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Reverse Mathematics of Ramsey's Theorem

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A Thesis

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Abstract

Reverse mathematics aims to determine which set theoretic axioms are necessary to prove the theorems outside of the set theory. Since the 1970's, there has been an interest in applying reverse mathematics to study combinatorial principles like Ramsey's theorem to analyze its strength and relation to other theorems. Ramsey's theorem for pairs states that for any infinite complete graph with a finite coloring on edges, there is an infinite subset of nodes all of whose edges share one color. In this thesis, we introduce the fundamental terminology and techniques for reverse mathematics, and demonstrate their use in proving Kőnig's lemma and Ramsey's theorem over RCA_0.

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Chapter 1

Introduction

Reverse mathematics is a novel field of logic, related directly computability theory. Per Hirschfeldt [Hir15], the primary question in the field of reverse mathematics is “What are the necessary axioms in mathematics?” To find an answer, we must reason in the language of second order arithmetic–the weakest possible language that still retains the ability to express most mathematical definitions. The existence of such approach to studying mathematics was foreshadowed by the equivalence of the Axiom of Choice (AC) to Zorn’s Lemma under the Zermelo-Fraenkel (ZF) axiom system. This equivalence shows that Zorn’s Lemma is a fundamental notion in the given axiom system and that set theory is used in the said areas of mathematics implicitly. Similar results for other various theorems let us compare them by their relative strength. In such process, we can better understand the importance and the connections between them as they are applied to their each respective subfield of mathematics, motivating further research into reverse mathematics.

In this paper, we aim to provide an introduction to assessing provability of Ramsey’s theorem, an important statement for the field of combinatorics. All our work will be based off of the limited sets of axioms from the subsystems of second order arithmetic, such as ACA_0 and RCA_0. Throughout this process, we will introduce some notions from computability theory such as encodings, though the majority of our focus will remain on obtaining Ramsey’s theorem through several proofs with limited accessibility to other theorems.

Most of our material comes from [Sim09], [Hir15], and [Soa16]. We also assume the reader knows some of the basics of first-order logic, including the recursive definition of a first-order language, though this document is designed to provide a "ground-up" approach to the topic. For additional reference on this point, please see [End01].

Chapter 2

Models and $\mathbf{ACA}_0$

2.1 Models

The development of theorems in this document will be following the manner as they are presented in Simpson, Soare, and Hirschfeldt where relevant. In order to start proving theorems with a limited "toolbox" of mathematical axioms, we first must define the environment in which we will be proving the said theorems. In this section, we will extrapolate on Z_2, the formal system of second order arithmetic and define the relevant terminology. We will use $\mathbb{N}$ for the natural numbers, though some texts on this topic will use ω.

Definition 2.1. *If a variable $i \in \mathbb{N} = \{0, 1, 2, ...\}$, then i is a number variable or, alternatively termed, a variable of the first sort.*

Definition 2.2. *If a variable $X \subseteq \mathbb{N} = \{0, 1, 2, ...\}$, then X is a set variable or, alternatively termed, a variable of the second sort.*

For minimization of our "toolbox," we must define the symbolic "set" of our given language–many symbols in mathematics have various uses. If we define "addition" with the plus symbol, we can no longer use the plus symbol to show group action in the same language. As such, we proceed with the following definition:

Definition 2.3. *We define the language of the second order arithmetic, or L_2 as follows:*

- *Numerical terms are defined as the number variables from the Definition 1, including the terms defined by binary operations.*

- *Constant symbols are defined as "0" and "1", respectively meaning the empty set and the unit of the natural numbers.*
- *We allow "+" and "·" to represent addition and multiplication of natural numbers. It follows that* $t_1 + t_2$ *and* $t_1 \cdot t_2$ *are number variables as well, allowing the numerical terms denote every element of natural numbers.*
- *For numeric terms* t_1, t_2 *and set variable* X*, we define atomic formulas of* L_2 *as* $t_1 = t_2$*,* $t_1 < t_2$*, and* $t_1 \in X$*. The respective intended meanings are of equivalence between the two terms,* t_1 *being less than* t_2*, and* t_1 *being an element of* X*.*
- *A formula in* L_2 *is built up from atomic formulas, connected with propositional connective of* $\wedge$ *(and),* $\vee$ *(or),* $\neg$ *(not),* $\rightarrow$ *(implies), and* $\leftrightarrow$ *(if and only if).*
- *To make additional statements in* L_2*, we can employ number quantifiers* $\forall n$ *(for all* n*) and* $\exists n$ *(There exists an* n*) and set quantifiers* $\forall X$ *(for all* X*) and* $\exists X$ *(there exists a set* X*).*
- *We define a sentence in* L_2 *as a formula with no free variables, meaning there is no bound on the given variable.*

Definition 2.4. *We define a model for* L_2 *(otherwise known as a structure for* L_2 *or an* L_2*-structure) as an ordered 7-tuple:*

$$M = (|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$$

- $|M|$ *is the range of number variables*
- $\mathcal{S}_M$ *is the set of subsets of* $|M|$ *that defines the range of set variables.*
- 0_M *and* 1_M *are distinguished elements or constant symbols of* M
- $<_M$ *is the binary relation on* $|M|$*.*

We always assume $|M|$ *and* $\mathcal{S}_M$ *are disjoint and nonempty.*

Definition 2.5. *We claim* M *models* $\phi(n)$ *for any formula of* L_2 *using a number variable* n*, if* $\phi(n)$ *holds in* M*. We denote it with* $M \vDash \phi(n)$*.*

The model determines which sentences are considered true and false. For investigating the natural numbers, it is necessary to define a type of models termed ω-models.

Definition 2.6. *For any subset $\mathcal{B} \subset |M| \cup \mathcal{S}_M$, we define $L_2(\mathcal{B})$ to be the extended language that has constant symbols representing all elements of $\mathcal{B}$. A formula in $L_2(\mathcal{B})$ is referred to as a* formula with parameters from $\mathcal{B}$.

Though we will not be mentioning it explicitly, we will be continuously using formulas with parameters from $\mathcal{B}$, such as when we will be defining trees of sequences.

Definition 2.7. *A set $A \subseteq |M|$ is definable over M allowing parameters from $\mathcal{B}$ if there exists a formula $\phi(n)$ with parameters from $\mathcal{B}$ and no free variables other than n such that:*

$$A = \{a \in |M| : M \vDash \phi(a)\}$$

Definition 2.8. *We define ω-model as an L_2-structure of the form:*

$$M = (\mathbb{N}, \mathcal{S}, +, \cdot, 0, 1, <)$$

We define $\mathcal{S}$ as $\mathcal{S} \subseteq P(\mathbb{N})$, where $P(\mathbb{N})$ is the power set of $\mathbb{N}$, the subset containing every possible subset of $\mathbb{N}$.

Other models for use with $\mathbb{N}$ are possible, like β-models, but they will not be explored within this work. So from now on, number variables are assumed to take integer values, and set variables are assumed to be interpreted as subsets of $\mathbb{N}$. Given an L_2-structure M, it is a question which subsets of $\mathbb{N}$ actually appear (or are guaranteed to appear by the axioms true in M).

Because we wish to minimize the amount of statements we need to derive statements of advanced mathematics, we must also consider the smallest set of sentences in L_2 that can be used to derive all the other desired statements. When we consider ω-models going forward, we will only consider models of these basic axioms that are widely accepted to give true statements about $\mathbb{N}$. These axioms constitute the first-order L_2-theory Z_2, sometimes referred to as *second order arithmetic*. We define Z_2 as follows:

Definition 2.9. *We define the set of axioms of the second order arithmetic P_0 as the universal closure of the following statements:*

1. *Basic axioms:*
   $n + 1 \neq 0$
   $n + 1 = n + 1 \to m = n$

$m + 0 = m$

$m + (n + 1) = (m + n) + 1$

$m \cdot 0 = 0$

$m \cdot (n + 1) = (m \cdot n) + m$

$\neg m < 0$

2. *Induction axiom:*

$$(0 \in X \wedge \forall n(n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n(n \in X)$$

3. *Comprehension scheme:*

$$\exists X \forall n(n \in X \leftrightarrow \phi(n))$$

The main reason for developing the subsystems of Z_2 is to provide different constraints for set existence. In particular, notice that the Comprehension scheme is the only vector for proving existence of certain subsets of $\mathbb{N}$. For example, developing an infinite tree with restrictions may be possible in one formal system, but not in the other, thus limiting which sentences can be derived. Of particular interest to us are the two systems ACA_0 and RCA_0, which are restricted in specific manner to deal exclusively with arithmetical formulas.

