&\Rightarrow {\Gamma_{\theta r}}^{r} = 0. \\
{\Gamma_{\theta r}}^{\theta}(r^2 \sin^2(\varphi)) &= \Gamma_{\theta r \theta} = r \sin^2(\varphi), \\
&\Rightarrow {\Gamma_{\theta r}}^{\theta} = \frac{r \sin^2(\varphi)}{r^2 \sin^2(\varphi)} = \frac{1}{r}. \\
{\Gamma_{\theta r}}^{\varphi} \cdot (r^2) &= \Gamma_{\theta r \varphi} = 0, \\
&\Rightarrow {\Gamma_{\theta r}}^{\varphi} = 0.
\end{aligned}$$

(d) *Lastly, we simplify:*

$$\begin{aligned}
\nabla_{\partial_r}\partial_\theta &= \nabla_{\partial_\theta}\partial_r = {\Gamma_{\theta r}}^{r}\partial_r + {\Gamma_{\theta r}}^{\theta}\partial_\theta + {\Gamma_{\theta r}}^{\varphi}\partial_\varphi, \\
\nabla_{\partial_r}\partial_\theta &= \nabla_{\partial_\theta}\partial_r = (0)\partial_r + \frac{1}{r}\partial_\theta + (0)\partial_\varphi, \\
\nabla_{\partial_r}\partial_\theta &= \nabla_{\partial_\theta}\partial_r = \frac{1}{r}\partial_\theta.
\end{aligned}$$

2. *Solving for:* $\nabla_{\partial_r}\partial_\varphi = \nabla_{\partial_\varphi}\partial_r$.

   By definition:

$$\nabla_{\partial_r}\partial_\varphi = \nabla_{\partial_\varphi}\partial_r = {\Gamma_{r\varphi}}^{r}\partial_r + {\Gamma_{r\varphi}}^{\theta}\partial_\theta + {\Gamma_{r\varphi}}^{\varphi}\partial_\varphi.$$

(a) *Christoffel symbols of the first kind:*

$$\begin{aligned}
g(\nabla_{\partial_r}\partial_\varphi, \partial_r) &= {\Gamma_{r\varphi}}^{r} g(\partial_r, \partial_r) + {\Gamma_{r\varphi}}^{\theta} g(\partial_\theta, \partial_r) + {\Gamma_{r\varphi}}^{\varphi} g(\partial_\varphi, \partial_r) = {\Gamma_{r\varphi}}^{r} \cdot (1). \\
g(\nabla_{\partial_r}\partial_\varphi, \partial_\theta) &= {\Gamma_{r\varphi}}^{r} g(\partial_r, \partial_\theta) + {\Gamma_{r\varphi}}^{\theta} g(\partial_\theta, \partial_\theta) + {\Gamma_{r\varphi}}^{\varphi} g(\partial_\varphi, \partial_\theta) = {\Gamma_{r\varphi}}^{\theta}(r^2 \sin^2(\varphi)). \\
g(\nabla_{\partial_r}\partial_\varphi, \partial_\varphi) &= {\Gamma_{r\varphi}}^{r} g(\partial_r, \partial_\varphi) + {\Gamma_{r\varphi}}^{\theta} g(\partial_\theta, \partial_\varphi) + {\Gamma_{r\varphi}}^{\varphi} g(\partial_\varphi, \partial_\varphi) = {\Gamma_{r\varphi}}^{\varphi} \cdot (r^2).
\end{aligned}$$

(b) *Christoffel symbols of the second kind:*

$$\begin{aligned}
\Gamma_{r\varphi r} &= \frac{1}{2}(g_{\varphi r/r} + g_{rr/\varphi} - g_{r\varphi/r}) = \frac{1}{2}(g_{rr/\varphi}) = \frac{1}{2}\partial_\varphi \cdot (1) = 0. \\
\Gamma_{r\varphi\theta} &= \frac{1}{2}(g_{\varphi\theta/r} + g_{r\theta/\varphi} - g_{r\varphi/\theta}) = \frac{1}{2} \cdot (0) = 0. \\
\Gamma_{r\varphi\varphi} &= \frac{1}{2}(g_{\varphi\varphi/r} + g_{r\varphi/\varphi} - g_{r\varphi/\varphi}) = \frac{1}{2}(g_{\varphi\varphi/r}) = \frac{1}{2} \cdot (r^2)\partial_r = r.
\end{aligned}$$

(c) Solve the Christoffel symbols of the first kind using the second kind:

$$\begin{aligned} \Gamma_{r\varphi}{}^{r} \cdot (1) &= \Gamma_{r\varphi r} = 0, \\ &\Rightarrow \Gamma_{r\varphi}{}^{r} = 0. \\ \Gamma_{r\varphi}{}^{\theta} \cdot (r^2 \sin^2(\varphi)) &= \Gamma_{r\varphi\theta} = 0, \\ &\Rightarrow \Gamma_{r\varphi}{}^{\theta} = 0. \\ \Gamma_{r\varphi}{}^{\varphi} \cdot (r^2) &= \Gamma_{r\varphi\varphi} = r = \frac{r}{r^2} = \frac{1}{r}, \\ &\Rightarrow \Gamma_{r\varphi}{}^{\varphi} = \frac{1}{r}. \end{aligned}$$

(d) Simplify.

$$\begin{aligned} \nabla_{\partial_r}\partial_\varphi &= \nabla_{\partial_\varphi}\partial_r = \Gamma_{r\varphi}{}^{r}\partial_r + \Gamma_{r\varphi}{}^{\theta}\partial_\theta + \Gamma_{r\varphi}{}^{\varphi}\partial_\varphi, \\ \nabla_{\partial_r}\partial_\varphi &= \nabla_{\partial_\varphi}\partial_r = (0)\partial_r + (0)\partial_\theta + \frac{1}{r}\partial_\varphi, \\ \nabla_{\partial_r}\partial_\varphi &= \nabla_{\partial_\varphi}\partial_r = \frac{1}{r}\partial_\varphi. \end{aligned}$$

The other three covariant derivatives are computed in the same way. We find the nonzero covariant derivatives for the given coordinate vector fields on $M = \mathbb{R}^+ \times S^2$ are:

1. $\nabla_{\partial_r}\partial_\theta = \nabla_{\partial_\theta}\partial_r = \frac{1}{r}\partial_\theta$.

2. $\nabla_{\partial_r}\partial_\varphi = \nabla_{\partial_\varphi}\partial_r = \frac{1}{r}\partial_\varphi$.

3. $\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = \frac{\cos(\varphi)}{\sin(\varphi)}\partial_\theta$.

4. $\nabla_{\partial_\theta}\partial_\theta = -r\sin^2(\varphi)\partial_r - \sin(\varphi)\cos(\varphi)\partial_\varphi$.

5. $\nabla_{\partial_\varphi}\partial_\varphi = -r\partial_r$.

Chapter 5

Curvature

5.1 Curvature Tensors

This chapter will provide us the formality needed to understand how to compute curvature on manifolds.

Definition 5.1. *[O'n83] Let M be a semi-Riemannian manifold with Levi-Civita connection ∇. The function $R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$ given by*

$$R_{XY}Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z.$$

is a $(1, 3)$ tensor field on M and is called the Riemannian curvature tensor of M.

Remark. *[O'n83] The term "curvature tensor" and "curvature operator" are sometimes used interchangeably, for reasons we will see shortly. O'Neil's definition of the Riemannian curvature tensor is not the only reasonable possibility; one can change its sign defining curvature to be $\mathscr{R} = -R$. For example, $\mathscr{R}_{XY}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.*

Curvature R on coordinate vector fields is computed in the following way:

$$R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k,$$

since $[\partial_i, \partial_j] = 0$.

The following two definitions regarding curvature on manifolds will play a pivotal role in the upcoming chapter on warped products.

A semi-Riemannian manifold is termed flat when there is a coordinate chart about each point for which the metric tensor entries on the coordinate vector fields is constant. The following is a classical result we shall use in the presentation of Theorem 6.9, one of our main results.

Theorem 5.2. *[Lee06] A semi-Riemannian manifold is flat if and only if its curvature tensor vanishes identically.*

That is, $R = 0$ if and only if for all $p \in M$, there exists a coordinate system $(x_1, \ldots, x_n)$ near p with $g(\partial_i, \partial_j) = \pm\delta_{ij}$. Recall Definitions 4.14 and 4.16. If $\nabla_{\partial_i}\partial_j = 0$ at every point, then the Christoffel symbols of the second kind in the form

$$\Gamma_{ijk} = \frac{1}{2}(g_{jk/i} + g_{ik/j} - g_{ij/k}) = 0$$

vanish. In turn, ${\Gamma_{ij}}^k = 0$. We refer to this metric as the flat metric.

Definition 5.3. *Suppose M is equipped with a metric g, and the curvature*

$$R_{XY}Z = \lambda[g(Y,Z)X - g(X,Z)Y],$$

for some $\lambda \in \mathbb{R}$. Then we say M has constant sectional curvature.

