8-2022

SYMMETRIC GENERATIONS AND AN ALGORITHM TO PROVE RELATIONS

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Symmetric Generations And an Algorithm to Prove Relations

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

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Abstract

In this thesis we have discovered homomorphic images of several progenitors such as 3^{*56}:(2^3:(3:7), 3^{*14}:(2^3:(3:7)), $5^{*24} : S_5$, $2^{*10} : (10 : 2)$, $5^{*24} : (A_5 : 2)$, and $11^{*12} :_m L_2(11)$. We give isomorphism types of each image that we have found. We then create a monomial representation of $L_2(11)$ by lifting 5:11 onto it.

We manually perform Double Coset Enumeration of 3:(2×S_5) over D_{12} to create its Cayley graph. This is achieved by solving many word problems. The Cayley graph is used to find a permutation representation of 3:(2×S_5). We also perform Double Coset Enumeration $S_3 \times A_5$ over 15:2 and 10:2, where 15:2 is a maximal subgroup containing 10:2.

Finally, a code based algorithm is included that solves all of the word problems used to perform Double Coset Enumeration of $L_2(25)$ over S_5:2 and Double Coset Enumeration of $(A_5)^2$:2 over A_5:2.

Acknowledgements

I would like to thank all my California State San Bernardino professors for always being very helpful, kind, and especially patient. A special thanks to Dr. Hasan for guiding me in the completion of this thesis and having the record of being my professor for more classes than anyone else. I would also like to thank Dr. Trapp, Dr. Chavez (the best mathemagician), and Dr. Ventura, all of whom I have known for over 24 years. I would like to thank Dr. Jeffrey Meyer, Dr. Charles Stanton, Dr. Lynn Scow, Dr. Laura Wallace, Dr. Corey Dunn, Dr. Jeremy Aikin, Dr. Bob Stein, Dr. Gary Griffing, and Dr. Peter Williams, all which I appreciate. From the Computer Science department I would like to thank Dr. Richard John Botting and Dr. Kay Zemoudeh for teaching me how to code, a skill that is still helping me today. I would like to thank my friends Mayra McGrath and Jessica Coen for spending many hours studying and proving all of those math problems with me. Math is easier and more fun with friends like you. And the biggest thank you goes to my loving and selfless wife Katie. Not only did she encourage me to come back to CSUSB for my Masters, but she also supported me unconditionally, I love you. Finally, to everyone above, you have made my journey at CSUSB a wonderful adventure that I will never forget, and for that I am grateful.

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Introduction

In group theory it is often necessary to prove that a finite image of a progenitor is isomorphic to a permutation group. To do this we perform double coset enumeration of the image over a transitive subgroup. However, this process requires solving many right coset relations between words, which is a formidable task. It is very difficult to determine whether a given word is the identity. In the area of combinatorial group theory, the word problem for a finitely generated group G is the algorithmic problem of deciding whether two words in the generators represent the same element. There does not exist an algorithm to give relations between words for an arbitrary finitely generated group [Coll86]. However, there do exist algorithms to establish relationships between all words of some types of groups [Khar13]. We have given an original algorithm, to give all relations between all words, for a large number of groups.

Our constructions of the homomorphic images of the progenitors that we have presented in this thesis are based on our technique of manual double coset enumeration. We provide a background to this below.

Symmetric Generation

A progenitor is the semi-direct product $m^{*n} : N$, where $m^{\star n}$ represents a free product of n copies of the cyclic group $\mathbb{Z}_m$, m being the order of t_i, and N is a group of automorphisms of $m^{\star n}$ which permutes the n cyclic subgroups by conjugation. Thus, for $n \in N$, we have $t_i^n = t_j^r$, where r is an integer coprime to m. When $m = 2$, N acts by conjugation as permutations of the n involutory symmetric generators. Now, the above elements of N can be gathered on the left, every element of the progenitor can be represented as nw, where $n \in N$ and w is a word in the symmetric generators. Thus any additional relation by which we must factor the progenitor to obtain a finite image G must have the form $nw(t_1, t_2, .., t_n)$, where $n \in N$ and w is a word in $T = \{t_1, t_2, \ldots, t_n\}$. The group $\frac{m^{*n}:N}{n_1w_1,..,n_sw_s}$ may be identified.

Double Coset Enumeration

Assume $G = \frac{m^{*n}:N}{n_1w_1,..,n_sw_s}$. The double coset NwN is given by $NwN = \{Nwn | n \in N\} = \{Nnn^{-1}wn | n \in N\} = \{n \in N : Nw^n = Nw\}$, where w is a word in the symmetric generator. Since G is finite, G may be decomposed as a union of double cosets $G = \cup_{i=1}^{k} Nw_iN$.

Define $N^i = C_N(t_i)$; $N^{ij} = C_N(\langle t_i, t_j \rangle)$, etc., single point and two point stabilizers in N respectively. The coset stabilizing subgroup, $N^{(w)}$, of N is given by, $N^{(w)} = \{n \in N | Nwn = Nw\} = \{n \in N | Nnn^{-1}wn\} = \{n \in N | Nw^n = Nw\}$, where w is a word in the symmetric generators. Clearly $N^w \leq N^{(w)}$.

In order to obtain the index of N in G we shall perform a manual double coset enumeration on G over N; thus we must find all of the double cosets $[w]$ and work out how many single cosets each of them contains. We shall know that we have completed the double coset enumeration when the set of right cosets obtained is closed under right multiplication. Moreover, the completion test above is best performed by obtaining the orbits of $N^{(w)}$ on the symmetric generators. We need only identify, for each $[w]$, the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit.

Finding isomorphism types serves two purposes in this thesis, first it helps to classify the groups, and secondly it is a necessary step in the construction of finite images of progenitors. In chapter 1 we find the isomorphism type of several transitive groups in order to classify them. In chapter 3 we find isomorphism types of the permutation groups used to construct the progenitors. In order to start the process, a transitive group N generated by a set of permutations on the set X = { 1, 2, ... n } is required, where n is a natural number.

In chapter 4 and chapter 5 we solve numerous word problems in order to perform double coset enumeration on 3:(2×S_5) over D_{12}, $L_2(5)$ over S_5, S_3×A_5 over 15:2 and 10:2, and $(A_5)^2$:2 over A_5:2.

In section 4.3 we assign a numeric label in order to find isomorphic permutations to the generators of the group G. To achieve this, every right coset will be multiplied by the generators of the group N. These products must be proven to be one of the right cosets in order to assign it a label. The mapping of the labeling of cosets to the labeling of the products is a permutation corresponding to the generator. To prove that the image is isomorphic to a permutation group, the mapping of the relations to elements of the permutation group must be shown to have equivalent properties to the relations of the image.

Finally, in chapter 6 we explore an algorithm that can be used to solve double coset enumeration relation word problems. This algorithm is used to prove all word problems for $L_2(5)$ over S_5, $(A_5)^2{:}2$ over $A_5 : 2$. A MAGMA implementation of the algorithm can be found in Appendix G.

Chapter 1

Isomorphic Types

1.1 Definitions and Theorems

Definition 1.1. ***Definition 1.1.4.*** ***(Semigroup)*** A **semigroup** $(G, *)$ is a nonempty set G equipped with an associative operation $*$. [Rot95]

Definition 1.2. ***Definition 1.1.6.*** ***(Group).*** A **group** is a semigroup G containing an element e such that
(i) $e * a = a = a * e$ for all $a \in G$
(ii) for every $a \in G$, there is an element $b \in G$ with $a * b = e = b * a$. [Rot95]

Definition 1.3. *A subgroup $K \leq G$ is a normal subgroup,denoted by $k \trianglelefteq G$, if $gKg^1 = K$ for every $g \in G$. [Rot95]*

Definition 1.4. ***Definition 1.1.7.*** ***(Order)*** If G is a group, then the **order** of

G, dentoted $|G|$, is the number of elements in G. [Rot95]

Definition 1.5. ***Definition 1.1.5***. ***(Symmetric Group)*** The **symmetric group**, denoted S_n or S_X is the set of all permutations of the nonempty set $X = \{1, 2, ..., n\}$. S_n is a group of order $n!$ on n letters. [Rot95]

Definition 1.1.31. (**semi-direct product**) *A group G is a* ***semi-direct product*** *of K by Q, denoted by G = K:Q, if K* $\lhd$ *G and K has a complement* $Q_1 \cong$ *Q. One also says that G splits over K. [Rot95]*

Definition 1.1.29. (**direct product**) *If H and K are groups, then their* ***direct product****, denoted by H* $\times$ *K, is the group with elements all ordered pairs (h,k), where h* $\in$ *H and k* $\in$ *K, and with the operation (h,k)(h',k')=(hh',kk'). [Rot95]*

Definition 1.1.8. (**Free Group**) *If X is a subset of a group F, then F is a* ***free group*** *with basis X if, for every group G and every function* $f : X \to G$*, there exists a unique homomorphism* $\varphi : F \to G$ *extending f. [Rot95]*

Definition 1.6. *Let F be a field and G group. Then a presentation or G is a homomorphism* $\rho : G \to GL(n, F)$ *for some integer n. [Isaa76]*

Definition 1.1.1. (**Permutation**) *If X is a nonempty set, a* ***permutation*** *is the bijective mapping* $\alpha : X \to X$*. [Rot95]*

Definition 1.1.19. (**homomorphism**) *Let (G,*) and (H,$\circ$) be groups. A function f:G$\rightarrow$ is a **homomorphism** if, for all a, b $\in$ G, f(a*b) = f(a) $\circ$ f(b). [Rot95]*

Definition 1.1.20. *An **isomorphism** is a homomorphism that is also a bijection. We say that G is **isomorphic** to H, denoted G $\cong$ H, if there exists an isomorphism f:G $\rightarrow$ H. [Rot95]*

Theorem 1.7. ***Theorem 1.1.25**. **(Third Isomorphism Theorem)*** Let K$\leq$H$\leq$G, where both K and H are normal subgroups of G. Then H/K is a normal subgroup of G/K and (G/K)(H/K) $\cong$ G/H. [Rot95]

1.2 46:11

Notation: if x is a word in a presentation of a group and the corresponding permutation is xx then we express this by writing x $\sim$ xx.

Given a group N, we demonstrate our method of finding the isomorphism type of N. We give the following example to illustrate the process.

Let N = $\langle x, y\rangle$, where x = (1, 20, 9, 35, 42, 8, 32, 33, 38, 46, 15)(2, 19, 10, 36, 41, 7, 31, 34, 37, 45, 16)(3, 23, 18, 5, 27, 25, 22, 13, 43, 11, 40)(4, 24, 17, 6, 28, 26, 21, 14, 44, 12, 39) and y = (1, 36, 4, 25, 8, 5, 15, 11, 32, 23, 17, 2, 35, 3, 26, 7, 6, 16, 12, 31, 24, 18)(9, 41, 20, 37, 39, 30, 33, 13, 21, 27, 44, 10, 42, 19, 38, 40, 29, 34, 14, 22, 28, 43)(45, 46). We note that N < S_{46}. It is given that $|N|$

= 506.

We use MAGMA to compute the normal lattice of N and summarize the information in the following diagram in which every group is a normal subgroup of all the groups that it is connected by one or more upward lines. Each vertex of the diagram is labeled by the order of the normal subgroup.

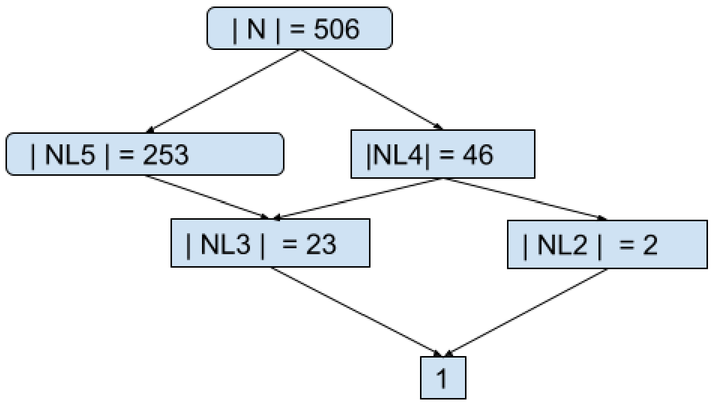

Figure 1.1: Normal Lattice Diagram N.

We will follow the following path to arrive at the isomorphism class.

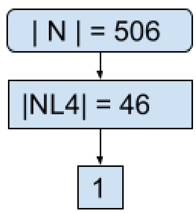

Figure 1.2: Relevant Path of N.

We use MAGMA to see that the largest normal subgroup of N is NL4 =

$\langle a \rangle$, where a = (1, 37, 28, 18, 8, 43, 33, 23, 14, 3, 39, 30, 20, 10, 46, 36, 26, 16, 6, 41, 32, 22, 12, 2, 38, 27, 17, 7, 44, 34, 24, 13, 4, 40, 29, 19, 9, 45, 35, 25, 15, 5, 42, 31, 21, 11). Thus, NL4 ≅ 46 since $|a| = 46$.

Let Q ≅ N/NL4. N cannot be the direct product of NL4 and Q since N has no normal subgroup of order 11, which means N is a semi-direct product of 46 by Q. Now Q ≅ N/NL4 and MAGMA gives us N/NL4 = $\{NL4b\}$, where b = (1, 20, 9, 35, 42, 8, 32, 33, 38, 46, 15)(2, 19, 10, 36, 41, 7, 31, 34, 37, 45, 16)(3, 23, 18, 5, 27, 25, 22, 13, 43, 11, 40)(4, 24, 17, 6, 28, 26, 21, 14, 44, 12, 39). Thus Q ≅ 11. Since Q ≅ 11 and NL4 ≅ 46, we find the action 46 on 11 is $a^b = a^{25}$. We verify in MAGMA that $\langle a, b | a^{46}, b^{11}, a^b = a^{25} \rangle$. Thus N ≅ 46:11. Please see Appendix B for the MAGMA code.

1.3 (46:11):2

Given a group N, we demonstrate our method of finding the isomorphism type of N. We give another example to illustrate the process.

Let N = $\langle x, y \rangle$, where x = (1, 42, 46, 28, 40, 32, 21, 44, 13, 34, 35, 4, 9, 6, 24, 11, 19, 30, 7, 38, 17, 15)(2, 41, 45, 27, 39, 31, 22, 43, 14, 33, 36, 3, 10, 5, 23, 12, 20, 29, 8, 37, 18, 16)(25, 26) and y = (1, 39, 21, 15, 13, 44, 7, 42, 37, 6, 25, 2, 40, 22, 16, 14, 43, 8, 41, 38, 5, 26)(3, 10, 12, 28, 17, 30, 33, 20, 46, 23, 31, 4, 9, 11, 27, 18, 29, 34, 19, 45, 24, 32)(35, 36). We note that N $< S_{46}$. It is given that $|N|$ = 1012.

We use MAGMA to compute the normal lattice of N and summarize the information in the following diagram in which each group is a normal subgroup of all the groups that is connected by one or more upward lines. Each vertex of the diagram are labeled by the order of the normal subgroup.

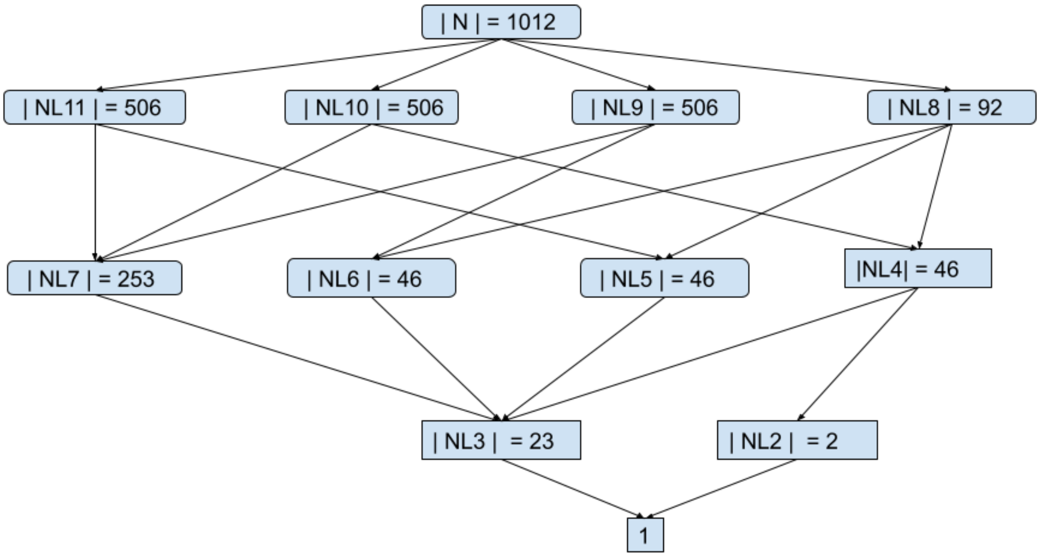

Figure 1.3: Normal Lattice Diagram of N.

We will follow the following path to arrive at the isomorphism class.

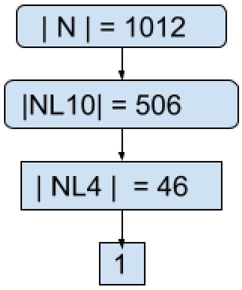

Figure 1.4: Relevant Path of 46:(11:2).

We use MAGMA to see that NL10 is isomorphic to the group in the previous section. MAGMA gives us NL10 = $\langle a, b \rangle$, where a = (1, 32, 16, 45, 29, 14, 43, 28, 12, 42, 25, 10, 40, 23, 7, 38, 21, 6, 35, 20, 3, 34, 17, 2, 31, 15, 46, 30, 13, 44, 27, 11, 41, 26, 9, 39, 24, 8, 37, 22, 5, 36, 19, 4, 33, 18) and b = (3, 5, 9, 17, 33, 19, 37, 27, 7, 13, 25)(4, 6, 10, 18, 34, 20, 38, 28, 8, 14, 26)(11, 22, 42, 36, 23, 45, 44, 39, 32, 15, 30)(12, 21, 41, 35, 24, 46, 43, 40, 31, 16, 29). Thus NL10 ≅ 46:11.

Let Q = N/NL10 ≅ 2. N cannot be the direct product of NL10 and any other subgroup since N has no normal subgroup of order 2 outside of NL10. This means N is a semi-direct product of 2 and Q. Q ≅ N/NL10 and MAGMA gives us N/NL10 = $\{NL10c\}$, where c = (1, 42, 46, 28, 40, 32, 21, 44, 13, 34, 35, 4, 9, 6, 24, 11, 19, 30, 7, 38, 17, 15)(2, 41, 45, 27, 39, 31, 22, 43, 14, 33, 36, 3, 10, 5, 23, 12, 20, 29, 8, 37, 18, 16)(25, 26). Using MAGMA we confirm Q ≅ 2. Since Q ≅ 2 and NL10 ≅ 46:11, we find the action of $a^c = a^{45}$ and the action $b^c = a^{36}b$. We verify in MAGMA that the presentation of N = $\langle a^{46}, b^{11}, a^b = a^{25}, c^2, a^c = a^{45}, b^c = a^{36}b \rangle$. Thus N ≅ (23 : 2):11. Please see Appendix A for the MAGMA proof.

Chapter 2

A Degree 12 Monomial Representation of $L_2(11)$

2.1 Definition and Theorems

Definition 2.1. *The set of all invertible $n \times n$ matrices with entries in field F, under matrix multiplication, forms a group. This group is called the general linear group of degree n over F, and is denoted by GL(n, F). [Jame01]*

Definition 2.2. *Let p be a prime number. SL (2, p) the set of all 2×2 matrices M with entries in $\mathbb{Z}_p$ such that det(M) = 1. Then SL(2, p) is a group under matrix multiplication, and is called the 2-dimensional special linear group over $\mathbb{Z}_p$. [Jame01]*

Definition 2.3. *The factor group SL(2, p)/Center(SL(2, p)) is called the 2-*

dimensional projective special linear group, and is written as PSL(2, p) or $L_2(p)$*. [Jame01]*

Definition 2.4. *If X is a set and G is a group, then X is a* ***G-set*** *if there is a function* α*:* $G\times X \longrightarrow X$ *(called an* ***action****), denoted by* α*:* $(g, x) \mapsto gx$*, such that:*

(i) $1x = x$ *for all* $x \in X$*; and*

(ii) $g(hx) = (gh)x$ *for all* $g, h \in G$ *and* $x \in X$*. [Rot95]*

Definition 2.5. *G* ***acts*** *on X, if* $|X| = n$*, then n is called the* ***degree*** *of the G-set X. [Rot95]*

Definition 2.6. *If X is a G-set and* $x \in X$*, then the* ***G-orbit*** *of x is* $Gx = \{gx : g \in G\} \subset X$*. [Rot95]*

Definition 2.7. *A G-set X is* ***transitive*** *if it has only one orbit; that is for every* $x, y \in X$*, there exists* $\sigma \in G$ *with* $y = \sigma x$*. [Rot95]*

Definition 2.8. *A representation of G over field F is a homomorphism* ρ *from G to GL(n, F), for some* ρ*. The degree of* ρ *is the integer n.*

Definition 2.9. *Let* ρ *be an representation of G. Then the character* χ *of afforded* *by ρ is the function given by $\chi(g) = tr\ (\rho(g))$, where tr is the trace. [Isaa76]*

Theorem 2.10. *Theorem 1.1.13. The number of irreducible characters of G is equal to the number of conjugacy classes of G. [Led77]*

Definition 2.11. ***Definition 1.1.14**. **(Degree of a Character)*** The sum of squares of the degrees of the distinct irreducible characters of G is equal to $|G|$. The **degree of a character** χ is $\chi(1)$. Note that a character whose degree is 1 is called a **linear character**.[Led77]

Definition 2.12. ***Definition 1.1.15**. **(Lifting Process)*** Let N be a normal subgroup of G and suppose that $A_0(Nx)$ is a representation of degree m of the group G/N. Then $A(x) = A_0(Nx)$ defines a representation of G/N lifted from G/N. If $\varphi_0(Nx)$ is a character of $A_0(Nx)$, then $\varphi(x) = \varphi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = Im, \varphi(u) = m = \varphi(1)$. The lifting process preserves irreducibility.[Led77]

Definition 2.13. ***Definition 1.1.16**. **(Induced Character)*** The character of A(x), which is called the **induced character** of ϕ, will be dentoted by ϕ^G. Thus, $\phi^G = \text{trA(x)} = \sum_{i=1}^{n} \phi(t_i x t_i{}^{-1})$. [Led77]

Definition 2.14. *Let n be a positive integer, and denote by $\mathbb{C}$ the set of all complex numbers. The set of nth roots of unity in $\mathbb{C}$, with the usual multiplication of* *complex numbers, is a group of order n. It is written as $\mathbb{C}_n$ and is called the cyclic group of order n. [Jame01]*

Definition 2.15. ***Definition 1.1.10.*** ***(Progenitor)*** A **progenitor** is a semi-direct product of the following form: P≅ 2^{*n}:N = {πw | π ∈ N, and w is a word in the t_i}, where 2^{*n} denotes a free product of n copies of a cyclic group of order 2 generated by involutions t_i for i=1,...,n; and N is a transitive permutation group of degree n which acts on the free product by permuting the involutory generators. [Curt96]

2.2 Induction of 5:11 Onto $L_2(11)$

Let G = ⟨ (3, 7, 9, 4, 5)(6, 8, 12, 10, 11), (1, 8, 2)(3, 4, 7)(5, 12, 11)(6, 9, 10)⟩. MAGMA gives us that the order of G = 660 and G is isomorphic $L_2(11)$. Let x = (3, 7, 9, 4, 5)(6, 8, 12, 10, 11) and y = (1, 8, 2)(3, 4, 7)(5, 12, 11).

Given that G has a faithful character χ of degree 12. In the table below z_5 is a primitive 5th root of unity.

Class Representative	Id(G)	$(xy^{-1})^3$	y	yx^2	$(yx^2)^2$	yx^{-1}	xy	$(yx)^2$
size	1	55	110	132	132	110	60	60
χ	12	0	0	$-z_5^3-z_5^2-1$	$-z_5^3-z_5^2-1$	0	1	1

Table 2.1: The Classes of G.

We need to find a subgroup, H, of G with index 12. If we can find a character of H that induces to χ then we can construct a faithful irreducible monomial representation of G.

Let H = ⟨ z, w ⟩, where z = (1, 4, 3, 10, 8)(2, 7, 5, 12, 11) and w = (1, 2, 12, 11, 3, 7, 6, 4, 5, 8, 10). Given ⟨ z ⟩ is normal in H and ⟨ w ⟩ is normal in H, We have H ≅ ⟨ z ⟩:⟨ w ⟩; that is, H ≅ 5:11.

We want to induce the following character of H up to G:

Class Representative	Id(H)	z	z^2	z^3	z^4	w	w^2
Size	1	11	11	11	11	5	5
χ_2	1	z_5	${z_5}^2$	${z_5}^3$	${z_5}^4$	1	1

Table 2.2: The Character χ_2 of 5:11.

Definition 2.16. ***(Formula for Induced Character)*** *Let G be a finite group and H be a subgroup such that [G : H] = $|G|/|H|$ = n. Let C_α, α = 1, 2, . . . m be the conjugacy classes of G with $|C_\alpha| = h_\alpha$. Let ϕ be a character of H and ϕ^G be the character of G induced from the character ϕ of H up to G. The values of ϕ^G on the m classes of G are given by:*

$$\phi_\alpha^G(x) = \frac{n}{h_\alpha} \sum_{\omega \in C_\alpha \cap H} = \phi(\omega).$$

In our case n = 660/55 = 12 and $\phi = \chi_2$. The conjugacy class of G are as follows:

Class Repre- sentative	Id(G)	$(xy^{-1})^3$	y	yx^2	$(yx^2)^2$	yx^{-1}	xy	$(yx)^2$
size	1	55	110	132	132	110	60	60

Table 2.3: The Conjugacy Classes of $L_{\mathbf{2}}\mathbf{(11)}$.

We will use MAGMA to look up the elements of the group and of the conjugacy classes.

$C_1 \cap H = \{id(H)\} \implies \chi_2(Id(H)) = 1 \implies \phi_1^G = 12/1(1) = 12.$

$C_2 \cap H = \{\} \implies \phi_2^G = 0$, where $\{\}$ is the empty set.

$C_3 \cap H = \{\} \implies \phi_3^G = 0.$

$C_4 \cap H = \{$ (1, 7, 5, 2, 10)(3, 11, 8, 6, 12), (1, 12, 2, 4, 3)(5, 8, 11, 6, 10), (1, 10, 2, 5, 7)(3, 12, 6, 8, 11), (1, 6, 7, 4, 11)(2, 3, 8, 10, 5), (1, 8, 12, 7, 10)(2, 4, 6, 5, 3), (1, 12, 8, 6, 2)(3, 7, 11, 4, 10), (1, 11, 4, 7, 6)(2, 5, 10, 8, 3), (1, 2, 6, 8, 12)(3, 10, 4, 11, 7), (1, 5, 11, 12, 3)(2, 6, 4, 7, 8), (1, 3, 12, 11, 5)(2, 8, 7, 4, 6), (1, 3, 4, 2, 12)(5, 10, 6, 11, 8), (1, 5, 3, 6, 7)(2, 12, 4, 10, 11), (1, 4, 8, 5, 11)(3, 7, 10, 12, 6), (1, 7, 6, 3, 5)(2, 11, 10, 4, 12), (1, 6, 11, 10, 2)(4, 5, 12, 7, 8), (1, 10, 7, 12, 8)(2, 3, 5, 6, 4), (1, 11, 5, 8, 4)(3, 6, 12, 10, 7), (1, 8, 10, 3, 4)(2, 11, 12, 5, 7), (2, 7, 11, 3, 8)(4, 12, 10, 6, 5), (1, 2, 10, 11, 6)(4, 8, 7, 12, 5), (2, 8, 3, 11, 7)(4, 5, 6, 10, 12),(1, 4, 3, 10, 8)(2, 7, 5, 12, 11) $\} \implies \phi_4^G =$ (12/132)(χ_2((1, 7, 5, 2, 10)(3, 11, 8, 6, 12)) + χ_2((1, 12, 2, 4, 3)(5, 8, 11, 6, 10))+ χ_2((1, 10, 2, 5, 7)(3, 12, 6, 8, 11)) +χ_2((1, 6, 7, 4, 11)(2, 3, 8, 10, 5)) + χ_2((1, 8, 12, 7, 10)(2, 4, 6, 5, 3)) + χ_2((1, 12, 8, 6, 2)(3, 7, 11, 4, 10)) + χ_2((1, 11, 4, 7, 6)(2, 5, 10, 8, 3)) + χ_2((1, 2, 6, 8, 12)(3, 10, 4, 11, 7)) + χ_2((1, 5, 11, 12, 3)(2, 6, 4, 7, 8)) + χ_2((1, 3, 12, 11, 5)(2, 8, 7, 4, 6)) + χ_2((1, 3, 4, 2, 12)(5, 10, 6, 11, 8)) + χ_2((1, 5, 3, 6, 7)(2, 12, 4, 10, 11)) + χ_2((1, 4, 8, 5, 11)(3, 7, 10, 12, 6) + χ_2((1, 7, 6, 3, 5)(2, 11, 10, 4, 12)) + χ_2((1, 6, 11, 10, 2)(4, 5, 12, 7, 8)) + χ_2((1, 10, 7, 12, 8)(2, 3, 5, 6, 4),) + χ_2((1, 11, 5, 8, 4)(3, 6, 12, 10, 7)) + χ_2((1, 8, 10, 3, 4)(2, 11, 12, 5, 7)) + χ_2((2, 7, 11, 3, 8)(4, 12, 10, 6, 5)) + χ_2((1, 2, 10, 11, 6)(4, 8, 7, 12, 5)) + χ_2((2, 8, 3, 11, 7)(4, 5, 6, 10, 12)) + χ_2((1, 4, 3, 10, 8)(2, 7, 5, 12, 11))) $\Longrightarrow$ (12/132)($z_5{}^4$ + $z_5{}^4$ + z_5 + z_5 +z_5 + z_5 + $z_5{}^4$ +$z_5{}^4$ + z_5 + $z_5{}^4$ + z_5 + $z_5{}^4$ + $z_5{}^4$ + z_5 + $z_5{}^4$ + $z_5{}^4$ + z_5 + $z_5{}^4$ + $z_5{}^4$ + $z_5{}^4$ + $z_5{}^4$ + $z_5{}^4$) = (12/132)(11)(-$z_5{}^3$ - $z_5{}^2$ - 1) = -$z_5{}^3$ - $z_5{}^2$ - 1.

$C_5 \cap H$ = { (1, 11, 2, 6, 10)(4, 12, 8, 5, 7), (1, 4, 12, 3, 2)(5, 6, 8, 10, 11), (1, 5, 10, 7, 2)(3, 8, 12, 11, 6), (1, 2, 7, 10, 5)(3, 6, 11, 12, 8), (1, 8, 2, 12, 6)(3, 11, 10, 7, 4), (1, 6, 5, 7, 3)(2, 10, 12, 11, 4), (1, 10, 6, 2, 11)(4, 7, 5, 8, 12), (1, 2, 3, 12, 4)(5, 11, 10, 8, 6), (1, 6, 12, 2, 8)(3, 4, 7, 10, 11), (1, 7, 8, 10, 12)(2, 5, 4, 3, 6), (1, 5, 4, 11, 8)(3, 12, 7, 6, 10), (1, 3, 8, 4, 10)(2, 5, 11, 7, 12), (2, 3, 7, 8, 11)(4, 6, 12, 5, 10), (1, 7, 11, 6, 4)(2, 8, 5, 3, 10), (1, 11, 3, 5, 12)(2, 4, 8, 6, 7), (2, 11, 8, 7, 3)(4, 10, 5, 12, 6), (1, 12, 10, 8, 7)(2, 6, 3, 4, 5), (1, 4, 6, 11, 7)(2, 10, 3, 5, 8), (1, 10, 4, 8, 3)(2, 12, 7, 11, 5), (1, 8, 11, 4, 5)(3, 10, 6, 7, 12), (1, 12, 5, 3, 11)(2, 7, 6, 8, 4),(1, 3, 7, 5, 6)(2, 4, 11, 12, 10) $\Longrightarrow$ ϕ_5^G = (12/132)(χ_2((1, 11, 2, 6, 10)(4, 12, 8, 5, 7)) + χ_2((1, 4, 12, 3, 2)(5, 6, 8, 10, 11)) + χ_2((1, 5, 10, 7, 2)(3, 8, 12, 11, 6)) + χ_2((1, 2, 7, 10, 5)(3, 6, 11, 12, 8)) + χ_2((1, 8, 2, 12, 6)(3, 11, 10, 7, 4)) + χ_2((1, 6, 5, 7, 3)(2, 10, 12, 11, 4)) + χ_2((1, 10, 6, 2, 11)(4, 7, 5, 8, 12)) + χ_2((1, 2, 3, 12, 4)(5, 11, 10, 8, 6)) + χ_2((1, 6, 12, 2, 8)(3, 4, 7, 10, 11)) + χ_2((1, 7, 8, 10, 12)(2, 5, 4, 3, 6)) + χ_2((1, 5, 4, 11, 8)(3, 12, 7, 6, 10)) + χ_2((1, 3, 8, 4, 10)(2, 5, 11, 7, 12)) + χ_2((2, 3, 7, 8, 11)(4, 6, 12, 5, 10)) + χ_2((1, 7, 11, 6, 4)(2, 8, 5, 3, 10)) + χ_2((1, 11, 3, 5, 12)(2, 4, 8, 6, 7)) + χ_2((2, 11, 8, 7, 3)(4, 10, 5, 12, 6)) + χ_2((1, 12, 10, 8, 7)(2, 6, 3, 4, 5)) + χ_2((1, 4, 6, 11, 7)(2, 10, 3, 5, 8)) + χ_2((1, 10, 4, 8, 3)(2, 12, 7, 11, 5)) + χ_2((1, 8, 11, 4, 5)(3, 10, 6, 7, 12)) + χ_2((1, 12, 5, 3, 11)(2, 7, 6, 8, 4)) +χ_2((1, 3, 7, 5, 6)(2, 4, 11, 12, 10))) = (12/132)($z_5{}^3$ + $z_5{}^2$ + $z_5{}^3$ + $z_5{}^2$ + $z_5{}^2$ + $z_5{}^2$ + $z_5{}^2$ + $z_5{}^3$ + $z_5{}^3$ + $z_5{}^3$ + $z_5{}^2$ + $z_5{}^2$ + $zz_5{}^2$ + $z_5{}^2$ + $z_5{}^2$ + $z_5{}^3$ + $z_5{}^2$ + $z_5{}^3$ + $z_5{}^3$ + $z_5{}^3$ + $z_5{}^3$ + $z_5{}^3$) = (12/132)(11)($z_5{}^3$ + $z_5{}^2$) = $z_5{}^3$ + $z_5{}^2$.

$C_6 \cap H = \{\} \implies \phi_6^G = 0$.

$C_7 \cap H$ = {(1, 12, 3, 6, 5, 10, 2, 11, 7, 4, 8), (1, 6, 2, 4, 12, 5, 11, 8, 3, 10, 7), (1, 4, 11, 10, 6, 12, 8, 7, 2, 5, 3), (1, 5, 7, 12, 10, 4, 3, 2, 8, 6, 11), (1, 10, 8, 5, 4, 6, 7, 3, 11, 12, 2)} $\implies \phi_7^G$ = (12/132)(χ_2((1, 12, 3, 6, 5, 10, 2, 11, 7, 4, 8)) + χ_2((1, 6, 2, 4, 12, 5, 11, 8, 3, 10, 7)) + χ_2((1, 4, 11, 10, 6, 12, 8, 7, 2, 5, 3)) + χ_2((1, 5, 7, 12, 10, 4, 3, 2, 8, 6, 11)) + χ_2((1, 10, 8, 5, 4, 6, 7, 3, 11, 12, 2))) = (60/12)(1 + 1 + 1 + 1 + 1) = 1.

$C_8 \cap H$ = {(1, 2, 12, 11, 3, 7, 6, 4, 5, 8, 10), (1, 11, 6, 8, 2, 3, 4, 10, 12, 7, 5), (1, 7, 10, 3, 8, 11, 5, 12, 4, 2, 6), (1, 3, 5, 2, 7, 8, 12, 6, 10, 11, 4), (1, 8, 4, 7, 11, 2, 10, 5, 6, 3, 12)} $\implies \phi_8^G$ = (12/60)(χ_2((1, 2, 12, 11, 3, 7, 6, 4, 5, 8, 10)) + χ_2((1, 11, 6, 8, 2, 3, 4, 10, 12, 7, 5)) + χ_2((1, 7, 10, 3, 8, 11, 5, 12, 4, 2, 6)) + χ_2((1, 3, 5, 2, 7, 8, 12, 6, 10, 11, 4)) + χ_2((1, 8, 4, 7, 11, 2, 10, 5, 6, 3, 12))) = (12/60)(1 + 1 + 1 + 1 + 1) = 1.

The values of character of ϕ^G are summarized below:

Class Representative	Id(G)	$(xy^{-1})^3$	y	yx^2	$(yx^2)^2$	yx^{-1}	xy	$(yx)^2$
size	1	55	110	132	132	110	60	60
ϕ^G	12	0	0	$-z_5^3 - z_5^2 - 1$	$-z_5^3 - z_5^2 - 1$	0	1	1

Table 2.4: The Induced Character of $L_2(11)$.

Note that χ is the induced character ϕ^G. Since we are able to induce character χ_2 of H to obtain character χ of G, we can now construct a faithful monomial representation of G.

2.3 A Monomial Representation of $L_2(11)$.

Given the following ten transversals of H in G:

t_1 = Id(G),

$t_2 = (3, 7, 9, 4, 5)(6, 8, 12, 10, 11)$,

$t_3 = (1, 8, 2)(3, 4, 7)(5, 12, 11)(6, 9, 10)$,

$t_4 = (1, 7, 12, 4, 8)(3, 11, 10, 9, 5)$,

$t_5 = (3, 5, 4, 9, 7)(6, 11, 10, 12, 8)$,

$t_6 = (1, 4)(2, 8)(3, 12)(5, 7)(6, 10)(9, 11)$,

$t_7 = (1, 10, 12, 8, 2)(5, 11, 7, 9, 6)$,

$t_8 = (3, 4, 7, 5, 9)(6, 10, 8, 11, 12)$,

$t_9 = (1, 5, 9, 12, 10)(3, 11, 7, 4, 6)$,

$t_{10} = (1, 11, 9, 8, 2)(3, 7, 4, 5, 6)$,

$t_{11} = (1, 5, 9, 2, 8)(4, 12, 11, 10, 6)$,

$t_{12} = (1,\ 12,\ 5,\ 10,\ 9)(3,\ 4,\ 11,\ 6,\ 7)$.

Recall the character χ_2 of H = ϕ.

Let A(x) =

$$\begin{pmatrix}
\phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) & \phi(t_1xt_3^{-1}) & \dots & \phi(t_1xt_{10}^{-1}) & \phi(t_1xt_{11}^{-1}) & \phi(t_1xt_{12}^{-1}) \\
\phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}) & \phi(t_2xt_3^{-1}) & \dots & \phi(t_2xt_{10}^{-1}) & \phi(t_2xt_{11}^{-1}) & \phi(t_2xt_{12}^{-1}) \\
\phi(t_3xt_1^{-1}) & \phi(t_3xt_2^{-1}) & \phi(t_3xt_3^{-1}) & \dots & \phi(t_3xt_{10}^{-1}) & \phi(t_3xt_{11}^{-1}) & \phi(t_3xt_{12}^{-1}) \\
\phi(t_4xt_1^{-1}) & \phi(t_4xt_2^{-1}) & \phi(t_4xt_3^{-1}) & \dots & \phi(t_4xt_{10}^{-1}) & \phi(t_4xt_{11}^{-1}) & \phi(t_4xt_{12}^{-1}) \\
\phi(t_5xt_1^{-1}) & \phi(t_5xt_2^{-1}) & \phi(t_5xt_3^{-1}) & \dots & \phi(t_5xt_{10}^{-1}) & \phi(t_5xt_{11}^{-1}) & \phi(t_5xt_{12}^{-1}) \\
\phi(t_6xt_1^{-1}) & \phi(t_5xt_2^{-1}) & \phi(t_5xt_3^{-1}) & \dots & \phi(t_5xt_{10}^{-1}) & \phi(t_5xt_{11}^{-1}) & \phi(t_5xt_{12}^{-1}) \\
\phi(t_7xt_1^{-1}) & \phi(t_5xt_2^{-1}) & \phi(t_5xt_3^{-1}) & \dots & \phi(t_5xt_{10}^{-1}) & \phi(t_5xt_{11}^{-1}) & \phi(t_5xt_{12}^{-1}) \\
\phi(t_8xt_1^{-1}) & \phi(t_5xt_2^{-1}) & \phi(t_5xt_3^{-1}) & \dots & \phi(t_5xt_{10}^{-1}) & \phi(t_5xt_{11}^{-1}) & \phi(t_5xt_{12}^{-1}) \\
\phi(t_9xt_1^{-1}) & \phi(t_9xt_2^{-1}) & \phi(t_9xt_3^{-1}) & \dots & \phi(t_9xt_{10}^{-1}) & \phi(t_9xt_{11^{-1}}) & \phi(t_9xt_{12}^{-1}) \\
\phi(t_{10}xt_1^{-1}) & \phi(t_{10}xt_2^{-1}) & \phi(t_{10}xt_3^{-1}) & \dots & \phi(t_{10}xt_{10}^{-1}) & \phi(t_{10}xt_{11}^{-1}) & \phi(t_{10}xt_{12}^{-1}) \\
\phi(t_{11}xt_1^{-1}) & \phi(t_{11}xt_2^{-1}) & \phi(t_{11}xt_3^{-1}) & \dots & \phi(t_{11}xt_{10}^{-1}) & \phi(t_{11}xt_{11}^{-1}) & \phi(t_{11}xt_{12}^{-1}) \\
\phi(t_{12}xt_1^{-1}) & \phi(t_{12}xt_2^{-1}) & \phi(t_{12}xt_3^{-1}) & \dots & \phi(t_{12}xt_{10}^{-1}) & \phi(t_{12}xt_{11}^{-1}) & \phi(t_{12}xt_{12}^{-1})
\end{pmatrix}$$

$$\Longrightarrow \text{A(x)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_5^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_5^2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_5^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_5^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_5^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_5^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_5^2 \end{pmatrix}$$

We need to find a prime p such that $\mathbb{Z}_\text{p}$ has elements of order 5 and 5|(p-1). This means $p-1 = 5k$. Now, p = $5k+1$. We find that k = 2 gives the smallest such p and p = 11. Since 5 is of order 5 in $\mathbb{Z}_{11}$, we replace z_5 with 5 and change the entries in A(x) so that they belong to $\mathbb{Z}_{11}$.

$$A(x) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\mathrm{B(y)} = \begin{pmatrix} \phi(t_1yt_1^{-1}) & \phi(t_1yt_2^{-1}) & \dots & \phi(t_1yt_{11}^{-1}) & \phi(t_1yt_{12}^{-1}) \\ \phi(t_2yt_1^{-1}) & \phi(t_2yt_2^{-1}) & \dots & \phi(t_2yt_{11}^{-1}) & \phi(t_2yt_{12}^{-1}) \\ \phi(t_3yt_1^{-1}) & \phi(t_3yt_2^{-1}) & \dots & \phi(t_3yt_{11}^{-1}) & \phi(t_3yt_{12}^{-1}) \\ \phi(t_4yt_1^{-1}) & \phi(t_4yt_2^{-1}) & \dots & \phi(t_4yt_{11}^{-1}) & \phi(t_4yt_{12}^{-1}) \\ \phi(t_5yt_1^{-1}) & \phi(t_5yt_2^{-1}) & \dots & \phi(t_5yt_{11}^{-1}) & \phi(t_5yt_{12}^{-1}) \\ \phi(t_6yt_1^{-1}) & \phi(t_5yt_2^{-1}) & \dots & \phi(t_5yt_{11}^{-1}) & \phi(t_5yt_{12}^{-1}) \\ \phi(t_7yt_1^{-1}) & \phi(t_5yt_2^{-1}) & \dots & \phi(t_5yt_{11}^{-1}) & \phi(t_5yt_{12}^{-1}) \\ \phi(t_8yt_1^{-1}) & \phi(t_5yt_2^{-1}) & \dots & \phi(t_5yt_{11}^{-1}) & \phi(t_5yt_{12}^{-1}) \\ \phi(t_9yt_1^{-1}) & \phi(t_9yt_2^{-1}) & \dots & \phi(t_9yt_{11^{-1}}) & \phi(t_9yt_{12}^{-1}) \\ \phi(t_{10}yt_1^{-1}) & \phi(t_{10}yt_2^{-1}) & \dots & \phi(t_{10}yt_{11}^{-1}) & \phi(t_{10}yt_{12}^{-1}) \\ \phi(t_{11}yt_1^{-1}) & \phi(t_{11}yt_2^{-1}) & \dots & \phi(t_{11}yt_{11}^{-1}) & \phi(t_{11}yt_{12}^{-1}) \\ \phi(t_{12}yt_1^{-1}) & \phi(t_{12}yt_2^{-1}) & \dots & \phi(t_{12}yt_{11}^{-1}) & \phi(t_{12}yt_{12}^{-1}) \end{pmatrix}$$

$$
B(y) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & z_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & z_5^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & z_5^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
z_5^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & z_5^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & z_5^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

We must replace z_5 with 5 and change the entries of B(y) so that they belong to $\mathbb{Z}_{11}$.

$$B(x) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Since our field of entries is $\mathbb{Z}_{11}$ and the dimension of the entries is 12, the monomial progenitor is $11^{*12} :_m L_2(11)$.

Since $|t_i| = 11$, each t_i has 10 distinct powers, we label the 12 t_is and their powers as follows:

label	1	2	3	4	5	6	7	8	9	10	11	12
element	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}
label	13	14	15	16	17	18	19	20	21	22	23	24
element	${t_1}^2$	${t_2}^2$	${t_3}^2$	${t_4}^2$	${t_5}^2$	${t_6}^2$	${t_7}^2$	${t_8}^2$	${t_9}^2$	${t_{10}}^2$	${t_{11}}^2$	${t_{12}}^2$
label	25	26	27	28	29	30	31	32	33	34	35	36
element	${t_1}^3$	${t_2}^3$	${t_3}^3$	${t_4}^3$	${t_5}^3$	${t_6}^3$	${t_7}^3$	${t_8}^3$	${t_9}^3$	${t_{10}}^3$	${t_{11}}^3$	${t_{12}}^3$
label	37	38	39	40	41	42	43	44	45	46	47	48
element	${t_1}^4$	${t_2}^4$	${t_3}^4$	${t_4}^4$	${t_5}^4$	${t_6}^4$	${t_7}^4$	${t_8}^4$	${t_9}^4$	${t_{10}}^4$	${t_{11}}^4$	${t_{12}}^4$
label	49	50	51	52	53	54	55	56	57	58	59	60
element	${t_1}^5$	${t_2}^5$	${t_3}^5$	${t_4}^5$	${t_5}^5$	${t_6}^5$	${t_7}^5$	${t_8}^5$	${t_9}^5$	${t_{10}}^5$	${t_{11}}^5$	${t_{12}}^5$
label	61	62	63	64	65	66	67	68	69	70	71	72
element	${t_1}^6$	${t_2}^6$	${t_3}^6$	${t_4}^6$	${t_5}^6$	${t_6}^6$	${t_7}^6$	${t_8}^6$	${t_9}^6$	${t_{10}}^6$	${t_{11}}^6$	${t_{12}}^6$
label	73	74	75	76	77	78	79	80	81	81	83	84
element	${t_1}^7$	${t_2}^7$	${t_3}^7$	${t_4}^7$	${t_5}^7$	${t_6}^7$	${t_7}^7$	${t_8}^7$	${t_9}^7$	${t_{10}}^7$	${t_{11}}^7$	${t_{12}}^7$
label	85	86	87	88	89	90	91	92	93	94	95	96
element	${t_1}^8$	${t_2}^8$	${t_3}^8$	${t_4}^8$	${t_5}^8$	${t_6}^8$	${t_7}^8$	${t_8}^8$	${t_9}^8$	${t_{10}}^8$	${t_{11}}^8$	${t_{12}}^8$
label	97	98	99	100	101	102	103	104	105	106	107	108
element	${t_1}^9$	${t_2}^9$	${t_3}^9$	${t_4}^9$	${t_5}^9$	${t_6}^9$	${t_7}^9$	${t_8}^9$	${t_9}^9$	${t_{10}}^9$	${t_{11}}^9$	${t_{12}}^9$
label	109	110	111	112	113	114	115	116	117	118	119	120
element	${t_1}^{10}$	${t_2}^{10}$	${t_3}^{10}$	${t_4}^{10}$	${t_5}^{10}$	${t_6}^{10}$	${t_7}^{10}$	${t_8}^{10}$	${t_9}^{10}$	${t_{10}}^{10}$	${t_{11}}^{10}$	${t_{12}}^{10}$

Table 2.5: Labeling of 120 t_is on Which $L_2(11)$ Acts.

The matrices of $L_2(11)$ act as monomial automorphisms on the 120 letters (t_is) given above. The monomial action is given by: $a_{i,j}$ = b $\iff$ $t_i \mapsto t_j^b$. We now want to compute the permutation representation of the monomial representation of $L_2(11)$ to give a symmetric presentation of the monomial progenitor $11^{*12} :_m L_2(11)$.

From the 1st row of matrix A(x) we get $a_{1,2} = 1$, which leads to all of the following:

$t_1 = t_2^1 \implies 1 \to 2$

$t_1^2 = (t_2^1)^2 = t_2^2 \implies 13 \to 14$

$t_1^3 = (t_2^1)^3 = t_2^3 \implies 25 \to 26$

$t_1^4 = (t_2^1)^4 = t_2^4 \implies 37 \to 38$

$t_1^5 = (t_2^1)^5 = t_2^5 \implies 49 \to 50$

$t_1^6 = (t_2^1)^6 = t_2^6 \implies 61 \to 62$

$t_1^7 = (t_2^1)^7 = t_2^7 \implies 73 \to 74$

$t_1^8 = (t_2^1)^8 = t_2^8 \implies 85 \to 86$

$t_1^9 = (t_2^1)^9 = t_2^9 \implies 97 \to 98$

$t_1^{10} = (t_2^1)^{10} = t_2^{10} \implies 109 \to 110$

$a_{2,4} = 4 \implies$

$t_2 = t_4^4 \implies 2 \to 40$

$t_2^2 = (t_4^4)^2 = t_4^8 \implies 14 \to 88$

$t_2^3 = (t_4^4)^3 = t_4^{12} = t_4^1 \implies 26 \to 4$

$t_2^4 = (t_4^4)^4 = t_4^{16} = t_4^5 \implies 38 \to 52$

$t_2^5 = (t_4^4)^5 = t_4^{20} = t_4^9 \implies 50 \to 100$

$t_2^6 = (t_4^4)^6 = t_4^{24} = t_4^2 \implies 62 \to 16$

$t_2^7 = (t_4^4)^7 = t_4^{28} = t_4^6 \implies 74 \to 64$

$t_2^8 = (t_4^4)^8 = t_4^{32} = t_4^{10} \implies 86 \to 112$

$t_2^9 = (t_4^4)^9 = t_4^{36} = t_4^3 \implies 98 \to 28$

$t_2^{10} = (t_4^4)^{10} = t_4^{40} = t_4^7 \implies 110 \to 76$

$a_{3,6} = 4 \implies$

$t_3 = t_6^4 \implies 3 \to 42$

$t_3^2 = (t_6^4)^2 = t_6^8 \implies 15 \to 90$

$t_3^3 = (t_6^4)^3 = t_6^{12} = t_6^1 \implies 27 \to 6$

$t_3^4 = (t_6^4)^4 = t_6^{16} = t_6^5 \implies 39 \to 54$

$t_3^5 = (t_6^4)^5 = t_6^{20} = t_6^9 \implies 51 \to 102$

$t_3^6 = (t_6^4)^6 = t_6^{24} = t_6^2 \implies 63 \to 18$

$t_3^7 = (t_6^4)^7 = t_6^{28} = t_6^6 \implies 7566$

$t_3^8 = (t_6^4)^8 = t_6^{32} = t_6^{10} \implies 87 \to 114$

$t_3^9 = (t_6^4)^9 = t_6^{36} = t_6^3 \implies 99 \to 30$

$t_3^{10} = (t_6^4)^{10} = t_6^{40} = t_6^7 \implies 111 \to 78$

$a_{4,8} = 3 \implies$

$t_4 = t_8^3 \implies 4 \to 32$

$t_4^2 = (t_8^3)^2 = t_8^6 \implies 16 \to 68$

$t_4^3 = (t_8^3)^3 = t_8^9 = t_8^9 \implies 28 \to 104$

$t_4^4 = (t_8^3)^4 = t_8^{12} = t_8^1 \implies 40 \to 8$

$t_4^5 = (t_8^3)^5 = t_8^{15} = t_8^4 \implies 52 \to 44$

$t_4^6 = (t_8^3)^6 = t_8^{18} = t_8^7 \implies 64 \to 80$

$t_4^7 = (t_8^3)^7 = t_8^{21} = t_8^{10} \implies 76 \to 116$

$t_4^8 = (t_8^3)^8 = t_8^{24} = t_8^2 \implies 88 \to 20$

$t_4^9 = (t_8^3)^9 = t_8^{27} = t_8^5 \implies 100 \to 56$

$t_4^{10} = (t_8^3)^{10} = t_8^{30} = t_8^8 \implies 112 \to 92$

$a_{5,1} = 1 \implies$

$t_5 = t_1^1 \implies 5 \implies 1$

$t_5^2 = (t_1^1)^2 = t_1^2 \implies 17 \to 13$

$t_5^3 = (t_1^1)^3 = t_1^3 \implies 29 \to 25$

$t_5^4 = (t_1^1)^4 = t_1^4 \implies 41 \to 37$

$t_5^5 = (t_1^1)^5 = t_1^5 \implies 53 \to 49$

$t_5^6 = (t_1^1)^6 = t_1^6 \implies 65 \to 61$

$t_5^7 = (t_1^1)^7 = t_1^7 \implies 77 \to 73$

$t_5^8 = (t_1^1)^8 = t_1^8 \implies 89 \to 85$

$t_5^9 = (t_1^1)^9 = t_1^9 \implies 101 \to 97$

$t_5^{10} = (t_1^1)^{10} = t_1^{10} \implies 113 \to 109$

$a_{6,7} = 3 \implies$

$t_6 = t_7^3 \implies 6 \to 31$

$t_6^2 = (t_7^3)^2 = t_7^6 \implies 18 \to 67$

$t_6^3 = (t_7^3)^3 = t_7^9 \implies 30 \to 103$

$t_6^4 = (t_7^3)^4 = t_7^{12} = t_7^1 \implies 42 \to 7$

$t_6^5 = (t_7^3)^5 = t_7^{15} = t_7^4 \implies 54 \to 43$

$t_6^6 = (t_7^3)^6 = t_7^{18} = t_7^7 \implies 66 \to 79$

$t_6^7 = (t_7^3)^7 = t_7^{21} = t_7^{10} \implies 78 \to 115$

$t_6^8 = (t_7^3)^8 = t_7^{24} = t_7^2 \implies 90 \to 19$

$t_6^9 = (t_7^3)^9 = t_7^{27} = t_7^5 \implies 102 \to 55$

$t_6^{10} = (t_7^3)^{10} = t_7^{30} = t_7^8 \implies 114 \to 91$

$a_{7,10} = 1 \implies$

$t_7 = t_{10}^1 \implies 7 \to 10$

$t_7^2 = (t_{10}^1)^2 = t_{10}^2 \implies 19 \to 22$

$t_7^3 = (t_{10}^1)^3 = t_{10}^3 \implies 31 \to 34$

$t_7^4 = (t_{10}^1)^4 = t_{10}^4 \implies 43 \to 46$

$t_7^5 = (t_{10}^1)^5 = t_{10}^5 \implies 55 \to 58$

$t_7^6 = (t_{10}^1)^6 = t_{10}^6 \implies 67 \to 70$

$t_7^7 = (t_{10}^1)^7 = t_{10}^7 \implies 79 \to 82$

$t_7^8 = (t_{10}^1)^8 = t_{10}^8 \implies 91 \to 94$

$t_7^9 = (t_{10}^1)^9 = t_{10}^9 \implies 103 \to 106$

$t_7^{10} = (t_{10}^1)^{10} = t_{10}^{10} \implies 115 \to 118$

$a_{8,5} = 1 \implies$

$t_8 = t_5^1 \implies 8 \to 5$

$t_8^2 = (t_5^1)^2 = t_5^2 \implies 20 \to 17$

$t_8^3 = (t_5^1)^3 = t_5^3 \implies 32 \to 29$

$t_8^4 = (t_5^1)^4 = t_5^4 \implies 44 \to 41$

$t_8^5 = (t_5^1)^5 = t_5^5 \implies 56 \to 53$

$t_8^6 = (t_5^1)^6 = t_5^6 \implies 68 \to 65$

$t_8^7 = (t_5^1)^7 = t_5^7 \implies 80 \to 77$

$t_8^8 = (t_5^1)^8 = t_5^8 \implies 92 \to 89$

$t_8^9 = (t_5^1)^9 = t_5^9 \implies 104 \to 101$

$t_8^{10} = (t_5^1)^{10} = t_5^{10} \implies 116 \to 113$

$a_{9,3} = 3 \implies$

$t_9 = t_3^3 \implies 9 \to 27$

$t_9^2 = (t_3^3)^2 = t_3^6 \implies 21 \to 63$

$t_9^3 = (t_3^3)^3 = t_3^9 \implies 33 \to 99$

$t_9^4 = (t_3^3)^4 = t_3^{12} = t_3^1 \implies 45 \to 3$

$t_9^5 = (t_3^3)^5 = t_3^{15} = t_3^4 \implies 57 \to 39$

$t_9^6 = (t_3^3)^6 = t_3^{18} = t_3^7 \implies 69 \to 75$

$t_9^7 = (t_3^3)^7 = t_3^{21} = t_3^{10} \implies 81 \to 111$

$t_9^8 = (t_3^3)^8 = t_3^{24} = t_3^2 \implies 93 \to 15$

$t_9^9 = (t_3^3)^9 = t_3^{27} = t_3^5 \implies 105 \to 51$

$t_9^{10} = (t_3^3)^{10} = t_3^{30} = t_3^8 \implies 117 \to 87$

$a_{10,9} = 4 \implies$

$t_{10} = t_9^4 \implies 10 \to 45$

$t_{10}^2 = (t_9^4)^2 = t_9^8 \implies 22 \to 93$

$t_{10}^3 = (t_9^4)^3 = t_9^{12} = t_9^1 \implies 34 \to 9$

$t_{10}^4 = (t_9^4)^4 = t_9^{16} = t_9^5 \implies 46 \to 57$

$t_{10}^5 = (t_9^4)^5 = t_9^{20} = t_9^9 \implies 58 \to 105$

$t_{10}^6 = (t_9^4)^6 = t_9^{24} = t_9^2 \implies 70 \to 21$

$t_{10}^7 = (t_9^4)^7 = t_9^{28} = t_9^6 \implies 82 \to 69$

$t_{10}^8 = (t_9^4)^8 = t_9^{32} = t_9^{10} \implies 94 \to 117$

$t_{10}^9 = (t_9^4)^9 = t_9^{36} = t_9^3 \implies 106 \to 33$

$t_{10}^{10} = (t_9^4)^{10} = t_9^{40} = t_9^7 \implies 118 \to 81$

$a_{11,11} = 4 \implies$

$t_{11} = t_{11}^4 \implies 11 \to 47$

$t_{11}^2 = (t_{11}^4)^2 = t_{11}^8 \implies 23 \to 95$

$t_{11}^3 = (t_{11}^4)^3 = t_{11}^{12} = t_{11}^1 \implies 35 \to 11$

$t_{11}^4 = (t_{11}^4)^4 = t_{11}^{16} = t_{11}^5 \implies 47 \to 59$

$t_{11}^5 = (t_{11}^4)^5 = t_{11}^{20} = t_{11}^9 \implies 59 \to 107$

$t_{11}^6 = (t_{11}^4)^6 = t_{11}^{24} = t_{11}^2 \implies 71 \to 23$

$t_{11}^7 = (t_{11}^4)^7 = t_{11}^{28} = t_{11}^6 \implies 83 \to 71$

$t_{11}^8 = (t_{11}^4)^8 = t_{11}^{32} = t_{11}^{10} \implies 95 \to 119$

$t_{11}^9 = (t_{11}^4)^9 = t_{11}^{36} = t_{11}^3 \implies 107 \to 35$

$t_{11}^{10} = (t_{11}^4)^{10} = t_{11}^{40} = t_{11}^7 \implies 119 \to 83$

$a_{12,12} = 3 \implies$

$t_{12} = t_{12}^3 \implies 12 \to 36$

$t_{12}^2 = (t_{12}^3)^2 = t_{12}^6 \implies 24 \to 72$

$t_{12}^3 = (t_{12}^3)^3 = t_{12}^9 = t_{12}^9 \implies 36 \to 108$

$t_{12}^4 = (t_{12}^3)^4 = t_{12}^{12} = t_{12}^1 \implies 48 \to 12$

$t_{12}^5 = (t_{12}^3)^5 = t_{12}^{15} = t_{12}^4 \implies 60 \to 48$

$t_{12}^6 = (t_{12}^3)^6 = t_{12}^{18} = t_{12}^7 \implies 72 \to 84$

$t_{12}^7 = (t_{12}^3)^7 = t_{12}^{21} = t_{12}^{10} \implies 84 \to 120$

$t_{12}^8 = (t_{12}^3)^8 = t_{12}^{24} = t_{12}^2 \implies 96 \to 24$

$t_{12}^9 = (t_{12}^3)^9 = t_{12}^{27} = t_{12}^5 \implies 108 \to 60$

$t_{12}^{10} = (t_{12}^3)^{10} = t_{12}^{30} = t_{12}^8 \implies 120 \to 96$

From matrix B(y) we obtain all of the following:

$b_{1,3} = 1 \implies$

$t_1 = t_3^1 \implies 1 \to 3$

$t_1^2 = (t_3^1)^2 = t_3^2 \implies 13 \to 15$

$t_1^3 = (t_3^1)^3 = t_3^3 \implies 25 \to 27$

$t_1^4 = (t_3^1)^4 = t_3^4 \implies 37 \to 39$

$t_1^5 = (t_3^1)^5 = t_3^5 \implies 49 \to 51$

$t_1^6 = (t_3^1)^6 = t_3^6 \implies 61 \to 63$

$t_1^7 = (t_3^1)^7 = t_3^7 \implies 73 \to 75$

$t_1^8 = (t_3^1)^8 = t_3^8 \implies 85 \to 87$

$t_1^9 = (t_3^1)^9 = t_3^9 \implies 97 \to 99$

$t_1^{10} = (t_3^1)^{10} = t_3^{10} \implies 109 \to 111$

$b_{2,5} = 5 \implies$

$t_2 = t_5^5 \implies 2 \to 53 \implies$

$t_2^2 = (t_5^5)^2 = t_5^{10} \implies 14 \to 113$

$t_2^3 = (t_5^5)^3 = t_5^{15} = t_5^4 \implies 26 \to 41$

$t_2^4 = (t_5^5)^4 = t_5^{20} = t_5^9 \implies 38 \to 101$

$t_2^5 = (t_5^5)^5 = t_5^{25} = t_5^3 \implies 50 \to 29$

$t_2^6 = (t_5^5)^6 = t_5^{30} = t_5^8 \implies 62 \to 89$

$t_2^7 = (t_5^5)^7 = t_5^{35} = t_5^2 \implies 74 \to 17$

$t_2^8 = (t_5^5)^8 = t_5^{40} = t_5^7 \implies 86 \to 77$

$t_2^9 = (t_5^5)^9 = t_5^{45} = t_5^1 \implies 98 \to 5$

$t_2^{10} = (t_5^5)^{10} = t_5^{50} = t_5^6 \implies 110 \to 65$

$b_{3,7} = 3 \implies$

$t_3 = t_7^3 \implies 3 \to 31$

$t_3^2 = (t_7^3)^2 = t_7^6 \implies 15 \to 67$

$t_3^3 = (t_7^3)^3 = t_7^9 \implies 27 \to 103$

$t_3^4 = (t_7^3)^4 = t_7^{12} = t_7^1 \implies 39 \to 7$

$t_3^5 = (t_7^3)^5 = t_7^{15} = t_7^4 \implies 51 \to 43$

$t_3^6 = (t_7^3)^6 = t_7^{18} = t_7^7 \implies 63 \to 79$

$t_3^7 = (t_7^3)^7 = t_7^{21} = t_7^{10} \implies 75 \to 115$

$t_3^8 = (t_7^3)^8 = t_7^{24} = t_7^2 \implies 87 \to 19$

$t_3^9 = (t_7^3)^9 = t_7^{27} = t_7^5 \implies 99 \to 55$

$t_3^{10} = (t_7^3)^{10} = t_7^{30} = t_7^8 \implies 111 \to 91$

$b_{4,9} = 3 \implies$

$t_4 = t_9^3 \implies 4 \to 33$

$t_4^2 = (t_9^3)^2 = t_9^6 \implies 16 \to 69$

$t_4^3 = (t_9^3)^3 = t_9^9 \implies 28 \to 105$

$t_4^4 = (t_9^3)^4 = t_9^{12} = t_9^1 \implies 40 \to 9$

$t_4^5 = (t_9^3)^5 = t_9^{15} = t_9^4 \implies 52 \to 45$

$t_4^6 = (t_9^3)^6 = t_9^{18} = t_9^7 \implies 64 \to 81$

$t_4^7 = (t_9^3)^7 = t_9^{21} = t_9^{10} \implies 76 \to 117$

$t_4^8 = (t_9^3)^8 = t_9^{24} = t_9^2 \implies 88 \to 21$

$t_4^9 = (t_9^3)^9 = t_9^{27} = t_9^5 \implies 100 \to 57$

$t_4^{10} = (t_9^3)^{10} = t_9^{30} = t_9^8 \implies 112 \to 93$

$b_{5,8} = 1 \implies$

$t_5 = t_8^1 \implies 5 \to 8$

$t_5^2 = (t_8^1)^2 = t_8^2 \implies 17 \to 20$

$t_5^3 = (t_8^1)^3 = t_8^3 \implies 29 \to 32$

$t_5^4 = (t_8^1)^4 = t_8^4 \implies 41 \to 44$

$t_5^5 = (t_8^1)^5 = t_8^5 \implies 53 \to 56$

$t_5^6 = (t_8^1)^6 = t_8^6 \implies 65 \to 68$

$t_5^7 = (t_8^1)^7 = t_8^7 \implies 77 \to 80$

$t_5^8 = (t_8^1)^8 = t_8^8 \implies 89 \to 92$

$t_5^9 = (t_8^1)^9 = t_8^9 \implies 101 \to 104$

$t_5^{10} = (t_8^1)^{10} = t_8^{10} \implies 113 \to 116$

$b_{6,4} = 1 \implies$

$t_6 = t_4^1 \implies 6 \to 4$

$t_6^2 = (t_4^1)^2 = t_4^2 \implies 18 \to 16$

$t_6^3 = (t_4^1)^3 = t_4^3 \implies 30 \to 28$

$t_6^4 = (t_4^1)^4 = t_4^4 \implies 42 \to 40$

$t_6^5 = (t_4^1)^5 = t_4^5 \implies 54 \to 52$

$t_6^6 = (t_4^1)^6 = t_4^6 \implies 66 \to 64$

$t_6^7 = (t_4^1)^7 = t_4^7 \implies 78 \to 76$

$t_6^8 = (t_4^1)^8 = t_4^8 \implies 90 \to 88$

$t_6^9 = (t_4^1)^9 = t_4^9 \implies 102 \to 100$

$t_6^{10} = (t_4^1)^{10} = t_4^{10} \implies 114 \to 112$

$b_{7,1} = 4 \implies$

$t_7 = t_1^4 \implies 7 \to 37$

$t_7^2 = (t_1^4)^2 = t_1^8 \implies 19 \to 85$

$t_7^3 = (t_1^4)^3 = t_1^{12} = t_1^1 \implies 31 \to 1$

$t_7^4 = (t_1^4)^4 = t_1^{16} = t_1^5 \implies 43 \to 49$

$t_7^5 = (t_1^4)^5 = t_1^{20} = t_1^9 \implies 55 \to 97$

$t_7^6 = (t_1^4)^6 = t_1^{24} = t_1^2 \implies 67 \to 13$

$t_7^7 = (t_1^4)^7 = t_1^{28} = t_1^6 \implies 79 \to 61$

$t_7^8 = (t_1^4)^8 = t_1^{32} = t_1^{10} \implies 91 \to 109$

$t_7^9 = (t_1^4)^9 = t_1^{36} = t_1^3 \implies 103 \to 25$

$t_7^{10} = (t_1^4)^{10} = t_1^{40} = t_1^7 \implies 115 \to 73$

$b_{8,2} = 9 \implies$

$t_8 = t_2^9 \implies 8 \to 98$

$t_8^2 = (t_2^9)^2 = t_2^{18} = t_2^7 \implies 20 \to 74$

$t_8^3 = (t_2^9)^3 = t_2^{27} = t_2^5 \implies 32 \to 50$

$t_8^4 = (t_2^9)^4 = t_2^{36} = t_2^3 \implies 44 \to 26$

$t_8^5 = (t_2^9)^5 = t_2^{45} = t_2^1 \implies 56 \to 2$

$t_8^6 = (t_2^9)^6 = t_2^{54} = t_2^{10} \implies 68 \to 110$

$t_8^7 = (t_2^9)^7 = t_2^{63} = t_2^8 \implies 80 \to 86$

$t_8^8 = (t_2^9)^8 = t_2^{72} = t_2^6 \implies 92 \to 62$

$t_8^9 = (t_2^9)^9 = t_2^{81} = t_2^4 \implies 104 \to 38$

$t_8^{10} = (t_2^9)^{10} = t_2^{90} = t_2^2 \implies 116 \to 14$

$b_{9,6} = 4 \implies$

$t_9 = t_6^4 \implies 9 \to 42$

$t_9^2 = (t_6^4)^2 = t_6^8 \implies 21 \to 90$

$t_9^3 = (t_6^4)^3 = t_6^{12} = t_6^1 \implies 33 \to 6$

$t_9^4 = (t_6^4)^4 = t_6^{16} = t_6^5 \implies 45 \to 54$

$t_9^5 = (t_6^4)^5 = t_6^{20} = t_6^9 \implies 57 \to 102$

$t_9^6 = (t_6^4)^6 = t_6^{24} = t_6^2 \implies 69 \to 18$

$t_9^7 = (t_6^4)^7 = t_6^{28} = t_6^6 \implies 81 \to 66$

$t_9^8 = (t_6^4)^8 = t_6^{32} = t_6^{10} \implies 93 \to 114$

$t_9^9 = (t_6^4)^9 = t_6^{36} = t_6^3 \implies 105 \to 30$

$t_9^{10} = (t_6^4)^{10} = t_6^{40} = t_6^7 \implies 117 \to 78$

$b_{10,11} = 1 \implies$

$t_{10} = t_{11}^1 \implies 10 \to 11$

$t_{10}^2 = (t_{11}^1)^2 = t_{11}^2 \implies 22 \to 23$

$t_{10}^3 = (t_{11}^1)^3 = t_{11}^3 \implies 34 \to 35$

$t_{10}^4 = (t_{11}^1)^4 = t_{11}^4 \implies 46 \to 47$

$t_{10}^5 = (t_{11}^1)^5 = t_{11}^5 \implies 58 \to 59$

$t_{10}^6 = (t_{11}^1)^6 = t_{11}^6 \implies 70 \to 71$

$t_{10}^7 = (t_{11}^1)^7 = t_{11}^7 \implies 82 \to 83$

$t_{10}^8 = (t_{11}^1)^8 = t_{11}^8 \implies 94 \to 95$

$t_{10}^9 = (t_{11}^1)^9 = t_{11}^9 \implies 106 \to 107$

$t_{10}^{10} = (t_{11}^1)^{10} = t_{11}^{10} \implies 118 \to 119$

$b_{11,12} = 1 \implies$

$t_{11} = t_{12}^1 \implies 11 \to 12$

$t_{11}^2 = (t_{12}^1)^2 = t_{12}^2 \implies 23 \to 24$

$t_{11}^3 = (t_{12}^1)^3 = t_{12}^3 \implies 35 \to 36$

$t_{11}^4 = (t_{12}^1)^4 = t_{12}^4 \implies 47 \to 48$

$t_{11}^5 = (t_{12}^1)^5 = t_{12}^5 \implies 59 \to 60$

$t_{11}^6 = (t_{12}^1)^6 = t_{12}^6 \implies 71 \to 72$

$t_{11}^7 = (t_{12}^1)^7 = t_{12}^7 \implies 83 \to 84$

$t_{11}^8 = (t_{12}^1)^8 = t_{12}^8 \implies 95 \to 96$

$t_{11}^9 = (t_{12}^1)^9 = t_{12}^9 \implies 107 \to 108$

$t_{11}^{10} = (t_{12}^1)^{10} = t_{12}^{10} \implies 119 \to 120$

$b_{12,10} = 1 \implies$

$t_{12} = t_{10}^1 \implies 12 \to 10$

$t_{12}^2 = (t_{10}^1)^2 = t_{10}^2 \implies 24 \to 22$

$t_{12}^3 = (t_{10}^1)^3 = t_{10}^3 \implies 36 \to 34$

$t_{12}^4 = (t_{10}^1)^4 = t_{10}^4 \implies 48 \to 46$

$t_{12}^5 = (t_{10}^1)^5 = t_{10}^5 \implies 60 \to 58$

$t_{12}^6 = (t_{10}^1)^6 = t_{10}^6 \implies 72 \to 70$

$t_{12}^7 = (t_{10}^1)^7 = t_{10}^7 \implies 84 \to 82$

$t_{12}^8 = (t_{10}^1)^8 = t_{10}^8 \implies 96 \to 94$

$t_{12}^9 = (t_{10}^1)^9 = t_{10}^9 \implies 108 \to 106$

$t_{12}^{10} = (t_{10}^1)^{10} = t_{10}^{10} \implies 120 \to 118$

Using the above mapping we get A(x) $\sim$ xx = (1, 3, 31)(2, 53, 56)(4, 33, 6)(5, 8, 98)(7, 37, 39)(9, 42, 40)(10, 11, 12)(13, 15, 67)(14, 113, 116)(16, 69, 18)(17, 20, 74)(19, 85, 87)(21, 90, 88)(22, 23, 24)(25, 27, 103)(26, 41, 44)(28, 105, 30)(29, 32, 50)(34, 35, 36)(38, 101, 104)(43, 49, 51)(45, 54, 52)(46, 47, 48)(55, 97, 99)(57, 102, 100)(58, 59, 60)(61, 63, 79)(62, 89, 92)(64, 81, 66)(65, 68, 110)(70, 71, 72)(73, 75, 115)(76, 117, 78)(77, 80, 86)(82, 83, 84)(91, 109, 111)(93, 114, 112)(94, 95, 96)(106, 107, 108)(118, 119, 120).

We apply the same method to matrix B(y) and obtain yy =(1, 27, 31)(2, 101, 56)(3, 7, 37)(4, 9, 54)(5, 32, 50)(6, 100, 105)(8, 98, 41)(10, 11, 12)(13, 63, 67)(14, 77, 116)(15, 19, 85)(16, 21, 114)(17, 68, 110)(18, 76, 81)(20, 74, 89)(22, 23, 24)(25, 99, 103)(26, 53, 44)(28, 33, 42)(29, 104, 38)(30, 52, 57)(34, 35, 36)(39, 43, 49)(40, 45, 102)(46, 47, 48)(51, 55, 97)(58, 59, 60)(61, 75, 79)(62, 113, 92)(64, 69, 90)(65, 80, 86)(66, 112, 117)(70, 71, 72)(73, 111, 115)(78, 88, 93)(82, 83, 84)(87, 91, 109)(94, 95, 96)(106, 107, 108)(118, 119, 120).

Now, $N = \langle xx, yy \rangle \cong L_2(11)$. Thus, N is a monomial permutation presentation on $L_2(11)$. It is now possible to create a monomial progenitor $G \cong 11^{*12} :_m L_2(11)$. There is a MAGMA example of this progenitor in the Appendix E.

Chapter 3

Progenitors

3.1 Progenitor 3^{*56}:(2^3:(3:7))

The purpose of this section is to show some examples of progenitor images that we found using Magma.