Definition 2.10. *We claim that a formula $\phi(n)$ of L_2 is arithmetical if it has no set quantifiers i.e. $\phi(n)$ has only number quantifiers.*

Remark: arithmetical formulas can contain free set variables (i.e. unbound, without a set quantifier variables) and any kind of number variables and quantifiers. An example of an arithmetic formula can be an asserting that all elements n of X are odd:

$$\forall n(n \in X \rightarrow \exists m((m + m) + 1 = n)$$

Further notable assertions that can be done using arithmetic formulas involve defining sets that consist of even numbers, prime numbers, and differences of distinct natural numbers.

Definition 2.11. *We define the arithmetical comprehension scheme as a restriction to the comprehension scheme outlined in the Definition* (2.9)*, where $\exists X \forall n(n \in x \leftrightarrow \phi(n))$ for $\phi(n)$ being arithmetical. Similarly, an arithmetical induction scheme is an induction* *axiom restricted to arithmetical formulas, where* $\phi(0) \wedge \forall n(\phi(n) \to \phi(n+1))) \to \forall n \phi(n)$ *holds only for arithmetic* $\phi(n)$.

Definition 2.12. *We define the formal system* ACA_0 *as a subsystem of* Z_2*, which employs the language* L_2 *and uses basic and induction axioms of Definition* (2.9)*, as well as arithmetical comprehension schema.*

Through its language and axioms, ACA_0 comes equipped with the theories we need to assert most of the facts about a natural number system. For example, within the system, we are able to define numerous properties like the uniqueness and divisibility with respect to certain elements of natural numbers. Notably, $n \in \mathbb{N}$ being prime is also a statement we can express as an arithmetic formula:

$$\forall m \forall k(n = m \cdot k \to (m = 1 \vee k = 1)) \wedge n > 1 \wedge n \in X$$

With these tools, we are able to start defining more sophisticated sets, like the set $\mathbb{Z}$ of integers and the set $\mathbb{Q}$ of rational numbers. While we cannot define the set $\mathbb{R}$ of real numbers through the set comprehension schemes, we are able to define them as a Cauchy sequence of rational numbers, i.e. for $x = \langle q_n : n \in \mathbb{N} \rangle$ and ϵ ranging over $\mathbb{Q}$:

$$\forall \epsilon(\epsilon > 0 \to \exists m \forall n(m < n \to |q_m - q_n| < \epsilon))$$

Further notable results include defining complete separable metric spaces, separable Banach spaces, and continuous functions. While we will not explore these results in detail, we will use ACA_0 to define sequences necessary to discuss the provability of Ramsey theory in the following sections. Notably, ACA_0 is not the strongest "smallest" formal system that we can define using the axioms of the second order arithmetic. Additional restrictions can be posed onto formulas with respect to their set quantifiers, allowing us to define Σ^1_k and Π^1_k formulas.

Definition 2.13. *Let* θ *be an arithmetic formula and* X *be a set quantifier. Then, a formula of the form* $\exists X\theta$ *is a* Σ^1_1 *formula, while a formula of the form* $\forall X\theta$ *is a* Π^1_1 *formula. For* $0 \leq k \in \mathbb{N}$*, we claim that a formula* ϕ *of the following form is* Π^1_k*:*

$$\forall X_1 \exists X_2 \forall X_3 \exists X_4 \cdots X_k \theta$$

Similarly, we claim that a formula ϕ *of the following form is* Σ^1_k*:*

$$\exists X_1 \forall X_2 \exists X_3 \forall X_4 \cdots X_k \theta$$

Additionally, the Π^0_n *or* Σ^0_n *formulas are arithmetic.*

Example 2.14. *Let* θ *be an arbitrary arithmetic formula. Then, a formula of the form* $\exists X_1 \forall X_2 \exists X_3 \theta$ *is a* σ^1_3 *formula. A formula of the form* $\forall X_1 \exists X_2 \forall X_3 \exists X_4 \theta$ *is a* Π^1_4 *formula.*

Definition 2.15. *For* X *as a set variable which does not occur freely in* $\phi(n)$*, with* $\phi(n)$ *as any* Σ^0_1 *formula and* $\psi(n)$ *as any* Π^0_1 *formula, we define a* Δ^0_1 *comprehension scheme as universal closures for the formulas of the following form:*

$$\forall n(\phi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \forall \phi(n))$$

The definitions for Σ^0_k and Π^0_k come from compatibility theory. By Turing's Thesis (otherwise known as Turing's Theorem), we claim a function to be computable if and only if it can be computed by a Turing machine, an abstract machine capable of implementing all programming algorithms. Any Turing machine utilizes some number of two-way infinite tape consisting of cells with entries and produces an output through a set of algorithmic actions that can move, read, or change the contents of the cells.

Definition 2.16. *A Turing machine* M *computes the partial function* $f : A \rightarrow \mathbb{N}$ *for* $A \subseteq \mathbb{N}$ *if and only if* M *with input* $x \in A$ *eventually halts and outputs* $f(x)$*. We say the function* f *is partially computable.*

Definition 2.17. *A subset* $A \subseteq \mathbb{N}$ *is computably enumerable if it is the domain of a partially computable function.*

In less formal terms, we define sets A and $\bar{A}$ to be *computably enumerable* if there is a Turing machine such that for every $x \in \mathbb{N}$, the machine halts on input x with "yes" if $x \in A$ and halts with "no" if $x \notin A$. Another word for computable is *recursive*. From Simpson, we have the following characterization:

Corollary 2.18. *The minimum* ω*-model of* RCA_0 *is such that the subsets of* $\mathbb{N}$ *are exactly the recursive subsets.*

Thus, Δ^0_1-comprehension always affords us all the recursive subsets of $\mathbb{N}$. At this stage of developing our definitions, we emphasize that the notions of Σ^1_k and Π^1_k formulas can be used to restrict the axioms of ACA_0 even further, allowing us to define restricted inductions schemes:

Definition 2.19. *We define Σ_1^0 induction scheme as a universal closure of $\phi(0) \wedge \forall n(\phi(n) \rightarrow \phi(n+1))) \rightarrow \forall n \phi(n)$, where $\phi(n)$ is any Σ_1^0 formula of L_2. Π_1^0 and Δ_1^0 induction schemas is defined similarly for $\phi(n)$ respectively being any Π_1^0 or Δ_1^0 formula.*

Generally, the notion of formulas being Σ_k^0, Π_k^0, and Δ_k^0 can be extended to sets, where such classes define an arithmetical hierarchy corresponding to the sets' computability strength. The resulting hierarchy is referred to as Kleene-Mostowski hierarchy and is also of interest in the philosophical studies associated with logic. To provide examples of how we can reason about organizing such sets, we can take note of the Hierarchy Theorem from Chapter 4 of [Soa16].

Definition 2.20. *The Hierarchy Theorem states that for all $n \in \mathbb{N}^+$ and the collections of corresponding formulas by type Δ_n, Σ_n, and Π_n, $\Delta_n \subset \Sigma_n$ and $\Delta_n \subset \Pi_n$ such that $\Sigma_n \not\subset \Delta_n$.*

In this context, as a consequence of reverse mathematics, we can discuss how computationally simple or complex specific proofs are, connecting various theorems with the concepts of relativized halting problems (how and when a Turing program can end) and Turing degrees (measures of computational difficulty). Though we will discuss the implications of certain theorems being provable in ACA_0 or RCA_0, we will opt to handwave towards these implications as opposed to formalizing them within this work. Interested readers are encouraged to further read Chapters 3 and 4 of [Soa16] and the first two sections of [Hir15] for a deeper insight into the matters of computability theory.
We now have enough tools to start defining an additional subsystem of interest termed RCA_0. This subsystem which concerns itself with being limited to recursive functions while providing us with tools to work with infinite sets. RCA_0 is substantially weaker than ACA_0 and can be used as a limited foundation for reconstruction of proofs of Ramsey theory. This system, along with its importance, will be defined in the next sections, where we will take a closer look at some foundational results, like Kőnig's lemma along with the Erdős/Rado trees, and how they connect to our main tool of defining tree structures - recursive functions.

Chapter 3

Finite Sequences

3.1 Setting up the Number System

We now established that we are working in the subsystem of the second order arithmetic Z_2, which allows us to select a very specific environment with limited axioms, thereby limiting our ability to prove certain theorems. To demonstrate how fundamental certain theorems are, we aspire to prove them in the most practically limited environment possible: RCA_0. In order to do so, it is helpful to define several lemmas and enable ourselves to regard most functions as sequences, which will be the main point of this section. The material in this chapter will thoroughly reference Chapters II.1-II.3 in [Sim09]

We will first define RCA_0, a subsystem of ACA_0 which has further limitations on the types of formulas it can prove.