This definition will be of use in the subsequent chapter involving the curvature of warped products.

The tensor R can be considered as an $\mathbb{R}$-multilinear function on individual tangent vectors. The curvature operator R is a (1,3) tensor. If the given the coordinates are $(x^1, \ldots, x^n)$, then R is

$$R = {R_{ijk}}^l dx^i \otimes dx^j \otimes dx^j \otimes dx^k \otimes \partial_l.$$

The components of ${R_{ijk}}^l$ are as follows:

$$R(\partial_i, \partial_j)\partial_k = {R_{ijk}}^l \partial_l.$$

The following identities are symmetries of curvature.

Proposition 5.4. *(Symmetries)[O'n83] For any indices i, j, k, l*

1. ${R_{ijk}}^l = -{R_{jik}}^l$,
2. ${R_{ijk}}^l = -{R_{jil}}^k$,
3. ${R_{ijk}}^l = {R_{kli}}^j$,

4. $R_{ijk}{}^{l} = R_{lkj}{}^{i}$,

5. $R_{ijk}{}^{l} + R_{jki}{}^{l} + R_{kli}{}^{j} = 0$.

Using Proposition 4.7, we can change the type of the curvature operator of type (1,3) to a tensor of type (0,4), sometimes also referred to as the curvature tensor. This process is called "type changing" and is described below.

The operation of lowering an index is denoted $\downarrow^a_b$: $\mathfrak{T}^r_s(M) \to \mathfrak{T}^{r-1}_{s+1}(M)$. The inverse operation of raising an index $\uparrow^a_b$, extracts the ath one-form and inserts its metrically equivalent vector field in the bth slot among the vector fields[1]. In coordinates, the vector field metrically equivalent to dx^i is $g^{ij}\partial_j$ where g^{ij} is the (i,j) entry for (g^{-1}).

The inherent nature of transforming tensors through index manipulation is so intuitive that it often takes place in practical applications without drawing explicit attention. Since tensors obtained from a given tensor by the raising and lowering operations are metrically equivalent, then they contain all the same information, and hence can be viewed as different manifestations of a single object.

When the curvature tensor $R : \mathfrak{X}(M)^3 \to \mathfrak{X}(M)$, as described in Lemma 5.1, is written in the ordinary way as a function of three vector fields, the classical index pattern demands $R(X,Y,Z) = R_{XY}Z$. The components of the $(0,4)$ tensor $\downarrow^1_1 R$ are then given by [O'n83]:

$$R_{ijkl} = (\downarrow^1_1 R)(\partial_i, \partial_j, \partial_k, \partial_l) = \langle \partial_i, R_{\partial_k \partial_l}(\partial_j)\rangle = g_{lm}R_{ijk}{}^{m}.$$

Let us revisit Lemma 3.20 presented in Chapter 3, where the concept of tensor contraction was mentioned. This process can be extended to the realm of smooth manifolds, providing a broader context and application. To clarify, contraction is a mathematical operation applied to a tensor, specifically affecting one contravariant and one covariant slot, with the outcome being a reduction from an (r,s) tensor to an $(r-1,s-1)$ tensor. On a semi-Riemannian manifold we can metrically contract two covariant indices by first raising either one of them and then contracting in the usual way. Thus for $1 \le a < b \le s$ and an arbitrary r, the metric contraction $C_{ab} : \mathfrak{T}^r_s(M) \to \mathfrak{T}^r_{s-2}(M)$ is given in coordinates by [O'n83]:

$$(C_{ab}A)^{i_1 \ldots i_r}_{j_1 \ldots j_{s-2}} = g^{pq} A^{i_1 \ldots i_r}_{j_1 \ldots p \ldots q \ldots j_{s-2}}.$$

[1] This raising and lowering operation is often referred to as Index Gymnastics and designated by musical notations $\flat$ and $\sharp$. One says that $X^\flat$ is obtained from X by lowering an index and $\omega^\sharp$ is obtained by raising the index. The reader can find an explanation of this in [Lee12].

Type changing and covariant differentiation commute with contraction and contraction of Riemannian curvature yields simpler invariants as we will see in the following section.

5.2 Ricci Tensor

In order to retrieve important information in the curvature tensor, it is useful to construct tensors that are simpler due to the complexity of 4 tensors. One such tensor is the Ricci curvature tensor, denoted *Rc* or *Ric* depending on the literature. The Ricci curvature tensor is a symmetric covariant 2-tensor field defined as the contraction of the curvature operator on its first and last indices. The components of *Ric* are usually denoted R_{ij} such that:

$$R_{ij} := {R_{kij}}^k = g^{km}R_{kijm},$$

and can be expressed in any of the following ways [Lee06]:

$$R_{ij} = {R_{kij}}^k = {R_{ikj}}^k = -{R_{kij}}^k = -{R_{ikj}}^k.$$

Lemma 5.5. *[O'n83] The Ricci curvature tensor Ric is symmetric, and relative to an orthonormal frame field* $\{E_i\}$ *as,*

$$\sum_m \epsilon_m R(E_m, Y, X, E_m) = Ric(X, Y) = \sum_m \epsilon_m R(E_m, X, Y, E_m),$$

where $\epsilon_m = \langle E_m, E_m \rangle = \pm 1.$

Proof. Consider an orthonormal frame field denoted by $\{E_i\}$. We express the curvature tensor as $R_{ijkl} = R(E_i, E_j, E_k, E_l)$. The associated metric components for this basis are given by $g_{ij} = g^{ij} = \partial_{ij}\epsilon_i$. So, R_{ij} can be represented as $R_{ij} = g^{km}R_{kijm} = \sum \epsilon_m R_{mijm}$. The lemma is confirmed to be true in the case where $X = E_i$ and $Y = E_j$. However, the Ricci tensor is multilinear so the lemma follows for all X and Y since it is true for a basis. □

Morgan presents an alternate perspective on this matter. They emphasize that while the Ricci curvature is a symmetric bilinear form on T_pM, it can also be defined as a trace over the Riemannian curvature tensor. Encouraging a deeper intuition, Morgan suggests envisioning R_{ijkl} as a "matrix of matrices". Consequently, R_{jl} would represent the matrix of these traces [Mor98].

$$\begin{bmatrix} [R_{i1k1}] & [R_{i2k2}] & \dots & [R_{i1km}] \\ \vdots & \vdots & \vdots & \vdots \\ [R_{imk1}] & [R_{imk2}] & \dots & [R_{imkm}] \end{bmatrix}.$$

If the Ricci tensor is identically zero, M is said to be Ricci flat. A flat manifold is certainly Ricci flat, but the converse does not hold, as we will see in Chapter 7. This fact, and the Ricci curvature tensor in general, holds a predominant role in the field of differential geometry, particularly in general relativity. We will illustrate the significance in Chapter 7.

Chapter 6

Warped Products

As highlighted by the title of this thesis, we have arrived at our main goal, where we unveil the intricate curvature characteristics inherent to warped product manifolds. Leveraging the extensive groundwork laid in preceding chapters, we are now prepared to articulate a precise definition of warped product manifolds, and subsequently, conduct a comprehensive investigation into their curvature properties.

Let B and F be manifolds, and let π and σ be projections from $B \times F$ onto the first and second factor, respectively. Let g_B and g_F be metrics on B and F respectively, with Levi-Civita connections ${}^B\nabla$ and ${}^F\nabla$ on B and F respectively. The metric tensor defined as $\pi^*(g_B) + \sigma^*(g_F)$ is called the product metric on $B \times F$. In this context, B is named the base while F is the fiber. The term fibers refers to $\{p\} \times F = \pi^{-1}(\{p\})$, whereas leaves are designated as $B \times \{q\} = \sigma^{-1}(\{q\})$. We imagine the leaves are orientated horizontally, while we imagine vectors tangent to the fibers as vertical.

A central aim here is to shed light on the geometric attributes of the manifold M by harnessing the power of a function $f \in \mathfrak{F}(B)$, combined with the geometries of both B and F.

Definition 6.1. *[O'n83] Suppose B and F are semi-Riemannian manifolds, and let $f > 0$ be a smooth function on B. The warped product $M = B \times_f F$ is the product manifold $B \times F$ furnished with the metric tensor*

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F).$$

The function f is sometimes called the warping function.

That is, a warped product is a product manifold equipped with a metric g on both manifolds and a smooth function f, denoted more informally as $g = g_B + f^2 g_F$, indicating that the tangent space of B and F are orthogonal, and distances in the fiber are scaled as a function of the base. We start with the following observations:

1. For each $q \in F$, the restricted map $\pi|_{B\times\{q\}}$ is an isometry onto B.

2. For each $p \in B$, the restricted map $\sigma|_{\{p\}\times F}$, that is, for every p in the set B, the restriction of the map σ to the product $p \times F$ constitutes a direct similarity transformation onto F, that is, the map scales distances in F.

3. For each $(p, q) \in M$, the tangent spaces of the leaf $B \times q$ and the fiber $p \times F$ are orthogonal at (p, q).