Notation: If x is a word in a presentation of a group, and the corresponding permutation is xx, then we express this by writing x $\sim$ xx.

x $\sim$ (1, 28, 30)(2, 40, 39)(3, 55, 54)(4, 26, 29)(5, 25, 10)(6, 31, 9)(7, 19, 18)(8, 16, 21)(11, 17, 24)(12, 41, 50)(13, 52, 42)(14, 51, 34)(15, 56, 33)(20, 53, 36)(22, 32, 38)(23, 49, 46)(35, 37, 44)(45, 48, 47).

y $\sim$ (1, 54, 45)(2, 25, 56)(3, 20, 22)(4, 34, 49)(5, 38, 27)(6, 35, 48)(7, 44, 10)(8,51, 19)(9, 39, 55)(11, 31, 52)(12, 43, 17)(14, 15, 41)(16, 24, 53)(18, 42,46)(21, 36, 47)(23, 40, 33)(26, 32, 30)(28, 50, 29).

N = $\langle$ x, y$\rangle$. Magma gives N $\cong$ (2^3:(3:7).

G = $\langle$ x^3, y^3, $yx^{-1}y^{\hat{}-1}x^{-1}y^{-1}xyxy^{-1}x$,

t^3,(t, $xy^{-1}x^{-1}y^{-1}x$),

$((x^{-1}t^{y\ x})^a$, $(xt^{yx})^b$, $((xy^{-1})^3t)^c$, $(xy^{-1}t)^d$, $(y^{-1}x^{-1}t)^e$, $((xy)^3t)^f$, $(xyt)^g\rangle$.

Table of images:

a	b	c	d	e	f	g	Order of G	Shape of G
0	0	0	0	0	2	6	367416	$3^7{\cdot}N$
0	0	0	0	3	0	3	4032	$2^6{\cdot}N$
0	0	0	2	0	0	0	1512	$3{\cdot}L_2(8)$
3	3	0	9	0	4	8	304819200	A_{10}

Table 3.1: Images of Progenitor 3^{*56}:(2^3:(3:7)).

Please Appendix C for the corresponding MAGMA code.

3.2 Progenitor 3^{*14}:($2^3{\cdot}(3{\cdot}7)$)

The purpose of this section is to show some examples of progenitor images that we found using Magma.

x ∼ (1, 2, 4)(3, 7, 6) (8, 9, 11)(10, 14, 13);

y ∼ (1, 3, 5)(2, 6, 4)(8,10,12)(9,13,11);

$N = \langle x, y\rangle \implies N \cong (2^3{:}(3{:}7)$.

$G = \langle\ x^3, y^3$, $yx^{-1}y^{\hat{}-1}x^{-1}y^{-1}xyx\ y^{-1}x$,$(xy)^3$,

t^3, (t, $xy^{-1}x^{-1}y^{-1}x$),

($x^{-1}t^{y\ x})^a$, $(xt^{yx})^b$, $((xy^{-1})^3t)^c$, $(xy^{-1}t)^d$, $(y^{-1}x^{-1}t)^e$, $((xy)^3t)^f$, $(xyt)^g\rangle$

Table of images:

a	b	c	d	e	f	g	Order of G	Shape of G
0	0	4	0	0	3	4	2520	A_7

Table 3.2: Images of Progenitor 3^{*14}:(2^3:(3:7))

3.3 Progenitor 2^{*56}:(2^3:((3:7))

Let x ∼ (1, 28, 30)(2, 40, 39)(3, 55, 54)(4, 26, 29)(5, 25, 10)(6, 31, 9)(7, 19, 18)(8, 16, 21)(11, 17, 24)(12, 41, 50)(13, 52, 42)(14, 51, 34)(15, 56, 33)(20, 53, 36)(22, 32, 38)(23, 49, 46)(35, 37, 44)(45, 48, 47).

Let y ∼ (1, 54, 45)(2, 25, 56)(3, 20, 22)(4, 34, 49)(5, 38, 27)(6, 35, 48)(7, 44, 10)(8,51, 19)(9, 39, 55)(11, 31, 52)(12, 43, 17)(14, 15, 41)(16, 24, 53)(18, 42,46)(21, 36, 47)(23, 40, 33)(26, 32, 30)(28, 50, 29).

$N = \langle x, y\rangle$. Using MAGMA we get N ≅ 2^3:(3:7).

$G = \langle x^3, y^3, yx^{-1}y^{-1}x^{-1}y^{-1}xyxy^{-1}x,$

$\mathrm{t}^2, (t, xy^{-1}x^{-1}y^{-1}x),$

$(\mathrm{xyt}^{(xy^{-1}x)}t^{(y^{-1}x^{-1}yxy^{-1})})^a, (xytt^{(y^-1,x)})^b, (xt^((x,y)))^c, (y^-1x^{-1}t^{(y^-1,x)})^d, (xyt^{(y^-1,x)}t^{yx})^e,$

$((\mathrm{xy})^3t^{y^{-1}xy^{-1}x^{-1}y^{-1}})^f, ((xy^{-1})^3t)^g, (xy^{-1}t)^h\rangle.$

a	b	c	d	e	f	g	h	Order of G	Shape of G
0	0	0	0	0	3	3	0	96768	$2^6{\cdot}(3{\cdot}L_2(8))$
0	0	0	0	0	6	3	0	12386304	$2^{13}{\cdot}(3{\cdot}L_2(8))$
0	0	0	0	0	9	3	0	211631616	$(2^6{\cdot}3^7){\cdot}(3{\cdot}L_2(8))$
0	0	0	0	3	0	3	0	1512	$3{\cdot}L_2(8)$
0	0	6	0	6	0	3	0	193536	$2^7{\cdot}(3{\cdot}L_2(8))$
0	0	6	0	9	0	3	0	3306744	$3^7{\cdot}(3{\cdot}L_2(8))$
0	0	6	3	0	2	0	0	5376	$2^5{\cdot}(2^3{\cdot}\ (7{\cdot}3))$
0	0	6	3	0	4	7	0	86016	$2^9{\cdot}(2^3{\cdot}(7{\cdot}3))$
0	0	6	3	6	4	0	0	1376256	$2^{10}{\cdot}\ (2^6{\cdot}(2{\cdot}7))$

Table 3.3: Images of Progenitor $2^{*56}{:}(2^3{:}((3{:}7))$

Please see Appendix D for the corresponding MAGMA code.

3.4 Progenitor $2^{*20}{:}A_5$

The purpose of this section is to show some examples of progenitor images that we found using Magma.

Let x $\sim$ (2, 3, 5)(4, 6, 9)(7, 10, 8)(11, 13, 16)(12, 14, 17)(15, 18, 19) and let y $\sim$ (1, 2)(3, 4)(5, 7)(6, 8)(9, 11)(10, 12)(13, 15)(14, 16)(17, 18)(19, 20).

$A_5 \sim \langle x, y\rangle$.

G = $\langle x^3, y^2, (yx^{-1})^5,$

t$^2, (t, x),$

(yt$^{yx^{-1}})^a, (yxyt^{yx})^b, ((yx)^2t^{yxy})^c, (yt^{yx})^d, (yxyt^y)^e, (yxyt)^f,$

$\langle\ (\mathrm{yx})^2t)^g, (yxt)^h, (yt)^i\rangle$.

Table of images:

a	b	c	d	e	f	g	h	i	j	Order of G	Shape of G
0	0	0	0	0	0	0	4	6	0	14400	$2{\cdot}(2{\cdot}A_5{}^2)$
0	0	0	0	0	0	0	5	4	0	30720	$2^9{\cdot}A_5$
0	0	0	0	0	0	0	6	3	0	660	$L_2(11)$
0	0	0	0	0	0	0	7	3	0	2520	A_7
0	0	0	0	0	0	4	4	0	0	8160	$2{\cdot}L_2(16)$
0	0	0	0	0	0	5	5	8	0	31457280	$2^{19}{\cdot}A_5$
0	0	0	0	0	0	6	5	0	0	675840	$2^{10}{\cdot}L_2(11)$
0	0	0	0	0	3	0	0	0	0	960	$2^4{\cdot}(A_5)$
0	0	0	0	0	4	0	6	6	0	1555200	$2{\cdot}(3{\cdot}(2{\cdot}A_6)))$
0	0	0	0	0	5	5	0	9	0	38972340	$3^{10}{\cdot}L_2(11)$
0	0	0	0	0	6	0	5	0	0	983040	$2^{14}{\cdot}A_5$
0	0	0	0	0	9	5	5	0	0	56687040	$3^{10}{\cdot}(2^4{\cdot}A_5)$
0	0	0	2	0	0	0	0	4	0	1920	$2^5{\cdot}A_5$
0	0	0	2	0	0	0	7	0	0	175560	J_1
0	0	0	2	0	0	0	8	8	0	117600	$2{\cdot}L_2(49)$
0	0	0	2	0	0	0	10	6	0	9720	$3^4{\cdot}(2{\cdot}A_5)$
0	0	0	4	0	0	0	5	4	0	15360	$5^4{\cdot}(2{\cdot}A_5)$
0	0	0	4	0	4	0	0	6	0	368640	$2^9{\cdot}(2{\cdot}A_6)$
0	0	0	4	0	4	6	0	0	0	11520	$2^4{\cdot}(2{\cdot}A_6)$

Table 3.4: Images of Progenitor $2^{*20} : A_5$

Chapter 4

Construction of $3 : (2 \times S_5)$ Over D_{12}

4.1 Definitions and Theorems

Definition 4.1. *If X is a G-set and $x \in X$, then the **stabilizer** of x, denoted by G_x, is the subgroup $G_x = \{g \in G : gx = x\} \leq G$. [Rot95]*

Definition 4.2. *If S is a subgroup of G and if $t \in G$, then a **right coset** of $S \in G$ is the subset of G: $St = \{st{:}s \in S\}$ (a **left coset** is $tS = \{ts{:}s{\in}S\}$). One calls t a **representative** of St (and also tS). [Rot95]*

Theorem 4.3. *If $S \leq G$, then any two right (or any two left) cosets of S in G are either identical or disjoint. [Rot95]*

Theorem 4.4. *If $S \leq G$, then the number right cosets of S in G is equal to the number of left cosets of S in G. [Rot95]*

Theorem 4.5. *If $S \leq G$, then the **index** of S in G, denoted [G:S], is the number of right cosets of S in G. [Rot95]*

Definition 4.6. *If $x \in G$, then a **conjugate** of x in G is an element of the form axa^{-1} for some $a \in G$. [Rot95]*

Definition 4.7. *If S and T are subgroups of G, then a **double coset** is a subset of G of the form SgT, where $g \in G$. [Rot95]*

Theorem 4.8. ***(First Isomorphism Theorem)*** *Let f: G⟶H be a homomorphism with kernel K. Then K is a normal subgroup of G and $G/K \cong$ image(f). [Rot95]*

4.2 Double Coset Enumeration of 3:(2× S_5) Over D_{12}

A symmetric presentation of the progenitor $3^{*6} : D_{12}$ is G = ⟨ x^6, y^2 , $(xy)^2$, t^3, $t^{x^2y} = t^2$, (t, $(xy)^x$), $(t^x t^y)^2$⟩, where $D_{12} = \langle x, y \rangle$. Let xx = (1, 2, 3, 4, 5, 6) and yy = (1, 2, 3, 4, 5, 6). Let N = $\langle xx, yy \rangle$. Magma gives N $\cong D_{12}$. Let t = t_1.

The following table gives the correspondence between the words and permutations of N.

words of N	Permutations of N
x	(1, 2, 3, 4, 5, 6)
y	(1, 6)(2, 5)(3, 4)
xy	(1, 5)(2, 4)
x^2	(1, 3, 5)(2, 4, 6)
x^2y	(1, 4)(2, 3)(5, 6)
x^3	(1, 4)(2, 5)(3, 6)
x^3y	(1, 3)(4, 6)
x^4	(1, 5, 3)(2, 6, 4)
x^4y	(1, 2)(3, 6)(4, 5)
x^5	(1, 6, 5, 4, 3, 2)
x^4y	(2, 6)(3, 5)
e = Identity of G	Id(N)

Table 4.2: Words and Permutations of N.

We perform our technique of manual double coset enumeration to G over N to construct G and prove that G $\cong$ 3:(2$\times S_5$). Before we start our double coset enumeration, the relations need to be expanded.

From the progenitor we get $t^{x^2y} = t^2$. Therefore, $t_4 = {t_1}^2$, which produces the following notation: $t_5 = {t_2}^2$, ${t_5}^2 = t_2$, $t_6 = {t_3}^2$, ${t_6}^2 = t_3$, and ${t_4}^2 = t_1$.

Now $t_4 = {t_1}^2$ by notation, which means $(t_4 = {t_1}^2)^{\mathrm{n}}$ $\forall$ n $\in$ N. Therefore, ${t_4}^{(1,\ 2)(3,\ 6)(4,\ 5)} = ({t_1}^2)^{(1,\ 2)(3,\ 6)(4,\ 5)}$ since (1, 2)(3, 6)(4, 5) $\in$ N. Hence,

$t_5 = {t_2}^2 \implies (t_5)^2 = ({t_2}^2)^2 \implies t_5t_5 = t_2t_2t_2t_2 = t_2$ since $\mid t_2 \mid = 3$.

Similarly, ${t_4}^{(1, 3, 5)(2, 4, 6)} = ({t_1}^2)^{(1, 3, 5)(2, 4, 6)}$ since $(1, 3, 5)(2, 4, 6) \in$ N $\implies t_6 = {t_3}^2 \implies {t_6}^2 = t_3$. Similarly, $t_4 = {t_1}^2 \implies {t_4}^2 = t_1$.

Lemma 4.10. $(t_2t_6)^2 = e$.

Proof. From the progenitor we get $(t^xt^y)^2 = e \implies ({t_1}^{(1, 2, 3, 4, 5, 6)}{t_1}^{(1, 6)(2, 5)(3, 4)})^2$ $\implies (t_2t_6)^2 = e$.

Lemma 4.11. $t_4 = {t_1}^{-1}$, $t_5 = {t_2}^{-1}$, *and* $t_6 = {t_3}^{-1}$.

Proof. $t_4 = {t_1}^2$ by notation.

$t_4t_1 = {t_1}^2t_1 = e$ since since $|t_1| = 3$.

$t_4t_1 = e \implies t_4 = {t_1}^{-1}$.

Similarly $t_5 = {t_2}^{-1}$ and $t_6 = {t_3}^{-1}$. □

Lemma 4.12. $t_2t_6 = t_3t_5$.

Proof. $(t_2t_6)^2 = e$ (Lemma 4.10). $\implies t_2t_6t_2t_6 = e \implies t_2t_6t_{t6}{t_6}^2{t_2}^2 = e{t_6}^2{t_2}^2$ $\implies t_2t_6t_2{t_2}^2 = t_2t_6{t_2}^3 = t_2t_6 = {t_6}^2{t_2}^2 \implies t_2t_6 = ({t_3}^2)^2t_5 = {t_3}^4t_5 = t_3t_5$ since $\mid t_3 \mid = 3 \implies t_2t_6 = t_3t_5$. □

Lemma 4.13. $t_3t_1 = t_4t_6$

Proof. $t_3t_1 = t_4t_6$ since $(t_2t_6)^{(1, 2, 3, 4, 5, 6)} = (t_3t_5)^{(1, 2, 3, 4, 5, 6)}$ (Lemma 4.12). □

Lemma 4.14. $t_5t_3 = t_6t_2$.

Proof. $t_5t_3 = t_6t_2$ since $(t_2t_6)^{(1, 4)(2, 5)(3, 6)} = (t_3t_5)^{(1, 4)(2, 5)(3, 6)}$ (Lemma 4.12). □

Lemma 4.15. $t_1t_3 = t_6t_4$.

Proof. $t_1t_3 = t_6t_4$ since $(t_2t_6)^{1,\ 2)(3,\ 6)(4,\ 5)} = (t_3t_5)^{(1,\ 2)(3,\ 6)(4,\ 5)}$ (Lemma 4.12). □

Lemma 4.16. $t_1t_5 = t_2t_4$.

Proof. $t_1t_5 = t_2t_4$ since $(t_2t_6)^{(1,\ 6,\ 5,\ 4,\ 3,\ 2)} = (t_3t_5)^{(1,\ 6,\ 5,\ 4,\ 3,\ 2)}$ (Lemma 4.12). □

Lemma 4.17. $t_5t_1 = t_4t_2$.

Proof. $t_5t_1 = t_4t_2$ since $(t_2t_6)^{(1,\ 6)(2,\ 5)(3,\ 4)} = (t_3t_5)^{(1,\ 6)(2,\ 5)(3,\ 4)}$ (Lemma 4.12). □

4.2.1 Double Coset [*]

The only right coset of NeN = [*] is N $\implies$ $|[*]| = 1$, which is symbolized by placing "1" inside the circle representing [*]. The stabilizer of N is N, therefore the only orbit of N is $\{1, 2, 3, 4, 5, 6\}$ which yields the cosets Nt_1, Nt_2, Nt_3, Nt_4 Nt_5, Nt_6 $\in$ [1]. This is symbolized by placing a "6" next to the circle representing [*] in the diagram.

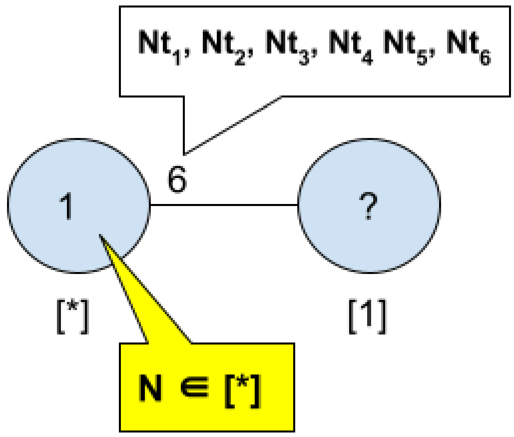

Figure 4.1: Double Cosets [∗]

4.2.2 Double Coset [1]

Let N^1 be the stabilizer of {1} over N. Therefore, $N^1 = \langle (2,6)(3,5) \rangle$. $N^{(1)} = N^1$ since there are no relations such that $t_i = t_j$ where $0 < i,j \leq 6$. Hence, $|[1]| = \frac{|N|}{|N^{(1)}|} \leq \frac{12}{2} = 6$. The right cosets of [1] are Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, and Nt_6 which verifies $|[1]| = 6$. This is symbolized by placing a "6" inside the circle representing [1] in the diagram. Now the obits of $N^{(1)}$ on X = { 1, 2, 3, 4 ,5, 6 } are {1}, {4}, {2,6}, and {3, 5}.

For the double coset [w], we need only determine the double coset of the right coset Nwt_i for one representative t_i for each orbit of the stabilizing group $N^{(w)}$ of the coset Nw. We start with $Nt_1t_1 = Nt_4 \in [1]$. Now $Nt_1t_4 = Nt_1{t_1}^2 = N{t_1}^3 = N \in [*]$. Therefore, $t_1t_4 \in [*]$, since $t_1t_4 = e$. The third set of orbits yields Nt_1t_2, $Nt_1t_6 \in [1, 2]$. Similarly, Nt_1t_3, $Nt_1t_5 \in [1, 3]$.

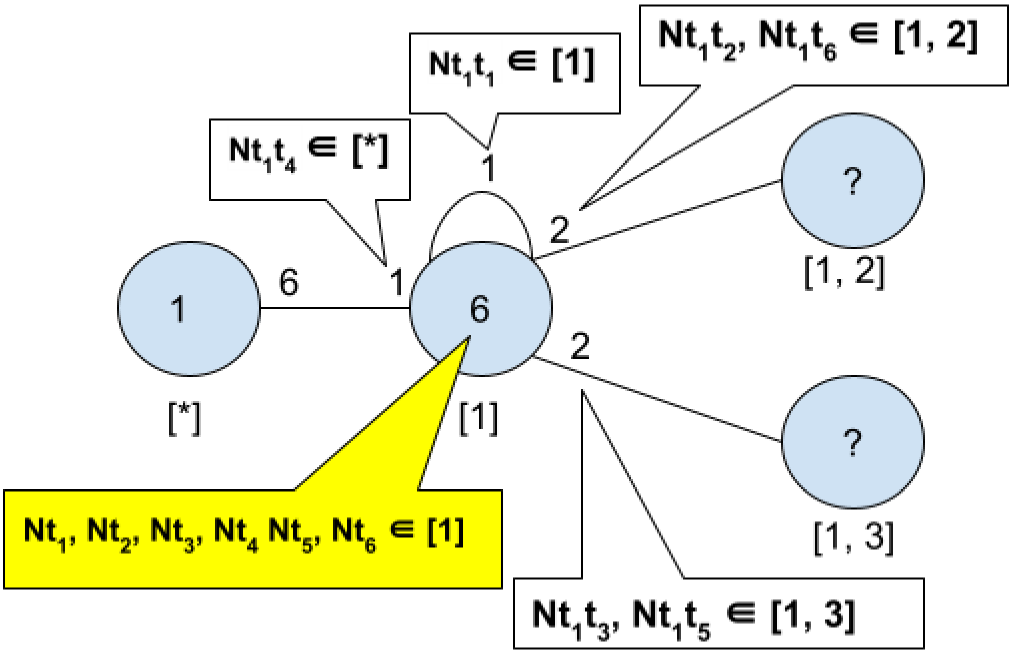

Figure 4.2: Double Cosets of [1]

4.2.3 Double Coset [1,3]

The $N^{(1,3)} \geq \langle$ e $\rangle$. By Lemma 4.12, $t_2t_6 = t_3t_5$, therefore $(t_2t_6)^{(1,\ 2)(3,\ 6)(4,\ 5)} = (t_3t_5)^{(1,\ 2)(3,\ 6)(4,\ 5)}$. This yields $t_1t_3 = t_6t_4$. $(t_1t_3\ )^{(1,\ 6)(2,\ 5)(3,\ 4)} = t_6t_4 = t_1t_3$, which means $(1,6)(2,5)(3,4) \in N^{(1,3)}$. This implies, $N^{(1,3)} \geq \langle\ (1,\ 6)(2,\ 5)(3,\ 4)\ \rangle$. Consequently, $N^{(1,3)} \geq \langle\ (1,\ 6)(2,\ 5)(3,\ 4)\ \rangle$, therefore $|[1,\ 3]| = \frac{|N|}{|N^{(1,3)}|} \leq \frac{12}{2} = 6$. The orbits of $\langle\ (1,\ 6)(2,\ 5)(3,\ 4)\ \rangle$ on X are {1, 6}, {2,5}, and {3,4}.

The first set of orbits yield $Nt_1t_3t_1$ and $Nt_1t_3t_6$, which belong to the same double coset. Since $Nt_1t_3t_6 = Nt_1t_3{t_3}^2 = Nt_1$, we now get $Nt_1t_3t_1$, $Nt_1t_3t_6 \in [1]$.

The second set of orbits yield $Nt_1t_3t_2$, $Nt_1t_3t_5 \in [1,3,\ 2]$. The third set of orbits yield $Nt_1t_3t_3$ and $Nt_1t_3t_4$, which belong to the same double coset. Since

N$t_1t_3t_3$ = Nt_1t_6 = N$(t_1t_2)^{(2,\ 6)(3,\ 5)}$ $\in$ [1, 2], therefore N$t_1t_3t_3$, N$t_1t_3t_4$ $\in$ [1, 2].

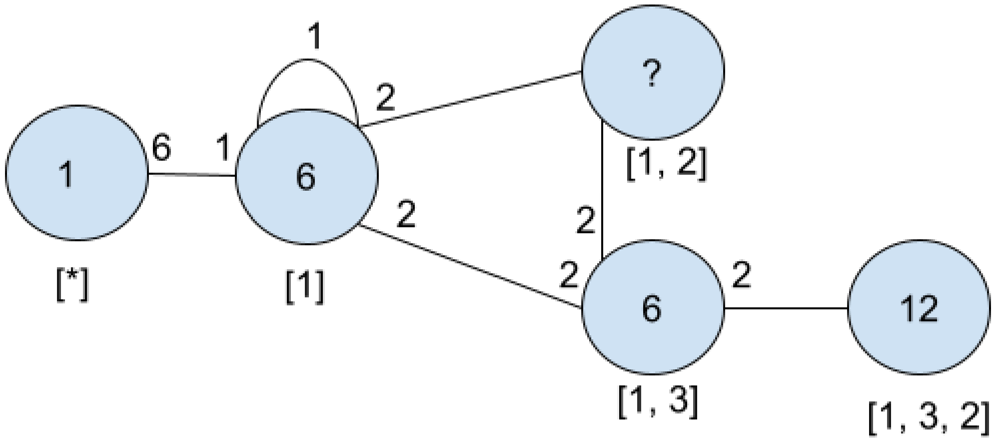

Figure 4.3: Double Cosets of $[1, 3]$

4.2.4 Double Coset [1,3,2]

The orbits of N$^{(1,3,2)}$ on X are {1}, { 2}, {3} ,{4}, {5}, and {6}. N$^{(1,3,2)}$ $\geq \langle e \rangle$. Therefore, $\frac{|N|}{|N^{(1,3,2)}|} \leq \frac{12}{1} = 12$. The first set of orbits yield N$t_1t_3t_2t_1$ $\in$ [1,3, 2,1], which is a new Double Coset.

From the second orbit we get the right coset N$t_1t_3t_2t_2$ = N$t_1t_3t_5$ since ${t_2}^2$ = t_5. The third orbit of N$^{(1,3)}$ yields that N$t_1t_3t_2$ and N$t_1t_3t_5$ belong to the same right coset. Therefore, N$t_1t_3t_5$ = N$t_1t_3t_2t_2$ $\in$ [1,3, 2].

From the third set of orbits of N$^{(1,3,2)}$ we get the right coset N$t_1t_3t_2t_3$ $\in$ [1,3, 2,3], which is a new double coset.

Lemma 4.18. $t_1t_3t_2t_4 = t_6t_4t_2t_4 = t_6t_4t_1t_5 = t_6t_5$.

Proof. $t_1t_3t_2t_4 = t_6t_4t_2t_4$ by Lemma 4.15.

$t_6t_4t_2t_4 = t_6t_4t_1t_5$ by Lemma 4.16.

$t_6t_4t_1t_5 = t_6t_5$ by Lemma 4.11.

$\implies t_1t_3t_2t_4 = t_6t_5$ □

From the fourth set of orbits of $N^{(1,3,2)}$ we get the right coset $Nt_1t_3t_2t_4$. $Nt_1t_2{}^{(1,\,6)(2,\,5)(3,\,4)} = Nt_6t_5 = Nt_1t_3t_2t_4$ by Lemma 4.18, therefore $Nt_1t_3t_2t_4 \in [\ 1,\ 2\ ]$.

From the fifth set of orbits of $N^{(1,3,2)}$ we get the right coset $Nt_1t_3t_2t_5$. $Nt_1t_3t_2t_5 = Nt_1t_3 \in [\ 1,\ 3\ ]$ since $t_2t_5 = t_2t_2t_2 = e$.

Lemma 4.19. $t_1t_3t_2t_6 = t_1t_6t_5 = t_1t_5t_3t_2 = t_2t_4t_3t_2$.

Proof. $t_1t_3t_2t_6 = t_1t_3t_3t_5$ by Lemma 4.12.

$t_1t_3t_3t_5 = t_1t_6t_5$ since $t_3^2 = t_6$.

$t_1t_6t_5 = t_1t_6t_2t_2$ since $t_2^2 = t_5$.

$t_1t_6t_2t_2 = t_1t_5t_3t_2$ by Lemma 4.14.

$t_1t_5t_3t_2 = t_2t_4t_3t_2$ by Lemma 4.16. □

From the sixth set of orbits of $N^{(1,3,2)}$ we get the right coset $Nt_1t_3t_2t_6$. This gives us $N(t_1t_3t_2t_1)^{(1,\,2,\,3,\,4,\,5,\,6)} = Nt_2t_4t_3t_2 = Nt_1t_3t_2t_6$ by Lemma 4.19, therefore $N\mathrm{t}_1t_3t_2t_6 \in [1,\ 3,\ 2,\ 1]$.

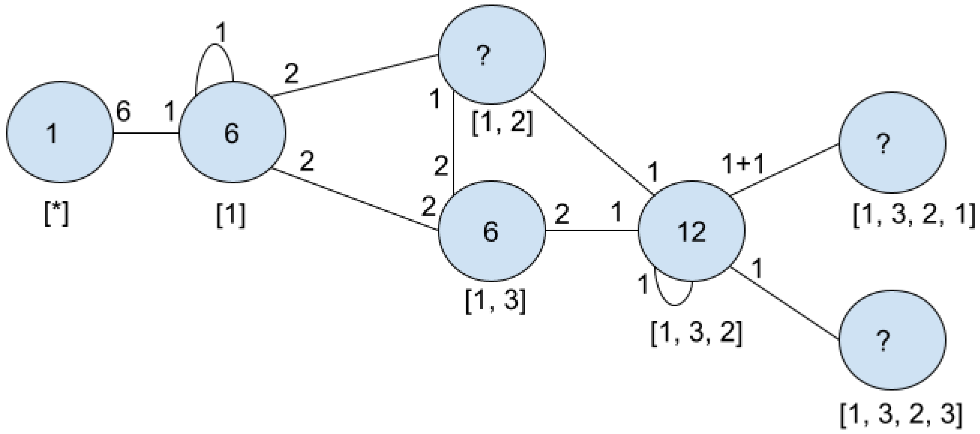

Figure 4.4: Double Cosets of $[1, 3, 2]$

4.2.5 Double Coset [1,2]

The stabilizing group if Nt_1t_2, denoted by N,$^{(1,2)} \geq \langle$ e $\rangle$, which means $|[1,\ 2]| = |\mathrm{N}| \div |\mathrm{N}^{(1,2)}| \leq 12 \div 1 = 12$. We know the orbits of $\langle$ e $\rangle$ on X are {1}, {2}, {3}, {4}, {5}, and {6}. From the first set of orbits we get the N$t_1t_2t_1$ right coset. Therefore, N$t_1t_2t_1 \in$ [1, 2, 1], which is a new double coset.

From the second set of orbits we get N$t_1t_2t_2$ right coset and N$t_1t_2t_2$ = Nt_1t_5. $(\mathrm{N}t_1t_3)^{(2,\ 6)(3,\ 5)} = \mathrm{N}t_1t_5$, therefore N$t_1t_3 \in$ [1, 3].

Lemma 4.20. *N$t_1t_2t_3$ right coset $\in$ [1,3, 2, 1].*

Proof. $t_1t_2t_3 = t_1t_2t_6t_6$ since $t_3 = t_6t_6$.

$t_1t_2t_6t_6 = t_1t_3t_5t_6$ by Lemma 4.12.

$t_1t_3t_5t_6 = t_6t_4t_5t_6$ by Lemma 4.15.

$\mathrm{N}t_6t_4t_5t_6{}^{(1,\ 6)(2,\ 5)(3,\ 4)} = \mathrm{N}t_1t_3t_2t_1 \implies \mathrm{N}t_6t_4t_5t_6 \in [1,\ 3,\ 2,\ 1]$. □

From the third set of orbits we get the N$t_1t_2t_3$ right coset $\in$ [1, 3, 2, 1]. The fourth set of orbits yield N$t_1t_2t_4$. N$t_1t_2t_4^{(1, 2, 3, 4, 5, 6)}$ = N$t_1t_3t_4$ $\in$ [1, 2] as seen before. Therefore, N$t_1t_2t_4$ $\in$ [1, 2].

The fifth set of orbits yield N$t_1t_2t_5$. N$t_1t_2t_5$ = Nt_1 since $t_2^{-1} = t_5$, which implies N$t_1t_2t_5$ $\in$ [1].

The sixth orbit yields N$t_1t_2t_6$. N$t_1t_2t_6$ = N$t_1t_3t_5$ by Lemma 4.12. N$t_1t_3t_5$ $\in$ [1,3, 2] as seen before. Hence, Nt$_1t_2t_6$ $\in$ [1, 3, 2].

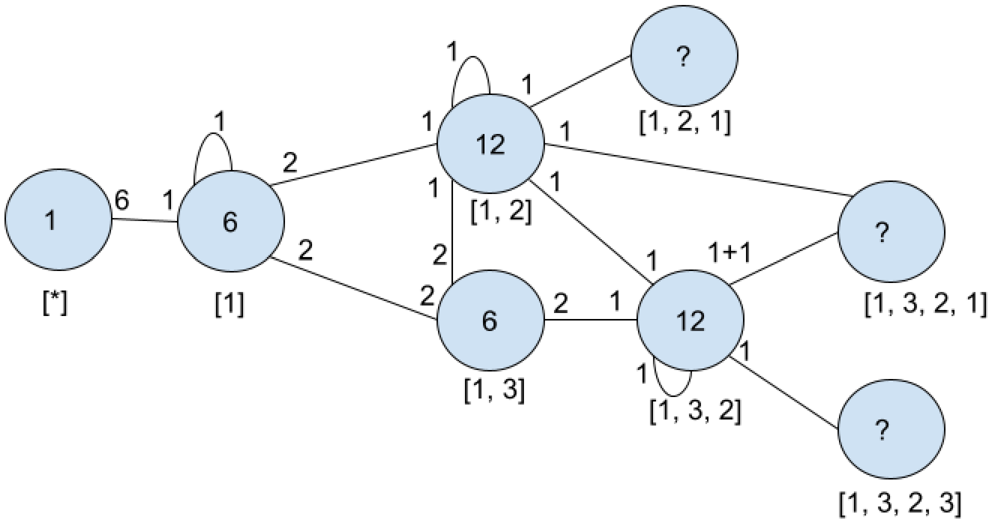

Figure 4.5: Double Coset $[1, 2]$

4.2.6 Double Coset [1,2,1]

Lemma 4.21. $t_1t_2t_1 = t_4t_5t_4$.

Proof. $t_1t_2t_1 = t_1t_2$e$t_1 = t_1t_2t_4t_1t_1$ by Lemma 2.

$t_1t_2t_4t_1t_1 = t_1t_1t_5t_1t_1$by Lemma 4.16.

$t_1t_1t_5t_1t_1 = t_1t_1t_5et_1t_1 = t_1t_1t_5t_5t_2t_1t_1$ by Lemma 4.11.

$t_1t_1t_5t_5t_2t_1t_1 = t_1t_1t_5et_1t_1$ by Lemma 4.11.

$t_1t_1t_5t_1t_1 = t_4t_5t_4$ (Lemma 4.10) $t_1t_1 = t_4 \implies t_1t_2t_1 = t_4t_5t_4$. □

Lemma 4.22. $t_1t_2t_1 = t_2t_1t_2 = t_5t_4t_5$.

Proof. $t_1t_2t_1 = t_1t_5t_5t_1$ since $t_5^2 = t_2$.

$t_1t_5t_5t_1 = t_2t_4t_5t_1$ (Lemma 4.16).

$t_2t_4t_5t_1 = t_2t_4t_4t_2$ (Lemma 4.17).

$t_2t_4t_4t_2 = t_2t_1t_2$ since $t_4 = t_1t_1$.

$\implies t_1t_2t_1 = t_2t_1t_2$.

$t_1t_2t_1 = t_4t_5t_4$ (Lemma 4.21). $\implies (t_1t_2t_1 = t_4t_5t_4)^{(1,\ 2)(3,\ 6)(4,\ 5)} \implies$

$t_2t_1t_2 = t_5t_4t_5$. □

Let the stabilizing group of N$t_1t_2t_1$ be denoted by $N^{(1,2,1)}$. $N(t_1t_2t_1)^{(1,\ 4)(2,\ 5)(3,\ 6)}$ = N$t_5t_4t_5$ = N$t_1t_2t_1$ by Lemma 4.22, therefore $(1, 4)(2, 5)(3, 6) \in N^{(1,2,1)}$. $N(t_1t_2t_1)^{(1,\ 2)(3,\ 6)(4,\ 5)}$ = N$t_4t_5t_4$ = N$t_1t_2t_1$ by Lemma 4.21, this proves $(1, 2)(3, 6)(4, 5) \in N^{(1,2,1)}$. $N^{(1,2,1)} \geq \langle (1, 4)(2, 5)(3, 6), (1, 2)(3, 6)(4, 5) \rangle$, which has an order of 4. Consequently, $|[1,2,1]| = \frac{|N|}{|N^{(1,2,1)}|} \leq \frac{12}{4} = 3$. The orbits of $N^{(1,2,1)}$ on X are = {3, 6}, and {1,5,4,2}.

The second set of orbits yield N$t_1t_2t_1t_1$, N$t_1t_2t_1t_2$, N$t_1t_2t_1t_4$, and N$t_1t_2t_1t_5$, which belong to the same double coset.

Lemma 4.23. *N$t_1t_2t_1t_1$, N$t_1t_2t_1t_2$, N$t_1t_2t_1t_4$, N$t_1t_2t_1t_5$ ∈ [1, 2].*

Proof. N$t_1t_2t_1t_4$ = Nt_1t_2e (Lemma 4.11). $\implies$

N$t_1t_2t_1t_4$ ∈ [1, 2] $\implies$

N$t_1t_2t_1t_1$, N$t_1t_2t_1t_2$, N$t_1t_2t_1t_4$, and N$t_1t_2t_1t_5$ ∈ [1, 2]. □

The first set of orbits yield N$t_1t_2t_1t_3$ and N$t_1t_2t_1t_6$ cosets, which belong to the same double coset.

Lemma 4.24. *$Nt_1t_2t_1t_3$, $Nt_1t_2t_1t_6 \in$ [1, 3, 2, 3].*

Lemma 4.25. *$Nt_1t_2t_1t_3 \in$ [1, 3, 2, 3].*

Proof. $t_1t_2t_1t_3 = t_1t_2t_6t_4$ by Lemma 4.15.

$t_1t_2t_6t_4 = t_1t_3t_5t_4$ by Lemma 4.12.

$t_1t_3t_5t_4 = t_6t_4t_5t_4$ by Lemma 4.15. $\implies$

$N(t_6t_4t_5t_4)^{(1,\ 6)(2,\ 5)(3,\ 4)} = Nt_1t_3t_2t_3$. $\implies$

$Nt_1t_2t_1t_3 = Nt_6t_4t_5t_4 \in [1, 3, 2, 3] \implies$

$Nt_1t_2t_1t_3$ and $Nt_1t_2t_1t_6 \in [1, 3, 2, 3]$. □

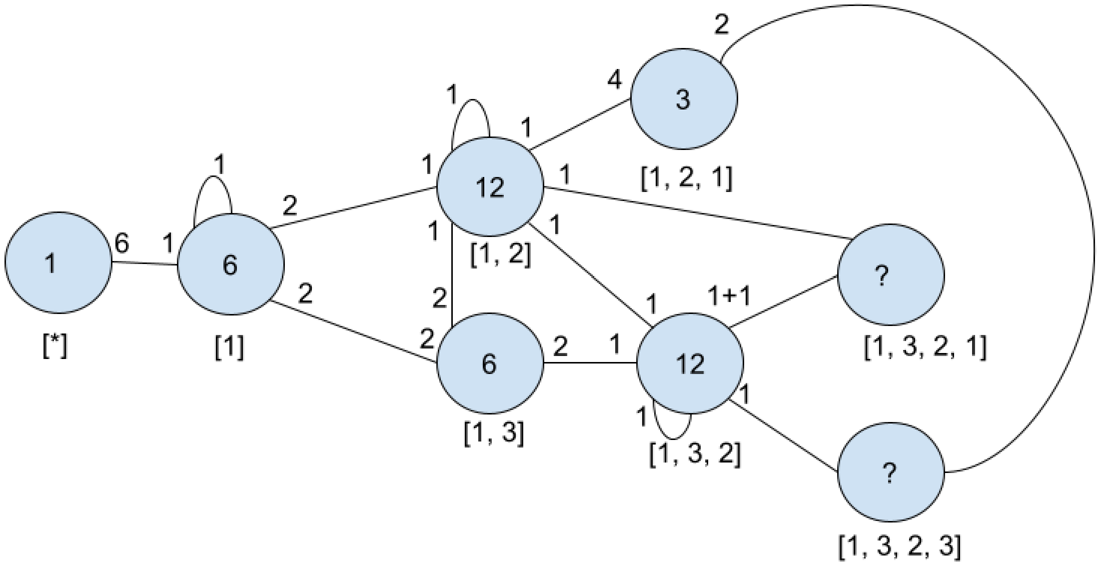

Figure 4.6: Double Coset $[1, 2, 1]$

Lemma 4.26. $t_5t_6t_5 = t_2t_3t_2$.

Proof. By Lemma 4.21 $t_1t_2t_1 = t_4t_5t_4 \implies$

$(t_1t_2t_1 = t_4t_5t_4)^{(1,\ 5,\ 3)(2,\ 6,\ 4)} \implies$

$t_5t_6t_5 = t_2t_3t_2$. □

Lemma 4.27. $t_1t_3t_2t_3 = t_1t_5t_6t_5$.

Proof. $t_1t_3t_2t_3 = t_1t_3t_5t_5t_3$ since $t_2 = t_5^2$.

$t_1t_3t_5t_5t_3 = t_1t_2t_6t_5t_3$ (Lemma 4.12).

$t_1t_2t_6t_5t_3 = t_1t_2t_6t_6t_2$ (Lemma 4.14).

$t_1t_2t_6t_6t_2 = t_1t_2t_3t_2$ since $t_6 = t_3^2$.

$t_1t_2t_3t_2 = t_1t_5t_6t_5$ (Lemma 4.26).

$t_1t_3t_2t_3 = t_1t_5t_6t_5$. □

Let the stabilizing group if N$t_1t_3t_3t_3$ be denoted by $N^{(1,3,2,3)}$. $N(t_1t_3t_2t_3)^{(2,\ 6)(3,\ 5)}$ = N$t_1t_5t_6t_5$ = N$t_1t_3t_2t_3$ by Lemma 4.27, therefore $(2,\ 6)(3,\ 5) \in N^{(1,3,2,3)}$. $N^{(1,3,2,3)} \geq \langle\ (2,\ 6)(3,\ 5)\ \rangle$, which has an order of 2. This proves $|[1,2,1]| = \frac{|N|}{|N^{(1,3,2,3)}|} \leq \frac{12}{2} = 6$. Hence, the orbits of $N^{(1,3,2,3)}$ on X are {1}, {4}, {2, 6}, and {3,5}. The first orbit yields the N$t_1t_3t_2t_3t_1$ coset.

Lemma 4.28. *N$t_1t_3t_2t_3t_1 \in$ [1, 2, 1].*

Proof. $t_1t_3t_2t_3t_1 = t_6t_4t_2t_3t_1$ (Lemma 4.15).

$t_6t_4t_2t_3t_1 = t_6t_5t_1t_3t_1$ (Lemma 4.17).

$t_6t_5t_1t_3t_1 = t_6t_5t_6t_4t_1$ (Lemma 4.15).

$t_6t_5t_6t_4t_1 = t_6t_5t_6$e (Lemma 4.11). $t_4t_1 =$ e.

$\implies$ N$t_1t_3t_2t_3t_1$ = N$t_6t_5t_6$.

$\Longrightarrow$ Note that $\mathrm{N}(t_1t_2t_1)^{(1,\ 6)(2,\ 5)(3,\ 4)} = \mathrm{N}t_6t_5t_6 = \ \mathrm{N}t_1t_3t_2t_3t_1$

$\Longrightarrow$ $\mathrm{N}t_1t_3t_2t_3t_1 \in [1,\ 2,\ 1]$. □

The second orbit yields the $\mathrm{N}t_1t_3t_2t_3t_4$ coset.

Lemma 4.29. *$Nt_1t_3t_2t_3t_4 \in$ [1, 3, 2, 3].*

Proof. $t_1t_3t_2t_3t_4 = t_1t_3t_2t_3t_1t_1$ by notation.

$t_1t_3t_2t_3t_1t_1 = t_1t_3t_2t_4t_6t_1$ (Lemma 4.13).

$t_1t_3t_2t_4t_6t_1 = t_1t_3t_1t_5t_6t_1$ (Lemma 4.16).

$t_1t_3t_1t_5t_6t_1 = t_1t_4t_6t_5t_6t_1$ (Lemma 4.13).

$t_1t_4t_6t_5t_6t_1 = \mathrm{e}t_6t_5t_6t_1$ (Lemma 4.11).

$t_6t_5t_6t_1 = t_6t_5t_3t_3t_1$ by notation.

$t_6t_5t_3t_3t_1 = t_6t_6t_2t_3t_1$ (Lemma 4.12).

$t_6t_6t_2t_3t_1 = t_3t_2t_3t_1$ by notation.

$t_3t_2t_3t_1 = t_3t_2t_4t_6$ (Lemma 4.11).

$t_3t_2t_4t_6 = t_3t_1t_5t_6$ (Lemma 4.14).

$t_3t_1t_5t_6 = t_4t_6t_5t_6$ (Lemma 4.11).

$t_4t_6t_5t_6 = t_1t_3t_2t_3t_4$. □

Now $\mathrm{N}(t_1t_3t_2t_3)^{(1,\ 4)(2,\ 5)(3,\ 6)} = \mathrm{N}t_4t_6t_5t_6$, therefore $\mathrm{N}t_4t_6t_5t_6 = \mathrm{N}t_1t_3t_2t_3t_4$ $\in [1,\ 3,\ 2,\ 3]$.

Lemma 4.30. *$Nt_1t_3t_2t_3t_6$, $Nt_1t_3t_2t_3t_2 \in$ [1, 3, 2].*

The third set of orbits yield $\mathrm{N}t_1t_3t_2t_3t_2$ and $\mathrm{N}t_1t_3t_2t_3t_6$ cosets, which belong to the same double coset.

Proof. $t_1t_3t_2t_3t_6 = t_1t_3t_2$e since $t_3t_6 = 1$. $\implies$

N$t_1t_3t_2t_3t_6 =$ N$t_1t_3t_2 \in [1, 3, 2]$. $\implies$

N$t_1t_3t_2t_3t_2 \in [1, 3, 2]$. □

The fourth set of orbits yield N$t_1t_3t_2t_3t_3$ and N$t_1t_3t_2t_3t_5$ cosets, which belong to the same double coset.

Lemma 4.31. *$Nt_1t_3t_2t_3t_3$, $Nt_1t_3t_2t_3t_5 \in$ [1, 3, 2, 1].*

Proof. $t_1t_3t_2t_3t_3 = t_1t_3t_2t_6$ since $t_3^2 = t_6$.

N$(t_1t_3t_2t_1)^{(1, 2, 3, 4, 5, 6)} =$ N$t_2t_4t_3t_2 =$ N$t_1t_3t_2t_6$ (Lemma 4.19). $\implies$

$Nt_1t_3t_2t_6 \in [1, 3, 2, 1]$. $\implies$

N$t_1t_3t_2t_6 =$ N$t_1t_3t_2t_3t_3 \in [1, 3, 2, 1]$. $\implies$

N$t_1t_3t_2t_3t_5 \in [1, 3, 2, 1]$. □

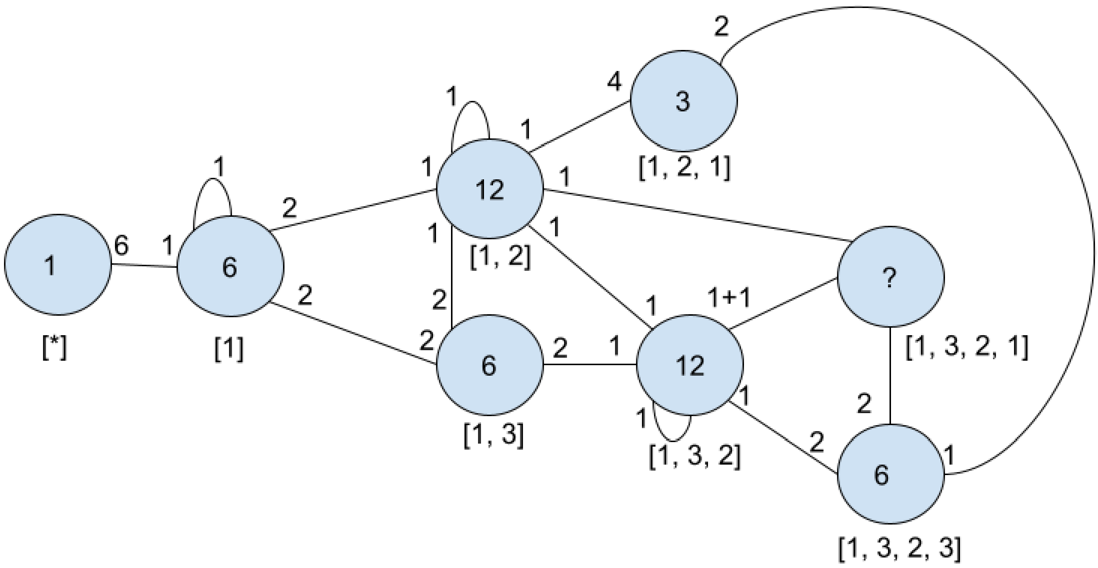

Figure 4.7: Double Cosets of $[1, 3, 2, 3]$

4.2.7 Double Coset [1,3,2,1]

The stabilizing group if Nt_1t_2, denoted by $\mathrm{N}^{(1,3,2,1)} \geq \langle$ e $\rangle$, therefore |[1, 3, 2, 1]| = |N|÷|$\mathrm{N}^{(1,3,2,1)}$| $\leq$ 12÷1 = 12. The orbits $\langle$ e $\rangle$ on X are {1}, { 2} , {3}, {4}, {5}, {6}.

From the first set of orbits we get N$t_1t_3t_2t_1t_1$ right coset.

Lemma 4.32. *N$t_1t_3t_2t_1t_1 \in$ [1, 2].*

Proof. N$t_1t_3t_2t_1t_1$ = N$t_1t_3t_2t_4$ since $t_1^2 = t_4$.
N$t_1{t_2}^{(1,\,6)(2,\,5)(3,\,4)}$ = Nt_6t_5 = N$t_1t_3t_2t_4$ (Lemma 4.18) $\implies$
N$t_1t_3t_2t_4$ = N$t_1t_3t_2t_1t_1 \in$ [1, 2]. □

From the second set of orbits we get N$t_1t_3t_2t_1t_2$ right coset.

Lemma 4.33. *$Nt_1t_3t_2t_1t_2 \in$ [1, 3, 2].*

Proof. $t_1t_3t_2t_6 = t_2t_4t_3t_2$ (Lemma 4.19) $\implies$

$(t_1t_3t_2t_6)^{(1,\ 6,\ 5,\ 4,\ 3,\ 2)} = (t_2t_4t_3t_2)^{(1,\ 6,\ 5,\ 4,\ 3,\ 2)} \implies$

$t_6t_2t_1t_5 = t_1t_3t_2t_1$

$t_1t_3t_2t_1t_2 = t_6t_2t_1t_5t_2$

$t_6t_2t_1t_5t_2 = t_6t_2t_1\text{e} = t_6t_2t_1 \implies$

$t_1t_3t_2t_1t_2 = t_6t_2t_1$.

$\text{N}t_1t_3t_2{}^{(1,\ 2,\ 3,\ 4,\ 5,\ 6)} = \text{N}t_6t_2t_1 \implies$

$\text{N}t_6t_2t_1 \in$ [1, 3, 2] $\implies$

$\text{N}t_6t_2t_1 = \text{N}t_1t_3t_2t_1t_2 \in$ [1, 3, 2]. □

From the third set of orbits we get the $\text{N}t_1t_3t_2t_1t_3$ right coset and $\text{N}t_1t_3t_2t_1t_3$ $\in$ [1, 3, 2, 1, 3].

From the fourth set of orbits we get the the $\text{N}t_1t_3t_2t_1t_4$ right coset. Now $t_1t_3t_2t_1t_4 = t_1t_3t_2\text{e} \implies \text{N}t_1t_3t_2t_1t_4 = \text{N}t_1t_3t_2 \in$ [1, 3, 2]. □

From the fifth set of orbits we get the $\text{N}t_1t_3t_2t_1t_5$ right coset.

Lemma 4.34. *$Nt_1t_3t_2t_1t_5 \in$ [1, 3, 2, 3].*

Proof. $t_1t_3t_2t_1t_5 = t_1t_3t_2t_2t_4$ (Lemma 4.16) .

$t_1t_3t_2t_2t_4 = t_1t_3t_5t_4$ since $t2^2 = t_5$.

$t_1t_3t_5t_4 = t_1t_2t_6t_4$ (Lemma 4.12).

$t_1t_2t_6t_4 = t_1t_2t_1t_3$ (Lemma 4.15). $\implies$

$\text{N}t_1t_3t_2t_1t_5 = \text{N}t_1t_2t_1t_3 \in$ [1, 3, 2, 3] (Lemma 4.24). □

From the sixth set of orbits we get the $\text{N}t_1t_3t_2t_1t_6$ right coset.

Lemma 4.35. *$Nt_1t_3t_2t_1t_6 \in$ [1, 3, 2, 1].*

Proof. $t_1t_3t_2t_1t_6 = t_6t_4t_2t_1t_6$ (Lemma 4.15).

$t_6t_4t_2t_1t_6 = t_6t_5t_1t_1t_6$ (Lemma 4.17).

$t_6t_5t_1t_1t_6 = t_6t_5t_4t_6$ since $t_1^2 = t_4$.

$t_6t_5t_4t_6 = t_6t_5t_3t_1$ (Lemma 4.13).

$t_6t_5t_3t_1 = t_6t_6t_2t_1$ (Lemma 4.14).

$t_6t_6t_2t_1 = t_3t_2t_1$ by since $t_6^2 = t_2$.

$t_3t_2t_1 = t_3t_5t_5t_1$ since $t_2 = t_5^2$.

$t_3t_5t_5t_1 = t_3t_5t_4t_2$ (Lemma 4.17).

By Lemma 4.19 $t_1t_3t_2t_6 = t_1t_5t_3t_2 \implies$

$(t_1t_3t_2t_6)^{(1,3,5)(2,4,6)} = (t_1t_5t_3t_2)^{(1,3,5)(2,4,6)} \implies$

$t_3t_5t_4t_2 = t_4t_6t_5t_4 \implies t_4t_6t_5t_4 = t_1t_3t_2t_1t_6$.

$\mathrm{N}t_1t_3t_2{t_1}^{(1,4)(2,5)(3,6)} = \mathrm{N}t_4t_6t_5t_4 \implies$

$\mathrm{N}t_4t_6t_5t_4 = \mathrm{N}t_1t_3t_2t_1t_6 \in [\ 1,\ 3,\ 2,\ 1\ ]$. □

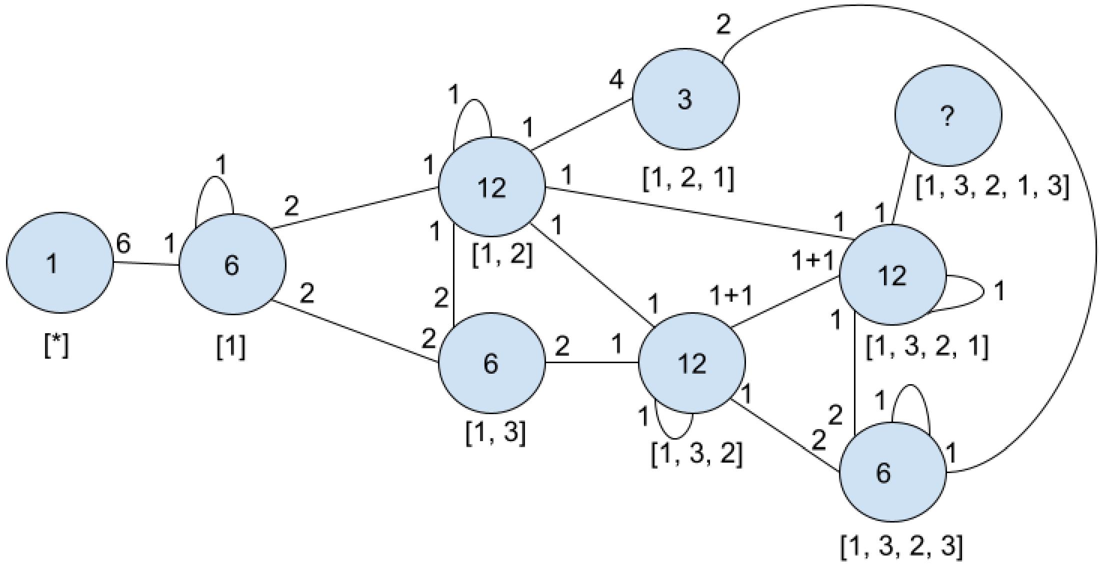

Figure 4.8: Double Cosets of $[1, 3, 2, 1,]$

4.2.8 Double Coset [1,3,2,1,3]

Lemma 4.36. $t_1t_3t_2t_1t_3 = t_6t_2t_1t_6t_2$.

Proof. $t_1t_3t_2t_6 = t_2t_4t_3t_2$ (Lemma 4.19) $\implies$ $(t_1t_3t_2t_6 = t_2t_4t_3t_2)^{(1,\ 6,\ 5,\ 4,\ 3,\ 2)}$

$\implies$ $t_6t_2t_1t_5 = t_1t_3t_2t_1$.

$t_1t_3t_2t_1t_3 = t_6t_2t_1t_5t_3$

$t_6t_2t_1t_5t_3 = t_6t_2t_1t_6t_2$ (Lemma 4.14). □

Let $N^{(1,\ 3,\ 2,\ 1,\ 3)}$ be the group stabilizer of $Nt_1t_3t_2t_1t_3$. $Nt_1t_3t_2t_1t_3{}^{(1,\ 6,\ 5,\ 4,\ 3,\ 2)}$ $= Nt_6t_2t_1t_6t_2 = Nt_1t_3t_2t_1t_3$, therefore $(1, 6, 5, 4, 3, 2) \in N^{(1,\ 3,\ 2,\ 1,\ 3)}$. Hence, $N^{(1,\ 3,\ 2,\ 1,\ 3)} \geq$ $\langle\ (1, 6, 5, 4, 3, 2)\rangle$, which proves $|\ [1, 3, 2, 1, 3]\ | = \frac{|N|}{|N^{(1,3,2,13)}|} \leq \frac{12}{6} = 2$. The orbit of $\langle\ (1, 6, 5, 4, 3, 2)\rangle$ is $\{1, 2, 3, 4, 5, 6\}$ which yields $Nt_1t_3t_2t_1t_3t_1$, $Nt_1t_3t_2t_1t_3t_1$, N$t_1t_3t_2t_1t_3t_4$, N$t_1t_3t_2t_1t_3t_5$, and N$t_1t_3t_2t_1t_3t_6$ right cosets, all which belong to the same double coset.

Now N$t_1t_3t_2t_1t_3t_6$ = N$t_1t_3t_2t_1$e by notation, therefore N$t_1t_3t_2t_1t_3t_6$ $\in$ [1, 3, 2, 1]. This proves Nt$_1t_3t_2t_1t_3t_1$, N$t_1t_3t_2t_1t_3t_1$, N$t_1t_3t_2t_1t_3t_4$, N$t_1t_3t_2t_1t_3t_5$ $\in$ [1, 3, 2, 1].

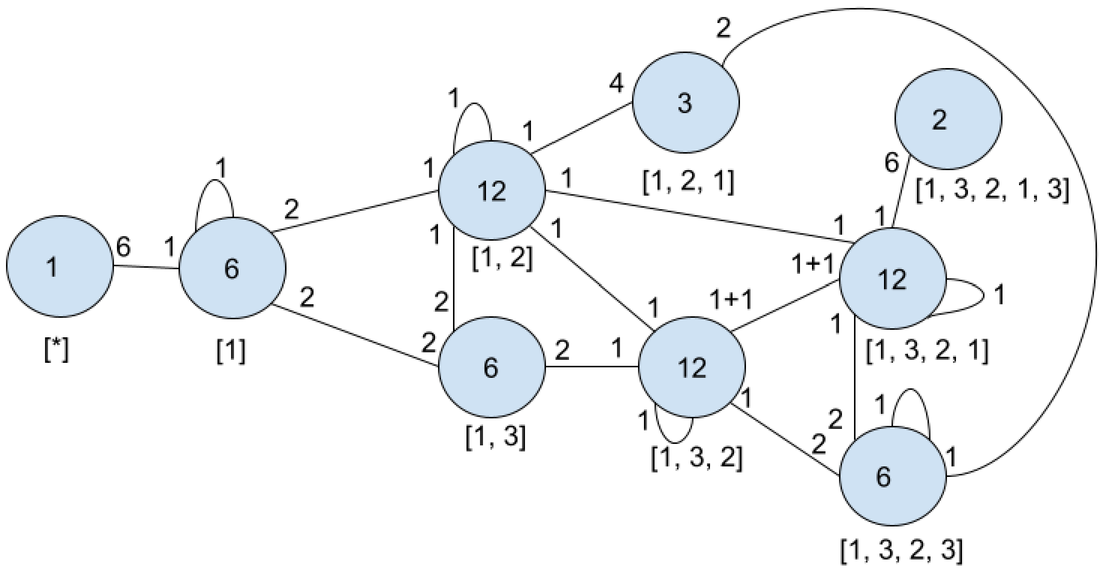

Figure 4.9: Complete Cayley Diagram of 3:(2×S_5) Over D_{12}.

From the Cayley diagram of G over N shown above we conclude |G| = (|[*]| + |[1]| + |[1, 2]| + |[1, 3]| + |[1, 2, 1]| + |[1, 3, 2]| + |[1, 3, 2, 1]| + |[1, 3, 2, 1]| + |[1, 3, 2, 1, 3]|)|N|. Therefore |G| $\leq$ (1 + 6 + 12 + 6 + 3 + 12 +12 + 6 + 2)(12) = 60(12) = 720.

4.3 A Permutation Representation of 3:(2× S_5)

The Cayley diagram of G over N shows that there are 60 distinct right cosets. To find a permutation representation of G we assign a numeric label from 1 to 60 to each right cosets obtained from Cayley diagram. Next, conjugate all of the 60 right cosets by x to get a permutation of the 60 right cosets (corresponding numeric values are recorded) . Of course, this gives a permutation of S_{60}. Repeating the process for y will produce a second permutation of S_{60}. To find a third permutation, multiply each right coset by t, and find the equivalent coset to the products.

The following proof is an example of how to find all equivalent right cosets conjugated by x. All of the proofs for the mappings in the table below are similar to the proof below.

Lemma 4.37. $Nt_6t_1t_6 = Nt_1t_6t_1 = Nt_3t_4t_3 = Nt_4t_3t_4$.

Proof. $\mathrm{N}t_1t_2t_1 = \mathrm{N}(t_1t_2t_1)^{\mathrm{n}}\ \forall \mathrm{n} \in \mathrm{N}^{(1,2,1)} \geq \langle$ e, (1, 4)(2, 5)(3, 6), (1, 2)(3, 6)(4, 5), (1, 5)(2, 4)$\rangle \implies \mathrm{N}t_1t_2t_1 = \mathrm{N}t_5t_4t_5 = \mathrm{N}t_4t_5t_4 = \mathrm{N}t_2t_1t_2 \implies$
$\mathrm{N}(t_1t_2t_1)^{\mathrm{x}} = \mathrm{N}t_5t_4t_5)^{\mathrm{x}} = \mathrm{N}(t_4t_5t_4)^{\mathrm{x}} = \mathrm{N}(t_2t_1t_2)^{\mathrm{x}} \implies$
$\mathrm{N}t_6t_1t_6 = \mathrm{N}t_1t_6t_1 = \mathrm{N}t_3t_4t_3 = \mathrm{N}t_4t_3t_4$. □

Next we will construct f(x) by conjugating the right cosets of our group by x.

Label	Right Coset	Right Coset conjugated by x.	Label
1	N	N	1
2	$\mathrm{N}t_1$	$\mathrm{N}t_2$	4

3	Nt_4	Nt_5	6
4	Nt_2	Nt_3	7
5	Nt_6	Nt_1	2
6	Nt_5	Nt_6	5
7	Nt_3	Nt_4	3
8	Nt_2t_1	Nt_3t_2	16
9	$Nt_1t_5 = Nt_2t_4$	$Nt_2t_6 = Nt_3t_5$	19
10	Nt_6t_1	Nt_1t_2	21
11	$Nt_6t_4 = Nt_1t_3$	$Nt_1t_5 = Nt_2t_4$	9
12	$Nt_5t_1 = Nt_4t_2$	$Nt_6t_2 = Nt_5t_3$	20
13	Nt_5t_4	Nt_6t_5	22
14	$Nt_3t_1 = Nt_4t_6$	$Nt_4t_2 = Nt_5t_1$	12
15	Nt_3t_4	Nt_4t_5	25
16	Nt_3t_2	Nt_4t_3	24
17	Nt_5t_6	Nt_6t_1	10
18	Nt_1t_6	Nt_2t_1	8
19	$Nt_2t_6 = Nt_3t_5$	$Nt_3t_1 = Nt_4t_6$	14
20	$Nt_6t_2 = Nt_5t_3$	$Nt_1t_3 = Nt_6t_4$	11
21	Nt_1t_2	Nt_2t_3	23
22	Nt_6t_5	Nt_1t_6	18
23	Nt_2t_3	Nt_3t_4	15
24	Nt_4t_3	Nt_5t_4	13
25	Nt_4t_5	Nt_5t_6	17

26	N$t_4t_6t_5t_4$	N$t_5t_1t_6t_5$	40
27	N$t_4t_6t_5$	N$t_5t_1t_6$	43
28	N$t_4t_2t_3t_4$	N$t_5t_3t_4t_5$	45
29	N$t_4t_2t_3$	N$t_5t_3t_4$	34
30	N$t_6t_1t_6$ = N$t_1t_6t_1$ = N$t_3t_4t_3$ = N$t_4t_3t_4$	N$t_1t_2t_1$ = N$t_5t_4t_5$ = N$t_4t_5t_4$ = N$t_2t_1t_2$	35
31	N$t_2t_6t_1$	N$t_3t_1t_2$	44
32	N$t_3t_5t_4$	N$t_4t_6t_5$	27
33	N$t_6t_2t_1$	N$t_1t_3t_2$	36
34	N$t_5t_3t_4$	N$t_6t_4t_5$	48
35	N$t_2t_1t_2$ = N$t_4t_5t_4$ = N$t_5t_4t_5$ = N$t_1t_2t_1$	N$t_3t_2t_3$ = N$t_6t_5t_6$ = N$t_5t_6t_5$ = N$t_2t_3t_2$	46
36	N$t_1t_3t_2$	N$t_2t_4t_3$	47
37	N$t_1t_3t_2t_1$	N$t_2t_4t_3t_2$	49
38	N$t_1t_5t_6$	N$t_2t_6t_1$	31
39	N$t_1t_5t_6t_1$	N$t_2t_6t_1t_2$	52
40	N$t_5t_1t_6t_5$	N$t_6t_2t_1t_6$	51
41	N$t_3t_1t_2t_3$	N$t_4t_2t_3t_4$	28
42	N$t_3t_5t_4t_3$	N$t_4t_6t_5t_4$	26
43	N$t_5t_1t_6$	N$t_6t_2t_1$	33
44	N$t_3t_1t_2$	N$t_4t_2t_3$	29
45	N$t_5t_3t_4t_5$	N$t_6t_4t_5t_6$	50
46	N$t_3t_2t_3$ = N$t_6t_5t_6$ = N$t_5t_6t_5$ = N$t_2t_3t_2$	N$t_4t_3t_4$ = N$t_3t_4t_3$ = N$t_1t_6t_1$ = N$t_6t_1t_6$	30

47	N$t_2t_4t_3$	N$t_3t_5t_4$	32
48	N$t_6t_4t_5$	N$t_1t_5t_6$	38
49	N$t_2t_4t_3t_2$	N$t_3t_5t_4t_3$	42
50	N$t_6t_4t_5t_6$	N$t_1t_5t_6t_1$	39
51	N$t_6t_2t_1t_6$	N$t_1t_3t_2t_1$	37
52	N$t_2t_6t_1t_2$	N$t_3t_1t_2t_3$	41
53	N$t_5t_1t_6t_5t_1$ = N$t_6t_2t_1t_6t_2$ = N$t_1t_3t_2t_1t_3$ = N$t_2t_4t_3t_2t_4$ = N$t_4t_6t_5t_4t_6$ = N$t_3t_5t_4t_3t_5$	N$t_6t_2t_1t_6t_2$ = N$t_1t_3t_2t_1t_3$ = N$t_2t_4t_3t_2t_4$ = N$t_3t_5t_4t_3t_5$ = N$t_5t_1t_6t_5t_1$ = N$t_4t_6t_5t_4t_6$	53
54	N$t_3t_1t_2t_3t_1$ = N$t_4t_2t_3t_4t_2$ = N$t_5t_3t_4t_5t_3$ = N$t_6t_4t_5t_6t_4$ = N$t_2t_6t_1t_2t_6$ = N$t_1t_5t_6t_1t_5$	N$t_4t_2t_3t_4t_2$ = N$t_5t_3t_4t_5t_3$ = N$t_6t_4t_5t_6t_4$ = N$t_1t_5t_6t_1t_5$ = N$t_3t_1t_2t_3t_1$ = N$t_2t_6t_1t_2t_6$	54
55	N$t_2t_6t_1t_6$ = N$t_2t_4t_3t_4$	N$t_3t_1t_2t_1$ = N$t_3t_5t_4t_5$	57
56	N$t_5t_1t_6t_1$ = N$t_5t_3t_4t_3$	N$t_6t_2t_1t_2$ = N$t_6t_4t_5t_4$	58
57	N$t_3t_1t_2t_1$ = N$t_3t_5t_4t_5$	N$t_4t_2t_3t_2$ = N$t_4t_6t_5t_6$	59
58	N$t_6t_2t_1t_2$ = N$t_6t_4t_5t_4$	N$t_1t_3t_2t_3$ = N$t_1t_5t_6t_5$	60
59	N$t_4t_2t_3t_2$ = N$t_4t_6t_5t_6$	N$t_5t_3t_4t_3$ = N$t_5t_1t_6t_1$	56
60	N$t_1t_5t_6t_5$ = N$t_1t_3t_2t_3$	N$t_2t_6t_1t_6$ = N$t_2t_4t_3t_4$	55

Table 4.3: Construction of f(x).

There exists a homomorphism f(G) $\rightarrow$ S_{60}. From the table we extract the permutation f(x) = (2, 4, 7, 3, 6, 5)(8, 16, 24, 13, 22, 18)(9, 19, 14, 12, 20, 11)(10, 21, 23, 15, 25, 17)(26, 40, 51, 37, 49, 42)(27, 43, 33, 36, 47, 32)(28, 45, 50, 39, 52, 41)(29, 34, 48, 38, 31, 44)(30, 35, 46)(55, 57, 59, 56, 58, 60).

Next we will construct f(y) by conjugating the right cosets of our group by y.

Label	Right Coset	Right Coset conjugated by y.	Label
1	N	N	1
2	Nt_1	Nt_6	5
3	Nt_4	Nt_3	7
4	Nt_2	Nt_5	6
5	Nt_6	Nt_1	2
6	Nt_5	Nt_2	4
7	Nt_3	Nt_4	3
8	Nt_2t_1	Nt_5t_6	17
9	Nt_2t_4 = Nt_1t_5	Nt_5t_3 = Nt_6t_2	20
10	Nt_6t_1	Nt_1t_6	18
11	Nt_1t_3 = Nt_6t_4	Nt_6t_4 = Nt_1t_3	11
12	Nt_5t_1 = Nt_4t_2	Nt_2t_6 = Nt_3t_5	19
13	Nt_5t_4	Nt_2t_3	23
14	Nt_4t_6 = Nt_3t_1	Nt_3t_1 = Nt_4t_6	14
15	Nt_3t_4	Nt_4t_3	24
16	Nt_3t_2	Nt_4t_5	25
17	Nt_5t_6	Nt_2t_1	8
18	Nt_1t_6	Nt_6t_1	10
19	Nt_3t_5 = Nt_2t_6	Nt_4t_2 = Nt_5t_1	12
20	Nt_6t_2 = Nt_5t_3	Nt_1t_5 = Nt_2t_4	9

21	Nt_1t_2	Nt_6t_5	22
22	Nt_6t_5	Nt_1t_2	21
23	Nt_2t_3	Nt_5t_4	13
24	Nt_4t_3	Nt_3t_4	15
25	Nt_4t_5	Nt_3t_2	16
26	$Nt_4t_6t_5t_4$	$Nt_3t_1t_2t_3$	41
27	$Nt_4t_6t_5$	$Nt_3t_1t_2$	44
28	$Nt_4t_2t_3t_4$	$Nt_3t_5t_4t_3$	42
29	$Nt_4t_2t_3$	$Nt_3t_5t_4$	32
30	$Nt_6t_1t_6 = Nt_1t_6t_1 = Nt_3t_4t_3 = Nt_4t_3t_4$	$Nt_1t_6t_1 = Nt_6t_1t_6 = Nt_4t_3t_4 = Nt_3t_4t_3$	30
31	$Nt_2t_6t_1$	$Nt_5t_1t_6$	43
32	$Nt_3t_5t_4$	$Nt_4t_2t_3$	29
33	$Nt_6t_2t_1$	$Nt_1t_5t_6$	38
34	$Nt_5t_3t_4$	$Nt_2t_4t_3$	47
35	$Nt_1t_2t_1 = Nt_5t_4t_5 = Nt_4t_5t_4 = Nt_2t_1t_2$	$Nt_6t_5t_6 = Nt_3t_2t_3 = Nt_2t_3t_2 = Nt_5t_6t_5$	46
36	$Nt_1t_3t_2$	$Nt_6t_4t_5$	48
37	$Nt_1t_3t_2t_1$	$Nt_6t_4t_5t_6$	50
38	$Nt_1t_5t_6$	$Nt_6t_2t_1$	33
39	$Nt_1t_5t_6t_1$	$Nt_6t_2t_1t_6$	51
40	$Nt_5t_1t_6t_5$	$Nt_2t_6t_1t_2$	52
41	$Nt_3t_1t_2t_3$	$Nt_4t_6t_5t_4$	26
42	$Nt_3t_5t_4t_3$	$Nt_4t_2t_3t_4$	28

43	N$t_5t_1t_6$	N$t_2t_6t_1$	31
44	N$t_3t_1t_2$	N$t_4t_6t_5$	27
45	N$t_5t_3t_4t_5$	N$t_2t_4t_3t_2$	49
46	N$t_5t_6t_5$ = N$t_2t_3t_2$ = N$t_3t_2t_3$ = N$t_6t_5t_6$	N$t_2t_1t_2$ = N$t_4t_5t_4$ = N$t_5t_4t_5$ = N$t_1t_2t_1$	35
47	N$t_2t_4t_3$	N$t_5t_3t_4$	34
48	N$t_6t_4t_5$	N$t_1t_3t_2$	36
49	N$t_2t_4t_3t_2$	N$t_5t_3t_4t_5$	45
50	N$t_6t_4t_5t_6$	N$t_1t_3t_2t_1$	37
51	N$t_6t_2t_1t_6$	N$t_1t_5t_6t_1$	39
52	N$t_2t_6t_1t_2$	N$t_5t_1t_6t_5$	40
53	N$t_1t_3t_2t_1t_3$ = N$t_2t_4t_3t_2t_4$ = N$t_3t_5t_4t_3t_5$ = N$t_4t_6t_5t_4t_6$ = N$t_6t_2t_1t_6t_2$ = N$t_5t_1t_6t_5t_1$	N$t_6t_4t_5t_6t_4$ = N$t_1t_5t_6t_1t_5$ = N$t_2t_6t_1t_2t_6$ = N$t_3t_1t_2t_3t_1$ = N$t_5t_3t_4t_5t_3$ = N$t_4t_2t_3t_4t_2$	54
54	N$t_6t_4t_5t_6t_4$ = N$t_1t_5t_6t_1t_5$ = N$t_2t_6t_1t_2t_6$ = N$t_3t_1t_2t_3t_1$ = N$t_5t_3t_4t_5t_3$ = N$t_4t_2t_3t_4t_2$	N$t_1t_3t_2t_1t_3$ = N$t_2t_4t_3t_2t_4$ = N$t_3t_5t_4t_3t_5$ = N$t_4t_6t_5t_4t_6$ = N$t_6t_2t_1t_6t_2$ = N$t_5t_1t_6t_5t_1$	53
55	N$t_2t_4t_3t_4$ = N$t_2t_6t_1t_6$	N$t_5t_3t_4t_3$ = N$t_5t_1t_6t_1$	56
56	N$t_5t_3t_4t_3$ = N$t_5t_1t_6t_1$	N$t_2t_4t_3t_4$ = N$t_2t_6t_1t_6$	55
57	N$t_3t_5t_4t_5$ = N$t_3t_1t_2t_1$	N$t_4t_2t_3t_2$ = N$t_4t_6t_5t_6$	59
58	N$t_6t_4t_5t_4$ = N$t_6t_2t_1t_2$	N$t_1t_3t_2t_3$ = N$t_1t_5t_6t_5$	60
59	N$t_4t_6t_5t_6$ = N$t_4t_2t_3t_2$	N$t_3t_1t_2t_1$ = N$t_3t_5t_4t_5$	57
60	N$t_1t_3t_2t_3$ = N$t_1t_5t_6t_5$	N$t_6t_4t_5t_4$ = N$t_6t_2t_1t_2$	58

Table 4.4: Construction of f(y).