Definition 3.1. *We define P_0 as a set of following first-order axioms*

1. $\forall x(x + 1 \neq 0)$
2. $m + 1 = n + 1 \rightarrow m = n$
3. $m + 0 = m$
4. $m + (n + 1) = (m + n) + 1$
5. $m * 0 = 0$
6. $m \cdot (n + 1) = (m \cdot n) + m$

7. $\forall m \neg m < 0$

8. $\forall n \forall m (m < n + 1 \leftrightarrow (m < n \vee m = n))$

Note that these axioms are some of the few tools that we have at the moment to start proving mathematical statements of major importance–the P_0 set is extremely limited. Previously, we defined the Δ^0_1 comprehension scheme, where $\phi(n)$ is any Σ^1_0-formula, $\psi(n)$ is any Π^0_1, n is any number variable, and set is a set variable that is not free in $\phi(n)$:

$$\forall n(\phi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \phi(n)))$$

.

Alongside this scheme, another crucial component of RCA_0 is the Σ^0_1-induction scheme, previously defined as a restriction of the second order induction scheme, which provides a universal closure for any $\phi(n)$ such that $\phi(n)$ is a Σ^0_1-formula of L_2:

$$(\phi(0) \wedge \forall n(\phi(n) \rightarrow \phi(n+1))) \rightarrow \forall(n)\phi(n)$$

Definition 3.2. *RCA_0 is a subsystem of the second order arithmetic, consisting of P_0, Δ^0_1-comprehension scheme, and Σ^0_1-induction scheme.*

To investigate any possible mathematical statements that concern natural numbers or operations on them, we must properly define the system of natural numbers first. To do so, we fix any model of RCA_0 as in (2.4) such that P_0 holds for all $m, n \in M$:

$$M = (|M|, \mathcal{S}_M, +_M, \cdot_M, 0_M, 1_M, <_M)$$

Our first task is to show that the first-order component $N = (|M|, +_M, \cdot_M, 0_M, 1_M, <_M$) is a commutative ordered semi-ring with cancellation, meaning our construction M is isomorphic to $\mathbb{N}$. Note that this statement can be unraveled into 25 distinct properties for any elements $n, m, p \in N$.

Theorem 3.3. *The following statements are provable in RCA_0:*

1. $(m + n) + p = m + (n + p)$

2. $0 + m = m$

3. $1 + m = m + 1$

4. $m + n = n + m$

5. $m \cdot (n + p) = m \cdot n + m \cdot p$

6. $(m \cdot n) \cdot p = m \cdot (n \cdot p)$

7. $(m + n) \cdot p = m \cdot p + n \cdot p$

8. $0 \cdot m = 0$

9. $1 \cdot m = m$

10. $m \cdot n = n \cdot m$

11. $(m < n \wedge n < p) \rightarrow m < p$

12. $m < n \rightarrow m + 1 < n + 1$

13. $n \neq 0 \rightarrow 0 < n$

14. $m < n \wedge m = n \wedge n < m$

15. $\neg n < n$

16. $m + p < n + p \rightarrow m < n$

17. $m < m + n + 1$

18. $m + p = n + p \rightarrow m = n$

19. $(p \neq 0 \wedge m < n) \rightarrow m \cdot p < n \cdot p$

20. $(p \neq 0 \wedge m \cdot p < n \cdot p) \rightarrow m < n$

21. $(p \neq 0 \wedge m \cdot p = n \cdot p) \rightarrow m = n$

22. $m < n \rightarrow (\exists k < n) m + k + 1 = n$

23. $n \neq 0 \rightarrow (\exists m < n) m + 1 = n$

All such properties would use induction and periodically require the use of other properties of the said semi-ring for some associated proofs. We will present an example as follows:

Lemma 3.4. $\forall n \in \mathbb{N}, 1 + n = n + 1$

For this proof, we will need to assume that $\forall m, n, p \in \mathbb{N}, (m+n)+p = m+(n+p)$ (associative property for $\mathbb{N}$). In the context of Lemma 1, this property would normally be proven beforehand.

Proof. We proceed by induction.
Base Case: Let n = 0. By an L_2 axiom, we know $\forall n \in \mathbb{N}, n + 0 = 0$.
So, $1 + 0 = 1 = 1 + 0$. Base case holds.
Inductive Assumption: $1 + (n - 1) = (n - 1) + 1$
Inductive Proof: Using the inductive assumption, we want to show that $1 + n = n + 1$.
Assume that $\forall m, n, p \in \mathbb{N}, (m + n) + p = m + (n + p)$.
It follows that $(n-1)+1 = n+(-1+1) = n$, so $1+(n-1) = n$. Then, $n+1 = 1+(n-1)+1$.
By the associative property for $\mathbb{N}$, $1 + (n - 1) + 1 = 1 + n + (-1 + 1) = 1 + n$.
Thus, $n + 1 = 1 + n$. □

In a manner similar to this proof, we can cascade our results to build up all the axioms of $\mathbb{N}$, allowing us to employ all the properties of natural numbers to work with functions and sequences. For the future purposes, it is also helpful to define a pairing map:

Definition 3.5. *We define a pairing map for* $i, j \in \mathbb{N}$ *as* $(i, j) = (i + j)^2 + i$.

This pairing map can also be defined as $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and will help us develop an encoding method to represent information in the form of natural number sequences. We will also include an additional lemma concerning some properties in RCA_0:

Lemma 3.6. *The following have a pairwise equivalence over* RCA_0*:*

1. ACA_0
2. Σ^0_1 *comprehension*
3. *For all one-to-one function* $f : \mathbb{N} \to \mathbb{N}$ *there exists a set* $X \subseteq \mathbb{N}$ *such that* $(n \in X \leftrightarrow (f(m) = n))$. *In other words,* X *is the range of* f.

We will examine the relevance of this lemma in the proof of Ramsey's theorem. In the following section, we will introduce a fundamental concept for getting to that point concerning encoding mathematical information using natural numbers.

3.2 Encoding Algorithms

Reverse mathematics, as a field of study, needs a way to manipulate and obtain information from various mathematical objects, such modules, rings, and topological spaces. However, given that our methodology thus far employs numerous techniques from computability theory, it becomes increasingly difficult to study structures that are essentially uncountable, like particular topological spaces, while limited to the second order arithmetic alone. To circumvent this issue, we employ *encoding* as a way to express information concerning any mathematical object using natural numbers. Following, we present a method of encoding *finite sets as natural numbers.* To do so, we must first define what a finite set is and then provide two lemmas that come from basic number theory and are necessary to show that there exists a unique encoding for each such set.

Definition 3.7. *In RCA_0, X such that $\exists k \forall i (i \in X \to i < k)$ is a finite set.*

Definition 3.8. *We say that m_1 is prime relative to m_2 if $\forall n(m_2 | m_1 n \to m_2 | n)$.*

Lemma 3.9. *The following fact is provable in RCA_0: for all $m_1, m_2 \in M$ if m_1 is prime relative to m_2, m_2 is prime relative to m_1.*

Lemma 3.10. *The following is provable in RCA_0:*

1. *Given k, there exists $m > 0$ such that $\forall i < k(i + 1 | m)$*
2. *Let k and m be as in (1). Then $m(i + 1) + 1$ and $m(j + 1) + 1$ are relatively prime to each other for all $i < j < k$.*

Theorem 3.11. *For any finite set $X \subseteq \mathbb{N}, \exists n, m, k \in \mathbb{N}$ such that $\forall i((i \in X) \iff ((i < k) \wedge ((m(i + 1) + 1)|n)))$*

Before engaging with the proof of this statement, it is helpful to look at a few examples to parse the importance of this theorem.

Example 3.12. *For our first example, suppose $X = \{0, 3\}$, where all of the elements of X are denoted as i. We fix k such that $\forall i \in X, k > i$, so one possible value is $k = 4$. Following, we define m consistently with Lemma 3.1. In this case, to allow $\forall i < k(k{+}1|m)$ to hold, we will define m as follows:*

$$m = \prod_{i=\{0,3\}} (i+1) = (0+1)(3+1) = 4$$

For $i = 0$, $m(i+1)+1 = 5$. For $i = 3, m(i+1)+1 = 17$. The two values are relatively prime, thus satisfying the lemma 3.2. Additionally, $n = 85$ satisfies the condition that $m(i+1)+1|n$ for all i.

