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KNOT EQUIVALENCE

Jacob Trubey

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KNOT EQUIVALENCE

A Thesis
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Mathematics

by
Jacob Trubey
May 2023
Knot Equivalence

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Approved by:

Dr. Giovanna Llosent, Committee Chair
Dr. Corey Dunn, Committee Member
Dr. Bronson Lim, Committee Member

Dr. Madeleine Jetter, Chair, Department of Mathematics
Dr. Corey Dunn, Graduate Coordinator
Abstract

A knot is a closed curve in $\mathbb{R}^3$. Alternatively, we say that a knot is an embedding $f : S^1 \to \mathbb{R}^3$ of a circle into $\mathbb{R}^3$. Analogously, one can think of a knot as a segment of string in a three-dimensional space that has been knotted together in some way, with the ends of the string then joined together to form a knotted loop. A link is a collection of knots that have been linked together.

An important question in the mathematical study of knot theory is that of how we can tell when two knots are, or are not, equivalent. That is, we would like to know when two seemingly different looking knots are, or are not, in fact the same knot. In this thesis, we discuss knots, links, and the concepts necessary for establishing knot equivalence.

We seek to show when two knots are equivalent through a process known as ambient isotopy. We seek to show when two knots are not equivalent through a process known as knot invariance. We will discuss special types of isotopies, such as Reidemeister moves, and we will cover various types of invariants, such as linking number, tricolorability, and the Jones polynomial.
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Chapter 1

Knots and Knot Equivalence

1.1 Knots

In this chapter, we will introduce various basic definitions about knots and cover
some background topological information in order to establish the concepts of ambient
isotopy and knot equivalence. We begin Section 1.1 by defining a knot as a closed curve
in $\mathbb{R}^3$ [Ada04]. Alternatively, we say that a knot is an embedding $f : S^1 \to \mathbb{R}^3$ of a circle
into $\mathbb{R}^3$ [AF08]. One can think of such a curve as a piece of string that has been knotted
up in some way with the two ends of the string glued together to form a knotted loop of
sorts [Als19]. Some common examples of knots include the unknot, the trefoil knot, and
the figure-eight knot, as depicted respectively in Figure 1.1.

Figure 1.1: The unknot, trefoil knot, and figure-eight knot.

One thing we would like to know about knots is whether or not two seemingly
different looking knots are actually the same knot, that is, we would like to know when
two knots are, or are not, equivalent. However, to understand knot equivalence is to
first understand how one knot can be deformed into another knot through a topological process known as ambient isotopy. In the next section, we explore the topics that will lead us to ambient isotopy.

1.2 Ambient Isotopy

To show knot equivalence, we must introduce the idea of ambient isotopy, a topological process that deforms one knot into another by deforming the surrounding space, but to define ambient isotopy, we must first begin with some background topological information. We begin with the concept of a homotopy, and then proceed to build upon this definition until we arrive at that of an ambient isotopy.

We know from topology that, given two topological spaces, $X$ and $Y$, a homotopy is a continuous function $F : X \times I \to Y$ that we can think of as an interval $I$ of maps from one of the spaces to the other space [AF08]. As we travel along the interval, we see a continuous family of maps that appear to deform $X$ into $Y$, much like the still images on a reel of video tape for a film, as depicted in Figure 1.2.

![Figure 1.2: Film reel.](image)

We also know from topology that two spaces are topologically equivalent if there exists a homeomorphism between them, that is, if there exists a continuous bijective function between the spaces whose inverse is also continuous [AF08]. Now we want a more rigorous definition of a function between spaces that will bring us closer to showing
equivalence between knots.

Using the concept of topologically equivalent spaces in combination with our concept of a homotopy, we define an isotopy as a homotopy, $F : X \times I \to Y$, where $F |_{X \times \{t\}}$ is a homeomorphism for all $t \in I$ [AF08]. Now recall from topology that an embedding of $Y$ in $X$ is a function $f : Y \to X$ that maps $Y$ homeomorphically to a closed subspace $f(Y)$ in $X$ [AF08]. We now further refine our definition of an isotopy to that of an ambient isotopy by combining the concept of isotopy with the concept of embeddings.

Let $f : Y \to X$ and $g : Y \to X$ be embeddings of $Y$ into $X$. An ambient isotopy is an isotopy, $F : X \times I \to X$, where $F(f(y), 1) = g(y)$ for all $y \in Y$. In such an ambient isotopy, we say that $f$ and $g$ are ambient isotopic and that the space $X$ is the ambient space [AF08].

We can see this definition of ambient isotopy put to use in the following two examples, where we find an ambient isotopy, $F$, between two embeddings. Our first example will be finding an isotopy between the embeddings $f, g : I \to \mathbb{R}^2$ given by $f(x) = (x, x)$ and $g(x) = (x^2, x)$.

Let $S = \{(x, y) \in \mathbb{R}^2| x > 0, y > 0\}$, and consider $F : S \times I \to S$ given by $F((x, y), t) = (x^{(t+1)}, y)$, which takes vectors in $\mathbb{R}^2$ to vectors in $\mathbb{R}^2$, and notice that $F((x, y), 0) = (x, y)$ and $F(f(x), 1) = F((x, x), 1) = (x^2, x) = g(x)$. Since our values for $x$ are restricted to $S$, we don’t have any discontinuity issues that would otherwise arise, so $F$ is a continuous function where $I = [0, 1]$ inherits the subspace topology from $\mathbb{R}$, $X \times I$ has the product topology, $F((x, y), 0) = (x, y)$ for all $(x, y) \in \mathbb{R}^2$, and $F(f(x), 1) = g(x)$ for all $x \in I$. So $F$ is a homotopy, and since $f$ and $g$ are embeddings, we know that $F |_{S \times t}$ for all $t \in I$, so $F$ is an ambient isotopy.