From the table we extract the permutation f(y) = (2, 5)(3, 7)(4, 6)(8, 17)(9, 20)(10, 18)(12, 19)(13, 23)(15, 24)(16, 25)(21, 22)(26, 41)(27, 44)(28, 42)(29, 32)(31, 43)(33, 38)(34, 47)(35, 46)(36,48)(37, 50)(39, 51)(40, 52)(45, 49)(53, 54)(55, 56)(57, 59)(58, 60).

Next we will construct f(t) by multiplying the right cosets of our group by t_1.

Label	Right Coset	Right Coset multiplied by t	Label
1	N	Nt_1	1
2	Nt	Nt_1t_1 = Nt_4	3
3	Nt_4	Nt_4t_1 = N	1
4	Nt_2	Nt_2t_1	8
5	Nt_6	Nt_6t_1	10
6	Nt_5	Nt_5t_1 = Nt_4t_2	12
7	Nt_3	Nt_3t_1 = Nt_4t_6	14
8	Nt_2t_1	Nt_2t_4 = Nt_1t_5	9
9	Nt_2t_4 = Nt_1t_5	N$t_2t_4t_1$ = N$t_2t_6t_3$	4
10	Nt_6t_1	Nt_6t_4 = Nt_1t_3	11
11	Nt_1t_3 = Nt_6t_4	N$t_1t_3t_1$ = N$t_5t_3t_5$	5
12	Nt_5t_1 = Nt_4t_2	N$t_5t_1t_1$= Nt_5t_4	13
13	Nt_5t_4	N$t_5t_4t_1$ = N$t_5t_6t_3$ = Nt_5	6
14	Nt_4t_6 = Nt_3t_1	N$t_4t_6t_1$	15
15	Nt_3t_4	N$t_3t_4t_1$ = N$t_3t_2t_5$ = Nt_3	7
16	Nt_3t_2	N$t_3t_2t_1$	26

17	Nt_5t_6	$Nt_5t_6t_1$	28
18	Nt_1t_6	$Nt_1t_6t_1 = Nt_6t_1t_6 = Nt_4t_3t_4 = Nt_3t_4t_3$	30
19	$Nt_3t_5 = Nt_2t_6$	$Nt_3t_5t_1$	31
20	$Nt_6t_2 = Nt_5t_3$	$Nt_6t_2t_1$	33
21	Nt_1t_2	$Nt_1t_2t_1 = Nt_5t_4t_5 = Nt_4t_5t_4 = Nt_2t_1t_2$	35
22	Nt_6t_5	$Nt_6t_5t_1$	36
23	Nt_2t_3	$Nt_2t_3t_1$	38
24	Nt_4t_3	$Nt_4t_3t_1$	18
25	Nt_4t_5	$Nt_4t_5t_1$	21
26	$Nt_4t_6t_5t_4$	$Nt_4t_6t_5t_4t_1 = Nt_4t_6t_5$	27
27	$Nt_4t_6t_5$	$Nt_4t_6t_5t_1$	16
28	$Nt_4t_2t_3t_4$	$Nt_4t_2t_3t_4t_1 = Nt_4t_2t_3$	29
29	$Nt_4t_2t_3$	$Nt_4t_2t_3t_1$	17
30	$Nt_6t_1t_6 = Nt_1t_6t_1 = Nt_3t_4t_3 = Nt_4t_3t_4$	$Nt_6t_1t_6t_1$	24
31	$Nt_2t_6t_1$	$Nt_2t_6t_1t_1 = Nt_2t_6t_4$	32
32	$Nt_3t_5t_4$	$Nt_3t_5t_4t_1 = Nt_2t_6t_1t_4 = Nt_3t_5 = Nt_2t_6$	19
33	$Nt_6t_2t_1$	$Nt_6t_2t_1t_1 = Nt_6t_2t_4$	34
34	$Nt_5t_3t_4$	$Nt_5t_3t_4t_1 = Nt_6t_2t_1t_4 = Nt_5t_3 = Nt_6t_2$	20
35	$Nt_1t_2t_1 = Nt_5t_4t_5 = Nt_4t_5t_4 = Nt_2t_1t_2$	$Nt_1t_2t_1t_1 = Nt_1t_2t_4$	25
36	$Nt_1t_3t_2$	$Nt_1t_3t_2t_1$	37

37	$Nt_1t_3t_2t_1$	$Nt_1t_3t_2t_1t_1$	22
38	$Nt_1t_5t_6$	$Nt_1t_5t_6t_1$	39
39	$Nt_1t_5t_6t_1$	$Nt_1t_5t_6t_1t_1 = Nt_1t_5t_6t_4$	23
40	$Nt_5t_1t_6t_5$	$Nt_5t_1t_6t_5t_1 = Nt_6t_2t_1t_6t_2 = Nt_1t_3t_2t_1t_3 = Nt_2t_4t_3t_2t_4 = Nt_4t_6t_5t_4t_6 = Nt_3t_5t_4t_3t_5$	53
41	$Nt_3t_1t_2t_3$	$Nt_3t_1t_2t_3t_1 = Nt_4t_2t_3t_4t_2 = Nt_5t_3t_4t_5t_3 = Nt_6t_4t_5t_6t_4 = Nt_2t_6t_1t_2t_6 = Nt_1t_5t_6t_1t_5$	54
42	$Nt_3t_5t_4t_3$	$Nt_3t_5t_4t_3t_1 = Nt_1t_5t_6t_1t_3$	55
43	$Nt_5t_1t_6$	$Nt_5t_1t_6t_1 = Nt_5t_3t_4t_3$	56
44	$Nt_3t_1t_2$	$Nt_3t_1t_2t_1 = Nt_3t_5t_4t_5$	57
45	$Nt_5t_3t_4t_5$	$Nt_5t_3t_4t_5t_1 = Nt_1t_3t_2t_1t_5$	58
46	$Nt_5t_6t_5 = Nt_2t_3t_2 = Nt_3t_2t_3 = Nt_6t_5t_6$	$Nt_5t_6t_5t_1 = Nt_3t_2t_3t_1$	59
47	$Nt_2t_4t_3$	$Nt_2t_4t_3t_1$	42
48	$Nt_6t_4t_5$	$Nt_6t_4t_5t_1$	45
49	$Nt_2t_4t_3t_2$	$Nt_2t_4t_3t_2t_1$	40
50	$Nt_6t_4t_5t_6$	$Nt_6t_4t_5t_6t_1$	41
51	$Nt_6t_2t_1t_6$	$Nt_6t_2t_1t_6t_1$	43
52	$Nt_2t_6t_1t_2$	$Nt_2t_6t_1t_2t_1$	44
53	$Nt_1t_3t_2t_1t_3 = Nt_2t_4t_3t_2t_4 = Nt_3t_5t_4t_3t_5 = Nt_4t_6t_5t_4t_6 = Nt_6t_2t_1t_6t_2 = Nt_5t_1t_6t_5t_1$	$Nt_1t_3t_2t_1t_3t_1$	49

54	$Nt_6t_4t_5t_6t_4 = Nt_1t_5t_6t_1t_5 = Nt_2t_6t_1t_2t_6 = Nt_3t_1t_2t_3t_1 = Nt_5t_3t_4t_5t_3 = Nt_4t_2t_3t_4t_2$	$Nt_6t_4t_5t_6t_4t_1 = Nt_6t_4t_5t_6$	50
55	$Nt_2t_4t_3t_4 = Nt_2t_6t_1t_6$	$Nt_2t_4t_3t_4t_1 = Nt_2t_4t_3$	47
56	$Nt_5t_3t_4t_3 = Nt_5t_1t_6t_1$	$Nt_5t_3t_4t_3t_1$	51
57	$Nt_3t_5t_4t_5 = Nt_3t_1t_2t_1$	$Nt_3t_5t_4t_5t_1$	52
58	$Nt_6t_4t_5t_4 = Nt_6t_2t_1t_2$	$Nt_6t_4t_5t_4t_1$	48
59	$Nt_4t_6t_5t_6 = Nt_4t_2t_3t_2$	$Nt_4t_6t_5t_6t_1 = Nt_4t_2t_3t_2t_1$	60
60	$Nt_1t_3t_2t_3 = Nt_1t_5t_6t_5$	$Nt_1t_3t_2t_3t_1 = Nt_4t_6t_5t_6t_4 = Nt_1t_5t_6t_5t_1 = Nt_4t_2t_3t_2t_4$	46

Table 4.5: Construction of f(t).

From the table we extract the permutation f(t)= (1, 2, 3)(4, 8, 9)(5, 10, 11)(6, 12, 13)(7, 14, 15)(16, 26, 27)(17, 28, 29)(18, 30, 24)(19, 31, 32)(20, 33, 34)(21, 35, 25)(22, 36, 37)(23, 38, 39)(40, 53, 49)(41, 54, 50)(42, 55, 47)(43, 56, 51)(44, 57, 52)(45, 58, 48)(46, 59, 60).

The permutations f(x), f(y), f(t) $\in S_{60}$ represent x, y, t $\in$ G, therefore $|G| \geq |\langle$ f(x), f(y), f(t) $\rangle| = 720$. Combining the previous with that fact $|G| \leq 720$ to obtain $|G| = 720$. Now $f(t^xt^y)^2 = [f(t^x)f(t^y)]^2 = [f(t)^{f(x)}f(t)^{f(y)}]^2$ = [(1, 2, 3)(4, 8, 9)(5, 10, 11)(6, 12, 13)(7, 14, 15)(16, 26, 27)(17, 28, 29)(18, 30, 24)(19, 31, 32)(20, 33, 34)(21, 35, 25)(22, 36, 37)(23, 38, 39)(40, 53, 49)(41, 54, 50)(42, 55, 47)(43, 56, 51)(44, 57, 52)(45, 58, 48)(46, 59, 60)$^{(2, 4, 7, 3, 6, 5)(8, 16, 24, 13, 22, 18)(9, 19, 14, 12, 20, 11)(10, 21, 23, 15, 25, 17)(26, 40, 51, 37, 49, 42)(27, 43, 33, 36, 47, 32)(28, 45, 50, 39, 52, 41)(29, 34, 48, 38, 31, 44)(30, 35, 46)(55, 57, 59, 56, 58, 60)}$(1, 2, 3)(4, 8, 9)(5, 10, 11)(6, 12, 13)(7, 14, 15)(16, 26, 27)(17, 28, 29)(18, 30, 24)(19, 31, 32)(20, 33, 34)(21, 35, 25)(22, 36, 37)(23, 38, 39)(40, 53, 49)(41, 54, 50)(42, 55, 47)(43, 56, 51)(44, 57, 52)(45, 58, 48)(46, 59, 60)$^{(2, 5)(3, 7)(4, 6)(8, 17)(9, 20)(10, 18)(12, 19)(13, 23)(15, 24)(16, 25)(21, 22)(26, 41)(27, 44)(28, 42)(29, 32)(31, 43)(33, 38)(34, 47)(35, 46)(36,48)(37, 50)(39, 51)(40, 52)(45, 49)(53, 54)(55,56)(57, 59)(58, 60)}$]^2 = [(1, 4, 6)(2, 21, 9)(3, 12, 25)(5, 20, 22)(7, 16, 19)(8, 35, 13)(10, 45, 34)(11, 36, 48)(14, 44, 27)(15, 31, 52)(17, 23, 46)(18, 47, 49)(24, 40, 43)(26, 57, 32)(28, 54, 39)(29, 59, 41)(30, 56, 55)(33, 58, 37)(38, 50, 60)(42, 51, 53)*(1, 5, 7)(2, 18, 11)(3, 14, 24)(4, 19, 23)(6, 17, 20)(8, 42, 32)(9, 38, 47)(10, 30, 15)(12, 43, 29)(13, 33, 51)(16, 22, 46)(21, 48, 50)(25, 41, 44)(26, 53, 37)(27, 59, 40)(28, 56, 34)(31, 55, 39)(35, 57, 58)(36, 49, 60)(45, 52, 54)]^2 = [(1, 19)(2, 48)(3, 43)(4, 17)(5, 6)(7, 22)(8, 57)(9, 18)(10, 52)(11, 49)(12, 41)(13, 42)(14, 25)(15, 55)(16, 23)(20, 46)(21, 38)(24, 27)(26, 58)(28, 45)(29, 40)(30, 34)(31, 54)(32, 53)(33, 35)(36, 50)(37, 51)(39, 56)(44, 59)(47, 60)] = e. Hence, G $\cong \langle$ f(x), f(y), f(t) $\rangle$.

Chapter 5

Double Coset Enumerations

5.1 Definitions and Theorems

Definition 5.1. *A subgroup H≤G is a* ***maximal normal subgroup*** *of G if there is no normal subgroup N of G with H<N<G. [Rot95]*

Lemma 5.2. *If H and K are finite Groups of G and x is an element of G, then* $|HxK| = |K| \times |K|/|K^x \cap H|$. *[Curt07]*

5.2 Double Coset Enumeration of $L_2(25)$ Over S_5

A symmetric presentation of the progenitor $5^{*24} : S_5$ is G = ⟨ x^5, y^2,$(x^{-1}y)^4$, $(xyx^{-2}yx)^2$, t^5, (t, xyx^2yx^{-1}), $t^{x^{-1}y}= t^3$, $(x,tt^{x^2})^2$, $(x^y t)^4$, $x^{-1}yx^{-1}yxyxtxt^2xt$⟩. Let xx = (1, 2, 4, 18, 9)(3, 13, 14, 16, 24)(6, 21, 7, 8, 10)(12, 15, 19, 20, 22) and let yy = (1, 21)(2, 5)(3, 19)(4, 6)(7, 15)(8, 11)(9, 13)(10, 12)(14, 17)(16, 18)(20, 23)(22, 24). Let N = $\langle xx, yy \rangle$. MAGMA gives N ≅ S_5. Let t = t_1.

Theorem 5.3. $L_2(25) \cong G = \langle x^5, y^2, (x^{-1}y)^4, (xyx^{-2}yx)^2, t^5, (t, xyx^2yx^{-1}), t^{x^{-1}y} = t^3, (x, tt^{x^2})^2, (x^yt)^4, x^{-1}yx^{-1}yxyxtxt^2xt\rangle$.

We perform our technique of manual double coset enumeration to G over N to construct G and prove that G $\cong L_2(25)$. Before we start our double coset enumeration, the relations need to be expanded.

Now $|t| = |t_1| = 5$ and $t_{13} = t^{x^{-1}y} = t^3$, therefore $(t_1{}^3)^N = (t_{13})^N$, which yields the following:

Inverses:	Powers:	Splits:
$t_1 = t_{19}{}^{-1}$	$t_1{}^2 = t_7$, $t_1{}^3 = t_{13}$, $t_1{}^4 = t_{19}$	$t_1 = t_7t_{19}$
$t_2 = t_{20}{}^{-1}$	$t_2{}^2 = t_8$, $t_2{}^3 = t_{14}$, $t_2{}^4 = t_{20}$	$t_2 = t_8t_{20}$
$t_3 = t_{21}{}^{-1}$	$t_3{}^2 = t_9$, $t_3{}^3 = t_{15}$, $t_3{}^4 = t_{21}$	$t_3 = t_9t_{21}$
$t_4 = t_{22}{}^{-1}$	$t_4{}^2 = t_{10}$, $t_4{}^3 = t_{16}$, $t_4{}^4 = t_{22}$	$t_4 = t_{10}t_{22}$
$t_5 = t_{23}{}^{-1}$	$t_5{}^2 = t_{11}$, $t_5{}^3 = t_{17}$, $t_5{}^4 = t_{23}$	$t_5 = t_{11}t_{23}$
$t_6 = t_{24}{}^{-1}$	$t_6{}^2 = t_{12}$, $t_6{}^3 = t_{18}$, $t_6{}^4 = t_{24}$	$t_6 = t_{12}t_{24}$
$t_7 = t_{13}{}^{-1}$	$t_7{}^2 = t_{19}$, $t_7{}^3 = t_1$, $t_7{}^4 = t_{13}$	$t_7 = t_{19}t_{13}$
$t_8 = t_{14}{}^{-1}$	$t_8{}^2 = t_{20}$, $t_8{}^3 = t_2$, $t_8{}^4 = t_{14}$	$t_8 = t_{20}t_{14}$
$t_9 = t_{15}{}^{-1}$	$t_9{}^2 = t_{21}$, $t_9{}^3 = t_3$, $t_9{}^4 = t_{15}$	$t_9 = t_{21}t_{15}$
$t_{10} = t_{16}{}^{-1}$	$t_{10}{}^2 = t_{22}$, $t_{10}{}^3 = t_4$, $t_{10}{}^4 = t_{16}$	$t_{10} = t_{22}t_{16}$
$t_{11} = t_{17}{}^{-1}$	$t_{11}{}^2 = t_{23}$, $t_{11}{}^3 = t_5$, $t_{11}{}^4 = t_{17}$	$t_{11} = t_{23}t_{17}$
$t_{12} = t_{18}{}^{-1}$	$t_{12}{}^2 = t_{24}$, $t_{12}{}^3 = t_6$, $t_{12}{}^4 = t_{18}$	$t_{12} = t_{24}t_{18}$
$t_{13} = t_7{}^{-1}$	$t_{13}{}^2 = t_1$, $t_{13}{}^3 = t_{19}$, $t_{13}{}^4 = t_7$	$t_{13} = t_1t_7$
$t_{14} = t_8{}^{-1}$	$t_{14}{}^2 = t_2$, $t_{14}{}^3 = t_{20}$, $t_{14}{}^4 = t_8$	$t_{14} = t_2t_8$
$t_{15} = t_9{}^{-1}$	$t_{15}{}^2 = t_3$, $t_{15}{}^3 = t_{21}$, $t_{15}{}^4 = t_9$	$t_{15} = t_3t_9$

$t_{16} = t_{10}^{-1}$	$t_{16}^2 = t_4$, $t_{16}^3 = t_{22}$, $t_{16}^4 = t_{10}$	$t_{16} = t_4 t_{10}$
$t_{17} = t_{11}^{-1}$	$t_{17}^2 = t_5$, $t_{17}^3 = t_{23}$, $t_{17}^4 = t_{11}$	$t_{17} = t_5 t_{11}$
$t_{18} = t_{12}^{-1}$	$t_{18}^2 = t_6$, $t_{18}^3 = t_{24}$, $t_{18}^4 = t_{12}$	$t_{18} = t_6 t_{12}$
$t_{19} = t_1^{-1}$	$t_{19}^2 = t_{13}$, $t_{19}^3 = t_7$, $t_{19}^4 = t_1$	$t_{19} = t_{13} t_1$
$t_{20} = t_2^{-1}$	$t_{20}^2 = t_{14}$, $t_{20}^3 = t_8$, $t_{20}^4 = t_2$	$t_{20} = t_{14} t_2$
$t_{21} = t_3^{-1}$	$t_{21}^2 = t_{15}$, $t_{21}^3 = t_9$, $t_{21}^4 = t_3$	$t_{21} = t_{15} t_3$
$t_{22} = t_4^{-1}$	$t_{22}^2 = t_{16}$, $t_{22}^3 = t_{10}$, $t_{22}^4 = t_4$	$t_{22} = t_{16} t_4$
$t_{23} = t_5^{-1}$	$t_{23}^2 = t_{17}$, $t_{23}^3 = t_{11}$, $t_{23}^4 = t_5$	$t_{23} = t_{17} t_5$
$t_{24} = t_6^{-1}$	$t_{24}^2 = t_{18}$, $t_{24}^3 = t_{12}$, $t_{24}^4 = t_6$	$t_{24} = t_{18} t_6$

Table 5.1: Powers of t.

Lemma 5.4. $t_9 t_2 t_{22} t_{19} = t_1 t_4 t_{20} t_{15}$.

Proof. From the relation $(xt_1t_4x^{-1}t_4^{-1}t_1^{-1})^2 = e \implies$

$(xt_1t_4x^{-1}t_4^{-1}t_1^{-1})^2 = (xt_1t_4x^{-1}t_{22}t_{19})^2 \implies$

$(x(x^{-1}(t_1t_4)t^{x^{-1}})t_{22}t_{19})^2 = (t_9t_2t_{22}t_{19})^2 = e \implies$

$t_9t_2t_{22}t_{19}t_9t_2t_{22}t_{19} = e \implies$

$t_9t_2t_{22}t_{19} = t_{19}^{-1}t_{22}^{-1}t_2^{-1}t_9^{-1} \implies$

$t_9t_2t_{22}t_{19} = t_1t_4t_{20}t_{15}$ by substituting the inverses. □

Lemma 5.5. *$t_{19}t_9 = yx^{-1}yt_{12}t_{11}$ and $t_{12}t_{11} = yxyt_{19}t_9$.*

Proof. From the relation $(x^y t_1)^4 = e \implies$

Let $p = x^y \sim (1,15,11,12,4)(3,23,24,10,7)(5,6,16,13,21)(9,17,18,22,19) \implies$

$pt_1pt_1p(t_1p)t_1 = pt_1pt_1p(pt_1{}^{p})t_1 \implies$

$pt_1p(t_1p^2)t_1{}^{p}t_1 = pt_1p(p^2t_1{}^{p^2})t_1{}^{p}t_1 \implies$

$p(t_1p^3)t_1{}^{p^2}t_1{}^{p}t_1 = p(p^3t_1{}^{p^3})t_1{}^{p^2}t_1{}^{p}t_1 \implies$

$p^4t_1{}^{p^3}t_1{}^{p^2}t_1{}^{p}t_1 = yx^{-1}yt_{12}t_{11}t_{15}t_1 = e \implies$

$yx^{-1}yt_{12}t_{11}t_{15}t_1t_1{}^{-1}t_{15}{}^{-1} = et_1{}^{-1}t_{15}{}^{-1} \implies$

$yx^{-1}yt_{12}t_{11} = t_1{}^{-1}t_{15}{}^{-1} \implies$

$yx^{-1}yt_{12}t_{11} = t_{19}t_9 \implies$

$(yx^{-1}y)^{-1}yx^{-1}yt_{12}t_{11} = (yx^{-1}y)^{-1}t_{19}t_9 \implies t_{12}t_{11} = yxyt_{19}t_9.$ □

Lemma 5.6. *$t_4t_8 = yx^2yx^2yt_{19}$ and $t_{19} = yx^2yx^{-2}yt_4t_8$.*

Proof. From the relation $x^{-1}yx^{-1}yxyxt_1xt_1t_1xt_1 = e \implies$

$x^{-1}yx^{-1}yxyx(t_1\ x)t^2xt_1 = x^{-1}yx^{-1}yxyx(xt_1{}^{x})t_1t_1xt_1 \implies$

$x^{-1}yx^{-1}yxyxx(t_1{}^{x})t_1t_1xt_1 = x^{-1}yx^{-1}yxyxx(t_2)t_1t_1xt_1 \implies$

$x^{-1}yx^{-1}yxyxx(t_2t_1t_1x)t_1 = x^{-1}yx^{-1}yxyxx(x(t_2t_1t_1)^{x})t_1 \implies$

$x^{-1}yx^{-1}yxyxxxt_4t_2t_2t_1 = x^{-1}yx^{-1}yxyxxxt_4t_8t_1 \implies$

$x^{-1}yx^{-1}yxyxxxt_4t_8t_1 = y\ x^{-2}yx^{-2}yt_4t_8t_1 = e \implies$

$yx^{-2}yx^{-2}yt_4t_8t_1t_{19} = et_{19} \implies$

$(yx^{-2}yx^{-2}y)^{-1}yx^{-2}yx^{-2}yt_4t_8t_1t_{19} = (yx^{-2}yx^{-2}y)^{-1}t_{19} \implies$

$t_4t_8 = yx^2yx^2yt_{19} \implies$

$(yx^2yx^2y)^{-1}t_4t_8 = (yx^2yx^2y)^{-1}yx^2yx^2yt_{19} \implies yx^2yx^{-2}yt_4t_8 = t_{19}.$ □

In what follows, we will denote the number of right cosets in the double coset [w] by $|[w]|$. The only right coset of NeN = [*] is N, therefore |[*]| = 1, which is symbolized by placing "1" inside the circle representing [*]. The stabilizer of N is N. The only orbit of N is {1, 2, 3, ..., 23, 24}, which yields the cosets Nt_1, Nt_2, Nt_3, ... Nt_{23}, Nt_{24} ∈ [1]. This is symbolized by placing a "24" next to the circle representing [*] in the diagram below.

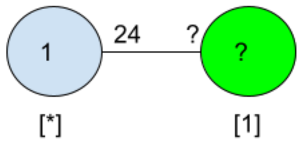

Figure 5.1: Double Cosets [∗]

Let N^1 be the point stabilizer of 1 over N, therefore $N^1 = \langle$ (2, 11, 9, 22, 12)(3, 10, 6, 14, 5)(4, 18, 20, 17, 15)(8, 23, 21, 16, 24)$\rangle$. $N^{(1)} = N^1$ since there are no relations such that $t_i = t_j$ where $0 < i,j \leq 6$. The double cosets of [1] are Nt_1, Nt_2, Nt_3, ... Nt_{23}, and Nt_{24} which verifies |[1]| = 24. This is symbolized by placing a "24" inside the circle representing [1] in the diagram. Now the orbits of $N^{(1)}$ on X = { 1, 2,... ,23, 24 } are { 1 }, { 7 }, { 13 }, { 19 }, { 2, 12, 22, 9, 11 }, { 3, 5, 14, 6, 10 }, { 4, 15, 17, 20, 18 }, { 8, 24, 16, 21, 23 }.

The first orbit yields $Nt_1t_1 = Nt_7 \in$ [1], therefore $Nt_1t_1 \in$ [1]. The second orbit yields $Nt_1t_7 = Nt_{13} \in$ [1], which means $Nt_1t_7 \in$ [1]. The third orbit yields $Nt_1t_{13} = Nt_{19} \in$ [1], which shows $Nt_1t_{13} \in$ [1]. The fourth orbit yields Nt_1t_{19} = Ne = N ∈ [*], which proves $Nt_1t_{19} \in$ [*].

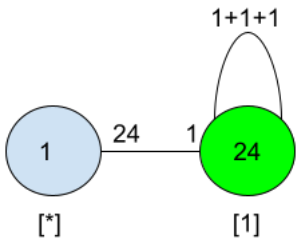

Figure 5.2: DCE up to the Fourth Orbit of [1].

For the double coset [w], we need only determine the double coset of the right coset Nwt_i for one representative t_i for each orbit of the stabilizer group $N^{(w)}$ of the coset Nw. The fifth orbit yields Nt_1t_2, Nt_1t_{12}, Nt_1t_{22}, Nt_1t_9, $Nt_1t_{11} \in [1,2]$, since $Nt_1t_2 \in [1,2]$.

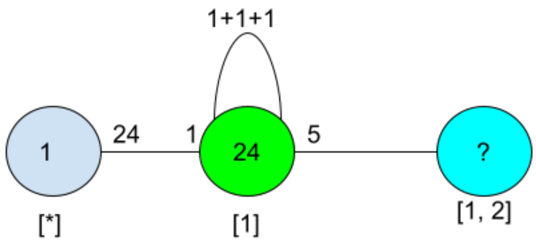

Figure 5.3: DCE up to the Fifth Orbit of [1].

The sixth orbit yields Nt_1t_3, Nt_1t_5, Nt_1t_{14}, Nt_1t_{16}, $Nt_1t_{10} \in [1,3]$.

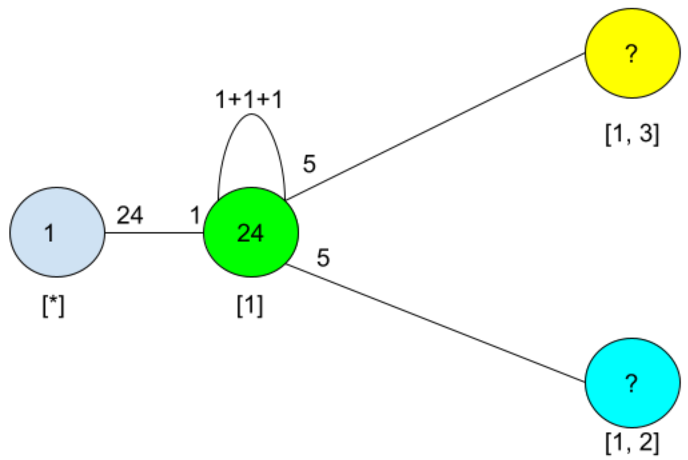

Figure 5.4: DCE up to the Sixth Orbit of [1].

The seventh orbit yields Nt_1t_4, Nt_1t_{15}, Nt_1t_{17}, Nt_1t_{20}, $Nt_1t_{18} \in [1,3]$. It suffices to show $Nt_1t_4 \in [1,3]$.

Lemma 5.7. *$Nt_1t_4 \in [1, 3]$.*

Proof. Let p = (1, 19)(2, 4)(3, 23)(5, 21)(6, 24)(7, 13)(8, 10)(9, 17)(11, 15)(12, 18)(14, 16)(20, 22) ∈ N. $t_{19} = yx^2yx^{-2}yt_4t_8$ (Lemma 5.6) then $(t_{19})^p = (xyx^2yx^{-1}yyt_4t_8)^p$

$\implies t_1 = xyx^2yx^{-1}yt_2t_{10}$.

$t_1t_4 = xyx^2yx^{-1}yt_2t_{10}t_4$

$xyx^2yx^{-1}yt_2t_{10}t_4 = xyx^2yx^{-1}yt_2t_{16}$ since $t_{16} = t_{10}t_4 \implies$

$Nt_1t_4 \in [\,1, 3\,]$ since $N(t_2t_{16})^n = Nt_1t_3$, where n = (1,12, 2)(3, 23, 16)(4, 9, 17)(5, 10, 21)(6, 14, 13)(7, 24, 8)(11, 22, 15)(18, 20,19). □

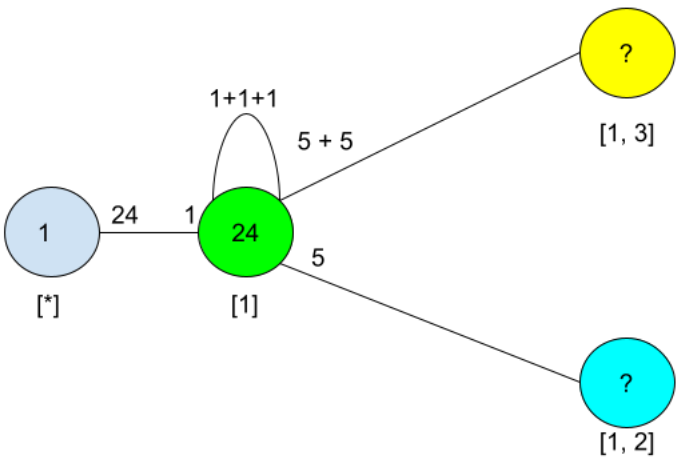

Figure 5.5: DCE up to the Eighth Orbit of [1].

The eighth orbit yields Nt_1t_8, Nt_1t_{24}, Nt_1t_{16}, Nt_1t_{21}, $Nt_1t_{23} \in [1]$.

Lemma 5.8. *$Nt_1t_8 \in$ [1].*

Proof. Let p = (1, 15, 11, 12, 4)(3, 23, 24, 10, 7)(5, 6, 16, 13, 21)(9, 17, 18, 22, 19) $\in$ N. $t_4t_8 = yx^2yx^2yt_{19}$ (Lemma 5.6) then $(t_4t_8)^p = (yx^2yx^2yt_{19})^p \implies t_1t_8 = yxyx^{-2}yt_9 \implies$

$Nt_1t_8 \in [\ 1\ ]$ since $N(t_9)^n \in [\ 1\ ]$, where n = (1,2, 4, 18, 9)(3, 13, 14, 16, 24)(6, 21, 7, 8, 10)(12, 15, 19, 20, 22). □

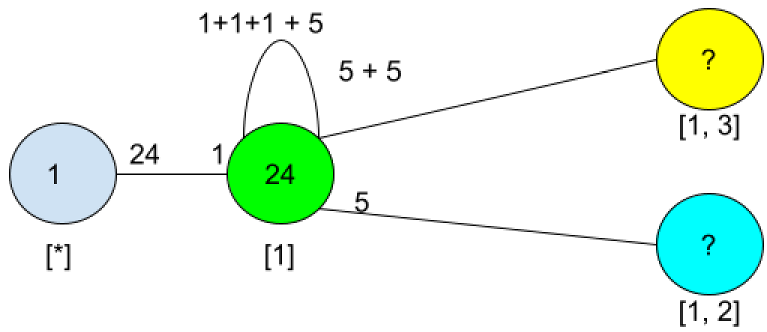

Figure 5.6: Double Cosets [1]

Let $N^{1,2}$ be the point stabilizer of 1 and 2 over N, then $N^{1,2} = \langle e \rangle$.

Lemma 5.9. *(1, 6, 15, 23, 20, 10)(2, 16, 19, 24, 9, 5)(3, 17, 14, 22, 7, 12)(4, 13, 18, 21, 11, 8), (1,16)(2, 6)(3, 11)(4, 7)(5, 15)(8, 12)(9, 23)(10, 19)(13, 22)(14, 18)(17, 21)(20,24) and (1, 6, 15, 23, 20, 10)(2, 16, 19, 24, 9, 5)(3, 17, 14, 22, 7, 12)(4, 13, 18, 21, 11, 8)* $\in N^{(1,2)}$.

Proof. Let p = (1, 20, 12, 9, 17)(2, 18, 15, 11, 19)(3, 23, 13, 8, 6)(5, 7, 14, 24, 21) $\in$ N. $t_{19} = yx^2yx^{-2}yt_4t_8$ (Lemma 5.6) then $(t_{19})^p = (yx^{-2}yx^{-2}yt_4t_8)^p \implies$

$t_2 = yx^2yx^{-1}yxt_4t_6 \implies$

$t_1t_2 = yx^2yx^{-1}yxt_{10}t_4t_6 \implies$

$yx^2yx^{-1}yxt_{10}t_4t_6 = yx^2yx^{-1}yxt_{16}t_6$ since $t_{10}t_4 = t_6 \implies$

$Nt_1t_2 = Nt_{16}t_6$.

Let n = (1,16)(2, 6)(3, 11)(4, 7)(5, 15)(8, 12)(9, 23)(10, 19)(13, 22)(14, 18)(17, 21)(20,24) $\in N \implies Nt_1t_2 = (Nt_{16}t_6)^n \implies n \in N^{(1,2)}$. □

Proof. Let n = (1, 6, 15, 23, 20, 10)(2, 16, 19, 24, 9, 5)(3, 17, 14, 22, 7, 12)(4, 13, 18, 21, 11, 8) $\in N^{(1,2)}$.

Let p = (1, 19)(2, 4)(3, 23)(5, 21)(6, 24)(7, 13)(8, 10)(9, 17)(11, 15)(12,18)(14, 16)(20, 22) $\in$ N.

$t_{19} = yx^2yx^{-2}yt_4t_8$ (Lemma 5.6) then $(t_{19})^p = (yx^{-2}yx^{-2}yt_4t_8)^p \implies$

$t_1 = xyx^2yx^{-1}yt_2t_{10} \implies$

$t_1t_2 = xyx^2yx^{-1}yt_2t_{10}t_2$

Let p = (1, 23, 18, 21)(2, 14, 20, 8)(3, 19, 5, 12)(4, 10, 22, 16)(6, 15, 7, 17)(9, 13, 11, 24) $\in$ N.

$t_4t_8 = yx^2yx^2yt_{19}$ (Lemma 5.6) then $(t_4t_8)^p = (yx^2yx^2yt_{19})^p \implies$

$t_{10}t_2 = yx^2yxyx^{-1}t_5 \implies$

$xyx^2yx^{-1}yt_2t_{10}t_2 = xyx^2yx^{-1}yt_2tyx^2yxyx^{-1}t_5 \implies$

$t_1t_2 = yx^{-1}yx^2t_{10}t_5 \implies$

$Nt_1t_2 = Nt_{10}t_5$.

Let n = (1, 6, 15, 23, 20, 10)(2, 16, 19, 24, 9, 5)(3, 17, 14, 22, 7, 12)(4, 13, 18, 21, 11, 8) $\in$ N. $Nt_1t_2 = (N_{10}t_5) \implies$ n $\in N^{(1,2)}$. □

$N^{(1,2)} \geq \langle$ (1, 6, 15, 23, 20, 10)(2, 16, 19, 24, 9, 5)(3, 17, 14, 22, 7, 12)(4, 13, 18, 21, 11, 8), (1,16)(2, 6)(3, 11)(4, 7)(5, 15)(8, 12)(9, 23)(10, 19)(13, 22)(14, 18)(17, 21)(20,24) $\rangle$, therefore |P| = 12. This proves $|[1,2]| = |N| \div |N^{(1,2)}| \leq 120 \div 12 = 10$.

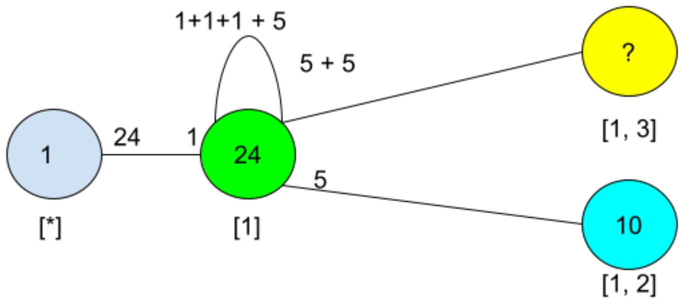

Figure 5.7: Size of [1,2].

Now the obits of $N^{(1,2)}$ on X = { 1, 2,... ,23, 24 } are {1, 24, 5, 20, 9, 19, 23, 15, 10, 2, 6, 16} and {3, 4, 18, 7, 13, 8, 22, 14, 12, 21, 17, 11}.

The first orbit yields $Nt_1t_2t_1$, $Nt_1t_2t_{24}$, $Nt_1t_2t_5$, $Nt_1t_2t_{20}$, $Nt_1t_2t_9$, $Nt_1t_2t_{19}$, $Nt_1t_2t_{23}$, $Nt_1t_2t_{15}$, $Nt_1t_2t_{10}$, $Nt_1t_2t_2$, $Nt_1t_2t_6$, and $Nt_1t_2t_{16}$. Since $Nt_1t_2t_{20} = Nt_1 \in [1]$, because $Nt_1t_2t_{20} = Nt_1t_2t_2^{-1}$ (since $t_2^{-1} = t_{20}$) $= Nt_1 \in [1]$.

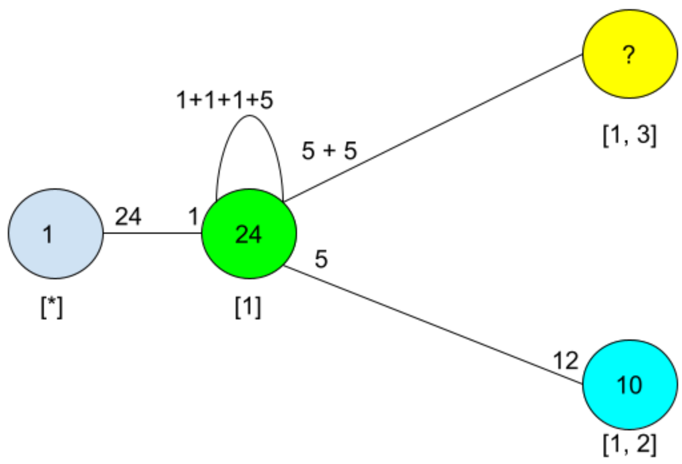

Figure 5.8: DCE up to the First Orbit of [1,2].

The second orbit yields $Nt_1t_2t_3$, $Nt_1t_2t_4$, $Nt_1t_2t_{18}$, $Nt_1t_2t_7$, $Nt_1t_2t_{13}$, $Nt_1t_2t_8$, $Nt_1t_2t_{22}$, $Nt_1t_2t_{14}$, $Nt_1t_2t_{12}$, $Nt_1t_2t_{21}$, $Nt_1t_2t_{17}$, $Nt_1t_2t_{11} \in [1, 3]$.

Lemma 5.10. *$Nt_1t_2t_3 \in [1, 3]$.*

Proof. Let p = (1, 17, 18)(2, 15, 4)(3, 10, 8)(5, 6, 7)(9, 22, 20)(11, 12, 19)(13, 23, 24)(14, 21, 16) $\in$ N.

$t_4t_8 = yx^2yx^2yt_{19}$ (Lemma 5.6) then $(t_4t_8)^p = (yx^2yx^2yt_{19})^p$. $\implies$

$t_2t_3 = x^{-2}yx^{-2}t_{11}$. $\implies$

$t_1t_2t_3 = x^{-2}yx^{-2}t_{13}t_{11}$.

Let n = (1, 7, 19, 13)(2, 14, 20, 8)(3, 18, 16, 11)(4, 23,9, 6)(5, 15, 24, 22)(10, 17, 21, 12). $\implies$ $(t_1t_2t_3)^n = (x^{-2}yx^{-2}t_{13}t_{11})^n$ $\implies$

$t_7t_{14}t_{18} = yx^{-1}yx^2yt_1t_3$.

Let p = (1, 13, 19, 7)(2, 8, 20, 14)(3, 11, 16, 18)(4, 6, 9, 23)(5, 22, 24,15)(10, 12, 21, 17) = n^{-1}. $\implies$

$(t_7t_{14}t_{18})^p = (yx^{-1}yx^2yt_1t_3)^p \implies$

$t_1t_2t_3 = x^{-2}yx^{-2}t_{13}t_{11} = yx^{-1}yx^2y(t_1t_3)^p$. □

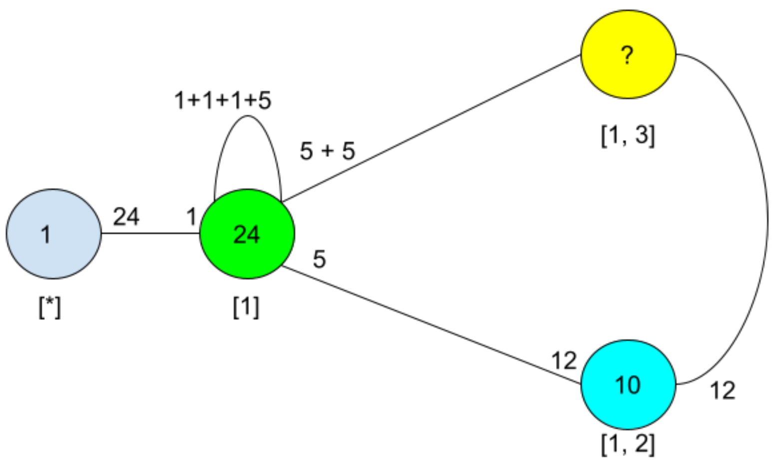

Figure 5.9: DCE of $[1, 2]$

Lemma 5.11. $Nt_7t_4 = Nt_1t_3$.

Proof. Let p = (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9, 20)(10, 24)(11, 17)(14, 21)(18, 22) ∈ N.

$t_{19} = yx^2yx^{-2}yt_4t_8$ (Lemma 5.6) where $yx^2yx^{-2}y \sim$ (1, 8)(2, 13)(3, 17)(4, 24)(5, 9)(6, 22)(7, 20)(10, 18)(11, 21)(12, 16)(14, 19)

(15, 23) $\implies$ $(t_{19})^p = (yx^{-2}yx^{-2}yt_4t_8)^p$ $\implies$

$t_1 = yxyx^2yx^{-1}t_{12}t_3$ where $yxyx^2yx^{-1} \sim$ (1, 23, 18, 21)(2, 14, 20, 8)(3, 19, 5, 12)(4, 10, 22, 16)(6, 15, 7, 17)(9, 13, 11, 24) $\implies$ $t_1t_3 = yxyx^2yx^{-1}t_{12}t_3t_3$.

Let p = (1, 18, 15)(2, 11, 4)(3, 7, 6)(5, 16, 14)(8, 23, 10)(9, 19, 12)(13, 24,21)(17, 22, 20) $\in$ N $\implies$

$t_{19} = yx^2yx^{-2}yt_4t_8$ (Lemma 5.6) then $(t_{19})^p = (yx^{-2}yx^{-2}yt_4t_8)^p$ $\implies$

$t_{12} = x^2yxyx^{-1}yt_2t_{23}$ where $x^2yxyx^{-1}y \sim$ (1, 13, 19, 7)(2, 23, 18, 10)(3, 9, 21, 15)(4, 14, 11, 24)(5, 12, 16, 20)(6, 22, 8, 17) $\implies$ $yxyx^2yx^{-1}t_{12}t_3t_3 =$ $(yxyx^2yx^{-1})(x^2yxyx^{-1}y)$

$t_2t_{23}t_3t_3 = x^{-1}yx^2yt\ _2t_{23}t_3t_3$, where $x^{-1}yx^2y \sim$ (1, 18,15)(2, 11, 4)(3, 7, 6) (5, 16, 14)(8, 23, 10)(9, 19, 12)(13, 24, 21)(17, 22, 20).

Let p = (1, 5, 4, 24, 20, 3)(2, 21, 19, 23, 22, 6)(7, 11, 10, 18, 14, 9)(8, 15, 13, 17, 16, 12) $\in$ N.

$t_{19} = yx^2yx^{-2}yt_4t_8$ (Lemma 5.6) then $(t_{19})^p = (yx^{-2}yx^{-2}yt_4t_8)^p$ $\implies$

$t_{23} = yx^{-1}yxyx^{-2}t_{24}t_{15}$ where $yx^{-1}yxyx^{-2} \sim$ (1, 13, 19, 7)(2, 8, 20, 14)(3, 11, 16, 18)(4, 6, 9, 23)(5, 22, 24, 15)(10, 12, 21, 17) $\implies$ $x^{-1}yx^2yt_2t_{23}t_3t_3 =$ $x^{-1}yx^2yt_2yx^{-1}yxyx^{-2}$

$t_{24}t_{15}t_3t_3$ $\implies$

$x^{-1}yx^2yt_2t_{23}t_3t_3 = x^{-2}yx^2t_8t_{24}t_{15}t_3t_3 = x^{-2}yx^2t_8t_{24}t_{24}t_3 =$ $x^{-2}yx^2t_8t_{24}$ since $t_{24}t_3$ = Id(N).

Let p = (1, 6, 15, 23, 20, 10)(2, 16, 19, 24, 9, 5)(3, 17, 14, 22, 7, 12)(4, 13, 18, 21, 11, 8) $\in$ N.

$t_{19} = yx^2yx^{-2}yt_4t_8$ (Lemma 5.6) then $(t_{19})^p = (yx^{-2}yx^{-2}yt_4t_8)^p$ $\implies$

$t_{24} = yx^2yx^{-1}yxt_{13}t_4$ where $yx^2yx^{-1}yx \sim$ (1, 10, 12, 14)(2, 7, 22, 24)(3, 9, 21, 15)(4, 6, 20, 13)(5, 17, 23, 11)(8, 19, 16, 18) $\Longrightarrow$ $x^{-2}yx^2t_8t_{24}t_{15}t_3t_3$ = $x^{-2}yx^2t_8yx^2yx^{-1}yxt_{13}t_4 = yx^{-1}yx^2t_{19}t_{13}t_4 = \; = yx^{-1}yx^2t_7t_4 \Longrightarrow$ $Nt_1t_3 = Nt_7t_4$. □

Let $N^{1,3}$ be the point stabilizer of 1 and 3 over N, therefore $N^{1,3} = \langle e \rangle$. Let p = (1, 7, 19, 13)(2, 6, 15, 16)(3, 4, 8, 12)(5, 17, 23, 11)(9, 10, 20, 24)(14, 18, 21, 22). $N(t_1t_3)^p = Nt_7t_4 \in N^{(1,3)}$. Now $N^{(1,3)} \geq P = \langle$ (1, 7, 19, 13)(2, 6, 15, 16)(3, 4, 8, 12)(5, 17, 23, 11)(9, 10, 20, 24)(14, 18, 21, 22) $\rangle$, this means |P| = 4. $\Longrightarrow |[1,3]| = |N| \div |N^{(1,3)}| \leq 120 \div 4 = 30$.

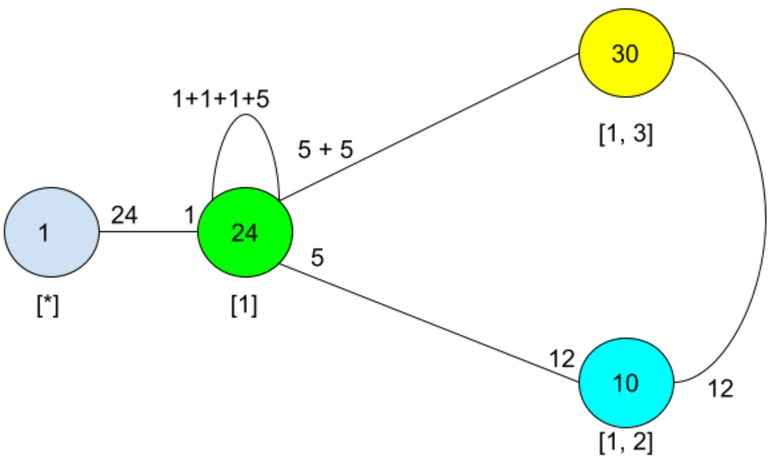

Figure 5.10: Size of [1, 3].

The orbits of $N^{(1,3)}$ on X = { 1, 7, 19, 13}, { 2, 6, 15, 16 }, { 3, 4, 8, 12 }, { 5, 17, 23, 11 }, { 9, 10, 20, 24 }, and { 14, 18, 21, 22 }.

Lemma 5.12. *$Nt_1t_3t_1 \in$ [1, 3].*

Proof. Let p = (1, 19)(2, 18)(3, 21)(4, 11)(5, 16)(6, 8)(7, 13)(9, 15)(10, 23)(12, 20)(14, 24)(17, 22) $\in$ N.

$t_{19} = yx^2yx^{-2}yt_4t_8$ (Lemma 5.6) then $(t_{19})^p = (yx^{-2}yx^{-2}yt_4t_8)^p \implies$

$t_1 = x^{-1}yx^2t_{11}t_6 \implies$

$t_1t_3 = x^{-1}yx^2t_{11}t_6t_3$.

Let p = (1, 21, 22, 5, 12, 8)(2, 13, 9, 10, 17, 6)(3, 4, 23, 18, 14, 19)(7, 15, 16, 11, 24, 20) $\in$ N.

$t_{19} = yx^2yx^{-2}yt_4t_8$ (Lemma 5.6) then $(t_{19})^p = (yx^{-2}yx^{-2}yt_4t_8)^p \implies$

$t_3 = xyx^{-1}yxyxt_{23}t_1 \implies$

$x^{-1}yx^2t_{11}t_6t_3 = x^{-1}yx^2t_{11}t_6xyx^{-1}yxyxt_{23}t_1$, where $xyx^{-1}yxyx \sim (1,21,18,23)(2,8,20,14)(3,12,5,19)(4,16,22,10)(6,17,7,15)(9,24,11,13) \implies$

$t_1t_3 = x^2yx^2yxyx^{-1}yt_{13}t_{17}t_{23}t_1$.

Let n = (1, 7, 19, 13)(2, 14, 20, 8)(3, 18, 16, 11)(4, 23, 9, 6)(5, 15, 24, 22)(10, 17, 21, 12). $\implies t_7t_{18} = (x^{-1}yx^2)^2t_1t_{21}t_9t_7 \implies$

$(x^{-1}yx^2)^2t_1t_{21}t_9t_7 = (x^{-1}yx^2)^2t_1t_3t_7 \implies$

Let $p = n^{-1}$ = (1, 13, 19, 7)(2, 8, 20, 14)(3, 11, 16, 18)(4, 6, 9, 23)(5, 22, 24, 15)(10, 12, 21, 17) $\implies$

$(t_7t_{18})^p = ((x^{-1}yx^2)^2t_1t_3t_7)^p \implies t_1t_3 = (x^{-1}yx^2)^2(t_1t_3t_7)^p$. □

The first orbit yields $Nt_1t_3t_1$, $Nt_1t_3t_7$, $Nt_1t_3t_{19}$, $Nt_1t_3t_{13} \in [\ 1, 3\ ]$.

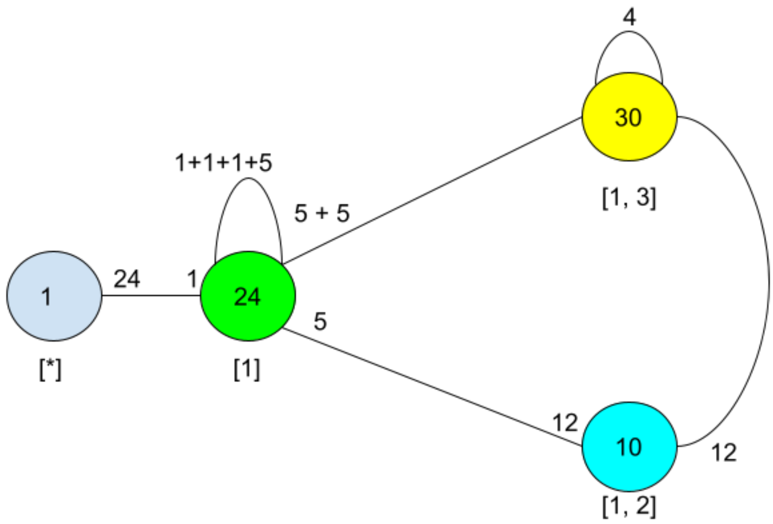

Figure 5.11: DCE up to the First Orbit of [1,3].

The second orbit yields $Nt_1t_3t_2$, $Nt_1t_3t_6$, $Nt_1t_3t_{15}$, $Nt_1t_3t_{16} \in [\ 1\ ]$. Since $t_1t_3t_{15} = t_1t_{21} \in [\ 1\ ]$ as seen before.

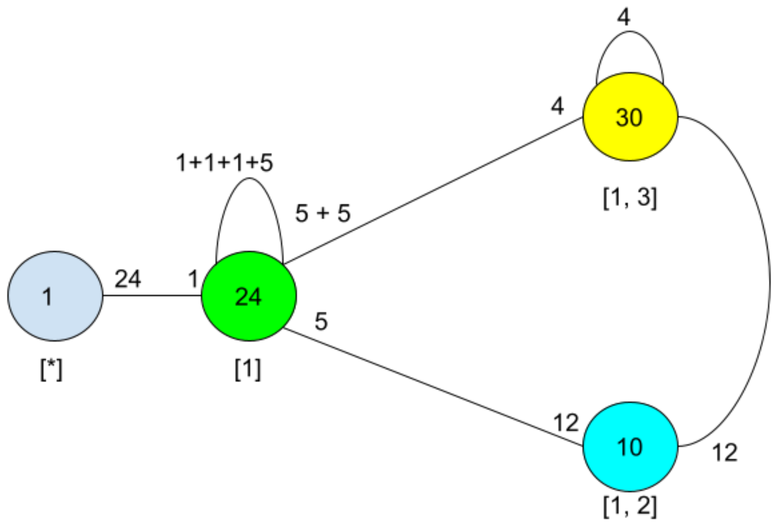

Figure 5.12: DCE up to the Second Orbit of [1,3].

The third orbit yields $Nt_1t_3t_3$, $Nt_1t_3t_4$, $Nt_1t_3t_8$, $Nt_1t_3t_{12}$, $t_1t_3t_3 = t_1t_9 \in [\ 1,\ 2\ ]$ as seen before. Therefore, $Nt_1t_3t_3$, $Nt_1t_3t_4$, $Nt_1t_3t_8$, $Nt_1t_3t_{12} \in [\ 1,\ 2\ ]$.

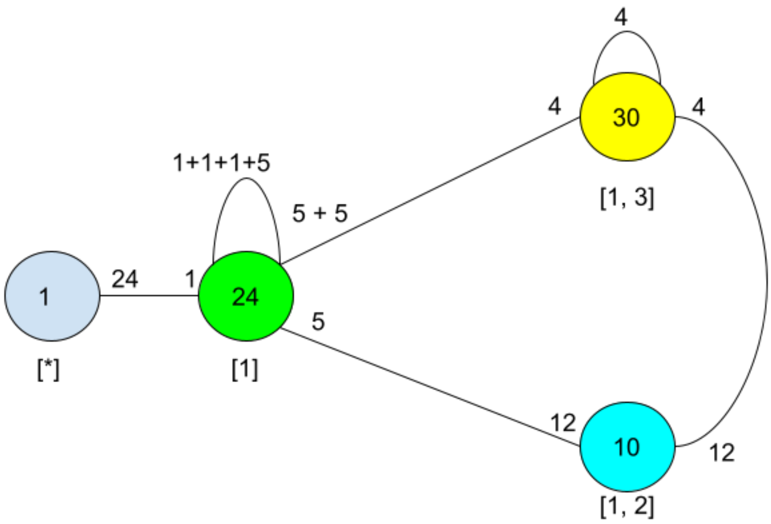

Figure 5.13: DCE up to the Third Orbit of [1,3].

Lemma 5.13. *$Nt_1t_3t_5 \in$ [1, 3].*

Proof. Let p = (1, 21, 18, 23)(2, 8, 20, 14)(3, 12, 5, 19)(4, 16, 22, 10)(6, 17, 7, 15)(9, 24, 11, 13) $\in$ to N.

$t_{19} = yx^2yx^{-2}yt_4t_8$ (Lemma 5.6) then $(t_{19})^p = (yx^{-2}yx^{-2}yt_4t_8)^p \implies$

$t_3 = xyx^{-1}yx^2yt_{16}t_{20} \implies$

$t_1t_3t_5 = t_1xyx^{-1}yx^2yt_{16}t_{20}t_5 = xyx^{-1}yx^2yt_{16}t_{16}t_{20}t_5$.

Let p = (1, 15, 22, 2, 17)(3, 16, 8, 5, 7)(4, 20, 11, 19, 9)(10, 14, 23, 13, 21)$\in$ to N.

$t_4t_8 = yx^2yx^2yt_{19}$ (Lemma 5.6) then $(t_4t_8)^p = (yx^2yx^2yt_{19})^p \implies$

$t_{20}t_5 = yx^{-1}yxyx^2t_9 \implies$

$xyx^{-1}yx^2yt_{16}t_{16}t_{20}t_5 = yx^{-1}yx^2yt_{16}t_{16}yx^{-1}yxyx^2t_9 = xyx^{-2}yxt_4t_4t_9$.

We now have $t_1t_3t_5 = xyx^{-2}yxt_4t_4t_9$.

Let n = (1, 14, 12, 10)(2, 24, 22, 7)(3, 15, 21, 9)(4, 13, 20, 6)(5, 11, 23, 17)(8, 18, 16, 19). $\Longrightarrow$ $(t_1t_3t_5)^n = (xyx^{-2}yxt_4t_4t_9)^n$ $\Longrightarrow$

$t_{14}t_{15}t_{11} = (xy)^2t_{13}t_{13}t_3 = (xy)^2t_1t_3$ since $t_{13}t_{13} = t_1$.

Let p = (1, 10, 12, 14)(2, 7, 22, 24)(3, 9, 21, 15)(4, 6, 20, 13)(5, 17, 23, 11)(8, 19, 16, 18) = n^{-1}. $\Longrightarrow$

$(t_{14}t_{15}t_{11})^p = ((xy)^2t_1t_3)^p$ $\Longrightarrow$

$t_1t_3t_5 = xyx^{-2}yxt_4t_4t_9 = (xy)^2(t_1t_3)^p$. $\Longrightarrow$ $Nt_1t_3t_5 \in [\ 1,\ 3\ ]$. □

The fourth orbit yields $Nt_1t_3t_5$, $Nt_1t_3t_{17}$, $Nt_1t_3t_{23}$, $Nt_1t_3t_{11} \in [1, 3]$.

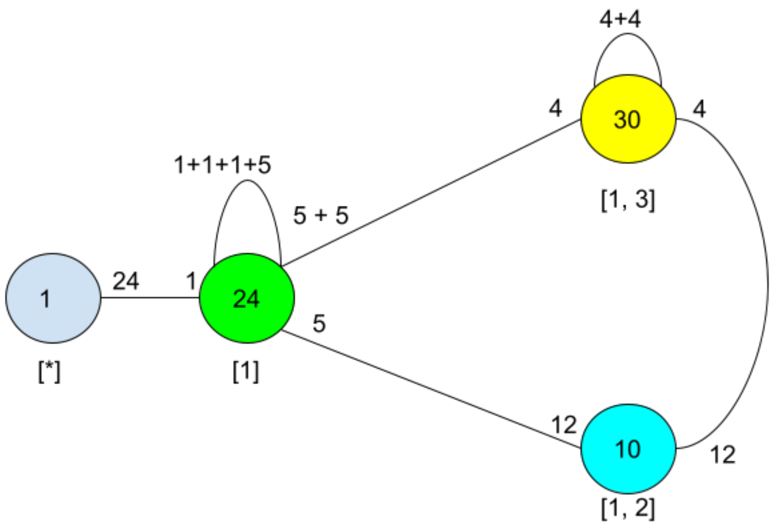

Figure 5.14: DCE up to the Fourth Orbit of [1,3].

Lemma 5.14. *$Nt_1t_3t_9 \in$ [1, 3].*

Proof. Let p = (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9, 20)(10, 24)(11,17)(14, 21)(18, 22) $\in$ N.

$t_{19} = yx^2yx^{-2}yt_4t_8$ (Lemma 5.6) then $(t_{19})^p = (yx^{-2}yx^{-2}yt_4t_8)^p \implies$

$t_1 = yxyx^2yx^{-1}t_{12}t_3 \implies$

$t_1t_3t_9 = yxyx^2yx^{-1}t_{12}t_3t_3t_9$.

Let n = (1, 12)(2, 22)(3, 21)(4, 20)(5, 23)(6, 13)(7, 24)(8,16)(9, 15)(10, 14)(11, 17)(18, 19). $\implies$ $(t_1t_3t_9)^n = (yxyx^2yx^{-1}t_{12}t_3t_3t_9)^n \implies$

$t_{12}t_{21}t_{15} = xyx^{-1}yxyxt_1t_{21}t_{21}t_{15}$.

$xyx^{-1}yxyxt_1t_{21}t_{21}t_{15} = xyx^{-1}yxyxt_1t_{15}t_{15}$

$xyx^{-1}yxyxt_1t_{15}t_{15} = xyx^{-1}yxyxt_1t_3$

Let p = (1, 12)(2, 22)(3, 21)(4, 20)(5, 23)(6, 13)(7, 24)(8, 16)(9, 15)(10,14)(11, 17)(18, 19) = n^{-1}. $\implies$

$(t_{12}t_{21}t_{15})^p = (xyx^{-1}yxyxt_1t_3)^p \implies$

$t_1t_3t_9 = yxyx^2yx^{-1}t_{12}t_{21} = xyx^{-1}yxyx(t_1t_3)^p$. □

The fifth orbit yields $Nt_1t_3t_9$, $Nt_1t_3t_{10}$, $Nt_1t_3t_{20}$, $Nt_1t_3t_{24} \in [1, 3]$.

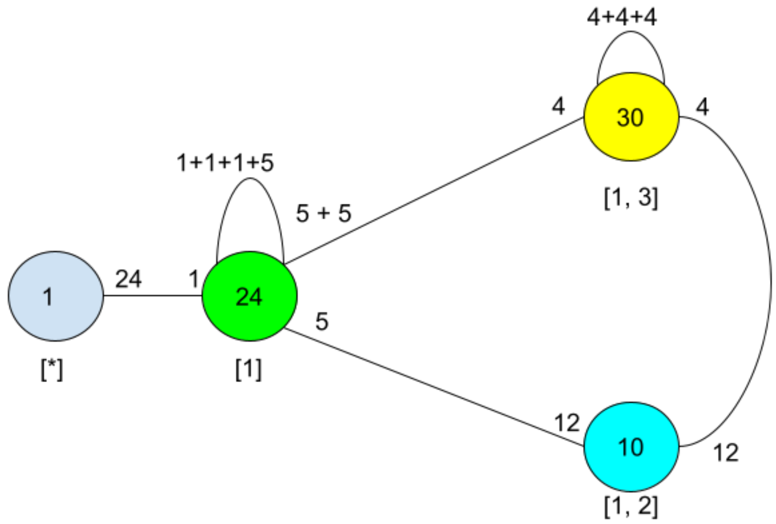

Figure 5.15: DCE up its Fifth Orbit of [1,3].

Lemma 5.15. *$Nt_1t_3t_{14} \in$ [1].*

Proof. Let p = (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9, 20)(10, 24)(11,17)(14, 21)(18, 22) $\in$ N.

$t_{19} = yx^2yx^{-2}yt_4t_8$ (Lemma 5.6) then $(t_{19})^p = (yx^{-2}yx^{-2}yt_4t_8)^p \implies$

$t_1 = yxyx^2yx^{-1}t_{12}t_3 \implies$

$t_1t_3t_{14} = yxyx^2yx^{-1}t_{12}t_3t_3t_{14}$.

Let p = (1, 8)(2, 13)(3, 17)(4, 24)(5, 9)(6, 22)(7, 20)(10, 18)(11, 21)(12, 16)(14, 19)(15, 23) $\in$ N.

$t_{19} = yx^2yx^{-2}yt_4t_8$ (Lemma 5.6) then $(t_{19})^p = (yx^{-2}yx^{-2}yt_4t_8)^p \implies$

$t_{14} = yx^2yx^{-1}yt_{24}t_1 \implies$

$yxyx^2yx^{-1}t_{12}t_3t_3t_{14} = yxyx^2yx^{-1}t_{12}t_3t_3yx^2yx^{-1}yt_{24}t_1 = x^2yx^{-2}yt_{13}t_{18}t_{18}t_{24}t_1$.

Let $n = (1, 19)(2, 4)(3, 23)(5, 21)(6, 24)(7, 13)(8, 10)(9, 17)(11, 15)(12, 18)(14, 16)(20, 22)$. $\Longrightarrow$

$(t_1t_3t_{14})^n = (x^2yx^{-2}yt_{13}t_{18}t_{18}t_{24}t_1)^n \Longrightarrow$

$t_{19}t_{23}t_{16} = (x^2yx^{-1})^2t_7t_{12}t_{12}t_6t_{19}$.

$(x^2yx^{-1})^2t_7t_{12}t_{12}t_6t_{19} = (x^2yx^{-1})^2t_7t_{24}t_6t_{19} \Longrightarrow$

$(x^2yx^{-1})^2t_7t_{24}t_6t_{19} = (x^2yx^{-1})^2t_7t_{19}$ since $t_{24}t_6 = e \Longrightarrow$

$(x^2yx^{-1})^2t_7t_{19} = (x^2yx^{-1})^2t_1$.

Let $p = (1, 19)(2, 4)(3, 23)(5, 21)(6, 24)(7, 13)(8, 10)(9, 17)(11, 15)(12, 18)(14, 16)(20, 22) = n^{-1}$. $\Longrightarrow$

$(t_{19}t_{23}t_{16})^p = ((x^2yx^{-1})^2t_1)^p \Longrightarrow$

$t_1t_3t_{14} = x^2yx^{-2}yt_{19} = (x^2yx^{-1})^2t_1{}^p \in [1]$. $\square$

The sixth orbit yields $Nt_1t_3t_{14}$, $Nt_1t_3t_{18}$, $Nt_1t_3t_{22}$, $Nt_1t_3t_{22} \in [1]$.

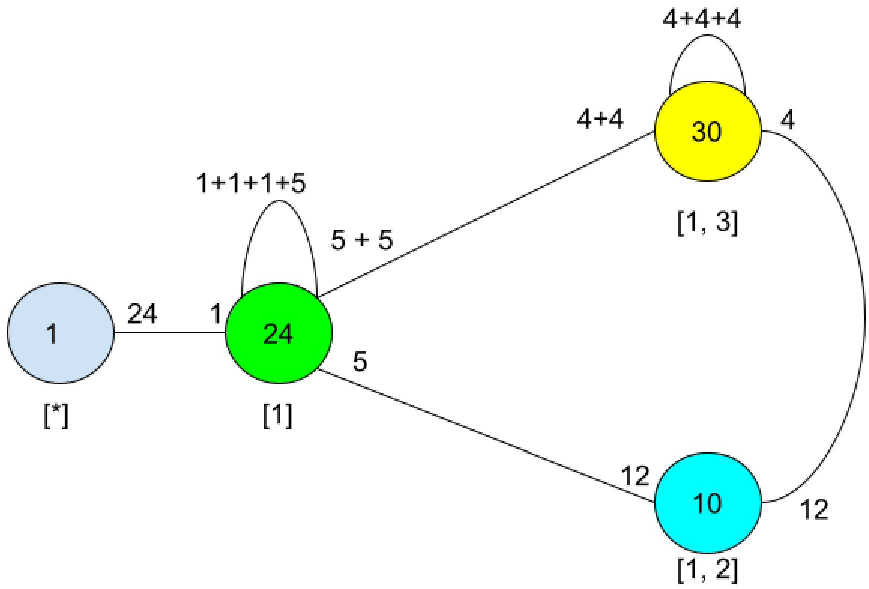

Figure 5.16: DCE up to the Sixth Orbit of [1,3].

Our augment shows that $|G| \leq 65(120) = 7800$. Let X be the set of the 65 right cosets that we have found. Now $\langle x, y, t \rangle$ act on X and it readily checks that $|f(x), f(y), f(t)| = 7800$. By the First Isomorphism Theorem, there exist a, b in G that satisfy a presentation of $L_2(25)$. Thus, $L_2(25) \leq 7800$. Now $|G| = |L_2(25)| = 7800$. So $G \cong L_2(25)$.

5.3 DCE $S_3 \times A_5$ Over 15:2 and 10:2

A symmetric presentation of the progenitor $2^{*10} : (10 : 2)$ is $G = \langle$ x^{10}, y^2, $(xy)^2$, t^2, (t,y),$(x^2ytt^x)^3$, $(xt^x)^3\rangle$. Let xx = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10) , yy = (2, 10)(3, 9)(4, 8)(5, 7), and N = ⟨ xx, yy ⟩. Magma gives | N | = 20 and N ≅

$(10:2)$. Let t = t_1.

Given H = ⟨ x, y, tx^5tx⟩, is a maximal subgroup of G and $|H| = 60$.

Theorem 5.16. $S_3 \times A_5 \cong G = \langle\ x^{10},\ y^2,\ (xy)^2,\ t^2,\ (t,y),\ (x^2ytt^x)^3,\ (xt^x)^3 \rangle$.

We perform our technique of manual double coset enumeration to G over H and N to construct G and prove that G ≅ $S_3 \times A_5$. Before we start our double coset enumeration, the relations need to be expanded.

Lemma 5.17. $x^2yt_1t_2t_9\ = t_2t_1t_8$.

Proof. $(x^2ytt^x)^3$ = e (Definition of G). $\implies$ $(x^2yt_1t_2)^3$ = e. Let $\pi = x^2y \sim$ (1, 9)(2, 8)(3, 7)(4, 6). $\implies$ $(\pi t_1t_2)^3 = \pi^3(t_1t_2)^{\pi^2}(t_1t_2)^{\pi}t_1t_2 = x^2yt_1t_2t_9t_8t_1t_2 = 1$. $x^2yt_1t_2t_9t_8t_1t_2(t_2t_1t_8) = e(t_2t_1t_8) \implies x^2yt_1t_2t_9 = t_2t_1t_8$, which also means $Nt_1t_2t_9 = Nt_2t_1t_8$. □

Lemma 5.18. $x^3t_4t_3\ = t_2$.

Proof. $(xt^x)^3$= e (Definition of G). $\implies$ $(xt_2)^3$ = e $\implies$ $xt_2xt_2xt_2 = x^3t_4t_3t_2$ = e $\implies$ $x^3t_4t_3t_2(t_2)\ = e(t_2) \implies\ x^3t_4t_3 = t_2$, which also means $Ht_2 = Ht_4t_3$. □

Lemma 5.19. $Ht_7\ = Ht_2$.

Proof. Given H = ⟨ x, y, tx^5tx⟩, is a maximal subgroup of G and $|H| = 60$. $\implies$ $tx^5tx \sim t_1x^5tx \implies Hx^6t_7t_2 = H \implies$
$Hx^6t_7t_2 = H \implies Ht_7t_2 = H \implies Ht_7 = Ht_2$. □

The only right coset of HeN = [*] is H, therefore|[*]| = 1, which is symbolized by placing "1" inside the circle representing [*]. The stabilizer of N is N, which means the only orbit of N is {1, 2, 3, . . ., 10} which yields the cosets Nt_1, Nt$_2$, Nt$_3$, ... Nt$_{10}$ $\in$ [1]. This is symbolized by placing a "10" next to the circle representing [1] in the diagram below.

Figure 5.17: DCE [$*$]

Lemma 5.20. *$N^{(W)} = w^{-1}Nw\cap$ N.*

Proof. $N^{(W)} = \{n \in N \mid Nwn \in Nw\} \implies$
$N^{(W)} = \{n \in N \mid wn \in Nw\} \implies$
$N^{(W)} = \{n \in N \mid n \in w^{-1}Nw\} \implies$
$N^{(W)} = w^{-1}Nw\cap$ N. $\square$

Lemma 5.21. *$(w^{-1}Hw) \cap N = wNw^{-1} \cap N$.*

Proof. Let the right coset decomposition of H over N $H = N \cup Nw_1 \cup Nw_2 ... \cup Nw_k$, where $N \cap Nw_i = \{\}$, $Nw_j \cap Nw_i = \{\} \implies$, for i $\neq$ j.
$(w^{-1}Hw) \cap N = (wHw^{-1}) \cap N = w(N \cup Nw_1 \cup Nw_2 ... \cup Nw_k)w^{-1} \cap N \implies$
$(w^{-1}Hw) \cap N = (wNw^{-1} \cup wNw_1w^{-1} \cup wNw_2w^{-1} ... \cup Nw_kw^{-1}) \cap N = wNw^{-1} \cap N$ since $N \cap Nw_i = \{\}$. $\square$

Theorem 5.22. *Let $w \in G$, $N \leq$, $H \leq G$ then the number of right cosets of H in the double coset $HwN = |N|/|N^{(w)}$, were $w \in G$ and H,K are normal subgroups of G.*

Proof. Let w $\in G$. Lemma 5.2, the number of single cosets of H in HwK = $|H| \times |N|/|wHw^{-1} \cap N| = |N|/|wNw^{-1} \cap N|$ by Lemma 5.20. Now $|HwK| = |H| \times |N|/|N^{(w)}|$ by Lemma 5.21.
Now divide by $|H|$ to obtain $|[w]| = |N|/|N^{(w)}|$. □

Let N^1 be the stabilizer of {1} over N. Therefore, $N^1 = \langle$ (2,10)(3, 9)(4, 8)(5, 7)$\rangle$. Let p = (1, 5, 9, 3, 7)(2, 6, 10, 4, 8). Hence, $Ht_7{}^p = Ht_2{}^p$, which means $Ht_1 = Ht_6$. Let n = (1, 6)(2, 7)(3, 8)(4, 9)(5, 10) or (1, 6)(2, 5)(3, 4)(7, 10)(8, 9). $Ht_1{}^n = Ht_6 = Ht_1$, which proves n $\in N^{(1)}$. $N^{(1))} \geq P = \langle$ (2, 10)(3, 9)(4, 8)(5, 7), (1, 6)(2, 7)(3, 8)(4, 9)(5, 10)$\rangle$. Consequently, |P| = 12, which results in |[1]| = $|N| \div |N^{(1)}| \leq 20/4 = 5$.

Figure 5.18: Size of [1].

Now the obits of $N^{(1)}$ on X = { 1 }, { 6 }, { 2, 10 }, {3, 9}, { 4, 8}, {5, 7}. The first orbits yields the right coset Ht_1t_1 = H $\in$ [*] since $|t_1| = 2$.

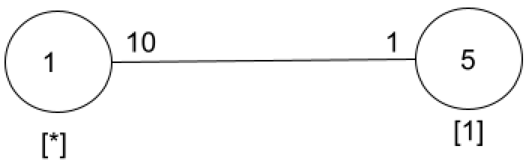

Figure 5.19: DCE up to the First Orbit of [1].

The second orbit yields Ht_1t_6 = Ht_6t_6 since $Ht_1 = Ht_6$. Therefore, Ht_6t_6 = H. This proves $Ht_1t_6 \in [*]$.

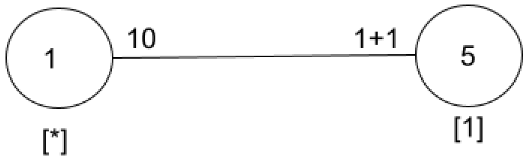

Figure 5.20: DCE up to the Second Orbit of [1].

The third orbit yields Ht_1t_2 and Ht_1t_{10}.

Lemma 5.23. *$Ht_1t_{10} \in [1]$.*

Proof. Let p = (1, 9, 7, 5, 3)(2, 10, 8, 6, 4).

$x^3t_4t_3 = t_2$ (Lemma 5.18) $\implies (x^3t_4t_3)^p = t_2{}^p \implies$

$(x^3)^pt_2t_1 = t_{10}$, where $(x^3)^p$ = (1, 4, 7, 10, 3, 6, 9, 2, 5, 8).