Example 3.13. *Another example satisfying both the Theorem 4 and Lemma 3 is as follows. Consider $X = \{2, 4, 5\}, m = (2+1)(4+1)(5+1) = 90$. Then, the outputs of $m(i+1)+1$ for each $i \in X$ are, respectively, 271, 451, and 541. Remarkably, all of the values are relatively prime. Then, $n = 271 * 451 * 541 = 66121561$.*

Proof. (of Theorem 3.11) Let k satisfy $\forall i(i \in X \to i < k)$. By Lemma 3, we can fix m such that $m(i+1)+1$ for $i < k$ are pairwise relatively prime. From construction of RCA_0, we can create a Σ^0_1-formula as follows:

$$\varphi(j) = j > k \vee \exists n \forall i < k\,[(m(i+1)+1)|n \leftrightarrow (i \in X \wedge i < j)]$$

We can verify that this formula is Σ^0_1 by writing out an equivalent formula:

$$\varphi(j) = j > k \vee \exists n \forall i < k\,[(\exists t < n)((m(i+1)+1) \cdot t = n) \leftrightarrow (i \in X \wedge i < j)]$$

In turn, by standard first-order logical equivalences (see [End01]), this is equivalent to:

$$\varphi(j) = \exists n[j > k \vee \forall i < k\,[(\exists t < n)((m(i+1)+1) \cdot t = n) \leftrightarrow (i \in X \wedge i < j)]]$$

All possible cases for values of j can be summarized as follows:

(1) $j = 0$

(2) $j > k$

(3) $0 < j \leq k$

We will aim to show that $\varphi(j)$ holds for all $j \in \mathbb{N}$. In the case of (2), $\varphi(j)$ is trivially true. To show that $\varphi(j)$ holds in the other cases, we will focus on proving that

$$\exists n \forall i < k\, [(m(i+1)+1)|n \leftrightarrow (i \in X \wedge i < j)] \tag{3.1}$$

holds by induction while using (1) as a base case for such a statement. Note that we may assume $j < k$ in our induction hypothesis, because the induction hypothesis for $j = k$ immediately yields $\varphi(j+1)$ since we fall into case (2).

Note that we can avoid claiming that $j = k$ fixing $k = \Pi_{i<k}(i+1)$.

Let $j = 0$. Then, clearly $j \leq k$, meaning we only need to show that $m(i+1)+1 \nmid n$. Assume $m(i+1)+1|n$ where $n = (\Pi_{i<k} m(i+1)+1)+1$. It follows that $m(i+1)+1|n - \prod_{i<k}(m(i+1)+1)$. Then $m(i+1)+1|1$, but $m(i+1)+1 \geq 2$ for any $i \in \mathbb{N}$. This is a contradiction, so $m(i+1)+1 \nmid n$ and the base case holds. For an inductive assumption, suppose that $\exists n \forall i < k\, [(m(i+1)+1)|n \leftrightarrow (i \in X \wedge i < j)]$. We must now show that for $j' = j+1 \leq k$, $n' = n(m(j+1)+1)$.
Note that if $j \notin X$, $j > k$ or $j = k$.
If $j \notin X$, we set $n' = n$ If $j > k$, $\varphi(j)$ is true. By inductive assumption, $((i \in X) \wedge i < j') \rightarrow \forall i < k[((m(i+1)+1)|n]$, meaning $\varphi(j)$ holds. In either case, $\varphi(j)$ holds for all $j \in \mathbb{N}$. This concludes the proof. □

While we have a way to encode finite sets now, we do not directly have a way to start defining operations on natural numbers. This can be done by defining functions as sequences of natural numbers which can be encoded with the method provided above. By Σ^0_1-comprehension, as all of the possible sequences we can consider can be described in a formula using an existential quantifier, we can define a set containing all the codes of finite sequences. We will denote it as either "Seq" or $\mathbb{N}^{<\mathbb{N}}$. Note that while each natural number that corresponds to a unique sequence is a regular, finite value, the sequence of natural numbers that corresponds to it and encodes information is not always finite.
We will denote the sequence S as one of the following:

$$s = \langle s(0), s(1), ..., s(\mathrm{lh}(s)-1) \rangle$$

$$s = \langle s(i) : i < \mathrm{lh}(s) \rangle$$

For $s, t \in$ Seq, s being concatenated with t as denoted as follows:

$$s\hat{\ }t = \langle s(0), ..., (s(\mathrm{lh}(s)-1), t(0), ..., t(\mathrm{lh}(s)-1) \rangle$$

Note that $\mathrm{lh}(s\hat{\ }t) = \mathrm{lh}(s) + \mathrm{lh}(t)$.

3.3 Recursion

In order to prove certain statements, we must introduce the concept of functions and primitive recursion. The purpose of defining the latter here is for developing a theorem that can generate successors to certain inputs on a sequence, thus allowing us to prove Kőnig's lemma and other mathematical statements within a limited system like RCA_0.

To proceed, we first definite functions within RCA_0:

Definition 3.14. *Assume RCA_0 and let $X, Y \subset \mathbb{N}$. We claim $X \subseteq Y$ if $\forall n(n \in X \to n \in Y)$.*

Definition 3.15. *$X \times Y$ is the set of all k such that $\exists t \leq k \exists j \leq k[(i \in X) \wedge (j \in Y) \wedge ((i,j) = k)]$*

Definition 3.16. *Let $f \subset (X \times Y)$. $f : X \to Y$ is a function if the following two formulas hold:*

$$\forall i \forall j \forall k[(((i,j) \in f) \wedge ((i,k) \in f)) \to (j = k)]$$

$$\forall i \exists j[(i \in X) \to ((i,j) \in f)]$$

If $f : X \to Y$ and $i \in X$, $f(i) = j$ such that $(i,j) \in f$.

Theorem 3.17. *Under RCA_0, if $f : X \to Y$, $g : Y \to Z$, then there is $h = gf : X \to Z$ such that $h(i) = g(f(i))$.*

Proof. By the definition of a function, since we are given g and f, we know:

$$[(\exists j((i,j) \in f \wedge (j,k) \in g)) \leftrightarrow ((i \in X) \wedge (\forall j(((i,j) \in f \to (j,k)) \in g))]$$

Let $m = (i,k) = (i+k)^2 + i$. Then, let $\varphi(m) = \exists j[((i,j) \in f \wedge (j,k) \in g)$ and $\psi(m) = \forall j(((i,j) \in f \to (j,k)) \in g$. Notably $\psi(m) \leftrightarrow \varphi(m)$, as g and f would contradict the given restrictions otherwise. By Δ^0_1-comprehension, there exists a set h witnessed by the formula $\theta(m)$ such that:

$$\theta(m) = [\forall m(\phi(m) \leftrightarrow \psi(m)) \leftrightarrow \exists h \forall m((m \in h) \wedge (\phi(m))$$

This formula satisfies the conditions for the definition of $h(i,k)$ as a function, where $h = gf$. $\square$

Definition 3.18. *By Σ^0_0-comprehension, there exists a set of all $s \in Seq$ such that $lh(s) = k$, denoted $\mathbb{N}^k$. For $f : \mathbb{N}^k \to \mathbb{N}$ and $s = \langle n_1, ..., n_k \rangle \in \mathbb{N}^k$, we can write $f(n_1, ..., n_m) = f(s)$.*

Definition 3.19. *We define the successor function as $S(n) = n + 1$ for $n \in \mathbb{N}$.*

Example 3.20. *One of the most primitive recursive functions one can define is the addition function $A(n,m) = h(m) = m+n$ for $n, m \in \mathbb{N}$. We can define h as a combination of two functions f and g:*

$$h(0,n) = f(n) = n + 0 = n$$

$$h(m+1,n) = g(h(m,n),m,n) = S(h(m,n)) = S(n+m)$$

It is evident that if we add 0 to any n, n is the output of the addition. In case of $m \neq 0$, suppose $A(2,3)$, the following occurs:

$$A(2,3) = g(2) = S(1) + 3 = S(1+3) = S(4) = 5$$

Theorem 3.21. *In RCA_0, given $f : \mathbb{N}^k \to \mathbb{N}$ and $g : \mathbb{N}^{k+2} \to \mathbb{N}$, there exists a unique $h : \mathbb{N}^{k+1} \to N$ defined as:*

$$h(0, n_1, ..., n_k) = f(n_1, ..., n_k)$$

$$h(m+1, n_1, ..., n_k) = g(h(m, n_1, ...n_k), m, n_1, ..., n_k)$$

Proof. Consider a formula $\theta(s, m, \langle n_1, ..., n_k \rangle)$ as follows:

$$\theta(s, m, \langle n_1, ..., n_k \rangle) = [((s \in \text{Seq}) \wedge ((lh(s) = m+1) \wedge (s(0) = f(n_1, ..., n_m)) \wedge (\forall i < m(s(i+1) = g(s(i), i, n_1, ..., n_k)))))]$$