We denote by $\mathscr{H}$ the orthogonal projection of $T_{(p,q)}(M)$ onto its horizontal subspace $T_{(p,q)}(B \times q)$ and by $\mathscr{V}$ the projection onto the vertical subspace $T_{(p,q)}(p \times F)$. Recall the notion of lift of a vector field on B or F to $B \times F$, the set of all such lifts are denoted $\mathfrak{L}(B)$ and $\mathfrak{L}(F)$ respectively as previously discussed in Definition 3.6 [O'n83].

Note that if the warping function $f = 1$, then $B \times_f F$ reduces to a semi-Riemannian product manifold. The less general form of warped product manifold is a product manifold $M \times N$ which allows the metric on N to vary in a more general way[1].

6.1 The Levi-Civita Connection on Warped Products

The Levi-Civita connection of M, discussed subsequently, can be related to those of B and F. We will use the following common definition in Proposition 6.3.

Definition 6.2. *The gradient of a function $f \in \mathfrak{F}(M)$ denoted as grad(f), is the vector field metrically equivalent to the differential $df \in \mathfrak{X}^*(M)$. Thus, by Proposition 4.7,*

$$\langle \mathrm{grad}(f), X\rangle = df(X) = Xf \quad \textit{for all } X \in \mathfrak{X}(M).$$

In terms of a coordinate system $df = \sum(\partial f/\partial x^i)dx^i$, hence

$$\mathrm{grad}(f) = \sum_{i,j} g^{ij}\frac{\partial f}{\partial x^i}\partial_j.$$

[1]This subject is widely recognized and constitutes a prominent theme extensively explored in the work of [Bes07].

Proposition 6.3. *[O'n83] On $M = B \times_f F$, if $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$, then*

1. $\nabla_X Y \in \mathfrak{L}(B)$ *is the lift of* ${}^B\nabla_X Y$ *on* B.
2. $\nabla_X V = \nabla_V X = \left(\frac{X(f)}{f}\right) V$.
3. $\nabla_V W = \left(\frac{\langle V,W\rangle}{f}\right) grad(f) + {}^F\nabla_V W$.

In the ensuing proof, we will illustrate the explicit computation resulting in the three covariant derivatives given above.

Proof. This proof requires the employment of the Koszul formula appearing after Theorem 4.13. To ensure the reader with clarity and avoid ambiguity, we rewrite the Koszul formula with the vector fields denoted by a, b, c.

$$2\langle \nabla_a b, c\rangle = a\langle b, c\rangle + b\langle c, a\rangle - c\langle a, b\rangle - \langle a, [b, c]\rangle + \langle b, [c, a]\rangle + \langle c, [a, b]\rangle.$$

1. For $\nabla_X Y$ let us first consider the inner product with a vector field in the fiber F, $\langle \nabla_X Y, V\rangle$:

   $$2\langle \nabla_X Y, V\rangle = X\langle Y, V\rangle + Y\langle V, X\rangle - V\langle X, Y\rangle - \langle X, [Y, V]\rangle + \langle Y, [V, X]\rangle + \langle V, [X, Y]\rangle.$$

   Given the orthogonality of the base and fiber, the inner products $\langle Y, V\rangle$ and $\langle V, X\rangle$ are both zero. Consequently, their inner product evaluates to zero. The brackets $[Y, V]$ and $[V, X]$ are both equal to zero by Corollary 3.11. Since $X, Y \in \mathfrak{L}(B)$, then $[X, Y] \in \mathfrak{L}(B)$. We know $\mathfrak{L}(B)$ is perpendicular to $\mathfrak{L}(F)$, therefore the inner product $\langle V, [X, Y]\rangle$ is equal to zero. Thus our inner product $2\langle \nabla_X Y, V\rangle$ reduces to

   $$2\langle \nabla_X Y, V\rangle = -V\langle X, Y\rangle.$$

   The inner product $\langle X, Y\rangle$ in coordinates is derived exclusively from the coordinates of the base, as seen by $\langle X, Y\rangle = g_B(X, Y)$. The vector field $V \in F$ expressed in coordinates, involves differentiation solely in the direction tangent to the fiber, without any influence from the base. As a result, we have $V\langle X, Y\rangle = 0$. After simplifying, we find that $\langle \nabla_X Y, V\rangle = 0$. This statement implies that the inner product of $\nabla_X Y$ with a vector field V has no contributions from the fiber component. Consequently, the only component of $\nabla_X Y$ must originate from the base. This leads us to conclude that $\nabla_X Y$ is an element of $\mathfrak{L}(B)$.

2. Recall the definition of the Levi-Civita connection, Definition 4.12. Given the vector fields $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$, then $\nabla_V X - \nabla_X V = [X, V]$ is torsion free and metrically compatible. Leveraging these facts we assert a proof of what $\nabla_X V$ and $\nabla_V X$ must be.

Let us begin by analyzing the inner product given by $\langle \nabla_X V, Y \rangle$. Invoking the principle of metric compatibility, we have $X\langle V, Y\rangle = \langle \nabla_X V, Y\rangle + \langle V, \nabla_X Y\rangle$. Given that the base and fiber are orthogonal, the inner products $\langle V, Y\rangle$ and $\langle V, \nabla_X Y\rangle$ are both identically zero. This simplifies our expression to $\langle \nabla_X V, Y\rangle = 0$. Consequently, this inner product is inherently associated with the base manifold.

We now turn our focus to the inner product $\langle \nabla_X V, W\rangle$. As above, we shall employ the Koszul formula to elucidate the properties of our connection.

$$\begin{aligned} 2\langle \nabla_X V, W\rangle &= X\langle V, W\rangle + V\langle W, X\rangle - W\langle X, V\rangle - \langle X, [V, W]\rangle \\ &\quad + \langle V, [W, X]\rangle + \langle W, [X, V]\rangle. \end{aligned}$$

The following inner products are all zero by perpendicularity of the base and fiber: $\langle W, X\rangle$, $\langle X, V\rangle$, and $\langle X, [V, W]\rangle$. By Corollary 3.11, inner products $\langle V, [W, X]\rangle$ and $\langle W, [X, V]\rangle$ are also zero. This simplifies the Koszul formula to,

$$2\langle \nabla_X V, W\rangle = X\langle V, W\rangle.$$

The inner product denoted by $\langle V, W\rangle$ corresponds to the warped product metric on F. Hence, we can express $\langle V, W\rangle$ in the form $f^2 \cdot g_F(V, W)$. We will employ the product rule in the computation below.

$$\begin{aligned} 2\langle \nabla_X V, W\rangle &= X\langle V, W\rangle, \\ &= X(f^2 \cdot g_F(V, W)), \\ &= X(f^2) \cdot g_F(V, W) + f^2 \cdot X(g_F(V, W)). \end{aligned}$$

Given that the derivative $X(g_F(V, W))$ pertains to a function with coordinates exclusively in F and not in B and X is a vector field on B, then $X(g_F(V, W)) = 0$. This implies that

$$\begin{aligned} X(f^2) \cdot g_F(V, W) + f^2 \cdot X(g_F(V, W)) &= 2f \cdot X(f) g_F(V, W), \\ &= \frac{2X(f)}{f} \cdot f^2 \cdot g_F(V, W). \end{aligned}$$

Replacing $f^2 \cdot g_F(V,W)$ with $\langle V,W\rangle$ once again, we find that this equals

$$\frac{2X(f)}{f} \cdot \langle V,W\rangle.$$

Now,

$$\begin{aligned}
2\langle \nabla_X V, W\rangle &= \frac{2X(f)}{f} \cdot \langle V,W\rangle, \\
\langle \nabla_X V, W\rangle &= \frac{X(f)}{f} \cdot \langle V,W\rangle, \\
\langle \nabla_X V, W\rangle &= \left\langle \frac{X(f)}{f} V, W \right\rangle, \\
\langle \nabla_X V, W\rangle - \left\langle \frac{X(f)}{f} V, W \right\rangle &= 0 \quad \text{ for all } W, \\
\left\langle \nabla_X V - \frac{X(f)}{f} V, W \right\rangle &= 0.
\end{aligned}$$

This is true by Lemma 4.3 and the corresponding Corollary 4.4, since this is true for all vectors W. We deduce that $\nabla_X V = \frac{X(f)}{f} V$.