Our second example will be that of finding an isotopy between the embeddings $f, g : I \to \mathbb{R}^2$ given by $f(x) = (x, x)$ and $h(x) = (0, x)$. Consider $F : \mathbb{R}^2 \times I \to \mathbb{R}^2$ given by $F((x, y), t) = ((1-t)x, y)$, which takes vectors in $\mathbb{R}^2$ to vectors in $\mathbb{R}^2$, and notice
that \( F((x,y),0) = (x,y) \) and \( F(f(x),1) = F((x,x),1) = (0,x) = h(x) \). Since \( F \) is a continuous function where \( I = [0,1] \) inherits the subspace topology from \( \mathbb{R} \), \( X \times I \) has the product topology, \( F((x,y),0) = (x,y) \) for all \( (x,y) \in \mathbb{R}^2 \), and \( F(f(x),1) = h(x) \) for all \( x \in I \), we have that \( F \) is a homotopy, and since \( f \) and \( g \) are embeddings, we know that \( F|_{\mathbb{R}^2 \times t} \) for all \( t \in I \), so \( F \) is an ambient isotopy.

So we have developed a means of showing equivalence between embeddings. We would like to apply this concept to knots, so as to determine knot equivalence. In the next section, we explore the topics necessary for establishing knot equivalence.

### 1.3 Knot Equivalence

We have used ambient isotopy to show equivalence between two embeddings by deforming the surrounding space, but how can we apply this concept to knots? Knots themselves are embeddings, but we want to abide by the contextual circumstances that apply specifically to knots, that is, we want to make sure that we do not encounter situations that would not make sense for knots, such as knots with infinitely many loops and crossings known as wildly embedded knots, as depicted in Figure 1.3 [AF08].

![Figure 1.3: A wildly embedded knot.](image)

We can avoid such fruitless circumstances by considering knots that can be viewed as a **polygonal knot**, which is a knot made of a finite number of line segments that we call **edges** connected by points that we call **vertices**, as seen in Figure 1.4 [AF08].
We also would not want our knots to be deformed in such a way as to pass through themselves or to cause a part of the knot to collapse to a point, as shown in Figure 1.5.

However, by applying our concept of ambient isotopy to deform the space surrounding the knot, we ensure that the space, and consequently the knot, does not pass through itself or collapse to a point. With these circumstances covered, we are now ready to define knot equivalence. Two knots $f, g : S^1 \to \mathbb{R}^3$ are equivalent if they are ambient isotopic. A collection of equivalent knots is called a knot type [AF08].

Now, what if we wanted to convey the process of an ambient isotopy between two knots by showing a series of pictures instead of trying to physically move a piece of
string in $\mathbb{R}^3$? This would be a convenient option, but we would need to make sure that our pictures show that the process abides by the concepts of ambient isotopy. We will develop a rigorous definition of such pictures in the next section, but we will end this section by introducing an example of such a series of pictures known as a triangle move.

First, we consider a polygonal knot in $\mathbb{R}^2$ and focus on one edge of the knot. We want to move the edge in $\mathbb{R}^2$ to mimic the motion of moving a part of a knot during an ambient isotopy in $\mathbb{R}^3$. We can accomplish this by ensuring that our polygonal knot is piecewise-linear, which allows us to replace our edge with two adjacent edges as if we were pulling the first edge at a point, as depicted in Figure 1.6.

![Figure 1.6: Triangle move.](image)

Similarly, we can reverse this process to transition from two adjacent edges to one edge. These kinds of piecewise-linear moves in our polygonal knot projection in $\mathbb{R}^2$ are known as triangle moves and we use this concept to establish a definition of a series of pictures in $\mathbb{R}^2$ showing knot equivalence known as a planar isotopy, which we discuss in the following section [AF08].

### 1.4 Planar Isotopy

Now, we could show knot equivalence by physically moving a knot in $\mathbb{R}^3$ until it is deformed into another knot, but we would like a more convenient approach of communicating that two knots are equivalent through some easily interpreted pictures of sorts. Pictures that would retain the information we would want to know about our knots so
as to imagine the knots being deformed in $\mathbb{R}^3$. We can keep track of information about our knots, like which parts of the knot are above other parts in our pictures and how two equivalent knots are deformed into one another, by rigorously defining such pictures of knots as what we call knot projections, which depict a process known as planar isotopy. We begin with a definition of a series of pictures known as a regular projection and then proceed to build upon this definition [AF08]. A regular projection is a projection of a knot that meets the following three conditions:

(i) No point in the projection corresponds to more than two points on the knot;

(ii) There are only finitely many points in the projection that correspond to two points on the knot. These are called double points of the projection;

(iii) No double point corresponds to a vertex of the knot.

We can see an example of a regular projection in Figure 1.7.
Figure 1.7: A regular projection.

Although the above knot depicted in $\mathbb{R}^3$ may seem at first to be the figure-eight knot, a close inspection will reveal that there is a subtle difference between which edges lie above and below each other in this knot versus the figure-eight knot. However, this information appears to be lost when we project our knot to its regular projection in $\mathbb{R}^2$, particularly at the double points of the projection. In order to properly interpret the information provided so as to imagine the knots in $\mathbb{R}^3$, we will need to refine our definition of a regular projection to a projection that provides more information, which we simply call a knot projection.

A knot projection is a regular projection that indicates which edges are above or below one another at the double points of the projection. We show this by relabeling our double points as crossings, and interpret such crossings as we would interpret one part of our knot in $\mathbb{R}^3$ crossing over another part [AF08]. We can see an example of a knot projection in Figure 1.8, using the same knot from the previous example. Note the difference between our knot projection and the previous regular projection.
With the information provided by a knot projection, we can begin the process of representing ambient isotopies of our knot in $\mathbb{R}^3$ by what is known as a planar isotopy of a knot projection in $\mathbb{R}^2$. A **planar isotopy** is a piecewise-linear isotopy of the plane, like the piecewise-linear isotopies that we saw before called triangle moves [AF08]. We can show equivalence between knots by showing a planar isotopy from one knot projection to another, much like we would show equivalence between two knots via an ambient isotopy.