Note that for $\langle n_1, ..., n_k \rangle \in \mathbb{N}^k$, The formula $\varphi(s) = \exists s(\theta(s, m, \langle n_1, ..., n_k \rangle))$ is provable by Σ^0_1-induction on m. Note that our underlying model $M \vDash \text{RCA}_0$. We proceed by exploring the base case of $m = 0$. Then, for $\theta(0, \langle n_1, ..., n_k \rangle)$, $lh(s) = 1$ and i must be less than 0. Hence, θ holds vacuously true for $m = 0$. For the inductive assumption, we assume RCA_0 proves $\theta(m, \langle n_1, ..., n_k \rangle)$ and now proceed to show that RCA_0 proves $\theta(m+1, \langle n_1 \rangle)$. Consider a sequence $s_2 \in M$ such that $s_2 = s\hat{\ }t$ for some $t \in \mathbb{N}$ where $lh(t) = 1$. Then, it follows that:

1. $s_2 \in \text{Seq}$
2. $lh(s_2) = m + 2$
3. $s_2(0) = f(n_1, ..., n_k) = s(0)$
4. $\forall i$ such that $i < m + 1$, by inductive assumption, $s(i + 1) = g(s(i), i, n_1, ..., n_k)$. Note that if $i = m$, then $s_2(m + 2) = t(0)$, meaning $s(i + 1) = g(s(i), i, n_1, ..., n_k)$ holds by the base case.

Hence, $\varphi(s) = \exists s(\theta(s, m, \langle n_1, ..., n_k\rangle))$ holds for all lengths finite lengths m. By a similar approach, $\text{RCA}_0 \vDash (\theta(s, m, \langle n_1, ..., n_k\rangle) = \theta(s', m, \langle n_1, ..., n_k\rangle \leftrightarrow (s(i) = s'(i)))$ by induction on i, for all $i < m+1$. Then, for all $m, j \in \mathbb{N}$, $\langle n_1, ..., n_k\rangle \in \mathbb{N}^k$, the following formula holds:

$$((\exists s((\theta(s, m, \langle n_1, ..., n_k\rangle) \wedge (s(m) = j)))) \leftrightarrow (\forall s((s, m, \langle n_1, ..., n_k\rangle) \rightarrow (s(m) = j))))$$

Note that this formula satisfies the antecedent for a Δ^0_1-comprehension. Thus, by Δ^0_1-comprehension, there exists a finite set $h \subseteq \mathbb{N}^{k+1} \times \mathbb{N}$, a function by definition of the term, such that $h(m, n_1, ..., n_k) = j$ if and only if $\exists s(\theta(s, m, \langle n_1, ..., n_k\rangle) \wedge (s(m) = j))$. □

It is worth further discussing that RCA_0 proves the closure under minimization, that is that functions have the smallest element that they hold for.

Theorem 3.22. *Under RCA_0, let $f : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$ such that for all $\langle n_1, ..., n_k\rangle \in \mathbb{N}^k$, there exists $m \in \mathbb{N}$ such that $f(m, n_1, ..., n_k) = 1$. It follows that there exists a function $g : \mathbb{N}^k \rightarrow \mathbb{N}$ such that $g(n_1, ..., n_k) =$ least m and $f(m, n_1, ..., n_k) = 1$.*

Proof. Note that to define a subset of $\mathbb{N}^k \times \mathbb{N}$, we do not require any quantifiers. Thus, by Σ^0_0-comprehension, there exists a set $g \subseteq \mathbb{N}^k \times \mathbb{N}$ such that $((\langle n_1, ..., n_k\rangle, m) \in g) \leftrightarrow (((\langle m, n_1, ..., n_k\rangle, 1) \in f) \wedge (\neg(\exists j < m(\langle j, n_1, ..., n_k\rangle, 1) \in f))$. The given conditions for the set g satisfy the definition of a function that holds consistent with closure under minimization as described above. □

Closure under minimization yields useful theorems, such as closure under ordering and infinite recursively enumerable set being a range of a one-to-one recursive function.

Lemma 3.23. *In RCA_0, for any infinite set $X \subseteq \mathbb{N}$, there exists a function $\pi_X : \mathbb{N} \to \mathbb{N}$ such that $\forall k \forall m((k < m) \to (\pi_X(k) < \pi_X(m)))$ and $\forall n((n \in X) \leftrightarrow (\exists m(\pi_X(m) = n)))$.*

Lemma 3.24. *Let $\varphi(n)$ be a Σ^0_1-formula such that X and f do not occur freely. Then, the following is provable in RCA_0. Either there is a finite set X such that $\forall n((n \in X) \leftrightarrow (\varphi(n)))$ or there exists a one-to-one $f : \mathbb{N} \to \mathbb{N}$ such that $\forall n((\varphi(n) \leftrightarrow (\exists m(f(m) = n)))$.*

While these results are notable on their own, we can use them to expand the range of theorems provable in RCA_0 by developing additional induction and comprehension schemes. Some of such notable results are as follows:

Theorem 3.25. *RCA_0 proves bounded Σ^0_1-comprehension.*

Theorem 3.26. *RCA_0 proves the Π^0_1-induction scheme for any Π^0_1-formula (n):*

$$(\psi(0) \wedge \forall n(\psi(n) \to \psi(n+1))) \to \forall n(\psi(n))$$

Thus far, we were able to define several notions with respect to recursive character of various functions for natural numbers. Our ability to discuss these functions in RCA_0 also holds some implications for the provided functions through computability concepts.

Lemma 3.27. *Let $X, Y \subset \mathbb{N}$. The following are equivalent:*

1. *X is recursively enumerable (i.e. X serves as a range of some recursive function) in Y.*

2. *X is definable in some model of Z_2 by a Σ^0_1 formula with parameter Y.*

Additionally, we can discuss the collection of all possible recursive functions for $\mathbb{N}$ as follows:

Lemma 3.28. *The minimum ω-model of RCA_0 is the collection $REC = \{X \subseteq \omega : X \text{ is recursive}\}$.*

The nature of recursive sets as a minimum ω-model carries significant mathematical power for provability of theorems and organization of sets in model theory and are studied for theorems related to degrees of unsolvability. For deeper insight, we recommend reading Chapters I II, specifically p. 64 of [Sim09] for further direction. We will now transition from recursive functions to their application in proving a major fundamental result for our goal: Kőnig's lemma.

3.4 Kőnig's Lemma

Our work in RCA_0 thus far allowed us to analyze and assert the existence of certain functions using the provided set of axioms. It should be obvious by now that existence of a set is rarely a trivial matter. This matter requires us to employ RCA_0 over ACA_0 to show the existence of certain infinite sets - a property that is essential to obtaining Ramsey's Theory through reverse mathematics. Through the following definitions, we hope to prove Kőnig's lemma and show that given RCA_0, there exists an infinite subset in a provided collection of subsets. We will employ the methodology as Simpson does in Chapter III.7 of [Sim09]. We begin our method with defining trees.

Definition 3.29. *A tree is a set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that $\forall\sigma\forall\tau[((\sigma \in \mathbb{N}^{<\mathbb{N}}) \wedge (\sigma \subseteq \tau) \wedge (\tau \in T)) \to (\sigma \in T)]$.*

Remark: we will typically refer to trees as structures that have downward closure.

Definition 3.30. *If $\forall\sigma[(\sigma \in T) \to (\exists n\forall m((\sigma^\frown\langle m\rangle \in T) \to (m < n)))]$, we say that T is finitely branching.*

Definition 3.31. *A path through T is a function $g : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathbb{N}(g[n] \in T)$. We write $g[n] = \langle g(0), g(1), ..., g(n-1)\rangle$.*

Definition 3.32. *We define the set T^* as the set of all $\tau \in T$ such that there exist infinitely many $\sigma \in T$ such that $\sigma \supseteq \tau$.*

Lemma 3.33. *T^* is a tree.*

Proof. Assume for arbitrary ρ and τ that $[((\rho \in \mathbb{N}^{<\mathbb{N}}) \wedge (\rho \subseteq \tau) \wedge (\tau \in T^*))]$. We want to show that $\rho \in T*$. By definition of $T*$, τ is in $T*$ if and only if

1. $\tau \in T$
2. There exist infinitely many $\sigma \in T$ such that $\sigma \supseteq \tau$.

