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Simple Groups and Related Topics

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A Thesis

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In Partial Fulfillment

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in

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by

Simrandeep Kaur

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Abstract

Since every nonabelian simple group is a homomorphic image of an involutory progenitor $2^{*n} : N$, where $N \leq S_n$ is transitive, our motivation for the thesis has been to seek finite homomorphic images of such progenitors and construct them using our technique of double coset enumeration.

We have constructed $U_3(3) : 2$ over $5^2 : S_3$, $2 \times (A_5 \times A_5)$ over $D_5 \times D_5$, S_6 over S_5, $2^5 : S_5$ over S_5, and $3^3 : 2^3$ over $3^2 : 2$.

We have discovered original symmetric presentations numerous group as homomorphic images various progenitors. We have also found new monomial representations of groups and given monomial progenitors. We have given isomorphism class of every image that we have discovered.

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Chapter 1

Introduction

Let G be a finite group generated by n involutions; that is, $G =< t_i | 1 \leq I \leq n >$ and let $N \leq S_n$. Then $G = \cup_i^m N w_i N$, where w_i is a word in the t_iS.

Subject to certain conditions, G is a homomorphic image of a progenitor of the form $2^{*n} : N$, where $N \leq S_n$.

Since a simple group satisfies the conditions satisfied by G above, every non-abelian simple group is a homomorphic image of $2^{*n} : N$. Every element of G can be written as nw, where $n \in N$ and w is a word in the t_is.

Now the double coset, NwN, is given by $\{Nwn | n \in N\} = \{Nnn^{-1}wn | n \in N\} = \{Nw^n | n \in N\}$ for a word w in the t_i's and the coset stabilising group of the coset Nw is $N^(w) = \{$n $\in N | Nw^n = Nw\}$. Double coset enumeration of G over N is performed to construct G.

We need to compute the number of right cosets in each double coset by using the formula $\frac{|N|}{|N^{(w)}|}$.

For the right coset Nw, it suffices to determine the double coset of Nwt_i for one representative t_i from each orbit of the stabilising group $N^{(w)}$ on $\{t_1, t_2, \ldots, t_n\}$. The double coset enumeration is complete if the set of right cosets is closed under the right multiplication by the t_is.

In chapter 2 we will define the important definitions and theorems which are the bases for our research.

In the chapter 3 we will demonstrate the technique of double coset enumeration which we use to construct finite images.

In Chapter 4, we will introduce wreath products.

In chapter 5, we will demonstrate our method of finding symmetric presentations of progenitors as well as additional relations. We will also list some of the finite images that we have discovered.

In Chapter 6, we will demonstrate, with examples, how to determine the isomorphism class of a group given in terms of its permutation generators.

In Chapter 7, we will discuss two linear groups.

In Chapter 8, we construct groups using our technique of double coset enumeration.

In Chapter 9, we give Magma codes to establish isomorphism classes of groups.

In Chapter 10, we have given tables of finite homomorphic images of progenitors.

Chapter 2

Preliminaries

2.1 Definitions

Definition 2.1. *If X is a nonempty set, a permutation of X is a bijection $\alpha : X \to X$ We denote the set of all permutations of X by S_x.*

Definition 2.2. *S_n is a symmetric group that composed by all bijective mapping $\phi : X \to X$, where X is a nonempty set.*

Definition 2.3. *If $x \in X$ and $\alpha \in S_x$, then α fixes x if $\alpha(x) = x$ and α moves x if $\alpha(x) \neq x$.*

Definition 2.4. *A permutation is said to be transposition if it changes two elements and fixes the rest.*

Definition 2.5. *The alternating group A_n is a subgroup of S_n with order equal to $\frac{n!}{2}$.*

Definition 2.6. *A (binary) operation on a nonempty set G is a function $\mu : G \times G \Rightarrow G$.*

Definition 2.7. *A semigroup $(G; *)$ is a nonempty set G equipped with an associative operation $*$.*

Definition 2.8. *A group is a semigroup G containing an element e such that:*
(i) $e * a = a = a *$ for all $a \in G$;
(ii) for every $a \in G$; there is an element $b \in G$ with $a * b = e = b * a$.

Definition 2.9. *A group G is abelian if every pair a,b $\in$ G commutes such as* $a * b = b * a$.

Definition 2.10. *If G is a group, there is a unique element e with* $e * a = a = a * e$ *for all* $a \in G$.
Moreover, for each $a \in G$, *there is a unique* $b \in G$ *with* $a * b = e = b * a$.
We call e the identity of G and, if $a * b = e = b * a$, *then we call b the inverse of a and denote it by* a^{-1}.

Definition 2.11. *(order of permutation) Let* $\alpha = (x_1, \ldots , x_i)(x_1, \ldots, x_j) \in S_x$, *where* α *is a multiple of two disjoint cycle. The order of* α *is the least common multiple of the i-cycle and the j-cycle.*

$$|\alpha| = lcm(i,j).$$

Definition 2.12. *If G is a group and* $a \in G$, *then*

$$(a^{-1})^{-1} = a.$$

Definition 2.13. *Let* $(G, *)$ *and* $(H, \circ)$ *be groups. A function* $f : G \Rightarrow H$ *is a homomorphism if, for all a,b* $\in$ *G,*

$$f(a * b) = f(a) \circ f(b).$$

Definition 2.14. *An isomorphism is a homomorphism that is also a bijection. We say that G is isomorphic to H, denoted by* $G \cong H$, *if there exists an isomorphism* $f : G \Rightarrow H$.

Definition 2.15. *A nonempty subset H of a group G is a subgroup of G if* $h \in H$ *implies* $h^{-1} \in H$, *and h,* $k \in H$ *implies* $hk \in H$. $H \leq G$.

Definition 2.16. *If H is any subgroup other than G, H is a proper subgroup of G.*

Definition 2.17. *If H is the subgroup generated by the identity of group G, H is a trivial subgroup of G.*

Definition 2.18. *If G is a group and* $a \in G$*, then the Cyclic subgroup generated by a is the set of all powers of a and it is denoted by* $< a >$.

Definition 2.19. *Let* $f : (G; *) \Rightarrow (G', \circ)$ *be a homorphism.*
(i) $f(e) = e'$, *where e' is the identity in G' (ii) If* $a \in G$, *then* $f(a^{-1}) = f(a)^{-1}$.
(iii) If $a \in G$ *and* $n \in Z$, *then* $f(a^n) = f(a)^n$.

Definition 2.20. *Let G be a group and* $K \leq G$. *K is a maximal subgroup of G if there is no normal subgroup* $N \leq G$ *such that* $K < N < G$.

Definition 2.21. *A subset S of a group G is a subgroup if and only if 1* $\in S$ *and s,t* $\in S$ *imply* $st^{-1} \in S$.

Definition 2.22. *If* $g \in G$ *and* $\phi \in S_X$, *then* ϕ *fixes g if* $\phi(x) = g$, ϕ *moves g if* $\phi(x) \neq g$.

Definition 2.23. *If G is a group and* $a \in G$, *then the cyclic subgroup generated by a, denoted by* $< a >$, *is the set of all powers of a. A group G is called cyclic if there is* $a \in G$ *with* $G =< a >$; *that is, G consists of all the powers of a.*

Definition 2.24. *If S is a subgroup of G and if t* $\in$*G, then a right coset of S in G is the subset of G*

$$St = st : s \in S$$

(a left coset is $tS = ts : s \in S$*). One calls t a representative of St (and also of tS).*

Definition 2.25. *If* $\alpha, \beta \in S_n$, $\alpha and \beta$ *are disjoint if every element moved by one permutation is fixed by the other. if*

$\alpha(n) \neq n$, *then* $\beta(m) = m$ *and if* $\alpha(x) = x$, *then* $\beta(x) \neq x$.

Definition 2.26. *If a permutation interchanges a pair of elements, it is called a transposition.*

Definition 2.27. *Let H be a nonempty subset of a group G. Let* $w \in G$ *where* $w = h_1^{e1} h_2^{e2} \cdots h_n^{en}$, *with* $h_i \in H$ *and* $e_i = \pm 1$. *We say that w is a word on H.*

Definition 2.28. *Let H be a group. We say H is a direct product of two subgroups G and K if:*

- $H = GK$;
- $G \cap K = 1$,

Definition 2.29. *If* $H \leq G$ *and* $x \in G$, *the subset of G,* $Hx = \{xh : x \in H\}$ *is the right coset of H in G.*

Definition 2.30. *If* $H \leq G$ *and* $x \in G$, $HxH = \{ HxH \mid x \in H\}$ *is the double coset of G.*

Definition 2.31. *If* $h^n = 1$ *for all* $h \in G$, *the group G has an exponent n*

Definition 2.32. *If G is a finite group and* $a \in G$. *Then the order of a divides* $|G|$.

Definition 2.33. *If p is a prime and* $|G| = p$, *then G is a cyclic group.*

Definition 2.34. *Let* $x \in G$, *the for* $x^{-1}gx$ *for* $x \in G$ *is the conjugate of g in G.*

Definition 2.35. *If x,* $y \in G$, *the commutator of x and y,* $[x, y] is [x, y] = xyx^{-1}y^{-1}$.

Definition 2.36. *A group H is a p-group if the order of every element of H is a power of p.*
Let H be a finite group. If it is an abelian, it is called elementary abelian group and every nontrivial element $x \in H$ *has a prime order p.*

Definition 2.37. *We call* X^g *stabiliser.*$X^g = \{x \in X | g^x = g\}$, *where x is a word of* $t_i's$.
$X^{(g)} = \{x \in X | Xg^x\}$ *where g is a word of* $t_i's$. *We call X(g) a coset stabiliser.*

Definition 2.38. *If X is a set and G be a group. We say X is a G-set if there exists a function* $\phi : G \times X \to X$ *and the following hold for* $\phi : (g; x) \to gx$.

- $1x = x$, *for all* $x \in X$.

- $g(hx) = (gh)x$, for $g,h \in G$ and $x \in X$.

Definition 2.39. *A projective special linear group, PSL(n , F) is the set of all $n \times n$ matrices with determinant 1 over field F factored by its center:*

$$PSL(n, F) = L_n(F) = \frac{SL(n,F)}{Z(SL(n,F)}.$$

Definition 2.40. *For the group G, (Z(G)) is a center of G. The set of all $g \in G$ commute with every elements of G.*

Definition 2.41. *We call D_2n Dihedral Group. Dihedral group generated by two elements x and y with presentation $< x, y|x^n = y^2 = (xy)^2 = 1 >$. The order of D_2n is equal to 2n and $2n \geq 4$.*

Definition 2.42. *If group G has a composition series, the factor groups of its series are the Composition Factors of G.*

Definition 2.43. *Let X be a set and δ by a family of words on X. A group G has Generators X and Relations δ if $G \cong K/R$, where K is a free group with basis X and R is the normal subgroup of K generated by δ. We say $< X|\delta >$ is a Presentation of G.*

2.2 Group Extension Preliminaries

Definition 2.44. *The Group Extension is an extension of a group N by a group K with a normal subgroup H such that*

$$H \cong N and G/H \cong K.$$

Definition 2.45. *The Central Extension is the extension that N is the center of G if G is a central extension of N by K which is based on*

$$\psi : K \times K \to N.(n_1, k_1) * (n_2, k_2) = (n_1 * n_2 * \psi(k_1, k_2)k_1k_2).$$

Definition 2.46. *The Semi-direct Product is a group extension composed by H and Q. G = H : Q when $H \Delta G$. H has a complement $Q1 \cong Q$.*

Definition 2.47. *The Mixed Extension is the extension combined the properties of a semi-product and a central extension. (N is a normal subgroup and it is not a central of the group)*

$$\phi : K \to Aut(N) \text{ and } \psi K \times K \to N.$$
$$N \cdot K : (n_1, k_1) * (n_2, k_2) = (n_1 * k_2^{k1} * \psi(k_1, k_2), k_1 k_2).$$

2.3 Preliminary Theorems and Lemmas

Definition 2.48. ***First Isomorphism Theorem*** *Let $\phi : G \to H$ be a homomor- phism with ker ϕ then,*

- $[ker\phi \Delta G]$,
- $[G/ker\phi \cong im\phi]$.

Definition 2.49. *Let X be a G-set, and let $xy \in X$.*

- *If $K \leq G, then K_x \cap K_y \neq \varnothing$; implies $K_x = K_y$,*
- *If $K \Delta G$, then the subsets K_x are Blocks of X.*

Definition 2.50. ***(GrindStaff/ Factoring Lemma)****: Factoring the progenitor $m^{*n} : N$ by (t_i, t_j) for $1 \leq i \leq j \leq n$ gives the group $m^n : N$*

Chapter 3

Monomial Progenitors

3.1 Preliminaries

Definition 3.1. *A monomial representation of a group G is a homomorphism from G into GL(n,F),the group of nonsingular $n \times n$ matrices over the field F, in which the image of every element of G is a monomial matrix over F.*

Definition 3.2. (**MonomialCharacter**)*Let G be a finite group and $H \leq G$.The character X of G is monomial if $X = \lambda^G$, where λ is a linear character of H.*

Definition 3.3. *A* **matrix** *in which there is precisely one non-zero term in each row and in each column is said to be monomial.*

Definition 3.4. *Let $A(x) = (a_{ij}(x))$ be a matrix representation of G of degree m.We consider the characteristic polynomial of A(x), namely*

$$det(\lambda I - A(x)) = \begin{pmatrix} \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ -a_{21}(x) & \lambda - a_{22}(x) & \cdots & -a_{2m}(x) \\ \cdots & \cdots & \cdots & \cdots \\ -a_{m1}(x) & -a_{m2}(x) & \cdots & \lambda - a_{mm}(x) \end{pmatrix}.$$

This is a polynomial of degree m in λ, and inspection shows that the coefficient of $-\lambda^{m-1}$ equal to

$$\phi(x) = a_1(x) + a_{22}(x) + \cdots + a_{mm}(x).$$

It is customary to call the right-hand side of this equation the trace of A(x), abbreviated to trA(x), so that

$$\phi\ (x) = trA(x).$$

Definition 3.5. *The sum of squares of the degrees of the distinct irreducible characters of G is equal to* $|G|$. *The degree of a character* χ *is* χ *(1).Note that a character whose degree is 1 is called a linear character.*

Definition 3.6. *Let* $H \leq G$ *and* $\phi(u)$ *be a charcter of H and define* $\phi(x) = 0$ *if* $x \in H$*, then*

$$\phi^G(x) = \begin{cases} \phi(x) & ,x \in H; \\ x \notin H. \end{cases}$$

is an induced character of G.

Definition 3.7. FormulaforInducedCharacter *Let G be a finite group and H be a subgroup such that* $\frac{|G|}{|H|}$=*n.Let*C_α, $\alpha = 1, 2, \ldots, m$ *be the conjugacy classes of G with* $|C_\alpha| = h_\alpha$, $\alpha = 1, 2, \ldots, m$. *Let* ϕ *be a character of H and* ϕ^G *be the character of G induced from the character* ϕ *of H up to G.The values of* ϕ^G *on the m classes of G are given by:*

$$\phi_\alpha^G = \frac{n}{h_\alpha}$$

$$\sum_{w \in C_\alpha \cap H} \phi(w), \alpha = 1, 2, 3, \ldots, m.$$

3.2 Monomial Progenitors $(13^{*2} : m(12 : 2))$

Consider G $= < xx, yy, zz >$, where
$xx = (1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12)$,
$yy = (1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12)$
and $zz = (1, 11)(2, 10)(3, 9)(4, 8)(5, 7)$.

$G = (12 : 2)$ has a monomial irreducible representation of degree 2.
Since, $\frac{|G|}{|H|} = \frac{24}{12} = 2$. $|H| = 12$, we need to find a subgroup H of order 12 and induce a linear character of H up to G to obtain the irreducible character of degree 2 of G.
Consider the subgroup H of G generated by $< (1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12)$,
$(1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12), (1, 9, 5)(2, 10, 6)(3, 11, 7)(4, 12, 8) > \cong 12$. G has the following conjugacy classes.

$C_1 = \{e\}$

$C_2 = \{(1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12)\}$

$C_3 = \{(1, 11)(2, 10)(3, 9)(4, 8)(5, 7)\}$

$C_4 = \{(1, 2)(3, 12)(4, 11)(5, 10)(6, 9)(7, 8)\}$

$C_5 = \{(1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12)\}$

$C_6 = \{(1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12)\}$

$C_7 = \{(1, 3, 5, 7, 9, 11)(2, 4, 6, 8, 10, 12)\}$

$C_8 = \{(1, 8, 3, 10, 5, 12, 7, 2, 9, 4, 11, 6)\}$

$C_9 = \{(1, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)\}$.

The conjugacy classes of H are

$D_1 = \{Id(H)\}$

$D_2 = \{(1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12)\}$

$$D_3 = \{(1,9,5)(2,10,6)(3,11,7)(4,12,8)\}$$

$$D_4 = \{(1,5,9)(2,6,10)(3,7,11)(4,8,12)\}$$

$$D_5 = \{(1,4,7,10)(2,5,8,11)(3,6,9,12)\}$$

$$D_6 = \{(1,10,7,4)(2,11,8,5)(3,12,9,6)\}$$

$$D_7 = \{((1,11,9,7,5,3)(2,12,10,8,6,4)\}$$

$$D_8 = \{(1,3,5,7,9,11)(2,4,6,8,10,12)\}$$

$$D_9 = \{(1,12,11,10,9,8,7,6,5,4,3,2)\}$$

$$D_{10} = \{(1,8,3,10,5,12,7,2,9,4,11,6)\}$$

$$D_{11} = \{(1,6,11,4,9,2,7,12,5,10,3,8)\}$$

$$D_{12} = \{(1,2,3,4,5,6,7,8,9,10,11,12)\}.$$

We verify in Magma,

```
 Induction(CH[12],G) eq CG[9];\\
/*true*/
```

We know that the character table of H is given by

```
Character Table of Group H
--------------------------


---------------------------------------------------
Class |   1  2   3   4  5  6   7   8    9   10   11
Size  |   1  1   1   1  1  1   1   1    1    1    1
Order |   1  2   3   3  4  4   6   6   12   12   12
---------------------------------------------------
p  =  2   1  1   4   3  2  2   3   4    7    8    7
p  =  3   1  2   1   1  6  5   2   2    6    6    5
---------------------------------------------------
X.1   +   1  1   1   1  1  1   1   1    1    1    1
X.2   0   1 -1   1   1 -I  I  -1  -1   -I   -I    I
X.3   +   1  1   1   1 -1 -1   1   1   -1   -1   -1
X.4   0   1 -1   1   1  I -I  -1  -1    I    I   -I
X.5   0   1  1   J-1-J  1  1-1-J   J    J -1-J    J
X.6   0   1 -1   J-1-J -I  I 1+J  -J   Z1 Z1#5  -Z1
X.7   0   1  1   J-1-J -1 -1-1-J   J   -J  1+J   -J
X.8   0   1 -1   J-1-J  I -I 1+J  -J  -Z1-Z1#5   Z1
X.9   0   1  1-1-J   J  1  1   J-1-J -1-J    J -1-J
X.10  0   1 -1-1-J   J -I  I  -J 1+J Z1#5   Z1-Z1#5
X.11  0   1  1-1-J   J -1 -1   J-1-J  1+J   -J  1+J
X.12  0   1 -1-1-J   J  I -I  -J 1+J-Z1#5  -Z1 Z1#5


--------------
Class |     12
Size  |      1
Order |     12
--------------
p  =  2      8
p  =  3      5
--------------
X.1   +      1
X.2   0      I
X.3   +     -1
X.4   0     -I
X.5   0   -1-J
X.6   0  -Z1#5
X.7   0    1+J
X.8   0   Z1#5
```

```
X.9   0      J
X.10  0    -Z1
X.11  0     -J
X.12  0     Z1


Explanation of Character Value Symbols
--------------------------------------

# denotes algebraic conjugation, that is,
#k indicates replacing the root of unity w by w^k

J = RootOfUnity(3)

I = RootOfUnity(4)

Z1     = (CyclotomicField(12: Sparse := true)) ! [
RationalField() | 0, 0, 0, -1 ]
```

Consider the irreducible character ϕ of H and ϕ^G of G given below.

The output we have is given in the following table.