Now, if we have a planar isotopy showing equivalence between two knot projections in $\mathbb{R}^2$, we could construct a corresponding ambient isotopy showing equivalence between two knots in $\mathbb{R}^3$. If the same were true of a given ambient isotopy to a planar isotopy, then we could use the two interchangeably. However, it turns out that the same is not true for an ambient isotopy to a planar isotopy. In Figure 1.9, we show a representation in $\mathbb{R}^2$ of an ambient isotopy from $\mathbb{R}^3$ that fails to be a knot projection in Image 3, and thus fails to be a planar isotopy.
Figure 1.9: A projection that fails to be a regular projection.

Fortunately, there is a means of interpreting such a situation so as to still show equivalence through a process known as Reidemeister moves. These moves along with our planar isotopies make up a fundamental part of knot theory which allows us to show equivalence between knots and knot projections interchangeably. We discuss this process in detail in the following chapter.
Chapter 2

Reidemeister Moves

2.1 Reidemeister Moves

As stated in the last chapter, an important goal in knot theory is to show equivalence between knots. We have been able to show that two knots are equivalent in $\mathbb{R}^3$ through the process known as ambient isotopy, but we would like to have a convenient means of showing this process through a series of pictures in $\mathbb{R}^2$ as well. We can accomplish this via knot projections and planar isotopies up until we no longer have a knot projection, as was demonstrated at the end of the previous chapter. We will now introduce a fundamental concept of knot theory known as Reidemeister moves that, in combination with our planar isotopies, will allow us to show knot equivalence in $\mathbb{R}^2$ and $\mathbb{R}^3$ interchangeably [AF08].

A Reidemeister move is one of three different special ambient isotopies, RI, RII, and RIII such that

(i) RI is that of putting in or taking out a kink in a projection;

(ii) RII is that of sliding one of two adjacent strands under the other;

(iii) RIII is that of sliding a strand past a crossing, over the two strands that make up the crossing.
Recall from our definition of knot equivalence that two knots are equivalent if they are ambient isotopic, and that a collection of equivalent knots is called a knot type. As we explore each of the three Reidemeister moves, notice that all of the moves are ambient isotopies, and thus, all preserve the knot type.

Beginning with the Type I Reidemeister move, RI, we consider a single piece of string in our knot and proceed to add an overhand or underhand twist into the string so that we obtain one additional crossing, as shown in Figure 2.1.

![Figure 2.1: Type I Reidemeister move.](image)

Similarly, given such a crossing, we can untwist the knot at the crossing to obtain one less crossing and a string in our knot like the one we started with. Since this move is clearly an ambient isotopy, we know that the Type I Reidemeister move leaves the knot type unchanged, and the two projections shown are thus equivalent.

Moving on to the Type II Reidemeister move, RII, we consider two pieces of string in a knot and proceed to slide one string over or under the other, so that we obtain two additional crossings, as shown in Figure 2.2.
Once again, we can see that our move is an ambient isotopy and that the Type II Reidemeister move preserves the knot type.

Lastly, looking at the Type III Reidemeister move, RIII, we consider three strings in a knot as depicted in Figure 2.3, and proceed to slide the top or bottom string past the crossing made by the other two strings.

Since this move is an ambient isotopy, we can see that the Type III Reidemeister move leaves the knot type unchanged.

So, we have that all three Reidemeister moves preserve the knot type, and thus preserve equivalence between knots. We now introduce a very important theorem in knot theory known as **Reidemeister’s theorem**, which is stated as follows [AF08].

**Theorem 2.1** (Reidemeister’s theorem). *Two knots are equivalent if and only if there is a finite sequence of planar isotopies and Reidemeister moves taking a knot projection of one to a knot projection of the other.*
Proof. To prove Reidemeister’s theorem, we would need to consider the forward and reverse direction of the if and only if statement therein. Fortunately, the forward direction is already done for us, since we know that planar isotopies and Reidemeister moves leave the knot type unchanged. So we have that two knots are equivalent if there is a finite sequence of planar isotopies and Reidemeister moves taking a knot projection of one to a knot projection of the other.

For the reverse direction, we want to show that there is a finite sequence of planar isotopies and Reidemeister moves taking a knot projection of one knot to a knot projection of another if the two knots are equivalent. Suppose we have two equivalent knots. We can look at our two knots as polygonal knots made of line segments attached at vertices. Since these knots are equivalent, we know that there is an ambient isotopy taking one to the other.

Since we are viewing our knots as polygonal knots in \( \mathbb{R}^2 \), we will need to view our ambient isotopy as a planar isotopy, that is, we will need each stage of the isotopy to be a regular projection. In the cases where our projection is not regular, which are the three cases that would fail the three requirements of the definition of a regular projection, we will use our previously mentioned triangle moves to show that these stages are those of our three Reidemeister moves.

Consider the case presented in Figure 2.4 and notice that we fail to have a planar isotopy between steps 2 and 4, as this fails the requirement that no point in the projection corresponds to more than two points on the knot. As we travel between steps 1 and 4, we see that this is precisely the Reidemeister move RIII.

![Figure 2.4: This is precisely RIII.](image)
Next, we consider the case presented in Figure 2.5 and notice that we fail to have a planar isotopy again between steps 2 and 4, as this fails the requirement that no double point corresponds to a vertex of the knot. As we travel between steps 1 and 4, we see that this is precisely the Reidemeister move RII.

![Figure 2.5: This is precisely RII.](image)

Finally, we consider the case presented in Figure 2.6 and notice that we fail to have a planar isotopy between steps 2 and 4, as this fails the requirement that there are only finitely many points in the projection that correspond to two points on the knot. As we travel between steps 1 and 4, we see that this is precisely the Reidemeister move RI.

![Figure 2.6: This is precisely RI.](image)

So, we have that there is a finite sequence of planar isotopies and Reidemeister moves taking a knot projection of one knot to a knot projection of another if the two knots are equivalent. Thus, two knots are equivalent if and only if there is a finite sequence of planar isotopies and Reidemeister moves taking a knot projection of one to a knot projection of the other.