3. In this concluding section of our proof, we aim to determine the value of $\nabla_V W$. We begin by considering the property of metric compatibility.

$$V\langle W,X\rangle = \langle \nabla_V W, X\rangle + \langle W, \nabla_V X\rangle.$$

The inner product $\langle W,X\rangle$ we know to be zero. As established in Proof 2 above, it follows that $\nabla_V X = \nabla_X V = \frac{X(f)}{f} V$. Consequently,

$$\begin{aligned}
0 &= \langle \nabla_V W, X\rangle + \left\langle W, \frac{X(f)}{f} V \right\rangle, \\
\langle \nabla_V W, X\rangle &= - \left\langle W, \frac{X(f)}{f} V \right\rangle, \\
\langle \nabla_V W, X\rangle &= - \frac{X(f)}{f} \langle W, V\rangle.
\end{aligned}$$

Recall Definition 6.2 in the Notation Section 1.1, of the differential operator the gradient. The gradient is given by $\langle \text{grad}(f), X\rangle = df(X) = Xf$ for all $X \in \mathfrak{X}(M)$. Then,

$$\begin{aligned}
\langle \nabla_V W, X\rangle &= - \frac{\langle \text{grad}(f), X\rangle}{f} \langle W, V\rangle, \\
&= \left\langle - \frac{\text{grad}(f)}{f} \langle W, V\rangle, X \right\rangle.
\end{aligned}$$

Therefore $\nabla_V W = -\frac{\text{grad}(f)}{f}\langle W, V\rangle$ in the inner product of $\langle \nabla_V W, X\rangle$.

We also need to determine the inner product of $\nabla_V W$ with a vector field residing in the fiber. By expressing this in coordinates, we aim to highlight the central role that coordinate bases play throughout this thesis. Recall that

$$\nabla_{\partial_{v_i}}\partial_{v_j} = -\frac{\text{grad}(f)}{f}\langle \partial_{v_i}, \partial_{v_j}\rangle + {\Gamma_{ij}}^k \partial_{v_k}.$$

Using a slight abuse of notation in terms of coordinates, the inner product $\langle \nabla_V W, U\rangle$ can be expressed as follows:

$$\langle \nabla_{\partial_{v_i}}\partial_{v_j}, \partial_{v_l}\rangle = g\left({\Gamma_{ij}}^k \partial_{v_k}, \partial_{v_l}\right).$$

We can notice that the solution to ${\Gamma_{ij}}^k$ on M is exactly the solution of ${\Gamma_{ij}}^k$ on F. Therefore, the part of $\nabla_V W$ on the fiber is the lift of ${}^F\nabla_V W$. Let us say $U \in \mathfrak{L}(F)$. Then the the nonzero terms in the Koszul formula reduce to,

$$2\langle \nabla_V W, U\rangle = V\langle W, U\rangle,$$
$$2\left\langle -\frac{\langle V, W\rangle}{f}\text{grad}(f) + {}^F\nabla_V W, U\right\rangle = V\langle W, U\rangle,$$

Therefore $\nabla_V W = \left(-\frac{\langle V,W\rangle}{f}\right)\text{grad}(f) + {}^F\nabla_V W$.

Let us refer back our main Example 4.17. One can notice that this particular product manifold is the warped product $M = \mathbb{R}^+ \times_f S^2$, where $f = r$, is a linear warping function. We will use Proposition 6.3 to show that the covariant derivatives of $M = \mathbb{R}^+ \times_r S^2$ are exactly that of the product manifold $\mathbb{R}^+ \times S^2$. □

Example 6.4. *(Continuation of Example 4.17) Let $\mathbb{R}^+$ be the base and S^2 the fiber, and the warping function $f = r$, for the warped product. Let $X, Y \in \mathfrak{L}(\mathbb{R}^+)$ and $V, W \in \mathfrak{L}(S^2)$. Let $X = \partial_r, V = \partial_\theta, W = \partial_\varphi$. Let us recall our computed covariant derivatives.*

1. $\nabla_{\partial_r}\partial_\theta = \nabla_{\partial_\theta}\partial_r = \frac{1}{r}\partial_\theta$.

2. $\nabla_{\partial_r}\partial_\varphi = \nabla_{\partial_\varphi}\partial_r = \frac{1}{r}\partial_\varphi$.

3. $\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = \frac{\cos(\varphi)}{\sin(\varphi)}\partial_\theta$.

4. $\nabla_{\partial_\theta}\partial_\theta = -r\sin^2(\varphi)\partial_r - \sin(\varphi)\cos(\varphi)\partial_\varphi$.

5. $\nabla_{\partial_\varphi}\partial_\varphi = -r\partial_r$.

The first two $\nabla_{\partial_r}\partial_\theta = \nabla_{\partial_\theta}\partial_r$ *and* $\nabla_{\partial_r}\partial_\varphi = \nabla_{\partial_\varphi}\partial_r$, *are both of the form* $\nabla_X V = \nabla_V X$. *A straightforward substitution is as follows:*

$$\nabla_X V = \nabla_V X = \left(\frac{X(f)}{f}\right) V,$$
$$\Rightarrow \nabla_{\partial_r}\partial_\theta = \nabla_{\partial_\theta}\partial_r = \left(\frac{\partial_r(r)}{r}\right)\partial_\theta = \left(\frac{1}{r}\right)\partial_\theta.$$

Similarly,

$$\nabla_X V = \nabla_V X = \left(\frac{X(f)}{f}\right) V,$$
$$\Rightarrow \nabla_{\partial_r}\partial_\varphi = \nabla_{\partial_\varphi}\partial_r = \left(\frac{\partial_r(r)}{r}\right)\partial_\varphi = \left(\frac{1}{r}\right)\partial_\varphi.$$

This shows that Equations 1 *and* 2 *are as they should be according to our covariant derivative equations of Proposition 4.12.*

The last three equations $3, 4$, *and* 5 *are all of the form* $\nabla_V W$. *We note that* $\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta$. *By substitution,*

$$\nabla_V W = \left(\frac{\langle V, W\rangle}{f}\right)\mathrm{grad}(f) + {}^F\nabla_V W$$
$$\Rightarrow \nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = \left(\frac{\langle\partial_\theta, \partial_\varphi\rangle}{r}\right)\mathrm{grad}(r) + {}^F\nabla_{\partial_\theta}\partial_\varphi.$$

The inner product of $\langle\partial_\theta, \partial_\varphi\rangle$ *is zero by the metric. One sees that the computation of* ${}^F\nabla_{\partial_\theta}\partial_\varphi$ *on* S^2 *is precisely as shown in Example 4.17.*

6.2 Curvature of Warped Products

We will use the following common definition in Proposition 6.6.

Definition 6.5. *The Hessian* H^f *of* f *is the symmetric* $(0, 2)$ *tensor field such that,*

$$H^f(X, Y) = XYf - (\nabla_x Y)f = \langle\nabla_X(grad(f)), Y\rangle.$$

Recall the differential operations Definitions 6.2. These two differential operations will be leveraged to compute the tensor equations for a warped product following O'Neill [O'n83].

Proposition 6.6. *Let $M = B \times_f F$ be a warped product with Riemannian curvature tensor R. Let ${}^B R$ and ${}^F R$ be the curvature tensors of B and F respectively. If the vector fields $X, Y, Z \in \mathfrak{L}(B)$ and $U, V, W \in \mathfrak{L}(F)$, then*

1. *$R_{XY}Z \in \mathfrak{L}(B)$ is the lift of ${}^B R_{XY}Z$ on B.*
2. *$R_{VX}Y = \frac{H^f(X,Y)}{f}V$, where H^f is the Hessian of f.*
3. *$R_{XY}V = R_{VW}X = 0$.*
4. *$R_{XV}W = \frac{\langle V,W\rangle}{f}\nabla_X \mathrm{grad}(f)$.*
5. *$R_{VW}U = {}^F R_{VW}U - \frac{\langle \mathrm{grad}(f), \mathrm{grad}(f)\rangle}{f^2}(\langle V, U\rangle W - \langle W, U\rangle V)$.*

The reader may notice that there could be other combinations that need to be considered. By the symmetries of curvature (Proposition 5.4), all other curvatures are contained within these five equations, up to a sign.

Following O'Neill, we will explicitly compute the curvature equations.

Proof. Let us first recall in the proof of Proposition 6.3 concerning the covariant derivatives of warped products. We revisit the curvature formula as presented in Definition 5.1. For clarity, we will express this formula using vector fields a, b, c to ensure they are distinct from the vector fields discussed here:

$$R(a, b)c = \nabla_{[a,b]}c - \nabla_a \nabla_b c + \nabla_b \nabla_a c.$$

1. Property one follows directly from Proposition 4.12.
2. From the curvature formula, $R_{VX}Y = \nabla_{[V,X]}Y - \nabla_V \nabla_X Y + \nabla_X \nabla_V Y$. The bracket $[V, X] = 0$ by Corollary 3.11. This observation simplifies our expression considerably. We continue by using our computed covariant derivatives from Proposition

6.3.

$$\begin{aligned}
R_{VX}Y &= \nabla_{[V,X]}Y - \nabla_V\nabla_X Y + \nabla_X\nabla_V Y \\
&= 0 - \frac{(\nabla_X Y)(f)}{f}V + \nabla_X\left(\frac{Y(f)}{f}V\right) \\
&= -\frac{(\nabla_X Y)(f)}{f}V + X\left(\frac{Y(f)}{f}\right)V + \frac{Y(f)}{f}\nabla_X V \\
&= -\frac{(\nabla_X Y)(f)}{f}V + X\left(\frac{Y(f)}{f}\right)V + \frac{Y(f)}{f}\frac{X(f)}{f}V \\
&= \left[-\frac{(\nabla_X Y)(f)}{f} + X\left(\frac{Y(f)}{f}\right) + \frac{Y(f)}{f}\frac{X(f)}{f}\right]V \\
&= \left[-\frac{(\nabla_X Y)(f)}{f} + X\left(\frac{Y(f)}{f}\right) + \frac{Y(f)X(f)}{f^2}\right]V.
\end{aligned}$$