Irreducible Character of ϕ			
Class	Size	Representation	ϕ
D_1	1	$Id(H)$	1
D_2	1	$(1,7)(2,8)(3,9)(4,10)(5,11)(6,12)$	1
D_3	1	$(1,9,5)(2,10,6)(3,11,7)(4,12,8)$	$-1-w$
D_4	1	$(1,5,9)(2,6,10)(3,7,11)(4,8,12)$	w
D_5	1	$((1,4,7,10)(2,5,8,11)(3,6,9,12)$	1
D_6	1	$(1,10,7,4)(2,11,8,5)(3,12,9,6)$	1
D_7	1	$(1,11,9,7,5,3)(2,12,10,8,6,4)$	w
D_8	1	$(1,3,5,7,9,11)(2,4,6,8,10,12)$	$-1-w$
D_9	1	$(1,12,11,10,9,8,7,6,5,4,3,2)$	$-1-w$
D_{10}	1	$(1,8,3,10,5,12,7,2,9,4,11,6)$	w
D_{11}	1	$(1,6,11,4,9,2,7,12,5,10,3,8)$	$-1-w$
D_{12}	1	$(1,2,3,4,5,6,7,8,9,10,11,12)$	w

Table 3.1: Irreducible Character of ϕ

We have,

```
 CH[12];
 ( 1, -1, -zeta(12)_3 - 1, zeta(12)_3, zeta(12)_4,
    -zeta(12)_4, -zeta(12)_3, zeta(12)_3 + 1,
    -zeta(12)_4*zeta(12)_3 - zeta(12)_4,
    zeta(12)_4*zeta(12)_3, zeta(12)_4*zeta(12)_3 +
    zeta(12)_4, -zeta(12)_4*zeta(12)_3 )
```

```
 CG[9];
( 2, -2, 0, 0, -1, 0, 1, 2*zeta(12)_4*zeta(12)_3 +
    zeta(12)_4, -2*zeta(12)_4*zeta(12)_3 - zeta(12)_4 )
```

We verify by hand that we have a monomial representation by inducing $\phi = CH[12]$ up to $\phi^G = CG[9]$.

$\phi \uparrow_H^G$

$$\phi \uparrow_{\alpha}^{G}= \frac{n}{h_{\alpha}} \sum_{w \in C_{\alpha} \cap H} = \alpha(w), \alpha = 1, 2, 3, ..., m$$

Using

$$\phi \uparrow_{\alpha}^{G}= \frac{n}{h_{\alpha}} \sum_{w \in C_{\alpha} \cap H} = \phi(w)$$

where $n = \frac{|G|}{|H|} = \frac{24}{12} = 2$

$\phi \uparrow_{1}^{G}= \frac{n}{h_{\alpha}} \sum_{w \in C_1 \cap H} = 2(\phi Id(H)) = 2(1) = 2$

$\phi \uparrow_{2}^{G}= \frac{n}{h_{\alpha}} \sum_{w \in C_2 \cap H} = \frac{2}{1}(\phi(1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12)) = (2)(1) = 2$

$\phi \uparrow_{3}^{G}= \frac{n}{h_{\alpha}} \sum_{w \in C_3 \cap H} = \frac{2}{1}(\phi(1, 9, 5)(2, 10, 6)(3, 11, 7)(4, 12, 8) = \frac{1}{2}(-1 - w) = -2 - w$

$\phi \uparrow_{4}^{G}= \frac{n}{h_{\alpha}} \sum_{w \in C_4 \cap H} = \frac{2}{1}(\phi(1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12)) = \frac{1}{2}(w) = 2w$

$\phi \uparrow_{5}^{G}= \frac{n}{h_{\alpha}} \sum_{w \in C_5 \cap H} = \frac{2}{1}(\phi(1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12)) = 2(1) = 2$

$\phi \uparrow_{6}^{G}= \frac{n}{h_{\alpha}} \sum_{w \in C_6 \cap H} = \frac{2}{1}(\phi(1, 10, 7, 4)(2, 11, 8, 5)(3, 12, 9, 6)) = (2)(1) = 2$

$\phi \uparrow_{7}^{G}= \frac{n}{h_{\alpha}}(1, 11, 9, 7, 5, 3)(2, 12, 10, 8, 6, 4) = (1)(w) = w$

$\phi \uparrow_{8}^{G}= \frac{n}{h_{\alpha}} \sum_{w \in C_8 \cap H} = \frac{2}{1}(\phi(1, 3, 5, 7, 9, 11)(2, 4, 6, 8, 10, 12)) = (2)(-1 - w) = -2 - w$

$\phi \uparrow_{9}^{G}= \frac{n}{h_{\alpha}} \sum_{w \in C_9 \cap H} = \frac{2}{1}(\phi(1, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)) = (2)(-1 - w) = -2 - w$

$\phi \uparrow_{10}^{G}= \frac{n}{h_{\alpha}} \sum_{w \in C_9 \cap H} = \frac{2}{1}(\phi v) = (2)(w) = w$

$\phi \uparrow_{11}^{G}= \frac{n}{h_{\alpha}} \sum_{w \in C_9 \cap H} = \frac{2}{1}(\phi(1, 6, 11, 4, 9, 2, 7, 12, 5, 10, 3, 8)) = (2)(-1 - w) = -2 - w$

$\phi \uparrow_{12}^{G}= \frac{n}{h_{\alpha}} \sum_{w \in C_9 \cap H} = \frac{2}{1}(\phi(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)) = (2)(w) = 2w$

Thus, $CH[12] \uparrow_H^G = CG[9]$. Since CG[9] is faithful, our group has a faithful irreducible monomial representation of degree 2.
We now find an irreducible monomial representation of G. From Magma, we can find the transversals of H in G. Note that the number of transversals equals the order of G divided by the order of H. Below is the code:

```
T:=Transversal(G,H);
T;
{@
    Id(G),
    (1, 11)(2, 10)(3, 9)(4, 8)(5, 7)
@}
#T;
2
```

Now the matrix becomes: $A(xx) = \begin{bmatrix} \phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) \\ \phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}) \end{bmatrix}$,

A(xx) $= \begin{bmatrix} zeta_{12}^2 - 1 & 0 \\ 0 & -zeta_{12}^2 \end{bmatrix}$,

Similarily with A(yy) and A(zz),

$A(yy) = \begin{bmatrix} \phi(t_1yt_1^{-1}) & \phi(t_1yt_2^{-1}) \\ \phi(t_2yt_1^{-1}) & \phi(t_2yt_2^{-1}) \end{bmatrix}$,

A(yy)$= \begin{bmatrix} zeta_{12}^3 & 0 \\ 0 & -zeta_{12}^2 \end{bmatrix}$,

$A(zz) = \begin{bmatrix} \phi(t_1zt_1^{-1}) & \phi(t_1zt_2^{-1}) \\ \phi(t_2zt_1^{-1}) & \phi(t_2zt_2^{-1}) \end{bmatrix}$,

A(zz) $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, We verify these matrices by running the following loop.

$> C := CyclotomicField(12);$

$> A := [[C.1, 0] : iin[1..2]];$

¿for i,j in [1..2]$doA[i, j] := 0; endfor;$

¿ for i,j in [1..2]$doifT[i] * xx * T[j]^{-1}$ in H then

$> A[i, j] := CH[12](T[i] * xx * T[j]^{-1}); endif; endfor;$

$> GG := GL(2, C);$

$> GG!A;$

$$\begin{bmatrix} zeta_{12}^2 - 1 & 0 \\ 0 & -zeta_{12}^2 \end{bmatrix}$$

Order(xx);

/*3*/

Order($GG!A$);

/*3*/

$B := [[C.1, 0] : iin[1..2]];$

for i,j in [1..2]$doB[i, j] := 0; endfor;$

for i,j in [1..2]$doifT[i] * yy * T[j]^{-1}inHthen$

$B[i, j] := CH[12](T[i] * yy * T[j]^{-1}); endif; endfor;$

$GG!B;$

$$\begin{bmatrix} zeta_{12}^3 & 0 \\ 0 & -zeta_{12}^3 \end{bmatrix}$$

Order(yy);

/*4*/

$Order(GG!B);$

/*4*/

$D := [[C.1, 0] : iin[1..2]];$

$fori, jin[1..2]doD[i, j] := 0; endfor;$

$fori, jin[1..2]doifT[i] * zz * T[j]^{-1}inHthen$

$D[i, j] := CH[12](T[i] * zz * T[j]^{-1}); endif; endfor;$

$GG!D$;

A(zz) $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Order(zz);
/*2*/
$Order(GG!D)$;
/*2*/

The order of xx is 3 and the order of yy is 4 and the order of zz is 2. Now As 2 is a primitive root of 12.
So $zeta_{12} = 2$
$-zeta_{12}^2 = -2^2 = -4 = 9 mod 13$
$zeta_{12}^2 - 1 = 2^2 - 1 = 3$
$zeta_{12}^3 = 2^3 = 8$
$-{zeta_{12}}^2 = -8 = 5 mod 13$
Lowest relative prime is 13
$12|p-1 \Rightarrow 12|13-1$

The permutation representation of A(xx), A(yy) and A(zz) of the monomial reprsenatation are:
So,the matrix A(xx) is:

$= \begin{bmatrix} zeta_{12}^2 - 1 & 0 \\ 0 & -zeta_{12}^2 \end{bmatrix},$
And it becomes,

A(xx)$= \begin{bmatrix} 3 & 0 \\ 0 & 9 \end{bmatrix},$

where $a_{11} = 3, a_{22} = 9,$
therefore, $t_1 \to {t_1}^3$,$t_2 \to {t_2}^9$.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
t_1^3	t_2^9	t_1^6	t_2^5	t_1^9	t_2	t_1^{12}	t_2^{10}	t_1^2	t_2^6	t_1^5	t_2^2	t_1^8	t_2^{11}	t_1^{11}	t_2^7
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
5	18	11	10	17	2	23	20	3	12	9	4	15	22	21	14

17	18	19	20	21	22	23	24
t_1	t_2^3	t_1^4	t_2^{12}	t_1^7	t_2^8	$t_1^1 0$	t_2^4
↓	↓	↓	↓	↓	↓	↓	↓
1	6	7	24	13	16	19	8

Therefore,the new permutation of

A(xx) = (1,5,17)(2,18,6)(3,11,9)(4,10,12)(7,23,19) (8,20,24)(13,15,21)(14,22,16).

Similarly,the matrix A(yy) is:

$$\begin{bmatrix} zeta_{12}^3 & 0 \\ 0 & -zeta_{12}^3 \end{bmatrix},$$

and it becomes

$$A(yy) = \begin{bmatrix} 8 & 0 \\ 0 & 5 \end{bmatrix},$$

where $a_{11} = 8, a_{22} = 5$,

therefore, $t_1 \rightarrow {t_1}^5$,$t_2 \rightarrow {t_2}^5$.

Let us now verify if this representation is faithful.

```
IsIsomorphic(G,sub<GG|GG!A,GG!B,GG!D>);
true
```

Hence $\langle$ A(xx),A(yy),A(zz)$\rangle$ is a faithful monomial representation of $2^3 : 3$.

We first need to find a permutation representation using the field order and the degree of representation. By Euler's Formula, the primitive square root of unity is $e^{\frac{i2\pi}{2}} = \cos\left(\frac{2\pi}{2}\right) + i \sin\left(\frac{2\pi}{2}\right) = \cos(\pi) + i \sin(\pi) = -1$

The field order will be the smallest finite field that has square roots of unity. This will be a cyclic group of order $p-1$ where $12|p-1$. So the field order will be Z_12. We will use the matrices we created in order to label the automorphisms of t_i's.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
t_1^8	t_2^5	t_5^3	t_2^{10}	t_1^{11}	t_2^2	t_1^6	t_2^7	t_1^1	t_2^12	t_1^9	t_2^4	t_1^4	t_2^9	t_1^{12}	t_2^1
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
15	10	5	20	21	4	11	14	1	24	17	8	7	18	23	2

17	18	19	20	21	22	23	24
t_1^7	t_2^6	t_1^2	t_2^{11}	t_1^{10}	t_2^3	t_1^5	t_2^8
↓	↓	↓	↓	↓	↓	↓	↓
13	12	3	22	19	6	9	16

Therefore,the new permutation of A(yy)=(1,15,23,9)(2,10,24,16)(3,5,21,19) (4,20,22,6) (7,11,17,13)(8,14,18,12);

The matrix A(zz) is

$$A(zz) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where $a_{12} = 1, a_{21} = 1$,
therefore, $t_1 \to t_2, t_2 \to t_1$.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
t_2	t_1	t_2^2	t_1^2	t_2^3	t_1^3	t_2^4	t_1^4	t_2^5	t_1^5	t_2^6	t_1^6	t_2^7	t_1^7	t_2^8	t_1^8
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
2	1	4	3	6	5	8	7	10	9	12	11	14	3	16	15

17	18	19	20	21	22	23	24
t_2^9	t_1^9	t_2^{10}	t_1^0	t_2^{11}	t_1^{11}	t_2^{12}	t_1^{12}
↓	↓	↓	↓	↓	↓	↓	↓
18	17	20	19	22	21	24	23

Therefore A(zz) = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14) (15,16)(17,18)(19,20) (21,22)(23,24).

Since, the matrix representation has entries in Z_{13} our progenitor is $13^{*2} : (12 : 2)$. We show below using Magma, that a symmetric presentation of the progenitor is: 13 $(Z_{13})^{*2} : (12 : 2) = < x, y, z, t | x^3, y^4, z^2, (x, y), (x^{-1} * z)^2, (y^{-1} * z)^2, t^{13}, t^{(x^{-1})} = t^9, t^{(y^{-1})} = t^5 >$.

> S:=Sym(24);

> xx:=S!(1,5,17)(2,18,6)(3,11,9)(4,10,12)(7,23,19)(8,20,24)(13,15,21)(14,22,16);

>yy:=S!(1,15,23,9)(2,10,24,16)(3,5,21,19)(4,20,22,6)(7,11,17,13)(8,14,18,12);

> zz:=S!(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)(17,18)(19,20)(21,22)(23,24);

$N := sub < S|xx, yy, zz >$;

♯N;

/*24*/

Stabiliser(N,1,3,5,7,9,11,13,15,17,19,21,23);

/*

Permutation group acting on a set of cardinality 24

Order $= 12 = 2^2 * 3$

(1, 17, 5)(2, 6, 18)(3, 9, 11)(4, 12, 10)(7, 19, 23)(8, 24, 20)(13, 21, 15)(14, 16, 22)

(1, 9, 23, 15)(2, 16, 24, 10)(3, 19, 21, 5)(4, 6, 22, 20)(7, 13, 17, 11)(8, 12, 18, 14)

*/

```
 FPGroup(N);
/*
Finitely presented group on 3 generators
Relations
    $.1^{3} = Id($)
    $.2^{4} = Id($)
    $.3^{2} = Id($)
    ($.1, $.2) = Id($)
    ($.1^{-1} * $.3)^{2} = Id($)
    ($.2^{-1} * $.3)^{2} = Id($)
*/
```

$G < x, y, z, t >$:=Group< $x, y, z, t | x^3, y^4, z^2, (x, y), (x^{-1} * z)^2, (y^{-1} * z)^2, t^{13}, t^{(x^{-1})} = t^9, t^{(y^{-1})} = t^5 >$;

> $\sharp G$;

/* 4056 */

3.3 Monomial Progenitors A_4

Consider G = $< xx, yy >$, where
$xx = (1, 2, 3, 4)$,
$yy = (1, 2)$.
Since, $\frac{|G|}{|H|} = \frac{24}{12} = 2.|H| = 12$, we need to find a subgroup H of order 12 and induce a linear character of H up to G to obtain the irreducible character of degree 2 of G.
Consider the subgroup H of G generated by $< Id(G), (1, 2)(3, 4), (1, 2, 3, 4), (1, 3, 2) >$.
G has the following conjugacy classes.

$$C1 = Id(G),$$

$$C2 = (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3),$$

$$C3 = (1, 2), (1, 4), (3, 4), (2, 3), (1, 3), (2, 4),$$

$$C4 = (1, 2, 3), (1, 4, 2), (1, 3, 4), (1, 2, 4), (2, 4, 3), (1, 4, 3), (2, 3, 4), (1, 3, 2).$$

The conjugacy classes of H are

$$D1 = Id(G),$$

$$D2 = (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3),$$

$$D3 = (1, 2, 3), (1, 4, 2), (2, 4, 3), (1, 3, 4),$$

$$D4 = (1, 3, 2), (2, 3, 4), (1, 4, 3), (1, 2, 4).$$

Consider the irreducible characters ϕ (of H) and ϕ^G of G given below.

Irreducible Character of ϕ			
Class	Size	Representation	ϕ
D_1	1	$Id(H)$	1
D_2	3	$(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)$	1
D_3	4	$(1,2,3),(1,4,2),(2,4,3),(1,3,4)$	w
D_4	4	$(1,3,2),(2,3,4),(1,4,3),(1,2,4)$	$-w-1$

Table 3.2: Irreducible Character of ϕ

Irreducible Character of ϕ			
Class	Size	Representation	ϕ
C_1	1	$Id(H)$	2
C_2	3	$(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)$	2
C_3	6	$(1,2),(1,4),(3,4),(2,3),(1,3),(2,4)$	0
C_4	8	$(1,2,3),(1,4,2),(1,3,4),(1,2,4),(2,4,3),(1,4,3),(2,3,4),(1,3,2)$	-1

Table 3.3: Irreducible Character of ϕ

Induce the character ϕ of H up to G to obtain the character ϕ^G of G.

$\phi \uparrow_H^G$

$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w)$, where $n = \frac{|G|}{|H|} = \frac{24}{12} = 2$.

$\phi_1^G = \frac{2}{1} \sum_{w \in H \cap C_1} \phi(w)$

So, $\phi_1^G = 2(\phi(1)) = 2(1) = 2$.

$\phi_2^G = \frac{2}{3} \sum_{w \in H \cap C_2} \phi(w)$,

So, $\phi_2^G = \frac{2}{3}(\phi((1,2)(3,4) + (1,3)(2,4) + (1,4)(2,3)) = \frac{2}{3}(1+1+1) = 2$.

$\phi_3^G = \frac{2}{6} \sum_{w \in H \cap C_3} \phi(w)$, So, $\phi_3^G = \frac{2}{6}(\phi(0)) = \frac{2}{6}(0) = 0.$

$\phi_4^G = \frac{2}{8}\sum_{w\in H\cap C_\alpha}\phi(w),$

So, $\phi_4^G = \frac{2}{8}(4w + 4(-w-1)) = \frac{-8}{8} = -1.$

$\phi_5^G = \frac{2}{6}\sum_{w\in H\cap C_\alpha}\phi(w),$

So, $\phi_5^G = \frac{2}{6}(\phi(0)) = \frac{2}{6}(0) = 0.$

$\phi\uparrow_H^G = 2 \quad 2 \quad 0 \quad -1.$

(b) Show the monomial representation has the generators

$$A(xx) = \begin{bmatrix} \phi(t_1xt_1^{-1}) & \phi(t_1xt_2^{-1}) \\ \phi(t_2xt_1^{-1}) & \phi(t_2xt_2^{-1}) \end{bmatrix},$$
$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
$$A(yy) = \begin{bmatrix} \phi(t_1yt_1^{-1}) & \phi(t_1yt_2^{-1}) \\ \phi(t_2yt_1^{-1}) & \phi(t_2yt_2^{-1}) \end{bmatrix},$$
$$= \begin{bmatrix} 0 & w \\ w^2 & 0 \end{bmatrix}.$$

(c) Give a permutation representation of $A(xx)$ and $A(yy)$ of the monomial representation of part (b).

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
where $a_{12} = 1 = a_{21} = 1.$

$B = \begin{bmatrix} 0 & w \\ w^2 & 0 \end{bmatrix}$, where $a_{12} = 2,\ a_{21} = 1,$

Therefore,

$t_1 \rightarrow t_2^1,$

$t_2 \rightarrow t_1^1,$

1	2	3	4	5	6	7	8	9	10	11	12
t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5	t_1^6	t_2^6
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2	t_1	t_2^2	t_1^2	t_2^3	t_1^3	t_2^4	t_1^4	t_2^5	t_1^5	t_2^6	t_1^6

$A(yy) = \begin{bmatrix} 0w \\ w^2 0 \end{bmatrix},$

where $a_{12} = 2,\ a_{21} = 4.$

Therefore,

$t_1 \rightarrow t_2^2,$

$t_2 \rightarrow t_1^4.$

1	2	3	4	5	6	7	8	9	10	11	12
t_1	t_2	t_1^2	t_2^2	t_1^3	t_2^3	t_1^4	t_2^4	t_1^5	t_2^5	t_1^6	t_2^6
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2^2	t_1^4	t_1^4	$t_1^8 = t_1$	t_2^6	$t_1^1 = t_1^5$	t_2	t_1^2	t_2^3	t_1^3	t_2^2	t_1^2

(d) Show that the monomial representation in (2) is not faithful.