Let $n = (x^3)^p \implies nt_2t_1t_1 = t_{10}t_1 \implies nt_2 = t_{10}t_1 \implies$

$(nt_2)^{-1} = (t_{10}t_1)^{-1} \implies t_2n^{-1} = n^{-1}t_9 = t_1t_{10} \implies Ht_9 = Ht_1t_{10} \implies$

$Ht_9 = Ht_1t_{10} \in [1]$ since $Ht_9 \in [1]$.

Since Ht_1t_2 and Ht_1t_{10} come from the same orbit. $\implies$

Ht_1t_2 and $Ht_1t_{10} \in [1]$. □

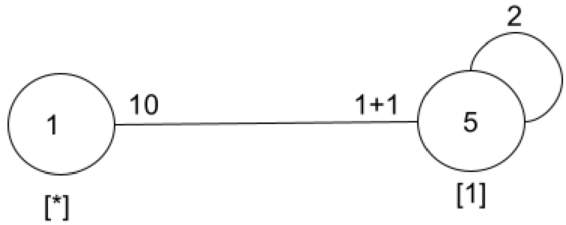

Figure 5.21: DCE up to the Third Orbit of [1].

The fifth orbit yields Ht_1t_4 and Ht_1t_8.

Lemma 5.24. *$Ht_1t_4 \in [1]$.*

Proof. Let p = (1, 2)(3, 10)(4, 9)(5, 8)(6, 7).

$t_2 = x^3t_4t_3$ (Lemma 5.18). $\implies (t_2)^p = (x^3t_4t_3)^p \implies t_1 = x^{-3}t_9t_{10} \implies$

$t_1t_4 = x^{-3}t_9t_{10}t_4$.

Let p = (1, 3, 5, 7, 9)(2, 4, 6, 8, 10).

$t_2 = x^3t_4t_3$ (Lemma 5.18). $\implies (t_2)^p = (x^3t_4t_3)^p \implies t_4 = x^3t_6t_5 \implies$

$x^{-3}t_9t_{10}t_4 = x^{-3}t_9t_{10}x^3t_6t_5 = t_2t_3t_6t_5$,

where $x^3 \sim (1, 4, 7, 10, 3, 6, 9, 2, 5, 8)$.

Let p = (1, 3)(4, 10)(5, 9)(6, 8).

$t_2t_1t_8 = x^2yt_1t_2t_9$ (Lemma 5.18). $\implies (t_2t_1t_8)^p = (x^2yt_1t_2t_9)^p$

$\implies t_2t_3t_6 = x^4yt_3t_2t_5 \implies$

$t_2t_3t_6t_5 = x^4yt_3t_2t_5t_5$.

Let p = (1, 10, 9, 8, 7, 6, 5, 4, 3, 2)

$t_4t_3 = x^{-3}t_2$ (Lemma 5.18). $\implies (t_4t_3)^p = (x^{-3}t_2)^p \implies t_3t_2 = x^{-3}t_1 \implies$

$x^4yt_3t_2t_5t_5 = x^4yx^{-3}t_1t_5t_5 = x^4yx^{-3}t_1 \implies$

$\mathrm{H}t_1t_4 = \mathrm{H}x^4yx^{-3}t_1 = \mathrm{H}t_1 \in [1]$. □

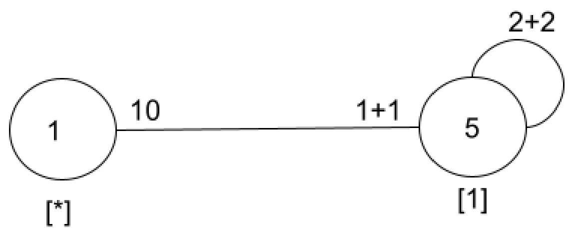

Figure 5.22: DCE up to the Fifth Orbit of [1].

The fourth orbit yields $\mathrm{H}t_1t_3$ and $\mathrm{H}t_1t_9$.

Lemma 5.25. *Ht_1t_3, $Ht_1t_9 \in$ [1].*

Proof. Let p = (1, 10, 9, 8, 7, 6, 5, 4, 3, 2).

$t_2 = x^3t_4t_3$ (Lemma 5.18). $\implies (t_2)^p = (x^3t_4t_3)^p \implies t_1 = x^3t_3t_2 \implies$

$t_1t_3 = x^3t_3t_2t_3$

Let p = (1, 5)(2, 4)(6, 10)(7, 9)

$t_4t_3 = x^{-3}t_2$ (Lemma 5.18). $\implies (t_4t_3)^p = (x^{-3}t_2)^p \implies t_2t_3 = x^3t_4 \implies$

$x^3t_3t_2t_3 = x^3t_3\, x^3t_4 = x^6t_6t_4 \implies$

$\mathrm{H}t_1t_3 = \mathrm{H}x^6t_6t_4$

Let p = (1, 9, 7, 5, 3)(2, 10, 8, 6, 4).

$\mathrm{H}t_7 = \mathrm{H}t_2 \implies \mathrm{H}t_7t_2 = \mathrm{H}$ (Lemma 5.19). $\implies (\mathrm{H}t_7t_2)^p = \mathrm{H}^p \implies \mathrm{H}t_5t_{10} =$

$H \implies$

$\mathrm{H}t_1t_3 = \mathrm{H}x^6t_6t_4 = \mathrm{H}t_5t_{10}x^6t_6t_4 = \mathrm{H}x^6t_1t_6t_6t_4 = \mathrm{H}t_1t_4 \in [1] \implies$

$\mathrm{H}t_1t_3 \in [1]$. □

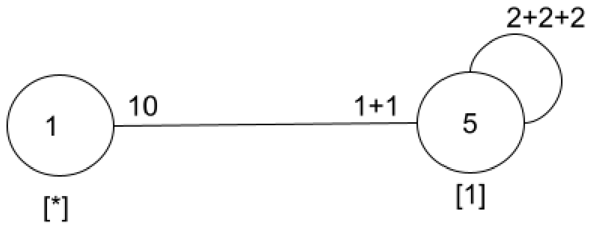

Figure 5.23: DCE up to the Fourth Orbit of [1].

The sixth orbit yields Ht_1t_5 and Ht_1t_7.

Lemma 5.26. *Ht_1t_5, Ht_1t_7 $\in$ [1].*

Proof. Let p = (1, 6)(2, 5)(3, 4)(7, 10)(8, 9).

$t_2 = x^3t_4t_3$ (Lemma 5.18). $\implies (t_2)^p = (x^3t_4t_3)^p \implies t_1 = t_5 = x^{-3}t_3t_4 \implies$

$t_1t_5 = t_1 \implies t_1x^{-3}t_3t_4 = x^{-3}t_8t_3t_4$ since $x^{-3} \sim (1, 8, 5, 2, 9, 6, 3, 10, 7, 4)$.

$Ht_1t_5 = Hx^{-3}t_8t_3t_4$.

Let p = (1, 10, 9, 8, 7, 6, 5, 4, 3, 2)

$Ht_7 = Ht_2$ (Lemma 5.18). $\implies Ht_7t_2 = H \implies$

$(Ht_7t_2)^p = H^p \implies Ht_6t_1 = H \implies$

$Ht_1t_3 = Ht_6t_1x^{-3}t_8t_3t_4 = Hx^{-3}t_3t_8t_8t_3t_4 = Ht_4 \in [1]$

$\implies Ht_1t_5, Ht_1t_7 \in [1]$. □

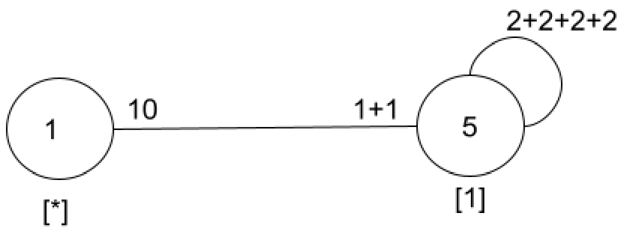

Figure 5.24: Cayley Diagram of $S_3 \times A_5$ Over 15:2 and 10:2.

Our augment shows that $|G| \leq 6(20) = 360$. Let X be the set of the 6 right cosets that we have found. Now $\langle x, y, t\rangle$ act on X and it readily checks that $|f(x), f(y), f(t)| = 360$. By the First Isomorphism Theorem, there exist a, b in G that satisfy a presentation of $S_3 \times A_5$. Thus, $S_3 \times A_5 \leq 360$. Now $|G| = |S_3 \times A_5|$ $= 360$. So $G \cong S_3 \times A_5$.

5.4 DCE of $(A_5)^2$:2 Over $A_5 : 2$

A symmetric presentation of the progenitor $5^{*24} : (A_5 : 2)$ is G = $\langle$ x, y, t | x^5, y^2, $(x^{-1}y)^4$, $(xyx^{-2}yx)^2$, t^5, (t, x^2yxy), $t^{yx} = t^2$, $(yx^2t^{yx^2})^4$, $(yx^2t^2)^4\rangle$. Let xx = (1, 2, 4, 12, 21)(3, 7, 8, 10, 18)(6, 15, 19, 20, 22)(9, 13, 14, 16, 24) and let yy = (1, 15)(2, 5)(3, 13)(4, 6)(7, 21)(8, 11)(9, 19)(10, 12)(14, 17)(16, 18)(20,23)(22, 24). Let N = $\langle$ xx, yy$\rangle$. Magma gives $|N| = 120$. Let t = t_1.

Let G = $\langle$ x, y, t | x^5, y^2, $(x^{-1}y)^4$, $(xyx^{-2}yx)^2$, t^5, (t, x^2yxy), $t^{yx} = t^2$, $(yx^2t^{yx^2})^4$, $(yx^2t^2)^4\rangle$.

We perform our technique of manual double coset enumeration to G over N to construct G and prove that G $\cong (A_5)^2 : 2$. Before we start our double coset enumeration, the relations need to be expanded.

Notation: $t = t_1$, $t_{19} = t_1^2$, $t_{20} = t_2^2$, $t_{21} = t_3^2$, $t_{22} = t_4^2$, $t_{23} = t_5^2$, $t_{24} = t_6^2$, $t_7 = t_1^3$, $t_8 = t_2^3$, $t_9 = t_3^3$, $t_{10} = t_4^3$, $t_{11} = t_5^3$, $t_{12} = t_6^3$, $t_{13} = t_1^4$, $t_{14} = t_2^4$, $t_{15} = t_3^4$, $t_{16} = t_4^4$, $t_{17} = t_5^4$, and $t_{18} = t_6^4$.

Additional Notation: $t_{19} = t_1^2 \implies (t_{19})^N = (t_1^2)^N \implies t_1 = t_{13}^{-1}$, $t_2 = t_{14}^{-1}$, $t_3 = t_{15}^{-1}$, $t_4 = t_{16}^{-1}$, $t_5 = t_{17}^{-1}$, $t_6 = t_{18}^{-1}$, $t_7 = t_{19}^{-1}$, $t_8 = t_{20}^{-1}$, $t_9 = t_{21}^{-1}$, $t_{10} = t_{22}^{-1}$, $t_{11} = t_{23}^{-1}$, $t_{12} = t_{24}^{-1}$, $t_{13} = t_1^{-1}$, $t_{14} = t_2^{-1}$, $t_{15} = t_3^{-1}$, $t_{16} = t_4^{-1}$, $t_{17} = t_5^{-1}$, $t_{18} = t_6^{-1}$, $t_{19} = t_7^{-1}$, $t_{20} = t_8^{-1}$, $t_{21} = t_9^{-1}$, $t_{22} = t_{10}^{-1}$, $t_{23} = t_{11}^{-1}$, $t_{24} = t_{12}^{-1}$.

Powers of t_i:

Inverses	Powers	Splits
$t_1 = t_{13}^{-1}$	$t_1^2 = t_{19}$, $t_1^3 = t_7$, $t_1^4 = t_{13}$	$t_1 = t_{19}t_{13}$
$t_2 = t_{14}^{-1}$	$t_2^2 = t_{20}$, $t_8 = t_2^3$, $t_{14} = t_2^4$	$t_2 = t_{20}t_{14}$
$t_3 = t_{15}^{-1}$	$t_3^2 = t_{21}$, $t_9 = t_3^3$,$t_{15} = t_3^4$	$t_3 = t_{21}t_{15}$
$t_4 = t_{16}^{-1}$	$t_4^2 = t_{22}$, $t_{10} = t_4^3$, $t_{16} = t_4^4$	$t_4 = t_{22}t_{16}$
$t_5 = t_{17}^{-1}$	$t_5^2 =t_{23}$, $t_{11} = t_5^3$, $t_{15} = t_3^4$	$t_5 = t_{23}t_{17}$
$t_6 = t_{18}^{-1}$	$t_6^2 = t_{24}$, $t_{12} = t_6^3$,$t_{18} = t_6^4$	$t_6 = t_{24}t_{18}$
$t_7 = t_{19}^{-1}$	$t_7 = t_1^3$	$t_7 = t_1t_{19}$
$t_8 = t_{20}^{-1}$	$t_8 = t_2^3$	$t_8 = t_2t_{20}$
$t_9 = t_{21}^{-1}$	$t_9 = t_3^3$	$t_9 = t_3t_{21}$
$t_{10} = t_{22}^{-1}$	$t_{10} = t_4^3$	$t_{10} = t_4t_{22}$
$t_{11} = t_{23}^{-1}$	$t_{11} = t_5^3$	$t_{11} = t_5t_{23}$
$t_{12} = t_{24}^{-1}$	$t_{12} = t_6^3$	$t_{12} = t_6t_{24}$

$t_{13} = t_1^{-1}$	$t_{13} = t_1^4$	$t_{13} = t_7t_1$
$t_{14} = t_2^{-1}$	$t_{14} = t_2^4$	$t_{14} = t_8t_2$
$t_{15} = t_3^{-1}$	$t_{15} = t_3^4$	$t_{15} = t_9t_3$
$t_{16} = t_4^{-1}$	$t_{16} = t_4^4$	$t_{16} = t_{10}t_4$
$t_{17} = t_5^{-1}$	$t_{17} = t_5^4$	$t_{17} = t_{11}t_5$
$t_{18} = t_6^{-1}$	$t_{18} = t_6^4$	$t_{18} = t_{12}t_6$
$t_{19} = t_7^{-1}$	$t_{19} = t_1^2$	$t_{19} = t_{13}t_7$
$t_{20} = t_8^{-1}$	$t_{20} = t_2^2$	$t_{20} = t_{14}t_8$
$t_{21} = t_9^{-1}$	$t_{21} = t_3^2$	$t_{21} = t_{15}t_9$
$t_{22} = t_{10}^{-1}$	$t_{22} = t_4^2$	$t_{22} = t_{16}t_{10}$
$t_{23} = t_{11}^{-1}$	$t_{23} = t_5^2$	$t_{23} = t_{17}t_{11}$
$t_{24} = t_{12}^{-1}$	$t_{24} = t_6^2$	$t_{24} = t_{18}t_{12}$

Table 5.2: Power of the ts.

Proof. $(w\pi)^4 = e \implies w^4\pi^{w^3}\pi^{w^2}\pi^{w}\pi = e$

$(w\pi)^4 = e \implies w\pi w\pi w\pi w\pi = e \implies$

$w(ww^{-1})\pi w\pi w\pi w\pi = ww(w^{-1}\pi w)\pi w\pi w\pi = w^2(\pi^w)\pi w\pi w\pi =$

$w^2\pi^w(ww^{-1})\pi w\pi w\pi = w^2\pi^w w(w^{-1}\pi w)\pi w\pi = w^2\pi^w w(\pi^w)\pi w\pi =$

$w^2\pi^w w\pi^w(ww^{-1})\pi w\pi = w^2\pi^w w\pi^w w(w^{-1}\pi w)\pi = w^2\pi^w w\pi^w w(\pi^w)\pi =$

$w^2(ww^{-1})\pi^w w(ww^{-1})\pi^w w\pi^w\pi = w^2w(w^{-1}\pi^w w)w(w^{-1}\pi^w w)\pi^w\pi =$

$w^3(\pi^{w^2})w\pi^{w^2}\pi^w\pi = w^3(ww^{-1})\pi^{w^2}w\pi^{w^2}\pi^w\pi = w^3w(w^{-1}\pi^{w^2}w)(\pi^{w^2}\pi^w\pi =$

$w^4\pi^{w^3}\pi^{w^2}\pi^w\pi.$ □

Let w = (1, 20, 23, 6, 21, 10)(2, 5, 12, 3, 16, 7)(4, 19, 14, 17, 24, 15) $\sim$ yx^2. (8, 11, 18, 9, 22, 13).

Additional Relation: $(wtt)^4 = e \implies w^4(t^2)^{w^3}(t^2)^{w^2}(t^2)^{w}tt = e \implies w^4(t_{19})^{w^3}(t_{19})^{w^2}(t_{19})^{w}t_{19} = e \implies w^4t_{24}t_{17}t_{14}t_{19} = e$.

Additional Relation: $(wt_{20})^4 = e \implies w^4{t_{20}}^{w^3}{t_{20}}^{w^2}{t_{20}}^{w}t_{20} = e \implies w^4t_{21}t_6t_{23}t_{20} = e$.

Lemma 5.27. $t_{24}t_{17} = w^2t_7t_2$.

Proof. $w^4t_{24}t_{17}t_{14}t_{19} = e \implies w^2w^4t_{24}t_{17}t_{14}t_{19}{t_{19}}^{-1}{t_{14}}^{-1} = w^2e{t_{19}}^{-1}{t_{14}}^{-1} \implies t_{24}t_{17} = w^2{t_{19}}^{-1}{t_{14}}^{-1} \implies t_{24}t_{17} = w^2t_7t_2$. □

Lemma 5.28. $t_{21}t_6 = w^2t_8t_{11}$.

Proof. $w^4t_{21}t_6t_{23}t_{20} = e \implies w^2w^4t_{21}t_6t_{23}t_{20}{t_{20}}^{-1}{t_{23}}^{-1} = w^2e{t_{20}}^{-1}{t_{23}}^{-1} \implies t_{21}t_6 = w^2{t_{20}}^{-1}{t_{23}}^{-1} \implies t_{21}t_6 = w^2t_8t_{11}$ by notation. □

Lemma 5.29. $t_{17}t_{14}t_{19} = w^2t_{16}$.

Proof. $t_{24}t_{17} = w^2t_7t_2$ 5.27. $\implies t_{24}t_{17}{t_2}^{-1} = w^2t_7t_2{t_2}^{-1} \implies t_{24}t_{17}t_{14} = w^2t_7$ since ${t_2}^{-1} = t_{14} \implies (t_{24}t_{17}t_{14})^p = (w^2t_7)^p$ where p = (1, 10, 21, 6, 23, 20)(2, 7, 16, 3, 12, 5)(4, 15, 24, 17, 14, 19)(8, 13, 22, 9, 18, 11) $\implies t_{17}t_{14}t_{19} = w^2t_{16}$. □

Lemma 5.30. $t_1t_2 = x^{-2}t_{16}$.

Proof. $t_1t_2 = t_{19}t_{13}t_{20}t_{14}$ by splitting t_1 and t_2.

Let p = (1, 7, 13, 19)(2, 20, 14, 8)(3, 23, 10, 12)(4, 6, 21, 17)(5, 16, 18, 9)(11, 22, 24, 15) $\in$ N $\implies$ (Lemma 5.27). $(t_{24}t_{17})^p = (w^2t_7t_2)^p \implies t_{13}t_{20} =$

$x^{-1}yxyx^{-1}t_{15}t_4$ where $x^{-1}yxyx^{-1} \sim (1, 11, 4)(2, 24, 9)(3, 20, 18)(5, 22, 19)(6, 15, 8)(7, 17, 10)(12, 21, 14)(13, 23, 16) \implies$

$t_{19}(t_{13}t_{20})t_{14} = t_{19}(x^{-1}yxyx^{-1}t_{15}t_4)t_{14} \implies$

$t_{19}(x^{-1}yxyx^{-1}t_{15}t_4)t_{14} = x^{-1}yxyx^{-1}t_5t_{15}t_4t_{14} \implies$

$x^{-1}yxyx^{-1}t_5(t_{15})t_4t_{14} = x^{-1}yxyx^{-1}t_5(t_9t_3)t_4t_{14}$.

Let $p = (1, 6, 9, 17, 14, 22)(2, 10, 13, 18, 21, 5)(3, 11, 8, 16, 19, 24)(4, 7, 12, 15, 23, 20) \in N \implies (t_{21}t_6)^p = (w^2t_8t_{11})^p$ (Lemma 5.28). $\implies$

$t_5t_9 = x^{-2}yxyt_{16}t_8$ where $x^{-2}yxy \sim (1, 16, 21)(2, 11, 12)(3, 7, 22)(4, 9, 13)(5, 6, 20)(8, 17, 18)(10, 15, 19)(14, 23, 24) \implies$

$x^{-1}yxyx^{-1}(t_5t_9)t_3t_4t_{14} = x^{-1}yxyx^{-1}(x^{-2}yxyt_{16}t_8)t_3t_4t_{14}$ where $x^{-1}yxyx^{-1}x^{-2}yxy = xyx^2yx^2$.

Let $p = (1, 8, 24, 22)(2, 18, 16, 19)(3, 9, 15, 21)(4, 7, 14, 6)(5, 23, 17, 11)(10, 13, 20, 12) \in N \implies (t_{21}t_6)^p = (w^2t_8t_{11})^p$ (Lemma 5.28) $\implies$

$t_3t_4 = x^2yx^{-1}yt_{24}t_5$ where $x^2yx^{-1}y \sim (1, 24, 16)(2, 11, 21)(3, 8, 17)(4, 13, 12)(5, 15, 20)(6, 22,7)(9, 14, 23)(10, 19, 18) \implies$

$xyx^2yx^2t_{16}t_8(t_3t_4)t_{14} = xyx^2yx^2t_{16}t_8(x^2yx^{-1}yt_{24}t_5)t_{14} \implies$

$xyx^2yx^2t_{16}t_8(x^2yx^{-1}yt_{24}t_5)t_{14} = xyx^2yx^2x^2yx^{-1}yt_1t_{17}t_{24}t_5t_{14}$ where $xyx^2yx^2x^2yx^{-1}y = xyx^{-2}yx$.

Let $p = (1, 19, 13, 7)(2, 3, 11, 18)(4, 10, 16, 22)(5, 12, 20, 21)(6, 14, 15, 23)(8, 9, 17, 24) \in N \implies$ (Lemma 5.27) $(t_{24}t_{17})^p = (w^2t_7t_2)^p \implies t_5t_{14} = yx^{-1}yxt_9t_{18}$ where $yx^{-1}yx \sim (1, 12, 11)(2, 4, 9)(3, 20, 22)(5, 19, 6)(7, 18, 17)(8, 10, 15)(13, 24, 23)(14, 16, 21) \implies$

$xyx^{-2}yxt_1t_{17}t_{24}(t_5t_{14}) = xyx^{-2}yxt_1t_{17}t_{24}(yx^{-1}yxt_9t_{18}) \implies$

$xyx^{-2}yxt_1t_{17}t_{24}(yx^{-1}yxt_9t_{18}) = xyx^{-2}yxyx^{-1}yxt_{12}t_7t_{23}t_9t_{18}$ where $xyx^{-2}yxyx^{-1}yx = (x^2y)^2$.

Let p = (1, 20, 9, 18)(2, 15, 24, 7)(3, 12, 19, 14)(4, 22, 16, 10)(5, 11, 17, 23)(6, 13, 8, 21) $\in$ N $\implies$ $(t_{24}t_{17})^p = (w^2t_7t_2)^p$ (Lemma 5.27). $\implies$

$t_7t_{23} = x^{-2}yxyt_2t_{15}$ where $x^{-2}yxy \sim$ (1, 16, 21)(2, 11, 12)(3, 7, 22)(4, 9, 13)(5, 6, 20)(8, 17, 18)(10, 15, 19)(14, 23, 24) $\implies$

$(x^2y)^2t_{12}(t_7t_{23})t_9t_{18} = (x^2y)^2t_{12}(x^{-2}yxyt_2t_{15})t_9t_{18} \implies$

$(x^2y)^2t_{12}(x^{-2}yxyt_2t_{15})t_9t_{18} = (x^2y)^2x^{-2}yxyt_2t_2t_{15}t_9t_{18}$ where $(x^2y)^2x^{-2}yxy = (yx)^2 \implies$

$(yx)^2(t_2t_2)t_{15}t_9t_{18} = (yx)^2(t_{20})t_{15}t_9t_{18} \implies$

$(yx)^2t_{20}(t_{15}t_9)t_{18} = (yx)^2t_{20}(t_{21})t_{18}$.

Let p = (1, 24, 2)(3, 17, 10)(4, 21, 11)(5, 22, 15)(6, 8, 7)(9, 23, 16)(12, 14, 13)(18, 20, 19).

$t_{17}t_{14}t_{19} = w^2t_{16}$ (Lemma 5.28). $\implies$

$(t_{17}t_{14}t_{19})^p = (w^2t_{16})^p \implies t_{20}t_{21}t_{18} = yxyx^{-1}t_{16} \implies$

$(yx)^2t_{20}t_{21}t_{18} = (yx)^2yxyx^{-1}t_{16} = x^{-2}t_{16} \implies t_1t_2 = x^{-2}t_{16}$. □

5.4.1 Double Cosets [*].

Let N = ⟨ xx, yy ⟩ and Let X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, . . . , 23, 24}. Orbits(N) = { 1, 2, 15, 4, 5, 19, 12, 6, 20, 9, 21, 10, 22, 23, 13, 7, 18, 24, 14, 3, 8, 16, 17, 11} = X $\implies$ $\forall i \in X, t_i \in [1]$.

The only right coset of NeN = [*] is N $\implies$ |[*]| = 1, which is symbolized by placing "1" inside the circle representing [*]. The coset stabilizer of coset N is N. Therefore, the only orbit of N on X = { 1, 2, 15, 4, 5, 19, 12, 6, 20, 9, 21, 10, 22, 23, 13, 7, 18, 24, 14, 3, 8, 16, 17, 11} = X , hence $\forall i \in X, t_i \in [1]$, which yields the cosets $Nt_1, Nt_2, \ldots, Nt_{24} \in [1]$. This is symbolized by placing a "24" next to the circle representing [*] in the diagram.

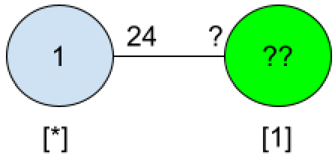

Figure 5.25: DCE of $[*]$.

5.4.2 Double Cosets [1].

Stabilizer of 1 denoted by $N^1 = \langle$ (2, 23, 21, 16, 24)(3, 22, 6, 8, 5)(4, 12, 14, 11, 9)(10, 18, 20, 17, 15)$\rangle$. Since there are no relations that take t_i to t_j for $0 \leq$ i,j ≤ 24, therefore the stabilizing group denoted by $N^{(1)} = N^1$. $|[1]| = |N| \div |N^{(1)}| = 120 \div 5 = 24$. This is symbolized by placing a "24" inside the circle representing [1] in the diagram.

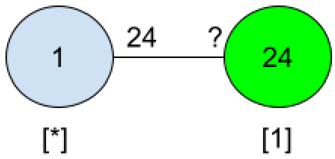

Figure 5.26: Size of [1].

Now the orbits $N^{(1)}$ on X = { 1 }, { 7 }, { 13 }, { 19 }, { 2, 23, 21, 16, 24 }, { 3, 22, 6, 8, 5 }, { 4, 12, 14, 11, 9 }, { 10, 18, 20, 17, 15 }. The first orbit gives us $Nt_1t_1 = Nt_{19} \in [1]$.

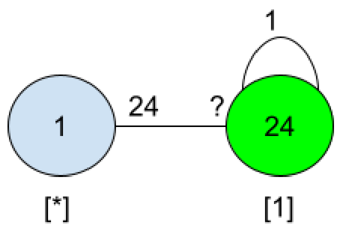

Figure 5.27: DCE up to the First Orbit of [1].

The second orbit gives us $Nt_1t_7 = Nt_1({t_1}^2) = N{t_1}^3 = Nt_8 \in [1]$.

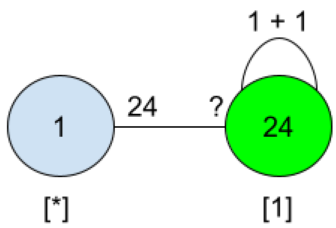

Figure 5.28: DCE up to the Second Orbit of [1].

The third orbit gives us $Nt_1t_{13} = N{t_1}^4t_1 = N{t_1}^5 = Ne = N \in [*]$.

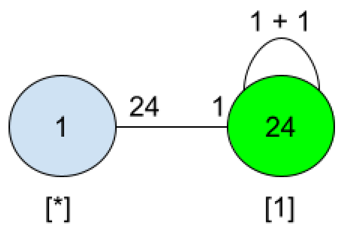

Figure 5.29: DCE up to the Third Orbit of [1].

The fourth orbit gives us $Nt_1t_{19} = Nt_1{t_1}^2 = N{t_1}^3 = Nt_7 \in [1]$.

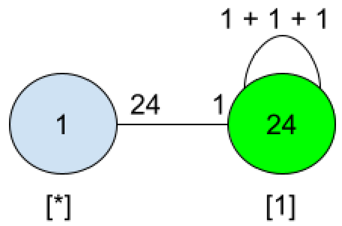

Figure 5.30: DCE up to the Fourth Orbit of [1].

The fifth orbit gives us $Nt_1t_2 = Nx^{-2}t_{16}$ (Lemma 5.30). $Nx^{-2}t_{16} = Nt_{16}$ $\in$ [1], therefore $Nt_1t_2 \in$ [1], which shows Nt_1t_{23}, Nt_1t_{21}, Nt_1t_{16}, $Nt_1t_{24} \in$ [1].

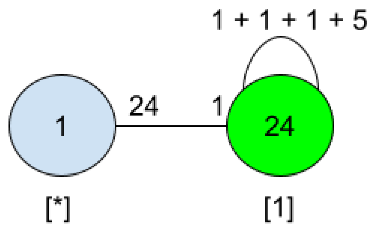

Figure 5.31: DCE up to the Fifth Orbit of [1].

The sixth orbit gives us $Nt_1t_3 \in$ [1, 3], therefore Nt_1t_{22}, Nt_1t_6, Nt_1t_8, $Nt_1t_5 \in$ [1, 3].

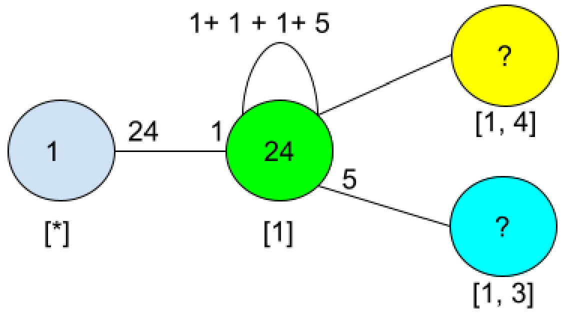

Figure 5.32: DCE up to the Sixth Orbit of [1].

The seventh orbit gives us $Nt_1t_4 \in [1, 4]$, therefore Nt_1t_{12}, Nt_1t_{14}, Nt_1t_{11}, $Nt_1t_9 \in [1, 4]$.

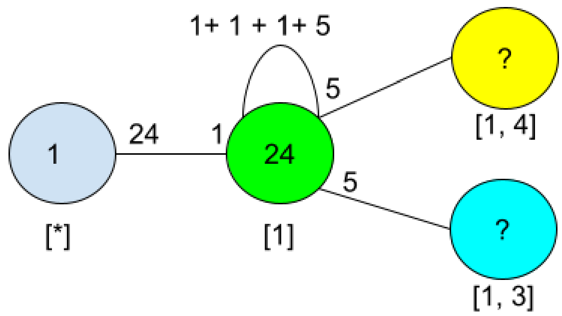

Figure 5.33: DCE up to the Seventh Orbit of [1].

From the eighth orbit we get Nt_1t_{20}, Nt_1t_{10}, Nt_1t_{18}, Nt_1t_{17}, Nt_1t_{15}.

Lemma 5.31. *Nt_1t_{20}, Nt_1t_{10}, Nt_1t_{18}, Nt_1t_{17}, $Nt_1t_{15} \in [1, 4]$.*

Proof. $Nt_1t_{20} = Nt_1t_2t_2$ since $t_{20} = t_2t_2$.

$N(t_1t_2)t_2 = N(x^{-2}t_{16})t_2$ (Lemma 5.30): $\implies$

$Nt_1t_{20} = Nt_{16}t_2$.

Let p =(1, 23, 12, 14, 16)(2, 4, 13, 11, 24)(5, 18, 20, 22, 7)(6, 8, 10, 19, 17) $\implies$ $N(t_{16}t_2)^p = Nt_1t_4 \implies Nt_1t_{20} \in [1, 4] \implies Nt_1t_{10}, Nt_1t_{18}, Nt_1t_{17}, Nt_1t_{17}, \in [1, 4]$. □

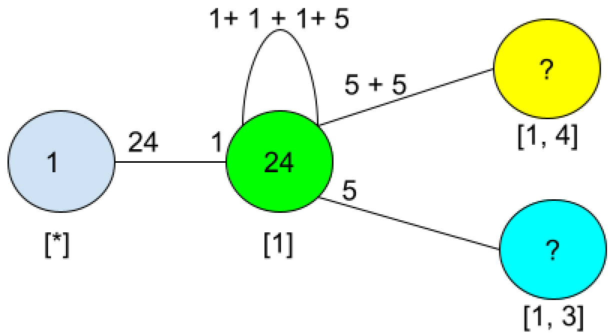

Figure 5.34: DCE [1].

5.4.3 Double Cosets [1,3]

The $N^{1,3} = \langle e \rangle$, therefore $N^{(1,3)} \geq \langle e \rangle$. Let p = (1, 2, 24)(3, 10, 17)(4, 11, 21)(5, 15, 22)(6, 7, 8)(9, 16, 23)(12, 13, 14)(18, 19, 20), hence $t_{24}t_{17} = w^2t_7t_2$ (Lemma 5.27). Now, $(t_{24}t_{17})^p = (w^2t_7t_2)^p$, which means $t_1t_3 = x^{-1}yxyt_8t_{24}$. This gives us $Nt_1t_3 = Nx^{-1}yxyt_8t_{24}$. Consequently, $Nt_1t_3 = Nt_8t_{24}$, resulting in $n = (1,8)(2,19)(3,24)(4,5)(6,9)(7,14)(10,11)(12,15)(13,20)(16,17)(18,21)(22,23) \in N^{(1,3)}$ since $N(t_1t_3)^n = Nt_8t_{24}$.

Now, $t_1(t_3) = t_1(t_{21}t_{15})$ by notation. Let p = (2, 21, 24, 23, 16)(3, 6, 5, 22, 8)(4, 14, 9, 12, 11)(10, 20, 15, 18, 17). Now, $t_1t_2 = x^{-2}t_{16}$(Lemma 5.27), therefore $(t_1t_2)^p = (x^{-2}t_{16})^p$. This gives us $t_1t_{21} = x^{-1}yxyxt_2$, we substitute to obtain $(t_1t_{21})t_{15} = (x^{-1}yxyxt_2)t_{15}$. Consequently, $Nt_1t_3 = Nt_2t_{15}$, which shows $n = (1,2)(3,15)(4,16)(5,6)(7,8)(9,21)(10,2)(11,12)(13,14)(17,18)(19,20)(23,24) \in N^{(1,3)}$

since $N(t_1t_3)^n = Nt_2t_{15}$.

Let p = (1, 7, 13, 19)(2, 20, 14, 8)(3, 23, 10, 12)(4, 6, 21, 17)(5, 16, 18, 9)(11, 22, 24, 15) $\in N^{(1,3)}$ since $Nt_1t_3 = Nt_2t_{15} = N(t_8t_{24})^p$. Therefore, $p^2 =$ (1, 13)(2, 14)(3, 10)(4, 21)(5, 18)(6, 17)(7, 19)(8, 20)(9, 16)(11, 24)(12, 23)(15, 22) $\in N^{(1,3)}$. Now, $N^{(1,3)} \geq P = \langle$ (1, 8)(2, 19)(3, 24)(4, 5)(6, 9)(7, 14)(10, 11)(12, 15)(13, 20)(16, 17)(18, 21)(22, 23), (1, 2)(3, 15)(4, 16)(5, 6)(7, 8)(9, 21)(10, 2)(11, 12)(13, 14)(17, 18)(19, 20)(23, 24), (1, 2)(3, 15)(4, 16)(5, 6)(7, 8)(9, 21)(10, 2)(11, 12)(13, 14)(17, 18)(19, 20)(23, 24)$\rangle$, which gives us $|[1, 3]| = |N| \div |N^{(1,3)}| \leq 120 \div 8 = 15$.

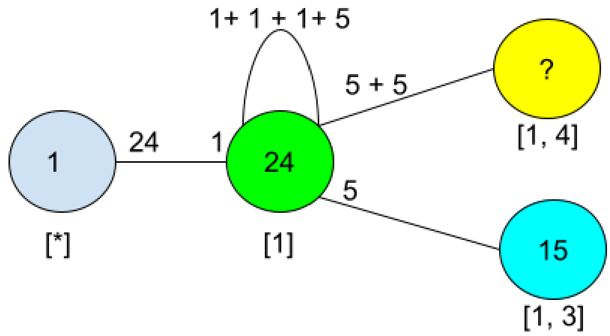

Figure 5.35: Size of [1,3].

The orbits of P are { 1, 20, 7, 14, 8, 13, 2, 19 }, { 3, 11, 23, 22, 24, 10, 15, 12}, and { 4, 18, 6, 9, 5, 21, 16, 17}. From the first orbit we choose ‘1’ as the representative:

Lemma 5.32. *$Nt_1t_3t_1$, $Nt_1t_3t_{20}$, $Nt_1t_3t_7$, $Nt_1t_3t_{14}$, $Nt_1t_3t_8$, $Nt_1t_3t_{13}$, $Nt_1t_3t_2$, and $Nt_1t_3t_{19} \in$ [1, 3].*

Proof. Let p = (1, 10, 24, 3, 2, 17)(4, 18, 21, 20, 11, 19)(5, 13, 22, 12, 15, 14)(6, 9, 8, 23, 7, 16).

$t_{24}t_{17} = w^2t_7t_2$ (Lemma 5.27). $\implies$ $(t_{24}t_{17})^p = (w^2t_7t_2)^p$ $\implies$ $t_3t_1 = (x^{-1}yx^{-1})^2t_{16}t_{17}$ where $x^{-1}yx^{-1})^2 \sim (1, 14, 4)(2, 16, 13)(3, 18, 5)(6, 17, 15)(7, 20, 10)(8, 22, 19)(9, 24, 11)(12, 23, 21)$ $\implies$

$t_1(t_3t_1) = t_1((x^{-1}yx^{-1})^2t_{16}t_{17})$ $\implies$

$t_1((x^{-1}yx^{-1})^2t_{16}t_{17}) = (x^{-1}yx^{-1})^2t_{14}t_{16}t_{17}$.

Let $p = (1, 14, 4)(2, 16, 13)(3, 18, 5)(6, 17, 15)(7, 20, 10)(8, 22, 19)(9, 24, 11)(12, 23, 21)$.

$t_1t_2 = x^{-2}t_{16}$ (Lemma 5.27). $\implies$ $(t_1t_2)^p = (x^{-2}t_{16})^p$ $\implies$ $t_{14}t_{16} = x^{-2}yxyx^{-1}t_{13}$ $\implies$

$(x^{-1}yx^{-1})^2(t_{14}t_{16})t_{17} = (x^{-1}yx^{-1})^2(x^{-2}yxyx^{-1}t_{13})t_{17} = w^2t_{13}t_{17}$ $\implies$ $Nt_1t_3t_1 = Nt_{13}t_{17}$.

Let $p = (1, 13)(2, 4)(3, 17)(5, 15)(6, 18)(7, 19)(8, 10)(9, 23)(11, 21)(12, 24)(14, 16)(20, 22)$. $\implies$

$N(t_1t_3t_1)^p = N(t_{13}t_{17})^p$ $\implies$ $N(t_1t_3t_1)^p = Nt_1t_3$ $\implies$ $Nt_1t_3t_1 \in [1, 3]$ $\implies$ $Nt_1t_3t_{20}$, $Nt_1t_3t_7$, $Nt_1t_3t_{14}$, $Nt_1t_3t_8$, $Nt_1t_3t_{13}$, $Nt_1t_3t_2$, and $Nt_1t_3t_{19} \in [1, 3]$ since 1, 20, 7, 14, 8, 13, 2, 19 belong to the same orbit. □

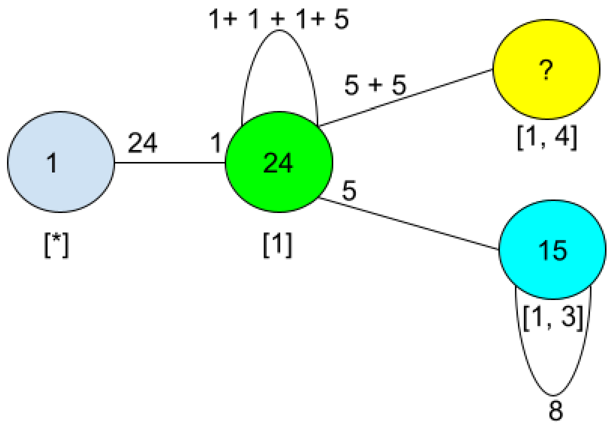

Figure 5.36: DCE up to the First Orbit of [1,3].

From the 2nd orbit we choose '3' as the representative.

Lemma 5.33. *$Nt_1t_3t_3 \in [1]$.*

Proof. $\mathrm{Nt_1t_3t_3 = Nt_1t_{21}}$ by notation $\mathrm{t_3t_3 = t_{21}} \implies \mathrm{Nt_1t_3t_3 = Nt_1t_{21}} \in [1]$ as seen before. $\implies$ $\mathrm{Nt_1t_3t_3}$, $\mathrm{Nt_1t_3t_{11}}$, $\mathrm{Nt_1t_3t_{23}}$, $\mathrm{Nt_1t_3t_{22}}$, $\mathrm{Nt_1t_3t_{24}}$, $\mathrm{Nt_1t_3t_{10}}$, $\mathrm{Nt_1t_3t_{15}}$, and $\mathrm{Nt_1t_3t_{12}} \in [1]$. □

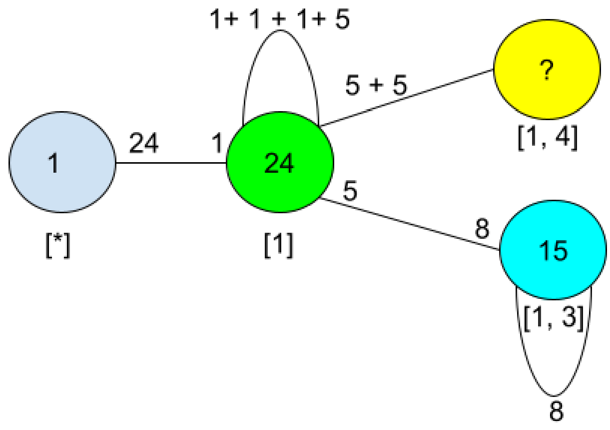

Figure 5.37: DCE [1,3].

From the 2nd orbit we choose '21' as the representative. $Nt_1t_3t_{21} = Nt_1t_9 \in [1,4]$ as seen before. Therefore, $Nt_1t_3t_4$, $Nt_1t_3t_{18}$, $Nt_1t_3t_6$, $Nt_1t_3t_9$, $Nt_1t_3t_5$, $Nt_1t_3t_{16}$, and $Nt_1t_3t_{17} \in [1,4]$.

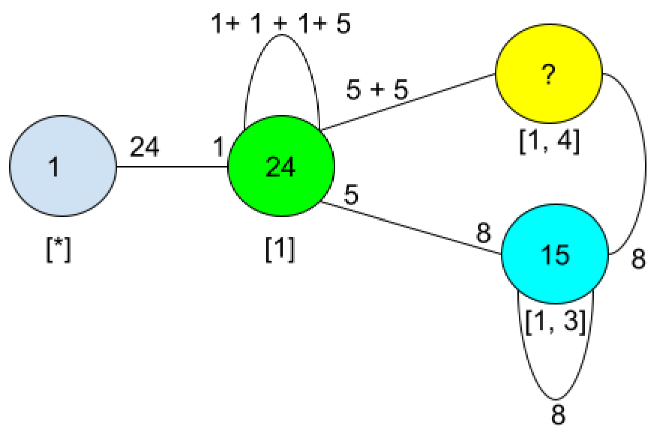

Figure 5.38: DCE of [1,3].

5.4.4 Double Cosets [1,4].

Lemma 5.34. $t_1t_4 = x^2yx^{-1}\ yxt_{23}t_{22} = \ xyx^{-1}yx^2t_6t_{14}$.

Proof. Let p = (1, 23, 12, 14, 16)(2, 4, 13, 11, 24)(5, 18, 20, 22, 7)(6, 8, 10, 19, 17) $\in$ N.

$t_1t_2 = x^{-2}t_{16}$ (Lemma 5.30) $\implies x^2t_1t_2 = t_{16} \implies (x^2t_1t_2)^p = (t_{16})^p \implies$

$t_1 = x^2yx^{-1}yxt_{23}t_4 \implies$

$(t_1)t_4 = (x^2yx^{-1}yxt_{23}t_4)t_4 \implies$

$x^2yx^{-1}yxt_{23}(t_4t_4) = x^2yx^{-1}yxt_{23}(t_{22})$.

Let p = (2, 24, 16, 21, 23)(3, 5, 8, 6, 22)(4, 9, 11, 14, 12)(10, 15, 17, 20, 18) $\in$ N.

$t_{21}t_6 = w^2t_8t_{11}$ (Lemma 5.28). $\implies (t_{21}t_6)^p = (w^2t_8t_{11})^p \implies t_{23}t_{22} = xyx^{-2}yt_6t_{14}$. $\implies x^2yx^{-1}yx(t_{23}t_{22}) = x^2yx^{-1}yx(xyx^{-2}yt_6t_{14}) = xyx^{-1}yx^2t_6t_{14}$. $\square$

The $N^{1,4} = \langle e \rangle \implies N^{(1,4)} \geq \langle e \rangle$. Now $t_1t_4 = x^2yx^{-1}\ yxt_{23}t_{22}$ (Lemma 5.34). $\implies Nt_1t_4 = Nx^2yx^{-1}\ yxt_{23}t_{22} = N_{23}t_{22}$. Now $t_1t_4 = xyx^{-1}yx^2t_6t_{14}$ (Lemma 5.34). $\implies Nt_1t_4 = Nxyx^{-1}yx^2t_6t_{14} \implies Nt_1t_4 = Nt_6t_{14}$. Let n = (1, 3)(2, 10)(4, 20)(5, 12)(6, 23)(7, 9)(8, 16)(11, 18)(13, 15)(14, 22)(17, 24)(19, 21) $\implies N(t_1t_4)^p = N(t_{23}t_{22})^p = Nt_6t_{14} = Nt_1t_4 \implies N(t_1t_4)^p = Nt_1t_4 \implies n \in N^{(1,4)}$.

Lemma 5.35. $t_1t_4 = (yx^{-2})^2t_{24}t_{21}$.

Proof. Let p = (1, 9, 16, 2, 11)(3, 10, 20, 5, 19)(4, 14, 23, 13, 21)(7, 15, 22, 8, 17) $\in$ N.

$t_1t_2 = x^{-2}t_{16}$ (Lemma 5.30). $\implies x^2t_1t_2 = t_{16} \implies (x^2t_1t_2)^p = (t_{16})^p \implies t_4 = xyxyx^{-1}t_{23}t_{21} \implies$

$t_1(t_4) = t_1(xyxyx^{-1}t_{23}t_{21}) \implies$

$(t_1xyxyx^{-1})t_{23}t_{21} = (xyxyx^{-1}t_1)t_{23}t_{21}$.

Let p = (1, 6, 11, 15, 4, 8)(2, 19, 24, 5, 9, 22)(3, 16, 20, 13, 18, 23)(7, 12, 17, 21, 10, 14) $\in$ N.

$t_1t_2 = x^{-2}t_{16}$ (Lemma 5.30). $(t_1t_2)^p = (x^{-2}t_{16})^p$ $t_1t_{23} = x^{-2}yxyx^{-1}t_{24}$. $\implies$ $xyxyx^{-1}(t_1t_{23})t_{21} = xyxyx^{-1}(x^{-2}yxyx^{-1}t_{24})t_{21} = t_1t_4 = (yx^{-2})^2t_{24}t_{21}$. □

Now, $t_1t_4 = (yx^{-2})^2t_{24}t_{21}$ (Lemma 5.34), therefore $Nt_1t_4 = N(yx^{-2})^2t_{24}t_{21}$. We now have $Nt_1t_4 = Nt_{24}t_{21}$. Let n = (1, 24, 2)(3, 17, 10)(4, 21, 11)(5, 22, 15)(6, 8, 7)(9, 23, 16)(12, 14, 13)(18, 20, 19), hence $N(t_1t_4)^n = Nt_{24}t_{21} = Nt_1t_4$. Consequently, $N(t_1t_4)^n = Nt_1t_4$, which means $n \in N^{(1,4)}$.

Let P = ⟨ (1, 3)(2, 10)(4, 20)(5, 12)(6, 23)(7, 9)(8, 16)(11, 18)(13, 15)(14, 22)(17, 24)(19, 21), (1, 24, 2)(3, 17, 10)(4, 21, 11)(5, 22, 15)(6, 8, 7)(9, 23, 16)(12, 14, 13)(18, 20, 19) ⟩ = ⟨ (1, 17, 2, 3, 24, 10)(4, 19, 11, 20, 21, 18)(5, 14, 15, 12, 22, 13)(6, 16, 7, 23, 8, 9) ⟩, therefore $N^{(1,4)} \geq P$. Consequently, $|[1, 4]| = |N| \div |N^{(1,4)}| \leq 120 \div 6 = 20$. Orbits of P on X are {1, 10, 3, 17, 2, 24}, { 4, 18, 20, 19, 11, 21}, { 5, 13, 12, 14, 15, 22 }, { 6, 9, 23, 16, 7, 8 }.

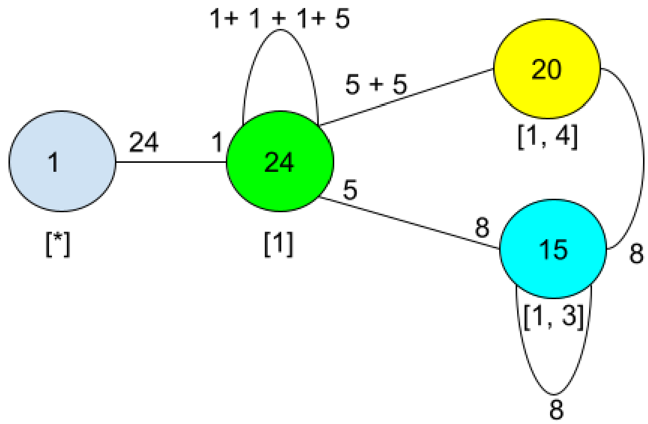

Figure 5.39: Size of [1,4].

From the 1st orbit we choose '10' as the representative. $Nt_1t_4t_{10} = Nt_1t_{16}$ since $t_{16} = t_4t_{10}$, therefore $Nt_1t_4t_{10} \in [1]$ since $Nt_1t_{16} \in [1]$. Consequently, $Nt_1t_4t_1$, $Nt_1t_4t_{10}$, $Nt_1t_4t_3$, $Nt_1t_4t_{17}$, $Nt_1t_4t_2$, $Nt_1t_4t_{24} \in [1]$.

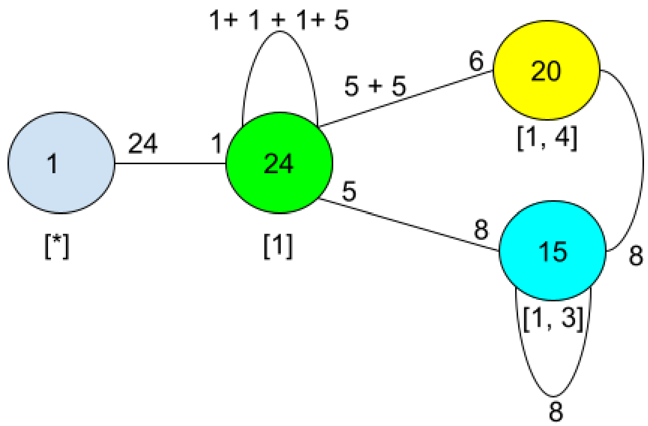

Figure 5.40: DCE up to the First Orbit of [1,4].

From the 2nd orbit we choose '4' as the representative. $Nt_1t_4t_4 = Nt_1t_{22}$

by notation, therefore $Nt_1t_4t_4 \in [1,3]$ since $Nt_1t_{22} \in [1,3]$. Consequently,

$$Nt_1t_4t_4,\ Nt_1t_4t_{18},\ Nt_1t_4t_{20},\ Nt_1t_4t_{19},\ Nt_1t_4t_{11},\ Nt_1t_4t_{21} \in [1,3].$$

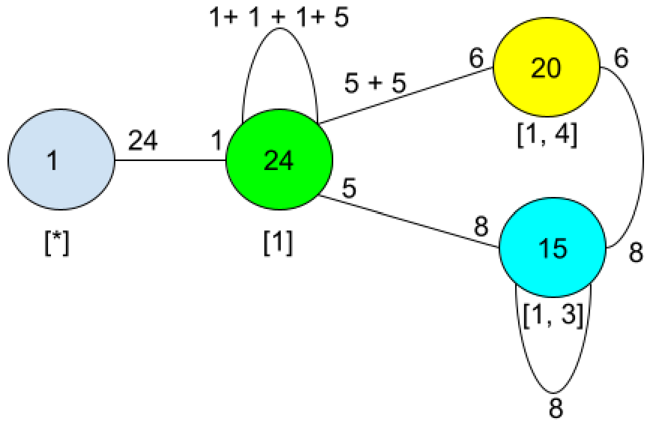

Figure 5.41: DCE up to the Second Orbit of [1,4].

From the 3rd orbit we choose '22' as the representative. $Nt_1t_4t_{22} = Nt_1t_{10}$ since $t_{10} = t_4t_{22}$, therefore $Nt_1t_4t_{22} \in [1,4]$ since $Nt_1t_{10} \in [1,4]$. Consequently, $Nt_1t_4t_5$, $Nt_1t_4t_{13}$, $Nt_1t_4t_{12}$, $Nt_1t_4t_{14}$, $Nt_1t_4t_{15}$, $Nt_1t_4t_{22} \in [1,4]$.

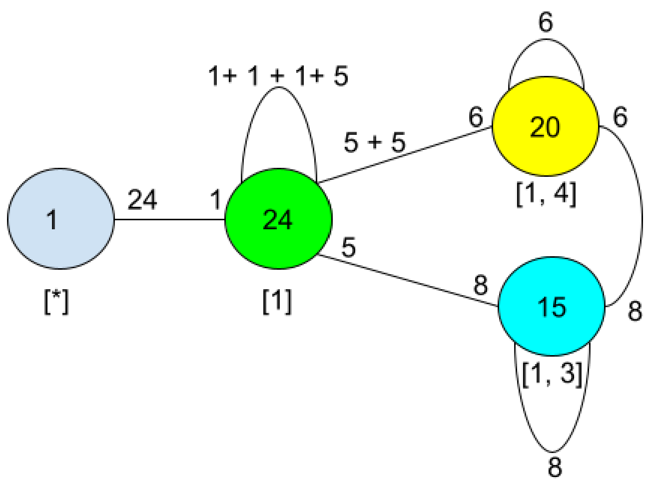

Figure 5.42: DCE up to the Third Orbit of [1,4].

From the 4rd orbit we choose '16' as the representative. $Nt_1t_4t_{16} = Nt_1$ since $t_4t_{16} = t_4{t_4}^4 = e$, therefore $Nt_1t_4t_{16} \in [1]$. Consequently, $Nt_1t_4t_6$, $Nt_1t_4t_9$, $Nt_1t_4t_{23}$, $Nt_1t_4t_{16}$, $Nt_1t_4t_7$, $Nt_1t_4t_8 \in [1]$.

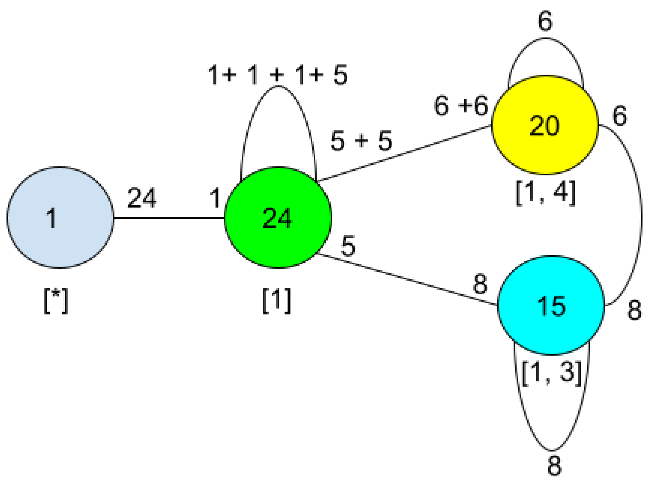

Figure 5.43: Cayley Diagram of $(A_5)^2$:2 Over $A_5 : 2$.

Our augment shows that $|G| \leq 60(120) = 7200$. Let X be the set of the 60 right cosets that we have found. Now $\langle x, y, t \rangle$ act on X and it readily checks that $|f(x), f(y), f(t)| = 7200$. By the First Isomorphism Theorem, there exist a, b in G that satisfy a presentation of $(A_5)^2 : 2$. Thus, $(A_5)^2 : 2 \leq 7200$. Now $|G| = |(A_5)^2 : 2| = 7200$. So $G \cong (A_5)^2 : 25$.

Chapter 6

An Algorithm To Prove Coset Relations

Let N be a transitive permutation group. Let G $\cong$ $(|t|)^{*n}$:N factor by a set of relations, where n is the number of letters of N.

Step 1: We extract the first and second relations from G and rewrite them in the form of no-empty t-word = permutation times another t-word. We save all relations into a list to be used later. Let's call this list LR. For example:

Let G = $\langle$ x^5, y^2,$(x^{-1}y)^4$,$(xyx^{-2}yx)^2$, t^5,(t, xyx^2yx^{-1}), $t^{x^{-1}y}$= t^3, $(x,tt^{x^2})^2$,$(x^y t)^4$, $X^{-1}yx^{-1}yxyxtxt^2xt\rangle$.

In chapter 5 we use the relations: $t_4t_8 = yx^2yx^2yt^{19}$, $t_{19} = yx^{-2}yx^{-2}yt_4t_8$, $t_{19}t_9 = yx^{-1}yt_{12}t_{11}$, $t_{12}t_{11} = yxyt_{19}t_9$, $t_9t_2t_{22}t_{19} = t_1t_4t_{20}t_{15}$, and $t_1t_4t_{20}t_{15} = t_9t_2t_{22}t_{19}$.

We must also include some of the labeling of the powers of t. In chapter 5 we use: $t_1^2 = t_7$, $t_7 = t_1^2$, t_1t_{19} = Id(N), and $t_1t_1t_1t_1t_1$ = Id(N). The equations $t_1 = t_7t_{19}$ and $t_7t_{19} = t_1$ also need to be included.

Step 2: We start with an equation that we want to prove. For example, $t_1t_3 = pt_2t_{23}t_9$, where p is allowed to be any permutation of N. t_1t_3 is the starting t-word and $pt_2t_{23}t_9$ is what we are looking for. We add the starting t-word to a tree structure, let's name this tree Elements. A function called GetReationsToUse is used to conjugate all the relations LR by all the permutations in N. This is done every time the function is used. Each time we conjugate a relation, we check to see if the resulting equation can be applied to the t-word being processed. If the equation can be applied, then we add it to an array along with the proof of the new relation. The following is the Magma output of the list of relations that can be applied to t_1t_3:

```
-----------------------------------------
Let p = (1, 19)(2, 4)(3, 23)(5, 21)(6, 24)(7, 13)(8, 10)(9,
    17)(11, 15)(12,
18)(14, 16)(20, 22) belong to N.
Lemma 3: t19 = yx2yx-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = x * y * x^2 * y * x^-1 * yt2t10. Apply at 1.
Found a new name:  x * y * x^2 * y * x^-1 * y*t2t10t3.
-----------------------------------------
Let p = (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9,
    20)(10, 24)(11,
```

17)(14, 21)(18, 22) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t1 = y ∗ x ∗ y ∗ x^2 ∗ y ∗ x^−1t12t3. Apply at 1.

Found a new name: y ∗ x ∗ y ∗ x^2 ∗ y ∗ x^−1∗t12t3t3.

Let p = (1, 19)(2, 20)(3, 16)(4, 9)(5, 24)(6, 23)(7, 13)(8, 14)(10, 21)(11,

18)(12, 17)(15, 22) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t1 = y ∗ x ∗ y ∗ x^−2 ∗ yt9t14. Apply at 1.

Found a new name: y ∗ x ∗ y ∗ x^−2 ∗ y∗t9t14t3.

Let p = (1, 19)(2, 18)(3, 21)(4, 11)(5, 16)(6, 8)(7, 13)(9, 15)(10, 23)(12,

20)(14, 24)(17, 22) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t1 = x^−1 ∗ y ∗ x^2t11t6. Apply at 1.

Found a new name: x^−1 ∗ y ∗ x^2∗t11t6t3.

Let p = (1, 19)(2, 17)(3, 24)(4, 22)(5, 8)(6, 21)(7, 13)(9, 18)(10, 16)(11,

```
20)(12, 15)(14, 23) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = y * x^2 * y * x^2 * yt22t5. Apply at 1.
Found a new name:  y * x^2 * y * x^2 * y*t22t5t3.
----------------------------------------
Let p = (1, 21)(2, 5)(3, 19)(4, 6)(7, 15)(8, 11)(9, 13)(10,
     12)(14, 17)(16,
18)(20, 23)(22, 24) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x^-2 * y * x^-2t6t11. Apply at 2.
Found a new name:  x^-2 * y * x^-2*t13t6t11.
----------------------------------------
Let p = (1, 21, 22, 5, 12, 8)(2, 13, 9, 10, 17, 6)(3, 4,
    23, 18, 14, 19)(7, 15,
16, 11, 24, 20) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x * y * x^-1 * y * x * y * xt23t1. Apply at 2.
Found a new name:  x * y * x^-1 * y * x * y * x*t21t23t1.
----------------------------------------
Let p = (1, 21, 20, 16)(2, 10, 19, 3)(4, 7, 15, 14)(5, 17,
    23, 11)(6, 12, 24,
```

```
18)(8, 22, 13, 9) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x^2 * y * x^-1t7t22. Apply at 2.
Found a new name:  x^2 * y * x^-1*t10t7t22.
----------------------------------------
Let p = (1, 21, 18, 23)(2, 8, 20, 14)(3, 12, 5, 19)(4, 16,
   22, 10)(6, 17, 7,
15)(9, 24, 11, 13) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x * y * x^-1 * y * x^2 * yt16t20. Apply at 2.
Found a new name:  x * y * x^-1 * y * x^2 * y*t16t16t20.
----------------------------------------
Let p = (1, 21, 11, 10, 2, 24)(3, 17, 16, 20, 6, 19)(4, 14,
    12, 13, 9, 5)(7, 15,
23, 22, 8, 18) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = y * x^-2 * y * x * yt14t18. Apply at 2.
Found a new name:  y * x^-2 * y * x * y*t7t14t18.
----------------------------------------
Let p = Id($) belong to N.
t1 = t7t19 ==>
```

```
(t1)^p = (Id($)t7t19)^p ==>
t1 = Id($)t7t19. Apply at 1.
Found a new name:  t7t19t3.
----------------------------------------
Let p = (1, 3, 20, 24, 4, 5)(2, 6, 22, 23, 19, 21)(7, 9,
    14, 18, 10, 11)(8, 12,
16, 17, 13, 15) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t3 = Id($)t9t21. Apply at 2.
Found a new name:  t1t9t21.
```

Step 3: Now we have an array of equations that can be applied to a particular t-word. We apply these equations to produce new t-words. If a new t-word is not found in the tree of Elements, then we add it to the tree. We ensure that when we add a new t-word to the tree that we link it to its parent t-word.

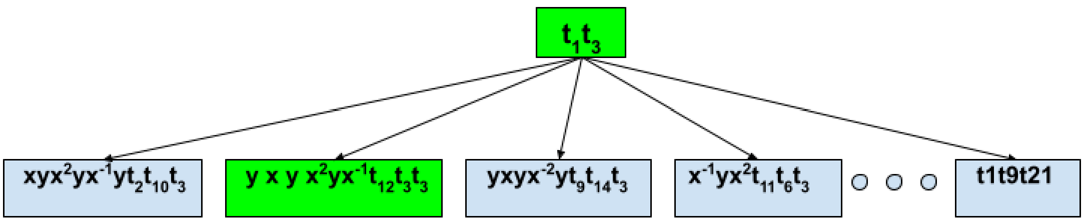

Figure 6.1: First Level of the Tree.

Step 4: We repeat the algorithm for every new t-word that has not been processed until we find the element we are looking for, in our case $pt_2t_{23}t_9$.

The following is the list of relations that can be applied to $yxyx^2yx^{-1}t_{12}t_3t_3$:

```
----------------------------------------
Let p = (1, 3, 20, 24, 4, 5)(2, 6, 22, 23, 19, 21)(7, 9,
   14, 18, 10, 11)(8, 12,
16, 17, 13, 15) belong to N.
t1^2 = t7 ==>
(t1t1)^p = (Id($)t7)^p ==>
t3t3 = Id($)t9. Apply at 2.
Found a new name:  y * x * y * x^2 * y * x^-1*t12t9
----------------------------------------
Let p = (1, 18, 17)(2, 4, 15)(3, 8, 10)(5, 7, 6)(9, 20, 22)
   (11, 19, 12)(13, 24,
23)(14, 16, 21) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t12 = y * x * y * x^-2 * y * xt15t10. Apply at 1.
Found a new name:  (x^-1 * y * x^2)^2*t15t10t3t3
----------------------------------------
Let p = (1, 18, 15)(2, 11, 4)(3, 7, 6)(5, 16, 14)(8, 23,
   10)(9, 19, 12)(13, 24,
21)(17, 22, 20) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t12 = x^2 * y * x * y * x^-1 * yt2t23. Apply at 1.
```

Found a new name: x^−1 ∗ y ∗ x^2 ∗ y∗t2t23t3t3

Let p = (1, 18, 22, 11, 20)(2, 19, 12, 4, 17)(5, 8, 13, 24, 10)(6, 16, 23, 14,

7) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t12 = y ∗ x^−1t17t13. Apply at 1.

Found a new name: x^2 ∗ y ∗ x^−1 ∗ y ∗ x^2∗t17t13t3t3

Let p = (1, 18, 2, 9, 4)(3, 16, 13, 24, 14)(6, 8, 21, 10, 7)(12, 20, 15, 22, 19)

belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t12 = x ∗ y ∗ x^−1 ∗ y ∗ x ∗ y ∗ xt1t21. Apply at 1.

Found a new name: t1t21t3t3

Let p = (1, 18)(2, 20)(3, 5)(4, 22)(6, 7)(8, 14)(9, 11)(10, 16)(12, 19)(13,

24)(15, 17)(21, 23) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t12 = y ∗ x^−2 ∗ y ∗ x ∗ yt22t14. Apply at 1.

```
Found a new name: (y * x^2)^2*t22t14t3t3
----------------------------------------
Let p = (1, 21)(2, 5)(3, 19)(4, 6)(7, 15)(8, 11)(9, 13)(10,
    12)(14, 17)(16,
18)(20, 23)(22, 24) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x^-2 * y * x^-2t6t11. Apply at 2.
Found a new name: y * x^2 * y * x * y * x^-1 * y*
   t23t6t11t3
----------------------------------------
Let p = (1, 21, 22, 5, 12, 8)(2, 13, 9, 10, 17, 6)(3, 4,
   23, 18, 14, 19)(7, 15,
16, 11, 24, 20) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x * y * x^-1 * y * x * y * xt23t1. Apply at 2.
Found a new name: t5t23t1t3
----------------------------------------
Let p = (1, 21, 20, 16)(2, 10, 19, 3)(4, 7, 15, 14)(5, 17,
   23, 11)(6, 12, 24,
18)(8, 22, 13, 9) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
```

```
t3 = x^2 * y * x^-1t7t22. Apply at 2.
Found a new name:  x^2 * y * x^2 * y * x*t24t7t22t3
-----------------------------------------
Let p = (1, 21, 18, 23)(2, 8, 20, 14)(3, 12, 5, 19)(4, 16,
   22, 10)(6, 17, 7,
15)(9, 24, 11, 13) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x * y * x^-1 * y * x^2 * yt16t20. Apply at 2.
Found a new name:  x^-1 * y * x * y*t6t16t20t3
-----------------------------------------
Let p = (1, 21, 11, 10, 2, 24)(3, 17, 16, 20, 6, 19)(4, 14,
    12, 13, 9, 5)(7, 15,
23, 22, 8, 18) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = y * x^-2 * y * x * yt14t18. Apply at 2.
Found a new name:  (y * x^2)^2*t3t14t18t3
-----------------------------------------
Let p = (1, 12, 2)(3, 23, 16)(4, 9, 17)(5, 10, 21)(6, 14,
   13)(7, 24, 8)(11, 22,
15)(18, 20, 19) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
```

```
t12 = Id($)t24t18. Apply at 1.
Found a new name:  y * x * y * x^2 * y * x^-1*t24t18t3t3
----------------------------------------
Let p = (1, 3, 20, 24, 4, 5)(2, 6, 22, 23, 19, 21)(7, 9,
    14, 18, 10, 11)(8, 12,
16, 17, 13, 15) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t3 = Id($)t9t21. Apply at 2.
Found a new name:  y * x * y * x^2 * y * x^-1*t12t9t21t3
```

Due to the limited space in the diagram, only some branches will be shown in the diagram. Please see Appendix G for the full output that was used to create the diagram.

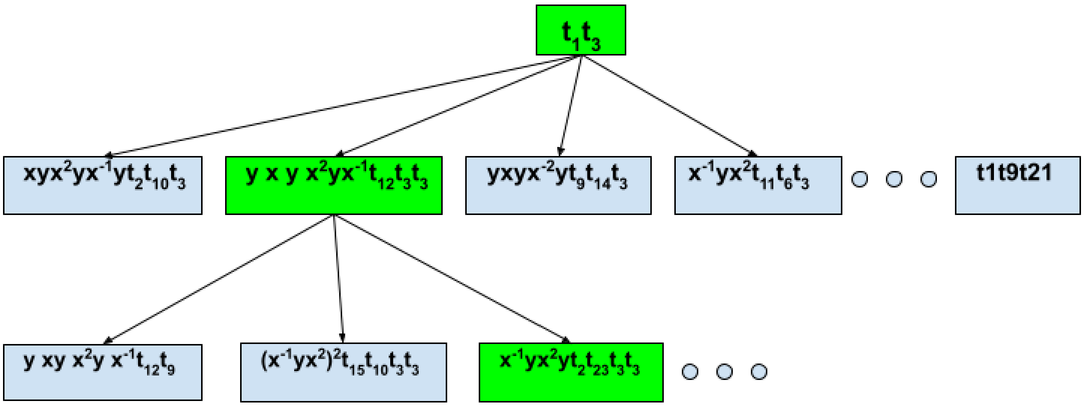

Figure 6.2: Second Level of the Tree.

Finally, a function named FindTwordWithReducing detects that $x^{-1}yx^2yt_2t_{23}t_3t_3$

is equivalent to $x^{-1}yx^2yt_2t_{23}t_9$, which triggers the algorithm to stop looking.

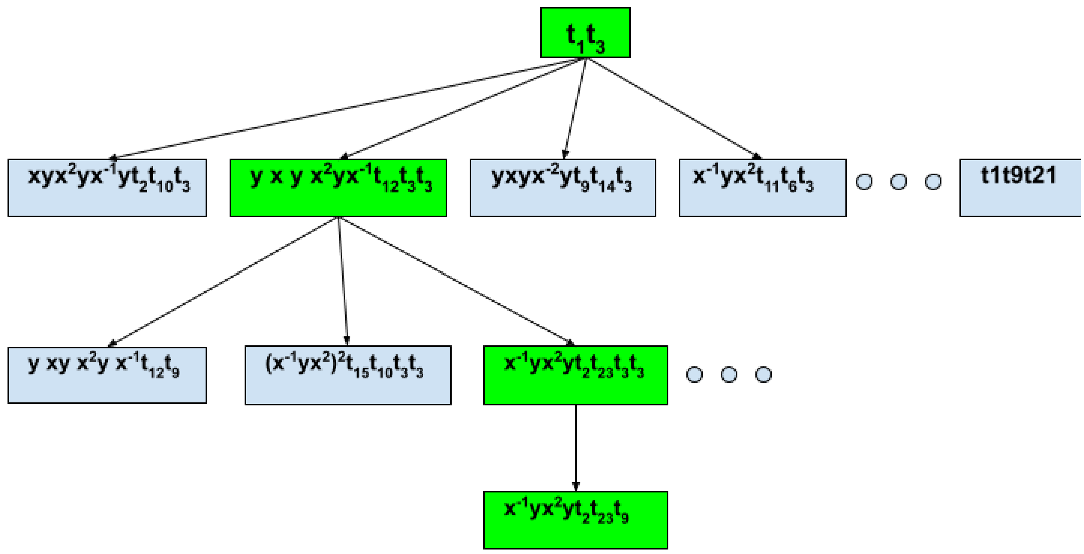

Figure 6.3: Third Level of the Tree.

Step 5: Now that we have found our t-word, all we have to do is to follow its path back to the top of the tree. Each time we move up the tree we print the proof for the t-word. If the algorithm does not find a solution in a timely manner, then the user must prove the relation by hand.

The following is the final output:

```
Start with  t1t3
-----------------------------------------------------
Apply the following at t word position 1:
Let p = (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9,
    20)(10, 24)(11,
17)(14, 21)(18, 22) belong to N.
```

```
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = y * x * y * x^2 * y * x^-1t12t3 ==>
t1t3 = y * x * y * x^2 * y * x^-1*t12t3t3.
---------------------------------------------------
Apply the following at t word position 1:
Let p = (1, 18, 15)(2, 11, 4)(3, 7, 6)(5, 16, 14)(8, 23,
    10)(9, 19, 12)(13, 24,
21)(17, 22, 20) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t12 = x^2 * y * x * y * x^-1 * yt2t23 ==>
y * x * y * x^2 * y * x^-1 * t12t3t3 = x^-1 * y * x^2 * y*
    t2t23t3t3.
---------------------------------------------------
Reduce in the following way:
x^-1 * y * x^2 * y*t2t23t3t3
x^-1 * y * x^2 * y*t2t23t9
```

This algorithm is used to prove all word problems for $L_2(5)$ over S_5, and $(A_5)^2{:}2$ over $A_5 : 2$, and it was tested on two additional groups not found in this thesis. A MAGMA implementation of the algorithm can be found in Appendix G.