The derivative $X\left(\frac{Y(f)}{f}\right) = \frac{fX(Y(f))-Y(f)X(f)}{f^2}$ by the quotient rule, which implies

$$\begin{aligned}
R_{VX}Y &= \left[-\frac{(\nabla_X Y)(f)}{f} + \frac{fX(Y(f)) - Y(f)X(f)}{f^2} + \frac{Y(f)X(f)}{f^2}\right]V \\
&= \left[-\frac{(\nabla_X Y)(f)}{f} + \frac{fX(Y(f))}{f^2}\right]V \\
&= \left[-\frac{(\nabla_X Y)(f)}{f} + \frac{X(Y(f))}{f}\right]V \\
&= \left[-\frac{(\nabla_X Y)(f) + X(Y(f))}{f}\right]V.
\end{aligned}$$

Recall the differential operator Definition 6.5 of the Hessian,

$$H^f(X,Y) = XYf - (\nabla_X Y)f = \langle\nabla_X(\mathrm{grad}(f)), Y\rangle.$$

By simplifying,

$$R_{VX}Y = \left[-\frac{(\nabla_X Y)(f) + X(Y(f))}{f}\right]V = \frac{H^f(X,Y)}{f}V.$$

3. For this curvature computation we will start with

$$R_{VW}X = \nabla_{[V,W]}X - \nabla_V\nabla_W X + \nabla_W\nabla_V X.$$

The bracket $[V,W]$ is a vector field in the lift of the fiber, so $\nabla_{[V,W]} = \frac{X(f)}{f}[V,W]$.

Then,

$$\begin{aligned} R_{VW}X &= \frac{X(f)}{f}[V,W] - \nabla_V\left(\frac{X(f)}{f}W\right) + \nabla_W\left(\frac{X(f)}{f}V\right) \\ &= \frac{X(f)}{f}[V,W] - \left(V\left(\frac{X(f)}{f}\right)W + \frac{X(f)}{f}\nabla_V W\right) \\ &\quad + \left(W\left(\frac{X(f)}{f}\right)V + \frac{X(f)}{f}\nabla_W V\right). \end{aligned}$$

Both derivatives $V\left(\frac{X(f)}{f}\right)W$ and $W\left(\frac{X(f)}{f}\right)V$ are zero since we are differentiating vector fields from the base in the direction of the fiber. This implies

$$\begin{aligned} R_{VW}X &= \frac{X(f)}{f}[V,W] - \frac{X(f)}{f}\nabla_V W + \frac{X(f)}{f}\nabla_W V \\ &= \frac{X(f)}{f}\left([V,W] - \nabla_V W + \nabla_W V\right). \end{aligned}$$

Recall the Levi-Civita connection is torsion free, as given in Property 5.1. This implies that $R_{VW}X = 0$. We will leverage this result in our subsequent analysis to compute the curvature of $R_{XY}V$.

Let us consider the inner product $\langle R(X,Y)V,W\rangle$. By symmetries of curvature Property 5.4,

$$\begin{aligned} \langle R(X,Y)V,W\rangle &= R(X,Y,V,W) \\ &= R(V,W,X,Y) \\ &= \langle R(V,W)X,Y\rangle. \end{aligned}$$

From our preceding proof, we have established that $\langle R(V,W)X,Y\rangle = 0$. Our following approach involves examining all possible inner products. To facilitate this, we invoke a result from linear algebra and Lemma 2.12: given that for any $(b,f) \in B \times F$, we have $T_{(b,f)}(B \times F) \cong T_bB \oplus T_fF$, for any vector field T in the tangent space of $B \times F$, there exists some $Z \in \mathfrak{L}(B)$ and $W \in \mathfrak{L}(F)$ such that $T = Z + W$. Exploiting the linearity of the inner product, we get the following results:

$$\begin{aligned} \langle R(X,Y)V,T\rangle &= \langle R(X,Y)V,Z+W\rangle \\ &= \langle R(X,Y)V,Z\rangle + \langle R(X,Y)V,W\rangle \\ &= \langle R(V,Z)X,Y\rangle + 0. \end{aligned}$$

Where $\langle R(X,Y)V,W\rangle = 0$ from our proof above. Recall again, the Hessian used previously.

$$\begin{aligned}\langle R(X,Y)V,T\rangle = \langle R(V,Z)X,Y\rangle &= \left\langle \frac{H^f(Z,X)}{f}V,Y\right\rangle\\ &= \frac{H^f(Z,X)}{f}\langle V,Y\rangle.\end{aligned}$$

Lastly, the inner product of $\langle V,Y\rangle = 0$ by perpendicularity of the base and fiber. Thus $R_{XY}V = R_{VW}X = 0$.

4. Again, by Corollary 3.11 and property 2 of Proposition 6.3, the bracket $[X,V]$ is zero and $\nabla_X W = \nabla_V(X(f)/f)W$. Explicitly,

$$\begin{aligned}R_{XV}W &= \nabla_{[X,V]}W - \nabla_X\nabla_V W + \nabla_V\nabla_X W\\ &= 0 - \nabla_X\nabla_V W + \nabla_V\left(\nabla_V\frac{X(f)}{f}W\right)\\ &= -\nabla_X(\nabla_V W) + \nabla_V\left(\nabla_V\frac{X(f)}{f}W\right)\\ &= -\nabla_X(\nabla_V W) + \left(V\left(\frac{X(f)}{f}\right)W + \left(\frac{X(f)}{f}\right)\nabla_V W\right).\end{aligned}$$

The derivative $V\left(\frac{X(f)}{f}\right)W = 0$ since we are differentiating vector fields in B in the direction of F. Recall that $\nabla_V W = \left(\frac{\langle V,W\rangle}{f}\right)\text{grad}(f) + {}^F\nabla_V W$. Then,

$$\begin{aligned}R_{XV}W &= -\nabla_X(\nabla_V W) + \left(\frac{X(f)}{f}\right)\nabla_V W\\ &= -\nabla_X\left({}^F\nabla_V W - \frac{\langle V,W\rangle}{f}\text{grad}(f)\right) + \left(\frac{X(f)}{f}\right){}^F\nabla_V W\\ &= -\nabla_X({}^F\nabla_V W) + \nabla_X\left(\frac{\langle V,W\rangle}{f}\text{grad}(f)\right) + \left(\frac{X(f)}{f}\right){}^F\nabla_V W\\ &= -\left(\frac{X(f)}{f}\right)({}^F\nabla_V W) + \nabla_X\left(\frac{\langle V,W\rangle}{f}\text{grad}(f)\right) + \left(\frac{X(f)}{f}\right){}^F\nabla_V W\\ &= \nabla_X\left(\frac{\langle V,W\rangle}{f}\text{grad}(f)\right)\\ &= X\left(\frac{\langle V,W\rangle}{f}\right)\text{grad}(f) + \frac{\langle V,W\rangle}{f}\nabla_X\text{grad}(f).\end{aligned}$$

Revisiting the relation for the warped product metric:

$$\langle V,W\rangle/f = f^2 g_F(V,W)/f = f g_F(V,W),$$

where the factor f^2 has been simplified. This implies

$$\begin{aligned} R_{XV}W &= X(fg_F(V,W)) + \frac{\langle V,W\rangle}{f}\nabla_X \mathrm{grad}(f), \\ &= g_F(V,W)X(f) + \frac{\langle V,W\rangle}{f}\nabla_X \mathrm{grad}(f) \\ &= 0 + \frac{\langle V,W\rangle}{f}\nabla_X \mathrm{grad}(f) \\ &= \frac{\langle V,W\rangle}{f}\nabla_X \mathrm{grad}(f). \end{aligned}$$

What is left to show is what is the inner product of $R_{XV}W$ with any vector. Let $T = U + Y$ once again, where $U \in \mathfrak{L}(F)$ and $Y \in \mathfrak{L}(B)$.

$$\langle R(X,V)W,T\rangle = \langle R(X,V)W,U\rangle + \langle R(X,V)W,Y\rangle.$$

First, we consider $\langle R(X,V)W,U\rangle$. By symmetry, $\langle R(X,V)W,U\rangle = \langle R(W,U)X,V\rangle$. Then from Assertion 3 in this proposition,

$$\langle R((X,V)W,U)\rangle = \langle R(W,U)X,V\rangle = 0.$$

Consider $\langle R(X,V)W,Y\rangle$. The symmetries of curvature then provide that by swapping the first two and second two vector fields, keeping their order provides equal curvatures, i.e $\langle R(X,V)W,Y\rangle = \langle R(W,Y)X,V\rangle$. From Assertion 2 in this proposition, $R(W,Y)X = (H^f(Y,X)/f)W$.