$|xx| = 4; .|yy| = 2$

$|xx * yy| = 3$

So,

$|A(xx)| = 4;$

$|A(yy)| = 2.$

Therefore; $|A(xx) * A(yy)| = 2.$

and

$\sharp sub < GG|GG!A; GG!B >;$

$/ * 6 * /.$

Chapter 4

Wreath Product

4.1 Define Wreath Product

The wreath product of the groups H by K, denoted $H \wr K$, is a semi-direct product composed of as many copies of H as the number of letters on which the permutation group K acts on. We define the wreath product below.

Definition 4.1. *Let X and Y be non-empty sets. Let H be a permutation group on X and K on Y. Let $Z = X \times Y$.* ***The wreath product*** *is a permutation group on Z. We define a permutation group on Z such that we let $\gamma \in H$ and define a permutation of $\gamma(y)$ of Z by*

$$\gamma(y) = \begin{Bmatrix} (x, y_1) \to (\gamma(x), y_1) \text{ if } y_1 = y \\ (x, y_1) \to (x, y_1) \text{ if } y_1 \neq y \end{Bmatrix}$$

Definition 4.2. *Also, for $k \in K$, define $k^* : (x; y) = (x; (y)k)$ such that $B = \Pi_{y \in Y} H(y)$ is a direct product of the group generated by the $\gamma(y)s$. Thus, $G = B : K^*$ is called a wreath product of H and K, where H is normal subgroup, denoted by $H \wr K$.*

4.2 Constructing Wreath product

The wreath product of H and K, written H≀K, is a permutation group G on Z = X×Y denoted by $\prod_{y \in Y}$ γ(y) : K^*

Consider H=$< (1,2,3),(1,2) > \cong S_3$ and K=$< (4,5,6,7,8) > \cong 5$.

We will construct permutation generators of the wreath product $H \wr K$ of H and K, as well as give its presentation.

Now a presentation of H is {x,y—x^3,y^2,$(x*y)^2$ } and a presentation of K is $|z^5\}$.

Let H and K be permutation group on the sets X = {1,2,3} and Y = {4,5,6,7,8}, respectively.

We defined a permutation group G on Z = $X \times Y$

Z = $X \times Y$ = {(x,y) — x ∈ X , y ∈ Y }

We have, $X * Y$= { (1,4), (1,5), (1,6), (1,7), (1,8), (2,4), (2,5), (2,6), (2,7), (2,8), (3,4), (3,5), (3,6), (3,7), (3,8) }.

Let $\gamma \in H$, define the permutation $\gamma(y_1)$, where $y_1 \in Y$, as follows.

Using wreath product definition we let $\gamma = (123) \in H$ and $y \in Y$. We will compute $\gamma(4), \gamma(5), \gamma(6), \gamma(7), \gamma(8)$.

Now,

$$\gamma(y) = \left\{ \begin{matrix} (x,y_1) \to (\gamma(x),y_1) \text{ if } y_1 = y \\ (x,y_1) \to (x,y_1) \text{ if } y_1 \neq y \end{matrix} \right\},$$

The compute $\gamma(4)$ in the following table.

Also, by definition this computation of $\gamma(4)$ will only change elements which contain 1, 2, and 3 as the x-coordinate and 4 as the y-coordinate.

Compute $\gamma(4)$				
Labeling	Element	Compute γ	Element	Labeling
(9)	$(1,4)$	$(\gamma(1),4)$	$(2,4)$	(14)
(10)	$(1,5)$	$(\gamma(1),5)$	$(1,5)$	(10)
(11)	$(1,6)$	$(\gamma(1),6)$	$(1,6)$	(11)
(12)	$(1,7)$	$(\gamma(1),7)$	$(1,7)$	(12)
(13)	$(1,8)$	$(\gamma(1),8)$	$(1,8)$	(13)
(14)	$(2,4)$	$(\gamma(2),4)$	$(3,4)$	(19)
(15)	$(2,5)$	$(\gamma(2),5)$	$(2,5)$	(15)
(16)	$(2,6)$	$(\gamma(2),6)$	$(2,6)$	(16)
(17)	$(2,7)$	$(\gamma(2),7)$	$(2,7)$	(17)

Table 4.1: Compute $\gamma(4)$

Compute $\gamma(4)$				
Labeling	Element	Compute γ	Element	Labeling
(18)	$(2,8)$	$(\gamma(2),8)$	$(2,8)$	(18)
(19)	$(3,4)$	$(\gamma(3),5)$	$(1,4)$	(9)
(20)	$(3,5)$	$(\gamma(3),5)$	$(3,5)$	(20)
(21)	$(3,6)$	$(\gamma(3),6)$	$(3,6)$	(21)
(22)	$(3,7)$	$(\gamma(3),7)$	$(3,7)$	(22)
(23)	$(3,8)$	$(\gamma(3),8)$	$(3,8)$	(23)

Table 4.2: Compute $\gamma(4)$

From above table we got $\gamma(4) = (9, 14, 19)$.

The below table compute the $\gamma(5)$.

Compute $\gamma(5)$				
Labeling	Element	Compute γ	Element	Labeling
(9)	$(1,4)$	$(\gamma(1),4)$	$(1,4)$	(9)
(10)	$(1,5)$	$(\gamma(1),5)$	$(2,5)$	(15)
(11)	$(1,6)$	$(\gamma(1),6)$	$(1,6)$	(11)
(12)	$(1,7)$	$(\gamma(1),7)$	$(1,7)$	(12)
(13)	$(1,8)$	$(\gamma(1),8)$	$(1,8)$	(13)
(14)	$(2,4)$	$(\gamma(2),4)$	$(2,4)$	(14)
(15)	$(2,5)$	$(\gamma(2),5)$	$(3,5)$	(20)
(16)	$(2,6)$	$(\gamma(2),6)$	$(2,6)$	(16)
(17)	$(2,7)$	$(\gamma(2),7)$	$(2,7)$	(17)
(18)	$(2,8)$	$(\gamma(2),8)$	$(2,8)$	(18)
(19)	$(3,4)$	$(\gamma(3),4)$	$(3,4)$	(19)
(20)	$(3,5)$	$(\gamma(1),5)$	$(1,5)$	(10)
(21)	$(3,6)$	$(\gamma(3),6)$	$(3,6)$	(21)
(22)	$(3,7)$	$(\gamma(3),7)$	$(3,7)$	(22)
(23)	$(3,8)$	$(\gamma(3),8)$	$(3,8)$	(23)

Table 4.3: Compute $\gamma(5)$

From above table we got $\gamma(5) = (10, 15, 20)$.

Also, by definition this computation of $\gamma(6)$ will only change elements which contain 1, 2, and 3 as the x-coordinate and 6 as the y-coordinate.
The below table compute the $\gamma(6)$.

Compute $\gamma(6)$				
Labeling	Element	Compute γ	Element	Labeling
(9)	$(1,4)$	$(\gamma(1),4)$	$(1,4)$	(9)
(10)	$(1,5)$	$(\gamma(1),5)$	$(1,5)$	(10)
(11)	$(1,6)$	$(\gamma(1),6)$	$(2,6)$	(16)
(12)	$(1,7)$	$(\gamma(1),7)$	$(1,7)$	(12)
(13)	$(1,8)$	$(\gamma(1),8)$	$(1,8)$	(13)
(14)	$(1,4)$	$(\gamma(1),4)$	$(2,4)$	(14)
(15)	$(2,5)$	$(\gamma(2),5)$	$(2,5)$	(15)
(16)	$(2,6)$	$(\gamma(2),6)$	$(3,6)$	(21)
(17)	$(2,7)$	$(\gamma(2),7)$	$(2,7)$	(17)
(18)	$(2,8)$	$(\gamma(2),8)$	$(2,8)$	(18)
(19)	$(3,4)$	$(\gamma(3),4)$	$(3,4)$	(19)
(20)	$(3,5)$	$(\gamma(3),5)$	$(3,5)$	(20)
(21)	$(3,6)$	$(\gamma(3),6)$	$(1,6)$	(11)
(22)	$(3,7)$	$(\gamma(3),7)$	$(3,7)$	(22)
(23)	$(3,8)$	$(\gamma(3),8)$	$(3,8)$	(23)

Table 4.4: Compute $\gamma(6)$

From above table we got $\gamma(6) = (11, 16, 21)$.

Also, by definition this computation of $\gamma(7)$ will only change elements which contain 1, 2, and 3 as the x-coordinate and 7 as the y-coordinate.
The below table compute the $\gamma(7)$.

Compute $\gamma(7)$				
Labeling	Element	Compute γ	Element	Labeling
(9)	$(1,4)$	$(\gamma(1),4)$	$(1,4)$	(9)
(10)	$(1,5)$	$(\gamma(1),5)$	$(1,5)$	(10)
(11)	$(1,6)$	$(\gamma(1),6)$	$(1,6)$	(11)
(12)	$(1,7)$	$(\gamma(1),7)$	$(2,7)$	(17)
(13)	$(1,8)$	$(\gamma(1),8)$	$(1,8)$	(13)
(14)	$(1,4)$	$(\gamma(1),4)$	$(2,4)$	(14)
(15)	$(2,5)$	$(\gamma(2),5)$	$(3,5)$	(20)
(16)	$(2,6)$	$(\gamma(2),6)$	$(2,6)$	(16)
(17)	$(2,7)$	$(\gamma(2),7)$	$(3,7)$	(22)
(18)	$(2,8)$	$(\gamma(2),8)$	$(2,8)$	(18)
(19)	$(3,4)$	$(\gamma(3),4)$	$(3,4)$	(9)
(20)	$(3,5)$	$(\gamma(1),5)$	$(1,5)$	(10)
(21)	$(3,6)$	$(\gamma(3),6)$	$(3,6)$	(21)
(22)	$(3,7)$	$(\gamma(3),7)$	$(1,7)$	(12)
(23)	$(3,8)$	$(\gamma(3),8)$	$(3,8)$	(23)

Table 4.5: Compute $\gamma(7)$

From above table we got $\gamma(7) = (12, 17, 22)$.

Also, by definition this computation of $\gamma(8)$ will only change elements which contain 1, 2, and 3 as the x-coordinate and 8 as the y-coordinate.
The below table compute the $\gamma(8)$.

Compute $\gamma(8)$				
Labeling	Element	Compute γ	Element	Labeling
(9)	$(1,4)$	$(\gamma(1),4)$	$(1,4)$	(9)
(10)	$(1,5)$	$(\gamma(1),5)$	$(1,5)$	(10)
(11)	$(1,6)$	$(\gamma(1),6)$	$(1,6)$	(11)
(12)	$(1,7)$	$(\gamma(1),7)$	$(1,7)$	(12)
(13)	$(1,8)$	$(\gamma(1),8)$	$(2,8)$	(18)
(14)	$(1,4)$	$(\gamma(1),4)$	$(2,4)$	(14)
(15)	$(2,5)$	$(\gamma(2),5)$	$(3,5)$	(20)
(16)	$(2,6)$	$(\gamma(2),6)$	$(2,6)$	(16)
(17)	$(2,7)$	$(\gamma(2),7)$	$(2,7)$	(17)
(18)	$(2,8)$	$(\gamma(2),8)$	$(3,8)$	(23)
(19)	$(3,4)$	$(\gamma(3),4)$	$(3,4)$	(9)
(20)	$(3,5)$	$(\gamma(1),5)$	$(1,5)$	(10)
(21)	$(3,6)$	$(\gamma(3),6)$	$(3,6)$	(21)
(22)	$(3,7)$	$(\gamma(3),7)$	$(3,7)$	(22)
(23)	$(3,8)$	$(\gamma(3),8)$	$(1,8)$	(13)

Table 4.6: Compute $\gamma(8)$

From above table we got $\gamma(8) = (13, 18, 23)$.

Let $k \in K$ then define the permutation k_1^* and k_2^* of z as follows:
Now we have $K = \langle (4,5,6,7,8) \rangle \cong S_3$. Let $k_1 = (4,5,6,7,8)$ and $k_2 = (4,5)$. Then as the definition shows, k_1^*, k_2^* will change all Y elements. Let compute k_1^* then we will get

Action of k_1^*				
Labeling	Element	Action k_1^*	Element	Labeling
(9)	$(1,4)$	$(1,k_1^*(4))$	$(1,5)$	(10)
(10)	$(1,5)$	$(1,k_1^*(5))$	$(1,6)$	(11)
(11)	$(1,6)$	$(1,k_1^*(4))$	$(1,7)$	(12)
(12)	$(1,7)$	$(1,k_1^*(5))$	$(1,8)$	(13)
(13)	$(1,8)$	$(1,k_1^*(8))$	$(1,4)$	(9)
(14)	$(2,4)$	$(2,k_1^*(4))$	$(2,5)$	(15)
(15)	$(2,5)$	$(2,k_1^*(5))$	$(2,6)$	(16)
(16)	$(2,6)$	$(2,k_1^*(6))$	$(2,7)$	(17)
(17)	$(2,7)$	$(2,k_1^*(7))$	$(2,8)$	(18)

Table 4.7: Action of k_1^*

Action of k_1^*				
Labeling	Element	Action k_1^*	Element	Labeling
(18)	$(2,8)$	$(2,k_1^*(8))$	$(2,4)$	(14)
(19)	$(3,4)$	$(3,k_1^*(4))$	$(3,5)$	(20)
(20)	$(3,5)$	$(3,k_1^*(5))$	$(3,6)$	(21)
(21)	$(3,6)$	$(3,k_1^*(6))$	$(3,7)$	(22)
(22)	$(3,7)$	$(3,k_1^*(7))$	$(3,8)$	(23)
(23)	$(3,8)$	$(3,k_1^*(8))$	$(3,4)$	(19)

Table 4.8: Action of k_1^*

So, $k_1^* = (9,10,11,12,13)(14,15,16,17,18)(19,20,21,22,23)$.

Then computing k_2^* we will get

Action of k_2^*				
Labeling	Element	Action of k_2^*	Element	Labeling
(9)	$(1,4)$	$(1,k_2^*(4))$	$(1,5)$	(10)
(10)	$(1,5)$	$(1,k_2^*(5))$	$(1,4)$	(9)
(11)	$(1,6)$	$(1,k_2^*(6))$	$(1,6)$	(11)
(12)	$(1,7)$	$(1,k_2^*(7))$	$(1,7)$	(12)
(13)	$(1,8)$	$(1,k_2^*(8))$	$(1,8)$	(13)
(14)	$(2,4)$	$(2,k_2^*(4))$	$(2,5)$	(15)
(15)	$(2,5)$	$(2,k_2^*(4))$	$(2,4)$	(14)
(16)	$(2,6)$	$(2,k_2^*(6))$	$(2,6)$	(16)
(17)	$(2,7)$	$(2,k_2^*(7))$	$(2,7)$	(17)
(18)	$(2,8)$	$(2,k_2^*(8))$	$(2,8)$	(15)
(19)	$(3,4)$	$(3,k_2^*(4))$	$(3,5)$	(20)
(20)	$(3,5)$	$(3,k_2^*(5))$	$(3,4)$	(19)
(21)	$(3,6)$	$(3,k_2^*(6))$	$(3,6)$	(21)
(22)	$(3,7)$	$(3,k_2^*(7))$	$(3,7)$	(22)
(23)	$(3,8)$	$(3,k_2^*(8))$	$(3,8)$	(23)

Table 4.9: Action of k_2^*

So, $k_2^* = (9,10),(14,15),(19,20)$.

Now we will write the presentation of this group. We will label them as follow,

a = (9,14,19),

b = (13,18),

c = (10,15,20),

d = (10,15),

e = (11,16,21),

f = (11,16),

g = (12,17,22),

h = (12,17),

i = (13,18,23),

j = (13,18).

We will conjugate the elements from H with the elements from K such as,

a = γ1(4),

(9,14,19) $\rightarrow$ (10,15,20)=γ1(5)=c.

c = γ1(5),

(10,15,20) $\rightarrow$ (11,16,21)=γ1(6)=e.

e = γ1(6),

(11,16,21) $\rightarrow$(12,17,22)=γ1(7)=g.

g = γ1(7),

(12,17,22) $\rightarrow$ (13,18,23)=γ1(8)=i.

i= γ1(8),

(13,18,23) $\rightarrow$ (9,14,19) =γ1(4)=a.

b= γ2(4),

(9,14) $\rightarrow$ (10,15)=γ2(5) = d.

d= γ2(5),

(10,15) $\rightarrow$ (11,16)=γ2(6) = f.

f= γ2(6),

(11,16) $\rightarrow$(12,17)=γ2(7) = h.

h= γ2(7),

(12,17) $\rightarrow$ (13,18)=γ2(8) = j.

j= γ2(8),

(13,18) $\rightarrow$ (9,14)=γ2(4) = b.

Presentation Of my group:

H=$< (1, 2, 3) >$,

K=$< (1, 2, 3, 4, 5) >$.

a^k = $x1^k$= $x1^{(1,2,3,4,5)}$ = x2 = c,

Similarly,

b^k= $y1^k$= $y1^{(1,2,3,4,5)}$ = y2 = d,

c^k= $x2^k$= $x2^{(1,2,3,4,5)}$ = x3 = e,

d^k= $y2^k$= $y2^{(1,2,3,4,5)}$ = y3 = f,

e^k= $x3^k$= $x3^{(1,2,3,4,5)}$ = x4 = g,

f^k= $y3^k$= $y3^{(1,2,3,4,5)}$ = y4 = h,

g^k= $x4^k$= $x4^{(1,2,3,4,5)}$ = x5 = i,

h^k= $y4^k$= $y4^{(1,2,3,4,5)}$ = y5 = j,

i^k= $x5^k$= $x5^{(1,2,3,4,5)}$ = x6=a,

j^k= $y5^k$= $y5^{(1,2,3,4,5)}$ = y6 = b.

k^5, a^k = c, c^k = e, e^k = g, g^k = i, i^k = a,

b^k = d, d^k = f, f^k = h, h^k = j, $j^k = b$.

Magma Code for Wreath Product

G$< a, b, c, d, e, f, g, h, i, j >$:= Group$< a, b, c, d, e, f, g, h, i, j | a^3, b^2, (a * b)^2,$

c^3, d^2, $(c * d)^2$,

e^3, f^2, $(e * f)^2$,

g^3, h^2, $(g * h)^2$,

i^3, j^2, $(i * j)^2$,

(a,c), (a,d), (a,e), (a,f), (a,g), (a,h), (a,i), (a,j),

(b,c), (b,d), (b,e), (b,f), (b,g), (b,h), (b,i), (b,j),

(c,e), (c,f), (c,g), (c,h), (c,i), (c,j),

(d,e), (d,f), (d,g), (d,h), (d,i), (d,j),

(e,g), (e,h), (e,i), (e,j),

(f,g), (f,h), (f,i), (f,j),

(g,i), (g,j),

(h,i), (h,j)>;

$\sharp$G;

/*7776*/

6^5;

/*7776*/

G$< a, b, c, d, e, f, g, h, i, j, k >$:= Group$< a, b, c, d, e, f, g, h, i, j, k | a^3, b^2, (a * b)^2,$

c^3, d^2, $(c * d)^2$,

e^3, f^2, $(e * f)^2$,

g^3, h^2, $(g * h)^2$,

i^3, j^2, $(i * j)^2$,

(a,c), (a,d), (a,e), (a,f), (a,g), (a,h), (a,i), (a,j),

(b,c), (b,d), (b,e), (b,f), (b,g), (b,h), (b,i), (b,j),

(c,e), (c,f), (c,g), (c,h), (c,i), (c,j),

(d,e), (d,f), (d,g), (d,h), (d,i), (d,j),

(e,g), (e,h), (e,i), (e,j),

(f,g), (f,h), (f,i), (f,j),

(g,i), (g,j),

(h,i), (h,j),

k^5, $a^k = c$, $c^k =$ e, $e^k =$ g, $g^k =$ i,$i^k =$ a,

$b^k =$ d, $d^k =$ f, $f^k =$ h, $h^k =$ j, $j^k =$ b>;

♯G;

/*38880*/

$6^5 * 5$;

/*38880*/

f,G1,k:=CosetAction($G, sub < G|Id(G) >$);

W:=WreathProduct(Sym(3),CyclicGroup(5));

Next we will find the isomorphism type of the wreath product and N .

We will use the following inputted into magma.

IsIsomorphic(G1,W);

/* true Mapping from: GrpPerm: G1 to $GrpPerm$:

Composition of Mapping from: GrpPerm: G1 to GrpPC and

Mapping from: GrpPC to GrpPC and

Mapping from: GrpPC to GrpPerm: W

*/

Chapter 5

Finite Homomorphic Images

In this chapter we will discuss four involutory four progenitors. We will factor these progenitors by the first order relations, nt_i, where $n \in N$, t_i is a symmetric generator, and determine finite homomorphic images.