Appendix A

MAGMA CODE: Isomorphism of (23 × 2):11

```
N:=TransitiveGroup(46,4);
CompositionFactors(N); /*
    G
    |  Cyclic(11)
    *
    |  Cyclic(23)
    *
    |  Cyclic(2)
    1 */
NL:=NormalLattice(N);
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
```

```
end for;
/*The relevant path of the normal lattice of N.
*/

NL4 :=NL[4];
NL4;

IsIsomorphic(NL4,AbelianGroup(GrpPerm,[46])); /* true */
A:=N!(1, 37, 28, 18, 8, 43, 33, 23, 14, 3, 39, 30, 20, 10,
    46, 36, 26, 16, 6, 41, 32, 22, 12, 2, 38, 27, 17, 7, 44,
     34, 24, 13, 4, 40, 29, 19, 9, 45, 35, 25, 15, 5, 42,
    31, 21, 11);
IsIsomorphic(NL4,sub<N|A>);
FPGroup(NL4);
Test<x>:=Group<x|x^46>;
f,Test1,k:=CosetAction(Test,sub<Test|Id(Test)>);
IsIsomorphic(NL4,Test1); /* true */
q,ff:=quo<N|NL4>;
q;
T:= Transversal(N,NL4);
ff(T[2]) eq q.1; /* true */
IsAbelian(q);
/* N cannot be the direct product to NL4 and q~<T[2]> since
    N has no normal subgroup of order 11. Which means N is
```

```
    an extension of NL4 by N/NL4. Let Q~N/NL4 = {NL[4]B}
   where T[2] = B = (1, 20, 9, 35, 42, 8, 32, 33, 38, 46,
   15)(2, 19, 10, 36, 41, 7, 31, 34, 37, 45, 16)(3, 23, 18,
    5, 27, 25, 22, 13, 43, 11, 40)(4, 24, 17, 6, 28, 26,
   21, 14, 44, 12, 39). N~ 46:11 ~ (2     23):11 */
B:=N!(1, 20, 9, 35, 42, 8, 32, 33, 38, 46, 15)(2, 19, 10,
   36, 41, 7, 31, 34, 37, 45, 16)(3, 23, 18, 5, 27, 25, 22,
    13, 43, 11, 40)(4, 24, 17, 6, 28, 26, 21, 14, 44, 12,
   39);

procedure FindFirstElementAsProductOfLast2(L,A,B)
for i1 in [0..(Order(A) -1)] do
        for i2 in [0..(Order(B) -1)] do
                if( L eq (A^i1*B^i2) ) then
                        i1, i2, A^i1*B^i2;
                end if;
        end for;
end for;
end procedure;

FindFirstElementAsProductOfLast2(A^B, A, Id(N));
A^B eq A^25;
G<x, y>:=Group<x, y | x^46, y^11, x^y = x^25>;
f,Test2,k:=CosetAction(G,sub<G|Id(G)>);
```

```
IsIsomorphic(N,Test2); /* True 8/

/* N ~ 46:11 */
```

Appendix B

MAGMA CODE: Isomorphic Type of ((23 $\times$ 2):11):2

```
N:=TransitiveGroup(46,6);
CompositionFactors(N);/*
    G
    |  Cyclic(11)
    *
    |  Cyclic(2)
    *
    |  Cyclic(23)
    *
    |  Cyclic(2)
    1*/
NL:=NormalLattice(N);
```

```
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for;  /* 4 */

/* NL[10] = TransitiveGroup(46,4) ~ (23  2  ):11 and has
   presentation Group< x, y | x^46, y^11, x^y = x^25> */
Test<x,y>:= Group< x, y | x^46, y^11, x^y = x^25>;
f,Test1,k:=CosetAction(Test,sub<Test|Id(Test)>);
IsIsomorphic(NL[10],Test1); /* true */
NL10:=NL[10];
temp:=TransitiveGroup(46,4);
s, m:=IsIsomorphic(temp, NL[10]);
A:= m(temp!(1, 37, 28, 18, 8, 43, 33, 23, 14, 3, 39, 30,
   20, 10, 46, 36, 26, 16, 6, 41, 32, 22, 12, 2, 38, 27,
   17, 7, 44, 34, 24, 13, 4, 40, 29, 19, 9, 45, 35, 25, 15,
    5, 42, 31, 21, 11));
B:=m(temp!(1, 20, 9, 35, 42, 8, 32, 33, 38, 46, 15)(2, 19,
   10, 36, 41, 7, 31, 34, 37, 45, 16)(3, 23, 18, 5, 27, 25,
    22, 13, 43, 11, 40)(4, 24, 17, 6, 28, 26, 21, 14, 44,
   12, 39));
A;
B;
AB:=sub<N|A,B>;
AB eq NL10;
```

```
q,ff:=quo<N|NL10>;
q;
T:= Transversal(N,NL10);
ff(T[2]) eq q.1; /* true */
C:=T[2]^11;

/* N cannot be the direct product to NL4 and any other
   subgroup since N has no normal subgroup of order 2
   outside of NL10.  Which means N is an extension of NL4
   by N/NL4. Let Q~N/NL10 = {NL[10]C} where C = (1, 42, 46,
    28, 40, 32, 21, 44, 13, 34, 35, 4, 9, 6, 24, 11, 19,
   30, 7, 38, 17, 15)(2, 41, 45, 27, 39, 31, 22, 43, 14,
   33, 36, 3, 10, 5, 23, 12, 20, 29, 8, 37, 18, 16)(25, 26)
   .  N~2:(11:(23  2  )) */

procedure FindFirstElementAsProductOfLast2(L,A,B)
for i1 in [0..(Order(A) -1)] do
        for i2 in [0..(Order(B) -1)] do
                if( L eq (A^i1*B^i2) ) then
                        i1, i2, A^i1*B^i2;
                end if;
        end for;
end for;
```

```
end procedure;

FindFirstElementAsProductOfLast2(A^C, A, B);
A^45 eq A^C; /* true */
FindFirstElementAsProductOfLast2(B^C, A, B);
B^C eq A^36*B;

G<x,y,z>:= Group< x, y, z | x^46, y^11, x^y = x^25, z^2, x^
    z = x^45, y^z = x^36*y>;
f,G1,k:=CosetAction(G,sub<G|Id(G)>);
IsIsomorphic(N,G1);
/* true */
```

Appendix C

MAGMA CODE: Shapes of Progenitor $3^{*56}:((3:7):2^3)$ Images

```
S:=Sym(56);
xx:=S!(1, 28, 30)(2, 40, 39)(3, 55, 54)(4, 26, 29)(5, 25,
   10)(6, 31, 9)(7, 19, 18)(8,
    16, 21)(11, 17, 24)(12, 41, 50)(13, 52, 42)(14, 51, 34)
       (15, 56, 33)(20, 53,
    36)(22, 32, 38)(23, 49, 46)(35, 37, 44)(45, 48, 47);
yy:=S!(1, 54, 45)(2, 25, 56)(3, 20, 22)(4, 34, 49)(5, 38,
   27)(6, 35, 48)(7, 44, 10)(8,
    51, 19)(9, 39, 55)(11, 31, 52)(12, 43, 17)(14, 15, 41)
```

```
        (16, 24, 53)(18, 42,
    46)(21, 36, 47)(23, 40, 33)(26, 32, 30)(28, 50, 29);
N:=sub<S|xx,yy>;

CompositionFactors(N);
NL:=NormalLattice(N);
NL;
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for;  /* 2*/

NL2:=NL[2];
IsIsomorphic(NL2, AbelianGroup(GrpPerm,[2,2,2]));
/* N~q:2^3

q,ff:=quo<N|NL[2]>;
q;

T:= Transversal(N,NL[2]);
T;
ff(T[2]) eq q.1;
ff(T[3]) eq q.2;
```

```
CompositionFactors(q);
NL:=NormalLattice(q);
NL;
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for; /* 2 */

q,ff:=quo<q|NL[2]>;
q;

N~ 2^3:(3:7)

/**********************************************************/

a:=0; b:=0; c:=0; d:=0; e:=0; f:=2; g:=6; index :=2187;
G<x,y,t>:=Group< x, y, t | x^3, y^3 , y * x^-1 * y^-1 * x
    ^-1 * y^-1 * x * y * x * y^-1 * x, t^3, (t, x * y^-1 * x
    ^-1 * y^-1 * x ), ( x^-1*t^(y * x))^a, (x*t^(y * x))^b
      , ((x * y^-1)^3*t)^c , (x * y^-1*t)^d , (y^-1 * x^-1*t
    )^e , ((x * y)^3*t)^f , (x * y*t)^g>;
f,G1,k:=CosetAction(G,sub<G|x,y>);

CompositionFactors(G1);
NL:=NormalLattice(G1);
```

```
NL;
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for; /* 2 */

IsIsomorphic(NL[2], AbelianGroup(GrpPerm,[3,3,3,3,3,3,3]));
     /* true */
q,ff:=quo<G1|NL[2]>;
 IsIsomorphic(N, q); /* true */
G1~N:3^7


/**********************************************************/

a:=0; b:=0; c:=0; d:=0; e:=3; f:=0; g:=3; index :=24;
G<x,y,t>:=Group< x, y, t | x^3, y^3 , y * x^-1 * y^-1 * x
    ^-1 * y^-1 * x * y * x * y^-1 * x, t^3, (t, x * y^-1 * x
    ^-1 * y^-1 * x ), ( x^-1*t^(y * x))^a, (x*t^(y * x))^b
      , ((x * y^-1)^3*t)^c , (x * y^-1*t)^d , (y^-1 * x^-1*t
    )^e , ((x * y)^3*t)^f , (x * y*t)^g>;
f,G1,k:=CosetAction(G,sub<G|x,y>);

CompositionFactors(G1);
NL:=NormalLattice(G1);
```

```
NL;
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for; /* 2 */

IsIsomorphic(NL[2], AbelianGroup(GrpPerm,[2,2,2,2,2,2]));
   /* true */
q,ff:=quo<N|NL[2]>;
q;
 IsIsomorphic(N, q);   /* true */

/*********************************************************/

a:=0; b:=0; c:=0; d:=0; e:=6; f:=2; g:=0; index :=2187;
G<x,y,t>:=Group< x, y, t | x^3, y^3 , y * x^-1 * y^-1 * x
   ^-1 * y^-1 * x * y * x * y^-1 * x, t^3, (t, x * y^-1 * x
   ^-1 * y^-1 * x ), ( x^-1*t^(y * x))^a, (x*t^(y * x))^b
      , ((x * y^-1)^3*t)^c , (x * y^-1*t)^d , (y^-1 * x^-1*t
   )^e , ((x * y)^3*t)^f , (x * y*t)^g>;
f,G1,k:=CosetAction(G,sub<G|x,y>);

CompositionFactors(G1);
NL:=NormalLattice(G1);
NL;
```

```
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for; /* 2 */

IsIsomorphic(NL[2], AbelianGroup(GrpPerm,[3,3,3,3,3,3,3]));
    /* true */
q,ff:=quo<G1|NL[2]>;
```

Appendix D

MAGMA CODE: Shapes of Progenitor $\mathbf{2}^{*56} : ((3:7):2^3)$ Images

```
S:=Sym(56);
xx:=S!(1, 28, 30)(2, 40, 39)(3, 55, 54)(4, 26, 29)(5, 25,
   10)(6, 31, 9)(7, 19, 18)(8,
    16, 21)(11, 17, 24)(12, 41, 50)(13, 52, 42)(14, 51, 34)
       (15, 56, 33)(20, 53,
    36)(22, 32, 38)(23, 49, 46)(35, 37, 44)(45, 48, 47);
yy:=S!(1, 54, 45)(2, 25, 56)(3, 20, 22)(4, 34, 49)(5, 38,
   27)(6, 35, 48)(7, 44, 10)(8,
    51, 19)(9, 39, 55)(11, 31, 52)(12, 43, 17)(14, 15, 41)
```

```
(16, 24, 53)(18, 42,
46)(21, 36, 47)(23, 40, 33)(26, 32, 30)(28, 50, 29);
N:=sub<S|xx,yy>;

CompositionFactors(N);
NL:=NormalLattice(N);
NL;
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for;  /* 2*/

NL2:=NL[2];
IsIsomorphic(NL2, AbelianGroup(GrpPerm,[2,2,2]));
/* N~q:2^3

q,ff:=quo<N|NL[2]>;
q;

T:= Transversal(N,NL[2]);
T;
ff(T[2]) eq q.1;
ff(T[3]) eq q.2;
```

```
CompositionFactors(q);
NL:=NormalLattice(q);
NL;
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for; /* 2 */

q,ff:=quo<q|NL[2]>;
q;

N~ 2^3:(3:7)

/**********************************************************/

a:=0; b:=0; c:=0; d:=0; e:=0; f:=3; g:=3; h:=7; GIndex
    :=576;
G<x,y,t>:=Group<x,y, t|x^3, y^3 , y * x^-1 * y^-1 * x^-1 *
    y^-1 * x * y * x *
y^-1 * x, t^2, (t, x * y^-1 * x^-1 * y^-1 * x ), (x * y*t^(
    x * y^-1 * x)*t^(y^-1
* x^-1 * y * x * y^-1))^a, (x * y*t*t^((y^-1, x)) )^b, (x*t
    ^((x, y)) )^c, (y^-1
* x^-1*t^((y^-1, x)))^d, (x * y*t^((y^-1, x))*t^(y * x) )^e
    , ( (x *
```

```
y)^3*t^(y^-1 * x * y^-1 * x^-1 * y^-1))^f, ( (x * y^-1)^3*t
    )^g, (x * y^-1*t)^h
>;
f,G1,k:=CosetAction(G,sub<G|x,y>);

CompositionFactors(G1);/*
    G
    |  Cyclic(3)
    *
    |  A(1, 8)                    = L(2, 8)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1 */
NL:=NormalLattice(G1);
```

```
NL;
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for; /* 2 */

IsIsomorphic(NL[2], AbelianGroup(GrpPerm,[2,2,2,2,2,2]));
   /* true*/
q,ff:=quo<G1|NL[2]>;
 CompositionFactors(q);/*
 G
    |   Cyclic(3)
    *
    |  A(1, 8)                    = L(2, 8)
    1
*/
/**********************************************************/

 a:=0; b:=0; c:=0; d:=0; e:=0; f:=6; g:=3; h:=0; GIndex
    :=73728;
G<x,y,t>:=Group<x,y, t|x^3, y^3 , y * x^-1 * y^-1 * x^-1 *
   y^-1 * x * y * x *y^-1 * x, t^2, (t, x * y^-1 * x^-1 * y
   ^-1 * x ), (x * y*t^(x * y^-1 * x)*t^(y^-1* x^-1 * y * x
    * y^-1))^a, (x * y*t*t^((y^-1, x)) )^b, (x*t^((x, y)) )
   ^c, (y^-1 * x^-1*t^((y^-1, x)))^d, (x * y*t^((y^-1, x))*
```

```
    t^(y * x) )^e,  ( (x *y)^3*t^(y^-1 * x * y^-1 * x^-1 * y
    ^-1))^f, ( (x * y^-1)^3*t)^g, (x * y^-1*t)^h>;
 f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);


NL:=NormalLattice(G1);
NL; /*
[6]  Order 12386304  Length 1  Maximal Subgroups: 5
---
[5]  Order 4128768   Length 1  Maximal Subgroups: 4
---
[4]  Order 8192      Length 1  Maximal Subgroups: 3
---
[3]  Order 128       Length 1  Maximal Subgroups: 2
---
[2]  Order 64        Length 1  Maximal Subgroups: 1
---
[1]  Order 1         Length 1  Maximal Subgroups:*/
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for; /* 4*/
NL[4];/*
Permutation group acting on a set of cardinality 73728
Order = 8192 = 2^13 */
```

```
IsIsomorphic(NL[3], AbelianGroup(GrpPerm
    ,[2,2,2,2,2,2,2,2,2,2,2,2,2])); /* false */
q,ff:=quo<G1|NL[4]>;
 CompositionFactors(q);
/* G ~ (2^7):(3:L(2, 8))  */

/*******************************************************/

a:=0; b:=0; c:=0; d:=0; e:=0; f:=9; g:=3; h:=0; GIndex
    :=1259712;
G<x,y,t>:=Group<x,y, t|x^3, y^3 , y * x^-1 * y^-1 * x^-1 *
    y^-1 * x * y * x * y^-1 * x, t^2, (t, x * y^-1 * x^-1 *
    y^-1 * x ), (x * y*t^(x * y^-1 * x)*t^(y^-1* x^-1 * y *
    x * y^-1))^a, (x * y*t*t^((y^-1, x)) )^b, (x*t^((x, y))
    )^c, (y^-1 * x^-1*t^((y^-1, x)))^d, (x * y*t^((y^-1, x))
    *t^(y * x) )^e,  ( (x * y)^3*t^(y^-1 * x * y^-1 * x^-1 *
     y^-1))^f, ( (x * y^-1)^3*t)^g, (x * y^-1*t)^h>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);/*
 G
     |  Cyclic(3)
     *
     |  A(1, 8)                 = L(2, 8)
```

```
*
|    Cyclic(3)
*
|    Cyclic(3)
*
|    Cyclic(3)
*
|    Cyclic(3)
*
|    Cyclic(3)
*
|    Cyclic(3)
*
|    Cyclic(3)
*
|    Cyclic(2)
*
|    Cyclic(2)
*
|    Cyclic(2)
*
|    Cyclic(2)
*
|    Cyclic(2)
```

```
    *
    |   Cyclic(2)
    1 */

NL:=NormalLattice(G1);
NL; /*
[6]  Order 211631616  Length 1  Maximal Subgroups: 5
---
[5]  Order 70543872   Length 1  Maximal Subgroups: 4
---
[4]  Order 139968     Length 1  Maximal Subgroups: 2 3
---
[3]  Order 2187       Length 1  Maximal Subgroups: 1
---
[2]  Order 64         Length 1  Maximal Subgroups: 1
---
[1]  Order 1          Length 1  Maximal Subgroups: */

for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for; /*  4*/
/* G ~ (2^6 * 3^7): q */

q,ff:=quo<G1|NL[4]>;
```

```
CompositionFactors(q); /*
 G
     |   Cyclic(3)
     *
     |   A(1, 8)                      = L(2, 8)
     1 */
/* G ~ (2^6 * 3^7): (3:L(2, 8)) */
```

```
/**********************************************************/

a:=0; b:=0; c:=0; d:=0; e:=6; f:=6; g:=3; h:=0; GIndex
    :=1152;
G<x,y,t>:=Group<x,y, t|x^3, y^3 , y * x^-1 * y^-1 * x^-1 *
    y^-1 * x * y * x *
y^-1 * x, t^2, (t, x * y^-1 * x^-1 * y^-1 * x ), (x * y*t^(
    x * y^-1 * x)*t^(y^-1
* x^-1 * y * x * y^-1))^a, (x * y*t*t^((y^-1, x)) )^b, (x*t
    ^((x, y)) )^c, (y^-1
* x^-1*t^((y^-1, x)))^d, (x * y*t^((y^-1, x))*t^(y * x) )^e
```

```
    ,   ( (x *
y)^3*t^(y^-1 * x * y^-1 * x^-1 * y^-1))^f, ( (x * y^-1)^3*t
    )^g, (x * y^-1*t)^h
>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);/*
 G
     |  Cyclic(3)
     *
     |  A(1, 8)                    = L(2, 8)
     *
     |  Cyclic(2)
     *
     |  Cyclic(2)
     *
     |  Cyclic(2)
     *
     |  Cyclic(2)
     *
     |  Cyclic(2)
     *
     |  Cyclic(2)
     *
     |  Cyclic(2)
```

```
    1*/

NL:=NormalLattice(G1);
NL;/*

[5]  Order 193536  Length 1  Maximal Subgroups: 4
---
[4]  Order 64512   Length 1  Maximal Subgroups: 3
---
[3]  Order 128     Length 1  Maximal Subgroups: 2
---
[2]  Order 2       Length 1  Maximal Subgroups: 1
---
[1]  Order 1       Length 1  Maximal Subgroups:*/

for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for;  /* 3 */
IsIsomorphic(NL[3], AbelianGroup(GrpPerm,[2,2,2,2,2,2,2]));
    /* true */

q,ff:=quo<G1|NL[3]>;
 CompositionFactors(q); /*
 G
```

```
   |   Cyclic(3)
   *
   |   A(1, 8)                        = L(2, 8)
   *
   1 */
/* G1 ~ (2^7):(3:L(2, 8)) */

/**********************************************************/

a:=0; b:=0; c:=0; d:=3; e:=0; f:=2; g:=0; h:=0; GIndex:=32;
G<x,y,t>:=Group<x,y, t|x^3, y^3 , y * x^-1 * y^-1 * x^-1 *
   y^-1 * x * y * x *
y^-1 * x, t^2, (t, x * y^-1 * x^-1 * y^-1 * x ), (x * y*t^(
   x * y^-1 * x)*t^(y^-1
* x^-1 * y * x * y^-1))^a, (x * y*t*t^((y^-1, x)) )^b, (x*t
   ^((x, y)) )^c, (y^-1
* x^-1*t^((y^-1, x)))^d, (x * y*t^((y^-1, x))*t^(y * x) )^e
   , ( (x *
y)^3*t^(y^-1 * x * y^-1 * x^-1 * y^-1))^f, ( (x * y^-1)^3*t
   )^g, (x * y^-1*t)^h
>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);/*
 G
```

```
  |  Cyclic(3)
  *
  |  Cyclic(7)
  *
  |  Cyclic(2)
  *
  |  Cyclic(2)
  *
  |  Cyclic(2)
  *
  |  Cyclic(2)
  *
  |  Cyclic(2)
  *
  |  Cyclic(2)
  *
  |  Cyclic(2)
  *
  |  Cyclic(2)
  1 */

NL:=NormalLattice(G1);
NL;/*
[7]  Order 5376  Length 1  Maximal Subgroups: 6
```

```
---
[6]   Order 1792   Length 1   Maximal Subgroups: 5
---
[5]   Order 256    Length 1   Maximal Subgroups: 3 4
---
[4]   Order 32     Length 1   Maximal Subgroups: 2
[3]   Order 32     Length 1   Maximal Subgroups: 2
---
[2]   Order 4      Length 1   Maximal Subgroups: 1
---
[1]   Order 1      Length 1   Maximal Subgroups: */

for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for;    /* 4 */
NL[4];
IsIsomorphic(NL[4], AbelianGroup(GrpPerm,[2,2,2,2,2])); /*
   true */
/* G ~ 2^5:q */

q,ff:=quo<G1|NL[4]>;
 CompositionFactors(q);/*
 G
    |  Cyclic(3)
```

```
    *
    |   Cyclic(7)
    *
    |   Cyclic(2)
    *
    |   Cyclic(2)
    *
    |   Cyclic(2)
    1 */

NL:=NormalLattice(q);
NL;/*
[4]  Order 168  Length 1  Maximal Subgroups: 3
---
[3]  Order 56   Length 1  Maximal Subgroups: 2
---
[2]  Order 8    Length 1  Maximal Subgroups: 1
---
[1]  Order 1    Length 1  Maximal Subgroups: */

 for i in [1..#NL] do
     if IsAbelian(NL[i]) then i; end if;
end for; /* 2 */
/* G ~ 2^5:(2^3: q)  */
```

```
N:=q;
q,ff:=quo<N|NL[2]>;
 CompositionFactors(q);
NL:=NormalLattice(q);
NL;
 for i in [1..#NL] do
     if IsAbelian(NL[i]) then i; end if;
end for; /*2 */
/* G ~ 2^5:(2^3: (7:3)) */
/*************************************************************/


a:=0; b:=0; c:=0; d:=3; e:=0; f:=4; g:=7; h:=0; GIndex
   :=512;
G<x,y,t>:=Group<x,y, t|x^3, y^3 , y * x^-1 * y^-1 * x^-1 *
   y^-1 * x * y * x * y^-1 * x, t^2, (t, x * y^-1 * x^-1 *
   y^-1 * x ), (x * y*t^(x * y^-1 * x)*t^(y^-1 * x^-1 * y *
    x * y^-1))^a, (x * y*t*t^((y^-1, x)) )^b, (x*t^((x, y))
    )^c, (y^-1 * x^-1*t^((y^-1, x)))^d, (x * y*t^((y^-1, x)
   )*t^(y * x) )^e,  ( (x * y)^3*t^(y^-1 * x * y^-1 * x^-1
   * y^-1))^f, ( (x * y^-1)^3*t)^g, (x * y^-1*t)^h >;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1); /*
 G
```

```
|    Cyclic(3)
*
|    Cyclic(7)
*
|    Cyclic(2)
*
|    Cyclic(2)
*
|    Cyclic(2)
*
|    Cyclic(2)
*
|    Cyclic(2)
*
|    Cyclic(2)
*
|    Cyclic(2)
*
|    Cyclic(2)
*
|    Cyclic(2)
*
|    Cyclic(2)
*
```

```
      |   Cyclic(2)
      *
      |   Cyclic(2)
      1 */
NL:=NormalLattice(G1);
NL;/*
[9]   Order 86016   Length 1   Maximal Subgroups: 8
---
[8]   Order 28672   Length 1   Maximal Subgroups: 7
---
[7]   Order 4096    Length 1   Maximal Subgroups: 5 6
---
[6]   Order 512     Length 1   Maximal Subgroups: 4
[5]   Order 512     Length 1   Maximal Subgroups: 4
---
[4]   Order 64      Length 1   Maximal Subgroups: 2 3
---
[3]   Order 8       Length 1   Maximal Subgroups: 1
[2]   Order 8       Length 1   Maximal Subgroups: 1
---
[1]   Order 1       Length 1   Maximal Subgroups: */
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for;   /* 5 */
```

```
NL[5]; /* 2^9 */
/* G ~ 2^9:q */
q,ff:=quo<G1|NL[5]>;
 NL:=NormalLattice(q);
NL;
  for i in [1..#NL] do
      if IsAbelian(NL[i]) then i; end if;
end for; /* 2 */
/* G ~ 2^9:(2^3:q) */
N:=q;
q,ff:=quo<N|NL[2]>;
NL:=NormalLattice(q);
NL; /* [3]  Order 21  Length 1  Maximal Subgroups: 2
---
[2]  Order 7   Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups: */


/* G ~ /* G ~ 2^9:(2^3:(7:3)) */


/**************************************************/
a:=0; b:=0; c:=0; d:=3; e:=6; f:=4; g:=0; h:=7; GIndex
    :=8192;
G<x,y,t>:=Group<x,y, t|x^3, y^3 , y * x^-1 * y^-1 * x^-1 *
```

```
    y^-1 * x * y * x *
y^-1 * x, t^2, (t, x * y^-1 * x^-1 * y^-1 * x ), (x * y*t^(
    x * y^-1 * x)*t^(y^-1 * x^-1 * y * x * y^-1))^a, (x * y*
    t*t^((y^-1, x)) )^b, (x*t^((x, y)) )^c, (y^-1 * x^-1*t
    ^((y^-1, x)))^d, (x * y*t^((y^-1, x))*t^(y * x) )^e,  (
    (x * y)^3*t^(y^-1 * x * y^-1 * x^-1 * y^-1))^f, ( (x * y
    ^-1)^3*t)^g, (x * y^-1*t)^h >;

f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1); /*
 G
     |   Cyclic(3)
     *
     |   Cyclic(7)
     *
     |   Cyclic(2)
     *
     |   Cyclic(2)
     *
     |   Cyclic(2)
     *
     |   Cyclic(2)
     *
     |   Cyclic(2)
```

```
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
*
|  Cyclic(2)
1 */
```

```
NL:=NormalLattice(G1);
NL;
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for;  /* 9 */
/* G ~ 2^10: q */

q,ff:=quo<G1|NL[9]>;
NL:=NormalLattice(q);
NL; /*
[6]  Order 1344  Length 1  Maximal Subgroups: 5
---
[5]  Order 448   Length 1  Maximal Subgroups: 4
---
[4]  Order 64    Length 1  Maximal Subgroups: 2 3
---
[3]  Order 8     Length 1  Maximal Subgroups: 1
[2]  Order 8     Length 1  Maximal Subgroups: 1
---
[1]  Order 1     Length 1  Maximal Subgroups: */
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for; /* 4 */
/* G ~ 2^10: (2^6:q) */
```

```
N:=q;
q,ff:=quo<N|NL[4]>;
NL:=NormalLattice(q);
NL; /*
3]  Order 21  Length 1  Maximal Subgroups: 2
---
[2]  Order 7   Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups: */
/* G ~ 2^10: (2^6:(2:7)) */
```

Appendix E

MAGMA CODE: Induction of 5:11 Onto $L_2(11)$

```
/* Character Induction */
```

```
MAGMA: Using Induction of 5:11 onto L2(11) To Find a
    Monomial Representation of  L2(11)

/* Character Induction */
S:=Sym(12);
xx:=S!(3, 7, 9, 4, 5)(6, 8, 12, 10, 11);
yy:=S!(1, 8, 2)(3, 4, 7)(5, 12, 11)(6, 9, 10);
G:=sub<S|xx,yy>;
```

```
H:=sub<G|(1, 4, 3, 10, 8)(2, 7, 5, 12, 11), (1, 2, 12, 11,
    3, 7, 6, 4, 5, 8, 10)>;
zz:= H!(1, 4, 3, 10, 8)(2, 7, 5, 12, 11);
ww:=H!(1, 2, 12, 11, 3, 7, 6, 4, 5, 8, 10);

FPGroup(H);
HH<z,w>:= Group<z, w | z^5, w^-2 * z^-1 * w^-1 * z, z^2 *
    w^-1 * z^2 * w^-1 * z * w>;
f,H1,k:=CosetAction(HH,sub<HH| Id(HH)>);
s:=IsIsomorphic(H1, H);
s; /* true */

SchH:=SchreierSystem(HH,sub<HH|Id(HH)>);
Hword:=function(Perm)
for w in SchH do
        seq := Eltseq(w);
        p:= Id(H);
        for j in seq do
                if j eq 1 then p:=p*zz; end if;
if j eq -1 then p:=p*zz^-1; end if;
if j eq 2 then p:=p*ww; end if;
if j eq -2 then p:=p*ww^-1; end if;
        end for;
        if Perm eq p then return w; end if;
```

```
end for;
end function;

ClassesH:=Classes(H);

FPGroup(G);
 /*   x^5,  y^3,  y * x^2 * y^-1 * x^2 * y * x^-1 * y^-1 *
     x^-1,    (x * y * x * y * x)^2, */
GG< x, y >:=Group<x, y | x^5,  y^3,  y * x^2 * y^-1 * x^2 *
     y * x^-1 * y^-1 * x^-1,    (x * y * x * y * x)^2>;
#GG; /* 660 */

f,G1,k:=CosetAction(GG,sub<GG| Id(GG)>);
s:=IsIsomorphic(G1, G);
s; /* true */

ClassesG:=Classes(G);
z5:=CH[2][2];
GCinterceptsH:=[Set(Class(G,ClassesG[i][3])) meet Set(H): i
     in [1..#ClassesG]];

str:= "";
for g in GCinterceptsH[4] do
        for i in [1..#ClassesH] do
```

```
                if g in Class(H, ClassesH[i][3]) then
c:= CH[2][i];
if c eq z5 then str:= str cat "z5 + "; end if;
                        if c eq z5^2 then  str:= str cat "
                           z5^2 + "; end if;
                        if c eq z5^3 then  str:= str cat "
                           z5^3 + "; end if;
if c eq z5^4 then  str:= str cat "z5^4 + "; end if;
end if;
        end for;
end for;
str;

for g in GCinterceptsH[4] do
        for i in [1..#ClassesH] do
                if g in Class(H, ClassesH[i][3]) then CH
                   [2][i];  end if;
                continue;
        end for;
end for;

GCinterceptsH[5];

for g in GCinterceptsH[5] do
```

```
        "kai(" cat  Sprint(g) cat ") + ";
end for;
str:= "";
for g in GCinterceptsH[5] do
        for i in [1..#ClassesH] do
                if g in Class(H, ClassesH[i][3]) then
c:= CH[2][i];
if c eq z5 then str:= str cat "z5 + "; end if;
                        if c eq z5^2 then  str:= str cat "
                           z5^2 + "; end if;
                        if c eq z5^3 then  str:= str cat "
                           z5^3 + "; end if;
if c eq z5^4 then  str:= str cat "z5^4 + "; end if;
end if;
        end for;
end for;
Str;

for g in GCinterceptsH[7] do
        "kai(" cat  Sprint(g) cat ") + ";
end for;
str:= "";
for g in GCinterceptsH[7] do
        for i in [1..#ClassesH] do
```

```
                    if g in Class(H, ClassesH[i][3]) then
c:= CH[2][i];
if c eq z5 then str:= str cat "z5 + "; end if;
                              if c eq z5^2 then  str:= str cat "
                                 z5^2 + "; end if;
                              if c eq z5^3 then  str:= str cat "
                                 z5^3 + "; end if;
if c eq z5^4 then  str:= str cat "z5^4 + "; end if;
if c eq 1 then  str:= str cat "1 + "; end if;
end if;
          end for;
end for;
str;

for g in GCinterceptsH[8] do
          "kai(" cat  Sprint(g) cat ") + ";
end for;
str:= "";
for g in GCinterceptsH[8] do
          for i in [1..#ClassesH] do
                    if g in Class(H, ClassesH[i][3]) then
c:= CH[2][i];
if c eq z5 then str:= str cat "z5 + "; end if;
if c eq z5^2 then  str:= str cat "z5^2 + "; end if;
```

```
if c eq z5^3 then  str:= str cat "z5^3 + "; end if;
if c eq z5^4 then  str:= str cat "z5^4 + "; end if;
if c eq 1 then  str:= str cat "1 + "; end if;
end if;
        end for;
end for;
str;

str:= "";
for g in GCinterceptsH[8] do
c:= CH[2](g);
if c eq z5 then str:= str cat "z5 + "; end if;
if c eq z5^2 then  str:= str cat "z5^2 + "; end if;
if c eq z5^3 then  str:= str cat "z5^3 + "; end if;
if c eq z5^4 then  str:= str cat "z5^4 + "; end if;
if c eq 1 then  str:= str cat "1 + "; end if;
end for;
str;
/*************************************************************/

S:=Sym(12);
xx:=S!(3, 7, 9, 4, 5)(6, 8, 12, 10, 11);
yy:=S!(1, 8, 2)(3, 4, 7)(5, 12, 11)(6, 9, 10);
G:=sub<S|xx,yy>;
```

```
S:=Sym(12);
xx:=S!(3, 7, 9, 4, 5)(6, 8, 12, 10, 11);
yy:=S!(1, 8, 2)(3, 4, 7)(5, 12, 11)(6, 9, 10);
G:=sub<S|xx,yy>;
CG:=CharacterTable(G);
H:=sub<G|(1, 4, 3, 10, 8)(2, 7, 5, 12, 11), (1, 2, 12, 11,
    3, 7, 6, 4, 5, 8, 10)>;
#H;
#G/#H;
CH:=CharacterTable(H);
CH[2];
C:=Classes(G);
#C;
Induction(CH[2],G) eq CG[7];
#CG[7];
CG[7](Id(G));

T:=Transversal(G,H);
#T eq Index(G,H);

/* The transversal changes. We save this transversal. */
T:=[Id(G),
G!(3, 7, 9, 4, 5)(6, 8, 12, 10, 11),
```

```
G!(1, 8, 2)(3, 4, 7)(5, 12, 11)(6, 9, 10),
G!(1, 7, 12, 4, 8)(3, 11, 10, 9, 5),
G!(3, 5, 4, 9, 7)(6, 11, 10, 12, 8),
G!(1, 4)(2, 8)(3, 12)(5, 7)(6, 10)(9, 11),
G!(1, 10, 12, 8, 2)(5, 11, 7, 9, 6),
G!(3, 4, 7, 5, 9)(6, 10, 8, 11, 12),
G!(1, 5, 9, 12, 10)(3, 11, 7, 4, 6),
G!(1, 11, 9, 8, 2)(3, 7, 4, 5, 6),
G!(1, 5, 9, 2, 8)(4, 12, 11, 10, 6),
G!(1, 12, 5, 10, 9)(3, 4, 11, 6, 7)];

C:=CyclotomicField(5);
A:=[[C.1,0,0,0,0,0,0,0,0,0,0,0]: i in [1..12]];
for i,j in [1..12] do A[i,j] :=0; end for;

for i,j in [1..12] do
if T[i]*xx*T[j]^-1 in H then
A[i,j] :=CH[2](T[i]*xx*T[j]^-1);
end if;
end for;
GG:=GL(12,C);
GG!A;

/* Print latex matrix */
```

```
strM:= "\\begin{pmatrix}\n";
matrix:= GG!A;
for i in [1.. 12] do
        for j in [1.. 12] do
                strM:= strM cat Sprint(matrix[i][j]) cat
                   "&";
        end for;
        strM:= strM cat " \\\\\n";
end for;
strM:= strM cat "\\end{pmatrix}";
strM;
/*\begin{pmatrix}
0&1&0&0&0&0&0&0&0&0&0&0& \\
0&0&0&z_5^3&0&0&0&0&0&0&0&0& \\
0&0&0&0&0&z_5^3&0&0&0&0&0&0& \\
0&0&0&0&0&0&0&z_5^2&0&0&0&0& \\
1&0&0&0&0&0&0&0&0&0&0&0& \\
0&0&0&0&0&0&z_5^2&0&0&0&0&0& \\
0&0&0&0&0&0&0&0&0&1&0&0& \\
0&0&0&0&1&0&0&0&0&0&0&0& \\
0&0&z_5^2&0&0&0&0&0&0&0&0&0& \\
0&0&0&0&0&0&0&0&z_5^3&0&0&0& \\
0&0&0&0&0&0&0&0&0&0&z_5^3&0& \\
0&0&0&0&0&0&0&0&0&0&0&z_5^2& \\
```

```
\\end{pmatrix}*/

B:=[[C.1,0,0,0,0,0,0,0,0,0,0,0]: i in [1..12]];
for i,j in [1..12] do B[i,j] :=0; end for;
for i,j in [1..12] do
if T[i]*yy*T[j]^-1 in H then
B[i,j] :=CH[2](T[i]*yy*T[j]^-1);
end if;
end for;
GG:=GL(12,C);
GG!B;

/* Print latex matrix */
strM:= "\\begin{pmatrix}\n";
matrix:= GG!B;
for i in [1.. 12] do
        for j in [1.. 12] do
                strM:= strM cat Sprint(matrix[i][j]) cat
                    "&";
        end for;
        strM:= strM cat " \\\\\n";
end for;
strM:= strM cat "\\end{pmatrix}";
strM;
```

```
primes := [ p : p in [7..1000] | IsPrime(p) ];
for p in primes do
        if ((p-1) mod 5) eq 0 then p; break; end if;
end for; /* 11 */

/* We change this rep of G over a finite field. Find a
   prime p such that GF(p) = Zp has elements of order 5. p
   is such that 5|(p-1). Then p-1 = 5k or p =5k+1. k=2
   gives the smallest such p. So p = 11 */

for n in [2..10] do n, n^5 mod 11; end for;
/* 5 is element of order 5 in Z11 */

Z5:=CH[2][2];
for i, j in  [1..12] do
        if A[i][j] eq Z5 then A[i][j]:= 5; end if;
        if A[i][j] eq Z5^2 then A[i][j]:= 5^2 mod 11; end
           if;
        if A[i][j] eq Z5^3 then A[i][j]:= 5^3 mod 11; end
           if;
if A[i][j] eq Z5^4 then A[i][j]:= 5^4 mod 11; end if;
end for;
```

```
/* Print latex matrix */
strM:= "\\begin{pmatrix}\n";
matrix:= GG!A;
for i in [1.. 12] do
        for j in [1.. 12] do
                strM:= strM cat Sprint(matrix[i][j]) cat
                   "&";
        end for;
        strM:= strM cat " \\\\\n";
end for;
strM:= strM cat "\\end{pmatrix}";
strM;

/*\begin{pmatrix}
0&1&0&0&0&0&0&0&0&0&0&0& \\
0&0&0&4&0&0&0&0&0&0&0&0& \\
0&0&0&0&0&4&0&0&0&0&0&0& \\
0&0&0&0&0&0&0&3&0&0&0&0& \\
1&0&0&0&0&0&0&0&0&0&0&0& \\
0&0&0&0&0&0&3&0&0&0&0&0& \\
0&0&0&0&0&0&0&0&0&1&0&0& \\
0&0&0&0&1&0&0&0&0&0&0&0& \\
0&0&3&0&0&0&0&0&0&0&0&0& \\
0&0&0&0&0&0&0&0&4&0&0&0& \\
```

```
0&0&0&0&0&0&0&0&0&0&4&0& \\
0&0&0&0&0&0&0&0&0&0&0&3& \\
\end{pmatrix} */

for i, j in  [1..12] do
        if B[i][j] eq Z5 then B[i][j]:= 5; end if;
        if B[i][j] eq Z5^2 then B[i][j]:= 5^2 mod 11; end
           if;
        if B[i][j] eq Z5^3 then B[i][j]:= 5^3 mod 11; end
           if;
if B[i][j] eq Z5^4 then B[i][j]:= 5^4 mod 11; end if;
end for;

/* Print latex matrix */
strM:= "\\begin{pmatrix}\n";
matrix:= GG!B;
for i in [1.. 12] do
        for j in [1.. 12] do
                strM:= strM cat Sprint(matrix[i][j]) cat
                   "&";
        end for;
        strM:= strM cat " \\\\\n";
end for;
strM:= strM cat "\\end{pmatrix}";
```

```
strM;

\begin{pmatrix}
0&0&1&0&0&0&0&0&0&0&0&0&  \\
0&0&0&0&5&0&0&0&0&0&0&0&  \\
0&0&0&0&0&0&3&0&0&0&0&0&  \\
0&0&0&0&0&0&0&3&0&0&0&0&  \\
0&0&0&0&0&0&0&1&0&0&0&0&  \\
0&0&0&1&0&0&0&0&0&0&0&0&  \\
4&0&0&0&0&0&0&0&0&0&0&0&  \\
0&9&0&0&0&0&0&0&0&0&0&0&  \\
0&0&0&0&0&4&0&0&0&0&0&0&  \\
0&0&0&0&0&0&0&0&0&1&0&0&  \\
0&0&0&0&0&0&0&0&0&0&1&  \\
0&0&0&0&0&0&0&0&1&0&0&0&  \\
\end{pmatrix}

GGG:=GL(12,11);
A:=GG!A;
A;/*
[0  1  0  0  0  0  0  0  0  0  0  0]
[0  0  0  4  0  0  0  0  0  0  0  0]
[0  0  0  0  0  4  0  0  0  0  0  0]
[0  0  0  0  0  0  0  3  0  0  0  0]
```

```
[1 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 3 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 1 0 0]
[0 0 0 0 1 0 0 0 0 0 0 0]
[0 0 3 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 4 0 0 0]
[0 0 0 0 0 0 0 0 0 0 4 0]
[0 0 0 0 0 0 0 0 0 0 0 3]
*/
B:=GG!B;
B;/*
[0 0 1 0 0 0 0 0 0 0 0 0]
[0 0 0 0 5 0 0 0 0 0 0 0]
[0 0 0 0 0 0 3 0 0 0 0 0]
[0 0 0 0 0 0 0 0 3 0 0 0]
[0 0 0 0 0 0 0 1 0 0 0 0]
[0 0 0 1 0 0 0 0 0 0 0 0]
[4 0 0 0 0 0 0 0 0 0 0 0]
[0 9 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 4 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 1 0]
[0 0 0 0 0 0 0 0 0 0 0 1]
[0 0 0 0 0 0 0 0 0 1 0 0]
*/
```

```
#sub<GGG|A,B>;
s:=IsIsomorphic(G,sub<GGG|A,B>);
S; /* true */

for i,j in  [1..12] do
        if A[i][j] ne 0 then
                "a_{" cat Sprint(i) cat "," cat Sprint(j)
                   cat "} = " cat Sprint(A[i][j]) cat " \\
                   implies " cat "t_{" cat Sprint(i) cat "}
                    = t_{" cat Sprint(j) cat "}^" cat
                   Sprint(A[i][j]) cat "\\\\" ;
        end if;
end for; /*
a_{1,2} = 1 \implies t_{1} = t_{2}^1\\
a_{2,4} = 4 \implies t_{2} = t_{4}^4\\
a_{3,6} = 4 \implies t_{3} = t_{6}^4\\
a_{4,8} = 3 \implies t_{4} = t_{8}^3\\
a_{5,1} = 1 \implies t_{5} = t_{1}^1\\
a_{6,7} = 3 \implies t_{6} = t_{7}^3\\
a_{7,10} = 1 \implies t_{7} = t_{10}^1\\
a_{8,5} = 1 \implies t_{8} = t_{5}^1\\
a_{9,3} = 3 \implies t_{9} = t_{3}^3\\
a_{10,9} = 4 \implies t_{10} = t_{9}^4\\
a_{11,11} = 4 \implies t_{11} = t_{11}^4\\
```

```
a_{12,12} = 3 \implies t_{12} = t_{12}^3\\   */

labeling:=[];
for i in [1..120] do
        tsubscript:= i mod 12;
power:= (i div 12) + 1;
if  tsubscript eq 0 then
 tsubscript:=12;
power:= i div 12;
end if;
labeling:=labeling cat [[tsubscript , power]];
end for;

S:=Sym(120);
arrX:=[0: i in [1..120]];

FindLabels:=procedure(row, pow1, col, pow2,  ~arr)

        /* Process labels to create an array that will be
           turned to a permutation */
        label1:= Position(labeling, [row, pow1 mod 11]);
        label2:= Position(labeling, [col, pow2*pow1 mod
           11]);
arr[label1]:= label2;
```

```
srow := Sprint(row); /* turn to a string */
scol := Sprint(col);
spow1 := Sprint(pow1);
spow2 := Sprint(pow2);

label1:= Sprint(label1);
label2:= Sprint(label2);

if pow1 eq 1 then
    /* a_{1,2} = 3      t_1 = t_2^3 */
   "a_{" cat  srow cat ", " cat scol cat "} = " cat spow2
      cat  " \\implies "  cat  "t_{" cat  srow cat "} = t_
      {" cat scol cat "}^{" cat spow2 cat "} \\implies
      \\\\ ";
end if;
        /* (t_1)^n = (t_2^3)^n   */
if pow1 gt 1 then
        "t_{" cat  srow cat "}^{" cat spow1 cat "} = " cat
           "(t_{" cat  scol cat "}^{" cat spow2 cat "})^{"
           cat spow1 cat "} =  " cat "t_{" cat  scol cat
           "}^{" cat Sprint(pow1*pow2) cat "} = " cat  "t_
           {" cat  scol cat "}^{" cat Sprint(pow1*pow2 mod
           11) cat "}. \\implies " cat label1 cat " \\
```

```
            implies " cat label2 cat "\\\\";

end if;
end procedure;

/* r for row in the matrix.*/
for r in [1..12] do
/* c for column in the matrix. */
for c in [1..12] do
                if A[r, c] ne 0 then
for p in [1..10] do
        value:= StringToInteger(Sprint(A[r, c]), 12);
        FindLabels(r, p, c, value, ~arrX);
end for;
                end if;
end for;
"-----------------------------------------------";
end for;

/*
a_{1, 2} = 1 \implies t_{1} = t_{2}^{1} \implies \\
t_{1}^{2} = (t_{2}^{1})^{2} =  t_{2}^{2} = t_{2}^{2}. \
   implies 13 \implies
14\\
```

```
t_{1}^{3} = (t_{2}^{1})^{3} =  t_{2}^{3} = t_{2}^{3}. \
   implies 25 \implies
26\\
t_{1}^{4} = (t_{2}^{1})^{4} =  t_{2}^{4} = t_{2}^{4}. \
   implies 37 \implies
38\\
t_{1}^{5} = (t_{2}^{1})^{5} =  t_{2}^{5} = t_{2}^{5}. \
   implies 49 \implies
50\\
t_{1}^{6} = (t_{2}^{1})^{6} =  t_{2}^{6} = t_{2}^{6}. \
   implies 61 \implies
62\\
t_{1}^{7} = (t_{2}^{1})^{7} =  t_{2}^{7} = t_{2}^{7}. \
   implies 73 \implies
74\\
t_{1}^{8} = (t_{2}^{1})^{8} =  t_{2}^{8} = t_{2}^{8}. \
   implies 85 \implies
86\\
t_{1}^{9} = (t_{2}^{1})^{9} =  t_{2}^{9} = t_{2}^{9}. \
   implies 97 \implies
98\\
t_{1}^{10} = (t_{2}^{1})^{10} =  t_{2}^{10} = t_{2}^{10}. \
   implies 109
\implies 110\\
```

```
-----------------------------------------------
a_{2, 4} = 4 \implies t_{2} = t_{4}^{4} \implies \\
t_{2}^{2} = (t_{4}^{4})^{2} =  t_{4}^{8} = t_{4}^{8}. \
   implies 14 \implies
88\\
t_{2}^{3} = (t_{4}^{4})^{3} =  t_{4}^{12} = t_{4}^{1}. \
   implies 26
\implies 4\\
t_{2}^{4} = (t_{4}^{4})^{4} =  t_{4}^{16} = t_{4}^{5}. \
   implies 38
\implies 52\\
t_{2}^{5} = (t_{4}^{4})^{5} =  t_{4}^{20} = t_{4}^{9}. \
   implies 50
\implies 100\\
t_{2}^{6} = (t_{4}^{4})^{6} =  t_{4}^{24} = t_{4}^{2}. \
   implies 62
\implies 16\\
t_{2}^{7} = (t_{4}^{4})^{7} =  t_{4}^{28} = t_{4}^{6}. \
   implies 74
\implies 64\\
t_{2}^{8} = (t_{4}^{4})^{8} =  t_{4}^{32} = t_{4}^{10}. \
   implies 86
\implies 112\\
t_{2}^{9} = (t_{4}^{4})^{9} =  t_{4}^{36} = t_{4}^{3}. \
```

```
   implies 98
\implies 28\\
t_{2}^{10} = (t_{4}^{4})^{10} =  t_{4}^{40} = t_{4}^{7}. \
   implies 110
\implies 76\\
-------------------------------------------
a_{3, 6} = 4 \implies t_{3} = t_{6}^{4} \implies \\
t_{3}^{2} = (t_{6}^{4})^{2} =  t_{6}^{8} = t_{6}^{8}. \
   implies 15 \implies
90\\
t_{3}^{3} = (t_{6}^{4})^{3} =  t_{6}^{12} = t_{6}^{1}. \
   implies 27
\implies 6\\
t_{3}^{4} = (t_{6}^{4})^{4} =  t_{6}^{16} = t_{6}^{5}. \
   implies 39
\implies 54\\
t_{3}^{5} = (t_{6}^{4})^{5} =  t_{6}^{20} = t_{6}^{9}. \
   implies 51
\implies 102\\
t_{3}^{6} = (t_{6}^{4})^{6} =  t_{6}^{24} = t_{6}^{2}. \
   implies 63
\implies 18\\
t_{3}^{7} = (t_{6}^{4})^{7} =  t_{6}^{28} = t_{6}^{6}. \
   implies 75
```

```
\implies 66\\
t_{3}^{8} = (t_{6}^{4})^{8} =  t_{6}^{32} = t_{6}^{10}. \
   implies 87
\implies 114\\
t_{3}^{9} = (t_{6}^{4})^{9} =  t_{6}^{36} = t_{6}^{3}. \
   implies 99
\implies 30\\
t_{3}^{10} = (t_{6}^{4})^{10} =  t_{6}^{40} = t_{6}^{7}. \
   implies 111
\implies 78\\
---------------------------------------------
a_{4, 8} = 3 \implies t_{4} = t_{8}^{3} \implies \\
t_{4}^{2} = (t_{8}^{3})^{2} =  t_{8}^{6} = t_{8}^{6}. \
   implies 16 \implies
68\\
t_{4}^{3} = (t_{8}^{3})^{3} =  t_{8}^{9} = t_{8}^{9}. \
   implies 28 \implies
104\\
t_{4}^{4} = (t_{8}^{3})^{4} =  t_{8}^{12} = t_{8}^{1}. \
   implies 40
\implies 8\\
t_{4}^{5} = (t_{8}^{3})^{5} =  t_{8}^{15} = t_{8}^{4}. \
   implies 52
\implies 44\\
```

```
t_{4}^{6} = (t_{8}^{3})^{6} =  t_{8}^{18} = t_{8}^{7}. \
   implies 64
\implies 80\\
t_{4}^{7} = (t_{8}^{3})^{7} =  t_{8}^{21} = t_{8}^{10}. \
   implies 76
\implies 116\\
t_{4}^{8} = (t_{8}^{3})^{8} =  t_{8}^{24} = t_{8}^{2}. \
   implies 88
\implies 20\\
t_{4}^{9} = (t_{8}^{3})^{9} =  t_{8}^{27} = t_{8}^{5}. \
   implies 100
\implies 56\\
t_{4}^{10} = (t_{8}^{3})^{10} =  t_{8}^{30} = t_{8}^{8}. \
   implies 112
\implies 92\\
---------------------------------------------
a_{5, 1} = 1 \implies t_{5} = t_{1}^{1} \implies \\
t_{5}^{2} = (t_{1}^{1})^{2} =  t_{1}^{2} = t_{1}^{2}. \
   implies 17 \implies
13\\
t_{5}^{3} = (t_{1}^{1})^{3} =  t_{1}^{3} = t_{1}^{3}. \
   implies 29 \implies
25\\
t_{5}^{4} = (t_{1}^{1})^{4} =  t_{1}^{4} = t_{1}^{4}. \
```

```
   implies 41 \implies
37\\
t_{5}^{5} = (t_{1}^{1})^{5} =  t_{1}^{5} = t_{1}^{5}. \
   implies 53 \implies
49\\
t_{5}^{6} = (t_{1}^{1})^{6} =  t_{1}^{6} = t_{1}^{6}. \
   implies 65 \implies
61\\
t_{5}^{7} = (t_{1}^{1})^{7} =  t_{1}^{7} = t_{1}^{7}. \
   implies 77 \implies
73\\
t_{5}^{8} = (t_{1}^{1})^{8} =  t_{1}^{8} = t_{1}^{8}. \
   implies 89 \implies
85\\
t_{5}^{9} = (t_{1}^{1})^{9} =  t_{1}^{9} = t_{1}^{9}. \
   implies 101
\implies 97\\
t_{5}^{10} = (t_{1}^{1})^{10} =  t_{1}^{10} = t_{1}^{10}. \
   implies 113
\implies 109\\
----------------------------------------------
a_{6, 7} = 3 \implies t_{6} = t_{7}^{3} \implies \\
t_{6}^{2} = (t_{7}^{3})^{2} =  t_{7}^{6} = t_{7}^{6}. \
   implies 18 \implies
```

```
67\\
t_{6}^{3} = (t_{7}^{3})^{3} =  t_{7}^{9} = t_{7}^{9}. \
   implies 30 \implies
103\\
t_{6}^{4} = (t_{7}^{3})^{4} =  t_{7}^{12} = t_{7}^{1}. \
   implies 42
\implies 7\\
t_{6}^{5} = (t_{7}^{3})^{5} =  t_{7}^{15} = t_{7}^{4}. \
   implies 54
\implies 43\\
t_{6}^{6} = (t_{7}^{3})^{6} =  t_{7}^{18} = t_{7}^{7}. \
   implies 66
\implies 79\\
t_{6}^{7} = (t_{7}^{3})^{7} =  t_{7}^{21} = t_{7}^{10}. \
   implies 78
\implies 115\\
t_{6}^{8} = (t_{7}^{3})^{8} =  t_{7}^{24} = t_{7}^{2}. \
   implies 90
\implies 19\\
t_{6}^{9} = (t_{7}^{3})^{9} =  t_{7}^{27} = t_{7}^{5}. \
   implies 102
\implies 55\\
t_{6}^{10} = (t_{7}^{3})^{10} =  t_{7}^{30} = t_{7}^{8}. \
   implies 114
```

```
\implies 91\\
-------------------------------------------
a_{7, 10} = 1 \implies t_{7} = t_{10}^{1} \implies \\
t_{7}^{2} = (t_{10}^{1})^{2} =  t_{10}^{2} = t_{10}^{2}. \
   implies 19
\implies 22\\
t_{7}^{3} = (t_{10}^{1})^{3} =  t_{10}^{3} = t_{10}^{3}. \
   implies 31
\implies 34\\
t_{7}^{4} = (t_{10}^{1})^{4} =  t_{10}^{4} = t_{10}^{4}. \
   implies 43
\implies 46\\
t_{7}^{5} = (t_{10}^{1})^{5} =  t_{10}^{5} = t_{10}^{5}. \
   implies 55
\implies 58\\
t_{7}^{6} = (t_{10}^{1})^{6} =  t_{10}^{6} = t_{10}^{6}. \
   implies 67
\implies 70\\
t_{7}^{7} = (t_{10}^{1})^{7} =  t_{10}^{7} = t_{10}^{7}. \
   implies 79
\implies 82\\
t_{7}^{8} = (t_{10}^{1})^{8} =  t_{10}^{8} = t_{10}^{8}. \
   implies 91
\implies 94\\
```

```
t_{7}^{9} = (t_{10}^{1})^{9} =  t_{10}^{9} = t_{10}^{9}. \
   implies 103
\implies 106\\
t_{7}^{10} = (t_{10}^{1})^{10} =  t_{10}^{10} = t_
   {10}^{10}. \implies 115
\implies 118\\
--------------------------------------------
a_{8, 5} = 1 \implies t_{8} = t_{5}^{1} \implies \\
t_{8}^{2} = (t_{5}^{1})^{2} =  t_{5}^{2} = t_{5}^{2}. \
   implies 20 \implies
17\\
t_{8}^{3} = (t_{5}^{1})^{3} =  t_{5}^{3} = t_{5}^{3}. \
   implies 32 \implies
29\\
t_{8}^{4} = (t_{5}^{1})^{4} =  t_{5}^{4} = t_{5}^{4}. \
   implies 44 \implies
41\\
t_{8}^{5} = (t_{5}^{1})^{5} =  t_{5}^{5} = t_{5}^{5}. \
   implies 56 \implies
53\\
t_{8}^{6} = (t_{5}^{1})^{6} =  t_{5}^{6} = t_{5}^{6}. \
   implies 68 \implies
65\\
t_{8}^{7} = (t_{5}^{1})^{7} =  t_{5}^{7} = t_{5}^{7}. \
```

```
   implies 80 \implies
77\\
t_{8}^{8} = (t_{5}^{1})^{8} =  t_{5}^{8} = t_{5}^{8}. \
   implies 92 \implies
89\\
t_{8}^{9} = (t_{5}^{1})^{9} =  t_{5}^{9} = t_{5}^{9}. \
   implies 104
\implies 101\\
t_{8}^{10} = (t_{5}^{1})^{10} =  t_{5}^{10} = t_{5}^{10}. \
   implies 116
\implies 113\\
--------------------------------------------
a_{9, 3} = 3 \implies t_{9} = t_{3}^{3} \implies \\
t_{9}^{2} = (t_{3}^{3})^{2} =  t_{3}^{6} = t_{3}^{6}. \
   implies 21 \implies
63\\
t_{9}^{3} = (t_{3}^{3})^{3} =  t_{3}^{9} = t_{3}^{9}. \
   implies 33 \implies
99\\
t_{9}^{4} = (t_{3}^{3})^{4} =  t_{3}^{12} = t_{3}^{1}. \
   implies 45
\implies 3\\
t_{9}^{5} = (t_{3}^{3})^{5} =  t_{3}^{15} = t_{3}^{4}. \
   implies 57
```

```
\implies 39\\
t_{9}^{6} = (t_{3}^{3})^{6} =  t_{3}^{18} = t_{3}^{7}. \
   implies 69
\implies 75\\
t_{9}^{7} = (t_{3}^{3})^{7} =  t_{3}^{21} = t_{3}^{10}. \
   implies 81
\implies 111\\
t_{9}^{8} = (t_{3}^{3})^{8} =  t_{3}^{24} = t_{3}^{2}. \
   implies 93
\implies 15\\
t_{9}^{9} = (t_{3}^{3})^{9} =  t_{3}^{27} = t_{3}^{5}. \
   implies 105
\implies 51\\
t_{9}^{10} = (t_{3}^{3})^{10} =  t_{3}^{30} = t_{3}^{8}. \
   implies 117
\implies 87\\
---------------------------------------------
a_{10, 9} = 4 \implies t_{10} = t_{9}^{4} \implies \\
t_{10}^{2} = (t_{9}^{4})^{2} =  t_{9}^{8} = t_{9}^{8}. \
   implies 22
\implies 93\\
t_{10}^{3} = (t_{9}^{4})^{3} =  t_{9}^{12} = t_{9}^{1}. \
   implies 34
\implies 9\\
```

```
t_{10}^{4} = (t_{9}^{4})^{4} =  t_{9}^{16} = t_{9}^{5}. \
   implies 46
\implies 57\\
t_{10}^{5} = (t_{9}^{4})^{5} =  t_{9}^{20} = t_{9}^{9}. \
   implies 58
\implies 105\\
t_{10}^{6} = (t_{9}^{4})^{6} =  t_{9}^{24} = t_{9}^{2}. \
   implies 70
\implies 21\\
t_{10}^{7} = (t_{9}^{4})^{7} =  t_{9}^{28} = t_{9}^{6}. \
   implies 82
\implies 69\\
t_{10}^{8} = (t_{9}^{4})^{8} =  t_{9}^{32} = t_{9}^{10}. \
   implies 94
\implies 117\\
t_{10}^{9} = (t_{9}^{4})^{9} =  t_{9}^{36} = t_{9}^{3}. \
   implies 106
\implies 33\\
t_{10}^{10} = (t_{9}^{4})^{10} =  t_{9}^{40} = t_{9}^{7}. \
   implies 118
\implies 81\\
---------------------------------------------
a_{11, 11} = 4 \implies t_{11} = t_{11}^{4} \implies \\
t_{11}^{2} = (t_{11}^{4})^{2} =  t_{11}^{8} = t_{11}^{8}. \
```

implies 23

\implies 95\\

t_{11}^{3} = (t_{11}^{4})^{3} = t_{11}^{12} = t_{11}^{1}.

\implies 35

\implies 11\\

t_{11}^{4} = (t_{11}^{4})^{4} = t_{11}^{16} = t_{11}^{5}.

\implies 47

\implies 59\\

t_{11}^{5} = (t_{11}^{4})^{5} = t_{11}^{20} = t_{11}^{9}.

\implies 59

\implies 107\\

t_{11}^{6} = (t_{11}^{4})^{6} = t_{11}^{24} = t_{11}^{2}.

\implies 71

\implies 23\\

t_{11}^{7} = (t_{11}^{4})^{7} = t_{11}^{28} = t_{11}^{6}.

\implies 83

\implies 71\\

t_{11}^{8} = (t_{11}^{4})^{8} = t_{11}^{32} = t_{11}^{10}.

\implies 95

\implies 119\\

t_{11}^{9} = (t_{11}^{4})^{9} = t_{11}^{36} = t_{11}^{3}.