$$\begin{aligned} \langle R(X,V)W,Y\rangle &= \langle R(W,Y)X,V\rangle \\ &= \left\langle \frac{H^f(Y,X)}{f}W,V\right\rangle \\ &= \frac{H^f(Y,X)}{f}\langle V,W\rangle \\ &= \frac{\langle V,W\rangle}{f}\langle \nabla_X \mathrm{grad}(f),Y\rangle. \end{aligned}$$

At this point, let us revisit the expression $\langle \nabla_X \mathrm{grad}(f),T\rangle$ which can be decomposed as $\langle \nabla_X \mathrm{grad}(f),T\rangle = \langle \nabla_X \mathrm{grad}(f),Y\rangle + \langle \nabla_X \mathrm{grad}(f),U\rangle$. Given the orthogonality, we have $\langle \nabla_X \mathrm{grad}(f),U\rangle = 0$. This simplifies our expression to the following, $\langle \nabla_X \mathrm{grad}(f),T\rangle = \langle \nabla_X \mathrm{grad}(f),Y\rangle$. With this result in hand, we can now proceed from where we left off.

$$\langle R(X,V)W,Y\rangle = \frac{\langle V,W\rangle}{f}\langle \nabla_X \mathrm{grad}(f), Y\rangle$$
$$\langle R(X,V)W,Y\rangle = \frac{\langle V,W\rangle}{f}\langle \nabla_X \mathrm{grad}(f), T\rangle.$$

Thus $R_{XV}W = \frac{\langle V,W\rangle}{f}\nabla_X \mathrm{grad}(f)$.

5. $R_{VW}U = {}^F R_{VW}U - (\langle \mathrm{grad}(f), \mathrm{grad}(f)\rangle / f^2)[\langle V,U\rangle W - \langle W,U\rangle V]$.

The proof of this result, while straightforward, is computationally intensive and beyond the scope of this thesis. We kindly direct the interested reader to consult Pages 210-211 in the reference [O'n83] for a comprehensive exposition. □

We have now developed the tools to compute the curvature of any warped product and analyze its results. To provide the reader with a comprehensive understanding, let us revisit out main example $M = \mathbb{R}^+ \times_r S^2$. It will suffice to compute $R_{\partial_r \partial_r}\partial_\varphi$ here, as the computations involved with the others are similar.

Example 6.7. *(Extension of Example 4.17) $M = \mathbb{R}^+ \times_r S^2$. Let us recall the curvature equation,*

$$R(a,b)c = \nabla_{[a,b]}c - \nabla_a \nabla_b c + \nabla_b \nabla_a c.$$

Applying the curvature equation,

$$\begin{aligned} R(\partial_r, \partial_r)\partial_\varphi &= \nabla_{[\partial_r,\partial_r]}\partial_\varphi - \nabla_{\partial_r}\nabla_{\partial_r}\partial_\varphi + \nabla_{\partial_r}\nabla_{\partial_r}\partial_\varphi, \\ &= -\nabla_{\partial_r}\nabla_{\partial_r}\partial_\varphi + \nabla_{\partial_r}\nabla_{\partial_r}\partial_\varphi, \\ &= -\nabla_{\partial_r}(\nabla_{\partial_r}\partial_\varphi) + \nabla_{\partial_r}(\nabla_{\partial_r}\partial_\varphi), \\ &= -\nabla_{\partial_r}\left(\frac{1}{r}\partial_\varphi\right) + \nabla_{\partial_r}\left(\frac{1}{r}\partial_\varphi\right), \\ &= 0. \end{aligned}$$

The bracket of $[\partial_r, \partial_r] = 0$ by Corollary 3.9, and the rest of the components sum to zero. By Proposition 6.6, $R_{XY}V = 0$. So our computation is as it should be.

6.3 Flat Manifolds

In Chapter 5 we discussed the curvature on manifolds. Recall Definition 6.8 stating that manifolds with vanishing curvature are flat manifolds. It is evident by Proposition 6.6 that when a flat manifold is warped with another flat manifold using a constant warping function, the result reduces to a product of two flat manifolds. Intuitively, the resulting product is flat.

Theorem 6.8. *Given two flat manifolds B and F, along with a constant function $f : B \to \mathbb{R}$, the warped product $B \times_f F$ is also flat.*

Proof. The proof of this theorem is intuitive and straightforward. Given that B and F are flat manifolds, we can establish that their respective curvature tensors are zero by definition of being flat. Recall the definition of a warped product Definition 6.1. The metric for a warped product is $g = g_B + f^2 g_F$. If the warping function f constant, it naturally follows that f^2 is constant, reducing the warped product to the less general form of a product manifold. And, the scaled metric $f^2 g_F$ is still flat because f is a constant. As a direct consequence of these conditions, the curvature tensor associated with the warped product manifold $B \times_f F$ is 0 according to Proposition 6.6. □

It is natural to inquire about the nature of flat warped products. Several example of questions that can emerge are: Is the converse of Proposition 6.8 true? Given a flat warped product $B \times_f F$, what can be deduced regarding the geometries of B, F, and the warping function f? In response to these queries, Akbar [Akb12] characterizes the flatness of warped products.

Theorem 6.9. *[Akb12] If $M = B \times_f F$ is flat, then B is flat, f is linear, and F is a manifold of constant sectional curvature.*

This proof will unfold in three distinct parts. Initially we will prove B is flat. Secondly we will show that f is linear. Lastly, we will exhibit that F is of constant curvature.

Proof. 1. Recall Theorem 5.2, M is flat if $R = 0$. By our curvature computations in Proposition 6.6, then $R_{XY}Z = {}^B R_{XY} Z = 0$, thus B is flat.

2. Since B is flat, then there exists a coordinate system $(x_1, \ldots, x_n)$ on B near any point with[2] $g_B(\partial_i, \partial_j) = \pm\delta_{ij}$. For ease of notation, we define $g_B(\partial_j, \partial_j) = \epsilon_j = \pm 1$. By our curvature computation, $R_{VX}Y = \left[\frac{H^f(X,Y)}{f}\right] \cdot V$. Since M is flat, its curvature is identically 0. Therefore $H^f(X,Y) = 0$. By definition of the Hessian, recall Definition 6.5, $H^f(X,Y) = g_B(\nabla_X(\text{grad}(f)), Y)$ for all X, Y. Then $H^f(X,Y) = g_B(\nabla_X(\text{grad}(f)), Y) = 0$. Recall that $df = \sum \frac{\partial f}{\partial x_i} dx_i$. Proposition 4.7 ensures that dx_i and $\epsilon_i \partial_i$ are metrically equivalent. Therefore, $\text{grad} f = \sum \epsilon_i \frac{\partial f}{\partial x_i} \partial_i$. Consequently, $H^f(X,Y) = g_B(\nabla_X(\text{grad}(f)), Y) = 0$. If $X = \partial_i$ and $Y = \partial_j$. Then,

$$\begin{aligned} 0 = H^f(X,Y) &= g_B\left(\nabla_{\partial_i}\left(\sum \epsilon_k \frac{\partial f}{\partial x_k} \partial x_k\right), \partial_j\right) \\ &= g_B\left(\sum \epsilon_k \nabla_{\partial_i} \frac{\partial f}{\partial x_k} \partial_k, \partial_j\right) \\ &= g_B\left(\sum \epsilon_k \left[\frac{\partial^2 f}{\partial x_i \partial x_k} \partial_k + \frac{\partial f}{\partial x_k} \nabla_{\partial_i} \partial_k\right], \partial_j\right). \end{aligned}$$

Recall that $\nabla_{\partial_i} \partial_k = 0$ since $g(\partial_i, \partial_k) = \pm\delta_{ij}$ and ∇ is the Levi-Civita connection. Therefore, continuing the above chain of equalities,

$$\begin{aligned} &= g_B\left(\sum \epsilon_k \frac{\partial^2 f}{\partial x_i \partial x_k} \partial x_k, \partial_j\right) \\ &= \epsilon_j \frac{\partial^2 f}{\partial x_i \partial x_j} g(\partial_j, \partial_j) \text{ so,} \\ 0 &= \frac{\partial^2 f}{\partial x_i \partial x_j} \quad \text{for all } i, j, k. \end{aligned}$$

Thus f is linear. That is, $f(x_1, \ldots, x_n) = c + \sum b_i x_i$, for some $c, b_i \in \mathbb{R}$.