To show that $\rho \in T*$ we must establish that ρ has the properties above.

1. First of all, to establish (1): $\rho \in T$ holds because we assumed $\rho \subseteq \tau$ and $\tau \in T$. Note, $\tau \in T$ because we assumed $\tau \in T*$ and as a consequence of that assumption, $\tau \in T$. So, by the property of trees being closed downward, $\rho \in T$.

2. Moreover, to establish (2): every σ that contains τ must also contain ρ if $\rho \subseteq \tau$. This is by transitivity of inclusion.

$$\rho \subseteq \tau \text{ (by assumption) } \wedge \tau \subseteq \sigma \text{ (by above) } \Rightarrow \rho \subseteq \sigma$$

Thus the infinitely many $\sigma \in T$ that witness that $\tau \in T*$ also witness that ρ is in $T*$.

□

Lemma 3.34. *Kőnig's lemma states that every infinite, finitely branching tree has at least one path.*

Theorem 3.35. *The following statements are pairwise equivalent over RCA_0:*

1. *ACA_0*
2. *Kőnig's lemma*
3. *Kőnig's lemma restricted to trees $T \subseteq \mathbb{N}$ such that for all subsequences $\sigma \subset T$, σ only has at most two immediate successors.*

Proof. We first prove that ACA_0 implies Kőnig's lemma. First, we assert that $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is an infinite tree with finite branching. By arithmetical comprehension, T^* exists in ACA_0. Note that for any $\tau \in T$, $\langle\rangle \subseteq \tau$. Thus, since T is infinite, $\langle\rangle$ has infinitely many extensions $\tau \in T$. Thus $\langle\rangle \in T*$. Additionally, for an empty sequence $\langle\rangle$, we can construct infinitely many finitely branching extensions in T, meaning $\langle\rangle \in T^*$. A path from an empty tree to a successor node can be represented with a function $f : \mathbb{N} \to \mathbb{N}$, which will vacuously satisfy the recursive function definition as empty set has zero length.

Additionally, because $T*$ is finitely branching, there must exist the least successor $\tau^\frown n \subset T^*$ for all $\tau \subset T^*$. We can represent a path from a node enumerated $n-1$ in τ to its successor enumerated n.

Thus, by Theorem 3.21, there exists a recursive function $h : \mathbb{N} \to \mathbb{N}$ such that $h(k) = m$, where m is the least possible value such that $h(k)^\frown m \in T^*$. Thus, there exists a path through any subsequence of an infinite, finitely branching tree. Thus, ACA_0 implies Kőnig's lemma.

Restricted Kőnig's lemma follows automatically from the general version of the said lemma. We only have to show that the restricted Kőnig's lemma implies ACA_0. We assume RCA_0. To achieve our goal, the easiest way is to show that for an arbitrary one-to-one function $f : \mathbb{N} \to \mathbb{N}$, the range exists as a set, as that would be equivalent to ACA_0 under Lemma 3.6:

$$\exists X \forall n((n \in X) \leftrightarrow (\exists m(f(m) = n)))$$

By Σ^0_0-comprehension, there exists a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ with the following conditions:

$$((\tau \in T) \leftrightarrow (\forall m < (\tau))(\forall n < lh(\tau))((f(m) = n) \leftrightarrow (\tau(n) = m + 1)))$$

$$(\forall n < lh(\tau))((\tau(n) > 0) \to (f(\tau(n) - 1) = n)$$

In simpler terms, we claim that for all preceding subtrees σ of τ, if $\tau(n) > 0$, then m is part of the domain of f. Furthermore, each m that is in domain of f has a successor node in T, meaning that T is infinite.

It then follows that for each $\sigma \in T$, there are only two possibilities for its immediate successors:

1. $\sigma^\frown(m + 1)$, where $f(m) = n$ for all $\tau(n) > 0$.
2. $\sigma^\frown(0)$, which falsifies the condition of T (7), meaning m does not belong to the domain of f

By bounded Σ^0_1-comprehension, we define Y to be the set of elements for the range of f–it is the set of all $n < k$ such that $\exists m(f(m) = n)$. Then, we fix $k \in \mathbb{N}$ such that for $\sigma \in \mathbb{N}^{<\mathbb{N}}$ and $lh(\sigma) = k$, the following holds for all $n < k$:

$$\sigma(n) = \begin{cases} 0 & \text{if } n \notin Y \\ m + 1 & \text{if } n \in Y \wedge f(m) = n \end{cases}$$

Note that either output is consistent with the conditions we provided for T earlier, meaning $\sigma \in T$. Furthermore, if there is at least one node in σ such that $\sigma(n) \neq 0$, then the $m + 1$ case applies and continues to apply for all successors by conditions of T. As such, T is infinite. Thus, by Restricted Kőnig's Lemma, there exists a path g through T.

By condition (number) of T, we claim $\forall m \forall n((f(m) = n) \leftrightarrow g(n) = m + 1)$. So, by Δ^0_1-comprehension, we define the set X to contain all n such that $g(n) > 0$. It follows that $\forall n((\exists m(f(m) = n)) \leftrightarrow (n \in X))$. It is worth noting that we were able to prove both weak and strong versions of Kőnig's lemma, with the primer being restricted to sequences of 0's and 1's. For the purposes of proving Ramsey's theorem, we will only use the strong version of the lemma. However, we can expand RCA_0 by including the weak Kőnig's lemma into its axiom list to define a new subsystem of Z_2 called WKL_0. This theory is particularly interesting due its strength being sufficient to prove *Heine/Borel theorem*:

Definition 3.36. *Heine/Borel theorem states that every covering of the closed unit interval* $0 \leq x \leq 1$ *by a sequence of open intervals has a finite subcovering.*

Furthermore, it is possible to obtain a reversal showing that WKL_0 is equivalent to Heine/Borel theorem over RCA_0 and compare that result to various other theorems, including Gödel's completeness theorem and Hahn/Banach theorem for separable Banach spaces. However, this investigation is not included in the scope of this work. Interested readers are encouraged to read Chapter IV of [Sim09] to obtain detailed information on the matter. Instead, we will now proceed to employ strong Kőnig's lemma to obtain Ramsey Theorem in the next chapter. □

Chapter 4

Ramsey's Theorem

4.1 Introducing the Theorem

Ramsey's theorem is a powerful combinatorial tool that deals with the idea of order. In layman's terms, this theorem implies that in a sufficiently large set, we can always find a "very orderly" subset that shares some sort of property like coloring. This is particularly evident in the context of graph theory, where Ramsey theorem is arguably the most prominent.

Definition 4.1. *Given integers n, m, the Ramsey number $p = r(m, n)$ is the least number p such that for any 2-coloring of the edges of K_p, there exists a subgraph isomorphic to K_m of color 0 under this coloring, or a subgraph isomorphic to K_n of color 1 under this coloring.*

The definition above is a direct application of Ramsey theorem, which makes a statement about the existence of order subsets. Well-known examples relevant to this definition include $R(3, 3) = 6$ and $R(3, 4) = 9$. The results in the field are continuously collected in [Rad94], with the most recent publicly available revision to the document published online in 2021.

We now proceed with attempting to prove Ramsey's theorem in RCA_0, using the theorems as provided in III.7 of [Sim09]. First, we define the theorem as follows:

Definition 4.2. *In RCA_0, for any $X \subseteq \mathbb{N}$ and $k \in \mathbb{N}$, we define $[X]^k$ to be the set of all increasing sequences of length k composed of elements from X.*

We assert that $s \in [X]^k$ *if and only if* $s \in \mathbb{N}^k$ *and for all* $j < k, s(j) \in X$ *and for all* $i < j, s(i) < s(j)$, *i.e. the sequences of* $[X]^k$ *are order-preserving.*
Ramsey theorem for exponent k, *denoted RT(k) states that for all* $l \in \mathbb{N}$ *and all* $f : [\mathbb{N}]^k \to \{0, 1, \cdots, l-1\}$, *there exists some* $i < l$ *and an infinite set* $X \subseteq \mathbb{N}$ *such that* $f(m_1, \cdots, m_k) = i$ *for all* $\langle m_1, \cdots, m_k \rangle \in [X]^k$.

As discussed before, provability of the theorem in RCA_0 demonstrates how essential it is to mathematics; the less equipped the subsystem we are working is, the more "primal" and necessary the theorem can be considered. To start proving certain statements with respect to Ramsey theory, we will assert definitions that define a possible ordering with a set/sequence using colors.