5.1 Progenitor $2^{*15} : (3^2 : S_3)$

We first give a symmetric presentation of the progenitor $2^{*15} : (3^2 : S_3)$.
N=$< xx, yy > \cong 3^2 : S_3$ where,
xx = (1, 15, 12, 8, 3, 9, 14, 13, 7, 4)(2, 11, 5, 6, 10), and
$yy = (1, 11, 14, 6, 12, 2, 7, 5, 3, 10)(4, 8, 13, 15, 9)$.
A presentataion of N is,
NN$< x, y, t >$:= Group$(y^{-1} * x^{-1})^3, (y^{-1} * x)^3, x^{-1} * y^{-1} * x^3 * y^{-1} * x^{-1} * y, x^2 * y * x^2 * y^3 >$.
N1 = Stabiliser(N,1).
$= < yy^{-1} * xx^2 * yy^{-1}, < xx^3 * yy^{-1} * xx >$.
Thus,
G = $2^{*15} : (3^2 : S_3)$ = G$< x, y, t >$:= Group$< x, y, t|$(y^{-1} $* \, x^{-1})^3$, $(y^{-1} * x)^3$, $x^{-1} * y^{-1}$ $* \, x^3 * y^{-1} * x^{-1} *$ y ,$x^2 *$ y $* \, x^2 * y^3$, (t,$y^{-1} * x^2 * y^{-1}$),(t,$x^3 * y^{-1} * x$) $>$.
We note that $|G| = \infty$.
We want to factor G by additional relations. There are many chooses for additional relations. A first order relations of the form $(nt_i)^a$ where n $\in$ N and A is a parameter.

We will factor G by the first order relations. We explain below how to obtain first order efficiently.

Class Representative	Elements of form πt_i
(2, 11, 5, 6, 10)(1, 9, 15, 14, 12, 13, 8, 7, 3, 4)	x
(1, 9, 6)(2, 14, 15)(3, 4, 11)(5, 12, 13), (7, 8, 10)	y
(2, 5, 10, 11, 6)(1, 12, 15, 3, 8, 9, 14, 13, 4, 7)	x^{-1}
(2, 10, 6, 5, 11)(1, 3, 15, 7, 9, 4, 12, 8, 13, 14)	y^{-1}
(2, 11, 5, 6, 10)(1, 14, 15, 12, 13, 8, 7, 4, 9, 3)	x^2
(2, 6, 11, 10, 5)(1, 7, 15, 14, 4, 13, 3, 9, 8, 12)	xy
(1, 14, 12, 7, 3)(2, 5, 10, 11, 6)(4, 8, 13, 15, 9)	xy^{-1}
(1, 7, 14, 3, 12)(2, 10, 6, 5, 11), (4, 13, 9, 8, 15)	yx
(2, 11, 5, 6, 10)(1, 9, 15, 14, 12, 13, 8, 7, 3, 4)	y^2
(2, 6, 11, 10, 5)(1, 9, 8, 7, 14, 15, 4, 3, 12, 13)	yx^{-1}
(2, 5, 10, 11, 6)(1, 9, 13, 12, 3, 4, 15, 14, 7, 8)	$x^{-1}y$
(2, 10, 6, 5, 11)(1, 9, 4, 3, 7, 8, 13, 12, 14, 15)	x^{-2}

Table 5.1: Conjugacy classes of elements of form πt_i

5.1.1 Progenitor $2^{*15} : N$ Factored By The First Order Relation

We run the following in magma to find finite homomorphic images. Some of the homomorphic images are given below.

Note that only the last image is a true image. for a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, u, v, w, z,

a1, b1, c1, d1 in [0..10] do

G$< x, y, t >$:= Group$< x, y, t|$ $(y^{-1} * x^{-1})^3$, $(y^{-1} * x)^3$,

$x^{-1} * y^{-1} * x^3 * y^{-1} * x^{-1} * y, x^2 * y * x^2 * y^3, t^2$,

(t, $y^{-1} * x^2 * y^{-1}$), (t,$x^3 * y^{-1} * x$),

$(x * t^{(y*x^{-1})})^a, (x * t)^b$, $(y * t)^c$, (y*$t^{(y*x^{-1})})^d$, (y*$t^{(y^{-2})})^e$,

$(x^{(-1)} * t^{(y*x^{-1})})^f$, $(x^{(-1)} * t)^g$, $(y^{(-1)} * t^{(y*x^{-1})})^h$, $(y^{(-1)} * t)^i$,

$(x^{(2)} * t^{(y*x^{-1})})^j$, $(x^{(2)} * t)^k$, (x*y*$t^{(y*x^{-1})})^l$,

$(x * y * t)^m$, (x*$y^{(-1)} * t)^n$,

$(x * y^{(-1)} * t^{(y*x^{(-1)})})^o$,

$(x * y^{(-1)} * t^{x^{(-1)p}}$,

$(y * x * t)^q$,

$(y * x * t^{(y*x^{(-1)})})^r$,

$(y * x * t^{x^{-1}})^s$,

$(y^2 * t^{y*x^{-1}})^u$,

$(y^{(2)} * t)^v$,

$(y * x^{-1} * t^{(y*x^{-1})})^w$,

$(y * x^{-1*t)})^z$,

$(x^{-1} * y * t^{(y*x^{(-1)})})^{a1}$,

$(x^{-1} * y * t)^{b1}$,

$(x^{-2} * t^{(y*x^{-1}})^{c1}$,

$(x^{-2} * t)^{d1} >$;

```
if Index (G, sub< G|x,y>) gt 1 then
a, b, c, d, e, f, g, h, i, j, k, l, m, n,
o, p,q, r, s, u, v, w, z, a1, b1, c1, d1,
Index (G, sub <G|x,y>); end if; end for;
```

1. $b1 := 3$; Index$(G, sub < G|x, y >)$;

   /*4*/

f,G1,k:=CosetAction($G, sub < G|x, y >$);

♯k;

/*25*/

♯sub$< G|x, y >$;

/*150*/

♯G1;

24/

CompositionFactors(G1);

```
    /*
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
*/
```

2.
```
a1:= 4; c1:= 2;
Inde(G,sub<G|x,y>);
/*8*/
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
/*25*/
#sub<G|x,y>;
/*150*/
#G1;
/*48*/
CompositionFactors(G1);
/*
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
```

```
    |  Cyclic(2)
    1
 */
```

3.
```
a1:= 5; c1:= 2;
Index(G,sub<G|x,y>);
/*20*/
f,G1,k:=CosetAction(G,sub<G|x,y>);
#k;
/*1*/
#sub<G|x,y>;
/*6*/
#G1;
/*120*/
CompositionFactors(G1);
/*
    G
    |  Alternating(5)
    *
    |  Cyclic(2)
    1
*/
```

4. $b1 := 5; c1 := 3;$

Index($G, sub < G|x, y >$);

/*832*/

♯sub$< G|x, y >$;

/*150*/

f,G1,k:=CosetAction($G, sub < G|x, y >$);

CompositionFactors(G1);

```
/*
    G
    |  Cyclic(2)
    *
    |  2A(2, 4)                      = U(3, 4)
    1
*/
```

♯DoubleCosets($G, sub < G|x, y >, sub < G|x, y >$);

/*12*/

```
♯k;
/*1*/
♯G1;
/*
124800
*/
```

5.2 Progenitor $2^{*25} : (D_5 \times D_5)$

We first give a symmetric presentation of the progenitor $2^{*25} : (D_5 \times D_5)$
N=$< xx, yy >$; where
xx = (1, 19, 11, 17, 2)(3, 16, 15, 5, 6, 9, 24, 8, 21, 22)(4, 18, 13, 25, 7, 14, 20, 10, 23, 12), and
$yy = (1,16)(2,8)(3,20)(4,24)(5,17)(6,13)(7,15)(9,19)(10,18)(11,22)(14,25)(21,23)$.
A presentataion of N is,
NN$< x, y, t >$:= Group$< x, y, t | y^2, (xyx)^2, x^{10}, x^{-1}yx^{-1}yx^{-1}yx^{-1}yx^{-1}yxyxyxyxyx^{-1}y >$.
N1 = Stabiliser(N,1),
$= yy*xx*yy*xx*yy*xx*yy*xx^-1*yy, xx*yy*xx*yy*xx*yy*xx^-1*yy*xx*yy >$.
Thus,
G = $2^{*25} : (D_5 \times D_5)$ = G$< x, y, t >$:=Group$< x, y, t | y^2, (x*y*x)^2, x^{10}, x^{-1}*y*x^{-1}*y*x^{-1}*y*x^{-1}*y*x^{-1}*y*x*y*x*y*x*y*x*y*x^{-1}*y$, t^2, $(t, y*x*y*x*y*x*y*x^{-1}*y)$, $(t, x*y*x*y*x*y*x^{-1}*y*x*y)$.
We note that $|G| = \infty$.
We want to factor G by additional relations. There are many chooses for additional relations. A first order relations of the form $(nt_i)^a$ where n $\in$ N and A is a parameter.

We will factor G by the first order relations. We explain below how to obtain first order efficiently.

Conjugacy classes of elements of form πt_i	
Class Representative	Elements of form πt_i
(1, 17, 2, 11, 19), (3, 6, 9, 22, 21, 5, 24, 15, 16, 8), (4, 23, 14, 7, 25, 13, 12, 10, 20, 18)	x
(6, 20, 19, 18, 22), (1, 5, 11, 9, 13, 4, 3, 14, 21, 10), (2, 8, 17, 16, 23, 7, 24, 12, 15, 25)	y
(7), (2, 15), (4, 23), (8, 12), (13, 20), (1, 21, 24, 17), (3, 11, 19, 6), (5, 14, 25, 16), (9, 10, 18, 22)	x^{-1}
(1, 5, 11, 18, 9, 20, 2, 25, 8, 23, 6, 19, 14,22, 4, 24, 3, 17, 10, 16, 13, 21, 15, 12, 7)	x^2
(1, 17, 4, 7, 23, 20, 21, 15, 24, 6, 13, 5, 8,16, 22, 3, 14, 12, 25, 18, 9, 2, 19, 10, 11)	xy
(1, 5, 2, 18, 8, 20, 17, 14, 11, 16, 4, 6, 9, 12, 19, 25, 7, 21, 22, 3, 23, 10, 15, 13, 24)	yx
(1, 17, 21, 7, 24, 20, 14, 8, 25, 22, 13, 4, 2, 23, 19, 9, 3, 5, 15, 16, 6, 11, 10, 12, 18)	yx^{-1}
(1, 13, 2, 15, 20, 17, 22, 24, 4, 11, 25, 5, 3, 7, 19, 10, 8, 6, 23, 18, 16, 21, 14, 9, 12)	$x^{-1}y$
(1, 15, 2, 25, 24, 17, 13, 10, 3, 11, 22, 20, 18, 6, 19, 5, 4, 14, 21, 8, 7, 12, 16, 23, 9)	x^{-2}
(1, 18, 2, 9, 14, 17, 24, 22, 12, 11, 7, 3, 5,25, 19, 23, 6, 8, 10, 13, 21, 16, 20, 15, 4)	x^3
(1, 9, 2, 7, 22, 17, 18, 23, 5, 11, 24, 14, 13, 8, 19, 3, 12, 20, 16, 6, 25, 4, 21, 10, 15)	x^2y
(1, 19, 11, 17, 2), (3, 9, 16, 24, 15, 8, 5, 21, 6, 22), (4, 14, 18, 20, 13,10, 25, 23, 7, 12)	xyx
(1, 17, 19, 2, 11), (3, 9, 5, 21, 24, 16, 22, 6, 15, 8), (4, 14, 25, 23, 20, 18, 12, 7, 13, 10)	xyx^{-1}
(6, 19, 22, 20, 18), (1, 11, 9, 5, 4, 13, 10, 14, 21, 3), (2, 17, 16, 8, 7, 23, 25, 12, 15, 24)	yx^2
(6, 20, 19, 18, 22), (1, 11, 10, 14, 5, 9, 3, 21, 4, 13), (2, 17, 25, 12, 8, 16, 24, 15, 7, 23)	yxy

Table 5.2: Conjugacy classes of elements of form πt_i

5.2.1 Progenitor $2^{*25} : N$ Factored By The First Order Relation

We run the following in magma to find finite homomorphic images. Some of the homomorphic images are given below.

Note that only the few images are a true images. for a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, u, v, w, z, a1, b1, c1, d1, e1, f1, g1, h1, i1, j1, k1, l1 in [0..10] do

G$< x, y, t >$:=Group$< x, y, t | y^2, (x*y*x)^2, x^{10},$

$x^{-1}*y*x^{-1}*y*x^{-1}*y*x^{-1}*y*x^{-1}*y*x*y*x*y*x*y*x*y*x^{-1}*y,$

$t^2, (t, y*x*y*x*y*x*y*x^{-1}*y),$

$(t, x*y*x*y*x*y*x^{-1}*y*x*y)$, $(x*t)^a,$

$(x*t^{(y*x^{-1}))})^b,$

$(x*t^{(x*y^2)})^c,$

$(y*t^{(x*y*x^{-1})})^d,$

$(y*t)^e,$

$(y*t^{(x^{-1})})^f,$

$(x^{(-1)}*t^{(y*x*y)})^g,$

$(x^{(-1)}*t^{x^{(-1)}})^h,$

$(x^{(-1)}*t^x*y^2)^i,$

$(x^{(-1)} * t^{(x^{(-1)} * y)})^j$,

$(x^{(-1)} * t^{(x * y * x^{(-1)} * y)})^k$,

$(x^{(-1)} * t)^l$,

$(x^{(-1)} * t^{(y * x^(-1))})^m$,

$(x^{(-1)} * t^{(y * x^(2))})^n$,

$(x^{(-1)} * t^{(x * y)})^o$,

$(x^2 * t)^p$,

$(x * y * t)^q$,

$(y * x * t)^r$,

$(y * x^{(-1)} * t)^s$,

$(x^{(-1)} * y * t)^u$,

$(x^{(-2)} * t)^v$,

$(x^3 * t)^w$,

$(x^2 * y * t)^z$,

$(x * y * x * t)^{a1}$,

$(x * y * x * t^{(y * x^{-1})})^{b1}$,

$(x * y * x * t^{(x * y)^2})^{c1}$,

$(x * y * x^{-1} * t)^{d1}$,

$(x * y * x^{-1} * t^{(y * x^{-1})})^{e1}$,

$(x * y * x^{-1} * t^{(x * y^2)})^{f1}$,

$(y * x^2 * t^{y * x})^{g1}$,

$(y * x^2 * t)^{h1}$,

$(y * x^{(2)} * t^{(x^{(-1)})})^{i1}$,

$(y * x * y * t^{(x * y * x^{-1})})^{j1}$,

$(y * x * y * t)^{h1}$,

$(y * x * y * t^{x^{-1}})^{k1} >$;

if $Index(G, sub < G|x, y >)$ gt 1 then

a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u,v,w,z,a1,b1,c1,d1,e1,f1,g1,h1,i1,j1,k1,l1,

$Index(G, sub < G|x, y >)$; end if; end for;

1. $j1 := 3;$
   Index$(G, sub < G|x, y >)$;
   /*144*/
   f,G1,k:=CosetAction$(G, sub < G|x, y >)$;
   $\sharp k$;
   /*1*/
   $\sharp$sub$< G|x, y >$;
   /*100*/
   $\sharp G1$;
   /*14400*/
   CompositionFactors(G1);
   /*

```
    G
    |  Alternating(5)
    *
    |  Alternating(5)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
*/
```

2. $i1 := 3; j1 := 3;$
   Index$(G, sub < G|x, y >)$;
   /*72*/
   f,G1,k:=CosetAction$(G, sub < G|x, y >)$;
   $\sharp k$;
   /*1*/
   $\sharp sub < G|x, y >$;
   /*100*/
   $\sharp G1$;
   /*7200*/
   CompositionFactors(G1);
   /*

```
     G
     |  Alternating(5)
     *
     |  Alternating(5)
     *
     |  Cyclic(2)
     1
*/
```

3. $h1 := 4; i1 := 4; j1 := 4;$

   Index($G, sub < G|x, y >$);

   /*384*/

   f,G1,k:=CosetAction($G, sub < G|x, y >$);

   $\sharp k$;

   /*1*/

   $\sharp sub < G|x, y >$;

   /*100*/

   $\sharp G1$;

   /*38400*/

   CompositionFactors(G1);

```
 /* G
     |  Cyclic(2)
     *
     |  Cyclic(5)
     *
     |  Cyclic(2)
     *
     |  Cyclic(2)
     *
     |  Cyclic(2)
     *
     |  Cyclic(2)
     *
     |  Cyclic(2)
     *
     |  Alternating(5)
     *
     |  Cyclic(2)
     1
*/
```

5.3 Progenitor $2^{*30} : S_5$

We first give a symmetric presentation of the progenitor $2^{*30} : S_5$

N=$< xx, yy >$; where

xx = (2, 3)(4, 6)(5, 8)(7, 11)(9, 14)(10, 15)(12, 18)(13, 20)(17, 24)(19, 26)(21,25)(23, 28), and

$yy = (1, 2, 4, 7, 12, 19)(3, 5, 9, 15, 22, 27)(6, 10, 16, 23, 18, 25)(8, 13, 21)$
$(11, 17, 14)(20, 26, 29, 30, 28, 24)$.

A presentataion of N is

NN$< x, y >$:= Group$< x, y|x^2, y^6, (y * x * y^{-1} * x)^2, (x * y^{-1})^5 >$.

N1 = Stabiliser(N,1),

$= < yy^2 * xx * yy^{-2} * xx * yy^2; yy^2 * xx * yy^{-2} * xx * yy^2 >$.

Thus,

G = $2^{*30} : S_5 =$ G$< x, y, t >$:=Group$< x, y, t|x^2, y^6, (y*x*y^{-1}*x)^2, (x*y^{-1})^5, t^2, (t, x), (t, y^2* x * y^{-2} * x * y^2) >$.

We note that $|G| = \infty$.

We want to factor G by additional relations. There are many chooses for additional relations. A first order relations of the form $(nt_i)^a$ where n $\in$ N and A is a parameter.

We will factor G by the first order relations. We explain below how to obtain first order efficiently.