We will now look at some examples of Reidemeister’s Theorem by showing knot equivalence through a finite sequence of planar isotopies and Reidemeister moves. First,
we show that the seemingly unfamiliar knot in Image 1 of Figure 2.7 can be deformed into our familiar friend, the trefoil knot, in the following six images, noting the Reidemeister moves used along the way.

Figure 2.7: The knot in Image 1 is equivalent to the trefoil knot in Image 6.

We can see that the knot in Image 1 is gradually deformed into the trefoil knot through a sequence of planar isotopies and Reidemeister moves, and by Reidemeister’s Theorem, we conclude that the knot in Image 1 is equivalent to the trefoil knot in Image 6.

How about an example with a bit more complexity? In the following two figures, or 18 images, we will show that the seemingly unfamiliar knot in Image 1 of Figure 2.8 can be deformed into the unknot in Image 18 of Figure 2.9.
Figure 2.8: A deformation of a knot.

Figure 2.9: A continued deformation of a knot, leading to the unknot.

Since the image of the unfamiliar knot in Image 1 is gradually deformed into
the unknot in Image 18 through a sequence of planar isotopies and Reidemeister moves, by Reidemeister’s Theorem, we conclude that the knot in Image 1 is equivalent to the unknot in Image 18.

Now, what if we were to consider two knots that were tangled together? How would our concepts and properties of single knots change to those of multiple knots? It turns out that everything covered so far is essentially the same, including the concepts of isotopies and Reidemeister’s theorem, and we call these groups of knots links, which we will explore in the next chapter.
Chapter 3

Links and Invariants

3.1 Links

A link is an embedding of a set of circles into $\mathbb{R}^3$. Two links are considered equivalent if one can be deformed to the other via ambient isotopy. Each embedded circle is called a component of the link and a link is called an $n$-component link if it has $n$ components [AF08]. In fact, the single knots that we have discussed thus far have actually all been 1-component links. Some common examples of 2-component links include the unlink of two components and the 2-component Whitehead link, as depicted respectively in Figure 3.1.

![Figure 3.1: The unlink and Whitehead link.](image)

Much like how ambient isotopic knots are equivalent, so too are ambient isotopic links. As an example of this, we will look at a deformation of the Whitehead link through ambient isotopy via the following 8 images of Figure 3.2.
Figure 3.2: The Whitehead link in Image 1 is equivalent to the link in Image 8.

Since the Whitehead link in Image 1 is gradually deformed into the link in Image 8, we conclude that these links are equivalent. In the next section, we will define one of the most effective means of determining knot equivalence called invariants, and we will introduce such an invariant for links known as linking number.

3.2 Invariance and Linking Number

In this section, and for the remainder of this chapter, we will discuss an important part of knot theory known as invariance. While we have been able to use isotopies to determine if two knots are equivalent, we have not yet established a means of determining whether two knots are not equivalent. We begin Section 3.2 with a definition of knot invariance and then proceed to explore one such invariant for links known as linking number.

A knot invariant is a function that, when applied to a knot, computes a value
for that knot. This value will be the same for equivalent knots, but may or may not be the same for non-equivalent knots. Alternatively, we say that a knot invariant is a value for a knot that is unchanged by ambient isotopy, Reidemeister moves, or choice of projection of the given knot [Ada04].

Now, to define one such invariant known as linking number, we will need a couple of preliminary definitions first, namely those of knot orientation and crossing labels. A **knot orientation** is a chosen direction of travel around a given knot. Upon choosing a direction, we indicate it on our knot projection with directed arrows as shown in Figure 3.3 [AF08].

![Figure 3.3: Knot orientation.](image)

Now, to define crossing label, we consider the following two types of crossings shown in Figure 3.4, which are in fact the only two types of crossings that can occur in an oriented knot projection.
Figure 3.4: Crossing labels.

Notice the orientations and +1 and −1 labels in the above figure and how they are used in the following definition.

A crossing label is a value of +1 or −1 attributed to a left-hand or right-hand crossing, respectively, of an oriented two-component link. We denote the label of a crossing, $c$, by $l(c)$ [AF08].

Now having defined knot orientation and crossing labels, we are ready to introduce our first invariant known as linking number.

The linking number of a knot projection of an oriented two-component link $L$ is given by

$$lk(L) = \frac{1}{2} \sum_c l(c),$$

where the sum is taken over all of the crossings $c$ involving both of the components in the link [AF08].

So how do invariants like this linking number help us to determine link equivalence? Unlike ambient isotopies, which only allowed us to show that two knots were
equivalent, linking number, and invariants in general for that matter, allows us to determine when two links are not equivalent. We use the following theorem for linking number [AF08].

**Theorem 3.1.** If two oriented links are equivalent, then all of their knot projections have the same linking number.

Notice that the contrapositive of this statement tells us that, if two knot projections of links have different linking numbers, then those two links are not equivalent.

**Proof.** To prove this theorem, we begin by assuming that we have two equivalent oriented links, and then proceed to show that all of their knot projections have the same linking number. As stated before, we know that Reidemeister’s theorem applies to links as well, so we know that two links are equivalent if and only if there is a finite sequence of planar isotopies and Reidemeister moves taking a knot projection of one to a knot projection of the other. Since linking number depends upon crossing labels, and since planar isotopies alone do not change crossings, we know that the linking number is unaffected by planar isotopy, so we need only show that linking number is also unaffected by Reidemeister moves.

First, we consider the Reidemeister move RI. Since linking number depends upon the crossings involving both components of a two-component link, and since RI only changes the crossings of one component of the link, namely adding or removing one crossing from one component, we know that linking number is unaffected by RI.