3. Using our curvature computations,

$$\begin{aligned} 0 = R_{VW}U &= {}^F R_{VW}U - g_B(\text{grad}(f), \text{grad}(f))[g_F(V,U)W - g_F(W,U)V], \\ &\Rightarrow {}^F R_{VW}U = g_B(\text{grad}(f), \text{grad}(f))[g_F(V,U)W - g_F(W,U)V]. \end{aligned}$$

We know that $f = \sum b_i x_i + c$ where $b_i, c \in \mathbb{R}$. Now we know $b_i = \frac{\partial f}{\partial x_i}$ and

[2]Notice the sign of the Kronecker delta here. The $\pm$ will be of use when we discuss spacetimes in Chapter 7.

grad $f = \sum \epsilon_k b_k \partial x_k$. By substitution, ${}^F R_{VW}U$ can be rewritten in the following way.

$$
\begin{aligned}
{}^F R_{VW}U &= g_B(\text{grad}(f), \text{grad}(f))[g_F(V,U)W - g_F(W,U)V], \\
&= g_B\left(\sum_k \epsilon_k b_k \partial x_k, \sum_l \epsilon_l b_l \partial x_l\right)[g_F(V,U)W - g_F(W,U)V], \\
&= \sum_k \epsilon_k \epsilon_k b_k b_k g\left(\partial x_k, \partial x_k\right)[g_F(V,U)W - g_F(W,U)V], \\
&= \sum_k \epsilon_k b_k^2 [g_F(V,U)W - g_F(W,U)V].
\end{aligned}
$$

The value $\sum_k \epsilon_k b_k^2 \in \mathbb{R}$. We can call it λ. Therefore,

$$
\begin{aligned}
{}^F R_{VW}U &= g_B(\text{grad}(f), \text{grad}(f))[g_F(V,U)W - g_F(W,U)V] = 0, \\
{}^F R_{VW}U &= \lambda[g_F(V,U)W - g_F(W,U)V] = 0.
\end{aligned}
$$

Therefore, by Definition 5.3, the fiber F is a manifold with constant sectional curvature.

□

Based on the proof, it is evident it is a trap to assume the outcome of a flat manifold lies solely in the warped product of two flat manifolds. This proof provides us the criteria under which a warped product gives rise to a flat manifold provided not both B and F are flat.

It is crucial to emphasize that the example we have revisited several times was chosen to illustrate the proof above. Specifically, in our example, we considered the manifold M defined as the warped product $M = \mathbb{R}^+ \times_r S^2$, where not both the base and the fiber are flat and a warping function $f(r) = r$ that is linear. By conducting a straightforward computation for each curvature components, it is not surprising that this manifold is flat. In fact, it is merely flat $\mathbb{R}^3 - \{0\}$ expressed in spherical coordinates.

This marks the end of our discussion on warped products and their associated curvature. The following chapter will delve into the significance and applications of warped products within the fascinating domain of general relativity.

Chapter 7

Applications to General Relativity and Cosmology: Schwarzschild Geometry

In this final chapter, we delve into the applications of warped products in general relativity, prefaced by a short historical overview to set the stage.

7.1 A Bit of History

When discussion of the cosmos is on the table, topics such as special and general relativity are at the forefront. Special relativity provides a comprehensive framework for understanding cosmic phenomena; however, it leaves a gap when it comes to notions of gravity. The relativistic melding of space and time left no room for Newton's gravitational law, which pertained solely to space. In the years post-1905, Einstein was was said to be convinced that gravity must be expressed in terms of curvature. By the year 1915, he had successfully unraveled this intuition, leading to the birth of the general theory of relativity [Nee21]. This groundbreaking theory replaced the "flat Minkowski spacetime" concept with spacetimes characterized by arbitrary curvature. Through the lens of relativity, the universe is perceived as spacetimes – time-oriented, four-dimensional Lorentz manifolds [O'n83].

The narrative continues with the introduction of Einstein's Vacuum Field Equations. For clarity's sake, these equations simply imply: In a vacuum, for the shaping forces to maintain their original dimensions, outward-bending curves need to be precisely offset by inward-bending curves, perfectly nullifying their collective impact. This equilibrium of forces is guided by the Ricci curvature. A central tenet of general relativity is that a region of spacetime devoid of matter will exhibit zero Ricci curvature. Einstein's Vacuum Field Equation is therefore given by: $Ric = 0 \iff R_{ik} = 0$.

A "solution" to this equation is the metric which satisfies it. Among the most significant solutions is one that portrays the spacetime outside a spherically symmetric (non-spinning) mass, represented by m. This solution was discovered by Karl Schwarzschild, almost immediately after Einstein revealed his theory. This spacetime model Schwarzschild discovered quickly became pertinent for astronomical objects that fit these parameters, our own sun being a prime example of this. Remarkably, when applied to our solar system, this framework provides an even more accurate representation than the revered Newtonian model. In the initial stages, physicists focused solely on one half of the Schwarzschild spacetime, deeming its exterior as the primary region of physical significance. Intriguingly, the previously overlooked half now stands as our most simple model of a black hole, which we will uncover here [Nee21].

7.2 The Relevance of Warped Products and Curvature

The scheme here is to unravel what we have learned thus far in our exposition of the curvature of warped product manifolds, and extend our understanding to the context of Schwarzschild spacetime. The following will provide an outline for the last sections of this thesis.

The Schwarzschild spacetime is characterized by a metric which encompasses a singular point mass—symbolizing the center of a black hole—yet this singularity is not represented within our model for reasons which will become clear later. We recognize this spacetime manifold as a warped product. Using a Weyl curvature invariant, as will be described in Section 7.5, we establish that the curvature approaches infinity near the singularity, thereby reinforcing the idea that such a point cannot be included into our spacetime model. Conversely, we will demonstrate that the curvature remains finite near the event horizon, suggesting that it can indeed be modeled. Using reasonable physical hypothesises we demonstrate for this spacetime what the metric is defined as.

The curvature formulas developed earlier give a complete description of the curvature of either spacetime inside or outside the event horizon, which we use to complete our last goal concerning the boundedness of curvature near the event horizon and singularity.

7.3 Schwarzschild Spacetime Metric

In alignment with the discussions in Chapter, a spacetime within the framework of general relativity takes the form of a 4-dimensional Lorentzian manifold [Bes07]. Let us first consider the manifold $\mathbb{R}^1 \times (\mathbb{R}^3 - \{0\})$, where $\mathbb{R}^1$ represents time and $\mathbb{R}^3 - \{0\}$ as a representation of space with a point mass at the origin removed. We consider four reasonable hypothesises to build our model. It should be: static, spherically symmetric, asymptotically flat, and vacuum. We will follow O'Neill satisfying these four conditions and derive the Schwarzschild metric [O'n83].

Static: We will use $\mathbb{R}^3 - \{0\}$ to be the rest space[1] with some line element q. Physical considerations not considered here imply that the manifold $\mathbb{R}^1 \times (\mathbb{R}^3 - \{0\})$ has a line element of the form:

$$A(x)dt^2 + q \quad (x \in \mathbb{R}^3 - \{0\}),$$

where q is the lifted from $\mathbb{R}^3$ [O'n83].

Spherically symmetric: It is natural to to use spherical coordinates on $\mathbb{R}^3 - \{0\}$ and view it as $\mathbb{R}^+ \times S^2$, where $\mathbb{R}^+ = \{\rho \in \mathbb{R} | \rho > 0\}$. Spherical symmetry means that any action of the orthogonal group $O(3)$ on our rest space is an isometry. Thus our line element for $\mathbb{R}^1 \times (\mathbb{R}^3 - \{0\})$ becomes:

$$A(\rho)dt^2 + B(\rho)d\rho^2 + C(\rho)d\sigma^2,$$

where the functions A, B, and C are only functions of ρ.

It will be convenient to change variables defining $C(\rho) = r^2$. Thus, $A(\rho)$ and $B(\rho)$ become some other functions E and G of r respectively. Consequently, this modification simplifies the line element to:

$$E(r)dt^2 + G(r)dr^2 + r^2 d\sigma^2.$$

[1] Rest space in the context presented here is referring to the space in the universe opposed to the space and time.

Note that our assumptions exhibit the manifold $\mathbb{R}^1 \times (\mathbb{R}^3 - \{0\})$ as a warped product, where $(B, g_B) = (\mathbb{R}^1 \times \mathbb{R}^+, E(r)dt^2 + G(r)dr^2)$ and $(F, g_F) = (S^2, d\sigma^2)$ is the standard unit sphere, and the warping function of B on F is $f(r,t) = r$. The keen-eyed reader may notice this as being strikingly similar to the central example we explored several times, our manifold: $M = \mathbb{R}^+ \times_r S^2$. Recalling Akbar's Theorem 6.9, it may initially appear that our manifold under consideration exhibits flatness, owing to two of the conditions that the fiber's constant sectional curvature and the linearity of our warping function. However the third condition is not met, the base manifold it not flat. This brings us back to a discussion in Chapter 5 Section 5.2, concerning Ricci flatness. While it is undeniably true that a flat manifold guarantees Ricci flatness, the converse does not hold; as evidenced by our current scenario, satisfying the conditions for Ricci flatness does not necessarily equate to the manifold being flat.