Conjugacy classes of elements of form πt_i	
Class Representation	πt_i
$(8, 13, 21), (11, 17, 14), (1, 7, 2, 12, 19, 4), (6, 18, 23, 25, 10, 16), (3, 26, 15, 29, 28, 5, 24, 27, 22, 20, 9, 30)$	x
$(7, 25), (1, 11, 5, 20), (2, 19, 23, 10), (9, 24, 16, 21), (3, 26, 6, 13, 18, 30, 22, 8), (4, 12, 15, 17, 28, 29, 27, 14)$	y
$(8, 21, 13), (11, 14, 17), (1, 4, 2, 12, 7, 19), (3, 9, 5, 22, 15, 27), (6, 16, 10, 18, 23, 25), (20, 29, 26, 28, 30, 24)$	y^{-1}
$(7, 25), (1, 11, 5, 20), (2, 23, 10, 19), (3, 6, 13, 22), (4, 15, 17, 27), (8, 30, 26, 18), (9, 24, 16, 21), (12, 28, 29, 14)$	xy
$(1, 3, 8, 20, 19), (2, 6, 15, 22, 27), (4, 11, 24, 13, 25), (5, 14, 7, 18, 21), (9, 10, 16, 28, 17), (12, 26, 29, 30, 23)$	xy^{-1}
$(8, 13, 21), (11, 17, 14), (1, 2, 4, 7, 12, 19), (3, 5, 9, 15, 22, 27), (6, 10, 16, 23, 18, 25), (20, 26, 29, 30, 28, 24)$	yx

Table 5.3: Conjugacy classes of elements of form πt_i

5.3.1 Progenitor $2^{*30} : N$ Factored By The First Order Relation

We run the following in magma to find finite homomorphic images. The some of the homomorphic images are below.
Note that only few of images are a true images. for a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s, u,v,w,z ,a1,b1,c1,d1,e1,f1,g1,h1,i1,j1,k1,l1,m1,n1 in $[0 \ldots 10]$ do

$G < x, y, t > :=$ Group $< x, y, t | x^2, y^6, (y * x * y^{-1} * x)^2, (x * y^{-1})^5, (t, x),$
$(t, y^2 * x * y^{-2} * x * y^2),$
$(x * t^{(y*x^2)})^a$,
$(x * t^{(y^3*x)})^b$,
$(x * t)^c,$
$(x * t^{(y^2*x)})^d,$
$(x * t^{(y*x)})^e,$
$(y * t^{(y^3)})^f,$
$(y * t)^g,$
$(y * t^{(b)})^h,$
$(y * t^{(y*x*y^2)})^i,$
$(y * t^{(y*x)})^j,$
$(y * t^{(y^2)})^k,$
$(y^{-1} * t^{((y*x)^2})^l,$
$(y^{-1} * t^{(y^3*x)})^m,$
$(y^{-1} * t)^n,$
$(y^{-1} * t^{(y*x)})^o,$
$(y^{-1} * t^{(y^2*x)})^p,$
$(y * x * t^{(y^{-1}*x*y^{-1})})^q,$
$(x * y * t^{(y^3)})^r,$
$(x * y * t)^s,$
$(x * y * t^{(y)})^u,$
$(x * y * t^{(y*x)})^v,$
$(x * y * t^{(y^2)})^w,$
$(x * y * t^{(y*x)^2})^z,$
$(x * y * t^{(y*x*y^2)})^{a1},$

$(x * y * t^{(y^{-2})})^{b1}$,
$(x * y^{-1} * t)^{c1}$,
$(x * y^{-1} * t^{(y)})^{d1}$,
$(x * y^{-1} * t^{(y^2)})^{e1}$,
$(x * y^{-1} * t^{(y*x*y)})^{f1}$,
$(x * y^{-1} * t^{(y*x*y^2)})^{g1}$,
$(x * y^{-1} * t^{(y^{-2})})^{h1}$,
$(y * x * t^{((y*x)^2})^{i1}$,
$(y * x * t^{(y^3*x)})^{j1}, (y * x * t)^{k1}$,
$(y * x * t^{(y*x)})^{l1}$,
$(y * x * t^{(y^2*x)})^{m1}$,
$(y * x * t^{(y^{-1}*x*y^{-1})})^{n1} >$;
if Index($G, sub < G|x, y >$) gt 1 then
a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,u,v,w,z,
a1,b1,c1,d1,e1,f1,g1,h1,i1,j1,k1,l1,m1,n1, Index($G, sub < G|x, y >$); end if; end for;

1. $l1 := 3; m1 := 4$;
   Index($G, sub < G|x, y >$);
   /*6*/
   f,G1,k:=CosetAction($G, sub < G|x, y >$);
   ♯ k;
   /*1*/
   ♯$sub < G|x, y >$;
   /*120*/
   ♯$G1$;
   /*720*/
   CompositionFactors(G1);
   /*
   G
   | Cyclic(2)
   *
   | Alternating(6)
   1

*/

2. $j1 := 3; n1 := 4;$
   Index($G, sub < G|x, y >$);
   /*32*/
   f,G1,k:=CosetAction($G, sub < G|x, y >$);
   ♯k;
   /*1*/
   ♯sub$< G|x, y >$;
   /*120*/
   ♯G1;
   /*3840*/
   CompositionFactors(G1);
   /*
   G
   | Cyclic(2)
   *
   | Alternating(5)
   *
   | Cyclic(2)
   *
   | Cyclic(2)
   *
   | Cyclic(2)
   *
   | Cyclic(2)
   *
   | Cyclic(2)
   1
   */

3. $j1 := 4; k1 := 4; l1 := 8; m1 := 4;$

Index($G, sub < G|x, y >$);

/*4*/

f,G1,k:=CosetAction($G, sub < G|x, y >$);

$\sharp$k;

/*0*/

$\sharp$sub$< G|x, y >$;

/*0*/

$\sharp$G1;

/*4*/

CompositionFactors(G1);

/*

G

| Cyclic(2)

*

| Cyclic(2)

1

*/

5.4 Progenitor $2^6 : (3^2 : 2)$

We first give a symmetric presenatation of thr progenitor $2^6 : (3^2 : 2)$

$N =< xx, yy >$; where

xx = (1, 4)(2, 5)(3, 6), and

yy = (1,2,3).

A presenatation of N is,

NN$< x, y >:= Group < x, y|x^2, y^3, y^{-1} * x * y^{-1} * x * y * x * y * x >$.

N1 = Stabiliser(N,1),

$= < xx * yy^{-1} * xx >$.

Thus,

G = 2^6 : $(3^2$: 2) = G$< x, y, t >$:= Group$< x, y, t|x^2, y^3, y^{-1} * x * y^{-1} * x * y * x * y * x, t^2, (t, y^x), (t, t^y), (t, t^{(y^2)}), (t, t^x) >$.

We note that $|G| = \infty$.

We want to factor G by additional relations. There are many chooses for additional relations. A first order relations of the form $(nt_i)^a$ where n ∈ N and A is a parameter.

We will factor G by the first order relations. We explain below how to obtain first order efficiently.

We find the class representative and elements by applying below loop in magma.

C:=Classes(N);

$\sharp C$;

/* 9 */.

for i in [2. . . $\sharp C$] do i;

for j in [1 . . . $\sharp N$] do

if ArrayP[j] eq C[i][3]

then Sch[i]; end if;

end for;

Orbits(Centraliser(N,C[i][3]));

end for;

for j in [2 . . . 9] do for i in [1. . . ♯Sch] do if $1^{ArrayP[i]}$ eq j then j, Sch[i]; break ; end if; end for ; end for;

5.4.1 Progenitor $2^{*6} : N$ Factored By The First Order Relation

We run the following in magma to find finite homomorphic images. The some of the homomorphic images are below.

Note that only few of images are a true images. for a,b,c,d,e,f,g,h,i,j,k [0 . . . 10] do,
G$< x, y, t >:= Group$¡x, y, t— x$^2, y^3, y^{-1} * x * y^{-1} * x * y * x * y * x, t^2, (t, y^x), (x * t)^a, (y * t)^b, (y^{(-1)} * t)^c, (x * y * t)^d, (x * y * t^{(x)})^e, (x * y^{-1} * t)^f, (x * y^{-1} * t^{(x)})^g, (y * x * t)^h, (y * x * t^{(x)})^i, (y^{-1} * x * t)^j, (x * y * x * t)^k > .$
if Index(G,sub$< G|x, y >$) gt 1 then
a, b, c, d, e, f, g, h, i, j, k, Index($G, sub < G|x, y >$); end if; end for;

1. $h := 4;$
   G$< x, y, t >$:=Group$< x, y, t|x^2, y^3, y^{-1} * x * y^{-1} * x * y * x * y * x, t^2, (t, y^x), (x * t)^a, (y * t)^b, (y^{(-1)} * t)^c, (x * y * t)^d, (x * y * t^{(x)})^e, (x * y^{-1} * t)^f, (x * y^{-1} * t^{(x)})^g, (y * x * t)^h, (y * x * t^{(x)})^i, (y^{-1} * x * t)^j, (x * y * x * t)^k >$;
   Index($G, sub < G|x, y >$);
   /*128*/
   f,G1,k:=CosetAction($G, sub < G|x, y >$);
   $\sharp k$;
   /*1*/
   $\sharp sub < G|x, y >$;
   /*18*/
   $\sharp G1$;
   /*216*/

```
CompositionFactors(G1);
   G
   |  Cyclic(2)
   *
   |  Cyclic(2)
   *
   |  Cyclic(2)
   *
   |  Cyclic(3)
   *
   |  Cyclic(3)
   *
   |  Cyclic(3)
```

```
    1
```

2. $h := 5;$

   Index$(G, sub < G|x, y >);$

   /*135*/

   $f, G1, k := CosetAction(G, sub < G|x, y >);$

   $\sharp k;$

   /*1*/

   $\sharp sub < G|x, y >;$

   /*18*/

   $\sharp G1;$

   /*2430*/

```
CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(5)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1
```

3. $j := 5; k := 2;$

   $Index(G, sub < G|x, y >);$

   /*5*/

   $f, G1, k := CosetAction(G, sub < G|x, y >);$

   $\sharp k;$

   /*1*/

   $\sharp sub < G|x, y >;$

/*2*/

$\sharp G1$;

/*10*/

```
  CompositionFactors(G1);
/*
    G
    |  Cyclic(2)
    *
    |  Cyclic(5)
    1
*/
```

4. $j := 7; k := 2;$

   $Index(G, sub < G|x, y >)$;

   /*7*/

   f,G1,k:=$CosetAction(G, sub < G|x, y >)$;

   $\sharp k$;

   /*1*/

   $\sharp sub < G|x, y >$;

   /*2*/

   $\sharp G1$;

   /*14*/

```
CompositionFactors(G1);
/*
    G
    |  Cyclic(2)
    *
    |  Cyclic(7)
    1
*/
```

5. $j := 10; k := 2;$

   Index$(G, sub < G|x, y >)$;

   /*10*/

   f,G1,k:=CosetAction$(G, sub < G|x, y >)$;

   $\sharp k$;

/*1*/

$\sharp sub < G|x, y >$;

/*2*/

$\sharp G1$;

/*20*/

```
  CompositionFactors(G1);
/*
    G
    |  Cyclic(2)
    *
    |  Cyclic(5)
    *
    |  Cyclic(2)
    1
*/
```

6. $j := 8; k := 2$;

   Index($G, sub < G|x, y >$);

   /*8*/

   f,G1,k:=CosetAction(G,sub$< G|x, y >$);

   $\sharp k$;

   /*1*/

   $\sharp sub < G|x, y >$;

   /*2*/

   $\sharp G1$;

   /*16*/

```
 CompositionFactors(G1);
/*
    G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
*/
```

7. $j := 9; k := 2;$ Index$(G, sub < G|x, y >);$

   /*9*/

   $f, G1, k := CosetAction(G, sub < G|x, y >);$

   $\sharp k;$

   /*1*/

   $\sharp sub < G|x, y >;$

   /*2*/

   $\sharp G1;$

   /*18*/

```
CompositionFactors(G1);
/*
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1
*/
```

Chapter 6

Isomorphism Types

Definition 6.1. *Let H and K be groups. The* ***direct product*** *of H and K, denoted* $H \times K$*, is the group with all elements as ordered pairs having the form (h, k) where* $h \in H$*, k* $\in K$*, and with operation* $(h, k)(h',k') = (hh',kk')$ *.*

Definition 6.2. *Let G be a group. Then G is a* ***semi-direct product*** *of K by Q if K* $\triangle$ *G and K has a complement of* $Q_1 \cong Q$ *.*

Definition 6.3. *Let K and Q be groups. Then a group G, with* $K_1 \triangle G$*, is an* ***extension*** *of K by Q where* $K_1 \cong K$ *and* $G/K_1 \cong Q$*.*

Definition 6.4. *Let G be a group with* $H \leq G$ *and* $N \leq G$ *such that* $|G| = |N||H|$*. Then G is a* ***central extension*** *by H, denoted* $G \cong N \cdot H$*, if N is the center of G .*

Definition 6.5. *Let G be a group with* $H \leq G$*,* $N \leq G$*, and* $N \triangle G$ *such that* $|G| = |N||H|$*. Then G is a* ***mixed extension*** *by H, denoted* $G \cong N{:}H$*, if G is formed by both central extension and semi-direct products .*

Largest Normal Abelian Subgroup In this chapter, we evaluate each image of the progenitor G that noted to be faithful and whose number of subgroups generated by x and y are equal to the order of our control group N, we will focus on the composition factors to find the rough shape of the images. We will regard the rough shape as the isomorphism type of the groups. There are four types of extension direct and semi-direct product, mixed and central extension. We will include examples of each type using similar road and most importantly using composition factors and normal lattice of each group.

6.1 Semi-Direct Product $3^2 : 2$

Consider the group G generated by

$xx = (1,4)(2,5)(3,6)$

and $yy = (1,2,3)$

A presentation of G is:

```
G<x,y>:=Group<x,y| x^2,y^3,y^-1 * x * y^-1 * x *y * x * y * x>.
```

We will find the isomorphisim type of G by first analyzing the composition factors and the normal lattice of G.

```
G = <xx,yy >;

CompositionFactors(G);
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1
```

<u>Normal Lattice</u>

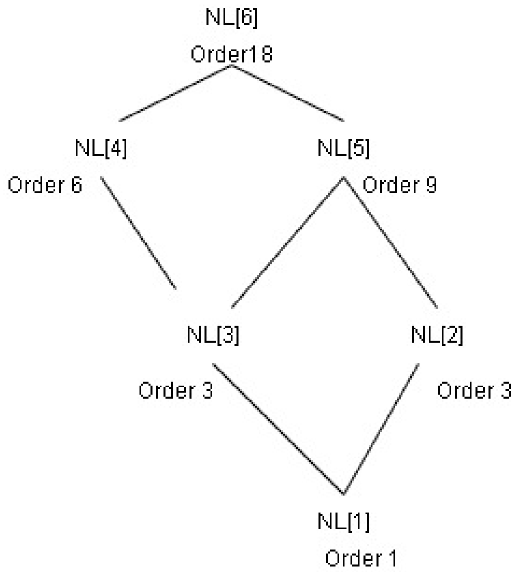

Figure 6.1: $3^2 : 2$

By looking at the composition factors of G it is not clear what the isomorphism type is. After analyzing the normal lattice of G, we see that $NL[5]$ is the largest normal abelian subgroup. The order of $NL[5]$ is 9 and the order of G is 18. Now $\frac{18}{9} = 2$. But G does not have a normal subgroup of order 2. Thus, G is an extension of $NL[5]$ by a group say $q = G/NL[5]$ but it is not a direct product because q is not isomorphic to a normal subgroup of G.

Now, we need to investigate to see whether G is the semi-direct product (split extension) of $NL[5] \cong 3^2$ by $q \cong 2$.

We note that NL[5] is isomorphic to 3×3, written 3^2, and NL[5] $= < A, B >$, where
A = (1,3,2) and
B = (1, 2, 3)(4, 6, 5).
q $= < (1, 2), Id(q) >$. Let T be the set of right coset representatives of N in G.
Now T[2] $\equiv$ q.1 and T[3] eq q.2.
We have $A^{T[2]}$ = AB and $B^{T[2]} = B^2$
Thus, we see $a^c = ab, b^c = b^2$.
We add these results in our presentation of H and verify that it is isomorphic to G to

confirm our presentation of G.

```
H<a,b,c>:=Group<a,b,c|a^3,b^3,(a,b),c^2,a^c=a*b,b^c=
b^2>;
f,H1,k:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(N,H1);
True
```

This $G \cong 3^2 : 2$.

6.2 Semi-Direct Product $2^2 : S_3$

We are given N is a transitive group on 8 letters which generated by

$xx = (1,6)(2,5)(3,7)(4,8)$,

$yy = (1,3,8)(4,5,7)$.

We begin by analyzing the composition factors and the normal lattice of N.

```
N = sub < S|xx,yy >,
CompositionFactors(N);
/*
G
| Cyclic(2)
*
| Cyclic(3)
*
| Cyclic(2)
*
| Cyclic(2)
1
*/
NL:=NormalLattice(N);
NL;
/*
Normal subgroup lattice
-----------------------
[4] Order 24 Length 1 Maximal Subgroups: 3
---
[3] Order 12 Length 1 Maximal Subgroups: 2
---
[2] Order 4 Length 1 Maximal Subgroups: 1
---
[1] Order 1 Length 1 Maximal Subgroups:
*/
```

Normal Lattice We then look for the largest abelian subgroup using the code.

foriin$[1 \cdots \sharp NL]$doif IsAbelian(N L[i])then i; end if ; end for;

/*

1

2

*/

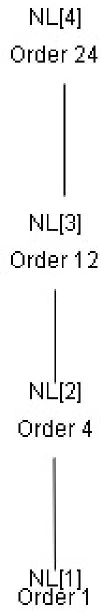

Figure 6.2: $2^3 : 3$

By looking at the composition factors of N, it is not clear what the isomorphism type may be.

After analyzing the normal lattice. we see that $NL[2]$ is the largest normal abelian subgroup. The order of $NL[2]$ is 4.

Now, we need to investigate to see the group is the semi-direct product (split extension) of $NL[2]$ by q. It is clear that $NL[2]$ has generators.

$A = (1,3)(2,8)(4,6)(5,7)$,

$B = (1,8)(2,3)(4,5)(6,7)$.

We see below that $NL[2] \cong 2^2$.

$X := [2,2]$;

IsIsomorphic(NL2,AbelianGroup(GrpPerm,X));

/* true */

FPGroup(q);

```
/*
Finitely presented group on 2 generators
```

```
Relations
£.1^2 = Id(£)
£.2^-3 = Id(£)
116
(£.2^-1 * £.1)^2 = Id(£)
*/
```

The group q has two generators say q.1 , q.2. We label them c , d, respectively. We note that $q \cong S_3$. Our next step is to find the action of the generators of q (c ,d) on the generators A , B of $NL[2]$. In order to do so, we need to look at the transversals of NL[2].

```
ff(T[1])eq q.1;

ff(T[2])eq q.2;

G< a, b, c, d >:= Group < a, b, c, d|a^2, b^2,
(a, b), c^2, d^-3, (d^-1* c)^2, a^c = b, a^d =b,
b^c = a,b^d = a * b >;
#G;
/* 24 */
f,G,K:=CosetAction(G, sub < G|Id(G) >);
#G1;
IsIsomorphic(N,G1);
/*
true Mapping from: GrpPerm: N to GrpPerm: G1
Composition of Mapping from: GrpPerm: N to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: G1
*/
```

This tells is that we have the semi-direct product NL[2]:q, where NL[2]$\cong 2^3$ and $q \cong S_3$. Thus we have N isomorphic to the semi-direct product $2^3 : S_3$.

6.3 Semi-Direct Product $S_5 = (A_5 : 2)$

We are given G is a transitive group on 30 letters, which is generated by
$xx = (2,3)(4,6)(5,8)(7,11)(9,14)(10,15)(12,18)(13,20)(17,24)(19,26)(21,25)(23,28)$
$yy = (1,2,4,7,12,19)(3,5,9,15,22,27)(6,10,16,23,18,25)(8,13,21)(11,17,14)$
$(20,26,29,30,28,24)$,

We will find the isomorphisim type of G by first analyzing the composition factors and the normal lattice of G:

```
CompositionFactors(G1);
  G
    |  Cyclic(2)
    *
    |  Alternating(5)
    1
```

Normal Lattice

Figure 6.3: s_6

By looking to the composition factors of G it is not clear what the isomorphism type. We see that NL[2] is normal in G and q = G/NL[2] $\cong$ 2. Thus we have $G \cong NL[2] : q$. Since NL[2] $\cong Alt(5)$ and $q \cong 2$, $G \cong A_5 : 2 \cong S_5$.

Chapter 7

Linear Maps

7.1 Linear Map of $PSL(2,7)$

Let X $=Z_7 \cup \{\infty\}$. The three linear maps on X that generates $PSL(2,7)$ are given by: α, β, γ where
$\alpha : x \rightarrow x+1$,
$\beta : x \rightarrow Kx$,
K is a nonzero square in F_7 whose powers give all of the squares of F_7, and
$\gamma : x \rightarrow \frac{-1}{x} = -x^{-1}$.
Then $\alpha = (\infty), (0,1,2,3,4,5,6)$.
In order to give the permutations for β , we need to find all nonzero squares for F_7

Square		
square power	$\cong 23$	Result
0^2	modulo 7	0
1^2	modulo 7	1
2^2	modulo 7	3
$3^2 = 9$	modulo 7	2
$4^2 = 16$	modulo 7	2
$5^2 = 25$	modulo 7	4
$6^2 = 36$	modulo 7	1
$7^2 = 49$	modulo 7	0

Table 7.1: Compute β

The squares we have are $(1, 2, 3, 4)$.

Now we need to find the smallest nonzero squares k whose power gives all of nonzero squares. We use 2, since

$2^0 = 1$,

$2^1 = 2$,

$2^2 = 4$,

$2^3 = 1$.

Thus, $\beta = (1, 2, 4)(3, 6, 5)$.

Now we will compute γ.

$\gamma : x \rightarrow \frac{-1}{x} = -x^{-1}$

We have $\gamma = (7, 8)(2, 3)(4, 5)(6, 1)$. We use magma to verify that $PSL(2, 7) =< \alpha, \beta, \gamma >$.

```
S:=Sym(8);
a:=S!(1,2,3,4,5,6,7);
b:=S!(1,2,4)(3,6,5);
g:=S!(7,8)(2,3)(4,5)(6,1);
psl27:=sub<S|a,b,g>;
#psl27;
IsIsomorphic(PSL(2,7),psl27);
true Homomorphism of GrpPerm: $, Degree 8, Order 2^3 * 3
* 7 into GrpPerm: psl27, Degree 8, Order 2^3 * 3 * 7
induced by
```

```
(3, 6, 7)(4, 5, 8) |--> (1, 7, 6)(2, 8, 5)
(1, 8, 2)(4, 5, 6) |--> (2, 8, 6)(3, 4, 5)
```

7.1.1 Linear Fractional Maps

We know that $PSL(2,7) = \{x \mapsto \frac{ax+b}{cx+d} | a,b,c,d \in Z_7, ad-bc = 1$ or $ad-bc$ is a squares $\}$, where $x \in X$.