Next, we consider the Reidemeister move RII. In the event that both strands used in RII are from the same component, we know that the linking number will be unaffected. If the two strands are from different components, then we either add a +1 and -1 crossing to the computed linking number or we remove a +1 and -1 crossing from the linking number. In both of these last cases, our net change on the computed linking number would be 0, so in any event, we know that the linking number would be unchanged by RII.
Finally, we consider the Reidemeister move RIII, which simply moves the +1 and -1 crossings while still retaining the total number of each, so we know that the linking number is unaffected by RIII as well. Since linking number is unaffected by all Reidemeister moves and planar isotopies, we have that, if two oriented links are equivalent, then all of their knot projections have the same linking number.

We can see how this theorem is applied in the next two examples. First we look at two projections of unfamiliar 2-component links, $L_1$ and $L_2$, as shown in Figure 3.5.

![Figure 3.5: Computing linking numbers.](image)

For all we know, the link from $L_1$ could just be a deformation of the link from $L_2$, or these could be projections of two entirely different links. We will proceed to investigate this uncertainty by computing the linking numbers of both projections.

$$lk(L_1) = \frac{1}{2} \sum_c l(c) = \frac{1}{2}(-1 - 1 - 1 - 1 - 1) = \frac{1}{2}(-6) = -3$$

$$lk(L_2) = \frac{1}{2} \sum_c l(c) = \frac{1}{2}(+1 + 1 + 1) = \frac{1}{2}(4) = 2$$

Since $-3$ does not equal $2$, by the contrapositive of our theorem, we have that $L_1$ and $L_2$ are not equivalent, and are thus two distinct links.

Next, we will look at two more projections of links, $L_1$ and $L_2$, where $L_1$ is that of another unfamiliar 2-component link and $L_2$ is that of the unlink of two components,
as shown in Figure 3.6. Note that the unlink of two components does not contain any crossings, so it is unnecessary to orient it and its only possible crossing label value is zero.

\[
\begin{align*}
\text{Figure 3.6: More linking number computation.}
\\
\text{Once again, we compute the linking numbers in an attempt to distinguish between the given links.}
\\
\text{Since } -2 \text{ does not equal 0, by the contrapositive of our theorem, we have that } L_1 \text{ and } L_2 \text{ are not equivalent, so the unfamiliar link from } L_1 \text{ is not the unlink of two components.}
\\
\text{Now, if these two links had the same linking number, we wouldn’t have enough information to distinguish between the two, that is, it could be that one link is a deformation of the other or that they are entirely different links. We would need to use ambient isotopy to determine if the two were equivalent, but we may not be able to feasably show this, depending on the complexity of the knot. However, although linking number has only allowed us to determine when some sets of links are not equivalent, it is still very useful in distinguishing between links and, as we will see in the rest of this chapter, the usefulness of invariants will become an indespensible tool in distinguishing between knots.}
\end{align*}
\]
3.3 Tricolorability

Recall from our definition of linking number that the linking number is determined by the crossings between both components of a 2-component link. This means that we cannot use linking number to distinguish between single knots like the unknot and the trefoil knot. So, although linking number can aid us greatly in telling two links apart, we have yet to develop any means of telling two single knots apart.

Fortunately, our scope of invariants is not limited to linking numbers and 2-component links. Throughout this chapter, we will discuss a variety of invariants that allow us to distinguish between different knots in different ways. We begin Section 3.3 by introducing an invariant known as tricolorability which, like linking number for 2-component links, will sometimes allow us to determine when two single knots are not equivalent.

Now, to define tricolorability, we must first introduce a formal definition of a strand. A **strand** in a projection of a link is a piece of the link that goes from one undercrossing to another with only overcrossings in between [Ada04]. Using this definition of a strand, we define tricolorability by the following two requirements.

The first requirement for a projection of a knot or link to be **tricolorable** is that each of the strands in the projection must be colored one of three different colors, so that at each crossing, either three different colors meet or only one color meets. The second requirement for a projection to be tricolorable is that at least two colors are used in our projection [Ada04].

We can see in Figure 3.7 that the trefoil knot meets the above two requirements, and is thus tricolorable. Notice that at least two colors are used in this projection and that three different colors meet at every crossing.
Figure 3.7: The trefoil knot is tricolorable.

Now, we want to use tricolorability to distinguish between knots, that is, we want to show that tricolorability is an invariant. To do this, we will need to show that tricolorability is unaffected by our three Reidemeister moves.

Proof. To show that tricolorability is unchanged by Reidemeister moves, suppose we have a tricolorable knot and we introduce the Type I Reidemeister move, RI, to one of the strands of our knot, resulting in one additional crossing, as illustrated in Figure 3.8. If we leave all strands at this crossing the same color as the previous strand, then our knot will still meet the requirements for tricolorability, so we have that RI leaves tricolorability unchanged.
Next, suppose we have a tricolorable knot and we introduce the Type II Reidemeister move, RII, to two of the strands of our knot, resulting in two additional crossings, as illustrated in Figure 3.9. If the two strands were the same color, then we could leave the new strand the same color and the knot would remain tricolorable. If the two strands were different colors, then we could color the new strand by the third color of the tricoloring and the knot would remain tricolorable, so we have that RII leaves tricolorability unchanged.

Figure 3.9: RII preserves tricolorability.

Now suppose we have a tricolorable knot and we introduce the Type III Reidemeister move, RIII, to three of the strands of our knot. We can see from the case exhaustive approach illustrated in Figure 3.10 that any coloring of the three strands can be colored in a way that preserves tricolorability, so we have that RIII leaves tricolorability unchanged.
Thus, we have that tricolorability is an invariant.

From this, we obtain the following theorem [Ada04].

**Theorem 3.2.** *Either every projection of a knot is tricolorable or no projection of that knot is tricolorable.*

We can see the theorem put to use in this example of the unknot. Consider the following projection of the unknot shown in Figure 3.11. Since it is only possible to color the unknot one color, it fails to meet the requirement that a tricolorable knot projection uses two colors, and is thus not tricolorable.
Now, by the above theorem, we know that, since this projection of the unknot is not tricolorable, then it must be that no projection of the unknot is tricolorable. Similarly, since our projection of the trefoil knot was tricolorable, it must be that all projections of the trefoil knot are tricolorable. These two results can only be true if the unknot and the trefoil knot are distinct knots. Though it may have seemed obvious before, this is the first time we have been able to prove that two single knots are not equivalent.