Asymptotically flat: As one moves progressively away from the gravitational source, its gravitational impact diminishes, eventually becoming negligible. This necessitates the condition $r \to \infty$ under which the Schwarzschild metric tensor converges towards a metric representative of empty flat spacetime. This is also known in literature as being Minkowski at infinity. That is, Minkowski space is a flat spacetime. As r approaches infinity, our metric on this spacetime tends to a flat one, so $E(r) \to -1$ and $G(r) \to +1$ as $r \to \infty$.

Vacuum: Lastly, the only source of gravitation in the Schwarzschild spacetime is the star itself. However this star is not modeled in the manifold therefore the spacetime is a vacuum. According to our earlier discussion, this means that our manifold is Ricci flat.

By contracting the curvature tensor, a routine but lengthy computation using contraction produces the following equations:

$$Ric(\partial_\varphi, \partial_\varphi) : \frac{G-1}{G} + \frac{rG'}{2G^2} - \frac{rE'}{2EG} = 0. \tag{7.1}$$

$$Ric(\partial_\theta, \partial_\theta) : 2G^2E - rE'G + rG'E - 2EG = 0. \tag{7.2}$$

$$\begin{aligned} Ric(\partial_r, \partial_r) : &-\frac{G'}{rG} + \\ &\left(\frac{1}{E^2G}\right) \cdot \left[\frac{E''EG}{2} - \frac{E'EG'}{4} - \frac{(E')^2G}{4}\right] = 0. \end{aligned} \tag{7.3}$$

$$\begin{aligned} Ric(\partial_t, \partial_t) : &G(E')^2r - 2GE''Rr \\ &+ E'EG'r - 4E'GE = 0. \end{aligned} \tag{7.4}$$

We recall the functions E and G are functions of r however, we ignore the r to illustrate the equations in a convenient way.

7.4 The Schwarzschild Solution

In addressing the term "solution" within the context of Einstein's equation, we are referring to a geometry of spacetime, defined by its metric, that satisfies the equation [Nee21]. To derive a solution for this metric we start with manipulating the Ricci equations above. If we multiply Equation (7.3) by $4E^2Gr$ and distribute, and we find that

$$-4G'E^2 + 2E''EGr - E'G'Er - (E')^2Gr = 0. \tag{7.5}$$

Now by adding Equations (7.4) and (7.5), we find:

$$-4G'E^2 - 4E'GE = 0.$$

However,

$$0 = -4G'E^2 - 4E'GE = -4E(G'E + E'G) = -4E\frac{d}{dr}(EG),$$

so $\frac{d}{dr}(EG) = 0$. Therefore EG is constant.

We are now in a position to draw substantial conclusions. Initially, observing that the limits as r approaches infinity are, $\lim_{r\to\infty} E(r) = -1$ and $\lim_{r\to\infty} G(r) = 1$, it logically follows that $\lim_{r\to\infty} E(r)G(r) = -1$. Given that EG maintains constancy across its domain, we deduce that $EG = -1$. Consequently, this implies that $G = \frac{-1}{E}$. Then the derivative with respect to r is $\frac{d}{dr}(EG) = E'G + EG' = 0$, we infer that $G' = \frac{-E'G}{E}$.

Simplifying further, we arrive at $E' = \frac{-EG'}{G} = \frac{G'}{G^2}$. We use this information in Equation (7.2):

$$\begin{aligned} 2G^2E - E'rG + rG'E - 2EG &= 0, \\ -2G - 2\tfrac{G'r}{G} + 2 &= 0, \\ G + \tfrac{G'}{G}r &= 1, \\ G'r &= G - G^2, \\ \tfrac{G'}{G-G^2} = \tfrac{G'}{G(1-G)} &= \tfrac{1}{r}. \end{aligned}$$

Using partial fractions, we integrate and then solve for G to find:

$$G = \frac{Cr}{1+Cr} = \frac{r}{\frac{1}{C}+r}.$$

In alignment with the existing literature on this derivation, we express this integration constant $C = -\frac{1}{2m}$ for some $m > 0$, so that $\frac{1}{C} = -2m$. Then,

$$\begin{aligned} G &= \frac{r}{\frac{1}{C}+r}, \\ G &= \left[\frac{r-2m}{r}\right]^{-1}, \\ G &= \left[1 - \frac{2m}{r}\right]^{-1}. \end{aligned}$$

Since $EG = -1$, it must then be that $E = -\left[1 - \frac{2m}{r}\right]$. We would like to draw the reader's attention to the noteworthy observation that if $r = 2m$, and then E and G are not defined. We have identified two distinct spacetimes! Since r represents a radial quantity from the point mass, $r > 2m$ is normal space outside the event horizon $r = 2m$, and $0 < r < 2m$ represents the interior of the event horizon.

In the literature regarding general relativity, m is representing the mass of a black hole under special physical circumstances. Schwarzschild at the time did not know that his discovery represented the pure vacuum gravitational field of a black hole. He simply was illustrating this models our own universe where m is the mass of any celestial body in our solar system [Nee21].

We pause to take note of a curious consequence of our efforts. Note that when $r > 2m$,

$$E(r) < 0, \text{ and } G(r) > 0.$$

But when $0 < r < 2m$, the signs of these change:

$$E(r) > 0, \text{ and } G(r) < 0.$$

A standard model for a spacetime is a four dimensional manifold M with a metric of index 1, where the one dimensional negative definite subspace in the manifold's tangent space represents the role of time in this spacetime. That $E(r) < 0$ for $r > 2m$ as the coefficient of dt^2 in our line element reinforces that the corresponding vector field ∂_t is the direction that time travels (forward) in, as we had always imagined it to. Since this line element is diagonal,i.e the coordinate vector fields are orthogonal, it makes sense that there are no other coefficients which are negative, in particular, for $r > 2m$ we have $G(r) > 0$.

When $0 < r < 2m$ however, the roles of the radial direction ∂_r and the time direction ∂_t reverse, as $E(r) > 0$ and $G(r) < 0$, suggesting that time flows according to your radius inside the event horizon instead of according to some clock. We do not attempt to explain this in more physical detail, but mention it as an intriguing artifact of this construction.

7.5 The Weyl Invariant

Using E and G are derived above, we close by considering the nature of the curvature of this spacetime. To do this we consider the Weyl invariant $||R||^2$, defined to be $R \otimes R$ contracted in the pairs of indices $(1,5), (2,6), (3,7)$ and $(4,8)$.

According to our curvature calculations in Proposition 6.6, and our metric above, we find the following to be the only nonzero curvature entries, up to the usual symmetries:

$$\begin{array}{ll} R(\partial_t, \partial_r, \partial_r, \partial_t) = \frac{2m}{r^3}, & R(\partial_t, \partial_\varphi, \partial_\varphi, \partial_t) = \frac{2m-r}{r^2} m, \\ R(\partial_t, \partial_\theta, \partial_\theta, \partial_t) = \frac{2m-r}{r^2} m \sin^2(\varphi), & R(\partial_r, \partial_\varphi, \partial_\varphi, \partial_r) = \frac{m}{r-2m}, \\ R(\partial_r, \partial_\theta, \partial_\theta, \partial_r) = \frac{m \sin^2(\varphi)}{r-2m}, & R(\partial_\varphi, \partial_\theta, \partial_\theta, \partial_\varphi) = -2rm \sin^2(\varphi). \end{array}$$

Using these entries and the metric tensor entries, after a somewhat lengthy computation which Maple assists us with, we find that,

$$||R||^2 = \frac{48m^2}{r^6}.$$

Thus, when $r = 2m$, the Weyl invariant $||R||^2 = \frac{3}{4m^4}$ has no special significance, suggesting that the event horizon of the black hole at the hypersurface $r = 2m$ is not unlike normal space. Indeed, a further study of this subject shows that particles may descend from outside the event horizon to inside the event horizon with no discernible problem at least described by this model. Intriguingly, this model allows for a transformation of coordinates to the “Kruskal plane”, a space that encompasses the event horizon itself. For a more comprehensive understanding, the reader is encouraged to refer to pages 386-398 in [O’n83], where this subject is explored in greater detail.

Simultaneously, we observe that,

$$\lim_{r \to 0^+} ||R||^2(r) = \infty,$$

so there is no smooth way to include the singularity of the black hole at $r = 0$ in this model. Thus, one expects that this singularity really is a “hole” in space.

In concluding our in-depth study of warped product manifolds’ curvature, we have unearthed a diverse array of findings that both resonate with and fortify the established paradigms within the realm of general relativity. Our work has served to forge meaningful connections between mathematical structures and a particular cosmological phenomena. This integration stands as a testament to the robustness and relevance of our approach. This effort lays a foundation for future research at the intersection of differential geometry and general relativity, as is evidenced by other literature in this area.

7.6 Closing Statement

Gratitude and appreciation extends to those who have taken the time to engage with this work. We hope that readers find this exploration of warped products to be both enlightening and intriguing. The central aim of this thesis is to provide clarity on some of the more complicated rigor often attached to understanding curvature on manifolds. And in turn hope that this exposition inspires readers to further explore curvature of warped product manifolds.

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