We need to compute the linear fractional maps and see what they give us using the induced permutations that magma gives us in the previous isomorphism command.

It is given that $PSL(2,7) \cong < A, B >$ where,

A $= (1,7,6)(2,8,5)$,

B $= (2,8,6)(3,4,5)$.

We will compute linear maps, $\frac{ax+b}{cx+d}$, for A and B.

First equation

$\frac{a+b}{c+d} \implies 7a+b = 7c+d$,

Second equation

$\frac{7a+b}{7c+d} = 6 \implies 7a+b = 142c+6d$,

First, we will calculate a linear map for A and check our results.

$\frac{ax+b}{cx+d}$,

$\frac{a+b}{c+d} = 2 \implies a+b = 2c+d$,

$\frac{2a+b}{2c+d} = 7 \implies 2a+b = 14c+7d$,

Similarly, we will find a linear map for B following the same process.

$\frac{a+b}{c+d} = 8 \implies a+b = 8c+d$,

$\frac{8a+b}{8c+d} = 2 \implies 8a+b = 16c+8d$,

We solve the above equations to get the linear maps for A and B and check if it works for all elements.

7.2 Linear Map of $PSL(2,13)$

Let $X = Z_{13} \cup \infty$. The threelinear maps on X that generates $PSL(2,13)$ are given by: α, β, γ where,

$\alpha : x \to x+1$,

$\beta : x \to Kx$,

K is a nonzero square in F_{13} whose powers give all of the squares of F_{13}, and

$\gamma : x \to \frac{-1}{x} = -x^{-1}$.

Then $\alpha = (\infty), (0,1,2,3,4,5,6,7,8,9,10,11,12)$.

In order to find β permutations, we need to find all nonzero squares for F_{13}.

Square		
square power	$\cong 23$	Result
0^2	modulo 13	0
1^2	modulo 13	1
2^2	modulo 13	1
3^2	modulo 13	4
$4^2 = 16$	modulo 13	3
$5^2 = 25$	modulo 13	12
$6^2 = 36$	modulo 13	10
$7^2 = 49$	modulo 13	10
$8^2 = 64$	modulo 13	12
$9^2 = 81$	modulo 13	3
$10^2 = 100$	modulo 13	9
$11^2 = 121$	modulo 13	4
$12^2 = 144$	modulo 13	1
$13^2 = 169$	modulo 13	0

Table 7.2: Compute β

The squares we have are $(1,3,4,9,10,12)$.

Now we need to find the smallest nonzero squares K whose power gives all of nonzero squares. We use 3, since

$3^0 = 1,$

$3^1 = 3,$

$3^2 = 9,$

$3^3 = 81 = 3,$

$3^4 = 243 = 9,$

$3^5 = 729 =, 1$

$3^6 = 2187 = 5,$

$3^7 = 6561 = 4,$

$2^8 = 19683 = 1.$

$\beta : x \to 2x,$

$1 \to 3 \to 9 \to 5 \to 4$

$(1, 3, 9, 5, 4).$

$2 \to 6 \to 7 \to 10 \to 8$

$(2, 6, 7, 10, 8).$

Thus, $\beta = (1, 3, 9, 5, 4)(2, 6, 7, 10, 8).$

Now we will compute γ.

$\gamma : x \mapsto \frac{-1}{x} = -x^{-1}$

$\gamma : x \to \frac{-1}{x} = -x^{-1}.$

We have $\gamma = (11, 12)(1, 10)(2, 5)(3, 7)(4, 8)(6, 9).$

We use magma to verify that $PSL(2, 13) =< \alpha, \beta, \gamma > .$

```
S:=Sym(12);
a:=S!(11,1,2,3,4,5,6,7,8,9,10);
b:=S!(1,3,9,5,4)(2,6,7,10,8);
g:=S!(11,12)(1,10)(2,5)(3,7)(4,8)(6,9);
psl211:=sub<S|a,b,g>;
#PSL(2,11);
IsIsomorphic(PSL(2,11),psl 211);
true Homomorphism of GrpPerm: $, Degree 12, Order 2^2 *
3 * 5 * 11 into GrpPerm: psl211, Degree 12, Order 2^2 *
3 * 5 * 11 induced by
(3, 7, 9, 4, 5)(6, 8, 12, 10, 11) |--> (1, 2, 9, 10,12)(3, 7, 4, 8, 11)
(1, 8, 2)(3, 4, 7)(5, 12, 11)(6, 9, 10) |--> (1, 3, 4)(2, 12, 9)
(5, 11, 6)(7, 8, 10).
```

7.2.1 Linear Fractional Maps

We know that $PSL(2, 13) = \{x \mapsto \frac{ax+b}{cx+d} | a, b, c, d \in Z_{13}, ad - bc = 1$ or $ad - bc$ is a squares $\}$, where $x \in X$.

We need to compute the linear fractional maps and see what they give us using the induced permutations that magma gives us in the previous isomorphism command.

It is given that $PSL(2, 13) \cong < A, B >$ where,

A = $(1, 2, 9, 10, 12)(3, 7, 4, 8, 11)$,

B = $(1, 3, 4)(2, 12, 9)(5, 11, 6)(7, 8, 10)$.

We will compute linear maps, $\frac{ax+b}{cx+d}$ for A and B.

First equation

$\frac{a+b}{c+d} \implies a + b = 2c + 2d$,

Second equation

$\frac{2a+b}{2c+d} = 9 \implies 2a + b = 18c + 9d$,

Third equation

$\frac{18a+b}{18c+9d} = 10 \implies 18a + b = 180 + 90d$.

First, we will calculate a linear map for A and check our results.

$\frac{ax+b}{cx+d}$

First, we will calculate a linear map for A and check our results.

$\frac{ax+b}{cx+d}$,

$\frac{a+b}{c+d} = 2 \implies a + b = 2c + 2d$,

$\frac{2a+b}{2c+d} = 5 \implies 2a + b = 10c + 5d$,

$\frac{10a+b}{10c+5d} = 3 \implies 10a + b = 30c + 15d$,

$\frac{30a+b}{30c+15d} = 8 \implies 30a + b = 180c + 120d$.

Similarly, we will find a linear map for B following the same process.

$\frac{a+b}{c+d} = 12 \implies a + b = 12c + 12d$,

$\frac{12a+b}{12c+d} = 6 \implies 12a + b = 72c + 6d$.

We solve the above equations to get the linear maps for A and B and check if it works for all elements.

Chapter 8

Double Coset Enumeration

8.1 Double Coset Enumeration Of S_6 Over $2^3 : 2$

Consider N=$< x, y >$; where x $\sim$ (1, 4)(2, 5)(3, 6), y $\sim$ (1,2,3).

Our progenitor is $2^{*6} : (2^3 : 2)$. We prove that $S_6 \cong$

$Group < x, y, t | x^2, y^3, y^{-1}xy^{-1}xyxyx, t^2, (t, y^x), (yxt)^5 >$.

We perform manual double coset enumeration of G over N. In order to find the order of G, we need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw.

Now, $\frac{|G|}{|N|} = \frac{2430}{18}$ =135. So, we have total 135 single cosets.

We expand the relation:$(y * x * t)^5$.

Now $(y * x)^5 = (y * x)^5 * t_1^{(y*x)^4} * t_1^{(y*x)^3} * t_1^{(y*x)^2} * t_1^{(y*x)} * t_1$,

So, above relation become:

$(y * x)^5 * t_3 * t_6 * t_2 * t_5 * t_1$.

• **First Double Coset [*]**

$NeN = \{Ne^n | n \in N\} = \{N\}$.

The double coset NeN is denoted by $[*]$ which contains 1 right coset. The coset stabiliser of the coset Ne is N.

The number of right cosets in [*] is equal to $\frac{|N|}{|N|} = \frac{18}{18} = 1$.

Since N is transitive on $X = \{1, 2, 3, 4, 5, 6\}$,

we need to determine the double coset of the right coset Nt_1.

Thus, the six cosets $\{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6\}$ extend to the new double coset [1], that means the six generators go forward to Nt_1

• **Second Double Coset** $Nt_1N = [1]$

$Nt_1N = \{Nt_1^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6\}$. Firstly, the point stabilizer of 1 in N,

$N^1 = \{n \in N | 1^n = 1\}$

We have, $N^1 = <(4,5,6)>$.

The number of right cosets in [1] is equal to

$\frac{|N|}{|N^{(1)}|} = \frac{18}{3} = 6.$

The orbits of $N^{(1)}$ on X $= \{1,2,3,4,5,6\}$ are $\{1\}, \{2\}, \{3\}, and \{4,5,6\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_1, Nt_1t_2, Nt_1t_3$, and Nt_1t_4 belong.

Now,

$Nt_1t_1 = Nt_1^2 = N \in [*]$.

Since the orbit $\{1\}$ contains one element, one symmetric generator goes back to the double coset [*].

One symmetric generator will go back to [1].

$Nt_1t_2 \in Nt_1t_2N$, which is a new double coset. We denote this double coset by [12].

One symmetric generators will go to the new double coset [12].

$Nt_1t_3 \in Nt_1t_3N$, which is a new double coset. We denote this double coset by [13].

One symmetric generators will go to the new double coset [13].

$Nt_1t_4 \in Nt_1t_4N$, is a new double coset which we will denote [14].

Three symmetric generators will go to the new double coset [14].

• **Third Double Coset** $Nt_1t_2N = [12]$

$Nt_1\ t_2 = \{N(t_1t_2)^n | n \in N\}$.

We now find the coset stabilizer $N^{(12)}$. Firstly, find the point stabilizer of 1 and 2 in N.

$N^{12} = \{n \in N | (12)^n = 12\}$.

Thus, $N^{12} = <(4,5,6)>$

Nt_1t_2N is denoted by [12].

The number of right cosets in [12] is equal to
$\frac{|N|}{|N|^{(12)}} = \frac{18}{3} = 6$.
The orbits of $N^{(12)}$ on X= $\{1,2,3,4,5,6\}$ are $\{1\},\{2\},\{3\}, and\{4,5,6\}$.
Now we select a represenatative from each orbit and determine to which double coset $Nt_1t_2t_1, Nt_1t_2t_3, Nt_1t_2t_4$ belong.
As $Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1]$.

$Nt_1t_2t_1 \in Nt_1t_2t_1N = [121]$ is a new double coset. We donate this double coset by [121] One symmetric generator will go to the new double coset [121].

$Nt_1t_2t_3 \in Nt_1t_2t_3N$ is a new double coset. We donate this double coset by [123] One symmetric generator will go to the new double coset [123]

$Nt_1t_2t_4 \in Nt_1t_2t_4N$ is a new double coset. We donate this double coset by [124] Three symmetric generator will go to the new double coset [124]

- **Fourth Double Coset** $Nt_1t_3N = [13]$

$Nt_1\ t_3 = \{N(t_1t_3)^n | n \in N\}$.
Firstly, the point stabilizer of 1 and 3 in N.
$N^{(13)} = \{n \in N | (13)^n = 13\}$
$Nt_1\ t_3N$ is denoted by [13]
The number of right cosets in [13] is equal to
$\frac{|N|}{|N|^{(13)}} = \frac{18}{3} = 6$.
The orbits of $N^{(13)}$ on X= $\{1,2,3,4,5,6\}$ are $\{1\},\{2\},\{3\},\{4,5,6\}$.
Now we select a representative from each orbit and determine to which double coset $Nt_1t_3t_1, Nt_1t_3t_3, Nt_1t_3t_4$ belong.
This shows us the following:
$Nt_1t_3t_3 = Nt_1t_3^2 = Nt_1 \in [1]$
Thus one symmetric generator will go back to [1].

$Nt_1t_3t_4 \in Nt_1t_3t_4N$ which is a new double coset. We donate this double coset by [134]. Three symmetric generator will go to the new double coset [134].

- **Fifth Double Coset** $Nt_1t_4N = [14]$

$Nt_1\ t_4 = \{N(t_1t_4)^n | n \in N\}$.
Firstly, the point stabilizer of 1 and 4 in N.
$N^{(14)} = \{n \in N | (14)^n = 14\}$
$Nt_1\ t_4N$ is denoted by [14]
The number of right cosets in [14] is equal to
$\frac{|N|}{|N|^{(14)}} = \frac{18}{1} = 18$.
The orbits of $N^{(14)}$ on X= $\{1,2,3,4,5,6\}$ are $\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_4t_1, Nt_1t_4t_3, Nt_1t_4t_4, Nt_1t_4t_5, Nt_1t_4t_6$ belong.
This shows us the following:

$Nt_1t_4t_1 \in Nt_1t_4t_1N$ which is a new double coset. We donate this double coset by [141]. One symmetric generator will go to the new double coset [1411].

- **Sixth Double Coset** $Nt_1t_2t_1N = [121]$

$Nt_1\ t_2t_1 = \{N(t_1t_2){t_1}^n | n \in N\}$.
Firstly, the point stabilizer of 1, 2 and 1 in N.
$N^{(121)} = \{n \in N | (121)^n = 121\}$
$Nt_1\ t_2t_1N$ is denoted by [121]
The number of right cosets in [121] is equal to

$\frac{|N|}{|N|^{(121)}} = \frac{18}{3} = 6$.
The orbits of $N^{(121)}$ on X= $\{1,2,3,4,5,6\}$ are $\{1\},\{2\},\{3\},\{4,5,6\}$.
Now we select a representative from each orbit and determine to which double coset

$Nt_1t_2t_1t_1, Nt_1t_2t_1t_2, Nt_1t_2t_1t_3, Nt_1t_2t_2t_4$ belongs.

This shows us the following:

$Nt_1t_2t_1t_3 \in Nt_1t_2t_1t_3N$ which is a new double coset. We donate this double coset by [1213]. One symmetric generator will go to the new double coset by [1213].

- **Seventh Double Coset** $Nt_1t_2t_3N = [123]$

$Nt_1\ t_2t_3 = \{N(t_1t_2){t_3}^n|n \in N\}$.

Firstly, the point stabilizer of 1, 2 and 3 in N.

$N^{(123)} = n \in N|(123)^n = 123$

$Nt_1\ t_2t_3N$ is denoted by [123]

The number of right cosets in [123] is equal to

$\frac{|N|}{|N|^{(123)}} = \frac{18}{3} = 6.$

The orbits of $N^{(123)}$ on X= $\{1, 2, 3, 4, 5, 6\}$ are $\{1\}, \{2\}, \{3\}, \{4, 5, 6\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_2t_3t_1, Nt_1t_2t_3t_2, Nt_1t_2t_3t_3, Nt_1t_2t_3t_4$ belong.

This shows us the following:

$Nt_1t_2t_3t_4 \in Nt_1t_2t_3t_4N$ which is a new double coset. We donate this double coset by [1234]. Three symmetric generator will go back to [1234].

- **Eighth Double Coset** $Nt_1t_2t_4N = [124]$

$Nt_1\ t_2t_4 = \{N(t_1t_2){t_4}^n|n \in N\}$.

Firstly, find the point stabilizer of 1, 2 and 4 in N.

$N^{(124)} = n \in N|(124)^n = 124$

$Nt_1\ t_2t_4N$ is denoted by [124]

The number of right cosets in [124] is equal to

$\frac{|N|}{|N|^{(124)}} = \frac{18}{1} = 18.$

The orbits of $N^{(123)}$ on X= $\{1, 2, 3, 4, 5, 6\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$.

Now we select a representative from each orbit and determine to which double coset

$Nt_1t_2t_4t_1, Nt_1t_2t_4t_2, Nt_1t_2t_4t_3, Nt_1t_2t_4t_4, Nt_1t_2t_4t54, Nt_1t_2t_4t_6$ belong.

This shows as following.

$Nt_1t_2t_4t_5 \in Nt_1t_2t_4t_5N$ which is a new double coset. We donate this double coset by [1245]. One symmetric generator will go to the new double coset [1245].

$Nt_1t_2t_4t_6 \in Nt_1t_2t_4t_6N$ which is a new double coset. We donate this double coset by [1246]. One symmetric generator will go to the new double coset [1246].

- **Ninth Double Coset** $Nt_1t_3t_4N = [134]$

$Nt_1t_3t_4N = \{N(t_1t_3t_4)^n | n \in N\}$.
Firstly, the point stabilizer of 1 , 3 and 4 in N.
$N^{(134)} = \{n \in N | (134)^n = 134\}$
$Nt_1\ t_3t_4N$ is denoted by [124]
The number of right cosets in [134] is equal to
$\frac{|N|}{|N|^{(134)}} = \frac{18}{1} = 18$.
The orbits of $N^{(134)}$ on X= $\{1, 2, 3, 4, 5, 6\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$.
Now we select a representative from each orbit and determine to which double coset $Nt_1t_3t_4t_1, Nt_1t_3t_4t_2, Nt_1t_3t_4t_3, Nt_1t_3t_4t_4, Nt_1t_3t_4t5, Nt_1t_3t_4t_6$ belong.
This shows us the following.

$Nt_1t_3t_4t_3 \in Nt_1t_3t_4t_3N$ is a new double coset. We donate this double coset by [1343]. One symmetric generator will go to the new double coset [1343].

$Nt_1t_3t_4t_6 \in Nt_1t_3t_4t_6N$ is a new double coset. We donate this double coset by [1346]. One symmetric generator will go to the new double coset [1346].

- **Tenth Double Coset** $Nt_1t_2t_1t_3N = [1213]$

$Nt_1\ t_2t_1t_3 = \{N(t_1t_2)t_1{t_3}^n | n \in N\}$.

Firstly, the point stabilizer of 1, 2, 1 and 3 in N.

$N^{(1213)} = \{n \in N | (1213)^n = 1213\}$

$Nt_1\ t_2t_1t_3N$ is denoted by [1213]

The number of right cosets in [1213] is equal to

$\frac{|N|}{|N|^{(1213)}} = \frac{18}{1} = 18.$

The orbits of $N^{(1213)}$ on X= $\{1, 2, 3, 4, 5, 6\}$ are $\{1\}, \{2\}, \{3, 4, 5, 6\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_2t_1t_3t_1, Nt_1t_2t_1t_3t_2, Nt_1t_2t_1t_3t_3, Nt_1t_2t_1t_3t_4t$ belong.

$Nt_1t_2t_1t_3t_1 \in Nt_1t_2t_1t_3t_1N$ is a new double coset. We donate this double coset by [12131]. One symmetric generator will go back to double coset [121].

$Nt_1t_2t_1t_3t_4 \in Nt_1t_2t_1t_3t_4N$ is a new double coset. We donate this double coset by [12134]. Three symmetric generator will go back to double coset [123].

- **Eleventh Double Coset** $Nt_1t_2t_3t_4N = [1234]$

$Nt_1\ t_2t_3t_4 = \{N(t_1t_2)t_3{t_4}^n | n \in N\}$.

Firstly, the point stabilizer of 1, 2, 3 and 4 in N.

$N^{(1234)} = \{n \in N | (1234)^n = 1234\}$

$Nt_1\ t_2t_3t_4N$ is denoted by [1234]

The number of right cosets in [1234] is equal to

$\frac{|N|}{|N|^{(1234)}} = \frac{18}{1} = 18.$

The orbits of $N^{(1234)}$ on X= $\{1, 2, 3, 4, 5, 6\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_2t_3t_4t_1, Nt_1t_2t_3t_4t_2, Nt_1t_2t_3t_4t_3,\ Nt_1t_2t_3t_4t_4, Nt_1t_2t_3t_4t_5,$
$Nt_1t_2t_3t_4t_6$ belong.

This shows us the following.

$Nt_1t_2t_3t_4t_1 \in Nt_1t_2t_3t_4N$ which is a new double coset. We donate this double coset by [1234]. Three symmetric generator will to the new double coset [1234].

• **Twelfth Double Coset** $Nt_1t_2t_4t_5N = [1245]$

$Nt_1\ t_2t_4t_5 = \{N(t_1t_2)t_4t_5{}^n | n \in N\}$.

Firstly, the point stabilizer of 1, 2, 4 and 5 in N.

$N^{(1245)} = \{n \in N | (1245)^n = 1245\}$

$Nt_1\ t_2t_4t_5N$ is denoted by [1245]

The number of right cosets in [1245] is equal to

$\frac{|N|}{|N|^{(1245)}} = \frac{18}{1} = 18$.