Furthermore, this implies that any knot that can be shown to be tricolorable is distinct from the unknot and any knot that can be shown to be not tricolorable is distinct from trefoil knot, or more generally, any knot that can be shown to be tricolorable is distinct from any knot that can be shown to be not tricolorable. It seems that invariants are proving quite useful now in determining knot equivalence. However, much like linking number for links, we have only been able to determine when two knots are not equivalent, and like all of our invariants, tricolorability will only get us this far.

If we were to look at two seemingly different knots that are both tricolorable, or both not tricolorable, we would not be able to determine whether they were equivalent or not equivalent from this information alone. Consider the following projections of the
figure-eight knot shown in Figure 3.12.

![Figure 3.12: The figure-eight knot is not tricolorable.](image)

If we start coloring the figure-eight knot three different colors at the top-most crossing, as in the image on the left, we will find by the bottom crossings that our projection fails to meet the requirement of every crossing having either only one or all three colors meet. No matter how we try to progress from that first crossing consisting of three colors, a short case exhaustive approach shows that this situation is unavoidable.

Now, if we were to consider the only other possibility for the top-most crossing, that of it having only one color meet as in the image on the right, we will find that the requirement of each crossing having only one or all three colors forces the rest of the crossings to all be colored with only one color, which then fails to meet the requirement of at least two colors being used in our projection.

So, it must be that the figure-eight knot is not tricolorable. By our theorem, this tells us that the figure-eight knot is not equivalent to the trefoil knot, but this alone will not determine the equivalence of the figure-eight knot and the unknot. Nonetheless, between ambient isotopies and more invariants which will be discussed in this chapter, we will still be able to make a great deal of progress in determining knot equivalence.
3.4 Unknotting Number

In this section, we consider an invariant that involves changing crossings in a knot so that the understrand becomes an overstrand or vice versa. We perform these crossing changes in combination with a series of ambient isotopies until our original knot is changed into the unknot. The invariant that concerns this process is known as the unknotting number and it is defined as follows.

A knot $K$ has **unknotting number** $n$ if there exists a projection of the knot such that changing $n$ crossings in the projection turns the knot into the unknot and there is no projection such that fewer changes would have turned it into the unknot. We denote the unknotting number of a knot by $u(K)$ [Ada04].

We can see this concept applied in the following example shown in Figure 3.13, where we find the unknotting number of the figure-eight knot.

![Figure 3.13: Calculating the unknotting number of the figure-eight knot.](image)

If we choose the bottom left crossing of the figure-eight knot, indicated by the dotted circle in Image 1 of the above figure, and proceed to change the understrand of that crossing into an overstrand, we can see through a series of ambient isotopies in the subsequent images that our knot becomes the unknot. Since we only had to make one crossing change to accomplish this, we conclude that the unknotting number of the figure-eight knot is 1. If we were to denote the figure-eight knot as $K$, we would say that $u(K) = 1$. 
There are many more invariants that could be explored in addition to linking number, tricolorability, and unknotting number, but we will conclude this thesis in the next section, where we discuss polynomial invariants and introduce one of the most successful such invariants known as the Jones polynomial.

3.5 The Jones Polynomial

We conclude our thesis with this section on the Jones polynomial. We start by introducing a polynomial known as the bracket polynomial and proceed to build upon this concept until we arrive at that of the Jones polynomial. A bracket polynomial of a knot or link projection $L$, denoted $\langle L \rangle$, is defined by the following three rules [Ada04]:

Rule 1: $\langle \bigcirc \rangle = 1$;

Rule 2: $\langle L \cup \bigcirc \rangle = C(L)$;

Rule 3: $\langle \bigtimes \rangle = A \langle \bigcirc \bigtimes \rangle + B \langle \bigtimes \bigtimes \rangle$;

This is a lot of new notation, so we will explain each of these three rules in more detail. Rule 1 states that the bracket polynomial of the unknot, denoted $\langle \bigcirc \rangle$, is equal to 1. As with all of these rules, we are simply defining Rule 1 as such for the time being with the goal of eventually finding a polynomial invariant for knots and links.

Rule 2 states that the bracket polynomial of a knot or link projection $L$ along
with an unlinked component of the unknot, denoted \( L \cup \bigcirc \) is equal to the bracket polynomial of the projection \( L \) multiplied by some coefficient \( C \).

Interpreting Rule 3 is a bit more involved. The two equations shown are actually two ways of looking at the same equation, which we call the skein relation. The skein relation states that a bracket polynomial containing the crossing shown by \( \bigotimes \) is equivalent to the sum of two alterations of that polynomial, where the crossing shown is split vertically \( \bigotimes \) and horizontally \( \bigotimes \) and multiplied by the coefficients \( A \) and \( B \), respectively. Again, the second equation shown in Rule 3 is just a variation of the first.

Now, we want our bracket polynomial to be an invariant for knots, so we will need for it to be unaffected by our three Reidemeister moves, RI, RII, and RIII. We will begin by showing that our bracket polynomial is unaffected by RII. In Figure 3.14, we use our skein relation, Rule 3, in an attempt to determine this.

To show that this equation is unchanged by RII, we need the right side of this equation to equal \( \bigotimes \). The last term in the second to last part of this equality would be equal to \( \bigotimes \) if the coefficient \( BA \) were equal to 1, that is, we need \( B = A^{-1} \), so we can start by substituting this value in for \( B \) to obtain the result shown in Figure 3.15.

![Figure 3.14: The effect of RII on a bracket polynomial.](image)
Figure 3.15: Substituting $B = A^{-1}$.

Looking good so far. Now all we need is for the coefficient $(A^2 + C + A^{-2})$ to be equal to 0, that is, we need $C = -A^2 - A^{-2}$, so we will substitute this value in for $C$ to obtain the result shown in Figure 3.16.