\implies 107

\implies 35\\

t_{11}^{10} = (t_{11}^{4})^{10} = t_{11}^{40} = t_

```
    {11}^{7}. \implies 119
\implies 83\\
-------------------------------------------
a_{12, 12} = 3 \implies t_{12} = t_{12}^{3} \implies \\
t_{12}^{2} = (t_{12}^{3})^{2} =  t_{12}^{6} = t_{12}^{6}. \
    implies 24
\implies 72\\
t_{12}^{3} = (t_{12}^{3})^{3} =  t_{12}^{9} = t_{12}^{9}. \
    implies 36
\implies 108\\
t_{12}^{4} = (t_{12}^{3})^{4} =  t_{12}^{12} = t_{12}^{1}.
    \implies 48
\implies 12\\
t_{12}^{5} = (t_{12}^{3})^{5} =  t_{12}^{15} = t_{12}^{4}.
    \implies 60
\implies 48\\
t_{12}^{6} = (t_{12}^{3})^{6} =  t_{12}^{18} = t_{12}^{7}.
    \implies 72
\implies 84\\
t_{12}^{7} = (t_{12}^{3})^{7} =  t_{12}^{21} = t_{12}^{10}.
     \implies 84
\implies 120\\
t_{12}^{8} = (t_{12}^{3})^{8} =  t_{12}^{24} = t_{12}^{2}.
    \implies 96
```

```
\implies 24\\
t_{12}^{9} = (t_{12}^{3})^{9} =  t_{12}^{27} = t_{12}^{5}.
   \implies 108
\implies 60\\
t_{12}^{10} = (t_{12}^{3})^{10} =  t_{12}^{30} = t_
   {12}^{8}. \implies 120
\implies 96\\
-----------------------------------------------*/

xxx:=S!arrX;
xxx; /*
(1, 2, 40, 8, 5)(3, 42, 7, 10, 45)(4, 32, 29, 25, 26)(6,
   31, 34, 9, 27)(11, 47,
    59, 107, 35)(12, 36, 108, 60, 48)(13, 14, 88, 20, 17)
       (15, 90, 19, 22,
    93)(16, 68, 65, 61, 62)(18, 67, 70, 21, 63)(23, 95,
       119, 83, 71)(24, 72, 84,
    120, 96)(28, 104, 101, 97, 98)(30, 103, 106, 33, 99)
       (37, 38, 52, 44, 41)(39,
    54, 43, 46, 57)(49, 50, 100, 56, 53)(51, 102, 55, 58,
       105)(64, 80, 77, 73,
    74)(66, 79, 82, 69, 75)(76, 116, 113, 109, 110)(78,
       115, 118, 81, 111)(85,
    86, 112, 92, 89)(87, 114, 91, 94, 117) */
```

```
for i,j in [1..12] do
        if B[i][j] ne 0 then
                "b_{" cat Sprint(i) cat "," cat Sprint(j)
                   cat "} = " cat Sprint(B[i][j]) cat " \\
                   implies " cat "t_{" cat Sprint(i) cat "}
                    \\implies t_{" cat Sprint(j) cat "}^"
                   cat Sprint(B[i][j]);
        end if;
end for;
/*b_{1,3} = 1 \implies t_{1} \implies t_{3}^1
b_{2,5} = 5 \implies t_{2} \implies t_{5}^5
b_{3,7} = 3 \implies t_{3} \implies t_{7}^3
b_{4,9} = 3 \implies t_{4} \implies t_{9}^3
b_{5,8} = 1 \implies t_{5} \implies t_{8}^1
b_{6,4} = 1 \implies t_{6} \implies t_{4}^1
b_{7,1} = 4 \implies t_{7} \implies t_{1}^4
b_{8,2} = 9 \implies t_{8} \implies t_{2}^9
b_{9,6} = 4 \implies t_{9} \implies t_{6}^4
b_{10,11} = 1 \implies t_{10} \implies t_{11}^1
b_{11,12} = 1 \implies t_{11} \implies t_{12}^1
b_{12,10} = 1 \implies t_{12} \implies t_{10}^1*/

/* Same as above but we use b  instead of a. */
```

```
FindLabels:=procedure(row, pow1, col, pow2, ~arr)

        /* Process labels to create an array that will be
           turned to a permutation */
        label1:= Position(labeling, [row, pow1 mod 11]);
        label2:= Position(labeling, [col, pow2*pow1 mod
           11]);
arr[label1]:= label2;

srow := Sprint(row); /* turn to a string */
scol := Sprint(col);
spow1 := Sprint(pow1);
spow2 := Sprint(pow2);

label1:= Sprint(label1);
label2:= Sprint(label2);

if pow1 eq 1 then
     /* b_{1,2} = 3       t_1 = t_2^3 */
    "b_{" cat  srow cat ", " cat scol cat "} = " cat spow2
       cat  " \\implies "  cat  "t_{" cat  srow cat "} = t_
       {" cat scol cat "}^{" cat spow2 cat "} \\implies
       \\\\ ";
end if;
```

```
        /* (t_1)^n = (t_2^3)^n  */
if pow1 gt 1 then
        "t_{" cat  srow cat "}^{" cat spow1 cat "} = " cat
           "(t_{" cat  scol cat "}^{" cat spow2 cat "})^{"
           cat spow1 cat "} =  " cat "t_{" cat  scol cat
           "}^{" cat Sprint(pow1*pow2) cat "} = " cat  "t_
           {" cat  scol cat "}^{" cat Sprint(pow1*pow2 mod
           11) cat "}. \\implies " cat label1 cat " \\
           implies " cat label2 cat "\\\\";

end if;
end procedure;

arrY:=[0: i in [1..120]];
/* r for row in the matrix.*/
for r in [1..12] do
/* c for column in the matrix. */
for c in [1..12] do
                if B[r, c] ne 0 then
for p in [1..10] do
        value:= StringToInteger(Sprint(B[r, c]), 12);
        FindLabels(r, p, c, value,  ~arrY);
end for;
                end if;
```

```
end for;
"-----------------------------------------";
end for;
/*
b_{1, 3} = 1 \implies t_{1} = t_{3}^{1} \implies \\
t_{1}^{2} = (t_{3}^{1})^{2} =  t_{3}^{2} = t_{3}^{2}. \
   implies 13 \implies
15\\
t_{1}^{3} = (t_{3}^{1})^{3} =  t_{3}^{3} = t_{3}^{3}. \
   implies 25 \implies
27\\
t_{1}^{4} = (t_{3}^{1})^{4} =  t_{3}^{4} = t_{3}^{4}. \
   implies 37 \implies
39\\
t_{1}^{5} = (t_{3}^{1})^{5} =  t_{3}^{5} = t_{3}^{5}. \
   implies 49 \implies
51\\
t_{1}^{6} = (t_{3}^{1})^{6} =  t_{3}^{6} = t_{3}^{6}. \
   implies 61 \implies
63\\
t_{1}^{7} = (t_{3}^{1})^{7} =  t_{3}^{7} = t_{3}^{7}. \
   implies 73 \implies
75\\
t_{1}^{8} = (t_{3}^{1})^{8} =  t_{3}^{8} = t_{3}^{8}. \
```

```
   implies 85 \implies
87\\
t_{1}^{9} = (t_{3}^{1})^{9} =  t_{3}^{9} = t_{3}^{9}. \
   implies 97 \implies
99\\
t_{1}^{10} = (t_{3}^{1})^{10} =  t_{3}^{10} = t_{3}^{10}. \
   implies 109
\implies 111\\
--------------------------------------------
b_{2, 5} = 5 \implies t_{2} = t_{5}^{5} \implies \\
t_{2}^{2} = (t_{5}^{5})^{2} =  t_{5}^{10} = t_{5}^{10}. \
   implies 14
\implies 113\\
t_{2}^{3} = (t_{5}^{5})^{3} =  t_{5}^{15} = t_{5}^{4}. \
   implies 26
\implies 41\\
t_{2}^{4} = (t_{5}^{5})^{4} =  t_{5}^{20} = t_{5}^{9}. \
   implies 38
\implies 101\\
t_{2}^{5} = (t_{5}^{5})^{5} =  t_{5}^{25} = t_{5}^{3}. \
   implies 50
\implies 29\\
t_{2}^{6} = (t_{5}^{5})^{6} =  t_{5}^{30} = t_{5}^{8}. \
   implies 62
```

```
\implies 89\\
t_{2}^{7} = (t_{5}^{5})^{7} =  t_{5}^{35} = t_{5}^{2}. \
   implies 74
\implies 17\\
t_{2}^{8} = (t_{5}^{5})^{8} =  t_{5}^{40} = t_{5}^{7}. \
   implies 86
\implies 77\\
t_{2}^{9} = (t_{5}^{5})^{9} =  t_{5}^{45} = t_{5}^{1}. \
   implies 98
\implies 5\\
t_{2}^{10} = (t_{5}^{5})^{10} =  t_{5}^{50} = t_{5}^{6}. \
   implies 110
\implies 65\\
---------------------------------------------
b_{3, 7} = 3 \implies t_{3} = t_{7}^{3} \implies \\
t_{3}^{2} = (t_{7}^{3})^{2} =  t_{7}^{6} = t_{7}^{6}. \
   implies 15 \implies
67\\
t_{3}^{3} = (t_{7}^{3})^{3} =  t_{7}^{9} = t_{7}^{9}. \
   implies 27 \implies
103\\
t_{3}^{4} = (t_{7}^{3})^{4} =  t_{7}^{12} = t_{7}^{1}. \
   implies 39
\implies 7\\
```

```
t_{3}^{5} = (t_{7}^{3})^{5} =  t_{7}^{15} = t_{7}^{4}. \
   implies 51
\implies 43\\
t_{3}^{6} = (t_{7}^{3})^{6} =  t_{7}^{18} = t_{7}^{7}. \
   implies 63
\implies 79\\
t_{3}^{7} = (t_{7}^{3})^{7} =  t_{7}^{21} = t_{7}^{10}. \
   implies 75
\implies 115\\
t_{3}^{8} = (t_{7}^{3})^{8} =  t_{7}^{24} = t_{7}^{2}. \
   implies 87
\implies 19\\
t_{3}^{9} = (t_{7}^{3})^{9} =  t_{7}^{27} = t_{7}^{5}. \
   implies 99
\implies 55\\
t_{3}^{10} = (t_{7}^{3})^{10} =  t_{7}^{30} = t_{7}^{8}. \
   implies 111
\implies 91\\
--------------------------------------------
b_{4, 9} = 3 \implies t_{4} = t_{9}^{3} \implies \\
t_{4}^{2} = (t_{9}^{3})^{2} =  t_{9}^{6} = t_{9}^{6}. \
   implies 16 \implies
69\\
t_{4}^{3} = (t_{9}^{3})^{3} =  t_{9}^{9} = t_{9}^{9}. \
```

```
   implies 28 \implies
105\\
t_{4}^{4} = (t_{9}^{3})^{4} =  t_{9}^{12} = t_{9}^{1}. \
   implies 40
\implies 9\\
t_{4}^{5} = (t_{9}^{3})^{5} =  t_{9}^{15} = t_{9}^{4}. \
   implies 52
\implies 45\\
t_{4}^{6} = (t_{9}^{3})^{6} =  t_{9}^{18} = t_{9}^{7}. \
   implies 64
\implies 81\\
t_{4}^{7} = (t_{9}^{3})^{7} =  t_{9}^{21} = t_{9}^{10}. \
   implies 76
\implies 117\\
t_{4}^{8} = (t_{9}^{3})^{8} =  t_{9}^{24} = t_{9}^{2}. \
   implies 88
\implies 21\\
t_{4}^{9} = (t_{9}^{3})^{9} =  t_{9}^{27} = t_{9}^{5}. \
   implies 100
\implies 57\\
t_{4}^{10} = (t_{9}^{3})^{10} =  t_{9}^{30} = t_{9}^{8}. \
   implies 112
\implies 93\\
--------------------------------------------
```

```
b_{5, 8} = 1 \implies t_{5} = t_{8}^{1} \implies \\
t_{5}^{2} = (t_{8}^{1})^{2} = t_{8}^{2} = t_{8}^{2}. \
    implies 17 \implies
20\\
t_{5}^{3} = (t_{8}^{1})^{3} = t_{8}^{3} = t_{8}^{3}. \
    implies 29 \implies
32\\
t_{5}^{4} = (t_{8}^{1})^{4} = t_{8}^{4} = t_{8}^{4}. \
    implies 41 \implies
44\\
t_{5}^{5} = (t_{8}^{1})^{5} = t_{8}^{5} = t_{8}^{5}. \
    implies 53 \implies
56\\
t_{5}^{6} = (t_{8}^{1})^{6} = t_{8}^{6} = t_{8}^{6}. \
    implies 65 \implies
68\\
t_{5}^{7} = (t_{8}^{1})^{7} = t_{8}^{7} = t_{8}^{7}. \
    implies 77 \implies
80\\
t_{5}^{8} = (t_{8}^{1})^{8} = t_{8}^{8} = t_{8}^{8}. \
    implies 89 \implies
92\\
t_{5}^{9} = (t_{8}^{1})^{9} = t_{8}^{9} = t_{8}^{9}. \
    implies 101
```

```
\implies 104\\
t_{5}^{10} = (t_{8}^{1})^{10} =  t_{8}^{10} = t_{8}^{10}. \
   implies 113
\implies 116\\
---------------------------------------------
b_{6, 4} = 1 \implies t_{6} = t_{4}^{1} \implies \\
t_{6}^{2} = (t_{4}^{1})^{2} =  t_{4}^{2} = t_{4}^{2}. \
   implies 18 \implies
16\\
t_{6}^{3} = (t_{4}^{1})^{3} =  t_{4}^{3} = t_{4}^{3}. \
   implies 30 \implies
28\\
t_{6}^{4} = (t_{4}^{1})^{4} =  t_{4}^{4} = t_{4}^{4}. \
   implies 42 \implies
40\\
t_{6}^{5} = (t_{4}^{1})^{5} =  t_{4}^{5} = t_{4}^{5}. \
   implies 54 \implies
52\\
t_{6}^{6} = (t_{4}^{1})^{6} =  t_{4}^{6} = t_{4}^{6}. \
   implies 66 \implies
64\\
t_{6}^{7} = (t_{4}^{1})^{7} =  t_{4}^{7} = t_{4}^{7}. \
   implies 78 \implies
76\\
```

```
t_{6}^{8} = (t_{4}^{1})^{8} =  t_{4}^{8} = t_{4}^{8}. \
   implies 90 \implies
88\\
t_{6}^{9} = (t_{4}^{1})^{9} =  t_{4}^{9} = t_{4}^{9}. \
   implies 102
\implies 100\\
t_{6}^{10} = (t_{4}^{1})^{10} =  t_{4}^{10} = t_{4}^{10}. \
   implies 114
\implies 112\\
--------------------------------------------
b_{7, 1} = 4 \implies t_{7} = t_{1}^{4} \implies \\
t_{7}^{2} = (t_{1}^{4})^{2} =  t_{1}^{8} = t_{1}^{8}. \
   implies 19 \implies
85\\
t_{7}^{3} = (t_{1}^{4})^{3} =  t_{1}^{12} = t_{1}^{1}. \
   implies 31
\implies 1\\
t_{7}^{4} = (t_{1}^{4})^{4} =  t_{1}^{16} = t_{1}^{5}. \
   implies 43
\implies 49\\
t_{7}^{5} = (t_{1}^{4})^{5} =  t_{1}^{20} = t_{1}^{9}. \
   implies 55
\implies 97\\
t_{7}^{6} = (t_{1}^{4})^{6} =  t_{1}^{24} = t_{1}^{2}. \
```

```
   implies 67
\implies 13\\
t_{7}^{7} = (t_{1}^{4})^{7} =  t_{1}^{28} = t_{1}^{6}. \
   implies 79
\implies 61\\
t_{7}^{8} = (t_{1}^{4})^{8} =  t_{1}^{32} = t_{1}^{10}. \
   implies 91
\implies 109\\
t_{7}^{9} = (t_{1}^{4})^{9} =  t_{1}^{36} = t_{1}^{3}. \
   implies 103
\implies 25\\
t_{7}^{10} = (t_{1}^{4})^{10} =  t_{1}^{40} = t_{1}^{7}. \
   implies 115
\implies 73\\
---------------------------------------------
b_{8, 2} = 9 \implies t_{8} = t_{2}^{9} \implies \\
t_{8}^{2} = (t_{2}^{9})^{2} =  t_{2}^{18} = t_{2}^{7}. \
   implies 20
\implies 74\\
t_{8}^{3} = (t_{2}^{9})^{3} =  t_{2}^{27} = t_{2}^{5}. \
   implies 32
\implies 50\\
t_{8}^{4} = (t_{2}^{9})^{4} =  t_{2}^{36} = t_{2}^{3}. \
   implies 44
```

```
\implies 26\\
t_{8}^{5} = (t_{2}^{9})^{5} =  t_{2}^{45} = t_{2}^{1}. \
   implies 56
\implies 2\\
t_{8}^{6} = (t_{2}^{9})^{6} =  t_{2}^{54} = t_{2}^{10}. \
   implies 68
\implies 110\\
t_{8}^{7} = (t_{2}^{9})^{7} =  t_{2}^{63} = t_{2}^{8}. \
   implies 80
\implies 86\\
t_{8}^{8} = (t_{2}^{9})^{8} =  t_{2}^{72} = t_{2}^{6}. \
   implies 92
\implies 62\\
t_{8}^{9} = (t_{2}^{9})^{9} =  t_{2}^{81} = t_{2}^{4}. \
   implies 104
\implies 38\\
t_{8}^{10} = (t_{2}^{9})^{10} =  t_{2}^{90} = t_{2}^{2}. \
   implies 116
\implies 14\\
---------------------------------------------
b_{9, 6} = 4 \implies t_{9} = t_{6}^{4} \implies \\
t_{9}^{2} = (t_{6}^{4})^{2} =  t_{6}^{8} = t_{6}^{8}. \
   implies 21 \implies
90\\
```

```
t_{9}^{3} = (t_{6}^{4})^{3} =  t_{6}^{12} = t_{6}^{1}. \
  implies 33
\implies 6\\
t_{9}^{4} = (t_{6}^{4})^{4} =  t_{6}^{16} = t_{6}^{5}. \
  implies 45
\implies 54\\
t_{9}^{5} = (t_{6}^{4})^{5} =  t_{6}^{20} = t_{6}^{9}. \
  implies 57
\implies 102\\
t_{9}^{6} = (t_{6}^{4})^{6} =  t_{6}^{24} = t_{6}^{2}. \
  implies 69
\implies 18\\
t_{9}^{7} = (t_{6}^{4})^{7} =  t_{6}^{28} = t_{6}^{6}. \
  implies 81
\implies 66\\
t_{9}^{8} = (t_{6}^{4})^{8} =  t_{6}^{32} = t_{6}^{10}. \
  implies 93
\implies 114\\
t_{9}^{9} = (t_{6}^{4})^{9} =  t_{6}^{36} = t_{6}^{3}. \
  implies 105
\implies 30\\
t_{9}^{10} = (t_{6}^{4})^{10} =  t_{6}^{40} = t_{6}^{7}. \
  implies 117
\implies 78\\
```

```
----------------------------------------------
b_{10, 11} = 1 \implies t_{10} = t_{11}^{1} \implies \\
t_{10}^{2} = (t_{11}^{1})^{2} =  t_{11}^{2} = t_{11}^{2}. \
   implies 22
\implies 23\\
t_{10}^{3} = (t_{11}^{1})^{3} =  t_{11}^{3} = t_{11}^{3}. \
   implies 34
\implies 35\\
t_{10}^{4} = (t_{11}^{1})^{4} =  t_{11}^{4} = t_{11}^{4}. \
   implies 46
\implies 47\\
t_{10}^{5} = (t_{11}^{1})^{5} =  t_{11}^{5} = t_{11}^{5}. \
   implies 58
\implies 59\\
t_{10}^{6} = (t_{11}^{1})^{6} =  t_{11}^{6} = t_{11}^{6}. \
   implies 70
\implies 71\\
t_{10}^{7} = (t_{11}^{1})^{7} =  t_{11}^{7} = t_{11}^{7}. \
   implies 82
\implies 83\\
t_{10}^{8} = (t_{11}^{1})^{8} =  t_{11}^{8} = t_{11}^{8}. \
   implies 94
\implies 95\\
t_{10}^{9} = (t_{11}^{1})^{9} =  t_{11}^{9} = t_{11}^{9}. \
```

```
    implies 106
\implies 107\\
t_{10}^{10} = (t_{11}^{1})^{10} =  t_{11}^{10} = t_
    {11}^{10}. \implies 118
\implies 119\\
----------------------------------------------
b_{11, 12} = 1 \implies t_{11} = t_{12}^{1} \implies \\
t_{11}^{2} = (t_{12}^{1})^{2} =  t_{12}^{2} = t_{12}^{2}. \
    implies 23
\implies 24\\
t_{11}^{3} = (t_{12}^{1})^{3} =  t_{12}^{3} = t_{12}^{3}. \
    implies 35
\implies 36\\
t_{11}^{4} = (t_{12}^{1})^{4} =  t_{12}^{4} = t_{12}^{4}. \
    implies 47
\implies 48\\
t_{11}^{5} = (t_{12}^{1})^{5} =  t_{12}^{5} = t_{12}^{5}. \
    implies 59
\implies 60\\
t_{11}^{6} = (t_{12}^{1})^{6} =  t_{12}^{6} = t_{12}^{6}. \
    implies 71
\implies 72\\
t_{11}^{7} = (t_{12}^{1})^{7} =  t_{12}^{7} = t_{12}^{7}. \
    implies 83
```

```
\implies 84\\
t_{11}^{8} = (t_{12}^{1})^{8} =  t_{12}^{8} = t_{12}^{8}. \
   implies 95
\implies 96\\
t_{11}^{9} = (t_{12}^{1})^{9} =  t_{12}^{9} = t_{12}^{9}. \
   implies 107
\implies 108\\
t_{11}^{10} = (t_{12}^{1})^{10} =  t_{12}^{10} = t_
   {12}^{10}. \implies 119
\implies 120\\
--------------------------------------------
b_{12, 10} = 1 \implies t_{12} = t_{10}^{1} \implies \\
t_{12}^{2} = (t_{10}^{1})^{2} =  t_{10}^{2} = t_{10}^{2}. \
   implies 24
\implies 22\\
t_{12}^{3} = (t_{10}^{1})^{3} =  t_{10}^{3} = t_{10}^{3}. \
   implies 36
\implies 34\\
t_{12}^{4} = (t_{10}^{1})^{4} =  t_{10}^{4} = t_{10}^{4}. \
   implies 48
\implies 46\\
t_{12}^{5} = (t_{10}^{1})^{5} =  t_{10}^{5} = t_{10}^{5}. \
   implies 60
\implies 58\\
```

```
t_{12}^{6} = (t_{10}^{1})^{6} =  t_{10}^{6} = t_{10}^{6}. \
   implies 72
\implies 70\\
t_{12}^{7} = (t_{10}^{1})^{7} =  t_{10}^{7} = t_{10}^{7}. \
   implies 84
\implies 82\\
t_{12}^{8} = (t_{10}^{1})^{8} =  t_{10}^{8} = t_{10}^{8}. \
   implies 96
\implies 94\\
t_{12}^{9} = (t_{10}^{1})^{9} =  t_{10}^{9} = t_{10}^{9}. \
   implies 108
\implies 106\\
t_{12}^{10} = (t_{10}^{1})^{10} =  t_{10}^{10} = t_
   {10}^{10}. \implies 120
\implies 118\\*/

S:=Sym(120);
yyy:=S!arrY;
yyy; /* (1, 3, 31)(2, 53, 56)(4, 33, 6)(5, 8, 98)(7, 37,
   39)(9, 42, 40)(10, 11, 12)(13,
    15, 67)(14, 113, 116)(16, 69, 18)(17, 20, 74)(19, 85,
       87)(21, 90, 88)(22,
    23, 24)(25, 27, 103)(26, 41, 44)(28, 105, 30)(29, 32,
       50)(34, 35, 36)(38,
```

```
    101, 104)(43, 49, 51)(45, 54, 52)(46, 47, 48)(55, 97,
        99)(57, 102, 100)(58,
    59, 60)(61, 63, 79)(62, 89, 92)(64, 81, 66)(65, 68,
        110)(70, 71, 72)(73, 75,
    115)(76, 117, 78)(77, 80, 86)(82, 83, 84)(91, 109, 111)
        (93, 114, 112)(94,
    95, 96)(106, 107, 108)(118, 119, 120) */

N:=sub<S| xxx, yyy>;
#N;

s:=IsIsomorphic(N,G);
s; /* true */

FPGroup(N);
 /*   x^5,  y^3,  y * x^2 * y^-1 * x^2 * y * x^-1 * y^-1 *
    x^-1,     (x * y * x * y * x)^2, */
NN< x, y >:=Group<x, y | x^5,  y^3,  y * x^2 * y^-1 * x^2 *
    y * x^-1 * y^-1 * x^-1,     (x * y * x * y * x)^2>;
#NN; /* 660 */

fn,N1,kn:=CosetAction(NN,sub<NN| Id(NN)>);
s:=IsIsomorphic(N1, N);
s; /* true */
```

```
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
word:=function(Perm)
for w in Sch do
        seq := Eltseq(w);
        p:= Id(N);
        for j in seq do
                if j eq 1 then p:=p*xxx; end if;
if j eq -1 then p:=p*xxx^-1; end if;
if j eq 2 then p:=p*yyy; end if;
if j eq -2 then p:=p*yyy^-1; end if;
        end for;
        if Perm eq p then return w; end if;
end for;
end function;

GetConjugationWordForSubscript:= function(subscript)
        size:=99999999;
        w:=word(Id(N));
for n in N do
        if n eq Id(N) then
                continue;
        end if;
if 1^n eq subscript then
```

```
        tmp:= word(n);
        if size gt #tmp then
                size:=#tmp;
                w:= tmp;
                end if;

end if;
end for;
return w;
end function;

 Orbits(Stabilizer(N,1));
/* GSet{@ 1 @},
    GSet{@ 13 @},
    GSet{@ 25 @},
    GSet{@ 37 @},
    GSet{@ 49 @},
    GSet{@ 61 @},
    GSet{@ 73 @},
    GSet{@ 85 @},
    GSet{@ 97 @},
    GSet{@ 109 @},
    GSet{@ 2, 104, 107, 58, 7, 102, 12, 28, 5, 57, 39 @},
    GSet{@ 3, 26, 56, 59, 46, 31, 54, 36, 100, 29, 45 @},
```

```
    GSet{@ 4, 41, 105, 51, 38, 32, 35, 106, 43, 30, 48 @},
    GSet{@ 6, 60, 40, 53, 33, 99, 50, 8, 11, 34, 55 @},
    GSet{@ 9, 27, 98, 44, 47, 10, 103, 42, 108, 52, 101 @},
    GSet{@ 14, 80, 83, 118, 19, 78, 24, 64, 17, 117, 87 @},
    GSet{@ 15, 62, 116, 119, 94, 67, 114, 72, 76, 65, 93 @
       },
    GSet{@ 16, 89, 81, 111, 86, 68, 71, 82, 91, 66, 96 @},
    GSet{@ 18, 120, 88, 113, 69, 75, 110, 20, 23, 70, 115 @
       },
    GSet{@ 21, 63, 74, 92, 95, 22, 79, 90, 84, 112, 77 @}
       */

/* To test progenitor (t1,t2) , (t1, t3), (t1,t4), (t1, t6)
   , (t1, t9) */

Stabilizer(N, {1,13,25,37,49,61,73,85,97,109});
gen:=Generators(Stabilizer(N,
   {1,13,25,37,49,61,73,85,97,109}));

g1:=N!(2, 57, 28, 102, 58, 104, 39, 5, 12, 7, 107)(3, 29,
   36, 31, 59, 26, 45, 100,
        54, 46, 56)(4, 30, 106, 32, 51, 41, 48, 43, 35, 38,
             105)(6, 34, 8, 99,
        53, 60, 55, 11, 50, 33, 40)(9, 52, 42, 10, 44, 27,
```

```
        101, 108, 103, 47,
    98)(14, 117, 64, 78, 118, 80, 87, 17, 24, 19, 83)
        (15, 65, 72, 67, 119,
    62, 93, 76, 114, 94, 116)(16, 66, 82, 68, 111, 89,
        96, 91, 71, 86,
    81)(18, 70, 20, 75, 113, 120, 115, 23, 110, 69, 88)
        (21, 112, 90, 22, 92,
    63, 77, 84, 79, 95, 74);

g2:= N!(1, 97, 37, 25, 49)(2, 8, 36, 105, 47)(3, 4, 10,
  102, 55)(5, 53, 29, 41,
    101)(6, 31, 51, 52, 58)(7, 99, 100, 106, 42)(9,
        107, 50, 56, 48)(11, 26,
    32, 108, 57)(12, 33, 59, 38, 44)(13, 73, 85, 61,
        109)(14, 20, 72, 81,
    95)(15, 16, 22, 78, 115)(17, 113, 65, 89, 77)(18,
        67, 111, 112, 118)(19,
    75, 76, 82, 90)(21, 83, 110, 116, 96)(23, 62, 68,
        84, 117)(24, 69, 119,
    86, 92)(27, 28, 34, 54, 43)(30, 103, 39, 40, 46)
        (35, 98, 104, 60,
    45)(63, 64, 70, 114, 91)(66, 79, 87, 88, 94)(71,
        74, 80, 120, 93);
```

```
found:=false;
for a, b in gen do
if a ne b and a ne Id(N) and b ne Id(N) and found eq false
    then
                Gen:=sub<N| a, b>;
                if Gen  eq sub<N|g1, g2> then
found := true;
break;
end if;
        end if;
end for;

/* These results changed from the last try, but they still
   work. */
word(g1); /*   g1 ~ x^-2 * y^-1 * x  ==> (t  , x^-2 * y^-1 *
    x) */
1^(xxx^-2*yyy^-1*xxx); /* 1 */

word(g2);/* g2 ~ x * y^-1 * x * y * x^2;   */
1^(xxx*yyy^-1*xxx*yyy*xxx^2);  /* 49 ==>  t1^( x * y^-1 * x
    * y * x^2) = t49 = t^5 */

IsIsomorphic(Gen,  sub<N| xxx^-2*yyy^-1*xxx, xxx*yyy^-1*xxx
   *yyy*xxx^2 >); /* true */
```

```
/* To test progenitor (t1,t2) , (t1, t3), (t1,t4), (t1, t6)
    , (t1, t9) */
GetConjugationWordForSubscript(2); /* (t, t^x ) */
GetConjugationWordForSubscript(3); /* (t, t^y */
GetConjugationWordForSubscript(4); /* (t, t^(y^-1 * x^-1 *
    y)) */
GetConjugationWordForSubscript(6); /* (t, t^(y^-1 * x^-1))
    */
GetConjugationWordForSubscript(9); /* (t, t^(x^2 * y)) */

/*****************************************************************


S:=Sym(120);
yy:=S!(1, 3, 31)(2, 53, 56)(4, 33, 6)(5, 8, 98)(7, 37, 39)
    (9, 42, 40)(10, 11, 12)(13,
     15, 67)(14, 113, 116)(16, 69, 18)(17, 20, 74)(19, 85,
        87)(21, 90, 88)(22,
     23, 24)(25, 27, 103)(26, 41, 44)(28, 105, 30)(29, 32,
        50)(34, 35, 36)(38,
     101, 104)(43, 49, 51)(45, 54, 52)(46, 47, 48)(55, 97,
        99)(57, 102, 100)(58,
     59, 60)(61, 63, 79)(62, 89, 92)(64, 81, 66)(65, 68,
```

```
    110)(70, 71, 72)(73, 75,
  115)(76, 117, 78)(77, 80, 86)(82, 83, 84)(91, 109, 111)
    (93, 114, 112)(94,
  95, 96)(106, 107, 108)(118, 119, 120);

xx:=S!(1, 2, 40, 8, 5)(3, 42, 7, 10, 45)(4, 32, 29, 25, 26)
  (6, 31, 34, 9, 27)(11, 47,
  59, 107, 35)(12, 36, 108, 60, 48)(13, 14, 88, 20, 17)
    (15, 90, 19, 22,
  93)(16, 68, 65, 61, 62)(18, 67, 70, 21, 63)(23, 95,
    119, 83, 71)(24, 72, 84,
  120, 96)(28, 104, 101, 97, 98)(30, 103, 106, 33, 99)
    (37, 38, 52, 44, 41)(39,
  54, 43, 46, 57)(49, 50, 100, 56, 53)(51, 102, 55, 58,
    105)(64, 80, 77, 73,
  74)(66, 79, 82, 69, 75)(76, 116, 113, 109, 110)(78,
    115, 118, 81, 111)(85,
  86, 112, 92, 89)(87, 114, 91, 94, 117);

N:=sub<S|xx, yy>;

G<x,y,t>:=Group< x, y, t | x^5, y^3, y * x^2 * y^-1 * x^2
  * y * x^-1 * y^-1 * x^-1, (x * y * x * y * x)^2, t
  ^11, (t , x^-2 * y^-1 * x), t^( x * y^-1 * x * y * x^2)
```

```
= t^5, (t, t^x), (t, t^y), (t, t^(y^-1 * x^-1 * y)), (t
  , t^(y^-1 * x^-1)), (t, t^(x^2 * y)) > ;

print Index(G,sub<G|x,y>: CosetLimit:=9^10, Hard:=true,
  Print:=2);

/* nohup magma "Mon11_12Index"&>Mon11_12Index.out&  */
/***********************************************************************


C:=Classes(N);
C;

/* PRINT SINGLE ORDER RELATIONS */
for i in [2..#C] do
        "*************** NEW CLASS ***************";
OC:=Orbits(Centraliser(N,C[i][3]));
for orbit in OC do
        conjWord:= "";
classRepWord:= Sprint(word(C[i][3]));
if orbit[1] eq 1 then
        classRepWord cat "*t";
"---------------------------------------------";
else
```

```
        conjWord:= Sprint(GetConjugationWordForSubscript(
           orbit[1]));
classRepWord cat "*t^(" cat conjWord cat ")";
"-------------------------------------------";
end if;
end for;
 end for;

/*
*************** NEW CLASS ***************
y * x^-1 * y^-1 * x^-2 * y * x*t
-------------------------------------------
y * x^-1 * y^-1 * x^-2 * y * x*t^(x)
-------------------------------------------
y * x^-1 * y^-1 * x^-2 * y * x*t^(y)
-------------------------------------------
y * x^-1 * y^-1 * x^-2 * y * x*t^(y * x^2)
-------------------------------------------
y * x^-1 * y^-1 * x^-2 * y * x*t^(Id(NN))
-------------------------------------------
y * x^-1 * y^-1 * x^-2 * y * x*t^(Id(NN))
-------------------------------------------
y * x^-1 * y^-1 * x^-2 * y * x*t^(Id(NN))
-------------------------------------------
```

```
y * x^-1 * y^-1 * x^-2 * y * x*t^(Id(NN))
--------------------------------------------
y * x^-1 * y^-1 * x^-2 * y * x*t^(y^-1 * x^-2)
--------------------------------------------
y * x^-1 * y^-1 * x^-2 * y * x*t^(Id(NN))
--------------------------------------------
**************** NEW CLASS ****************
y*t
--------------------------------------------
y*t^(x)
--------------------------------------------
y*t^(y^-1 * x^-1 * y)
--------------------------------------------
y*t^(x^-1)
--------------------------------------------
y*t^(y * x^2)
--------------------------------------------
y*t^(Id(NN))
--------------------------------------------
y*t^(Id(NN))
--------------------------------------------
y*t^(Id(NN))
--------------------------------------------
y*t^(Id(NN))
```

```
-----------------------------------------
y*t^(Id(NN))
-----------------------------------------
y*t^(y^-1 * x^-1 * y * x^-1)
-----------------------------------------
y*t^(x^-1 * y^-1 * x)
-----------------------------------------
y*t^(y^-1 * x^-1 * y * x * y^-1)
-----------------------------------------
y*t^(y * x^-1 * y^-1 * x^-1)
-----------------------------------------
y*t^(x * y * x * y^-1)
-----------------------------------------
y*t^(Id(NN))
-----------------------------------------
y*t^(Id(NN))
-----------------------------------------
y*t^(Id(NN))
-----------------------------------------
y*t^(Id(NN))
-----------------------------------------
y*t^(Id(NN))
-----------------------------------------
*************** NEW CLASS ****************
```

```
x*t
------------------------------------------
x*t^(y)
------------------------------------------
x*t^(y^-1 * x^-1 * y)
------------------------------------------
x*t^(y^-1 * x^-1)
------------------------------------------
x*t^(y^-1 * x * y * x)
------------------------------------------
x*t^(y^-1 * x * y^-1 * x^-1)
------------------------------------------
x*t^(Id(NN))
------------------------------------------
x*t^(Id(NN))
------------------------------------------
x*t^(Id(NN))
------------------------------------------
x*t^(Id(NN))
------------------------------------------
x*t^(Id(NN))
------------------------------------------
x*t^(Id(NN))
------------------------------------------
```

```
x*t^(x^-1 * y^-1 * x)
------------------------------------------
x*t^(x^-1 * y^-1 * x * y^-1)
------------------------------------------
x*t^(y * x^2 * y)
------------------------------------------
x*t^((y * x * y^-1)^2)
------------------------------------------
x*t^(x * y * x)
------------------------------------------
x*t^((x * y)^2)
------------------------------------------
x*t^(Id(NN))
------------------------------------------
x*t^(Id(NN))
------------------------------------------
x*t^(Id(NN))
------------------------------------------
x*t^(Id(NN))
------------------------------------------
x*t^(Id(NN))
------------------------------------------
x*t^(Id(NN))
------------------------------------------
```

```
**************** NEW CLASS ****************
x^2*t
-------------------------------------------
x^2*t^(y)
-------------------------------------------
x^2*t^(y^-1 * x^-1 * y)
-------------------------------------------
x^2*t^(y^-1 * x^-1)
-------------------------------------------
x^2*t^(y^-1 * x * y * x)
-------------------------------------------
x^2*t^(y^-1 * x * y^-1 * x^-1)
-------------------------------------------
x^2*t^(Id(NN))
-------------------------------------------
x^2*t^(Id(NN))
-------------------------------------------
x^2*t^(Id(NN))
-------------------------------------------
x^2*t^(Id(NN))
-------------------------------------------
x^2*t^(Id(NN))
-------------------------------------------
x^2*t^(Id(NN))
```

```
-----------------------------------------
x^2*t^(x^-1 * y^-1 * x)
-----------------------------------------
x^2*t^(x^-1 * y^-1 * x * y^-1)
-----------------------------------------
x^2*t^(y * x^2 * y)
-----------------------------------------
x^2*t^((y * x * y^-1)^2)
-----------------------------------------
x^2*t^(x * y * x)
-----------------------------------------
x^2*t^((x * y)^2)
-----------------------------------------
x^2*t^(Id(NN))
-----------------------------------------
x^2*t^(Id(NN))
-----------------------------------------
x^2*t^(Id(NN))
-----------------------------------------
x^2*t^(Id(NN))
-----------------------------------------
x^2*t^(Id(NN))
-----------------------------------------
x^2*t^(Id(NN))
```

```
-----------------------------------------
*************** NEW CLASS ***************
x^2 * y * x^-1 * y * x^-1*t

-----------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(x)

-----------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(y)

-----------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(y^-1 * x^-1 * y)

-----------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(y^-1 * x^-1)

-----------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(y * x^2)

-----------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(x^2 * y)

-----------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(y^-1 * x * y * x)

-----------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(Id(NN))

-----------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(Id(NN))

-----------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(Id(NN))

-----------------------------------------
```

```
x^2 * y * x^-1 * y * x^-1*t^(Id(NN))
--------------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(Id(NN))
--------------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(Id(NN))
--------------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(Id(NN))
--------------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(Id(NN))
--------------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(y^-1 * x^-2)
--------------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(x^y)
--------------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(Id(NN))
--------------------------------------------
x^2 * y * x^-1 * y * x^-1*t^(Id(NN))
--------------------------------------------
**************** NEW CLASS ****************
x * y*t^(x^-1 * y)
--------------------------------------------
x * y*t^(Id(NN))
--------------------------------------------
x * y*t^((y, x))
```

```
------------------------------------------
x * y*t^((y^-1, x))
------------------------------------------
x * y*t^(x * y^-1)
------------------------------------------
x * y*t^(Id(NN))
------------------------------------------
x * y*t^(Id(NN))
------------------------------------------
x * y*t^(Id(NN))
------------------------------------------
x * y*t^(x^-1 * y^-1 * x^2)
------------------------------------------
x * y*t^(Id(NN))
------------------------------------------
x * y*t
------------------------------------------
x * y*t^(y)
------------------------------------------
x * y*t^(y^-1 * x^-1 * y)
------------------------------------------
x * y*t^(y^-1 * x * y^-1 * x^-1)
------------------------------------------
x * y*t^(Id(NN))
```

```
------------------------------------------
x * y*t^(Id(NN))
------------------------------------------
x * y*t^(Id(NN))
------------------------------------------
x * y*t^(Id(NN))
------------------------------------------
x * y*t^(y^-1 * x^-2 * y^-1)
------------------------------------------
x * y*t^(Id(NN))
------------------------------------------
**************** NEW CLASS ****************
(x * y)^2*t^(x^-1 * y)
------------------------------------------
(x * y)^2*t^(Id(NN))
------------------------------------------
(x * y)^2*t^((y, x))
------------------------------------------
(x * y)^2*t^((y^-1, x))
------------------------------------------
(x * y)^2*t^(x * y^-1)
------------------------------------------
(x * y)^2*t^(Id(NN))
------------------------------------------
```

```
(x * y)^2*t^(Id(NN))
-------------------------------------------
(x * y)^2*t^(Id(NN))
-------------------------------------------
(x * y)^2*t^(x^-1 * y^-1 * x^2)
-------------------------------------------
(x * y)^2*t^(Id(NN))
-------------------------------------------
(x * y)^2*t
-------------------------------------------
(x * y)^2*t^(y)
-------------------------------------------
(x * y)^2*t^(y^-1 * x^-1 * y)
-------------------------------------------
(x * y)^2*t^(y^-1 * x * y^-1 * x^-1)
-------------------------------------------
(x * y)^2*t^(Id(NN))
-------------------------------------------
(x * y)^2*t^(Id(NN))
-------------------------------------------
(x * y)^2*t^(Id(NN))
-------------------------------------------
(x * y)^2*t^(Id(NN))
-------------------------------------------
```

```
(x * y)^2*t^(y^-1 * x^-2 * y^-1)
-------------------------------------------
(x * y)^2*t^(Id(NN))
------------------------------------------- */

/******************************************************************

StrN:="S:=Sym(120);
xx:=S!(1, 2, 40, 8, 5)(3, 42, 7, 10, 45)(4, 32, 29, 25, 26)
   (6, 31, 34, 9, 27)(11, 47,
    59, 107, 35)(12, 36, 108, 60, 48)(13, 14, 88, 20, 17)
       (15, 90, 19, 22,
    93)(16, 68, 65, 61, 62)(18, 67, 70, 21, 63)(23, 95,
       119, 83, 71)(24, 72, 84,
    120, 96)(28, 104, 101, 97, 98)(30, 103, 106, 33, 99)
       (37, 38, 52, 44, 41)(39,
    54, 43, 46, 57)(49, 50, 100, 56, 53)(51, 102, 55, 58,
       105)(64, 80, 77, 73,
    74)(66, 79, 82, 69, 75)(76, 116, 113, 109, 110)(78,
       115, 118, 81, 111)(85,
    86, 112, 92, 89)(87, 114, 91, 94, 117);

yy:=S!(1, 3, 31)(2, 53, 56)(4, 33, 6)(5, 8, 98)(7, 37, 39)
   (9, 42, 40)(10, 11, 12)(13,
```

```
    15, 67)(14, 113, 116)(16, 69, 18)(17, 20, 74)(19, 85,
       87)(21, 90, 88)(22,
    23, 24)(25, 27, 103)(26, 41, 44)(28, 105, 30)(29, 32,
       50)(34, 35, 36)(38,
    101, 104)(43, 49, 51)(45, 54, 52)(46, 47, 48)(55, 97,
       99)(57, 102, 100)(58,
    59, 60)(61, 63, 79)(62, 89, 92)(64, 81, 66)(65, 68,
       110)(70, 71, 72)(73, 75,
    115)(76, 117, 78)(77, 80, 86)(82, 83, 84)(91, 109, 111)
       (93, 114, 112)(94,
    95, 96)(106, 107, 108)(118, 119, 120);

N:=sub<S|xx,yy>;";
StrN;

S:=Sym(120);
xx:=S!(1, 2, 40, 8, 5)(3, 42, 7, 10, 45)(4, 32, 29, 25, 26)
   (6, 31, 34, 9, 27)(11, 47,
    59, 107, 35)(12, 36, 108, 60, 48)(13, 14, 88, 20, 17)
       (15, 90, 19, 22,
    93)(16, 68, 65, 61, 62)(18, 67, 70, 21, 63)(23, 95,
       119, 83, 71)(24, 72, 84,
    120, 96)(28, 104, 101, 97, 98)(30, 103, 106, 33, 99)
       (37, 38, 52, 44, 41)(39,
```

```
    54, 43, 46, 57)(49, 50, 100, 56, 53)(51, 102, 55, 58,
       105)(64, 80, 77, 73,
    74)(66, 79, 82, 69, 75)(76, 116, 113, 109, 110)(78,
       115, 118, 81, 111)(85,
    86, 112, 92, 89)(87, 114, 91, 94, 117);

yy:=S!S!(1, 3, 31)(2, 53, 56)(4, 33, 6)(5, 8, 98)(7, 37,
   39)(9, 42, 40)(10, 11, 12)(13,
    15, 67)(14, 113, 116)(16, 69, 18)(17, 20, 74)(19, 85,
       87)(21, 90, 88)(22,
    23, 24)(25, 27, 103)(26, 41, 44)(28, 105, 30)(29, 32,
       50)(34, 35, 36)(38,
    101, 104)(43, 49, 51)(45, 54, 52)(46, 47, 48)(55, 97,
       99)(57, 102, 100)(58,
    59, 60)(61, 63, 79)(62, 89, 92)(64, 81, 66)(65, 68,
       110)(70, 71, 72)(73, 75,
    115)(76, 117, 78)(77, 80, 86)(82, 83, 84)(91, 109, 111)
       (93, 114, 112)(94,
    95, 96)(106, 107, 108)(118, 119, 120);

N:=sub<S|xx,yy>;

Indices:=[];
first:=true;
```

```
for a, b, c, d, e, f, g in [0..10] do
if a eq 0 and b eq 0 and c eq  0 and d eq 0 and e eq 0 and
   f eq 0 and g eq 0 then
continue;
end if;
G<x,y,t>:=Group< x, y, t | x^5,  y^3,  y * x^2 * y^-1 * x^2
    * y * x^-1 * y^-1 * x^-1, t^12, (x * y * x * y * x)^2,(
   t,x^-2 * y^-1 * x), t^( x * y^-1 * x^2) = t^5 , t^11, (x
   ^2 * y * x^-1 * y * x^-1*t)^a, (y * x^-1 * y^-1 * x^-2 *
    y * x*t^y)^b, ((x^2*t)^c, (x * y*t)^d, ((x * y)^2*t^y))
   ^e ,((x*t))^f, (y*t)^g>;

index:=Index(G,sub<G|x,y>);
if index gt 2  then
         if not (index in Indices) then
Indices:=Indices cat [index];
printf "\"
   ---------------------------------------------------\";\n
   ";
printf "a:=%o; b:=%o; c:=%o; d:=%o; e:=%o; f:=%o; g:=%o;
   index :=%o;\n", a, b, c, d, e, f, g , index ;
printf "G<x,y,t>:=Group< x, y, t | x^5,  y^3,  y * x^2 * y
   ^-1 * x^2 * y * x^-1 * y^-1 * x^-1, t^12, (x * y * x * y
    * x)^2,(t,x^-2 * y^-1 * x), t^( x * y^-1 * x^2) = t^5 ,
```

```
    t^11, (x^2 * y * x^-1 * y * x^-1*t)^a, (y * x^-1 * y^-1
    * x^-2 * y * x*t^y)^b, ((x^2*t)^c, (x * y*t)^d, ((x * y
   )^2*t^y))^e  ,((x*t))^f, (y*t)^g>;\n";

f1,G1,k:=CosetAction(G,sub<G|x,y>);
printf "f,G1,k:=CosetAction(G,sub<G|x,y>);\n#k;\n /* %o */
   \n", #k;
printf "G1;\n"; G1;

printf "#N eq #sub<G1|f(x),f(y)>;\n /* %o */\n", #N eq #sub
   <G1|f1(x),f1(y)>;
printf "CompositionFactors(G1);/*\n";
CompositionFactors(G1);
                    else
                              SetOutputFile("Monomial11_12Nums3.
                                 txt");
                              if first then
                                       StrN;
                                       first:= false;
                              end if;
                              printf "\"
                                 ---------------------------------------------
                                 \";\n";
printf "a:=%o; b:=%o; c:=%o; d:=%o; e:=%o; f:=%o; g:=%o;
```

```
   index :=%o;\n", a, b, c, d, e, f, g , index ;
printf "G<x,y,t>:=Group< x, y, t | x^5, y^3, y * x^2 * y
   ^-1 * x^2 * y * x^-1 * y^-1 * x^-1, t^12, (x * y * x * y
    * x)^2,(t,x^-2 * y^-1 * x), t^( x * y^-1 * x^2) = t^5 ,
    t^11, (x^2 * y * x^-1 * y * x^-1*t)^a, (y * x^-1 * y^-1
    * x^-2 * y * x*t^y)^b, ((x^2*t)^c, (x * y*t)^d, ((x * y
   )^2*t^y))^e ,((x*t))^f, (y*t)^g>; \n";
printf "f,G1,k:=CosetAction(G,sub<G|x,y>);\n#k;\n";
printf "G1;\n";
printf "#N eq #sub<G1|f(x),f(y)>;\n";
printf "CompositionFactors(G1);\n";
UnsetOutputFile();
                    end if;
end if;
end for;
/*  nohup magma "Prog12^*11:_mN.1"&>Prog12^*11:_mN.1.out&
      */
```

Appendix F

MAGMA CODE: Double Coset Enumeration of 3:(2×S_5) Over D_{12}

```
S:=Sym(6);
  xx:=S!(1, 2, 3, 4, 5, 6);
yy:=S!(1, 6)(2, 5)(3, 4);

G<x,y,t>:=Group<x,y,t|x^6,y^2,(x*y)^2,t^3,(t,(x*y)^x),
t^(x^3)=t^2,(t^x*(t^y))^2>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#G1; /*720 */
```

```
IN:=sub<G1| f(x), f(y)>;
N:=sub<S|xx,yy>;

for i in [0..5] do
   for j in [0..1] do
        printf "%o", i;j,xx^i*yy^j;
   end for;
end for;
/*
00 Id(S)
01 (1, 6)(2, 5)(3, 4)
10 (1, 2, 3, 4, 5, 6)
11 (1, 5)(2, 4)
20 (1, 3, 5)(2, 4, 6)
21 (1, 4)(2, 3)(5, 6)
30 (1, 4)(2, 5)(3, 6)
31 (1, 3)(4, 6)
40 (1, 5, 3)(2, 6, 4)
41 (1, 2)(3, 6)(4, 5)
50 (1, 6, 5, 4, 3, 2)
51 (2, 6)(3, 5)
*/
CompositionFactors(G1); /*
```

```
  G
  |   Cyclic(2)
  *
  |   Alternating(5)
  *
  |   Cyclic(2)
  *
  |   Cyclic(3)
  1 */
NL:=NormalLattice(G1); NL;


IN:=sub<G1|f(x),f(y)>;
ts := [ Id(G1): i in [1 .. 6] ];
ts[1]:=f(t); ts[2]:=f(t^(x)); ts[3]:=f(t^(x^2));
ts[4]:=ts[1]^-1;ts[5]:=ts[2]^-1;ts[6]:=ts[3]^-1;
ts;
DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);/*
{ <GrpFP, Id(G), GrpFP>, <GrpFP, t * x * t * x * t * x^-1 *
    t, GrpFP>, <GrpFP, t* x * t * x * t * x * t, GrpFP>, <
   GrpFP, t * x * t * x^-1 * t, GrpFP>, <GrpFP, t* x * t
   ^-1, GrpFP>, <GrpFP, t, GrpFP>, <GrpFP, t * x * t, GrpFP
   >, <GrpFP, t * x* t * x * t, GrpFP>, <GrpFP, t * x * t *
    x * t^-1, GrpFP> }
```

```
*/

DC:= [f(Id(G)), f(t), f(t* x * t^-1), f(t * x * t), f( t *
    x * t * x^-1 * t), f( t * x* t * x * t), f(t * x * t * x
     * t^-1), f( t * x * t * x * t * x^-1 * t), f( t* x * t
    * x * t * x * t)  ];
DC;
 DoubleCosetRepresentatives(G1, IN, IN);
#N;
Index(G1,IN); /* 60 */

cst := [null : i in [1 .. Index(G1,IN)]] where null is [
    Integers() | ];
prodim := function(pt, Q, I)
    v := pt;
    for i in I do
        v := v^(Q[i]);
    end for;
    return v;
end function;
for i := 1 to 6 do
    cst[prodim(1, ts, [i])] := [i];
end for;
m:=0;
```

```
for i in [1..10] do
   if cst[i] ne [] then
      m:=m+1;
   end if;
end for;
m; /* 6 */

Orbits(N);
N1:= Stabilizer(N, 1);
Orbits(N1);

FindDCGroupNumber := function(DC, element, IN)
   for i in [1..#DC] do
      for m,n in IN do
            if element eq m*(DC[i])^n then
                  return i;
            end if;
      end for;
   end for;
end function;

/* ConvertToPermutation(ts, [1,2,3]); and returns a
   permutation of IN isomorphinc to t1t2t3 . Same as doing
   ts[1]*ts[2]*ts[3]*/
```

```
ConvertToPermutation:= function(ts, Tset)
   perm:= ts[Tset[1]];
   for i in [2..#Tset] do
     perm:= perm*ts[Tset[i]];
   end for;
   return perm;
end function;

FindDCGroupNumberOfTset := function(ts, DC, Tset, IN)
   IndPerm:= ConvertToPermutation(ts,Tset);
   return FindDCGroupNumber(DC,IndPerm, IN);
end function;

FindDCGroupNumber (DC, ts[1]*ts[1], IN); /*2        Nt1t1
   belongs to [1] which is #2 in magma*/
FindDCGroupNumber (DC, ts[1]*ts[4], IN); /* 1 */
FindDCGroupNumber (DC, ts[1]*ts[2], IN); /*4 */
FindDCGroupNumber (DC, ts[1]*ts[3], IN); /* 3 */

EquivalentCosets := function(N, IN, ts,Tset)
   perm:=ConvertToPermutation( ts, Tset);
   TSets :=[];
   for g in IN do
      for n in N do
```

```
          perm2:= ConvertToPermutation(ts, Tset^n);
          if perm eq g*perm2 then
             TSets:= TSets cat [Tset^n];
          end if;
       end for;
    end for;
    return TSets;
end function;

EquivalentCosets (N, IN, ts, [1,3]);
/* [1,3] and [6,4] */
S:={[1,3]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[3]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
 then print SSS[i];
end if; end for; end for;

StabilizingGroup := function(N, IN, ts, Tword)
    TWords:=EquivalentCosets(N, IN, ts, Tword);
    group:=Stabiliser(N, Tword);
    for i in [2..#TWords] do
```

```
        for n in N do
            if Tword^n eq TWords[i] then
                group:=sub<N|group,n>;
            end if;
        end for;
    end for;
    return group;
end function;
N13s:= StabilizingGroup(N, IN, ts, [1,3]);
N13s;
/* <(1, 6)(2, 5)(3, 4) > */
Orbits(N13s);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[1]
    eq m*(DC[i])^n then i; break; end if; end for;end for;
    /* 2 */
FindDCGroupNumber (DC,   ts[1]*ts[3]*ts[1], IN); /* 2 */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[2]
    eq m*(DC[i])^n then i; break; end if; end for;end for;
    /* 7*/
FindDCGroupNumber (DC,   ts[1]*ts[3]*ts[2], IN); /* 7 */

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[3]
```

```
    eq m*(DC[i])^n then i; break; end if; end for;end for;
    /* 4 */
FindDCGroupNumber (DC,  ts[1]*ts[3]*ts[3], IN);  /* 4 */

N132s:= StabilizingGroup(N, IN, ts, [1,3,2]);
N132s;

S:={[1,3,2]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[3]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])
    [2]]
 then print SSS[i];
end if; end for; end for; /* NONE */

N132:=Stabiliser(N,[1,3,2]);
#N132;
 Orbits(N132s);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[2]*
    ts[1] eq m*(DC[i])^n then i; break; end if; end for;end
    for; /* 6*/
```

```
FindDCGroupNumber (DC, ts[1]*ts[3]*ts[2]*ts[1], IN);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[2]*
    ts[2] eq m*(DC[i])^n then i; break; end if; end for;end
    for; /*7*/
FindDCGroupNumber (DC, ts[1]*ts[3]*ts[2]*ts[2], IN);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[2]*
    ts[3] eq m*(DC[i])^n then i; break; end if; end for;end
    for;
FindDCGroupNumber (DC, ts[1]*ts[3]*ts[2]*ts[3], IN);/*8*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[2]*
    ts[4] eq m*(DC[i])^n then i; break; end if; end for;end
    for;
FindDCGroupNumber(DC, ts[1]*ts[3]*ts[2]*ts[4], IN); /*4*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[2]*
    ts[5] eq m*(DC[i])^n then i; break; end if; end for;end
    for;
FindDCGroupNumber(DC, ts[1]*ts[3]*ts[2]*ts[5], IN); /*3*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[2]*
    ts[6] eq m*(DC[i])^n then i; break; end if; end for;end
```

```
   for;
FindDCGroupNumber(DC,   ts[1]*ts[3]*ts[2]*ts[6], IN); /*6*/

N12s:= StabilizingGroup(N, IN, ts, [1,2]);
N12s; /* <e> */

S:={[1,2]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
 then print SSS[i];
end if; end for; end for;

N12s:=Stabiliser(N,[1,2]);
#N12s; N12s;
 Orbits(N12s); /* {1}, { 2} , {3}, {4}, {5}, {6} */

FindDCGroupNumber (DC,  ts[1]*ts[2]*ts[1], IN); /* 5 */
FindDCGroupNumber (DC,  ts[1]*ts[2]*ts[2], IN); /*3 */
FindDCGroupNumber (DC,  ts[1]*ts[2]*ts[3], IN);/* 6 */
FindDCGroupNumber (DC,  ts[1]*ts[2]*ts[4], IN); /* 4 */
FindDCGroupNumber (DC,  ts[1]*ts[2]*ts[5], IN); /* 2 */
```

```
FindDCGroupNumber(DC,   ts[1]*ts[2]*ts[6], IN); /* 7 */

DoubleCosetRepresentatives(G1, IN, IN);
/* [ 12, 72, 144, 72, 144, 36, 144, 24, 72 ] */
[ 12/12, 72/12, 144/12, 72/12, 144/12, 36/12, 144/12,
   24/12, 72/12 ];
/* Numbers in the circles [ 1, 6, 12, 6, 12, 3, 12, 2, 6 ]
   */

N121s:= StabilizingGroup(N, IN, ts, [1,2, 1]);
N121s;
Orbits(N121s); {3, 6 }, {1,5,4,2};
FindDCGroupNumber(DC,   ts[1]*ts[2]*ts[1]*ts[1], IN); /*4*/
FindDCGroupNumber(DC,   ts[1]*ts[2]*ts[1]*ts[3], IN); /*8*/

N1323s:= StabilizingGroup(N, IN, ts, [1,3, 2,3]);
N1323s;
Orbits(N1323s); /* {1}, {4}, {2, 6}, {3, 5} */
FindDCGroupNumber(DC,   ts[1]*ts[3]*ts[2]*ts[3]*ts[1], IN);
   /* 5*/
FindDCGroupNumber(DC,   ts[1]*ts[3]*ts[2]*ts[3]*ts[4], IN);
   /* 8 */

FindDCGroupNumber(DC,   ts[1]*ts[3]*ts[2]*ts[3]*ts[2], IN)
```

```
    ;/* 7 */
FindDCGroupNumber(DC,  ts[1]*ts[3]*ts[2]*ts[3]*ts[3], IN);
    /* 6 */

N1321s:= StabilizingGroup(N, IN, ts, [1,3, 2,1]);
N1321s;
Orbits(N1321s); /* {1}, {2}, {3}, {4}, {5}, {6} */
FindDCGroupNumber(DC,  ts[1]*ts[3]*ts[2]*ts[1]*ts[1], IN);
    /* 4*/
FindDCGroupNumber(DC,  ts[1]*ts[3]*ts[2]*ts[1]*ts[2], IN);
    /* 7 */
FindDCGroupNumber(DC,  ts[1]*ts[3]*ts[2]*ts[1]*ts[3], IN);
    /* 9 */
FindDCGroupNumber(DC,  ts[1]*ts[3]*ts[2]*ts[1]*ts[4], IN);
    /* 7 */
FindDCGroupNumber(DC,  ts[1]*ts[3]*ts[2]*ts[1]*ts[5], IN);
    /*8 */
FindDCGroupNumber(DC,  ts[1]*ts[3]*ts[2]*ts[1]*ts[6], IN);
    /*  6*/

N13213s:= StabilizingGroup(N, IN, ts, [1,3, 2,1,3]);
/*(1, 2, 3, 4, 5, 6)
     (1, 3, 5)(2, 4, 6)
     (1, 4)(2, 5)(3, 6)
```

```
     (1, 6, 5, 4, 3, 2)
     (1, 5, 3)(2, 6, 4) */
N13213s;
EquivalentCosets(N, IN, ts,[1,3,2,1,3]);
/* [ 1, 3, 2, 1, 3 ],
     [ 2, 4, 3, 2, 4 ],
     [ 3, 5, 4, 3, 5 ],
     [ 4, 6, 5, 4, 6 ],
     [ 6, 2, 1, 6, 2 ],
     [ 5, 1, 6, 5, 1 ] */

FindFromToPerms:= function(Tset1, Tset2, N)
   NPerms:=[];
   for n in N do
      if Tset1^n eq Tset2 then
         NPerms:= NPerms cat [n];


      end if;
   end for;
      return NPerms;
end function;

GetStringT := function(set)
            Str1 := "";
```

```
for i in [1 .. #set] do
        Str1:= Str1 cat "t";
Str1:=  Str1 cat IntegerToString(set[i]);
end for;
 return Str1;
end function;

for n in N do
        print ( GetStringT([2,6]^n) cat " = ") cat
           GetStringT([3,5]^n) cat " (  From R2 (t2t6 =
           t3t5)^" cat  Sprint(n) cat " )";
end for;

/*t2t6 = t3t5 (  From R2 (t2t6 = t3t5)^Id(N) )
t3t1 = t4t6 (  From R2 (t2t6 = t3t5)^(1, 2, 3, 4, 5, 6) )
t5t1 = t4t2 (  From R2 (t2t6 = t3t5)^(1, 6)(2, 5)(3, 4) )
t5t3 = t6t2 (  From R2 (t2t6 = t3t5)^(1, 4)(2, 5)(3, 6) )
t1t3 = t6t4 (  From R2 (t2t6 = t3t5)^(1, 2)(3, 6)(4, 5) )
t1t5 = t2t4 (  From R2 (t2t6 = t3t5)^(1, 6, 5, 4, 3, 2) )
   */

/*Magma proof of Nt1t2t3     [1, 3, 2, 1]: */
ConvertToPermutation(ts, [1,3,5,6]) eq ConvertToPermutation
   (ts, [1,2,6,6]);
```

```
ConvertToPermutation(ts, [1,3,5,6]) eq ConvertToPermutation
   (ts, [6,4,5,6]); FindFromToPerms([6,4,5,6],[1,3,2,1],N);

EquivalentCosetsInfo := function(N, IN, ts,Tset)
   perm:=ConvertToPermutation( ts, Tset);
   TStrs :=[];
   for g in IN do
      for n in N do
         perm2:= ConvertToPermutation(ts, Tset^n);
         if perm eq g*perm2 then
            TStrs:= TStrs cat ["\n" cat Sprint(g) cat
               GetStringT(Tset) cat "^" cat  Sprint(n) cat
               " = " cat GetStringT(Tset^n) ];
         end if;
      end for;
   end for;
   return TStrs;
end function;

/* proof t1t3t2t4 = t6t5 */
 ConvertToPermutation(ts,[1,3,2,4]) eq ConvertToPermutation
    (ts,[6,5]); /* true */
for n in N do
        print ( GetStringT([1,3,2,4]^n) cat " = " cat
```

```
            GetStringT([6,5]^n) cat " from (t1t3t2t4 = t6t5)
            ^" cat  Sprint(n));
end for;
/* proof t1t3t2 = t6t5t1 */
 ConvertToPermutation(ts,[1,3,2]) eq ConvertToPermutation(
    ts,[6,5,1]);
for n in N do
        print ( GetStringT([1,3,2]^n) cat " = " cat
            GetStringT([6,5,1]^n) cat " from (t1t3t2 =
            t6t5t1)^" cat  Sprint(n));
end for;

/* t1t3t2t4 = t6t4t2t4 */
 ConvertToPermutation(ts,[1,3,2,4]) eq ConvertToPermutation
    (ts,[6,4,2,4]);
PrintConjugatedRelations:=procedure(TSet1, TSet2, N)
   str:= " from (" cat  GetStringT(TSet1) cat " = " cat
      GetStringT(TSet2) cat ")^";
   for n in N do
      print ( GetStringT(TSet1^n) cat " = " cat GetStringT(
         TSet2^n) cat str cat  Sprint(n));
   end for;
end procedure;
```

```
/proof *t1t3t2t6 = t1t6t5 */
ConvertToPermutation(ts,[1,3,2,6]) eq ConvertToPermutation(
    ts,[1,6,5]);
PrintConjugatedRelations([1,3,2,6], [1,6,5], N);

/* t1t3t2t6 = t1t5t3t2 */
ConvertToPermutation(ts,[1,3,2,6]) eq ConvertToPermutation(
    ts,[1,5,3,2]);
PrintConjugatedRelations([1,3,2,6], [1,5,3,2], N);

/*  t1t3t2t6 = t2t4t3t2 */
ConvertToPermutation(ts,[1,3,2,6]) eq ConvertToPermutation(
    ts,[2,4,3,2]);
PrintConjugatedRelations([1,3,2,6], [2,4,3,2], N);

/* t1t2t1 = t4t5t4. */
ConvertToPermutation(ts,[1,2,1]) eq ConvertToPermutation(ts
    ,[4,5,4]);
PrintConjugatedRelations([1,2,1], [4,5,4], N)
FindFromToPerms([1,2,1], [4,5,4], N);

/* t1t3t2t3 = t1t5t6t5 */
ConvertToPermutation(ts,[1,3,2,3]) eq ConvertToPermutation(
    ts,[1,5,6,5]);
```

```
/* cst:=FindCosetOrder([1,2], N, IN, ts, cst); */
FindCosetOrder:= function(TSet, N, IN, ts, cst)
        stab:= StabilizingGroup(N, IN, ts, TSet);
        tr:=Transversal(N,stab);
        for i:=1 to #tr1 do
ss:=TSet^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
        return cst;
end function;
cst:= FindCosetOrder([1,2], N, IN, ts, cst);
cst:=FindCosetOrder([1,3], N, IN, ts, cst);
cst:=FindCosetOrder([1,3,2], N, IN, ts, cst);
cst:=FindCosetOrder([1,3,2,1], N, IN, ts, cst);
cst:=FindCosetOrder([1,3,2,3], N, IN, ts, cst);
cst:=FindCosetOrder([1,2,1], N, IN, ts, cst);
cst:=FindCosetOrder([1,3,2,1,3], N, IN, ts, cst);
cst;

/* PrintEquivalentCosetsTo(N, IN, ts,[1,2,1]); */
PrintEquivalentCosetsTo:= function(N, IN, ts,Tset)
        ECs:=EquivalentCosets(N, IN, ts,Tset);
str:= "N";
```

```
        for i in [1..#ECs] do
                str := str cat GetStringT(ECs[i]);
                edge:=" = N";
                if i eq #ECs then
                        edge:= "\t";
                end if;
str := str cat edge;
        end for;
        return str;
end function;

PrintEquivalentRelations := procedure(N, IN, ts,Tset)
perm:=ConvertToPermutation( ts, Tset);
TSets :=[];
for g in IN do
for n in N do
        if n eq Id(N) or g eq Id(IN) then continue;  end if
           ;
perm2:= ConvertToPermutation(ts, Tset^n);
if perm eq g*perm2 then
GetStringT(Tset) cat " = " cat Sprint(InWord(g)) cat
   GetStringT(Tset^n);
GetStringT(Tset^n) cat " = " cat Sprint(InWord(g^-1)) cat
   GetStringT(Tset);
```

```
end if;
end for;
end for;
end procedure;

PrintEquivalentRelations (N, IN, ts, [1,3]);

S:=Sym(6);
xx:=S!(1, 2, 3, 4, 5, 6);
yy:=S!(1, 6)(2, 5)(3, 4);
N:=sub<S|xx,yy>;

G<x,y,t>:=Group<x,y,t|x^6,y^2,(x*y)^2,t^3,(t,(x*y)^x),
t^(x^3)=t^2,(t^x*(t^y))^2>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#G1;

NL:=NormalLattice(G1);
NL;
for i in [1..#NL] do
        if IsAbelian(NL[i]) then
                i;
        end if;
end for; /* 2 */
```

```
NL[2];
 A:=G1!(1, 54, 53)(2, 50, 49)(3, 41, 40)(4, 39, 42)(5, 45,
    37)(6, 28, 51)(7, 52, 26)(8, 23, 55)(9, 38,47)(10, 58,
    22)(11, 48, 36)(12, 29, 43)(13, 17, 56)(14, 44, 27)(15,
     57, 16)(18, 21, 60)(19, 31, 32)(20, 34, 33)(24, 25,
    59)(30, 35, 46);

q,ff:=quo<G1|NL[2]>;
Order(q); /* 240. G1 has not normal subgroups of 240 ==>
       G1 ~ 3:Q */

T:= Transversal(G1,NL[2]);
ff(T[2])  eq q.1;
B:=T[2];

N:=q;
NL:=NormalLattice(N);/*
[7]  Order 240  Length 1  Maximal Subgroups: 4 5 6
---
[6]  Order 120  Length 1  Maximal Subgroups: 3
[5]  Order 120  Length 1  Maximal Subgroups: 2 3
[4]  Order 120  Length 1  Maximal Subgroups: 3
---
[3]  Order 60   Length 1  Maximal Subgroups: 1
```

```
---
[2]  Order 2     Length 1  Maximal Subgroups: 1
---
[1]  Order 1     Length 1  Maximal Subgroups: */

NL;
for i in [1..#NL] do
        if IsAbelian(NL[i]) then
                i;
        end if;
end for; /*  2 */ /* | NLq[2] | = 2 */

q, ff := quo<N|NL[2]>;
q; /*
Permutation group q acting on a set of cardinality 10
Order = 120 = 2^3 * 3 * 5
    (2, 3, 6, 5, 7, 4)(8, 9, 10)
    (2, 4)(3, 7)(5, 6)(9, 10)
    (1, 2, 5)(3, 7, 9)(4, 6, 8)*/
IsIsomorphic(NL[4],q); /* true */
IsIsomorphic(DirectProduct(NL[4],NL[2]), N); /* true */
/*     G ~ 3:(2  q  ) */

N:=q;
```

```
NL:=NormalLattice(N);
NL;
  IsIsomorphic(q, Sym(5)); /* true */
/* Isomorphism type: G~3:(2  S5  :) */
```

Appendix G

MAGMA CODE: Double Coset Enumeration of $L_2(25)$ Over S5

```
/*The following magma code was used to isolate L2(25) */
S:= Sym(24);
xx:=S!(1,2,4,18,9)(3,13,14,16,24)(6,21,7,8,10)
    (12,15,19,20,22);
yy:=S!(1,21)(2,5)(3,19)(4,6)(7,15)(8,11)(9,13)(10,12)
    (14,17)(16,18)(20,23)(22,24);
N:=sub<S|xx,yy>;
CompositionFactors(N);
IsIsomorphic(N,Sym(5));/* true */

FPGroup(N);
NN<x,y>:= Group<x,y| x^5, y^2,(x^-1*y)^4, (x * y * x^-2 * y
```

```
    * x)^2>;
fn,N1,kn:=CosetAction(NN, sub<NN|Id(NN)>);
IsIsomorphic(N,N1); /* true */

W:=WordGroup(N);
 rho:=InverseWordMap(N);

Stabilizer(N, {1,7,13,19});
(N!((2, 12, 22, 9, 11)(3, 5, 14, 6, 10)(4, 15, 17, 20, 18)
    (8, 24, 16, 21, 23)))@rho;

C:=function(W)
     w3 := W.1 * W.2; w4 := w3 * W.1; w5 := w4 * W.1; w6 :=
         w5 * W.2; w1 := W.1^-1; w7 := w6 * w1; return w7;
end function;

C(FPGroup(N));
xx * yy * xx^2 * yy * xx^-1; /* this what we need to use
    for (t, x * y * x^2 * y * x^-1)*/

a:= 0; b:=0; c:=0; d:= 2; e:= 4;
G<x,y,t>:=Group<x,y,t| x^5, y^2,(x^-1*y)^4,(x*y*x^-2*y*x)
    ^2,t^5,(t, x * y * x^2 * y * x^-1), t^(x^-1*y)= t^3, (x*
    t)^a, (x*y*t^x)^b,(y*t^(x*y))^c, (x,t*t^(x*x))^d,(x^y*t)
```

```
    ^e>;

#sub<G|x,y>;
Index(G,sub<G|x,y>);

ff,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);

NL:=NormalLattice(G1);
NL;
NL[2];
Order(NL[2].2); /* 2*/

W:=WordGroup(G1);
rho:=InverseWordMap(G1);
(NL[2].2)@rho;

C:=function(W)
w1 := w.1^-1; w4 := w1 * w.2; w5 := w4 * w1; w6 := w5 * w
    .2; w7 := w6 * w.1; w8 := w7 * w.2; w9 := w8 * w.1; w10
    := w9 * w.3; w11 := w10 * w.1; w12 := w11 * w.3; w13 :=
    w12 * w.3; w14:= w13 * w.1; w15 := w14 * w.3; return w15
    ;
end function;
```

```
C(G); /*    x^-1 * y * x^-1 * y * x * y * x * t * x * t^2 *
    x * t     */

/* Original
a:= 0; b:=0; c:=0; d:= 2; e:= 4;
G<x,y,t>:=Group<x,y,t| x^5, y^2,(x^-1*y)^4,(x*y*x^-2*y*x)
    ^2,t^5,(t, x * y * x^2 * y * x^-1), t^(x^-1*y)= t^3, (x*
    t)^a, (x*y*t^x)^b,(y*t^(x*y))^c, (x,t*t^(x*x))^d,(x^y*t)
    ^e, x^-1 * y * x^-1 * y * x * y * x * t * x * t^2 * x *
    t>;

Gb<x,y,t>:=Group<x,y,t| x^5, y^2,(x^-1*y)^4,(x*y*x^-2*y*x)
    ^2,t^5,(t, x * y * x^2 * y * x^-1), t^(x^-1*y)= t^3, (x*
    t*t^(x*x)*x^-1*t^(y * x * y * x^-2)*t^((y * x)^2))^2,(x^
    y*t)^4, x^-1 * y * x^-1 * y * x * y * x * t * x * t^2 *
    x * t>;
fb,G1b,kb:=CosetAction(Gb,sub<Gb|x,y>);
 IsIsomorphic(G1, G1b); */

/*=====================================================================
    The following magma code helps prove the DCE of L(2,25)
    .*/
S:= Sym(24);
```

```
xx:=S!(1,2,4,18,9)(3,13,14,16,24)(6,21,7,8,10)
    (12,15,19,20,22);
yy:=S!(1,21)(2,5)(3,19)(4,6)(7,15)(8,11)(9,13)(10,12)
    (14,17)(16,18)(20,23)(22,24);
N:=sub<S|xx,yy>;
NN<x,y>:= Group<x,y| x^5, y^2,(x^-1*y)^4, (x * y * x^-2 * y
     * x)^2>;

G<x,y,t>:=Group<x,y,t| x^5, y^2,(x^-1*y)^4,(x*y*x^-2*y*x)
    ^2,t^5,(t, x * y * x^2 * y * x^-1), t^(x^-1*y)= t^3, (x,
    t*t^(x*x))^2,(x^y*t)^4, x^-1 * y * x^-1 * y * x * y * x
    * t * x * t^2 * x * t>;
/* { x, y t| x5, y2, (x-1y)4, (xyx-2yx)2, t5, (t, xyx2yx-1)
    , tx^(-1)y  = t3, ( x, ttx*x)2, (xyt)4, x-1yx-1
    yxyxtxt2xt } */
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);

IN:=sub<G1|f(x),f(y)>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
word:=function(Perm)
for w in Sch do
          seq := Eltseq(w);
```

```
        p:= Id(N);
        for j in seq do
                if j eq 1 then p:=p*xx; end if;
if j eq -1 then p:=p*xx^-1; end if;
if j eq 2 then p:=p*yy; end if;
if j eq -2 then p:=p*yy^-1; end if;
        end for;
        if Perm eq p then return w; end if;
end for;
end function;

Word2NPerm:= function(letters)
        seq := Eltseq(letters);
p:= Id(N);
for j in seq do
                if j eq 1 then p:=p*xx; end if;
if j eq -1 then p:=p*xx^-1; end if;
if j eq 2 then p:=p*yy; end if;
if j eq -2 then p:=p*yy^-1; end if;
end for;
return p;
end function;

InWord:=function(Perm)
```

```
for w in Sch do
         seq := Eltseq(w);
         p:= Id(IN);
         for j in seq do
                  if j eq 1 then p:=p*f(x); end if;
if j eq -1 then p:=p*f(x)^-1; end if;
if j eq 2 then p:=p*f(y); end if;
if j eq -2 then p:=p*f(y)^-1; end if;
         end for;
         if Perm eq p then return w; end if;
end for;
end function;

GetConjugationWordForSubscript:= function(subscript)
         size:=9999999;
         w:=word(N.1);
for n in N do
         if n eq Id(N) then
                  continue;
         end if;
if 1^n eq subscript then
         tmp:= word(n);
         if size gt #tmp then
                  size:=#tmp;
```

```
                w:= tmp;
            end if;
    end if;
    end for;
    return w;
    end function;

    for i in [1..24] do
      "ts[" cat Sprint( i) cat "] := " cat  "f(t^(" cat Sprint
         (GetConjugationWordForSubscript(i)) cat "));";
    end for;

    ts:= [ Id(G1): i in [1 .. 24] ];
    ts[1] := f(t);
    ts[2] := f(t^(x));
    ts[3] := f(t^(x^-1 * y * x^-1));
    ts[4] := f(t^(x^2));
    ts[5] := f(t^(x * y));
    ts[6] := f(t^(y * x^-1));
    ts[7] := f(t^(y * x));
    ts[8] := f(t^(y * x^2));
    ts[9] := f(t^(x^-1));
    ts[10] := f(t^(y * x^-2));
    ts[11] := f(t^(y * x^2 * y));
```

```
ts[12] := f(t^(y * x * y * x^-1));
ts[13] := f(t^(x^-1 * y));
ts[14] := f(t^(y^x));
ts[15] := f(t^(y * x * y));
ts[16] := f(t^(x^-2 * y));
ts[17] := f(t^(x^-1 * y * x * y));
ts[18] := f(t^(x^-2));
ts[19] := f(t^((y * x)^2));
ts[20] := f(t^(y * x * y * x^2));
ts[21] := f(t^(y));
ts[22] := f(t^(y * x * y * x^-2));
ts[23] := f(t^(y * x * y * x^2 * y));
ts[24] := f(t^(x^-1 * y * x^-2));

StrPowers:="";
for i in  [1..24] do
        for k in [2..4] do
                for j in [1..24] do
                        if i eq j then continue; end if;
                        if ts[i]^k eq ts[j] then
                                StrPowers:= StrPowers cat "
                                   t" cat Sprint(i) cat
                                   "^" cat Sprint(k) cat "
                                   = t" cat Sprint(j) cat
```

```
                                        ", ";
                              end if;
                        end for;
                  end for;
StrPowers:= StrPowers cat "\n";
end for;
StrPowers; /*
t1^2 = t7,        t1^3 = t13,       t1^4 = t19,
t2^2 = t8,        t2^3 = t14,       t2^4 = t20,
t3^2 = t9,        t3^3 = t15,       t3^4 = t21,
t4^2 = t10,       t4^3 = t16,       t4^4 = t22,
t5^2 = t11,       t5^3 = t17,       t5^4 = t23,
t6^2 = t12,       t6^3 = t18,       t6^4 = t24,
t7^2 = t19,       t7^3 = t1,        t7^4 = t13,
t8^2 = t20,       t8^3 = t2,        t8^4 = t14,
t9^2 = t21,       t9^3 = t3,        t9^4 = t15,
t10^2 = t22,      t10^3 = t4,       t10^4 = t16,
t11^2 = t23,      t11^3 = t5,       t11^4 = t17,
t12^2 = t24,      t12^3 = t6,       t12^4 = t18,
t13^2 = t1,       t13^3 = t19,      t13^4 = t7,
t14^2 = t2,       t14^3 = t20,      t14^4 = t8,
t15^2 = t3,       t15^3 = t21,      t15^4 = t9,
t16^2 = t4,       t16^3 = t22,      t16^4 = t10,
t17^2 = t5,       t17^3 = t23,      t17^4 = t11,
```

```
t18^2 = t6,      t18^3 = t24,     t18^4 = t12,
t19^2 = t13,     t19^3 = t7,      t19^4 = t1,
t20^2 = t14,     t20^3 = t8,      t20^4 = t2,
t21^2 = t15,     t21^3 = t9,      t21^4 = t3,
t22^2 = t16,     t22^3 = t10,     t22^4 = t4,
t23^2 = t17,     t23^3 = t11,     t23^4 = t5,
t24^2 = t18,     t24^3 = t12,     t24^4 = t6, */

InverseIndex:= function(Index)
for i in [1..24]  do
                if ts[Index]^-1 eq ts[i] then return i; end
                    if;
        end for;
end  function;

for i in [1..24] do
        "t" cat Sprint(i) cat " = t" cat Sprint(
           InverseIndex(i)) cat "^-1";
end for;

SplitT:= function(tsIndex)
        for n in N do
                if 13^n eq tsIndex then
                        return [1,7]^n;
```

```
                end if;
        end for;
        return [0,0];
end  function;

/* Split in doubles */
SplitTd:= function(tsIndex)
for n in N do
                if 7^n eq tsIndex then
                        return [1,1]^n;
                end if;
        end for;
        return [0,0];
end  function;

TStr:= function(Tword)
           Str1 := "";
for i in [1 .. #Tword] do
        if Tword[i] ne 0 then
Str1:= Str1 cat "t";
Str1:=  Str1 cat IntegerToString(Tword[i]);
end if;
end for;
return Str1;
```

```
end function;

for i in [1..24] do
        "t" cat Sprint(i) cat " = " cat TStr(SplitTd(i));
end for;

DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);/*
{ <GrpFP, Id(G), GrpFP>, <GrpFP, t * x * t^-1, GrpFP>, <
    GrpFP, t, GrpFP>,
<GrpFP, t * x * t, GrpFP> } */
DC:=[ f(Id(G)), f(t),  f( t * x * t ), f( t * x * t^-1 ) ];
DC;

#N; /* 120 */
 DoubleCosetRepresentatives(G1, IN, IN); /*  [ 120, 2880,
     1200, 3600 ] */
[ 120/120, 2880/120, 1200/120, 3600/120 ];
/* [ 1, 24, 10, 30 ]  */

FindDCGroupNumber := function(element)
    for i in [1..#DC] do
        for m,n in IN do
            if element eq m*(DC[i])^n then
                return i;
```

```
            end if;
        end for;
    end for;
end function;

TPerm:= function(Tset)
    perm:= ts[Tset[1]];
    for i in [2..#Tset] do
      perm:= perm*ts[Tset[i]];
    end for;
    return perm;
end function;

Orbits(N);
N1:= Stabilizer(N, 1);
N1; /* (2, 9, 12, 11, 22)(3, 6, 5, 10, 14)(4, 20, 15, 18,
    17)(8, 21, 24, 23, 16) */
Orbits(N1); /*GSet{@ 1 @},
     GSet{@ 7 @},
     GSet{@ 13 @},
     GSet{@ 19 @},
     GSet{@ 2, 9, 12, 11, 22 @},
     GSet{@ 3, 6, 5, 10, 14 @},
     GSet{@ 4, 20, 15, 18, 17 @},
```

```
  GSet{@ 8, 21, 24, 23, 16 @} */

FindDCGroupNumber (ts[1]*ts[1]); /* 2 */
FindDCGroupNumber (ts[1]*ts[7]); /* 2 */
FindDCGroupNumber (ts[1]*ts[13]); /* 2 */
FindDCGroupNumber (ts[1]*ts[19]); /* 1 */
FindDCGroupNumber (ts[1]*ts[2]); /* 3 */
FindDCGroupNumber (ts[1]*ts[3]); /* 4 */
FindDCGroupNumber (ts[1]*ts[4]); /* 4  */
FindDCGroupNumber (ts[1]*ts[8]); / * 2 */

/* Equivalent right coset */Find
EquivalentCosets:= function(Tset)
  perm:=TPerm( Tset);
  TSets :=[];
  for g in IN do
    for n in N do
      perm2:= TPerm(Tset^n);
      if perm eq g*perm2 then
        TSets:= TSets cat [Tset^n];
      end if;
    end for;
  end for;
  return TSets;
```

```
end function;

StabilizingGroup := function(Tword)
   TWords:=EquivalentCosets(Tword);
   group:=Stabiliser(N, Tword);
   for i in [2..#TWords] do
      for n in N do
         if Tword^n eq TWords[i] then
            group:=sub<N|group,n>;
         end if;
      end for;
   end for;
   return group;
end function;

N12s:= StabilizingGroup([1,2]);
N12s;
/* (1, 24)(2, 23)(3, 4)(5, 20)(6, 19)(7, 18)(8, 17)(9, 10)
   (11, 14)(12, 13)(15, 16)(21, 22)
    (1, 5)(2, 10)(3, 18)(4, 14)(6, 9)(7, 11)(8, 22)(12, 21)
       (13, 17)(15, 24)(16, 20)(19, 23)
    (1, 20, 15)(2, 9, 19)(3, 7, 14)(4, 11, 18)(5, 24, 16)
       (6, 10, 23)(8, 21, 13)(12, 22, 17)
    (1, 9)(2, 20)(3, 13)(4, 17)(5, 10)(6, 24)(7, 21)(8, 14)
```

```
      (11, 22)(12, 18)(15,  19)(16, 23)
    (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9, 20)
      (10, 24)(11, 17)(14, 21)(18, 22)
    (1, 23)(2, 24)(3, 22)(4, 21)(5, 19)(6, 20)(7, 17)(8,
      18)(9, 16)(10, 15)(11, 13)(12, 14)
    (1, 15, 20)(2, 19, 9)(3, 14, 7)(4, 18, 11)(5, 16, 24)
      (6, 23, 10)(8, 13, 21)(12, 17, 22)
    (1, 10, 20, 23, 15, 6)(2, 5, 9, 24, 19, 16)(3, 12, 7,
      22, 14, 17)(4, 8, 11, 21, 18, 13)
    (1, 2)(3, 21)(4, 22)(5, 6)(7, 8)(9, 15)(10, 16)(11, 12)
      (13, 14)(17, 18)(19,  20)(23, 24)
    (1, 6, 15, 23, 20, 10)(2, 16, 19, 24, 9, 5)(3, 17, 14,
      22, 7, 12)(4, 13, 18,  21, 11, 8)
    (1, 16)(2, 6)(3, 11)(4, 7)(5, 15)(8, 12)(9, 23)(10, 19)
      (13, 22)(14, 18)(17,  21)(20, 24) */

test:=sub<N|N!(1, 6, 15, 23, 20, 10)(2, 16, 19, 24, 9, 5)
   (3, 17, 14, 22, 7, 12)(4, 13, 18, 21, 11, 8), N!(1, 24)
   (2, 23)(3, 4)(5, 20)(6, 19)(7, 18)(8, 17)(9, 10)(11, 14)
   (12, 13)(15, 16)(21, 22)>;
IsIsomorphic(N12s, test);
test:=sub<N|N!(1, 6, 15, 23, 20, 10)(2, 16, 19, 24, 9, 5)
   (3, 17, 14, 22, 7, 12)(4, 13, 18, 21, 11, 8), N!(1,9)(2,
    20)(3, 13)(4, 17)(5, 10)(6, 24)(7, 21)(8, 14)(11, 22)
```

```
    (12, 18)(15, 19)(16,23)>;
IsIsomorphic(N12s, test);

Orbits(N12s); /*
 GSet{@ 1, 24, 5, 20, 9, 19, 23, 15, 10, 2, 6, 16 @},
     GSet{@ 3, 4, 18, 7, 13, 8, 22, 14, 12, 21, 17, 11 @} */

FindDCGroupNumber (ts[1]*ts[2]*ts[1]); /* 2 */
FindDCGroupNumber (ts[1]*ts[2]*ts[3]); /* 4 */

N13s:= StabilizingGroup([1,3]);
N13s; /*
(1, 7, 19, 13)(2, 6, 15, 16)(3, 4, 8, 12)(5, 17, 23, 11)(9,
     10, 20, 24)(14, 18, 21, 22)
(1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9, 20)(10,
     24)(11, 17)(14, 21)(18, 22)
(1, 13, 19, 7)(2, 16, 15, 6)(3, 12, 8, 4)(5, 11, 23, 17)(9,
     24, 20, 10)(14, 22, 21, 18) */
Orbits(N13s); /*
GSet{@ 1, 7, 19, 13 @},
     GSet{@ 2, 6, 15, 16 @},
     GSet{@ 3, 4, 8, 12 @},
     GSet{@ 5, 17, 23, 11 @},
     GSet{@ 9, 10, 20, 24 @},
```

```
    GSet{@ 14, 18, 21, 22 @} */
FindDCGroupNumber (ts[1]*ts[3]*ts[1]); /* 4 */
FindDCGroupNumber (ts[1]*ts[3]*ts[2]); /* 2  */
FindDCGroupNumber (ts[1]*ts[3]*ts[3]); /* 3 */
FindDCGroupNumber (ts[1]*ts[3]*ts[5]); /* 4 */
FindDCGroupNumber (ts[1]*ts[3]*ts[9]); /*  4 */
FindDCGroupNumber (ts[1]*ts[3]*ts[14]);/* 2*/

FindEquivalentRelations:= function(Tset, Tset2)
perm:= TPerm( Tset);
TSets :=[];
str:="";
for g in IN do
perm2:= TPerm(Tset2);
if perm eq g*perm2 then
str:= str cat TStr(Tset) cat " = " cat Sprint(InWord(g))
   cat TStr(Tset2) cat "\n";
end if;
end for;
for n,m in IN do
perm2:= TPerm(Tset2);
if perm eq n*(perm2^m) then
str:= str cat TStr(Tset) cat " = " cat Sprint(InWord(n))
   cat "(" cat TStr(Tset2) cat ")^" cat  Sprint(InWord(m))
```

```
    cat "\n";
end if;
end for;
return str;
end function;


FindFromToPerms:= function(Tset1, Tset2)
    NPerms:=[];
    for n in N do
        if Tset1^n eq Tset2 then
            NPerms:= NPerms cat [n];
        end if;
    end for;
        return NPerms;
end function;

RELATION:= recformat< rightTword : SeqEnum,  leftPerm:
    GrpPermElt, leftTword:SeqEnum,  info, applyAt  >;
Relation:=function(rTword, lPerm, lTword, str, apply)
return rec< RELATION | rightTword := rTword, leftPerm:=
    lPerm, leftTword:=lTword, info:= str, applyAt:=apply>;
end function;
```

```
/* Permutation follow by a T word */
PermTword:= recformat< perm:GrpPermElt , tword: SeqEnum,
   name:MonStgElt ,parents:SeqEnum, appliedRelations ,
   children >;
/* example rec< PermTword | perm:= xx , tword:=[8, 9]> =
   xt8t9; */

PTW:=function(p, tw)
        meStr:="";
        if p ne Id(N) then
                meStr:= Sprint(word(p)) cat " * ";
        end if;
        meStr:= meStr cat TStr(tw);
ptw:= rec< PermTword | perm:= p, tword:=tw, name:=meStr ,
   parents :=[], children:=0>;
return ptw;
end function;

AddParent:=function(ptw, parentIndex, relation)

        if #ptw`parents eq 0 then
        ptw`appliedRelations:= [relation];
ptw`parents:= [parentIndex];
else
```

```
        ptw‘appliedRelations:= ptw‘appliedRelations  cat [
           relation];
ptw‘parents:= ptw‘parents cat [parentIndex];
end if;
return ptw;
end function;

PtwStr:=function(ptw)
        if ptw‘perm eq Id(N) then
               return TStr(ptw‘tword);
        end if;
        return Sprint(word(ptw‘perm)) cat "*" cat TStr(ptw‘
           tword);
end function;

LemmaCC:= Relation([4,8], yy*xx^2*yy*xx^2*yy, [19], "Lemma
    3: t4t8 = yx^2yx^2yt19",0);
LemmaCC2:= Relation([19], yy * xx^-2 * yy * xx^-2 *yy,
    [4,8], "Lemma 3: t19 = yx^2yx^-2yt4t8.",0);
LemmaBB:=Relation([19,9], yy*xx^-1*yy, [12,11], "Lemma 2:
    t19t9 = yx^-1yt12t11",0);
LemmaBB2:=Relation([12,11], yy*xx*yy, [19,9], "Lemma 2:
    t12t11 = yxyt19t9 ",0);
LemmaAA:=Relation([9,2,22,19], Id(N), [1,4,20,15], "Lemma
```

```
    1: t9t2t22t19 = t1t4t20t15",0);
LemmaAA2:=Relation([1,4,20,15], Id(N), [9,2,22,19], "Lemma
    1: t1t4t20t15 = t9t2t22t19",0);

/* Power rules */
t7t19:=Relation([7,19], Id(N), [1], "t7t19 = t1",0);
t1t1:=Relation([1,1], Id(N), [7], "t1^2 = t7",0);
InvRelation:=Relation([1,19], Id(N), [], "t1t19 = Id",0);

IdPower:=Relation( [1,1,1,1,1], Id(N), [], "Id = t1^5",0);
    /*TODO not sure if we need [0] or [] */
t1:=Relation([1], Id(N), [7,19], "t1 = t7t19",0);
t7:=Relation([7], Id(N), [1,1], "t7 = t1^2",0);

ListRelations:=[ InvRelation, t7t19, t1t1,  IdPower,
   LemmaCC, LemmaAA, LemmaAA2, LemmaBB, LemmaBB2, LemmaCC2,
    t1 ];

/* If the order of ts = 2 then this will not work. Modify
    for that case. */
Reduce2Ts:= function(tsIndex1, tsIndex2)
          if tsIndex1 eq 0 and tsIndex2 eq 0 then return [0];
              end if;
          if tsIndex1 eq 0 then return [tsIndex2]; end if;
```

```
if tsIndex2 eq 0 then return [tsIndex1]; end if;
for i in [1..24]  do
                if ts[tsIndex1]*ts[tsIndex2] eq ts[i] then
                   return [i]; end if;
if ts[tsIndex1]*ts[tsIndex2] eq ts[i]^5 then return [0];
   end if;
        end for;
        return [tsIndex1, tsIndex2];
end  function;

ReduceTs:= function(Tset)
        steps:=[Tset];
        OK:=true;
        while OK do
        size:=#Tset;
        if size lt 1 then return steps; end if;
        for i in [1..size] do
                j:=i+1;
                OK:=false;
                if j le size then
                        r:= Reduce2Ts(Tset[i],Tset[j]);
                        if #r eq 1 then
                                Tset[i]:= r[1];
                                Tset:=Remove(Tset, j);
```

```
                                steps:= steps cat [Tset];
OK:=true;
                                break;
                        end if;
                end if;
        end for;
end while;
return steps;
end function;

ReducePtw:=function(ptw)
        tword:=ptw`tword;
        steps:=ReduceTs(tword);
        if #steps gt 1 then
                for step in steps do
                        if 0 in step then continue; end if;
                        Sprint(word(ptw`perm)) cat "*" cat
                                TStr(step);
                end for;
                ptw`tword:=steps[#steps];
        end if;
        return ptw;
end function;
```

```
ReducePtwToSize:=function(ptw,size)
        tword:=ptw`tword;
        steps:=ReduceTs(tword);
        if #steps gt 1 then
                for i in [1..#steps] do
                        step:=steps[i];
                        if 0 in step then continue; end if;
                        Sprint(word(ptw`perm)) cat "*" cat
                                TStr(step);
                        if #step eq size then
                                ptw`tword:=steps[i];
                                return ptw;
                        end if;
                end for;
                ptw`tword:=steps[#steps];
        end if;
        return ptw;
end function;

/*start is the starting index. The relation has to of the
   form Tword = perm*Tword */
ApplyRelation:=function(ptw, relation)
        applyAt:=relation`applyAt; /* t index to start at
            */
```

```
        Tw:=ptw`tword;
rTw:= relation`rightTword;
lTw:= relation`leftTword;
p:= relation`leftPerm;
        if applyAt gt #Tw then return ptw; end if;
        arr:=[];
        /*  Make sure we can apply relation at the starting
            index */
        good:=true;

        /* make sure the relation can be applied */
        for i in [applyAt..#rTw] do
                if not good then return ptw; end if;
                if Tw[i] ne rTw[i - applyAt + 1] then good
                   := false; end if;
        end for;
if not good then return ptw; end if;

        for i in [1..#Tw] do
                /* move the permutation to the front */
                if i lt applyAt  then
                        arr:= arr cat [Tw[i]^p];
                end if;
        end for;
```

```
        arr:= arr cat lTw; /* replace by adding the left t
           word */
        next:= applyAt + #rTw; /* add the part of the t
           word that wasnt replaced */
        if next gt #Tw then  return PTW((ptw`perm)*p, arr);
            end if;
        for i in [next..#Tw] do
                arr:= arr cat [Tw[i]];
        end for;
 return PTW((ptw`perm)*p, arr);
end function;

GetReationsToUse:=function(ptw )
        Tword:=ptw`tword;
        num:=#Tword;
        arr:=[]; /* saves relations that can be used on the
            element ptw */
        arrStr:=[]; /* keep track of the results as strings
            to make we dont repeat */
for j in  [1..#ListRelations] do
rTword:= ListRelations[j]`rightTword;
lTword:= ListRelations[j]`leftTword;
lPerm:= ListRelations[j]`leftPerm;
info:=ListRelations[j]`info;
```

```
          if num ge #rTword then
                    for i in [1..(num - #rTword + 1)] do
                              temp:=[]; /* this is the a t word
                                  ??? */
                              for k in [i..(i + #rTword - 1)] do
                                        temp:= temp cat [Tword[k]];
                              end for;
                              for n in N do

                              if  temp eq rTword^n  then
                                        wp:=word(n);
                                        strLeft:="";
if #lTword ne 0 then
          strLeft:= TStr(lTword^n);
else
          rTword^n, ptw`tword;
end if;
newInfo:="Let p = " cat Sprint(n) cat " belong to N.\n" cat
     info cat  " ==>\n(" cat TStr(rTword) cat ")^p = (" cat
    Sprint(word(lPerm)) cat TStr(lTword) cat ")^p ==> \n"
    cat TStr(rTword^n) cat " = " cat Sprint(word(lPerm^n))
    cat strLeft;
                                        r:=Relation(rTword^n, lPerm
                                            ^n, lTword^n, newInfo,i)
```

```
                                        ;
rs:= TStr(rTword^n) cat " = " cat Sprint(word(lPerm^n)) cat
    strLeft; /* the result if relation is applied */
                                    found:= rs in arrStr;
                                    if not found then
                                          arrStr:= arrStr cat
                                              [rs];
                                          arr:= arr cat [r];
                                    end if;
                              end if;
                        end for;
                        end for;
end if;
end for; /* for j */
      return arr;
end function;

FindTword:=function(elements, tword) /* tword = [1,2,34]
   ==> t1t2t3t4 */
      for i in [1..#elements] do
            if  elements[i]`tword eq tword then return
               i; end if;
      end for;
      return 0;
```

```
end function;

/* see if we can find the tword after we apply a
   permutation and reduction. tword = [1,2,34] ==> t1t2t3t4
    */
FindTwordAfterProcessing:=function(elements, tword, startAt
   )
for i in [startAt..#elements] do
                i, #elements;
                for n in N do
                        "About to reduce ", elements[i]`
                           name;
                        steps:=ReduceTs((elements[i]`tword)
                           ^n);
                        if #steps gt 0 then
                                for step in steps do
                                        if 0 in step  then
                                           continue; end if
                                           ;
                                        if tword eq step
                                           then
                                                return i, n
                                                   ;
                                        end if;
```

```
                                        end for;
                                end if;
                        end for;
                end for;
                return 0, Id(N);
end function;

ApplyRelationsToElements:=function(elements)
                newElements:=[];
for i in [1..#elements] do
                        /* find an element that has not being
                           processed */
                        /* i, #elements; */

if (elements[i])`children eq 0 then
                ptw:=elements[i];
                "************************************************************

                "Looking for relations to apply to", PtwStr(ptw);
                                relations:=GetReationsToUse(ptw);
                                "Found " cat Sprint(#relations);
                                (elements[i])`children:=1; /* TODO
                                   is not correct but as long as
                                   its greater than 1 it works */
```

```
                    for j in [1..#relations] do
                            newPtw:= ApplyRelation(ptw,
                               relations[j]);
                            /* "Looking for:", PtwStr(
                               newPtw); */
                            /* i is the parent index */
/* only add to the list if its new t word */
                            if FindTword(elements,
                               newPtw`tword) eq 0 and
                               FindTword(newElements,
                               newPtw`tword) eq 0 then
                                    newPtw:= AddParent(
                                       newPtw, i,
                                       relations[j]);
                                    newElements:=
                                       newElements cat
                                       [newPtw];
                                    "
                                       --------------------------------
                                       ";
                                     relations[j]`info
                                        cat ". Apply at
                                         position " cat
                                         Sprint(
```

```
                                                    relations[j]`
                                                    applyAt) cat
                                                    ".";
                                                "Found a new name:
                                                   ", PtwStr(
                                                   newPtw);
                                        end if;
                                end for;
                        end if;
                end for;
                return elements cat newElements;
end function;

/* Testing */
t1t3:=PTW(Id(N), [1,3]);
t1t3ERs:=[t1t3];
t1t3ERs:=ApplyRelationsToElements(t1t3ERs);
FindTword(t1t3ERs, [7,4]);
t1t3ERs:=ApplyRelationsToElements(t1t3ERs);
FindTword(t1t3ERs, [7,4]);
t1t3ERs:=ApplyRelationsToElements(t1t3ERs);
FindTword(t1t3ERs, [7,4]);
t1t3ERs:=ApplyRelationsToElements(t1t3ERs);
FindTword(t1t3ERs, [7,4]);
```

```
t1t4:=PTW(Id(N), [1,4]);
t1t4Rs:=[t1t4];
t1t4Rs:=ApplyRelationsToElements(t1t4Rs);
t1t4Rs:=ApplyRelationsToElements(t1t4Rs);
#t1t4Rs;
[ 15, 5, 7, 9 ]^(xx^2*yy^-1);
FindTwordAfterProcessing(t1t4Rs, [ 23, 2, 12, 5 ], 1);
t1t4Rs:=ApplyRelationsToElements(t1t4Rs);
#t1t4Rs;
/* End of test */

/* The targetTword is what we are trying to get to. Example
    : we are trying to prove Nt1t4 = belongs to coset [1,3].
     Then origTword = 1t4 = [1,4] and targetTword = t1t3 =
    [1,3] */
ProveRelation:= function(origTword, targetTword)
        permarr:= FindFromToPerms(origTword, targetTword);
        if #permarr gt 0 then
                for p in permarr  do
                        Sprint(word(p)) cat " takes " cat
                           TStr(origTword) cat " to " cat
                           TStr(targetTword);
                        return true;
```

```
            end for;
        end if;

        elements:=[PTW(Id(N), origTword)];
elements:=ApplyRelationsToElements(elements);
        temparr:=[];
        pos:=1;
        pos, perm:= FindTwordAfterProcessing(elements,
            targetTword, 1);
        while pos eq 0 do
                start:=#elements;
                elements:=ApplyRelationsToElements(elements
                    );
                pos, perm:= FindTwordAfterProcessing(
                    elements, targetTword, start);
        end while;

        sep:="
            ----------------------------------------------------
            ";
if pos eq 1 then /* we found it right away */
        tempTword:=elements[1]`tword;
                steps:=ReduceTs(tempTword^perm);
                if perm ne Id(N) then
```

```
                    sep;
                    ptwConj:=PTW( Id(N), tempTword^perm );
                    "Conjugate by n = " cat Sprint(perm) cat ".
                        ==> ";
"(" cat TStr(origTword) cat ")^n = " cat  PtwStr(ptwConj)
    cat ".";
                              if #steps gt 2 then
                                    sep;
                                    "Reduce in the following
                                        way:";
                                    ptwConj:=ReducePtwToSize(
                                        ptwConj, #targetTword);
                              end if;
                              sep;
                              p:= perm^-1;
                              "Let p = " cat Sprint(perm^-1) cat
                                  " = n^-1. ==>";
"(" cat  TStr(origTword^perm) cat ")^p = (" cat  PtwStr(
    ptwConj) cat ")^p ==> ";
TStr(origTword) cat " = " cat Sprint(word(   (ptwConj`perm)
    ^p  )) cat "(" cat TStr(ptwConj`tword) cat ")^p" cat
    ".";
                              return true;
                    elif  #steps gt 2 then
```

```
                        sep;
                        "Reduce in the following way:";
                        temp:=elements[1];
                        junk:=ReducePtw(temp);
                        return true;
                end if;
        end if;
while pos gt 1 do
                        temparr:= temparr cat [elements[pos
                            ]];
                        if #(elements[pos]`parents) gt 0
                            then
                                pos:=elements[pos]`parents
                                    [1];
                        else
                                pos:=1; /* we made it to
                                    the top */
                        end if;
        end while;
        "***************************************************************

        "PROOF";
"******************************************************************
```

```
"Start with " cat TStr(origTword);
for i in [1..#temparr] do
        "
            ------------------------------------------------------------
            ";
        element:=temparr[#temparr - i + 1];
         if #(element`appliedRelations) gt 0 then
                parentName:=TStr(origTword);
                if #(element`parents) gt 0 then
                        parentName:= elements[
                           element`parents[1]  ]`
                           name;
                end if;
                "Apply the following at t word
                   position " cat  Sprint((element`
                   appliedRelations[1])`applyAt)
                   cat ":";
(element`appliedRelations[1])`info cat " ==> ";
parentName cat " = " cat  PtwStr(element) cat ".";
end  if;
        end for;
        steps:=ReduceTs((temparr[1]`tword)^perm);
        if #temparr gt 0 and perm ne Id(N) then
                "
```

```
        --------------------------------------------------
        ";
      "We now have " cat TStr(origTword) cat " =
        " cat PtwStr(temparr[1]) cat ".";
      ptwConj:=PTW( ((temparr[1])`perm)^perm, ((
        temparr[1])`tword)^perm );
      "Conjugate both sides by n = " cat Sprint(
        perm) cat ". ==> ";
TStr(origTword^perm) cat " = " cat PtwStr(ptwConj) cat
  ".";
      if #steps gt 1 then
          "
            --------------------------------------------------
            ";
          "Reduce in the following way:";
          ptwConj:=ReducePtwToSize(ptwConj, #
            targetTword);
      end if;
      "
        --------------------------------------------------
        ";
      p:= perm^-1;
      "Let p = " cat Sprint(perm^-1) cat " = n
        ^-1. ==>";
```

```
"(" cat  TStr(origTword^perm) cat ")^p = (" cat  PtwStr(
   ptwConj) cat ")^p ==> ";
 TStr(origTword) cat  " = " cat PtwStr(PTW( (ptwConj`perm)^
    p, (ptwConj`tword)^p )) cat  " = " cat Sprint(word(
    ptwConj`perm )) cat "(" cat TStr(ptwConj`tword) cat ")^
    p" cat ".";
                return true;
         end if;
         if #steps gt 1 then
                "
                    -----------------------------------------------------------
                    ";
                "Reduce in the following way:";
                temp:=temparr[1];
                temp`tword:=(temp`tword)^perm;
                temp`perm:=(temp`perm)^perm;
                junk:=ReducePtw(temp);
                return true;
         end if;
         if #temparr gt 0 then
                return true;
         end if;
         return false;
end function;
```

```
/* testing ProveRelation function */
[ 1,4 ]^(xx*yy^-1*xx^2); /* t1t5 = [5 ,3 ] */
ProveRelation([1,4], [5, 3] ); /*
x * y * x^2 takes t1t4 to t5t3 */

ProveRelation([1,2,20], [1]); /*
Reduce in the following way:
Id($)*t1t2t20
Id($)*t1     */

ProveRelation([1,19,2,20], [] ); /*
Start with  t1t19t2t20
-----------------------------------------------------
Apply the following at t word position 1:
Let p = Id($) belong to N.
t1t19 = Id ==>
(t1t19)^p = (Id($))^p ==>
t1t19 = Id($) ==>
t1t19t2t20 = t2t20.
-----------------------------------------------------
Apply the following at t word position 1:
Let p = (1, 2, 4, 18, 9)(3, 13, 14, 16, 24)(6, 21, 7, 8,
   10)(12, 15, 19, 20, 22)
```

```
belong to N.
t1t19 = Id ==>
(t1t19)^p = (Id($))^p ==>
t2t20 = Id($) ==>
t2t20 = . */ /* This result was before I updated the
   functition */

[1,2,20]^(xx^2*yy^-1);  /*  [ 6, 16, 10 ] */
ProveRelation([ 6, 16, 10 ], [1]); /*
Conjugate by n = (1, 10, 20, 23, 15, 6)(2, 5, 9, 24, 19,
   16)(3, 12, 7, 22, 14,
17)(4, 8, 11, 21, 18, 13). ==>
(t6t16t10)^n = t1t2t20.
----------------------------------------------------
Reduce in the following way:
Id($)*t1t2t20
Id($)*t1
----------------------------------------------------
Let p = (1, 6, 15, 23, 20, 10)(2, 16, 19, 24, 9, 5)(3, 17,
   14, 22, 7, 12)(4, 13,
18, 21, 11, 8) = n^-1. ==>
(t1t2t20)^p = (t1)^p ==>
t6t16t10 = Id($)(t1)^p.  */
```

```
FindEquivalentRelations([1,2,1],[1,2,20]);/*
t1t2t1 = x * y * x^2 * y * x^-1(t1t2t20)^y * x * y
t1t2t1 = x * y * x^2 * y * x^-1(t1t2t20)^y * x^2 * y * x *
    y * x^-1 * y
t1t2t1 = x * y * x^2 * y * x^-1(t1t2t20)^(x^2 * y)^2
t1t2t1 = x * y * x^2 * y * x^-1(t1t2t20)^x^2 * y * x^-1 * y
     * x
t1t2t1 = x * y * x^2 * y * x^-1(t1t2t20)^y * x^-2 * y * x
    */

ProveRelation([1,2,1],[1,2,20]);
/* Start with  t1t2t1
----------------------------------------------------
Apply the following at t word position 1:
Let p = (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9,
     20)(10, 24)(11,
17)(14, 21)(18, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = y * x * y * x^2 * y * x^-1t12t3 ==>
t1t2t1 = y * x * y * x^2 * y * x^-1*t12t3t2t1.
----------------------------------------------------
Apply the following at t word position 2:
Let p = (1, 10, 9, 5)(2, 14, 20, 8)(3, 17, 13, 4)(6, 12,
```