The orbits of $N^{(1245)}$ on X= $\{1,2,3,4,5,6\}$ are $\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_2t_4t_5t_1, Nt_1t_2t_4t_5t_2, Nt_1t_2t_4t_5t_3$, $Nt_1t_2t_4t_5t_4, Nt_1t_2t_4t_5t_5$, $Nt_1t_2t_4t_5t_6$ belong.

$Nt_1t_2t_4t_5t_1 \in Nt_1t_2t_4t_5N$ which is a new double coset. We donate this double coset by [1245]. One symmetric generator will go back to coset [124].

• **Thirteenth Double Coset** $Nt_1t_2t_4t_6N = [1246]$

$Nt_1\ t_2t_4t_6 = \{N(t_1t_2)t_4t_6{}^n | n \in N\}$.

Firstly, the point stabilizer of 1, 2, 4 and 6 in N.

$N^{(1246)} = \{n \in N | (1246)^n = 1246\}$

$Nt_1\ t_2t_4t_6N$ is denoted by [1246]

The number of right cosets in [1246] is equal to

$\frac{|N|}{|N|^{(1246)}} = \frac{18}{1} = 18$.

The orbits of $N^{(1246)}$ on X= $\{1,2,3,4,5,6\}$ are $\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_2t_4t_6t_1, Nt_1t_2t_4t_6t_2, Nt_1t_2t_4t_6t_3$, $Nt_1t_2t_4t_6t_4, Nt_1t_2t_4t_6t_5$, $Nt_1t_2t_4t_6t_6$ belong.

This shows us the following.

‘ $Nt_1t_2t_4t_6t_1 \in Nt_1t_2t_4t_6t_1N$ which is a new double coset. We donate this double coset by [12461]. One symmetric generator will go back to coset [134].

$Nt_1t_2t_4t_6t_4 \in Nt_1t_2t_4t_6t_4N$ which is a new double coset. We donate this double coset by [12464]. One symmetric generator will go back to coset [124].

- **Fourteenth Double Coset** $Nt_1t_3t_4t_3N = [1343]$

$Nt_1\ t_3t_4t_3 = \{N(t_1t_3)t_4{t_3}^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(1343)}$. Firstly, the point stabilizer of 1, 3, 4 and 3 in N.

$N^{(1343)} = \{n \in N | (1343^n = 1343\}$

$Nt_1\ t_3t_4t_3N$ is denoted by [1343]

The number of right cosets in [1343] is equal to

$\frac{|N|}{|N|^{(1343)}} = \frac{18}{1} = 18$.

The orbits of $N^{(1343)}$ on X= $\{1, 2, 3, 4, 5, 6\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_3t_4t_3t_1, Nt_1t_3t_4t_3t_2, Nt_1t_3t_4t_3t_3,\ Nt_1t_3t_4t_3t_4, Nt_1t_3t_4t_3t_5$, $Nt_1t_3t_4t_3t_6$ belong.

$Nt_1t_3t_4t_3t_2 \in Nt_1t_3t_4t_3t_2N$ which is a new double coset. We donate this double coset by [13432]. One symmetric generator will go back to coset [121].

$Nt_1t_3t_4t_3t_4 \in Nt_1t_3t_4t_3t_4N$ which is a new double coset. We donate this double coset by [13434]. Two symmetric generator will go back to coset [141].

- **Fifteenth Double Coset** $Nt_1t_3t_4t_6N = [1346]$

$Nt_1\ t_3t_4t_6 = \{N(t_1t_3)t_4{t_6}^n | n \in N\}$.

Firstly, the point stabilizer of 1, 3, 4 and 6 in N.

$N^{(1346)} = \{n \in N | (1346^n = 1346\}$

$Nt_1\ t_3t_4t_6N$ is denoted by [1346]

The number of right cosets in [1343] is equal to

$\frac{|N|}{|N|^{(1346)}} = \frac{18}{1} = 18$.

The orbits of $N^{(1346)}$ on X= $\{1, 2, 3, 4, 5, 6\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$.

Now we select a representative from each orbit and determine to which double coset

$Nt_1t_3t_4t_6t_1, Nt_1t_3t_4t_6t_2, Nt_1t_3t_4t_6t_3,$ $Nt_1t_3t_4t_6t_4, Nt_1t_3t_4t_6t_5,$
$Nt_1t_3t_4t_6t_6$ belong.
$Nt_1t_3t_4t_6t_3 \in Nt_1t_3t_4t_6t_3N$ which is a new double coset. We donate this double coset by [13463]. One symmetric generator will go back to coset [145].

Cayley Diagram

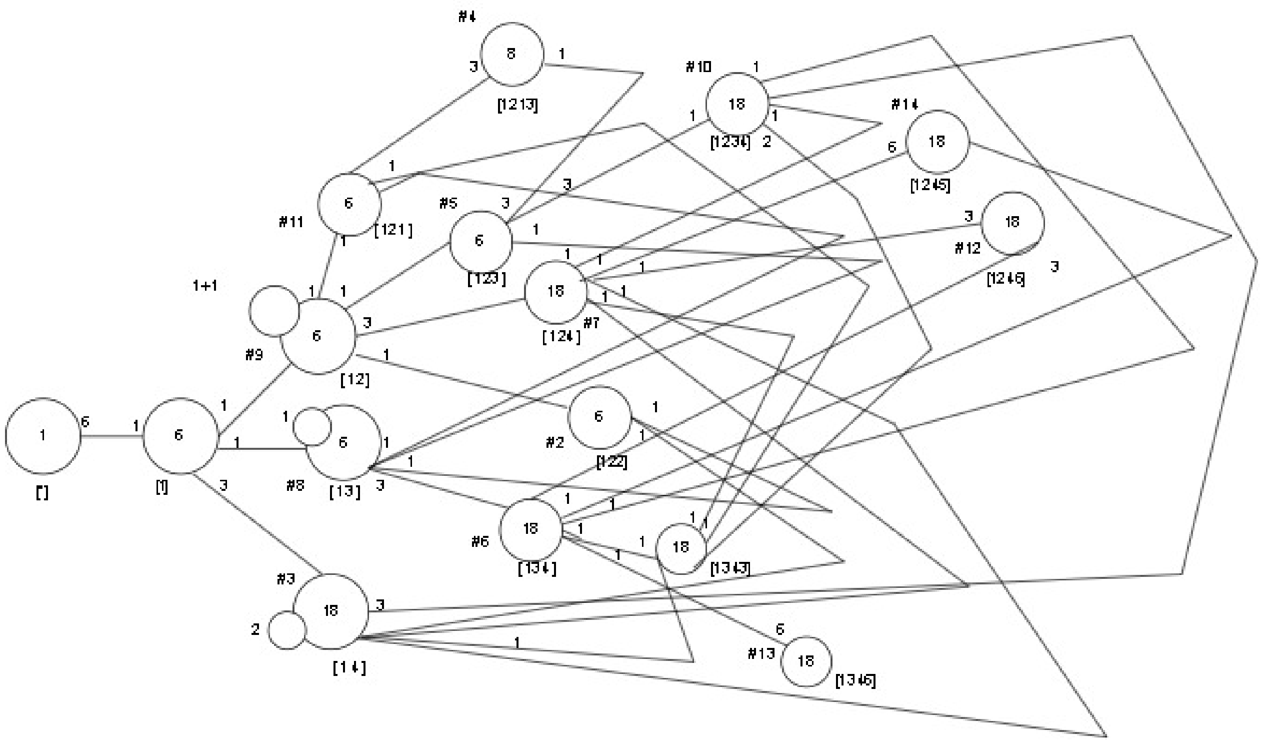

Figure 8.1: Cayley diagram for G over $3^2 : 2$

8.1.1 Magma Work $2 * 6 : 2^3 : 2$

S:=Sym(6);

$xx := S!(1,\ 4)(2,\ 5)(3,\ 6)$;

$yy := S!(1, 2, 3)$;

N:=sub< $S|xx, yy$ >;

$\sharp N$;

/*18*/

G< x, y, t >:=Group< $x, y, t|x^2, y^3, y^{-1} * x * y^{-1} * x * y * x * y * x, t^2, (t, y^x), (y * x * t)^5$ >;

$\sharp G$;

/* 2430*/

$\sharp sub < G|x, y >$;

/*18*/

f,G1,k:=CosetAction($G, sub < G|x, y >$);

$\sharp G1$;

/* 2430 */

```
CompositionFactors(G1);
/*
    G
    |  Cyclic(2)
    *
    |  Cyclic(5)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1
*/
  IN:=sub<G1|f(x),f(y)>;
ts := [ Id(G1): i in [1 .. 6] ];
ts[1]:=f(t);
ts[2]:=f(t^(y)); ts[3]:=f(t^(y^-1)); ts[4]:=f(t^(x)); ts[5]:=f(t^(y * x));
```

```
ts[6]:=f(t^(y^-1 * x));
#DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
/*15*/
DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
/*
{ <GrpFP, Id(G), GrpFP>, <GrpFP, t * y * t * y * t * y^-1 * t, GrpFP>,
<GrpFP, t* y * t * y * t, GrpFP>, <GrpFP, t * x * t * y^-1 * t, GrpFP>,
<GrpFP, t * x * t* y * t, GrpFP>, <GrpFP, t * y * t, GrpFP>,
<GrpFP, t * x * t, GrpFP>, <GrpFP, t, GrpFP>,
<GrpFP, t * y^-1 * t, GrpFP>,
<GrpFP, t * x * t * x * t, GrpFP>,
<GrpFP, t * y * t * y^-1 * t, GrpFP>,
<GrpFP, t * x * t * x * t * y * t, GrpFP>,
<GrpFP, t * x * t * y * t * x * t, GrpFP>,
<GrpFP, t * x * t * y^-1 * t * x * t,GrpFP>,
<GrpFP, t * y * t * x * t * y^-1 * t, GrpFP> }
*/
DC:=[Id(G1),f(t),f(t * x * t),f(t * y * t * y * t * y^-1 * t),
f(t* y * t * y * t),
f(t * x * t * y^-1 * t),f(t * x * t* y * t),f( t * y * t),f(t * y^-1 * t),
f(t * x * t * x * t),
f(t * y * t * y^-1 * t),f(t * y * t * x * t * y^-1 * t),
f(t * x * t * y^-1 * t * x * t),
f(t * x * t * y * t * x * t),
f(t * x * t * y * t * x * t) ];
```

Index(G1,IN);

/*135*/

```
cst := [null : i in $[1 ..Index(G1,IN)]] where null is [Integers() | ]$;
prodim := function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
for i := 1 to 6 do
cst[prodim(1, ts, [i])] := [i];
end for;
m:=0; for i in [1..135] do if cst[i] ne [] then m:=m+1; end if; end for;m;
/*6*/
Orbits(N);
```

```
/*
[
    GSet{@ 1, 4, 2, 5, 3, 6 @}
]
*/
N1:=Stabiliser(N,1);
Orbits(N1);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4, 6, 5 @}
]
*/
#N/#N1;
/*6*/
```

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1] * ts[1]$ eq $m * (DC[i])^n$ then i; break; end if; end for;end for;

/*1*/

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1] * ts[2]$ eq $m * (DC[i])^n$ then i; break; end if; end for;end for;

/*9*/

for i in [1 $\cdots\sharp$ DC] do for m,n in IN do if $ts[1] * ts[3]$ eq $m * (DC[i])^n$ then i; break; end if; end for;end for;

/*8*/

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1] * ts[4]$ eq $m * (DC[i])^n$ then i; break; end if; end for;end for;

/*3*/

```
S:={[1,2]};
SS:=S^N;SS;
/*
    {
        [ 1, 2 ]
```

```
    },
    {
        [ 4, 5 ]
    },
    {
        [ 2, 3 ]
    },
    {
        [ 5, 6 ]
    },
    {
        [ 3, 1 ]
    },
    {
        [ 6, 4 ]
    }
@}

*/
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
/*
{
    [ 1, 2 ]
}
*/
```

N12:=Stabiliser(N,[1,2]);

♯N12;

N12;

/*Permutation group N12 acting on a set of cardinality 6

Order = 3

(4, 5, 6)

*/

N12s:=N12;

$\sharp N12s$;

/*3*/

tr1:=Transversal(N,N12s);

for$i := 1 to \sharp$tr1 do

ss:=$[1, 2]^{tr1[i]}$;

cst[prodim(1,ts,ss)]:=ss;

end for;

m:=0; for i in $[1 \cdots 135] do if cst[i] ne []$

then m:=m+1;

end if; end for;m;

/*12*/

```
    Orbits(N12s);
/*[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4, 5, 6 @}
]
*/
#N/#N12s;
/*6*/
```

for i in [1

$\cdots \sharp$

DC] do for m,n in IN do if$ts[1] * ts[2] * ts[1]$ eq $m * (DC[i])^n$ then i; break; end if; end for;end for;

/*11*/

for i in [1$\cdots \sharp$ DC] do for m,n in IN do if $ts[1] * ts[2] * ts[2]$ eq $m * (DC[i])^n$ then i; break; end if; end for;end for;

/* 2 */

for i in [1$\cdots \sharp$ DC] do for m,n in IN do if $ts[1] * ts[2] * ts[3]$ eq $m * (DC[i])^n$ then i; break; end if; end for;end for;

/*5*/

for i in $[1 \cdots \sharp$DC] do for m,n in IN do if $ts[1]*ts[2]*ts[4]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*7*/

```
S:={[1,3]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[3]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
/*
{
    [ 1, 3 ]
}
*/
N13:=Stabiliser(N,[1,3]);
N13;
/*
Permutation group N13 acting on a set of cardinality 6
Order = 3
    (4, 5, 6)
*/
N13s:=N13;
#N13s;
/* 3 */
tr1:=Transversal(N,N13s);
for i:=1 to #tr1 do
ss:=[1,3]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..135] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 18*/
Orbits(N13s);
/*
[
    GSet{@ 1 @},
```

```
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4, 6, 5 @}
]
*/
#N/#N13s;
/*6*/
```

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[3]*ts[1]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/* 11*/

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[3]*ts[2]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*5*/

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[3]*ts[3]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*2*/

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[3]*ts[4]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*6*/

```
S:={[1,4]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
/*
{
[1,4]
}
*/
```

```
N14:=Stabiliser(N,[1,4]);
N14;
/*Permutation group N14 acting on a set of cardinality 6
Order = 1
*/

N14s:=N14;
#N14s;
/*1*/
tr1:=Transversal(N,N14s);
for i:=1 to #tr1 do
ss:=[1,4]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..135] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*36*/

 #N/#N14s;
/*18*/

Orbits(N14s);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @}
]
*/
```

for i in [1· · · ♯ DC] do for m,n in IN do if $ts[1] * ts[4] * ts[1]$ eq $m * (DC[i])^n$then i; break; end if; end for;end for;

/*10*/

for i in [1· · · ♯ DC] do for m,n in IN do if $ts[1] * ts[4] * ts[2]$ eq $m * (DC[i])^n$ then i; break; end if; end for;end for;

/*3*/

for i in [$1 \cdots \sharp$ DC] do for m,n in IN do if $ts[1] * ts[4] * ts[3]$ eq $m * (DC[i])^n$ then i; break; end if; end for;end for;
/*3*/

for i in [$1 \cdots \sharp$ DC] do for m,n in IN do if $ts[1] * ts[4] * ts[4]$ eq $m * (DC[i])^n$ then i; break; end if; end for;end for;
/*2*/

for i in [$1 \cdots \sharp$ DC] do for m,n in IN do if $ts[1] * ts[4] * ts[5]$ eq $m * (DC[i])^n$ then i; break; end if; end for;end for;
/*6*/

for i in [$1 \cdots \sharp$ DC] do for m,n in IN do if $ts[1] * ts[4] * ts[6]$ eq $m * (DC[i])^n$ then i; break; end if; end for;end for;
/*7*/

```
   S:={[1,2,1]};
 SS:=S^N;SS;
/*
GSet{@
    {
        [ 1, 2, 1 ]
    },
    {
        [ 4, 5, 4 ]
    },
    {
        [ 2, 3, 2 ]
    },
    {
        [ 5, 6, 5 ]
    },
    {
        [ 3, 1, 3 ]
    },
    {
        [ 6, 4, 6 ]
    }
```

```
@}
*/
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
 then print SSS[i];
end if; end for; end for;
N121:=Stabiliser(N,[1,2,1]);
#N121;
/*3*/

N121s:=N121;
[1,2,1]^N121s;
/*
GSet{@
    [ 1, 2, 1 ]
@}
*/
#N/#N121s;
/*6*/
tr1:=Transversal(N,N121s);
for i:=1 to #tr1 do
ss:=[1,2,1]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..135] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*42*/

 Orbits(N121s);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4, 6, 5 @}
]
*/
```

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[1]*ts[1]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*9*/

for i in $[1\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[1]*ts[2]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*8*/

for i in $[1\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[1]*ts[3]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*4*/

for i in $[1\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[1]*ts[4]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

```
S:={[1,2,3]};
SS:=S^N;SS;
/*
GSet{@
    {
        [ 1, 2, 3 ]
    },
    {
        [ 4, 5, 6 ]
    },
    {
        [ 2, 3, 1 ]
    },
    {
        [ 5, 6, 4 ]
    },
    {
        [ 3, 1, 2 ]
    },
    {
        [ 6, 4, 5 ]
    }
@}
*/
SSS:=Setseq(SS);
```

```
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[3]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
 then print SSS[i];
end if; end for; end for;
N123:=Stabiliser(N,[1,2,3]);
#N123;
/*3*/

N123s:=N123;
[1,2,3]^N123s;
/*
GSet{@
    [ 1, 2, 3 ]
@}

*/
tr1:=Transversal(N,N123s);
for i:=1 to #tr1 do
ss:=[1,2,3]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..135] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*48*/

#N/#N123s;
/*6*/
 Orbits(N123s);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4, 6, 5 @}
]
*/
```

for i in [$1 \cdots \sharp$ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[3]*ts[1]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*8*/

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[3]*ts[2]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*4*/

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[3]*ts[3]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*9*/

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[3]*ts[4]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*10*/

```
S:={[1,2,4]};
SS:=S^N;SS;
/*
GSet{@
    {
        [ 1, 2, 4 ]
    },
    {
        [ 4, 5, 1 ]
    },
    {
        [ 2, 3, 4 ]
    },
    {
        [ 4, 5, 2 ]
    },
    {
        [ 5, 6, 1 ]
    },
    {
        [ 3, 1, 4 ]
    },
    {
        [ 1, 2, 5 ]
    },
    {
        [ 4, 5, 3 ]
    },
    {
```

```
        [ 5, 6, 2 ]
    },
    {
        [ 6, 4, 1 ]
    },
    {
        [ 2, 3, 5 ]
    },
    {
        [ 1, 2, 6 ]
    },
    {
        [ 5, 6, 3 ]
    },
    {
        [ 6, 4, 2 ]
    },
    {
        [ 3, 1, 5 ]
    },
    {
        [ 2, 3, 6 ]
    },
    {
        [ 6, 4, 3 ]
    },
    {
        [ 3, 1, 6 ]
    }
@}

*/
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;
N124:=Stabiliser(N,[1,2,4]);
#N124;
/*1*/

N124s:=N124;
[1,2,4]^N124s;
```

```
/*
GSet{@
    [ 1, 2, 4 ]
@}

*/
tr1:=Transversal(N,N124s);
for i:=1 to #tr1 do
ss:=[1,2,4]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..135] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*66*/

#N/#N124s;
/*18*/
 Orbits(N124s);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @}
]
*/
```

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[4]*ts[1]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*3*/

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[4]*ts[2]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[4]*ts[3]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*10*/

for i in [1⋯♯ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[4]*ts[4]eq$m*(DC[i])^n then i; break; end if; end for;end for;
/*9*/

for i in [1⋯♯ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[4]*ts[5]eq$m*(DC[i])^n then i; break; end if; end for;end for;
/*14*/

for i in [1⋯♯ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[4]*ts[6]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;
/*12*/

```
S:={[1,3,4]};
SS:=S^N;SS;

SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[3]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
 then print SSS[i];
end if; end for; end for;
N134:=Stabiliser(N,[1,3,4]);
#N134;
/*1*/
N134s:=N134;
[1,3,4]^N134s;

tr1:=Transversal(N,N134s);
for i:=1 to #tr1 do
ss:=[1,3,4]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..135] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*84*/

#N/#N134s;
/*18*/
```

```
 Orbits(N134s);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @}
]
*/
```

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[3]*ts[4]*ts[1]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*3*/

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[3]*ts[4]*ts[2]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*10*/