Definition 4.3. *For a set* X, *let* $[X]^n$ *be the collection of* n*-element subsets of* X. *An* i*-coloring of* $[X]^n$ *is a map* $f : [X]^n \to \{0, \cdots, i\}$. *We claim set* $H \subseteq X$ *is homogeneous for* f *if there exists an* $l < i$ *such that for all* $s \in [H]^n, f(s) = l$. *We say that* H *is homogeneous to* l.

Remark: We can use sets and sequences interchangeably for defining homogeneity.
To proceed further, we define an additional type of a tree that must be constructed in the proof of the theorem.

Definition 4.4. *Given integers* l *and* k *and a function* $f : [\mathbb{N}]^{k+1} \to \{0, 1, ..., l-1\}$, *we define an Erdős/Rado tree for the tuple* (l, k, f) *to be the tree* $T := T_{(l,k,f)}$ *such that* $T \subseteq \mathbb{N}^{<\mathbb{N}}$, *where* $t \in T$ *if and only if for all* $n < lh(t)$, $t(n) =$ *the least* j *satisfying the following conditions:*

1. $t(m) < j$ *for all* $m < n$.
2. $f(t(m_1), \cdots, f(m_k), j) = f(t(m_1), \cdots, t(m_k), t(m))$,
   for all $m_1 < \cdots < m_k < m \leq n$.

Notably, we are guaranteed the existence of Erdős/Rado trees, given the satisfying l, k, and f, in RCA_0 and ACA_0 by Σ^0_0-comprehension. As such, we can invoke this type of trees in our proofs.

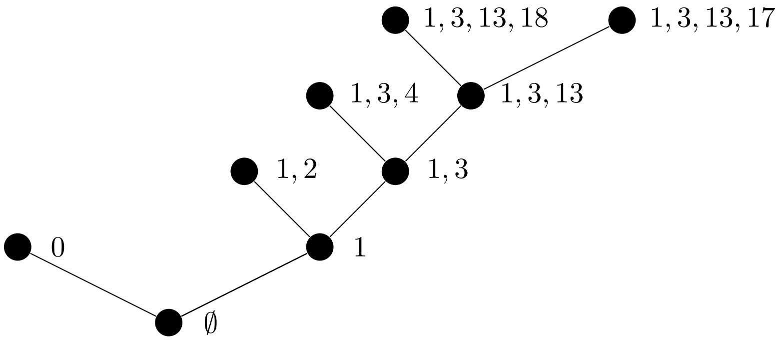

Figure 4.1: A tree structure presented in the Example 4.5.

Colors for pairs of T			
Outputs of the coloring f	0	1	2
Corresponding Sequences	[0,1]	[0,2]	[0,3]
	[0,4]	[2,3]	[3,4]
	[1,4]	[3,4]	[3,5]
	[0,5]	[4,7]	[5,6]
	[4,5]	[4,10]	[0,6]

Figure 4.2: Table for the pairs of T in the Example 4.6

Example 4.5. *Let the computable coloring* $f : [\mathbb{N}]^3 \to \{0, 1\}$ *be defined* $f(m_1, m_2, m_3) = i \in \mathbb{N}$ *where* $i = 0$ *if* $17 \nmid m_1 + m_2 + m_3$ *and* $i = 1$ *otherwise. Because the set of prime numbers is infinite, for any* m_1 *and* m_2 *we can consistently find* $m_3 > m_1, m_2$ *such that* $m_1 + m_2 + m_3 = n \in \mathbb{N}$*, where* n *is prime and, therefore, indivisible by 17. It is easy to define, by induction, an infinite sequence* $(n_1, n_2, \dots)$ *such that* $f(n_{i_1}, n_{i_2}, n_{i_3}) = 1$ *for all* $i_1 < i_2 < i_3$*. This set-up guarantees the existence of a homogeneous path in* T*. The resulting* T *is provided visually in Figure 4.1.*

Example 4.6. *Like before, we define a coloring* $f : [\mathbb{N}]^2 \to \{0, 1, 2\}$*. The outputs of* f *will be determined by the Table 4.1. It should be noted that this example presents a coloring that is not computable and still preserves prehomogeneity. The reader is encouraged to draw the graph in accordance with the table themselves to observe prehomogeneity directly.*

Lemma 4.7. *ACA_0 proves RT(0) and $\forall k(RT(k) \rightarrow RT(k+1))$.*

It should be evident that this lemma can be used as the two components necessary for the inductive proof of ACA_0 proving RT(k) for all $k \in \mathbb{N}$.

Proof. We first wish to show that ACA_0 proves RT(0). For this goal, We claim, in symbols, $s \in [X]^0$ if and only if $s = \emptyset$. Since there is only one element in $[X]^0$ for any $X \subseteq \mathbb{N}$, it is clear that we can achieve the monochromatic subset. We claim $s \in \mathbb{N}^0$ and $(\forall j < 0)(s(j)) \in X \wedge (\forall i < j)(s(i) < s(j)))$. However, there is no $s(j) \in \mathbb{N}$ such that $j < 0$, making the statement vacuously true. Thus, the case of RT(0) holds in ACA_0, vacuously.

We now wish to prove that $\forall k(\mathrm{RT}(k) \rightarrow \mathrm{RT}(k+1))$. We assume that RT($k$) holds, i.e. for all $l \in \mathbb{N}$ and all $f : [\mathbb{N}]^k \rightarrow \{0, 1, \cdots, l-1\}$, there exists some $i < l$ and an infinite set $X \subseteq \mathbb{N}$ such that $f(m_1, \cdots, m_k) = i$ for all $\langle m_1, \cdots, m_k \rangle \in [X]^k$. In order to show RT($k+1$), we fix a number of colors l, and an l-coloring of $k+1$-tuples $f : [\mathbb{N}]^{k+1} \rightarrow \{0, 1, \cdots, l-1\}$. For this proof, we will employ the Erdős/Rado tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that T is $T_{l,k+1,f}$.

First, we assert that T is infinite. To do so, we must show that for every $j \in \mathbb{N}$, there always exists some $t^\frown\langle j \rangle \in T$. The proof of this idea is to choose maximal $t \in T$ such that j has not yet been used. Then, by properties of T listed in definition (4.4), it follows that $t^\frown\langle j \rangle \in T$. We give some of the intuition for this result in the following paragraph.

1. The definition of T (as defined in definition (3.29)), the successor $\langle j \rangle$ to a node t must be such that it preserves prehomogeneity of the branch and is the smallest $j \in \mathbb{N}$ to do so, such that one of the colors is achieved in the "last sequence".

2. Then, for any $\langle j \rangle$ that succeeds an existing branch $t \in T$, there are two cases: it either satisfies the conditions of an Erdős/Rado tree or it fails to do so.

3. Take t of minimum length such that j fails to do so. We know we can do this due to additional colorings branching off at the least value starting from the base of T.

4. If the preceding branch to t, which we will term $t-1$, can be extended, there is no issue. However, it may not necessarily satisfy the coloring conditions. As such, we move down the nodes of t in succession until we get a branch that can be extended by $\langle j \rangle$.

5. Note that if there are no successors for some color in the range of f, there must exist a coloring with node values such that the properties of T hold somewhere in T. Otherwise, T is not an Erdős/Rado tree.

6. Through exclusion, we are guaranteed to be able to find a successor to some branch T for some output of f.

7. Consequently, since we have infinitely many successors, T must be infinite.

In addition, note that T must be finitely branching. Provided $t \in T$ of length n, $t^\frown\langle j\rangle$ needs to satisfy only two conditions for a finite number of possible successors and finitely many colors. Consequently, t can have $\leq l^{n^k}$ successors, which is finite. As a result T is an infinite, finitely branching tree.

By Kőnig's lemma in ACA_0, there exists a path $g \subseteq T$. g preserves the ordering of the sequence, so we can define an additional coloring function $f' : [\mathbb{N}]^k \to \{0, 1, ..., l-1\}$ and use it to map $f'(m_1, ..., m_k) = f(g(m_1), ..., g(m_k), g(m))$ where $m_1 < ... < m_k < m$. The function f' is well-defined by prehomogeneity of the tree.