24, 18)(7, 22, 21,

11)(15, 23, 19, 16) belong to N.

Lemma 3: t4t8 = yx^2yx^2yt19 ==>

(t4t8)^p = (y * x^2 * y * x^2 * yt19)^p ==>

t3t2 = x * y * x^-1 * y * x^2 * yt16 ==>

y * x * y * x^2 * y * x^-1 * t12t3t2t1 = x^-1 * y * x * y*
t6t16t1.

Apply the following at t word position 1:

Let p = (1, 24)(2, 23)(3, 4)(5, 20)(6, 19)(7, 18)(8, 17)(9,
10)(11, 14)(12,

13)(15, 16)(21, 22) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t6 = x^2 * y * x^2t3t17 ==>

x^-1 * y * x * y * t6t16t1 = x^2 * y * x^2 * y * x * y * x
^-1*t3t17t16t1.

Apply the following at t word position 2:

Let p = (2, 22, 11, 12, 9)(3, 14, 10, 5, 6)(4, 17, 18, 15,
20)(8, 16, 23, 24,

21) belong to N.

Lemma 3: t4t8 = yx^2yx^2yt19 ==>

(t4t8)^p = (y * x^2 * y * x^2 * yt19)^p ==>

```
t17t16 = x^-1 * y * x^2t19 ==>
x^2 * y * x^2 * y * x * y * x^-1 * t3t17t16t1 = x * y * x^2
    * y * x^-1*t15t19t1.
------------------------------------------------------
We now have t1t2t1 = x * y * x^2 * y * x^-1*t15t19t1.
Conjugate both sides by n = (1, 20, 15)(2, 9, 19)(3, 7, 14)
    (4, 11, 18)(5, 24,
16)(6, 10, 23)(8, 21, 13)(12, 22, 17). ==>
t20t9t20 = y * x^-2 * y*t1t2t20.
------------------------------------------------------
Reduce in the following way:
y * x^-2 * y*t1t2t20
------------------------------------------------------
Let p = (1, 15, 20)(2, 19, 9)(3, 14, 7)(4, 18, 11)(5, 16,
    24)(6, 23, 10)(8, 13,
21)(12, 17, 22) = n^-1. ==>
(t20t9t20)^p = (y * x^-2 * y*t1t2t20)^p ==>
t1t2t1 = x * y * x^2 * y * x^-1*t15t19t1 = y * x^-2 * y(
    t1t2t20)^p. */

ProveRelation([1,2,1],[1]);
/*Start with  t1t2t1
------------------------------------------------------
Apply the following at t word position 1:
```

Let p = (1, 19)(2, 4)(3, 23)(5, 21)(6, 24)(7, 13)(8, 10)(9, 17)(11, 15)(12,
18)(14, 16)(20, 22) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t1 = x * y * x^2 * y * x^−1 * yt2t10 ==>

t1t2t1 = x * y * x^2 * y * x^−1 * y*t2t10t2t1.

Apply the following at t word position 2:

Let p = (1, 23, 18, 21)(2, 14, 20, 8)(3, 19, 5, 12)(4, 10, 22, 16)(6, 15, 7,
17)(9, 13, 11, 24) belong to N.

Lemma 3: t4t8 = yx^2yx^2yt19 ==>

(t4t8)^p = (y * x^2 * y * x^2 * yt19)^p ==>

t10t2 = y * x^2 * y * x * y * x^−1t5 ==>

x * y * x^2 * y * x^−1 * y * t2t10t2t1 = y * x^−1 * y * x ^2*t10t5t1.

Apply the following at t word position 1:

Let p = (1, 16, 12, 3, 2, 23)(4, 24, 9, 8, 17, 7)(5, 19, 10, 18, 21, 20)(6, 15,
14, 11, 13, 22) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t10 = x^−2 * y * xt24t17 ==>

y * x^−1 * y * x^2 * t10t5t1 = y*t24t17t5t1.

Apply the following at t word position 4:

Let p = (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9, 20)(10, 24)(11,

17)(14, 21)(18, 22) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t1 = y * x * y * x^2 * y * x^−1t12t3 ==>

y * t24t17t5t1 = x * y * x^2 * y * x^−1*t9t6t12t12t3.

We now have t1t2t1 = x * y * x^2 * y * x^−1*t9t6t12t12t3.

Conjugate both sides by n = (1, 4, 12, 11, 15)(3, 7, 10, 24, 23)(5, 21, 13, 16,

6)(9, 19, 22, 18, 17). ==>

t4t2t4 = x^−1 * y * x^2 * y * x*t19t5t11t11t7.

Reduce in the following way:

x^−1 * y * x^2 * y * x*t19t5t11t11t7

x^−1 * y * x^2 * y * x*t19t17t11t7

x^−1 * y * x^2 * y * x*t19t7

x^−1 * y * x^2 * y * x*t1

```
Let p = (1, 15, 11, 12, 4)(3, 23, 24, 10, 7)(5, 6, 16, 13,
    21)(9, 17, 18, 22,
19) = n^-1. ==>
(t4t2t4)^p = (x^-1 * y * x^2 * y * x*t1)^p ==>
t1t2t1 = x * y * x^2 * y * x^-1*t15 = x^-1 * y * x^2 * y *
    x(t1)^p. */

ProveRelation([12, 9], [ 1,3 ]); /*
Start with  t12t9
--------------------------------------------------
Apply the following at t word position 1:
Let p = (1, 18, 2, 9, 4)(3, 16, 13, 24, 14)(6, 8, 21, 10,
    7)(12, 20, 15, 22, 19)
belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t12 = x * y * x^-1 * y * x * y * xt1t21 ==>
t12t9 = x * y * x^-1 * y * x * y * x * t1t21t9.
--------------------------------------------------
Reduce in the following way:
x * y * x^-1 * y * x * y * x*t1t21t9
x * y * x^-1 * y * x * y * x*t1t3 */

/*Proof Nt1t4, Nt1t15, Nt1t17, Nt1t20, Nt1t18      [1, 3]
```

```
        See above */
ProveRelation([1,4], [ 1,3 ]);/*
Start with  t1t4
-----------------------------------------------------
Apply the following at t word position 1:
Let p = (1, 19)(2, 4)(3, 23)(5, 21)(6, 24)(7, 13)(8, 10)(9,
    17)(11, 15)(12,
18)(14, 16)(20, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = x * y * x^2 * y * x^-1 * yt2t10 ==>
t1t4 = x * y * x^2 * y * x^-1 * y * t2t10t4.
-----------------------------------------------------
We now have t1t4 = x * y * x^2 * y * x^-1 * y*t2t10t4.
Conjugate both sides by n = (1, 12, 2)(3, 23, 16)(4, 9, 17)
    (5, 10, 21)(6, 14,
13)(7, 24, 8)(11, 22, 15)(18, 20, 19). ==>
t12t9 = x * y * x^-1 * y * x * y * x*t1t21t9.
-----------------------------------------------------
Reduce in the following way:
x * y * x^-1 * y * x * y * x*t1t21t9
x * y * x^-1 * y * x * y * x*t1t3
-----------------------------------------------------
Let p = (1, 2, 12)(3, 16, 23)(4, 17, 9)(5, 21, 10)(6, 13,
```

```
    14)(7, 8, 24)(11, 15,
22)(18, 19, 20) = n^-1. ==>
(t12t9)^p = (x * y * x^-1 * y * x * y * x*t1t3)^p ==>
t1t4 = x * y * x^2 * y * x^-1 * y*t2t16 = x * y * x^-1 * y
    * x * y * x(t1t3)^p. */

ProveRelation([1,4],[  23, 2, 12, 5 ]);/*
Start with  t1t4
--------------------------------------------------
Apply the following at t word position 1:
Let p = (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9,
     20)(10, 24)(11,
17)(14, 21)(18, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = y * x * y * x^2 * y * x^-1t12t3 ==>
t1t4 = y * x * y * x^2 * y * x^-1 * t12t3t4.
--------------------------------------------------
Apply the following at t word position 3:
Let p = (1, 22, 12)(2, 9, 17)(3, 23, 14)(4, 18, 19)(5, 8,
    21)(6, 13, 10)(7, 16,
24)(11, 20, 15) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
```

```
t4 = y * x^-2 * y * x * yt18t21 ==>
y * x * y * x^2 * y * x^-1 * t12t3t4 = (y * x^2)^2 *
    t3t4t18t21.
-----------------------------------------------------
We now have t1t4 = (y * x^2)^2*t3t4t18t21.
Conjugate both sides by n = (1, 19)(2, 4)(3, 23)(5, 21)(6,
    24)(7, 13)(8, 10)(9,
17)(11, 15)(12, 18)(14, 16)(20, 22). ==>
t19t2 = (x^-2 * y)^2*t23t2t12t5.
-----------------------------------------------------
Let p = (1, 19)(2, 4)(3, 23)(5, 21)(6, 24)(7, 13)(8, 10)(9,
     17)(11, 15)(12,
18)(14, 16)(20, 22) = n^-1. ==>
(t19t2)^p = ((x^-2 * y)^2*t23t2t12t5)^p ==>
t1t4 = (y * x^2)^2*t3t4t18t21 = (y * x^2)^2(t23t2t12t5)^p.
    */
/* End of ProveRelation function testing */

/* Proof of Lemma CC */
f(x^2)*ts[4]*ts[8]*ts[1] eq f(t * x * t^2 * x * t);
f(x^-1 * y * x^-1 * y * x * y * x *x*x )*ts[4]*ts[8]*ts[1];
ts[4]*ts[8] eq f(y * x^2 * y * x^2 * y)*ts[19];

/* Proof of Lemma AA */
```

```
TPerm([9,2,22,19]) eq TPerm([1,4,20,15]);

FindTwordWithReducing:=function(elements, tword, startAt)

for i in [startAt..#elements] do
            /*i, #elements; */
            /*"About to reduce ", elements[i]`name; */
            steps:=ReduceTs(elements[i]`tword);
            if #steps gt 0 then
                  for step in steps do
                        if 0 in step  then continue
                           ; end if;
                        if tword eq step then
                              return i;
                        end if;
                  end for;
            end if;
      end for;
      return 0;
end function;

ProveRightCosetRelation:= function(origTword, targetTword)
      elements:=[PTW(Id(N), origTword)];
elements:=ApplyRelationsToElements(elements);
```

```
        temparr:=[];
        pos:=1;
        pos:= FindTwordWithReducing(elements, targetTword,
            1);
        while pos eq 0 do
                start:=#elements;
                elements:=ApplyRelationsToElements(elements
                    );
                pos:= FindTwordWithReducing(elements,
                    targetTword, start);
        end while;
        sep:="
            --------------------------------------------------------
            ";
if pos eq 1 then /* we found it right away */
        tempTword:=elements[1]`tword;
                steps:=ReduceTs(tempTword);
                if #steps gt 2 then
                        sep;
                        "Reduce in the following way:";
                        temp:=elements[1];
                        junk:=ReducePtw(temp);
                        return true;
                end if;
```

```
        end if;
while pos gt 1 do
                        temparr:= temparr cat [elements[pos
                            ]];
                        if #(elements[pos]`parents) gt 0
                            then
                                pos:=elements[pos]`parents
                                    [1];
                        else
                                pos:=1; /* we made it to
                                    the top */
                        end if;
        end while;
        "*************************************************************

        "PROOF";
        "*************************************************************

        "Start with  " cat TStr(origTword);
        for i in [1..#temparr] do
                "

                    -----------------------------------------------------
                    ";
                element:=temparr[#temparr - i + 1];
```

```
if #(element‘appliedRelations) gt 0 then
        parentName:=TStr(origTword);
        if #(element‘parents) gt 0 then
                parentName:= elements[
                    element‘parents[1] ]‘
                    name;
        end if;
        "Apply the following at t word
            position " cat Sprint((element‘
            appliedRelations[1])‘applyAt)
            cat ":";
(element‘appliedRelations[1])‘info cat " ==> ";
parentName cat " = " cat PtwStr(element) cat ".";
end if;
          end for;
          steps:=ReduceTs(temparr[1]‘tword);
          if #steps gt 1 then
                    "
                        -----------------------------------------------------
                        ";
                    "Reduce in the following way:";
                    temp:=temparr[1];
                    temp‘tword:=(temp‘tword);
                    temp‘perm:=(temp‘perm);
```

```
                junk:=ReducePtw(temp);
                return true;
        end if;
        if #temparr gt 0 then
                return true;
        end if;
        return false;
end function;

/* Prove that Nt1t3 = Nt7t4 */
ProveRightCosetRelation([1,3], [1,3,7]);/*
Start with  t1t3
---------------------------------------------------
Apply the following at t word position 1:
Let p = (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9,
    20)(10, 24)(11,
17)(14, 21)(18, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = y * x * y * x^2 * y * x^-1t12t3 ==>
t1t3 = y * x * y * x^2 * y * x^-1*t12t3t3.
---------------------------------------------------
Apply the following at t word position 1:
Let p = (1, 18, 15)(2, 11, 4)(3, 7, 6)(5, 16, 14)(8, 23,
```

10)(9, 19, 12)(13, 24,
21)(17, 22, 20) belong to N.
Lemma 3: t19 = yx^2yx^−2yt4t8. ==>
(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>
t12 = x^2 ∗ y ∗ x ∗ y ∗ x^−1 ∗ yt2t23 ==>
y ∗ x ∗ y ∗ x^2 ∗ y ∗ x^−1 ∗ t12t3t3 = x^−1 ∗ y ∗ x^2 ∗ y∗
t2t23t3t3.

Apply the following at t word position 2:
Let p = (1, 5, 4, 24, 20, 3)(2, 21, 19, 23, 22, 6)(7, 11,
10, 18, 14, 9)(8, 15,
13, 17, 16, 12) belong to N.
Lemma 3: t19 = yx^2yx^−2yt4t8. ==>
(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>
t23 = y ∗ x^−1 ∗ y ∗ x ∗ y ∗ x^−2t24t15 ==>
x^−1 ∗ y ∗ x^2 ∗ y ∗ t2t23t3t3 = x^−2 ∗ y ∗ x^2∗
t8t24t15t3t3.

Apply the following at t word position 2:
Let p = (1, 6, 15, 23, 20, 10)(2, 16, 19, 24, 9, 5)(3, 17,
14, 22, 7, 12)(4, 13,
18, 21, 11, 8) belong to N.
Lemma 3: t19 = yx^2yx^−2yt4t8. ==>
(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

```
t24 = y * x^2 * y * x^-1 * y * xt13t4 ==>
x^-2 * y * x^2 * t8t24t15t3t3 = y * x^-1 * y * x^2*
    t19t13t4t15t3t3.
--------------------------------------------------------
Reduce in the following way:
y * x^-1 * y * x^2*t19t13t4t15t3t3
y * x^-1 * y * x^2*t7t4t15t3t3
y * x^-1 * y * x^2*t7t4t21t3
y * x^-1 * y * x^2*t7t4*/

/* Prove Nt1t3t7 belongs to DC [1,3] */
ProveRelation([1,3],[1,3,7]);/*
Start with  t1t3
--------------------------------------------------------
Apply the following at t word position 1:
Let p = (1, 19)(2, 18)(3, 21)(4, 11)(5, 16)(6, 8)(7, 13)(9,
     15)(10, 23)(12,
20)(14, 24)(17, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = x^-1 * y * x^2t11t6 ==>
t1t3 = x^-1 * y * x^2*t11t6t3.
--------------------------------------------------------
Apply the following at t word position 3:
```

Let p = (1, 21, 22, 5, 12, 8)(2, 13, 9, 10, 17, 6)(3, 4, 23, 18, 14, 19)(7, 15,
16, 11, 24, 20) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t3 = x * y * x^−1 * y * x * y * xt23t1 ==>

x^−1 * y * x^2 * t11t6t3 = x^2 * y * x^2 * y * x * y * x^−1 * y*t13t17t23t1.

We now have t1t3 = x^2 * y * x^2 * y * x * y * x^−1 * y* t13t17t23t1.

Conjugate both sides by n = (1, 7, 19, 13)(2, 14, 20, 8)(3, 18, 16, 11)(4, 23,
9, 6)(5, 15, 24, 22)(10, 17, 21, 12). ==>

t7t18 = (x^−1 * y * x^2)^2*t1t21t9t7.

Reduce in the following way:

(x^−1 * y * x^2)^2*t1t21t9t7

(x^−1 * y * x^2)^2*t1t3t7

Let p = (1, 13, 19, 7)(2, 8, 20, 14)(3, 11, 16, 18)(4, 6, 9, 23)(5, 22, 24,
15)(10, 12, 21, 17) = n^−1. ==>

(t7t18)^p = ((x^−1 * y * x^2)^2*t1t3t7)^p ==>

```
t1t3 = x^2 * y * x^2 * y * x * y * x^-1 * y*t13t11t1 = (x
    ^-1 * y *
x^2)^2(t1t3t7)^p. */

/* Output after a few minor modification some functions */
ProveRightCosetRelation([1,3], [2,23,9]);
/*****************************************************************

Looking for relations to apply to t1t3
Found 12
----------------------------------------
Let p = (1, 19)(2, 4)(3, 23)(5, 21)(6, 24)(7, 13)(8, 10)(9,
     17)(11, 15)(12,
18)(14, 16)(20, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = x * y * x^2 * y * x^-1 * yt2t10. Apply at position 1.
Found a new name:  x * y * x^2 * y * x^-1 * y*t2t10t3
----------------------------------------
Let p = (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9,
     20)(10, 24)(11,
17)(14, 21)(18, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
```

t1 = y ∗ x ∗ y ∗ x^2 ∗ y ∗ x^−1t12t3. Apply at position 1.

Found a new name: y ∗ x ∗ y ∗ x^2 ∗ y ∗ x^−1∗t12t3t3

Let p = (1, 19)(2, 20)(3, 16)(4, 9)(5, 24)(6, 23)(7, 13)(8, 14)(10, 21)(11,
18)(12, 17)(15, 22) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t1 = y ∗ x ∗ y ∗ x^−2 ∗ yt9t14. Apply at position 1.

Found a new name: y ∗ x ∗ y ∗ x^−2 ∗ y∗t9t14t3

Let p = (1, 19)(2, 18)(3, 21)(4, 11)(5, 16)(6, 8)(7, 13)(9, 15)(10, 23)(12,
20)(14, 24)(17, 22) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t1 = x^−1 ∗ y ∗ x^2t11t6. Apply at position 1.

Found a new name: x^−1 ∗ y ∗ x^2∗t11t6t3

Let p = (1, 19)(2, 17)(3, 24)(4, 22)(5, 8)(6, 21)(7, 13)(9, 18)(10, 16)(11,
20)(12, 15)(14, 23) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t1 = y * x^2 * y * x^2 * yt22t5. Apply at position 1.

Found a new name: y * x^2 * y * x^2 * y*t22t5t3

Let p = (1, 21)(2, 5)(3, 19)(4, 6)(7, 15)(8, 11)(9, 13)(10, 12)(14, 17)(16,
18)(20, 23)(22, 24) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t3 = x^-2 * y * x^-2t6t11. Apply at position 2.

Found a new name: x^-2 * y * x^-2*t13t6t11

Let p = (1, 21, 22, 5, 12, 8)(2, 13, 9, 10, 17, 6)(3, 4, 23, 18, 14, 19)(7, 15,
16, 11, 24, 20) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t3 = x * y * x^-1 * y * x * y * xt23t1. Apply at position 2.

Found a new name: x * y * x^-1 * y * x * y * x*t21t23t1

Let p = (1, 21, 20, 16)(2, 10, 19, 3)(4, 7, 15, 14)(5, 17, 23, 11)(6, 12, 24,
18)(8, 22, 13, 9) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

```
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x^2 * y * x^-1t7t22. Apply at position 2.
Found a new name:  x^2 * y * x^-1*t10t7t22
----------------------------------------
Let p = (1, 21, 18, 23)(2, 8, 20, 14)(3, 12, 5, 19)(4, 16,
    22, 10)(6, 17, 7,
15)(9, 24, 11, 13) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x * y * x^-1 * y * x^2 * yt16t20. Apply at position 2.
Found a new name:  x * y * x^-1 * y * x^2 * y*t16t16t20
----------------------------------------
Let p = (1, 21, 11, 10, 2, 24)(3, 17, 16, 20, 6, 19)(4, 14,
     12, 13, 9, 5)(7, 15,
23, 22, 8, 18) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = y * x^-2 * y * x * yt14t18. Apply at position 2.
Found a new name:  y * x^-2 * y * x * y*t7t14t18
----------------------------------------
Let p = Id($) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t1 = Id($)t7t19. Apply at position 1.
```

```
Found a new name:  t7t19t3
-----------------------------------------
Let p = (1, 3, 20, 24, 4, 5)(2, 6, 22, 23, 19, 21)(7, 9,
    14, 18, 10, 11)(8, 12,
16, 17, 13, 15) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t3 = Id($)t9t21. Apply at position 2.
Found a new name:  t1t9t21
*******************************************************************

Looking for relations to apply to x * y * x^2 * y * x^-1 *
    y*t2t10t3
Found 20
-----------------------------------------
Let p = (1, 16, 17, 24)(2, 8, 20, 14)(3, 15, 21, 9)(4, 5,
    18, 7)(6, 19, 10,
11)(12, 13, 22, 23) belong to N.
Lemma 2: t19t9 = yx^-1yt12t11 ==>
(t19t9)^p = (y * x^-1 * yt12t11)^p ==>
t10t3 = y * x^-2 * yt13t6. Apply at position 2.
Found a new name:  x * y * x^2 * y * x^2 * y*t2t13t6
-----------------------------------------
Let p = (1, 20, 12, 9, 17)(2, 18, 15, 11, 19)(3, 23, 13, 8,
```

6)(5, 7, 14, 24, 21)

belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t2 = y ∗ x^2 ∗ y ∗ x^−1 ∗ y ∗ xt4t6. Apply at position 1.

Found a new name: y ∗ x^2 ∗ y ∗ x ∗ y ∗ x^−1 ∗ y∗t4t6t10t3

Let p = (1, 20, 4)(2, 22, 19)(3, 24, 5)(6, 23, 21)(7, 14,
10)(8, 16, 13)(9, 18,

11)(12, 17, 15) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t2 = x ∗ y ∗ x^−1 ∗ y ∗ x^2 ∗ yt1t16. Apply at position 1.

Found a new name: t1t16t10t3

Let p = (1, 20, 11, 22, 18)(2, 17, 4, 12, 19)(5, 10, 24,
13, 8)(6, 7, 14, 23,

16) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t2 = y ∗ x^2 ∗ y ∗ x ∗ y ∗ x^−1t12t5. Apply at position 1.

Found a new name: y ∗ x^−1 ∗ y ∗ x^2∗t12t5t10t3

Let p = (1, 20, 15)(2, 9, 19)(3, 7, 14)(4, 11, 18)(5, 24,

16)(6, 10, 23)(8, 21,
13)(12, 22, 17) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t2 = x^-2 * y * x^-2t11t21. Apply at position 1.
Found a new name: (x^-1 * y * x^-1)^2*t11t21t10t3

Let p = (1, 20)(2, 19)(3, 10)(4, 15)(5, 23)(6, 24)(7, 14)
(8, 13)(9, 22)(11,
17)(12, 18)(16, 21) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t2 = x^-1 * y * x * y * x^2 * yt15t13. Apply at position 1.
Found a new name: (y * x)^2*t15t13t10t3

Let p = (1, 16)(2, 6)(3, 11)(4, 7)(5, 15)(8, 12)(9, 23)(10,
19)(13, 22)(14,
18)(17, 21)(20, 24) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t10 = x * y * x^-2 * y * x * yt7t12. Apply at position 2.
Found a new name: y * x^-1 * y * x * y * x^2 * y*t8t7t12t3

Let p = (1, 16, 17, 24)(2, 8, 20, 14)(3, 15, 21, 9)(4, 5,

18, 7)(6, 19, 10,

11)(12, 13, 22, 23) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t10 = y * x^2 * y * x * y * x^−1t5t20. Apply at position 2.

Found a new name: y * x^−1 * y * x^2*t10t5t20t3

Let p = (1, 16, 20, 21)(2, 3, 19, 10)(4, 14, 15, 7)(5, 11,

23, 17)(6, 18, 24,

12)(8, 9, 13, 22) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t10 = x^−1 * yt14t9. Apply at position 2.

Found a new name: x * y * x^−2 * y * x*t21t14t9t3

Let p = (1, 16, 9, 6, 11, 8)(2, 13, 22, 3, 18, 5)(4, 21,

12, 23, 20, 7)(10, 15,

24, 17, 14, 19) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t10 = x * y * x^−1 * y * x^2 * yt21t1. Apply at position 2.

Found a new name: t3t21t1t3

Let p = (1, 16, 12, 3, 2, 23)(4, 24, 9, 8, 17, 7)(5, 19,

10, 18, 21, 20)(6, 15,
14, 11, 13, 22) belong to N.
Lemma 3: t19 = yx^2yx^−2yt4t8. ==>
(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>
t10 = x^−2 * y * xt24t17. Apply at position 2.
Found a new name: (x^2 * y)^2*t14t24t17t3

Let p = (1, 21)(2, 5)(3, 19)(4, 6)(7, 15)(8, 11)(9, 13)(10, 12)(14, 17)(16,
18)(20, 23)(22, 24) belong to N.
Lemma 3: t19 = yx^2yx^−2yt4t8. ==>
(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>
t3 = x^−2 * y * x^−2t6t11. Apply at position 3.
Found a new name: (x^−1 * y * x^−1)^2*t24t22t6t11

Let p = (1, 21, 22, 5, 12, 8)(2, 13, 9, 10, 17, 6)(3, 4, 23, 18, 14, 19)(7, 15,
16, 11, 24, 20) belong to N.
Lemma 3: t19 = yx^2yx^−2yt4t8. ==>
(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>
t3 = x * y * x^−1 * y * x * y * xt23t1. Apply at position 3.
Found a new name: y * x^−1 * y * x*t8t4t23t1

Let p = (1, 21, 20, 16)(2, 10, 19, 3)(4, 7, 15, 14)(5, 17,
23, 11)(6, 12, 24,
18)(8, 22, 13, 9) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x^2 * y * x^-1t7t22. Apply at position 3.
Found a new name: x^2 * y * x^-2 * y*t14t9t7t22

Let p = (1, 21, 18, 23)(2, 8, 20, 14)(3, 12, 5, 19)(4, 16,
22, 10)(6, 17, 7,
15)(9, 24, 11, 13) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x * y * x^-1 * y * x^2 * yt16t20. Apply at position 3.
Found a new name: t3t2t16t20

Let p = (1, 21, 11, 10, 2, 24)(3, 17, 16, 20, 6, 19)(4, 14,
12, 13, 9, 5)(7, 15,
23, 22, 8, 18) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = y * x^-2 * y * x * yt14t18. Apply at position 3.
Found a new name: x^2 * y * x^2 * y * x * y * x^-1 * y*
t6t20t14t18

```
----------------------------------------
Let p = (1, 2, 4, 18, 9)(3, 13, 14, 16, 24)(6, 21, 7, 8,
    10)(12, 15, 19, 20, 22)
belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t2 = Id($)t8t20. Apply at position 1.
Found a new name:  x * y * x^2 * y * x^-1 * y*t8t20t10t3
----------------------------------------
Let p = (1, 10, 20, 23, 15, 6)(2, 5, 9, 24, 19, 16)(3, 12,
    7, 22, 14, 17)(4, 8,
11, 21, 18, 13) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t10 = Id($)t22t16. Apply at position 2.
Found a new name:  x * y * x^2 * y * x^-1 * y*t2t22t16t3
----------------------------------------
Let p = (1, 3, 20, 24, 4, 5)(2, 6, 22, 23, 19, 21)(7, 9,
    14, 18, 10, 11)(8, 12,
16, 17, 13, 15) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t3 = Id($)t9t21. Apply at position 3.
Found a new name:  x * y * x^2 * y * x^-1 * y*t2t10t9t21
```

```
*****************************************************************

Looking for relations to apply to y * x * y * x^2 * y * x
    ^-1*t12t3t3
Found 14
----------------------------------------
Let p = (1, 3, 20, 24, 4, 5)(2, 6, 22, 23, 19, 21)(7, 9,
    14, 18, 10, 11)(8, 12,
16, 17, 13, 15) belong to N.
t1^2 = t7 ==>
(t1t1)^p = (Id($)t7)^p ==>
t3t3 = Id($)t9. Apply at position 2.
Found a new name:  y * x * y * x^2 * y * x^-1*t12t9
----------------------------------------
Let p = (1, 18, 17)(2, 4, 15)(3, 8, 10)(5, 7, 6)(9, 20, 22)
    (11, 19, 12)(13, 24,
23)(14, 16, 21) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t12 = y * x * y * x^-2 * y * xt15t10. Apply at position 1.
Found a new name:  (x^-1 * y * x^2)^2*t15t10t3t3
----------------------------------------
Let p = (1, 18, 15)(2, 11, 4)(3, 7, 6)(5, 16, 14)(8, 23,
    10)(9, 19, 12)(13, 24,
```

```
21)(17, 22, 20) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t12 = x^2 * y * x * y * x^-1 * yt2t23. Apply at position 1.
Found a new name:  x^-1 * y * x^2 * y*t2t23t3t3
----------------------------------------
Let p = (1, 18, 22, 11, 20)(2, 19, 12, 4, 17)(5, 8, 13, 24,
     10)(6, 16, 23, 14,
7) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t12 = y * x^-1t17t13. Apply at position 1.
Found a new name:  x^2 * y * x^-1 * y * x^2*t17t13t3t3
----------------------------------------
Let p = (1, 18, 2, 9, 4)(3, 16, 13, 24, 14)(6, 8, 21, 10,
    7)(12, 20, 15, 22, 19)
belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t12 = x * y * x^-1 * y * x * y * xt1t21. Apply at position
    1.
Found a new name:  t1t21t3t3
----------------------------------------
Let p = (1, 18)(2, 20)(3, 5)(4, 22)(6, 7)(8, 14)(9, 11)(10,
```

16)(12, 19)(13,

24)(15, 17)(21, 23) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t12 = y * x^−2 * y * x * yt22t14. Apply at position 1.

Found a new name: (y * x^2)^2*t22t14t3t3

Let p = (1, 21)(2, 5)(3, 19)(4, 6)(7, 15)(8, 11)(9, 13)(10,

12)(14, 17)(16,

18)(20, 23)(22, 24) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t3 = x^−2 * y * x^−2t6t11. Apply at position 2.

Found a new name: y * x^2 * y * x * y * x^−1 * y*
t23t6t11t3

Let p = (1, 21, 22, 5, 12, 8)(2, 13, 9, 10, 17, 6)(3, 4,
23, 18, 14, 19)(7, 15,

16, 11, 24, 20) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t3 = x * y * x^−1 * y * x * y * xt23t1. Apply at position
2.

Found a new name: t5t23t1t3

```
----------------------------------------
Let p = (1, 21, 20, 16)(2, 10, 19, 3)(4, 7, 15, 14)(5, 17,
    23, 11)(6, 12, 24,
18)(8, 22, 13, 9) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x^2 * y * x^-1t7t22. Apply at position 2.
Found a new name:  x^2 * y * x^2 * y * x*t24t7t22t3
----------------------------------------
Let p = (1, 21, 18, 23)(2, 8, 20, 14)(3, 12, 5, 19)(4, 16,
    22, 10)(6, 17, 7,
15)(9, 24, 11, 13) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x * y * x^-1 * y * x^2 * yt16t20. Apply at position 2.
Found a new name:  x^-1 * y * x * y*t6t16t20t3
----------------------------------------
Let p = (1, 21, 11, 10, 2, 24)(3, 17, 16, 20, 6, 19)(4, 14,
     12, 13, 9, 5)(7, 15,
23, 22, 8, 18) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = y * x^-2 * y * x * yt14t18. Apply at position 2.
Found a new name:  (y * x^2)^2*t3t14t18t3
```

```
----------------------------------------
Let p = (1, 12, 2)(3, 23, 16)(4, 9, 17)(5, 10, 21)(6, 14,
   13)(7, 24, 8)(11, 22,
15)(18, 20, 19) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t12 = Id($)t24t18. Apply at position 1.
Found a new name:  y * x * y * x^2 * y * x^-1*t24t18t3t3
----------------------------------------
Let p = (1, 3, 20, 24, 4, 5)(2, 6, 22, 23, 19, 21)(7, 9,
   14, 18, 10, 11)(8, 12,
16, 17, 13, 15) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t3 = Id($)t9t21. Apply at position 2.
Found a new name:  y * x * y * x^2 * y * x^-1*t12t9t21t3
*****************************************************************

Looking for relations to apply to y * x * y * x^-2 * y*
   t9t14t3
Found 19
----------------------------------------
Let p = (1, 15, 11, 12, 4)(3, 23, 24, 10, 7)(5, 6, 16, 13,
   21)(9, 17, 18, 22,
```

```
19) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t9 = y * x^2 * y * x^-1 * yt1t8. Apply at position 1.
Found a new name:  t1t8t14t3
----------------------------------------
Let p = (1, 15)(2, 12)(3, 7)(4, 22)(5, 23)(6, 14)(8, 24)(9,
     19)(10, 16)(11,
17)(13, 21)(18, 20) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t9 = y * x^-1 * y * x * y * x^-2t22t24. Apply at position
    1.
Found a new name:  (x^2 * y)^2*t22t24t14t3
----------------------------------------
Let p = (1, 15, 20)(2, 19, 9)(3, 14, 7)(4, 18, 11)(5, 16,
    24)(6, 23, 10)(8, 13,
21)(12, 17, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t9 = y * x * y * x^-2 * y * xt18t13. Apply at position 1.
Found a new name:  y * x^-1 * y * x * y * x^2 * y*
    t18t13t14t3
----------------------------------------
```

```
Let p = (1, 15, 22, 2, 17)(3, 16, 8, 5, 7)(4, 20, 11, 19,
   9)(10, 14, 23, 13, 21)
belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t9 = x^-1 * y * x * y * x^2 * yt20t5. Apply at position 1.
Found a new name:  x * y * x^2 * y * x^2*t20t5t14t3
-----------------------------------------
Let p = (1, 15, 18)(2, 4, 11)(3, 6, 7)(5, 14, 16)(8, 10,
   23)(9, 12, 19)(13, 21,
24)(17, 20, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t9 = y * xt11t10. Apply at position 1.
Found a new name:  y * x * y * x^-1*t11t10t14t3
-----------------------------------------
Let p = (1, 8, 11, 6, 9, 16)(2, 5, 18, 3, 22, 13)(4, 7, 20,
    23, 12, 21)(10, 19,
14, 17, 24, 15) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t14 = x * yt7t11. Apply at position 2.
Found a new name:  x^2 * y * x^-1 * y * x^2*t21t7t11t3
-----------------------------------------
```

Let p = (1, 8, 12, 5, 22, 21)(2, 6, 17, 10, 9, 13)(3, 19, 14, 18, 23, 4)(7, 20,
24, 11, 16, 15) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t14 = y * x^−1 * y * x^2 * yt3t12. Apply at position 2.

Found a new name: y * x * y * x^2 * y * x^2 * y*t24t3t12t3

Let p = (1, 8, 15, 24)(2, 21, 12, 13)(3, 18, 7, 20)(4, 10, 22, 16)(5, 17, 23,
11)(6, 19, 14, 9) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t14 = y * xt10t15. Apply at position 2.

Found a new name: y * x * y * x^−1*t14t10t15t3

Let p = (1, 8)(2, 13)(3, 17)(4, 24)(5, 9)(6, 22)(7, 20)(10, 18)(11, 21)(12,
16)(14, 19)(15, 23) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t14 = y * x^2 * y * x^−1 * yt24t1. Apply at position 2.

Found a new name: t6t24t1t3

```
Let p = (1, 8, 4, 23)(2, 16, 11, 13)(3, 9, 21, 15)(5, 19,
   14, 22)(6, 18, 24,
12)(7, 20, 10, 17) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t14 = y * x^-2 * y * x^-2 * yt23t4. Apply at position 2.
Found a new name:  x^-1 * y * x^2 * y*t3t23t4t3
------------------------------------------
Let p = (1, 21)(2, 5)(3, 19)(4, 6)(7, 15)(8, 11)(9, 13)(10,
    12)(14, 17)(16,
18)(20, 23)(22, 24) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x^-2 * y * x^-2t6t11. Apply at position 3.
Found a new name:  x^-1 * y * x * y * x*t8t12t6t11
------------------------------------------
Let p = (1, 21, 22, 5, 12, 8)(2, 13, 9, 10, 17, 6)(3, 4,
   23, 18, 14, 19)(7, 15,
16, 11, 24, 20) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t3 = x * y * x^-1 * y * x * y * xt23t1. Apply at position
   3.
Found a new name:  x^2 * y * x^-2 * y*t24t2t23t1
```

Let p = (1, 21, 20, 16)(2, 10, 19, 3)(4, 7, 15, 14)(5, 17, 23, 11)(6, 12, 24,
18)(8, 22, 13, 9) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t3 = x^2 ∗ y ∗ x^−1t7t22. Apply at position 3.

Found a new name: y ∗ x^−1 ∗ y ∗ x∗t5t20t7t22

Let p = (1, 21, 18, 23)(2, 8, 20, 14)(3, 12, 5, 19)(4, 16, 22, 10)(6, 17, 7,
15)(9, 24, 11, 13) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t3 = x ∗ y ∗ x^−1 ∗ y ∗ x^2 ∗ yt16t20. Apply at position 3.

Found a new name: x ∗ y ∗ x^2 ∗ y ∗ x ∗ y ∗ x^−1 ∗ y∗ t13t15t16t20

Let p = (1, 21, 11, 10, 2, 24)(3, 17, 16, 20, 6, 19)(4, 14, 12, 13, 9, 5)(7, 15,
23, 22, 8, 18) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t3 = y ∗ x^−2 ∗ y ∗ x ∗ yt14t18. Apply at position 3.

```
Found a new name:  x^-1*t10t18t14t18
----------------------------------------
Let p = (1, 9, 22)(2, 18, 17)(3, 10, 13)(4, 19, 15)(5, 8,
   6)(7, 21, 16)(11, 20,
12)(14, 24, 23) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t9 = Id($)t21t15. Apply at position 1.
Found a new name:  y * x * y * x^-2 * y*t21t15t14t3
----------------------------------------
Let p = (1, 14)(2, 7)(3, 12)(4, 5)(6, 15)(8, 19)(9, 24)(10,
    11)(13, 20)(16,
17)(18, 21)(22, 23) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t14 = Id($)t2t8. Apply at position 2.
Found a new name:  y * x * y * x^-2 * y*t9t2t8t3
----------------------------------------
Let p = (1, 3, 20, 24, 4, 5)(2, 6, 22, 23, 19, 21)(7, 9,
   14, 18, 10, 11)(8, 12,
16, 17, 13, 15) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t3 = Id($)t9t21. Apply at position 3.
```

```
Found a new name:  y * x * y * x^-2 * y*t9t14t9t21
***********************************************************

Looking for relations to apply to x^-1 * y * x^2*t11t6t3
Found 19
-----------------------------------------
Let p = (1, 17, 18)(2, 15, 4)(3, 10, 8)(5, 6, 7)(9, 22, 20)
   (11, 12, 19)(13, 23,
24)(14, 21, 16) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t11 = x^2 * y * x^2t2t3. Apply at position 1.
Found a new name:  y * x * y * x^-2*t2t3t6t3
-----------------------------------------
Let p = (1, 17, 4)(2, 12, 15)(3, 8, 24)(5, 10, 7)(6, 21,
   14)(9, 20, 18)(11, 22,
19)(13, 23, 16) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t11 = x^-2 * y * xt1t24. Apply at position 1.
Found a new name:  t1t24t6t3
-----------------------------------------
Let p = (1, 17, 2, 22, 15)(3, 7, 5, 8, 16)(4, 9, 19, 11,
   20)(10, 21, 13, 23, 14)
```

belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t11 = x^−1 * yt9t16. Apply at position 1.

Found a new name: x^−1 * y * x * y*t9t16t6t3

Let p = (1, 17, 9, 12, 20)(2, 19, 11, 15, 18)(3, 6, 8, 13, 23)(5, 21, 24, 14, 7)

belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t11 = x * y * x^−2t4t13. Apply at position 1.

Found a new name: x^2 * y * x^2 * y * x*t4t13t6t3

Let p = (1, 17)(2, 20)(3, 21)(4, 18)(5, 7)(6, 10)(8, 14)(9, 15)(11, 19)(12,

22)(13, 23)(16, 24) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t11 = y * x^−1t18t14. Apply at position 1.

Found a new name: x * y * x^−2 * y * x*t18t14t6t3

Let p = (1, 24, 15, 8)(2, 13, 12, 21)(3, 20, 7, 18)(4, 16, 22, 10)(5, 11, 23,

17)(6, 9, 14, 19) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t6 = x^-2 * y * xt16t1. Apply at position 2.

Found a new name: t10t16t1t3

Let p = (1, 24, 22, 14, 9, 23)(2, 21, 17, 7, 18, 16)(3, 11, 13, 12, 10, 20)(4,
8, 15, 5, 19, 6) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t6 = y * x^2 * y * x^-1 * yt8t15. Apply at position 2.

Found a new name: x * y * x^-1 * y * x*t5t8t15t3

Let p = (1, 24, 2, 10, 11, 21)(3, 19, 6, 20, 16, 17)(4, 5, 9, 13, 12, 14)(7, 18,
8, 22, 23, 15) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t6 = y * x^-1 * y * x * y * x^-2t5t22. Apply at position 2.

Found a new name: x * y * x^2 * y * x^2*t16t5t22t3

Let p = (1, 24)(2, 23)(3, 4)(5, 20)(6, 19)(7, 18)(8, 17)(9, 10)(11, 14)(12,

13)(15, 16)(21, 22) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t6 = x^2 ∗ y ∗ x^2t3t17. Apply at position 2.

Found a new name: y ∗ x ∗ y ∗ x^−2∗t6t3t17t3

Let p = (1, 24, 17, 16)(2, 14, 20, 8)(3, 9, 21, 15)(4, 7, 18, 5)(6, 11, 10,

19)(12, 23, 22, 13) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t6 = y ∗ x^−1 ∗ y ∗ x^2 ∗ y ∗ xt7t2. Apply at position 2.

Found a new name: y ∗ x^−2 ∗ y ∗ x^2∗t23t7t2t3

Let p = (1, 21)(2, 5)(3, 19)(4, 6)(7, 15)(8, 11)(9, 13)(10, 12)(14, 17)(16,

18)(20, 23)(22, 24) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t3 = x^−2 ∗ y ∗ x^−2t6t11. Apply at position 3.

Found a new name: x^2∗t21t11t6t11

Let p = (1, 21, 22, 5, 12, 8)(2, 13, 9, 10, 17, 6)(3, 4, 23, 18, 14, 19)(7, 15,

16, 11, 24, 20) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t3 = x * y * x^-1 * y * x * y * xt23t1. Apply at position 3.

Found a new name: x^2 * y * x^2 * y * x * y * x^-1 * y* t13t17t23t1

Let p = (1, 21, 20, 16)(2, 10, 19, 3)(4, 7, 15, 14)(5, 17, 23, 11)(6, 12, 24,

18)(8, 22, 13, 9) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t3 = x^2 * y * x^-1t7t22. Apply at position 3.

Found a new name: y * x * y*t7t12t7t22

Let p = (1, 21, 18, 23)(2, 8, 20, 14)(3, 12, 5, 19)(4, 16, 22, 10)(6, 17, 7,

15)(9, 24, 11, 13) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t3 = x * y * x^-1 * y * x^2 * yt16t20. Apply at position 3.

Found a new name: (y * x^-2)^2*t23t18t16t20

Let p = (1, 21, 11, 10, 2, 24)(3, 17, 16, 20, 6, 19)(4, 14, 12, 13, 9, 5)(7, 15,
23, 22, 8, 18) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t3 = y * x^-2 * y * x * yt14t18. Apply at position 3.

Found a new name: x * y * x^-1 * y*t5t15t14t18

Let p = (1, 11, 4, 15, 12)(3, 24, 7, 23, 10)(5, 16, 21, 6, 13)(9, 18, 19, 17,
22) belong to N.

t1 = t7t19 ==>

(t1)^p = (Id($)t7t19)^p ==>

t11 = Id($)t23t17. Apply at position 1.

Found a new name: x^-1 * y * x^2*t23t17t6t3

Let p = (1, 6, 2, 5)(3, 15, 21, 9)(4, 10, 22, 16)(7, 12, 8, 11)(13, 18, 14,
17)(19, 24, 20, 23) belong to N.

t1 = t7t19 ==>

(t1)^p = (Id($)t7t19)^p ==>

t6 = Id($)t12t24. Apply at position 2.

Found a new name: x^-1 * y * x^2*t11t12t24t3

```
Let p = (1, 3, 20, 24, 4, 5)(2, 6, 22, 23, 19, 21)(7, 9,
    14, 18, 10, 11)(8, 12,
16, 17, 13, 15) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t3 = Id($)t9t21. Apply at position 3.
Found a new name:  x^-1 * y * x^2*t11t6t9t21
*************************************************************

Looking for relations to apply to y * x^2 * y * x^2 * y*
    t22t5t3
Found 20
--------------------------------------
Let p = (1, 23, 4, 8)(2, 13, 11, 16)(3, 15, 21, 9)(5, 22,
    14, 19)(6, 12, 24,
18)(7, 17, 10, 20) belong to N.
Lemma 2: t19t9 = yx^-1yt12t11 ==>
(t19t9)^p = (y * x^-1 * yt12t11)^p ==>
t5t3 = x * y * x^2 * y * x^-1t24t16. Apply at position 2.
Found a new name:  y * x^-1 * y * x * y * x*t9t24t16
--------------------------------------
Let p = (1, 4, 12, 11, 15)(3, 7, 10, 24, 23)(5, 21, 13, 16,
     6)(9, 19, 22, 18,
17) belong to N.
```

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t22 = y ∗ x^−1 ∗ y ∗ x^2 ∗ yt12t8. Apply at position 1.

Found a new name: x ∗ y ∗ x^−1 ∗ y ∗ x∗t12t8t5t3

Let p = (1, 4, 17)(2, 15, 12)(3, 24, 8)(5, 7, 10)(6, 14, 21)(9, 18, 20)(11, 19, 22)(13, 16, 23) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t22 = x ∗ y ∗ x^−2t17t3. Apply at position 1.

Found a new name: y ∗ x ∗ y ∗ x^2 ∗ y ∗ x^2 ∗ y∗t17t3t5t3

Let p = (1, 4, 20)(2, 19, 22)(3, 5, 24)(6, 21, 23)(7, 10, 14)(8, 13, 16)(9, 11, 18)(12, 15, 17) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t22 = y ∗ x^2 ∗ y ∗ x^−1 ∗ y ∗ xt20t13. Apply at position 1.

Found a new name: y ∗ x^−2 ∗ y ∗ x^2∗t20t13t5t3

Let p = (1, 4)(2, 11)(3, 21)(5, 14)(6, 24)(7, 10)(8, 23)(9, 15)(12, 18)(13,

16)(17, 20)(19, 22) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t22 = y * x^-2 * y * x^-2 * yt1t23. Apply at position 1.

Found a new name: t1t23t5t3

Let p = (1, 4, 9, 2, 18)(3, 14, 24, 13, 16)(6, 7, 10, 21, 8)(12, 19, 22, 15, 20)

belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t22 = x^2 * y * x^-1 * y * x * yt9t6. Apply at position 1.

Found a new name: x * y * x^-2 * y*t9t6t5t3

Let p = (1, 23, 9, 14, 22, 24)(2, 16, 18, 7, 17, 21)(3, 20, 10, 12, 13, 11)(4,

6, 19, 5, 15, 8) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t5 = x^2 * y * x^-1 * y * x * yt6t4. Apply at position 2.

Found a new name: x * y * x^-2 * y*t5t6t4t3

Let p = (1, 23)(2, 24)(3, 22)(4, 21)(5, 19)(6, 20)(7, 17)(8, 18)(9, 16)(10,

15)(11, 13)(12, 14) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t5 = x * y * x^-1 * y * x * y * xt21t18. Apply at position 2.

Found a new name: (x^-1 * y * x^-1)^2*t10t21t18t3

Let p = (1, 23, 18, 21)(2, 14, 20, 8)(3, 19, 5, 12)(4, 10, 22, 16)(6, 15, 7,

17)(9, 13, 11, 24) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t5 = x^2 * y * x * y * x^-1 * yt10t2. Apply at position 2.

Found a new name: (x^-1 * y * x^2)^2*t8t10t2t3

Let p = (1, 23, 4, 8)(2, 13, 11, 16)(3, 15, 21, 9)(5, 22, 14, 19)(6, 12, 24,

18)(7, 17, 10, 20) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t5 = y * x^-2 * y * x^-2 * yt8t1. Apply at position 2.

Found a new name: t14t8t1t3

Let p = (1, 23, 2, 3, 12, 16)(4, 7, 17, 8, 9, 24)(5, 20,

21, 18, 10, 19)(6, 22,
13, 11, 14, 15) belong to N.
Lemma 3: t19 = yx^2yx^−2yt4t8. ==>
(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>
t5 = y ∗ x^−1 ∗ y ∗ x ∗ y ∗ x^2t7t9. Apply at position 2.
Found a new name: (y ∗ x)^2∗t16t7t9t3

Let p = (1, 21)(2, 5)(3, 19)(4, 6)(7, 15)(8, 11)(9, 13)(10, 12)(14, 17)(16,
18)(20, 23)(22, 24) belong to N.
Lemma 3: t19 = yx^2yx^−2yt4t8. ==>
(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>
t3 = x^−2 ∗ y ∗ x^−2t6t11. Apply at position 3.
Found a new name: y ∗ x^−1 ∗ y ∗ x∗t16t9t6t11

Let p = (1, 21, 22, 5, 12, 8)(2, 13, 9, 10, 17, 6)(3, 4, 23, 18, 14, 19)(7, 15,
16, 11, 24, 20) belong to N.
Lemma 3: t19 = yx^2yx^−2yt4t8. ==>
(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>
t3 = x ∗ y ∗ x^−1 ∗ y ∗ x ∗ y ∗ xt23t1. Apply at position 3.
Found a new name: (x^−1 ∗ y ∗ x^−1)^2∗t10t19t23t1

Let p = (1, 21, 20, 16)(2, 10, 19, 3)(4, 7, 15, 14)(5, 17, 23, 11)(6, 12, 24,
18)(8, 22, 13, 9) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t3 = x^2 * y * x^−1t7t22. Apply at position 3.

Found a new name: x * y * x^2 * y*t21t1t7t22

Let p = (1, 21, 18, 23)(2, 8, 20, 14)(3, 12, 5, 19)(4, 16, 22, 10)(6, 17, 7,
15)(9, 24, 11, 13) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t3 = x * y * x^−1 * y * x^2 * yt16t20. Apply at position 3.

Found a new name: (x^2 * y * x^−1)^2*t8t11t16t20

Let p = (1, 21, 11, 10, 2, 24)(3, 17, 16, 20, 6, 19)(4, 14, 12, 13, 9, 5)(7, 15,
23, 22, 8, 18) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t3 = y * x^−2 * y * x * yt14t18. Apply at position 3.

Found a new name: y * x^−2 * y*t14t17t14t18

```
Let p = (1, 22, 12)(2, 9, 17)(3, 23, 14)(4, 18, 19)(5, 8,
   21)(6, 13, 10)(7, 16,
24)(11, 20, 15) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t22 = Id($)t16t4. Apply at position 1.
Found a new name:  y * x^2 * y * x^2 * y*t16t4t5t3
-----------------------------------------
Let p = (1, 5, 2, 6)(3, 9, 21, 15)(4, 16, 22, 10)(7, 11, 8,
    12)(13, 17, 14,
18)(19, 23, 20, 24) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t5 = Id($)t11t23. Apply at position 2.
Found a new name:  y * x^2 * y * x^2 * y*t22t11t23t3
-----------------------------------------
Let p = (1, 3, 20, 24, 4, 5)(2, 6, 22, 23, 19, 21)(7, 9,
   14, 18, 10, 11)(8, 12,
16, 17, 13, 15) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t3 = Id($)t9t21. Apply at position 3.
Found a new name:  y * x^2 * y * x^2 * y*t22t5t9t21
*****************************************************************
```

```
Looking for relations to apply to x^-2 * y * x^-2*t13t6t11
Found 20
----------------------------------------
Let p = (1, 7, 19, 13)(2, 14, 20, 8)(3, 18, 16, 11)(4, 23,
    9, 6)(5, 15, 24,
22)(10, 17, 21, 12) belong to N.
Lemma 2: t19t9 = yx^-1yt12t11 ==>
(t19t9)^p = (y * x^-1 * yt12t11)^p ==>
t13t6 = y * x^2 * yt10t3. Apply at position 1.
Found a new name:  x * y * x^2*t10t3t11
----------------------------------------
Let p = (1, 7, 19, 13)(2, 14, 20, 8)(3, 18, 16, 11)(4, 23,
    9, 6)(5, 15, 24,
22)(10, 17, 21, 12) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t13 = y * x^-1 * y * x * y * x^2t23t2. Apply at position 1.
Found a new name:  y * x^2 * y * x * y * x^-1 * y*
    t23t2t6t11
----------------------------------------
Let p = (1, 24, 15, 8)(2, 13, 12, 21)(3, 20, 7, 18)(4, 16,
    22, 10)(5, 11, 23,
17)(6, 9, 14, 19) belong to N.
```

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t6 = x^−2 ∗ y ∗ xt16t1. Apply at position 2.

Found a new name: x^2 ∗ y ∗ x^−1 ∗ y∗t12t16t1t11

Let p = (1, 24, 22, 14, 9, 23)(2, 21, 17, 7, 18, 16)(3, 11, 13, 12, 10, 20)(4,
8, 15, 5, 19, 6) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t6 = y ∗ x^2 ∗ y ∗ x^−1 ∗ yt8t15. Apply at position 2.

Found a new name: (x ∗ y)^2∗t2t8t15t11

Let p = (1, 24, 2, 10, 11, 21)(3, 19, 6, 20, 16, 17)(4, 5, 9, 13, 12, 14)(7, 18,
8, 22, 23, 15) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t6 = y ∗ x^−1 ∗ y ∗ x ∗ y ∗ x^−2t5t22. Apply at position 2.

Found a new name: y ∗ x^−1 ∗ y ∗ x ∗ y ∗ x^2 ∗ y∗ t19t5t22t11

Let p = (1, 24)(2, 23)(3, 4)(5, 20)(6, 19)(7, 18)(8, 17)(9, 10)(11, 14)(12,

13)(15, 16)(21, 22) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t6 = x^2 * y * x^2t3t17. Apply at position 2.

Found a new name: t1t3t17t11

Let p = (1, 24, 17, 16)(2, 14, 20, 8)(3, 9, 21, 15)(4, 7,
18, 5)(6, 11, 10,

19)(12, 23, 22, 13) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t6 = y * x^-1 * y * x^2 * y * xt7t2. Apply at position 2.

Found a new name: (x^-1 * y * x^-1)^2*t20t7t2t11

Let p = (1, 17, 18)(2, 15, 4)(3, 10, 8)(5, 6, 7)(9, 22, 20)
(11, 12, 19)(13, 23,

24)(14, 21, 16) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t11 = x^2 * y * x^2t2t3. Apply at position 3.

Found a new name: t1t20t2t3

Let p = (1, 17, 4)(2, 12, 15)(3, 8, 24)(5, 10, 7)(6, 21,
14)(9, 20, 18)(11, 22,

19)(13, 23, 16) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t11 = x^-2 * y * xt1t24. Apply at position 3.

Found a new name: x^2 * y * x^-1 * y*t12t11t1t24

Let p = (1, 17, 2, 22, 15)(3, 7, 5, 8, 16)(4, 9, 19, 11, 20)(10, 21, 13, 23, 14)

belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t11 = x^-1 * yt9t16. Apply at position 3.

Found a new name: y * x^-2 * y * x^2*t19t12t9t16

Let p = (1, 17, 9, 12, 20)(2, 19, 11, 15, 18)(3, 6, 8, 13, 23)(5, 21, 24, 14, 7)

belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t11 = x * y * x^-2t4t13. Apply at position 3.