Figure 3.16: Substituting $C = -A^2 - A^{-2}$.

So, with these modifications, our bracket polynomial is now unchanged by RII. Since we need these modifications for our bracket polynomial to be an invariant, we shall rewrite our three bracket polynomial rules as such.
Rule 1: \( \langle \bigcirc \rangle = 1; \)

Rule 2: \( \langle L \cup \bigcirc \rangle = (-A^2 - A^{-2}) \langle L \rangle; \)

Rule 3: \( \langle \times \rangle = A \langle \bigcirc \rangle + A^{-1} \langle \times \rangle; \)
\( \langle \times \rangle = A \langle \bigcirc \rangle + A^{-1} \langle \bigcirc \rangle. \)

With this notation, we can write any bracket polynomial that we calculate in terms of just one variable, \( A \). How convenient! Before we look at what kind of an affect our next Reidemeister move has on our bracket polynomial, we will take a moment to look at some examples of bracket polynomials in one variable that we may need for later. Our first examples will be the calculated bracket polynomials of the following two projections shown in Figure 3.17.

Figure 3.17: Calculating bracket polynomials.

We will use these bracket polynomials to calculate the bracket polynomial of the trefoil knot. We begin calculating the bracket polynomial of the trefoil knot, as illustrated in Figure 3.18.
Figure 3.18: Calculating the bracket polynomial of the trefoil knot.

Now, by substituting the results of the bracket polynomials calculated in our previous two examples, we obtain the result shown in Figure 3.19.

Figure 3.19: The calculated bracket polynomial of the trefoil knot.

So we have that the bracket polynomial of the trefoil knot is $A^7 - A^3 - A^{-5}$. We may still have further use for this yet, but in the meantime, we shall return to our investigation on the effects of the Reidemeister moves on bracket polynomials.

Let’s see what kind of an effect RIII has on our bracket polynomial. Since we know that our bracket polynomial is now unaffected by RII, we can see in Figure 3.20 that it follows that our bracket polynomial is also unaffected by RIII.

Figure 3.20: The effect of RIII on a bracket polynomial.

Now, we will just need to show that our polynomial is unaffected by RI, but it turns out that our bracket polynomial as it is will not be able to show this. In order
to find a polynomial that is unaffected by RI, as well as RII and RIII, we will need to introduce a new equation, the $X$ polynomial equation, which will consist of our bracket polynomial equation and another new equation known as a writhe. Notice in Figure 3.21 that we obtain two different looking bracket polynomials for the two different versions of RI. This will not do, but we will use the result of this equation to develop a polynomial that is unaffected by RI, RII, and RIII, and will thus be an invariant for knots and links.

\[
\begin{align*}
\langle\circ\rangle &= A\langle\circ\rangle + A^{-1}\langle\circ\rangle \\
&= A(-A^2-A^{-2})\langle\circ\rangle + A^{-1}\langle\circ\rangle \\
&= -A^2\langle\circ\rangle \\
\langle\circ\rangle &= A\langle\circ\rangle + A^{-1}\langle\circ\rangle \\
&= A\langle\circ\rangle + A^{-1}(-A^2-A^{-2})\langle\circ\rangle \\
&= -A^{-2}\langle\circ\rangle
\end{align*}
\]

Figure 3.21: The effect of RI on a bracket polynomial.

To introduce the $X$ polynomial equation, we begin by introducing an equation known as the writhe. Given a knot or link projection $L$, we denote the writhe of $L$ by $w(L)$. To calculate the writhe $w(L)$ we give an orientation to our link and assign a value of $+1$ or $-1$ to each left-hand or right-hand crossing, respectively, much like we did with linking number, and we take the sum of these values to obtain $w(L)$ [Ada04]. We can see from the following case exhaustive approaches shown in Figure 3.22 and Figure 3.23 that, like our bracket polynomial, our writhe is also unaffected by RII and RIII, as the total number of $+1$ and $-1$ crossings is left unchanged under these moves.
We are now ready to introduce our $X$ polynomial. Using our writhe and bracket polynomial, we define the $X$ polynomial by $X(L) = (-A^3)^{-w(L)} < L >$ [Ada04]. Since we have shown that both $w(L)$ and $< L >$ are unaffected by RII and RIII, it follows that $X(L)$ is also unaffected by RII and RIII, so in order to have a polynomial invariant for knots and links, we just need to show that $X(L)$ is unaffected by RI.
First, notice that adding a twist into our knot through RI creates a new crossing which will add a value of either +1 or −1 to \( w(L) \). We will consider the case of RI adding a +1 to \( w(L) \) and call this twisted link \( L' \), as illustrated in Figure 3.24.

![Figure 3.24: The effect of RI on \( X(L) \).](image)

If we think of starting with the twisted knot, \( L' \), we obtain the following calculations for \( X(L') \).

\[
X(L') = (-A^3)^{-w(L')} < L' > \\
= (-A^3)^{-(w(L)+1)} < L' > \\
= (-A^3)^{-w(L)-1} < L' > \\
= (-A^3)^{-w(L)}(-A^3)(-1) < L' >
\]

Now, we know from our attempt at showing the effect of RI on our bracket polynomial that \( < L' >= (-A^3) < L > \), so we have the following.
\[ X(L') = (-A^3)(-w(L))(-A^3)(-1) < L' > 
= (-A^3)(-w(L))(-A^3)(-1)((-A^3) < L >) 
= (-A^3)(-w(L)) < L > 
= X(L) \]

So \( X(L) \) is unaffected by the first version of RI. Repeating this process for the second version of RI, the case in which RI adds a \(-1\) to \( w(L) \), it would be shown similarly that \( X(L) \) is unaffected by both versions of RI and is consequently unaffected by RI in general. Thus, since \( X(L) \) is unaffected by RI, RII, and RIII, we have that \( X(L) \) is an invariant for knots and links. Excellent, we have our first polynomial invariant for knots and links.