The resulting coloring of the tree, which can also be described as an induced coloring, is guaranteed by g for a $k + 1$-tuple. We can use the inductive assumption, $\mathrm{RT}(k)$ here, to assert that there exist $i < l$ and $X' \subseteq \mathbb{N}$ such that X' is infinite and $f'(m_1, ..., m_k) = i$ for every $\langle m_1, ..., m_k\rangle \in [X']^k$. Then, since the existence of the successor $\langle m\rangle$ being guaranteed while preserving the coloring, $f(m_1, ..., m_k, m) = i$ for all $\langle m_1, ..., m_k, m\rangle$ for X being the set of all $g(m)$ for $m \in X'$. Thus, $X = \{g(m) : m \in X'\}$ is the infinite homogeneous subset that guarantees the conclusion of $\mathrm{RT}(k + 1)$. □

4.2 Alternative proofs

While Simpson's definition definition and proof are achievable with sequences, Ramsey theorem can also be proven with sets. Following, we will reference Dennis Hirschfeldt's writing [Hir15], specifically relying on Chapter 6.1. In his book, Hirschfeldt provides several alternatives proofs for Ramsey's theorem that are somewhat different from the methodology for proofs used by Simpson. For one, we can distinguish different versions of Ramsey's theorem by its scope of possible coloring. This is demonstrated in the formulation of the following theorem:

Theorem 4.8. *1. Ramsey's theorem for n-tuples and i-colors, denoted RT_i^n states that every i-coloring of $[\mathbb{N}]^n$ has an infinite homogeneous set.*

2. Ramsey's theorem for n-tuples, denoted $RT_{<\infty}^n$ states that for all i colorings such that $i \geq 1$, RT_i^n.

3. Ramsey's theorem states that for all i colorings such that $i \geq 1$, $RT_{<\infty}^n$.

Unlike in Simpson's definition, we are now working with sets and not sequences. Consequently, the tuples are unordered for RT. We will now proceed to produce a proof of RT_i^n using sets.

Proof. To prove RT_i^n, we proceed by induction under RCA_0. Like with sequences, the base case of RT_i^1 is trivial/vacuous, as the set contains only one element. For the inductive assumption, we assume RT_i^{n-1} holds and let $f : [\mathbb{N}]^n \to i$.

We first fix $a_0 = 0$ in our set $\mathbb{N}$. We map $d_0 : [\mathbb{N}\ \{a_0\}]^{n-1} \to i$ such that $d_0(s) = f(s \cup \{a_0\})$ for $s \in [A]^n$ where A is an infinite set. We also let H_0 be a set with homogeneous coloring for d_0 such that $a_0 < \min H_0$. We label the coloring for which H_0 is homogeneous as c_0.

We repeat the process with the least element of H_0 now. Let such element be labeled a_1. We define a mapping $d_1 : [H_0\ \{a_1\}]^{n-1} i$ such that $d_1(s) = f(s \cup \{a_1\}$. We also let H_1 be an infinite homogeneous set for d_1 such that $a_1 < \min H_1$ and define a_2 to be the least element of H_1. We continue recursively, defining an ordered set $A = \{a_0 < a_1 < \cdots\}$. If $s \in [A]^n$, we define $a_j \in s$ to be the least element. All other elements of s are in H_j, so $f(s) = f_j$. Consequently, there exists $h < k$ such that $c_j = h$ for infinitely many j. Let $H = \{a_j : f_j = h\}$. Then, $f(s) = h$ for all $s \in [H]^n$. Thus, H is an infinite homogeneous set for f. □

This proof is a version of Ramsey's original proof in his theorem. The reader is encouraged to read the original work [Ram29] and compare the approach themselves. This is, however, not the only possible proof of RT_i^n. We can construct an additional proof using the notion of set homogeneity.

Lemma 4.9. *Let $n \geq 2$ and $f : [\mathbb{N}]^n \to i$. If RT_i^{n-1} holds and f has an infinite prehomogeneous set, f must have an infinite homogeneous set.*

Proof. However, we will proceed by induction on RT. First, we regard the base case of RT_i^0. As in previous proofs, this case is vacuously true. For our inductive assumption, we assume RT_i^{n-1} holds and will aim to show the n-tuple case through an existence of an infinite prehomogeneous set.

We define infinite sets $I_0 \subset I_1 \subset \cdots$, where for each $i < n-1, I_i = \mathbb{N}\ [0, i]$. We let $a_m = minI_m$. Then, by set arrangement, it follows that $a_0 < a_1 < \cdots$. We let $m \geq n-3$ and a set $F = s \in [a_j : j \leq m\}]^{n-1}$. Then, we define a function $d : I_m\ \{a_m\} \to i^F$, where i^F is a set of functions from F into i and d is the function that maps s to the coloring $f(s \cup \{x\})$. Then, from the inductive assumption, d functions as an $i^{|F|}$-coloring for $I_m\ \{a_m\}$. If we let I_{m+1} be an infinite homogeneous set for d, for $s \in [\{a_j : j \leq m\}]^{n-1}$ and $x, y \in I_{m+1}$, $f(s \cup \{x\}) = f(s \cup \{y\})$. This satisfies the definition of prehomogeneity for sets. So, f has an infinite prehomogeneous set for every $i^{|F|}$-coloring. Ultimately, since RT_i^{n-1} holds from induction, it follows that f has an infinite homogeneous set. □

It is worth noting that the provided proofs do not complete the list of proofs possible for RT. In his book, Hirschfeldt presents an additional proof using set ultrafilters, which takes a very different approach from the previous two. While this proof is outside of our scope, the reader is encouraged to examine the approach themselves in [Hir15] on p. 73-74.

In this chapter, we proved Ramsey's theorem in RCA_0, indicating its importance to the formulaic structure in mathematics. Because the theorem is provable in the environment with minimal tools, we can reasonably claim that the theorem represents information that is, for lack of better terms, primal when compared to other theorems that are not provable in RCA_0. It must be noted that while Ramsey's theorem is strong and "primal" in mathematics, certain versions of it cannot be proven in certain other similarly limited systems like WKL_0. For example, on p. 75-76 of [Hir15], Hirschfeldt remarks that neither RCA_0 nor WKL_0 entail RT_2^2. Furthermore, larger tuples to work with imply larger computability complexity. This fact motivates further investigations into how RT fits into the general arithmetical hierarchy, carrying certain implications for computability theory. While the provability of RT in RCA_0 concludes our goal for this paper, we will showcase some recent results with respect to Ramsey's theorem in the next section.

Chapter 5

New Results

While Ramsey's theorem is a well-known topic in combinatorics, reverse mathematics is a relatively new field that started developing only in (approximately) 1970, due to the efforts in part of Charles Parsons [Par70] and continued by Harvey Friedman, Stephen Simpson and many others. At the time of this writing, research in reverse mathematics is continuing, including that related to Ramsey's theorem.

One notable example of recent publications on this matter is that of Chubb, Hirst, and McNicholl [CHM09]. Their work employs Ramsey's theorem limited to binary trees, which is described below:

Theorem 5.1. *Suppose that $[2^{<\mathbb{N}}]^n$ is colored with i colors. Then, there is a subtree S isomorphic to $2^{<\mathbb{N}}$ under RCA_0 such that $[S]^n$ is monochromatic. We denote this version of Ramsey's theorem as TT^n_i.*

Just as before, additional restrictions on Ramsey's theorem carry different implications.

Lemma 5.2. *Assume RCA_0 and let $f : 2^{<\mathbb{N}}$ be a two-coloring of the nodes for the full binary tree. Let these colors be red and blue. For any node σ of the tree either:*

1. *above σ, there is a subtree isomorphic to $2^{<\mathbb{N}}$ in which every nonempty node is red*
2. *σ can be extended to a node τ such that every node is properly extending τ is blue.*

Theorem 5.3. *Assume RCA_0 and Σ^0_2-induction. For all k, TT^1_i. That is, for any finite coloring of $2^{<\mathbb{N}}$, there is a monochromatic subtree isomorphic to $2^{<\mathbb{N}}$.*

Theorem 5.4. *Assume ACA_0. For all i, TT^2_i. hat is, for any finite coloring of pairs of comparable nodes of $2^{<\mathbb{N}}$, there is a monochromatic subtree isomorphic to $2^{<\mathbb{N}}$,*

Theorem 5.5. *Assume ACA_0. For all $n \geq 1$, TT^n implies TT^{n+1}.*

As usual, the reader is encouraged to read the original work to learn more about the methodology for the proofs.

Chapter 6

Conclusion

Throughout this thesis, we constructed the proof of Ramsey's theorem from employing only the most primitive subsystems of second order arithmetic possible. Starting from only P_0, a comprehension schema, and an induction schema, it is possible to prove critical combinatorial concepts, such as Kőnig's lemma and others.

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