Found a new name: x^-1 * y * x * y * x^-1*t17t18t4t13

Let p = (1, 17)(2, 20)(3, 21)(4, 18)(5, 7)(6, 10)(8, 14)(9, 15)(11, 19)(12,

```
22)(13, 23)(16, 24) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t11 = y * x^-1t18t14. Apply at position 3.
Found a new name:  x * y * x^2 * y * x^2*t18t2t18t14
----------------------------------------
Let p = (1, 13, 19, 7)(2, 16, 15, 6)(3, 12, 8, 4)(5, 11,
    23, 17)(9, 24, 20,
10)(14, 22, 21, 18) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t13 = Id($)t1t7. Apply at position 1.
Found a new name:  x^-2 * y * x^-2*t1t7t6t11
----------------------------------------
Let p = (1, 6, 2, 5)(3, 15, 21, 9)(4, 10, 22, 16)(7, 12, 8,
     11)(13, 18, 14,
17)(19, 24, 20, 23) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t6 = Id($)t12t24. Apply at position 2.
Found a new name:  x^-2 * y * x^-2*t13t12t24t11
----------------------------------------
Let p = (1, 11, 4, 15, 12)(3, 24, 7, 23, 10)(5, 16, 21, 6,
    13)(9, 18, 19, 17,
```

```
22) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t11 = Id($)t23t17. Apply at position 3.
Found a new name:  x^-2 * y * x^-2*t13t6t23t17
*************************************************************

Looking for relations to apply to x * y * x^-1 * y * x * y
    * x*t21t23t1
Found 20
-----------------------------------------
Let p = (1, 3, 11, 14)(2, 7, 9, 23)(4, 10, 22, 16)(5, 20,
    13, 15)(6, 18, 24,
12)(8, 19, 21, 17) belong to N.
Lemma 2: t19t9 = yx^-1yt12t11 ==>
(t19t9)^p = (y * x^-1 * yt12t11)^p ==>
t21t23 = x * y * x^-2 * y * x^-1t6t14. Apply at position 1.
Found a new name:  x * y * x^2 * y * x^2 * y*t6t14t1
-----------------------------------------
Let p = (1, 3, 20, 24, 4, 5)(2, 6, 22, 23, 19, 21)(7, 9,
    14, 18, 10, 11)(8, 12,
16, 17, 13, 15) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
```

```
t21 = y * x * y * x^2 * y * x^-1t5t12. Apply at position 1.
Found a new name: t5t12t23t1
----------------------------------------
Let p = (1, 5, 2, 6)(3, 9, 21, 15)(4, 16, 22, 10)(7, 11, 8,
    12)(13, 17, 14,
18)(19, 23, 20, 24) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t23 = y * x^2 * y * x * y * x^-1t16t12. Apply at position
   2.
Found a new name: (x^-2 * y)^2*t9t16t12t1
----------------------------------------
Let p = (1, 5, 4, 24, 20, 3)(2, 21, 19, 23, 22, 6)(7, 11,
   10, 18, 14, 9)(8, 15,
13, 17, 16, 12) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t23 = y * x^-1 * y * x * y * x^-2t24t15. Apply at position
   2.
Found a new name: (x^-1 * y * x^2)^2*t17t24t15t1
----------------------------------------
Let p = (1, 5)(2, 10)(3, 18)(4, 14)(6, 9)(7, 11)(8, 22)(12,
    21)(13, 17)(15,
24)(16, 20)(19, 23) belong to N.
```

```
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t23 = y * x^2 * y * x^2 * yt14t22. Apply at position 2.
Found a new name:  y * x^-2 * y * x*t15t14t22t1

----------------------------------------
Let p = (1, 5, 18, 16, 15, 14)(2, 7, 11, 6, 4, 3)(8, 19,
    23, 12, 10, 9)(13, 17,
24, 22, 21, 20) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t23 = y * x * y * x^2 * y * x^-1t3t19. Apply at position 2.
Found a new name:  t1t3t19t1

----------------------------------------
Let p = (1, 5, 9, 10)(2, 8, 20, 14)(3, 4, 13, 17)(6, 18,
    24, 12)(7, 11, 21,
22)(15, 16, 19, 23) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t23 = x^-1 * y * x * y * x^2 * yt13t20. Apply at position
    2.
Found a new name:  x^2 * y * x^-1 * y * x^2*t19t13t20t1

----------------------------------------
Let p = (1, 19)(2, 4)(3, 23)(5, 21)(6, 24)(7, 13)(8, 10)(9,
     17)(11, 15)(12,
```

```
18)(14, 16)(20, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = x * y * x^2 * y * x^-1 * yt2t10. Apply at position 3.
Found a new name:  (x^-1 * y * x^-1)^2*t20t11t2t10
-----------------------------------------
Let p = (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9,
    20)(10, 24)(11,
17)(14, 21)(18, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = y * x * y * x^2 * y * x^-1t12t3. Apply at position 3.
Found a new name:  t1t18t12t3
-----------------------------------------
Let p = (1, 19)(2, 20)(3, 16)(4, 9)(5, 24)(6, 23)(7, 13)(8,
    14)(10, 21)(11,
18)(12, 17)(15, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = y * x * y * x^-2 * yt9t14. Apply at position 3.
Found a new name:  (x * y)^2*t2t17t9t14
-----------------------------------------
Let p = (1, 19)(2, 18)(3, 21)(4, 11)(5, 16)(6, 8)(7, 13)(9,
    15)(10, 23)(12,
```

```
20)(14, 24)(17, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = x^-1 * y * x^2t11t6. Apply at position 3.
Found a new name: (x^2 * y * x^-1)^2*t9t12t11t6
----------------------------------------
Let p = (1, 19)(2, 17)(3, 24)(4, 22)(5, 8)(6, 21)(7, 13)(9,
    18)(10, 16)(11,
20)(12, 15)(14, 23) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = y * x^2 * y * x^2 * yt22t5. Apply at position 3.
Found a new name: y * x^-2 * y * x*t15t1t22t5
----------------------------------------
Let p = (1, 21)(2, 5)(3, 19)(4, 6)(7, 15)(8, 11)(9, 13)(10,
    12)(14, 17)(16,
18)(20, 23)(22, 24) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t21 = Id($)t15t3. Apply at position 1.
Found a new name: x * y * x^-1 * y * x * y * x*t15t3t23t1
----------------------------------------
Let p = (1, 23, 9, 14, 22, 24)(2, 16, 18, 7, 17, 21)(3, 20,
    10, 12, 13, 11)(4,
```

6, 19, 5, 15, 8) belong to N.

t1 = t7t19 ==>

(t1)^p = (Id($)t7t19)^p ==>

t23 = Id($)t17t5. Apply at position 2.

Found a new name: x * y * x^-1 * y * x * y * x*t21t17t5t1

Let p = Id($) belong to N.

t1 = t7t19 ==>

(t1)^p = (Id($)t7t19)^p ==>

t1 = Id($)t7t19. Apply at position 3.

Found a new name: x * y * x^-1 * y * x * y * x*t21t23t7t19

****

Looking for relations to apply to x^2 * y * x^-1*t10t7t22

Found 19

Let p = (1, 16, 12, 3, 2, 23)(4, 24, 9, 8, 17, 7)(5, 19,
10, 18, 21, 20)(6, 15,
14, 11, 13, 22) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t10 = x^-2 * y * xt24t17. Apply at position 1.

Found a new name: x^2 * y * x^2 * y * x*t24t17t7t22

```
Let p = (1, 13, 19, 7)(2, 16, 15, 6)(3, 12, 8, 4)(5, 11,
    23, 17)(9, 24, 20,
10)(14, 22, 21, 18) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t7 = x * y * x^-2t3t4. Apply at position 2.
Found a new name:  t1t3t4t22
-----------------------------------------
Let p = (1, 13, 19, 7)(2, 21, 4, 5)(3, 22, 23, 20)(6, 12,
    24, 18)(8, 15, 10,
11)(9, 16, 17, 14) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t7 = x^-1 * y * x * y * x^2 * yt5t15. Apply at position 2.
Found a new name:  x * y * x^-2 * y * x*t4t5t15t22
-----------------------------------------
Let p = (1, 13, 19, 7)(2, 8, 20, 14)(3, 11, 16, 18)(4, 6,
    9, 23)(5, 22, 24,
15)(10, 12, 21, 17) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t7 = y * x^2 * y * x^-1 * y * xt6t20. Apply at position 2.
Found a new name:  y * x * y * x^-1*t12t6t20t22
-----------------------------------------
```

```
Let p = (1, 13, 19, 7)(2, 23, 18, 10)(3, 9, 21, 15)(4, 14,
   11, 24)(5, 12, 16,
20)(6, 22, 8, 17) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t7 = y * x^-1t14t17. Apply at position 2.
Found a new name:  x^-2 * y * x * y*t22t14t17t22
----------------------------------------
Let p = (1, 13, 19, 7)(2, 24, 17, 3)(4, 10, 22, 16)(5, 9,
   8, 18)(6, 11, 21,
20)(12, 23, 15, 14) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t7 = y * x * y * x^-2 * y * xt10t18. Apply at position 2.
Found a new name:  x * y * x^2 * y * x^2*t18t10t18t22
----------------------------------------
Let p = (1, 4, 12, 11, 15)(3, 7, 10, 24, 23)(5, 21, 13, 16,
    6)(9, 19, 22, 18,
17) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t22 = y * x^-1 * y * x^2 * yt12t8. Apply at position 3.
Found a new name:  (x^-2 * y)^2*t9t1t12t8
----------------------------------------
```

Let p = (1, 4, 17)(2, 15, 12)(3, 24, 8)(5, 7, 10)(6, 14, 21)(9, 18, 20)(11, 19,
22)(13, 16, 23) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t22 = x * y * x^-2t17t3. Apply at position 3.

Found a new name: t1t11t17t3

Let p = (1, 4, 20)(2, 19, 22)(3, 5, 24)(6, 21, 23)(7, 10, 14)(8, 13, 16)(9, 11,
18)(12, 15, 17) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t22 = y * x^2 * y * x^-1 * y * xt20t13. Apply at position 3.

Found a new name: y * x * y * x^-1*t12t22t20t13

Let p = (1, 4)(2, 11)(3, 21)(5, 14)(6, 24)(7, 10)(8, 23)(9, 15)(12, 18)(13,
16)(17, 20)(19, 22) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t22 = y * x^-2 * y * x^-2 * yt1t23. Apply at position 3.

Found a new name: y * x * y * x^2 * y * x^2 * y*

```
  t20t17t1t23
-----------------------------------------
Let p = (1, 4, 9, 2, 18)(3, 14, 24, 13, 16)(6, 7, 10, 21,
  8)(12, 19, 22, 15, 20)
belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t22 = x^2 * y * x^-1 * y * x * yt9t6. Apply at position 3.
Found a new name:  (x^-1 * y * x^2)^2*t17t19t9t6
-----------------------------------------
Let p = (1, 10, 20, 23, 15, 6)(2, 5, 9, 24, 19, 16)(3, 12,
  7, 22, 14, 17)(4, 8,
11, 21, 18, 13) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t10 = Id($)t22t16. Apply at position 1.
Found a new name:  x^2 * y * x^-1*t22t16t7t22
-----------------------------------------
Let p = (1, 7, 19, 13)(2, 5, 4, 21)(3, 20, 23, 22)(6, 18,
  24, 12)(8, 11, 10,
15)(9, 14, 17, 16) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t7 = Id($)t19t13. Apply at position 2.
```

```
Found a new name:  x^2 * y * x^-1*t10t19t13t22
----------------------------------------
Let p = (1, 22, 12)(2, 9, 17)(3, 23, 14)(4, 18, 19)(5, 8,
   21)(6, 13, 10)(7, 16,
24)(11, 20, 15) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t22 = Id($)t16t4. Apply at position 3.
Found a new name:  x^2 * y * x^-1*t10t7t16t4
*****************************************************************

Looking for relations to apply to x * y * x^-1 * y * x^2 *
   y*t16t16t20
Found 14
----------------------------------------
Let p = (1, 16)(2, 6)(3, 11)(4, 7)(5, 15)(8, 12)(9, 23)(10,
    19)(13, 22)(14,
18)(17, 21)(20, 24) belong to N.
t1^2 = t7 ==>
(t1t1)^p = (Id($)t7)^p ==>
t16t16 = Id($)t4. Apply at position 1.
Found a new name:  x * y * x^-1 * y * x^2 * y*t4t20
----------------------------------------
Let p = (1, 10, 12, 14)(2, 7, 22, 24)(3, 9, 21, 15)(4, 6,
```

20, 13)(5, 17, 23,

11)(8, 19, 16, 18) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t16 = x^−1 * y * x^2t6t19. Apply at position 1.

Found a new name: x^−1 * y * x * y*t6t19t16t20

Let p = (1, 2, 4, 18, 9)(3, 13, 14, 16, 24)(6, 21, 7, 8,

10)(12, 15, 19, 20, 22)

belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t20 = x^2 * y * x * y * x^−1 * yt18t10. Apply at position

3.

Found a new name: (y * x^−2)^2*t20t20t18t10

Let p = (1, 2)(3, 21)(4, 22)(5, 6)(7, 8)(9, 15)(10, 16)(11,

12)(13, 14)(17,

18)(19, 20)(23, 24) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t20 = y * x^−1 * y * x^2 * y * xt22t7. Apply at position 3.

Found a new name: y * x^−1 * y * x * y * x^2 * y*

t19t19t22t7

```
----------------------------------------
Let p = (1, 2, 12)(3, 16, 23)(4, 17, 9)(5, 21, 10)(6, 13,
   14)(7, 8, 24)(11, 15,
22)(18, 19, 20) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t20 = x^2 * y * x^2t17t24. Apply at position 3.
Found a new name:  x^-2 * y * x * y*t22t22t17t24
----------------------------------------
Let p = (1, 2, 15, 17, 22)(3, 5, 16, 7, 8)(4, 19, 20, 9,
   11)(10, 13, 14, 21, 23)
belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t20 = x * y * x^2 * y * x^-1 * yt19t3. Apply at position 3.
Found a new name:  t1t1t19t3
----------------------------------------
Let p = (1, 2, 11)(3, 6, 16)(4, 9, 12)(5, 13, 14)(7, 8, 23)
   (10, 21, 24)(15, 18,
22)(17, 19, 20) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t20 = y * x^-1 * y * x * y * x^2t9t23. Apply at position 3.
Found a new name:  x * y * x^-2 * y * x*t4t4t9t23
```

```
-----------------------------------------
Let p = (1, 16)(2, 6)(3, 11)(4, 7)(5, 15)(8, 12)(9, 23)(10,
     19)(13, 22)(14,
18)(17, 21)(20, 24) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t16 = Id($)t4t10. Apply at position 1.
Found a new name:  x * y * x^-1 * y * x^2 * y*t4t10t16t20
-----------------------------------------
Let p = (1, 20, 12, 9, 17)(2, 18, 15, 11, 19)(3, 23, 13, 8,
     6)(5, 7, 14, 24, 21)
belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t20 = Id($)t14t2. Apply at position 3.
Found a new name:  x * y * x^-1 * y * x^2 * y*t16t16t14t2
*******************************************************************

Looking for relations to apply to y * x^-2 * y * x * y*
    t7t14t18
Found 19
-----------------------------------------
Let p = (1, 13, 19, 7)(2, 16, 15, 6)(3, 12, 8, 4)(5, 11,
    23, 17)(9, 24, 20,
```

10)(14, 22, 21, 18) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t7 = x * y * x^-2t3t4. Apply at position 1.

Found a new name: (y * x^2)^2*t3t4t14t18

Let p = (1, 8, 11, 6, 9, 16)(2, 5, 18, 3, 22, 13)(4, 7, 20, 23, 12, 21)(10, 19,

14, 17, 24, 15) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t14 = x * yt7t11. Apply at position 2.

Found a new name: y * x^2 * y * x^-1*t11t7t11t18

Let p = (1, 8, 12, 5, 22, 21)(2, 6, 17, 10, 9, 13)(3, 19, 14, 18, 23, 4)(7, 20,

24, 11, 16, 15) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t14 = y * x^-1 * y * x^2 * yt3t12. Apply at position 2.

Found a new name: t1t3t12t18

Let p = (1, 8, 15, 24)(2, 21, 12, 13)(3, 18, 7, 20)(4, 10, 22, 16)(5, 17, 23,

11)(6, 19, 14, 9) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t14 = y * xt10t15. Apply at position 2.

Found a new name: y * x^-2 * y * x^2*t19t10t15t18

Let p = (1, 8)(2, 13)(3, 17)(4, 24)(5, 9)(6, 22)(7, 20)(10,
18)(11, 21)(12,
16)(14, 19)(15, 23) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t14 = y * x^2 * y * x^-1 * yt24t1. Apply at position 2.

Found a new name: y * x * y * x^2 * y * x^2 * y*
t20t24t1t18

Let p = (1, 8, 4, 23)(2, 16, 11, 13)(3, 9, 21, 15)(5, 19,
14, 22)(6, 18, 24,
12)(7, 20, 10, 17) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t14 = y * x^-2 * y * x^-2 * yt23t4. Apply at position 2.

Found a new name: x^-1 * y * x * y * x^-1*t17t23t4t18

Let p = (1, 12, 2)(3, 23, 16)(4, 9, 17)(5, 10, 21)(6, 14,

13)(7, 24, 8)(11, 22,

15)(18, 20, 19) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t18 = x * y * x^−2 * y * x * yt9t7. Apply at position 3.

Found a new name: x * y * x^2 * y * x * y * x^−1 * y*
t12t2t9t7

Let p = (1, 12, 15, 4, 11)(3, 10, 23, 7, 24)(5, 13, 6, 21,
16)(9, 22, 17, 19,

18) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t18 = x * yt11t8. Apply at position 3.

Found a new name: y * x^2 * y * x^−1*t11t18t11t8

Let p = (1, 12)(2, 22)(3, 21)(4, 20)(5, 23)(6, 13)(7, 24)
(8, 16)(9, 15)(10,

14)(11, 17)(18, 19) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y * x^−2 * y * x^−2 * yt4t8)^p ==>

t18 = y * x^2 * y * x * y * x^−1t20t16. Apply at position
3.

Found a new name: x^2 * y * x^−1 * y * x^2*t19t4t20t16

Let p = (1, 12, 22)(2, 17, 9)(3, 14, 23)(4, 19, 18)(5, 21, 8)(6, 10, 13)(7, 24,
16)(11, 15, 20) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t18 = y * x * y * x^2 * y * x^-1t19t5. Apply at position 3.

Found a new name: x^-1 * y * x * y*t17t20t19t5

Let p = (1, 12, 17, 20, 9)(2, 15, 19, 18, 11)(3, 13, 6, 23, 8)(5, 14, 21, 7, 24)
belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t18 = y * x^-1 * y * x^2 * yt4t3. Apply at position 3.

Found a new name: t1t22t4t3

Let p = (1, 7, 19, 13)(2, 5, 4, 21)(3, 20, 23, 22)(6, 18, 24, 12)(8, 11, 10,
15)(9, 14, 17, 16) belong to N.

t1 = t7t19 ==>

(t1)^p = (Id($)t7t19)^p ==>

t7 = Id($)t19t13. Apply at position 1.

Found a new name: y * x^-2 * y * x * y*t19t13t14t18

```
----------------------------------------
Let p = (1, 14)(2, 7)(3, 12)(4, 5)(6, 15)(8, 19)(9, 24)(10,
    11)(13, 20)(16,
17)(18, 21)(22, 23) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t14 = Id($)t2t8. Apply at position 2.
Found a new name:  y * x^-2 * y * x * y*t7t2t8t18
----------------------------------------
Let p = (1, 18, 17)(2, 4, 15)(3, 8, 10)(5, 7, 6)(9, 20, 22)
   (11, 19, 12)(13, 24,
23)(14, 16, 21) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t18 = Id($)t6t12. Apply at position 3.
Found a new name:  y * x^-2 * y * x * y*t7t14t6t12
*****************************************************************

Looking for relations to apply to t7t19t3
Found 20
----------------------------------------
Let p = (1, 13, 19, 7)(2, 16, 15, 6)(3, 12, 8, 4)(5, 11,
   23, 17)(9, 24, 20,
10)(14, 22, 21, 18) belong to N.
```

```
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t7 = x * y * x^-2t3t4. Apply at position 1.
Found a new name:  x * y * x^-2*t3t4t19t3
----------------------------------------
Let p = (1, 13, 19, 7)(2, 21, 4, 5)(3, 22, 23, 20)(6, 12,
    24, 18)(8, 15, 10,
11)(9, 16, 17, 14) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t7 = x^-1 * y * x * y * x^2 * yt5t15. Apply at position 1.
Found a new name:  x^-1 * y * x * y * x^2 * y*t5t15t19t3
----------------------------------------
Let p = (1, 13, 19, 7)(2, 8, 20, 14)(3, 11, 16, 18)(4, 6,
    9, 23)(5, 22, 24,
15)(10, 12, 21, 17) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t7 = y * x^2 * y * x^-1 * y * xt6t20. Apply at position 1.
Found a new name:  y * x^2 * y * x^-1 * y * x*t6t20t19t3
----------------------------------------
Let p = (1, 13, 19, 7)(2, 23, 18, 10)(3, 9, 21, 15)(4, 14,
    11, 24)(5, 12, 16,
20)(6, 22, 8, 17) belong to N.
```

```
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t7 = y * x^-1t14t17. Apply at position 1.
Found a new name:  y * x^-1*t14t17t19t3
-----------------------------------------
Let p = (1, 13, 19, 7)(2, 24, 17, 3)(4, 10, 22, 16)(5, 9,
   8, 18)(6, 11, 21,
20)(12, 23, 15, 14) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t7 = y * x * y * x^-2 * y * xt10t18. Apply at position 1.
Found a new name:  y * x * y * x^-2 * y * x*t10t18t19t3
-----------------------------------------
Let p = Id($) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t19 = y * x^-2 * y * x^-2 * yt4t8. Apply at position 2.
Found a new name:  y * x^-2 * y * x^-2 * y*t17t4t8t3
-----------------------------------------
Let p = (2, 9, 12, 11, 22)(3, 6, 5, 10, 14)(4, 20, 15, 18,
   17)(8, 21, 24, 23,
16) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
```

```
t19 = x * y * x^-1 * y * x^2 * yt20t21. Apply at position
    2.
Found a new name:  x * y * x^-1 * y * x^2 * y*t4t20t21t3
----------------------------------------
Let p = (2, 12, 22, 9, 11)(3, 5, 14, 6, 10)(4, 15, 17, 20,
    18)(8, 24, 16, 21,
23) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t19 = y * x^2 * y * x^-1 * yt15t24. Apply at position 2.
Found a new name:  y * x^2 * y * x^-1 * y*t20t15t24t3
----------------------------------------
Let p = (2, 11, 9, 22, 12)(3, 10, 6, 14, 5)(4, 18, 20, 17,
    15)(8, 23, 21, 16,
24) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t19 = x * y * x^-1 * y * x * y * xt18t23. Apply at position
    2.
Found a new name:  x * y * x^-1 * y * x * y * x*t15t18t23t3
----------------------------------------
Let p = (2, 22, 11, 12, 9)(3, 14, 10, 5, 6)(4, 17, 18, 15,
    20)(8, 16, 23, 24,
21) belong to N.
```

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t19 = x^-2 * y * xt17t16. Apply at position 2.

Found a new name: x^-2 * y * x*t18t17t16t3

Let p = (1, 7, 19, 13)(2, 5, 4, 21)(3, 20, 23, 22)(6, 18, 24, 12)(8, 11, 10, 15)(9, 14, 17, 16) belong to N.

t1 = t7t19 ==>

(t1)^p = (Id($)t7t19)^p ==>

t7 = Id($)t19t13. Apply at position 1.

Found a new name: t19t13t19t3

Let p = (1, 19)(2, 4)(3, 23)(5, 21)(6, 24)(7, 13)(8, 10)(9, 17)(11, 15)(12, 18)(14, 16)(20, 22) belong to N.

t1 = t7t19 ==>

(t1)^p = (Id($)t7t19)^p ==>

t19 = Id($)t13t1. Apply at position 2.

Found a new name: t7t13t1t3

Let p = (1, 3, 20, 24, 4, 5)(2, 6, 22, 23, 19, 21)(7, 9, 14, 18, 10, 11)(8, 12, 16, 17, 13, 15) belong to N.

```
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t3 = Id($)t9t21. Apply at position 3.
Found a new name:  t7t19t9t21
************************************************************

Looking for relations to apply to t1t9t21
Found 19
----------------------------------------
Let p = (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9,
     20)(10, 24)(11,
17)(14, 21)(18, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = y * x * y * x^2 * y * x^-1t12t3. Apply at position 1.
Found a new name:  y * x * y * x^2 * y * x^-1*t12t3t9t21
----------------------------------------
Let p = (1, 15, 11, 12, 4)(3, 23, 24, 10, 7)(5, 6, 16, 13,
    21)(9, 17, 18, 22,
19) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t9 = y * x^2 * y * x^-1 * yt1t8. Apply at position 2.
Found a new name:  y * x^2 * y * x^-1 * y*t8t1t8t21
```

Let p = (1, 15)(2, 12)(3, 7)(4, 22)(5, 23)(6, 14)(8, 24)(9, 19)(10, 16)(11,
17)(13, 21)(18, 20) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t9 = y * x^-1 * y * x * y * x^-2t22t24. Apply at position 2.

Found a new name: y * x^-1 * y * x * y * x^-2*t13t22t24t21

Let p = (1, 15, 20)(2, 19, 9)(3, 14, 7)(4, 18, 11)(5, 16, 24)(6, 23, 10)(8, 13,
21)(12, 17, 22) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t9 = y * x * y * x^-2 * y * xt18t13. Apply at position 2.

Found a new name: y * x * y * x^-2 * y * x*t3t18t13t21

Let p = (1, 15, 22, 2, 17)(3, 16, 8, 5, 7)(4, 20, 11, 19, 9)(10, 14, 23, 13, 21)
belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t9 = x^-1 * y * x * y * x^2 * yt20t5. Apply at position 2.

Found a new name: x^−1 ∗ y ∗ x ∗ y ∗ x^2 ∗ y∗t14t20t5t21

Let p = (1, 15, 18)(2, 4, 11)(3, 6, 7)(5, 14, 16)(8, 10, 23)(9, 12, 19)(13, 21,
24)(17, 20, 22) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t9 = y ∗ xt11t10. Apply at position 2.

Found a new name: y ∗ x∗t7t11t10t21

Let p = (1, 3, 20, 24, 4, 5)(2, 6, 22, 23, 19, 21)(7, 9, 14, 18, 10, 11)(8, 12,
16, 17, 13, 15) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t21 = y ∗ x ∗ y ∗ x^2 ∗ y ∗ x^−1t5t12. Apply at position 3.

Found a new name: y ∗ x ∗ y ∗ x^2 ∗ y ∗ x^−1∗t23t13t5t12

Let p = (1, 3, 11, 14)(2, 7, 9, 23)(4, 10, 22, 16)(5, 20, 13, 15)(6, 18, 24,
12)(8, 19, 21, 17) belong to N.

Lemma 3: t19 = yx^2yx^−2yt4t8. ==>

(t19)^p = (y ∗ x^−2 ∗ y ∗ x^−2 ∗ yt4t8)^p ==>

t21 = x ∗ y ∗ x^2 ∗ y ∗ x^−1 ∗ yt10t19. Apply at position

3.

Found a new name: x * y * x^2 * y * x^-1 * y*t21t8t10t19

Let p = (1, 3, 18, 8, 17, 10)(2, 23, 4, 13, 15, 24)(5, 22, 7, 9, 6, 20)(11, 16,

19, 21, 12, 14) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t21 = x * y * x^-2t13t17. Apply at position 3.

Found a new name: x * y * x^-2*t5t10t13t17

Let p = (1, 3)(2, 16)(4, 8)(5, 18)(6, 11)(7, 9)(10, 20)(12, 23)(13, 15)(14,

22)(17, 24)(19, 21) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

t21 = y * x^-1 * y * x^2 * yt8t4. Apply at position 3.

Found a new name: y * x^-1 * y * x^2 * y*t13t24t8t4

Let p = (1, 3, 22, 6)(2, 14, 20, 8)(4, 24, 19, 21)(5, 11, 23, 17)(7, 9, 16,

12)(10, 18, 13, 15) belong to N.

Lemma 3: t19 = yx^2yx^-2yt4t8. ==>

(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>

```
t21 = x^2 * y * x^2t24t2. Apply at position 3.
Found a new name:  x^2 * y * x^2*t7t5t24t2
----------------------------------------
Let p = (1, 9, 22)(2, 18, 17)(3, 10, 13)(4, 19, 15)(5, 8,
   6)(7, 21, 16)(11, 20,
12)(14, 24, 23) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t9 = Id($)t21t15. Apply at position 2.
Found a new name:  t1t21t15t21
----------------------------------------
Let p = (1, 21)(2, 5)(3, 19)(4, 6)(7, 15)(8, 11)(9, 13)(10,
    12)(14, 17)(16,
18)(20, 23)(22, 24) belong to N.
t1 = t7t19 ==>
(t1)^p = (Id($)t7t19)^p ==>
t21 = Id($)t15t3. Apply at position 3.
Found a new name:  t1t9t15t3
************************************************************************
```

PROOF

```
************************************************************************

Start with  t1t3
```

```
-----------------------------------------------------
Apply the following at t word position 1:
Let p = (1, 19)(2, 15)(3, 8)(4, 12)(5, 23)(6, 16)(7, 13)(9,
    20)(10, 24)(11,
17)(14, 21)(18, 22) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t1 = y * x * y * x^2 * y * x^-1t12t3 ==>
t1t3 = y * x * y * x^2 * y * x^-1*t12t3t3.
-----------------------------------------------------
Apply the following at t word position 1:
Let p = (1, 18, 15)(2, 11, 4)(3, 7, 6)(5, 16, 14)(8, 23,
   10)(9, 19, 12)(13, 24,
21)(17, 22, 20) belong to N.
Lemma 3: t19 = yx^2yx^-2yt4t8. ==>
(t19)^p = (y * x^-2 * y * x^-2 * yt4t8)^p ==>
t12 = x^2 * y * x * y * x^-1 * yt2t23 ==>
y * x * y * x^2 * y * x^-1 * t12t3t3 = x^-1 * y * x^2 * y*
   t2t23t3t3.
-----------------------------------------------------
Reduce in the following way:
x^-1 * y * x^2 * y*t2t23t3t3
x^-1 * y * x^2 * y*t2t23t9
true */
```

Appendix H

MAGMA CODE: DCE $S_3 \times A_5$ Over 15:2 and 10:2

```
/*From DCE over N see CSUSB Thesis, December 2017 by
   Michelle Yeo */
/* Isomorphism type */
S:=Sym(10);
xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10);
yy:=S!(2, 10)(3, 9)(4, 8)(5, 7);
N:=sub<S|xx,yy>;
NN<x,y>:=Group<x,y|x^10,y^2,(x*y)^2>;

a:=0;b:=0;c:=3;d:=0;e:=3;f:=0;
```

```
G<x,y,t>:=Group<x,y,t|x^10,y^2,(x*y)^2,t^2,(t,y),
(x*t)^a, (x*y*t^x)^b, (x^2*y*t*t^x)^c, (t*t^x*t^(x^3))^d,
(x*t^x)^e,(y*t)^f>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
CompositionFactors(G1);
IN:=sub<G1|f(x),f(y)>;
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
M:=MaximalSubgroups(G1);
for i in [1..#M] do #M[i]`subgroup/20; end for;

for g in DD[6] do
if DD[6] eq sub<G1|f(x),f(y),g> then
Sch:=SchreierSystem(G, sub<G|Id(G)>);
for i in [1..#Sch] do
if g eq f(Sch[i]) then Sch[i]; end if;
end for;
break g;
end if;
end for;

H:=sub<G|x,y, t * x^5 * t * x>;
#DoubleCosets(G,H,sub<G|x,y>);
```

```
CompositionFactors(G1);
NL:=NormalLattice(G1);/*
[6]  Order 360  Length 1  Maximal Subgroups: 3 5
---
[5]  Order 180  Length 1  Maximal Subgroups: 2 4
---
[4]  Order 60   Length 1  Maximal Subgroups: 1
---
[3]  Order 6    Length 1  Maximal Subgroups: 2
---
[2]  Order 3    Length 1  Maximal Subgroups: 1
---
[1]  Order 1    Length 1  Maximal Subgroups:*/
NL;/*
 G
    |  Alternating(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
    1 */
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
```

```
end for;
/* 2 */
q,ff:=quo<G1|NL[2]>;
q;
/*q  = quotient group G1/N[2] of order 120. G1 cannot be
    the direct product to N[2] since G1 has no normal
    subgroup of order 120. Which means G1 is an extension of
     NL[2] by G1/NL[2] and G:=3:NL[2]. */

N:=q;
CompositionFactors(N);/*
G
     |   Alternating(5)
     *
     |   Cyclic(2)
     1    */
NL:=NormalLattice(N); /*
[4]   Order 120   Length 1   Maximal Subgroups: 2 3
---
[3]   Order 60    Length 1   Maximal Subgroups: 1
---
[2]   Order 2     Length 1   Maximal Subgroups: 1
---
[1]   Order 1     Length 1   Maximal Subgroups: */
```

```
for i in [1..#NL] do
if IsAbelian(NL[i]) then i; end if;
end for;  /* 2 */

q,ff:=quo<N|NL[2]>;
q;
/* TOD iso = s3xA5. And iso of IH = z15:2 */
/*Since the order of q = 60 and N has a normal subgroup of
   order 60 ==> N = 2  Alt  (5). */

/DCE Using Magma
   **********************************************************/



S:=Sym(10);
xx:=S!(1, 2, 3, 4, 5, 6, 7, 8, 9, 10);
yy:=S!(2, 10)(3, 9)(4, 8)(5, 7);
N:=sub<S|xx,yy>;
NN<x,y>:=Group<x,y|x^10,y^2,(x*y)^2>;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
word:=function(Perm)
for w in Sch do
```

```
        seq := Eltseq(w);
        p:= Id(N);
        for j in seq do
                if j eq 1 then p:=p*xx; end if;
if j eq -1 then p:=p*xx^-1; end if;
if j eq 2 then p:=p*yy; end if;
if j eq -2 then p:=p*yy^-1; end if;
        end for;
        if Perm eq p then return w; end if;
end for;
end function;
ConjugationWord:= function(subscript)
        size:=9999999;
        w:=word(N.1);
for n in N do
        if n eq Id(N) then
                continue;
        end if;
if 1^n eq subscript then
        tmp:= word(n);
        if size gt #tmp then
                size:=#tmp;
                w:= tmp;
        end if;
```

```
end if;
end for;
return w;
end function;

a:=0;b:=0;c:=3;d:=0;e:=3;f:=0;
G<x,y,t>:=Group<x,y,t|x^10,y^2,(x*y)^2,t^2,(t,y),
(x*t)^a, (x*y*t^x)^b, (x^2*y*t*t^x)^c, (t*t^x*t^(x^3))^d,
(x*t^x)^e,(y*t)^f>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);

ts:= [ Id(G1): i in [1 .. 10] ];
for i in [2..10] do
  "ts[" cat Sprint( i) cat "]  :=" cat  "f(t^(" cat Sprint
     (ConjugationWord(i)) cat "));";
end for;

ts[1] :=f(t);
ts[2]  :=f(t^(x));
ts[3]  :=f(t^(x^2));
ts[4]  :=f(t^(x^3));
ts[5]  :=f(t^(x^4));
ts[6]  :=f(t^(x^5));
```

```
ts[7]   :=f(t^(x^-4));
ts[8]   :=f(t^(x^-3));
ts[9]   :=f(t^(x^-2));
ts[10]   :=f(t^(x^-1));

H:=sub<G|x,y, t * x^5 * t * x>;
IN:=sub<G1| f(x), f(y)>;
IH:=sub<G1|f(x), f(y), f(t * x^5 * t * x)>;
#DoubleCosets(G,H,sub<G|x,y>);
DoubleCosets(G,H,sub<G|x,y>); /*
{ <GrpFP: H, Id(G), GrpFP>, <GrpFP: H, t, GrpFP> } */

DC:= [f(Id(G)), f(t)];
FindDCNumber:= function(element)
for i in [1..#DC] do
                for m in IH do
for n in IN do
if element eq m*(DC[i])^n then
                                        return i;
                                        end if;
end for;
   end for;
end for;
end function;
```

```
Orbits(N);/*
GSet{@ 1, 2, 3, 10, 4, 9, 5, 8, 6, 7 @} */
N1:= Stabilizer(N, 1);
N1;
Orbits(N1); /*
 GSet{@ 1 @},
     GSet{@ 6 @},
     GSet{@ 2, 10 @},
     GSet{@ 3, 9 @},
     GSet{@ 4, 8 @},
     GSet{@ 5, 7 @} */
FindDCNumber(ts[1]*ts[1]); /* 1 */
FindDCNumber(ts[1]*ts[6]); /* 1 */
FindDCNumber(ts[1]*ts[2]);  /* 2 */
FindDCNumber(ts[1]*ts[3]); /* 2 */
FindDCNumber(ts[1]*ts[4 ]); /* 2 */
FindDCNumber(ts[1]*ts[5]); /* 2 */

TPerm:= function(Tset)
    perm:= ts[Tset[1]];
    for i in [2..#Tset] do
      perm:= perm*ts[Tset[i]];
    end for;
```

```
    return perm;
end function;

EquivalentCosets:= function(Tset)
    perm:=TPerm( Tset);
    TSets :=[];
    for g in IN do
        for n in N do
            perm2:= TPerm(Tset^n);
            if perm eq g*perm2 then
                TSets:= TSets cat [Tset^n];
            end if;
        end for;
    end for;
    return TSets;
end function;

EquivalentCosetsH:= function(Tset)
    perm:=TPerm( Tset);
    TSets :=[];
    for g in IH do
        for n in N do
            perm2:= TPerm(Tset^n);
            if perm eq g*perm2 then
```

```
            TSets:= TSets cat [Tset^n];
         end if;
      end for;
   end for;
   return TSets;
end function;

FindFromToPerms:= function(Tset1, Tset2)
   NPerms:=[];
   for n in N do
      if Tset1^n eq Tset2 then
         NPerms:= NPerms cat [n];
      end if;
   end for;
      return NPerms;
end function;

EquivalentCosets([7,2]);/* [ 7, 2 ],
    [ 9, 4 ],
    [ 5, 10 ],
    [ 1, 6 ],
    [ 3, 8 ],
    [ 5, 10 ],
    [ 7, 2 ],
```

```
    [ 3, 8 ],
    [ 9, 4 ],
    [ 1, 6 ] */
FindFromToPerms([7,2], [1,6]); /*(1, 5, 9, 3, 7)(2, 6, 10,
  4, 8),
    (1, 7)(2, 6)(3, 5)(8, 10) */
```

Appendix I

MAGMA: DCE of $(A_5)^2$:2 Over $A_5 : 2$

```
S:=Sym(24);
xx:=S!(1, 2, 4, 12, 21)(3, 7, 8, 10, 18)(6, 15, 19, 20, 22)
    (9, 13, 14, 16, 24);
yy:=S!(1, 15)(2, 5)(3, 13)(4, 6)(7, 21)(8, 11)(9, 19)(10,
    12)(14, 17)(16, 18)(20,23)(22, 24);

N:=sub<S|xx,yy>;
NN<x,y>:=Group<x,y|x^5, y^2, (x^-1 * y)^4, (x * y * x^-2 *
    y * x)^2 >;

/*a:=0;b:=0;c:=0;d:=4;e:=0; f:=4;
```

```
G<x,y,t>:=Group<x,y,t|x^5,y^2,(x^-1*y)^4,(x*y*x^-2*y*x)^2,t
    ^5, (t,x^2*y*x*y),t^(y*x)= t^2, ((y*x*y)*t^(x^4*x^-1)^2)
    ^a, (y*x^2*t)^b,(y*x^2*t^(x^-1))^c,(y*x^2*t^(y*x^2))^d,(
    y*x^2*t^2)^f>;*/

G<x,y,t>:=Group<x,y,t|x^5,y^2,(x^-1*y)^4,(x*y*x^-2*y*x)^2,t
    ^5, (t,x^2*y*x*y),t^(y*x)= t^2,(y*x^2*t^(y*x^2))^4,(y*x
    ^2*t^2)^4>;

f,G1,k:=CosetAction(G,sub<G|x,y>);
G1;
IN:=sub<G1|f(x),f(y)>;
/*(2, 4, 12, 19, 7)(3, 8, 21, 28, 10)(5, 6, 18, 37, 16)(9,
    11, 29, 43, 25)(14, 15, 34, 48, 32)(17, 38, 50, 58,
        40)(20, 26, 31, 35, 30)(23, 51, 45, 49, 42)(24, 53,
            46, 59, 36)(27, 55, 44, 52, 57)(33, 54, 56, 60,
            39),

    (2, 5)(3, 9)(4, 13)(6, 10)(7, 11)(8, 22)(12, 16)(15,
       35)(17, 39)(18, 41)(19, 43)(20, 45)(21, 25)(23, 52)
       (24,
        53)(26, 30)(27, 48)(28, 37)(29, 47)(31, 32)(33, 40)
            (34, 57)(36, 58)(38, 46)(42, 55)(49, 51)(56, 59)
```

```
(1, 2, 6, 11, 3)(4, 12, 30, 33, 14)(5, 15, 31, 13, 17)
   (7, 8, 23, 46, 20)(9, 24, 43, 55, 26)(10, 27, 56,
   51,
    28)(16, 36, 18, 42, 32)(19, 44, 40, 35, 41)(21, 47,
        45, 58, 48)(22, 49, 57, 29, 50)(25, 34, 52, 37,
        54)(38,
    53, 60, 39, 59) */

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
word:=function(Perm)
for w in Sch do
        seq := Eltseq(w);
        p:= Id(N);
        for j in seq do
                if j eq 1 then p:=p*xx; end if;
if j eq -1 then p:=p*xx^-1; end if;
if j eq 2 then p:=p*yy; end if;
if j eq -2 then p:=p*yy^-1; end if;
        end for;
        if Perm eq p then return w; end if;
end for;
end function;

InWord:=function(Perm)
```

```
for w in Sch do
        seq := Eltseq(w);
        p:= Id(IN);
        for j in seq do
                if j eq 1 then p:=p*f(x); end if;
if j eq -1 then p:=p*f(x)^-1; end if;
if j eq 2 then p:=p*f(y); end if;
if j eq -2 then p:=p*f(y)^-1; end if;
        end for;
        if Perm eq p then return w; end if;
end for;
end function;

GetConjugationWordForSubscript:= function(subscript)
        size:=9999999;
        w:=word(N.1);
for n in N do
        if n eq Id(N) then
                continue;
        end if;
if 1^n eq subscript then
        tmp:= word(n);
        if size gt #tmp then
                size:=#tmp;
```

```
                w:= tmp;
            end if;
        end if;
        end for;
        return w;
        end function;

        ts:= [ Id(G1): i in [1 .. 24] ];
        ts[1]:=f(t);
        ts[2]:=f(t^(x));
        ts[3]:=f(t^((x*y*x)^-1));
        ts[4]:=f(t^(x^2));
        ts[5]:=f((t^x)^y);
        ts[6]:=f((t^(x^2))^y);
        ts[19]:= ts[1]^2;
        ts[20]:= ts[2]^2;
        ts[21]:= ts[3]^2;
        ts[22]:= ts[4]^2;
        ts[23]:= ts[5]^2;
        ts[24]:= ts[6]^2;
        ts[7]:= ts[1]^3;
        ts[8]:= ts[2]^3;
        ts[9]:= ts[3]^3;
        ts[10]:= ts[4]^3;
```

```
ts[11]:= ts[5]^3;
ts[12]:= ts[6]^3;
ts[13]:= ts[1]^4;
ts[14]:= ts[2]^4;
ts[15]:= ts[3]^4;
ts[16]:= ts[4]^4;
ts[17]:= ts[5]^4;
ts[18]:= ts[6]^4;

ts[19] eq ts[1]^2;
ts[20] eq ts[2]^2;
ts[21] eq ts[3]^2;
ts[22] eq ts[4]^2;
ts[23] eq ts[5]^2;
ts[24] eq ts[6]^2;
ts[7] eq ts[1]^3;
ts[8] eq ts[2]^3;
ts[9] eq ts[3]^3;
ts[10] eq ts[4]^3;
ts[11] eq ts[5]^3;
ts[12] eq ts[6]^3;
ts[13] eq ts[1]^4;
ts[14] eq ts[2]^4;
ts[15] eq ts[3]^4;
```

```
ts[16] eq ts[4]^4;
ts[17] eq ts[5]^4;
ts[18] eq ts[6]^4;

for i in [2..24] do
  "ts[" cat Sprint( i) cat "] eq " cat "f(t^(" cat
     Sprint(GetConjugationWordForSubscript(i)) cat ");";
end for;

ts[2]  eq f(t^(x));
ts[3]  eq f(t^(x^-1 * y * x^-1));
ts[4]  eq f(t^(x^2));
ts[5]  eq f(t^(x * y));
ts[6]  eq f(t^(y * x^-1));
ts[7]  eq f(t^(x^-1 * y));
ts[8]  eq f(t^(y^x));
ts[9]  eq f(t^(y * x * y));
ts[10]  eq f(t^(x^-2 * y));
ts[11]  eq f(t^(x^-1 * y * x * y));
ts[12]  eq f(t^(x^-2));
ts[13]  eq f(t^((y * x)^2));
ts[14]  eq f(t^(y * x * y * x^2));
ts[15]  eq f(t^(y));
ts[16]  eq f(t^(y * x * y * x^-2));
```

```
ts[17] eq f(t^(y * x * y * x^2 * y));
ts[18] eq f(t^(x^-1 * y * x^-2));
ts[19] eq f(t^(y * x));
ts[20] eq f(t^(y * x^2));
ts[21] eq f(t^(x^-1));
ts[22] eq f(t^(y * x^-2));
ts[23] eq f(t^(y * x^2 * y));
ts[24] eq f(t^(y * x * y * x^-1));

/* print the inverses */
for i,j in  [1..24] do
        if ts[i] eq ts[j]^-1 then
                "t" cat Sprint(i) cat " = " cat "t" cat
                   Sprint(j) cat "^-1";
        end if;
end for;

DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);

/*{ <GrpFP, Id(G), GrpFP>, <GrpFP, t * y * t^-1, GrpFP>, <
   GrpFP, t, GrpFP>,
<GrpFP, t * x * t^-1, GrpFP> }*/
```

```
DC:= [f(Id(G)), f(t), f(t * y * t^-1), f( t * x * t^-1)];
DC;
/*[     Id(G1), (1, 2, 6, 11, 3)(4, 12, 30, 33, 14)(5, 15,
   31, 13, 17)(7, 8, 23, 46, 20)(9, 24, 43, 55, 26)(10, 27,
    56, 51, 28)(16, 36, 18, 42, 32)(19, 44, 40, 35, 41)(21,
    47, 45, 58, 48)(22, 49, 57, 29, 50)(25, 34, 52, 37, 54)
   (38, 53,  60, 39, 59),
    (1, 17)(2, 28)(4, 32)(5, 40)(6, 20)(7, 50)(8, 34)(9,
       38)(10, 58)(11, 26)(12,  55)(13, 60)(14, 31)(15, 42)
       (16, 45)(18, 43)(21, 57)(22, 56)(23, 59)(24,  41)
       (25, 49)(27, 39)(30, 44)(35, 36)(46, 47)(48, 54)(51,
        52),
    (1, 14, 4, 41, 20)(2, 36, 52, 32, 17)(5, 25, 58, 42,
       33)(6, 57, 24, 54,
        26)(7, 48, 51, 28, 11)(8, 56, 47, 22, 18)(9, 38,
           23, 39, 16)(10, 43, 19,
        34, 49)(12, 46, 55, 15, 40)(13, 59, 29, 45, 44)(27,
            53, 60, 30, 37)]*/


#N; /* 120 */
[ 120/120, 2880/120, 2400/120, 1800/120 ];

FindDCGroupNumber := function(DC, element, IN)
```

```
    for i in [1..#DC] do
        for m,n in IN do
            if element eq m*(DC[i])^n then
                return i;
            end if;
        end for;
    end for;
end function;

ConvertToPermutation:= function(Tset)
    perm:= ts[Tset[1]];
    for i in [2..#Tset] do
      perm:= perm*ts[Tset[i]];
    end for;
    return perm;
end function;
TPerm:=ConvertToPermutation;

Orbits(N);
N1:= Stabilizer(N, 1);
N1;
Orbits(N1);
/* GSet{@ 1 @},
     GSet{@ 7 @},
```

```
    GSet{@ 13 @},
    GSet{@ 19 @},
    GSet{@ 2, 21, 24, 23, 16 @},
    GSet{@ 3, 6, 5, 22, 8 @},
    GSet{@ 4, 14, 9, 12, 11 @},
    GSet{@ 10, 20, 15, 18, 17 @} */

FindDCGroupNumber (DC,  ts[1]*ts[1], IN); /*2*/
FindDCGroupNumber (DC,  ts[1]*ts[7], IN);/*2*/
FindDCGroupNumber (DC,  ts[1]*ts[13], IN);/*1*/
FindDCGroupNumber (DC,  ts[1]*ts[19], IN);/*2*/
FindDCGroupNumber (DC,  ts[1]*ts[2], IN); /*2*/
FindDCGroupNumber (DC,  ts[1]*ts[3], IN);/*3*/
FindDCGroupNumber (DC,  ts[1]*ts[4], IN);/*4*/
FindDCGroupNumber (DC,  ts[1]*ts[10], IN);/*4*/

FindStabilizerOfNums:= function(num)
        ArrPerm:=[];
        for n in N do
                if num^n eq num then
        ArrPerm:= ArrPerm cat [n];
end if;
        end for;
        return ArrPerm;
```

```
end function;

EquivalentCosets:= function(Tset)
    perm:=ConvertToPermutation( Tset);
    TSets :=[];
    for g in IN do
        for n in N do
            perm2:= ConvertToPermutation(Tset^n);
            if perm eq g*perm2 then
                TSets:= TSets cat [Tset^n];
            end if;
        end for;
    end for;
    return TSets;
end function;

StabilizingGroup := function(N, IN, ts, Tword)
    TWords:=EquivalentCosets(Tword);
    group:=Stabiliser(N, Tword);
    for i in [2..#TWords] do
        for n in N do
            if Tword^n eq TWords[i] then
                group:=sub<N|group,n>;
            end if;
```

```
    end for;
  end for;
  return group;
end function;

N13s:= StabilizingGroup(N, IN, ts, [1,3]);
N13s;
/* < (1, 20)(2, 7)(3, 11)(4, 18)(5, 21)(6, 16)(8, 13)(9,
  17)(10, 24)(12, 22)(14, 19)(15, 23)
    (1, 7, 13, 19)(2, 20, 14, 8)(3, 23, 10, 12)(4, 6, 21,
      17)(5, 16, 18, 9)(11, 22, 24, 15)
    (1, 14)(2, 13)(3, 22)(4, 9)(5, 17)(6, 18)(7, 20)(8, 19)
      (10, 15)(11, 23)(12, 24)(16, 21)
    (1, 8)(2, 19)(3, 24)(4, 5)(6, 9)(7, 14)(10, 11)(12, 15)
      (13, 20)(16, 17)(18, 21)(22, 23)
    (1, 13)(2, 14)(3, 10)(4, 21)(5, 18)(6, 17)(7, 19)(8,
      20)(9, 16)(11, 24)(12, 23)(15, 22)
    (1, 2)(3, 15)(4, 16)(5, 6)(7, 8)(9, 21)(10, 22)(11, 12)
      (13, 14)(17, 18)(19, 20)(23, 24)
    (1, 19, 13, 7)(2, 8, 14, 20)(3, 12, 10, 23)(4, 17, 21,
      6)(5, 9, 18, 16)(11, 15, 24, 22) > */

Orbits(N13s);
/* GSet{@ 1, 20, 7, 14, 8, 13, 2, 19 @},
```

```
    GSet{@ 3, 11, 23, 22, 24, 10, 15, 12 @},
    GSet{@ 4, 18, 6, 9, 5, 21, 16, 17 @} */

FindDCGroupNumber (DC,  ts[1]*ts[3]*ts[1], IN); /*3*/
FindDCGroupNumber (DC,  ts[1]*ts[3]*ts[3], IN); /2*/
FindDCGroupNumber (DC,  ts[1]*ts[3]*ts[4], IN); /*4*/



N14s:= StabilizingGroup(N, IN, ts, [1,4]);
N14s;
/* (1, 10, 24, 3, 2, 17)(4, 18, 21, 20, 11, 19)(5, 13, 22,
   12, 15, 14)(6, 9, 8, 23, 7, 16)
    (1, 3)(2, 10)(4, 20)(5, 12)(6, 23)(7, 9)(8, 16)(11, 18)
       (13, 15)(14, 22)(17, 24)(19, 21)
    (1, 17, 2, 3, 24, 10)(4, 19, 11, 20, 21, 18)(5, 14, 15,
        12, 22, 13)(6, 16, 7, 23, 8, 9)
    (1, 2, 24)(3, 10, 17)(4, 11, 21)(5, 15, 22)(6, 7, 8)(9,
        16, 23)(12, 13, 14)(18, 19, 20)
    (1, 24, 2)(3, 17, 10)(4, 21, 11)(5, 22, 15)(6, 8, 7)(9,
        23, 16)(12, 14, 13)(18, 20, 19) */

Orbits(N14s);
/* GSet{@ 1, 10, 3, 17, 2, 24 @},
    GSet{@ 4, 18, 20, 19, 11, 21 @},
```

```
    GSet{@ 5, 13, 12, 14, 15, 22 @},
    GSet{@ 6, 9, 23, 16, 7, 8 @}*/

FindDCGroupNumber (DC,  ts[1]*ts[4]*ts[1], IN); /* 2 */
FindDCGroupNumber (DC,  ts[1]*ts[4]*ts[4], IN); /* 3 */
FindDCGroupNumber (DC,  ts[1]*ts[4]*ts[5], IN); /* 4 */
FindDCGroupNumber (DC,  ts[1]*ts[4]*ts[6], IN); /* 2*/
GetStringT := function(set)
          Str1 := "";
for i in [1 .. #set] do
       Str1:= Str1 cat "t";
Str1:=  Str1 cat IntegerToString(set[i]);
end for;
 return Str1;
end function;
TStr:=GetStringT;

/* PrintEquivalentCosetsTo([13]); */
PrintEquivalentCosetsTo:= function(N, IN, ts,Tset)
       ECs:=EquivalentCosets(Tset);
str:= "N";
       for i in [1..#ECs] do
              str := str cat GetStringT(ECs[i]);
              edge:=" = N";
```

```
                if i eq #ECs then
                        edge:= "\t";
                end if;
str := str cat edge;
        end for;
        return str;
end function;

/* Nt1t3 = Nt20t11 = Nt7t23 = Nt14t22 = Nt8t24 = Nt13t10 =
   Nt2t15 = Nt19t12 */

PrintEquivalentRelations := procedure(Tset)
perm:=ConvertToPermutation( ts, Tset);
TSets :=[];
for g in IN do
for n in N do
        if n eq Id(N) or g eq Id(IN) then continue;  end if
           ;
perm2:= ConvertToPermutation( Tset^n);
if perm eq g*perm2 then
GetStringT(Tset) cat " = " cat Sprint(InWord(g)) cat
   GetStringT(Tset^n);
GetStringT(Tset^n) cat " = " cat Sprint(InWord(g^-1)) cat
   GetStringT(Tset);
```

```
end if;
end for;
end for;
end procedure;

FindEquivalentRelations := function(Tset, Tset2)
perm:=ConvertToPermutation( Tset);
TSets :=[];
str:="";
for g in IN do
perm2:= ConvertToPermutation(Tset2);
if perm eq g*perm2 then
str:= str cat GetStringT(Tset) cat " = " cat Sprint(InWord(
   g)) cat GetStringT(Tset2) cat "\n";
end if;
end for;
return str;
end function;

FindFromToPerms:= function(Tset1, Tset2)
   NPerms:=[];
   for n in N do
      if Tset1^n eq Tset2 then
         NPerms:= NPerms cat [n];
```

```
        end if;
    end for;
        return NPerms;
end function;

Percent := procedure(curr, total)
          percent:= 100.00*curr/total;
printf "%.2o%%", percent;
end procedure;

FindDCRelations := function(tset, tset2)
          perm:= TPerm(tset);
          perm2 := TPerm(tset2);
          dc := FindDCGroupNumber(DC, perm2, IN);
          str:="\n";
          sep:="
              ---------------------------------------------\
              n";
          cur:= 0;          Total := #IN*#IN;          sep;
for m,n in IN do
          cur:=cur+1;
          printf "
                                                                    "
```

```
            cat CodeToString(13);
        Percent(cur,Total);
if perm eq m*perm2^n then
str:= str cat TStr(tset) cat " = " cat Sprint(InWord(m))
   cat TStr(tset2) cat "^"  cat Sprint(InWord(n));
str:= str cat  "\n" cat sep;
end if;
end for;
return str;
end function;

InverseIndex:= function(Index)
for i in [1..24]  do
                if ts[Index]^-1 eq ts[i] then return i; end
                    if;
        end for;
end  function;

SplitT:= function(tsIndex)
        for n in N do
                if 7^n eq tsIndex then
                        return [1,19]^n;
                end if;
        end for;
```

```
        return [0,0];
end function; /* SplitT(2); returns [20,14]     t2 =
   t20t14 */

Reduce2Ts:= function(tsIndex1, tsIndex2)
        if tsIndex1 eq 0 and tsIndex2 eq 0 then return [0];
            end if;
        if tsIndex1 eq 0 then return [tsIndex2]; end if;
if tsIndex2 eq 0 then return [tsIndex1]; end if;
for i in [1..24]  do
                if ts[tsIndex1]*ts[tsIndex2] eq ts[i] then
                   return [i]; end if;
if ts[tsIndex1]*ts[tsIndex2] eq ts[i]^5 then return [0];
   end if;
        end for;
        return [tsIndex1, tsIndex2];
end function;

ReduceTs:= function(Tset)
        steps:=[Tset];
        OK:=true;
        while OK do
        size:=#Tset;
        for i in [1..size] do
```

```
                j:=i+1;
                OK:=false;
                if j le size then
                        r:= Reduce2Ts(Tset[i],Tset[j]);
                        if #r eq 1 then
                                Tset[i]:= r[1];
                                Tset:=Remove(Tset, j);
                                steps:= steps cat [Tset];
OK:=true;
                                break;
                        end if;
                end if;
        end for;
end while;
return steps;
end  function;

Word2NPerm:= function(letters)
        seq := Eltseq(letters);
p:= Id(N);
for j in seq do
        if j eq 1 then p:=p*xx; end if;
if j eq -1 then p:=p*xx^-1; end if;
if j eq 2 then p:=p*yy; end if;
```

```
if j eq -2 then p:=p*yy^-1; end if;
end for;
return p;
 end function;

TryToReduce3to1:= procedure(Index1, Index2, Index3 )
        sep:="
           -------------------------------------------------------------
           ";
for n in N do
if [Index1, Index2, Index3] eq [17,14,19]^n then
                        GetStringT([Index1, Index2, Index3
                           ]) cat  " = (" cat GetStringT
                           ([17,14,19]) cat ")^" cat Sprint
                           (word(n));
FindEquivalentRelations([Index1, Index2, Index3],[16]^n);
n; sep;
                end if;
                if [Index1, Index2, Index3] eq [21,6,23]^n
                   then
                        GetStringT([Index1, Index2, Index3
                           ]) cat  " = (" cat GetStringT
                           ([21,6,23]) cat ")^" cat Sprint(
                           word(n));
```

```
FindEquivalentRelations([Index1, Index2, Index3],[8]^n);
n; sep;
                end if;
        end for;
end procedure;

/*Proofs DCE using magma */
ww:=yy*xx*xx;
/* Proof of addition relation w4t24t17t14t19 = e*/
 f(y*x*x)^4*ts[24]*ts[17]*ts[14]*ts[19]; /* Id(G1) */
/* Proof of addition relation  w4t21t6t23t20 = e*/
f(y*x*x)^4*ts[21]*ts[6]*ts[23]*ts[20]; /*Id(G ) */

/*Lemma 1:  t24t17 = w2t7t2*/
ts[24]*ts[17] eq f(y*x*x)^2*ts[7]*ts[2];  /* true */
/* Lemma 1 results */
for n in N do
"Conjugating Lemma 1 by " cat  Sprint(n) cat "  ==> "  cat
   GetStringT([24,17]^n) cat " = " cat   Sprint(word((ww^2)
   ^n)) cat GetStringT([7,2]^n) cat "."  ;
end for;

/* Lemma 2 results */
for n in N do
```

```
"Conjugating Lemma 2 by " cat  Sprint(n) cat " ==>  " cat
   GetStringT([21,6]^n) cat " = " cat   Sprint(word((ww^2)^
   n)) cat GetStringT([8,11]^n) cat "."  ;
end for;

/*Lemma 2:  t21t6 = w2t8t11*/
ts[21]*ts[6] eq f(y*x*x)^2*ts[8]*ts[11]; /* true */

/*Derived from Lemma 1 */
TPerm([24,17,14])  eq f(y*x*x)^2*ts[7]; /* true */
TPerm([15,24,17])  eq f(y*x*x)^2*ts[2]; /* true */
TPerm([7,2,5])  eq f(y*x*x)^4*ts[24]; /* true */
TPerm([16,7,2])  eq f(y*x*x)^4*ts[17]; /* true */
TPerm([17,14,19])  eq f(y*x*x)^2*ts[16]; /* true */

/*Proof Nt1t2      [1] */
/*for i in [1..24] do FindEquivalentRelations([1,2],[i]);
   end for;  */
SplitT(1);
SplitT(2);
FindEquivalentRelations([1,2], [5,15,4,14]);
FindEquivalentRelations([1,2], [16,8,3,4,14]);
FindEquivalentRelations([1,2], [1,17,24,5,14]);
FindEquivalentRelations([1,2], [12,7,23,9,18]);
```

```
FindEquivalentRelations([1,2], [2,2,15,9,18]);
FindEquivalentRelations([1,2], [20,21,18]);
FindEquivalentRelations([1,2], [6]);

/*Derived from Lemma 2*/
TPerm([21,6,23])  eq f(y*x*x)^2*ts[8]; /* true */
TPerm([10,21,6])  eq f(y*x*x)^2*ts[11]; /* true */
TPerm([8,11,18])  eq f(y*x*x)^4*ts[21]; /* true */
TPerm([13,8,11])  eq f(y*x*x)^4*ts[6];

 EquivalentCosets([1,3]); /* [[ 1, 3 ],    [ 20, 11 ],    [
     7, 23 ],    [ 14, 22 ],    [ 8, 24 ],    [ 13, 10 ],
       [ 2, 15 ],    [ 19, 12 ]] */

/* Lemma 2 results */
for n in N do
  GetStringT([21,6]^n) cat " = " cat   Sprint(word(ww^n))
     cat GetStringT([8,11]^n) cat ", conjugating by" cat
     Sprint(n) cat "." ;
end for;

/*Lemma 3: t1t2 = x-2t16 Results */
for n in N do
  GetStringT([1,2]^n) cat " = " cat   Sprint(word((xx^-2)^n
```

```
    )) cat GetStringT([16]^n) cat ", conjugating by" cat
    Sprint(n) cat "." ;
end for;

/*Right Coset already proven to be    [1,4] */
RCs14:= [ [1, 20], [1,10], [1,18] , [1,17], [1,15] , [1,4],
    [1,12], [1,11], [1, 9] ];
for n in N do
        if [17,19]^n in RCs14 then [17,19]^n, n; end if;
end for;
/* Double Cosets  [1,4] */

/* Isomorphic Types */
/*a:=0;b:=0;c:=0;d:=4;e:=0; f:=4;
OG<x,y,t>:=Group<x,y,t|x^5,y^2,(x^-1*y)^4,(x*y*x^-2*y*x)^2,
   t^5, (t,x^2*y*x*y),t^(y*x)= t^2, ((y*x*y)*t^(x^4*x^-1)
   ^2)^a, (y*x^2*t)^b,(y*x^2*t^(x^-1))^c,(y*x^2*t^(y*x^2))^
   d,(y*x^2*t^2)^f>;
Of,OG1,Ok:=CosetAction(G,sub<G|x,y>);*/

G<x,y,t>:=Group<x,y,t|x^5,y^2,(x^-1*y)^4,(x*y*x^-2*y*x)^2,t
   ^5, (t,x^2*y*x*y),t^(y*x)= t^2,(y*x^2*t^(y*x^2))^4,(y*x
   ^2*t^2)^4>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
```

```
G1;
CompositionFactors(G1);
NL:=NormalLattice(G1);
NL;

for i in [1..#NL] do
        if IsAbelian(NL[i]) then
                i;
        end if;
end for;

IN:=sub<G1|f(x),f(y)>;

#G1, #NL[4], #G1/#NL[4];
/* NL[4] is normal in G1 but there is no normal subgroup of
    order 2. This means that G1 = NL[4]:2 */

/*
Finitely presented group on 2 generators
Relations
    $.1^2 = Id($)
    $.2^5 = Id($)
    ($.2 * $.1 * $.2 * $.1 * $.2 * $.1 * $.2^-1 * $.1 * $
        .2)^2 = Id($)
```

```
    ($.2 * $.1 * $.2^-1 * $.1 * $.2 * $.1 * $.2 * $.1 * $
       .2)^2 = Id($)
    $.2^-1 * $.1 * $.2^-1 * $.1 * $.2 * $.1 * $.2 * $.1 * $
       .2^-2 * $.1 * $.2 * $.1 * $.2 * $.1 * $.2^-1 * $.1 *
    $.2^-1 * $.1 = Id($)
    $.2 * $.1 * $.2^-2 * $.1 * $.2^2 * $.1 * $.2^-2 * $.1 *
        $.2^-2 * $.1 * $.2^-1 * $.1 * $.2^-1 * $.1 * $.2^-2
        *
    $.1 * $.2^-2 * $.1 * $.2 = Id($)*/

q,ff:=quo<G1|NL[4]>;
q;
FPGroup(NL[4]);
T:= Transversal(G1,NL[4]);
T;

E:= G1!(2, 5)(3, 9)(4, 13)(6, 10)(7, 11)(8, 22)(12, 16)(15,
    35)(17, 39)(18, 41)(19, 43)(20, 45)(21, 25)(23, 52)(24,
    53)(26, 30)(27, 48)(28, 37)(29, 47)(31, 32)(33, 40)(34,
    57)(36, 58)(38, 46)(42, 55)(49, 51)(56, 59);

NL4:=NormalLattice(NL[4]);
NL4;
```

```
q2, ff2:=quo<NL[4]|NL4[3]>;
q2;
CompositionFactors(q2);

/*Permutation group q acting on a set of cardinality 6
Order = 60 = 2^2 * 3 * 5
    (1, 2)(3, 4)
    (2, 3, 5, 6, 4)*/
CompositionFactors(q2);/*
    G
    |   Alternating(5)
    1*/
FPGroup(q2);
/*Finitely presented group on 2 generators
Relations
    $.1^2 = Id($)
    $.2^5 = Id($)
    ($.1 * $.2^-1)^3 */
/* We know Alt(5) = < x^2, y^5, (x*y^-1)^3> */

/* Alt5<x,y> := Group<x,y|  x^2, y^5, (y^-2*x)^3>;
fa,Ga,ka:=CosetAction(Alt5,sub<Alt5|Id(Alt5)>);
Alt5b<x,y> := Group<x,y| x^2, y^5, (x*y^-1)^3>;
fab,Gab,kab:=CosetAction(Alt5b,sub<Alt5b|Id(Alt5b)>);
```

```
IsIsomorphic(Ga, Gab); */

/* NL[4]  s   NL4[3]  is normal in NL[4] and NL4[3] is a
    subgroup of order 60. This means that NL[4] = NL4[3]
      q2   */
GG<a,b,c,d>:=Group<a,b,c,d | a^2, b^5, (a*b^-1)^3, c^2, d
    ^5, (c*d^-1)^3, (a,c), (a,d), (b,c), (b,d)>;
f2,G2,k2:=CosetAction(GG,sub<GG|Id(GG)>);

t:=IsIsomorphic(NL[4],G2);
t;/* True */
FPGroup(G2);

/* Isomorphic type of G1 = 2:(Alt(5))2. */

FPGroup(NL[4]);
GGG<a,b>:= Group<a,b |    a^2, b^5, (b * a * b * a * b * a
    * b^-1 * a * b)^2, (b * a * b^-1 * a * b * a * b * a * b
    )^2,
b^-1 * a * b^-1 * a * b * a * b * a * b^-2 * a * b * a * b
    * a * b^-1 * a *
     b^-1 * a, b * a * b^-2 * a * b^2 * a * b^-2 * a * b^-2
         * a * b^-1 * a * b^-1 * a * b^-2 *
     a * b^-2 * a * b>;
```

```
f3 ,G3, k3:= CosetAction (GGG, sub<GGG| Id (GGG) >);
IsIsomorphic (GGG, DirectProduct ( Alt (5) , Alt (5) ) ;

q2 , ff2 :=quo<NL [ 4 ] | NL4[3] >;
q2 ;
T:= Transversal (NL[ 4] , NL4 [3] ) ;
ff2 (T[ 2] ) ;
ff2 (T[ 3] ) ;

A:=T [ 2 ] ;
A; /*(1, 14) (2, 56) (3, 41) (4, 34) (5, 59) (6, 45) (7, 22) (8,
    11) (9, 18) (10, 20) (12, 46) (13, 57) (15, 35) (16, 38) (17,
    23) (19,
     43) (24, 42) (26, 33) (27, 37) (28, 48) (29, 58) (30, 40) (31,
         49) (32, 51) (36, 47) (39, 52) (50, 60) (53, 55) */;
B:=T [ 3 ] ;
B; /*(1, 20, 38, 24, 32) (2, 51, 37, 33, 53) (3, 43, 16, 59,
    8) (4, 17, 25, 22, 10) (5, 44, 58, 57, 6) (7, 56, 50, 13,
    30) (9,
     47, 40, 42, 46) (11, 39, 27, 54, 12) (14, 31, 52, 29, 28)
        (15, 41, 21, 49, 60) (18, 26, 48, 34, 19) (23, 55, 45,
         35,
     36) */
```

```
A in NL4[3];
B in NL4[3];

bol,mp := IsIsomorphic(NL[4],G2); /* True */
bol; /* True */
InvMap:=Inverse(mp);
InvF2:=Inverse(f2);

A:=InvMap(f2(a));
B:=InvMap(f2(b));
(A*B^-1)^3;

Generators(NL4[3]);
C:= G1!(1, 53)(2, 39)(3, 59)(4, 22)(5, 8)(6, 38)(7, 32)(9,
    12)(10, 57)(11, 60)(13, 21)(14, 52)(15, 51)(16, 56)(17,
        33)(18, 48)(19, 36)(20, 24)(23, 37)(25, 40)(26, 58)
            (27, 43)(28, 54)(29, 35)(30, 42)(31, 44)(34, 47)
            (41,
        55)(45, 50)(46, 49);

D:=G1!(1, 55, 32)(2, 57, 9)(3, 18, 15)(4, 59, 41)(5, 30,
    12)(6, 37, 50)(7, 54, 8)(10, 47, 53)(11, 17, 25)(13, 34,
        19)(14, 49, 24)(16, 46, 43)(20, 60, 48)(21, 52, 22)
            (23, 44, 39)(26, 35, 40)(27, 36, 31)(28, 51, 29)
```

```
                (33, 56,
            38)(42, 58, 45);
/* We want Order(D) = 5 and (D^-2*C)^3 = Id(G1). We need to
     find a different way. */

D:=C*D;
D;
Order(D);
(C*D^-1)^3;
C in NL4[3];
D in NL4[3];

Temp:= sub<G1| A,B,C,D,E>;
IsIsomorphic(G1, Temp);

/*C:=InvMap(f2(c));
D:=InvMap(f2(d));*/

t:=IsIsomorphic(sub<G1| C, D>, NL4[3]);
t;

Sch:=SchreierSystem(GG,sub<GG|Id(GG)>);
Alt5SqrWord:=function(Perm);
for i in [2..#GG] do
```

```
P:=[Id(G1): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=A; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=A^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=B; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=B^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=C; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=C^-1; end if;
if Eltseq(Sch[i])[j] eq 4 then P[j]:=D; end if;
if Eltseq(Sch[i])[j] eq -4 then P[j]:=D^-1; end if;
end for;
PP:=Id(G1);
for k in [1..#P] do
PP:=PP*P[k];
end for;
if Perm eq PP then Ret:=Sch[i]; end if;
end for;
return Ret;
end function;

Sch:=SchreierSystem(GG,sub<GG|Id(GG)>);
PermToWord:=function(Perm)
for w in Sch do
        seq := Eltseq(w);
```

```
        p:= Id(G1);
        for j in seq do
                if j eq 1 then p:=p*A; end if;
if j eq -1 then p:=p*A^-1; end if;
if j eq 2 then p:=p*B; end if;
if j eq -2 then p:=p*B^-1; end if;
if j eq 3 then p:=p*C; end if;
if j eq -3 then p:=p*C^-1; end if;
if j eq 4 then p:=p*D; end if;
if j eq -4 then p:=p*D^-1; end if;
        end for;
        if Perm eq p then return w; end if;
end for;
end function;

PermToWord(A);
PermToWord(B);
PermToWord(C);
PermToWord(D);
PermToWord(E);

Alt5SqrWord(A^E);
Alt5SqrWord(B^E);
Alt5SqrWord(C^E);
```

```
Alt5SqrWord(D^E);

PermToWord(A^E); /*      a^e = a */
PermToWord(B^E); /*  b^e = a * b^-1 * a * b * a * b^-1  */
PermToWord(C^E); /* c^e = c * d^2 * c * d^-2 * c */
PermToWord(D^E); /* d^e = d^2 * c * d^-2 * c * d */

GGG<a,b,c,d,e>:=Group<a,b,c,d,e | a^2, b^5, c^2, d^5, (a*b
    ^-1)^3, (c*d^-1)^3, (a,c), (a,d), (b,c), (b,d), e^2, a^e
     = a,  b^e = a * b^-1 * a * b * a * b^-1, c^e = c * d^2
    * c * d^-2 * c, d^e = d^2 * c * d^-2 * c * d >;
f3,G3,k3:=CosetAction(GGG,sub<GGG|Id(GGG)>);
IsIsomorphic(G1, G3); /* true */
/* Isomorphic type of G1 = 2:m(Alt(5))2. */

/*Isomorphic type of N*/

S:= Sym(24);
xx:=S!(1,2,4,18,9)(3,13,14,16,24)(6,21,7,8,10)
    (12,15,19,20,22);
yy:=S!(1,21)(2,5)(3,19)(4,6)(7,15)(8,11)(9,13)(10,12)
    (14,17)(16,18)(20,23)(22,24);
N:=sub<S|xx,yy>;
```

```
FPGroup(N);
NN<x,y>:=Group<x,y|x^5, y^2, (x^-1 * y)^4, (x * y * x^-2 *
    y * x)^2 >;

CompositionFactors(N);
NL:= NormalLattice(N);/*
Normal subgroup lattice

[3]   Order 120   Length 1   Maximal Subgroups: 2
[2]   Order 60    Length 1   Maximal Subgroups: 1
[1]   Order 1     Length 1   Maximal Subgroups:*/

CompositionFactors(NL[2]);
q,ff:=quo<N|NL[2]>;
q;
T:= Transversal(N,NL[2]);
ff(T[2]);
K:=T[2];
K; /* (1, 21)(2, 5)(3, 19)(4, 6)(7, 15)(8, 11)(9, 13)(10,
    12)(14, 17)(16, 18)(20, 23)(22, 24)*/

Alt5<i,j> := Group<i,j|   i^2, j^5, (i*j^-1)^3>;
fa,Ga,ka:=CosetAction(Alt5,sub<Alt5|Id(Alt5)>);
bol,mp:= IsIsomorphic(Ga, NL[2]);
```

```
Bol; /* true */
I:= mp(fa(i));
I; /* (1, 17)(2, 20)(3, 21)(4, 18)(5, 7)(6, 10)(8, 14)(9,
   15)(11, 19)(12, 22)(13, 23)(16, 24) */
J:=mp(fa(j));
J;/*  (1, 9, 20, 17, 12)(2, 11, 18, 19, 15)(3, 8, 23, 6,
   13)(5, 24, 7, 21, 14) */
(I*J^-1)^3;

Sch:=SchreierSystem(Alt5,sub<Alt5|Id(Alt5)>);

/* a faster version of word */
word:=function(Perm)
for w in Sch do
        seq := Eltseq(w);
        p:= Id(N);
        for j in seq do
                if j eq 1 then p:=p*I; end if;
if j eq -1 then p:=p*I^-1; end if;
if j eq 2 then p:=p*J; end if;
if j eq -2 then p:=p*J^-1; end if;
        end for;
        if Perm eq p then return w; end if;
end for;
```

```
end function;

word(I^K); /* i^k =  j^2 * i * j^-2 * i * j^2   */
I*J^-2*I eq (J^I)^-2;
I*J^-2*I eq (J^I)^3;
I*J^-2*I eq ((J^I)^-1)^2;
word(J^K);/*  j^k = i * j^2 * i */
J^K eq (J^I)^2;
word((J^K)^(I^-1));

G6<i,j,k>:= Group< i, j, k |  i^2, j^5, (i*j^-1)^3, k^2, i^
   k =  j^2 * i * j^-2 * i * j^2,   j^k = (j^2) ^ i >;
f7,G7,k7:=CosetAction(G6,sub<G6|Id(G6)>);
IsIsomorphic(N,G7);

word(J^(K*I));
G6<i,j,k>:= Group< i, j, k |  i^2, j^5, (i*j^-1)^3, k^2, i^
   k =  j^2 * i * j^-2 * i * j^2,   j^(k*i) = (j^2) >;
f7,G7,k7:=CosetAction(G6,sub<G6|Id(G6)>);
IsIsomorphic(N,G7);

/* N ~ Alt(5):2 */
```

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