Let us look at an example of this by calculating the \( X \) polynomial for the trefoil knot. First, we calculate the writhe, \( w(L_1) \), of the trefoil knot, as depicted in Figure 3.25.

![Figure 3.25: Calculating the writhe of the trefoil knot.](image)
\[ w(L_1) = -1 - 1 - 1 = -3 \]

Now, from one of our previous examples, we know that the bracket polynomial of the trefoil knot is given by \( <L_1> = A^7 - A^3 - A^{-5} \). Substituting \( w(L_1) \) and \( <L_1> \) into our \( X \) polynomial equation, we obtain the following.

\[
X(L_1) = (-A^3)^{-w(L_1)} <L_1>
= (-A^3)^{-(-3)}(A^7 - A^3 - A^{-5})
= (-A^9)(A^7 - A^3 - A^{-5})
= -A^{16} + A^{12} + A^4
\]

So, we have that the \( X \) polynomial of the trefoil knot is \(-A^{16} + A^{12} + A^4\). Since we have shown that the \( X \) polynomial is an invariant for knots, we know that any knot that yields a different \( X \) polynomial from our result must be distinct from the trefoil knot. We will now introduce a tremendously successful variation of the \( X \) polynomial known as the Jones polynomial. To obtain the Jones polynomial, denoted \( V(t) \), we simply substitute a value of \( t^{-1/4} \) for \( A \) in our \( X \) polynomial [Ada04].

Let us look at some examples of Jones polynomials, starting with that of the trefoil knot. To obtain the Jones polynomial of the trefoil knot, we substitute \( t^{-1/4} \) into our \( X \) polynomial of the trefoil knot to obtain \(-t^{-4} + t^{-3} + t^{-1}\), which will distinguish the trefoil knot from any knot of a different Jones polynomial.
Now, let us calculate the Jones polynomial of the mirror image of the trefoil knot. First we will need to calculate the $X$ polynomial, as demonstrated in Figure 3.26 and Figure 3.27.

\[
V(t) = X(t^{-1/4})
\]
\[
= -(t^{-1/4})^{16} + (t^{-1/4})^{12} + (t^{-1/4})^{4}
\]
\[
= -t^{-4} + t^{-3} + t^{-1}
\]
\[ w(L_2) = +1 + 1 + 1 = 3 \]

\[ X(L_2) = (-A^3)^{-w(L_2)} < L_2 > \]
\[ = (-A^3)^{-3}(A^{-7} - A^{-3} - A^5) \]
\[ = (-A^{-9})(A^{-7} - A^{-3} - A^5) \]
\[ = -A^{-16} + A^{-12} + A^{-4} \]

So we have that the \( X \) polynomial of the mirror image of the trefoil knot is \(-A^{-16} + A^{-12} + A^{-4}\), which is the \( X \) polynomial of the trefoil knot with opposite signs on all of the exponents. Unsurprisingly, if we substitute \( t^{-1/4} \) into this polynomial to obtain the corresponding Jones polynomial, we get \(-t^4 + t^3 + t\), which is the Jones polynomial of the trefoil knot with opposite signs on the exponents as well. Since the Jones polynomial is an invariant, and the polynomial of the mirror image of the trefoil knot is distinct from that of the trefoil knot, we have that these two knots are distinct.

\[ V(t) = -(t^{-1/4})^{-16} + (t^{-1/4})^{-12} + (t^{-1/4})^{-4} \]
\[ = -t^4 + t^3 + t \]

Next, we will compute the Jones polynomial for the figure-eight knot, as demonstrated in Figure 3.28 and Figure 3.29.
Figure 3.28: The bracket polynomial of the figure-eight knot.

\[ w(L_3) = +1 + 1 - 1 - 1 = 0 \]
\[
X(L_3) = (-A^3)^{w(L_3)} < L_3 > \\
= (-A^3)^0(A^8 - A^4 + 1 - A^{-4} + A^{-8}) \\
= A^8 - A^4 + 1 - A^{-4} + A^{-8}
\]

\[
V(t) = (t^{-1/4})^8 - (t^{-1/4})^4 + 1 - (t^{-1/4})^{-4} + (t^{-1/4})^{-8} \\
= t^2 - t + 1 - t^{-1} + t^{-2}
\]

So, we have that the Jones polynomial of the figure-eight knot is \(t^2 - t + 1 - t^{-1} + t^{-2}\), which distinguishes the figure eight knot from the trefoil knot and its mirror image.

How about an example of a Jones polynomial for links? We can compute the Jones polynomial of the unlink of two components in the same way that we computed the previous Jones polynomials, as demonstrated in Figure 3.30 and Figure 3.31.

Figure 3.30: The bracket polynomial of the unlink of two components.
Figure 3.31: Calculating the writhe of the unlink of two components.

\[ w(L_4) = 0 \]

\[
X(L_4) = (-A^3)^{-w(L_4)} < L_4 > \\
= (-A^3)^0(-A^2 - A^{-2}) \\
= -A^2 - A^{-2}
\]

\[
V(t) = -(t^{-1/4})^2 - (t^{-1/4})^{-2} \\
= -t^{1/2} - t^{-1/2}
\]

So, we have that the Jones polynomial of the unlink of two components is \(-t^{1/2} - t^{-1/2}\). Now, since we have shown that the Jones polynomial is an invariant, and since all of the Jones polynomials we have computed are distinct from one another, we know that all of their corresponding knots are also distinct from one another. It seems that the Jones polynomial lives up to its reputation as a very successful means of distinguishing between knots.
So, as stated in Section 1.1, an important question in knot theory is how we can tell if two knots are, or are not, equivalent. We have learned that we can show when two knots are equivalent through ambient isotopy, planar isotopy, and Reidemeister moves. We have succeeded in showing when two knots are not equivalent through invariants like tricolorability, unknotting number, and the Jones polynomial. There are more invariants which provide more techniques to this end, and a great deal of work on knots that has yet to be realized, but with the material covered in this thesis, we have embarked upon a journey of discovery.
Bibliography