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[3]*ts[4]*ts[3]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[3]*ts[4]*ts[4]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*8*/

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[3]*ts[4]*ts[5]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*12*/

for i in [1$\cdots\sharp$ DC] do for m,n in IN do if $ts[1]*ts[3]*ts[4]*ts[6]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*13*/

```
S:={[1,2,1,3]};
SS:=S^N;SS;
```

```
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[1]*ts[3]
eq g*ts[Rep(SSS[i])[1]]
*ts[Rep(SSS[i])[2]]*
ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])[4]]
then print SSS[i];
end if; end for; end for;
N1213:=Stabiliser(N,[1,2,1,3]);
#N1213;
/*1*/
N1213s:=N1213;
[1,2,1,3]^N1213s;

tr1:=Transversal(N,N1213s);
for i:=1 to #tr1 do
ss:=[1,2,1,3]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..135] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*86*/

#N/#N1213s;
/*18*/
 Orbits(N1213s);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4, 5, 6 @}
]
*/
```

for i in $[1 \cdots \sharp$ DC$]$ do for m,n in IN do if $ts[1]*ts[2]*ts[1]*ts[3]*ts[1]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*11*/

for i in $[1 \cdots \sharp$ DC$]$ do for m,n in IN do if $ts[1]*ts[2]*ts[1]*ts[3]*ts[2]$ eq $m*(DC[i])^n$

then i; break; end if; end for;end for;

/*11*/

for i in [1⋯♯ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[1]*ts[3]*ts[3]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for;

/*11*/

for i in [1⋯♯ DC] do for m,n in IN do if $ts[1]*ts[2]*ts[1]*ts[3]*ts[4]$ eq $m*(DC[i])^n$ then i; break; end if; end for;end for; /*5*/

```
S:={[1,2,3,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[3]*ts[4]
eq g*ts[Rep(SSS[i])[1]]
*ts[Rep(SSS[i])[2]]*
ts[Rep(SSS[i])[3]]*
ts[Rep(SSS[i])[4]]
then print SSS[i];
end if; end for; end for;
N1234:=Stabiliser(N,[1,2,3,4]);
#N1234;
/*1*/
N1234s:=N1234;
[1,2,3,4]^N1234s;

tr1:=Transversal(N,N1234s);
for i:=1 to #tr1 do
ss:=[1,2,3,4]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..135] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*104*/

#N/#N1234s;
/*18*/
 Orbits(N1234s);
```

```
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @}
]
*/
```

for i in $[1 \cdots \sharp DC] do for m, n in IN do if ts[1]*ts[2]*ts[3]*ts[4]*ts[1] eq m*(DC[i])^n$ then i; break; end if; end for;end for;

for i in $[1 \cdots \sharp DC] do for m, n in IN do if ts[1]*ts[2]*ts[3]*ts[4]*ts[2] eq m*(DC[i])^n$ then i; break; end if; end for;end for;

/*3*/

for i in $[1 \cdots \sharp DC] do for m, n in IN do if ts[1]*ts[2]*ts[3]*ts[4]*ts[3] eq m*(DC[i])^n$ then i; break; end if; end for;end for;

for i in $[1 \cdots \sharp DC] do for m, n in IN do if ts[1]*ts[2]*ts[3]*ts[4]*ts[4] eq m*(DC[i])^n$ then i; break; end if; end for;end for;

/*5*/

for i in $[1 \cdots \sharp DC] do for m, n in IN do if ts[1]*ts[2]*ts[3]*ts[4]*ts[5] eq m*(DC[i])^n$ then i; break; end if; end for;end for;

/*6*/

for i in $[1 \cdots \sharp DC] do for m, n in IN do if ts[1]*ts[2]*ts[3]*ts[4]*ts[6] eq m*(DC[i])^n$then i; break; end if; end for;end for;

/*7*/

```
S:={[1,2,4,5]};
SS:=S^N;
SSS:=Setseq(SS);
```

```
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[4]*ts[5]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]\\
*ts[Rep(SSS[i])[4]]
then print SSS[i];
end if; end for; end for;
N1245:=Stabiliser(N,[1,2,4,5]);
#N1245;
/*1*/
N1245s:=N1245;
[1,2,4,5]^N1245s;

tr1:=Transversal(N,N1245s);
for i:=1 to #tr1 do
ss:=[1,2,4,5]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..135] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*107*/

#N/#N1245s;
/*18*/
 Orbits(N1245s);
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @}
]

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4]*ts[5]*ts[1] eq
m*(DC[i])^n then i; break; end if; end for;end for;

/*7*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4]*ts[5]*ts[2] eq
m*(DC[i])^n then i; break; end if; end for;end for;

/*7*/
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4]*ts[5]*ts[3] eq
m*(DC[i])^n then i; break; end if; end for;end for;

/*7*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4]*ts[5]*ts[4] eq
m*(DC[i])^n then i; break; end if; end for;end for;

/*7*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4]*ts[5]*ts[5] eq
m*(DC[i])^n then i; break; end if; end for;end for;

/*7*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4]*ts[5]*ts[6] eq
m*(DC[i])^n then i; break; end if; end for;end for;

/*7*/

S:={[1,2,4,6]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[4]*ts[6]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
*ts[Rep(SSS[i])[4]]
then print SSS[i];
end if; end for; end for;
N1246:=Stabiliser(N,[1,2,4,6]);
#N1246;
/*1*/
N1246s:=N1246;
[1,2,4,6]^N1246s;

tr1:=Transversal(N,N1246s);
for i:=1 to #tr1 do
ss:=[1,2,4,6]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..135] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*113*/
```

```
#N/#N1246s;
/*18*/
 Orbits(N1246s);
/*
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @}
]
*/
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4]*ts[6]*ts[1] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*6*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4]*ts[6]*ts[2] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*6*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4]*ts[6]*ts[3] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*6*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4]*ts[6]*ts[4] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*7*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4]*ts[6]*ts[5] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*7*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4]*ts[6]*ts[6] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*7*/

S:={[1,3,4,3]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[3]*ts[4]*ts[3]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*
ts[Rep(SSS[i])[3]]
```

```
*ts[Rep(SSS[i])[4]]
then print SSS[i];
end if; end for; end for;
N1343:=Stabiliser(N,[1,3,4,3]);
#N1343;
/*1*/
N1343s:=N1343;
[1,3,4,3]^N1343s;

tr1:=Transversal(N,N1343s);
for i:=1 to #tr1 do
ss:=[1,3,4,3]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..135] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*131*/

#N/#N1343s;
/*18*/
 Orbits(N1343s);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @}
]
*/
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[4]*ts[3]*ts[1] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*7*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[4]*ts[3]*ts[2] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*11*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[4]*ts[3]*ts[3] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*6*/
```

```
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[4]*ts[3]*ts[4] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*10*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[4]*ts[3]*ts[5] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*10*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[4]*ts[3]*ts[6] eq
m*(DC[i])^n then i; break; end if; end for;end for;


S:={[1,3,4,6]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[3]*ts[4]*ts[6]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
*ts[Rep(SSS[i])[4]]
then print SSS[i];
end if; end for; end for;
N1346:=Stabiliser(N,[1,3,4,6]);
#N1346;
/*1*/
N1346s:=N1346;
[1,3,4,6]^N1346s;

tr1:=Transversal(N,N1346s);
for i:=1 to #tr1 do
ss:=[1,3,4,6]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..135] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*134*/

#N/#N1346s;
/*18*/
 Orbits(N1346s);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
```

```
    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @}
]
*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[4]*ts[6]*ts[1] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*6*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[4]*ts[6]*ts[2] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*6*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[4]*ts[6]*ts[3] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*6*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[4]*ts[6]*ts[4] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*6*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[4]*ts[6]*ts[5] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*6*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]*ts[4]*ts[6]*ts[6] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*6*/
```

8.2 Construction Of $U(3,4):2$ Over $N \sim 5^2 : S_3$

Consider N $= < x, y >$; where
x $\sim$ (1, 15, 12, 8, 3, 9, 14, 13, 7, 4)(2, 11, 5, 6, 10) and
$\sim$ (1, 11, 14, 6, 12, 2, 7, 5, 3, 10)(4, 8, 13, 15, 9);
Our progenitor is $5^2 : S_3$. We prove that $U(3,4) : 2 \cong G < x, y, t >:=$ Group $< x, y, t|(y^{-1} * x^{-1})^3, (y^{-1} * x)^3, x^{-1} * y^{-1} * x^3 * y^{-1} * x^{-1} * y, x^2 * y * x^2 * y^3, t^2, (t, y^{-1} * x^2 * y^{-1}), (t, x^3 * y^{-1} * x), (x^{-1} * y * t)^5, (x^{-2} * t^{(y*x^{-1})})^3 >$;
We perform manual double coset enumeration of G over N. We need to determine all distinct double coset NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find the index which is the order of G over the order of N.
Hence,$\frac{|G|}{|N|} = \frac{124800}{150} = 832$. So, we have 832 single cosets.

- **<u>First Double Coset [*]</u>**

$NeN = \{Ne^n|n \in N\} = \{N\}$.
The double coset $NeN = [*]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.
The number of right coset in $*$ is equal to $\frac{|N|}{|N|} = \frac{150}{150} = 1$.
Since N is transitive on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$.
We need only determine the double coset of the right coset Nt_1.
Thus nine cosets extend to the new double coset [1], that mean the nine generators go forward to Nt_1.

Cayley Diagram

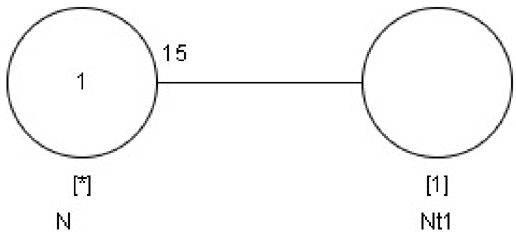

Figure 8.2: Cayley diagram for G over S_{15}

- **Second Double Coset [1]**

$Nt_1N = \{Nt_1^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12}, Nt_{13}, Nt_{14}, Nt_{15}\}$. We now find the Coset Stabilizer $N^{(1)}$. We first find the point stabilizer of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$

$N^1 =$ (2, 5, 10, 11, 6)(4, 8, 13, 15, 9) (2, 13)(4, 10)(5, 8)(6, 15)(9, 11)

Thus,$N^{(1)} \geq$ (2, 5, 10, 11, 6)(4, 8, 13, 15, 9) (2, 13)(4, 10)(5, 8)(6, 15)(9, 11)

The number of right cosets in [1] is equal to

$\frac{|N|}{|N|^{(1)}} = \frac{150}{10} = 15$.

The orbits of $N^{(1)}$ on X= $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ are

$\{1\}, \{3\}, \{7\}, \{12\}, \{14\}, \{2, 5, 13, 10, 8, 15, 11, 4, 9, 6\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_1, Nt_1t_3, Nt_1t_7, Nt_1t_12, Nt_1t_14, Nt_1t_2 belongs.$

This shows us the following:

$Nt_1t_1 = Nt_1^2 = N \in [*]$.

Since the orbit $\{1\}$ contains one element, then one symmetric generator goes back to the double coset [*].

$Nt_1 \in [1]$

One symmetric generator will go back to [1].

$Nt_1t_3N$is a new double coset which we will denote [13].

One symmetric generators will go to the new double coset [13].

Nt_1t_7N , Nt_1t_12N, Nt_1t_14N that each produce one symmetric generators which will all go to [13].

Nt_1t_2N is a new double coset which we will denote [12].
Ten symmetric generators will go to the new double coset [13].

Cayley Diagram

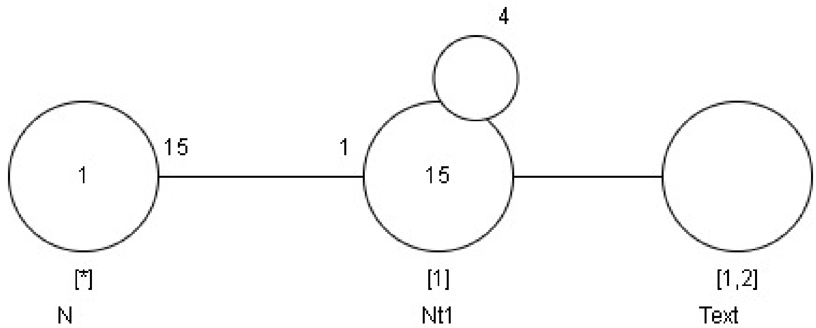

Figure 8.3: Cayley diagram for G over S_{15}

- **Third Double Coset** $Nt_1t_3N = [13]$

$Nt_1\ t_3 = N(t_1t_3)^n|n \in N$.
We now find the Coset Stabilizer $N^{(13)}$. Firstly, find the point stabilizer of 1 and 3 in N.
$N^{(13)} = n \in N|(13)^n = 13$
$Nt_1\ t_3N$ is denoted by [13]

Thus,$N^{13} = 1$
The number of right cosets in [13] is equal to
$\frac{|N|}{|N|^{(13)}} = \frac{150}{1} = 150.$
The orbits of $N^{(12)}$ on X= $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ are
$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}, \{15\}$.
We take $t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_10, t_11, t_12, t_13, t_14, t_15$ from each orbit respectively,

and determine to which double coset $Nt_1t_2t_1, Nt_1t_2t_2, Nt_1t_2t_3, Nt_1t_2t_4, Nt_1t_2t_5, Nt_1t_2t_6,$ $Nt_1t_2t_7, Nt_1t_2t_8, Nt_1t_2t_9, Nt_1t_2t_10, Nt_1t_2t_11, Nt_1t_2t_12, Nt_1t_2t_13, Nt_1t_2t_14, Nt_1t_2t_15$ belong.

$Nt_1t_2t_1$ is a new double coset which will donate by [121] One symmetric generator will go to [121].

$Nt_1t_2t_2$ is a new double coset which will donate by [122] One symmetric generator will go back to [1].

$Nt_1t_2t_3$ is a new double coset which will donate by [123] One symmetric generator will go to [13].

$Nt_1t_2t_4$ is a new double coset which will donate by [124] One symmetric generator will go to [124].

$Nt_1t_2t_5$ is a new double coset which will donate by [125] One symmetric generator will go to [123].

$Nt_1t_2t_6$ is a new double coset which will donate by [126] One symmetric generator will go to [123].

$Nt_1t_2t_7$ is a new double coset which will donate by [127] One symmetric generator will go to [127].

$Nt_1t_2t_8$ is a new double coset which will donate by [128] One symmetric generator will go to [123].

$Nt_1t_2t_9$ is a new double coset which will donate by [129] One symmetric generator will go to [129].

$Nt_1t_2t_{10}$ is a new double coset whose one symmetric generator will go to [123].

$Nt_1t_2t_{11}$ is a new double coset whose one symmetric generator will go to [123].

$Nt_1t_2t_{12}$ is a new double coset whose one symmetric generator will go to [127].

$Nt_1t_2t_{13}$ is a new double coset whose one symmetric generator will go to [124].

$Nt_1t_2t_{14}$ is a new double coset whose one symmetric generator will go to [123].

$Nt_1t_2t_{15}$ is a new double coset whose one symmetric generator will go to [129].

Cayley Diagram

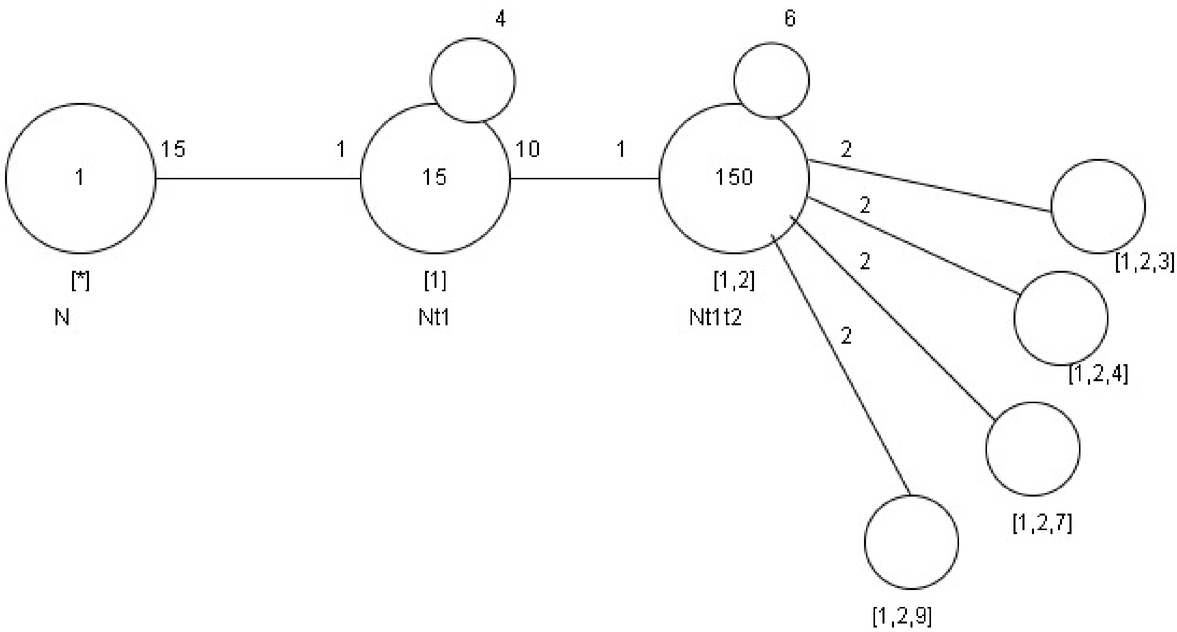

Figure 8.4: Cayley diagram for G over S_{15}

• **Fourth Double Coset** $Nt_1t_2t_9N = [129]$

$Nt_1t_2t_9 = \{N(t_1t_2t_9)^n | n \in N\}$. We now find the Coset Stabilizer $N^{(129)}$. We first find the point stabilizer of 1, 2 and 9 in N.

$N^{129} = \{n \in N | (129)^n = 129\} = \{e\}$

We have $N(t_1t_2t_9) = N(t_{13}t_2t_7)$.

Now $N(t_1t_2t_9)^{(1,13)(3,15)(4,12)(7,9)(8,14)} = N(t_{13}t_2t_7) = Nt_1t_2t_9$.

$\implies Nt_1t_2t_9^{(1,13)(3,15)(4,12)(7,9)(8,14)} = Nt_1t_2t_9$

$\implies (1,13)(3,15)(4,12)(7,9)(8,14) \in N^{(129)}$. So, $N^{((129)} \geq \langle(1,13)(3,15)(4,12)(7,9)(8,14)\rangle$.

The number of right cosets in $Nt_1t_2t_9N$ is calculated by the formula,

$\frac{|N|}{|N^{(129)}|} = \frac{150}{2} = 75.$

The orbits of $N^{(129)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ are orbits

$\{2\}, \{5\}, \{6\}, \{10\}, \{11\}, \{1, 13\}, \{3, 15\}, \{4, 12\}, \{7, 9\}, \{8, 14\}$

We will determine the double cosetes$Nt_1t_2t_9t_2, Nt_1t_2t_9t_5, Nt_1t_2t_9t_6, Nt_1t_2t_9t_{10}$,

$Nt_1t_2t_9t_{11}, Nt_1t_2t_9t_1, Nt_1t_2t_9t_3, Nt_1t_2t_9t_4, Nt_1t_2t_9t_7, Nt_1t_2t_9t_8$

by selecting one representative from this orbit such as,

$Nt_1t_2t_9t_2, Nt_1t_2t_9t_6, Nt_1t_2t_9t_4$ has four symmetric generators go to [1,2,9].

$Nt_1t_2t_9t_2N$is a new double coset which has One symmetric generators we will denote [1292].

$Nt_1t_2t_9t_5N$is a new double coset which has One symmetric generators we will denote [1295].

$Nt_1t_2t_9t_6N$is a new double coset which has One symmetric generators we will denote [1296].

$Nt_1t_2t_9t_{10}N$is a new double coset which has One symmetric generators we will denote [12910].

$Nt_1t_2t_9t_{11}N$is a new double coset which has One symmetric generators we will denote [12911].

$Nt_1t_2t_9t_1N$is a new double coset which has two symmetric generators we will denote [1291].

$Nt_1t_2t_9t_3N$is a new double coset which has two symmetric generators we will denote [1293].

$Nt_1t_2t_9t_4N$is a new double coset which has two symmetric generators we will denote [1294].

$Nt_1t_2t_9t_7N$is a new double coset which has two symmetric generators we will denote [1297].

$Nt_1t_2t_9t_8N$is a new double coset which has two symmetric generators we will denote [1298].

Cayley Diagram

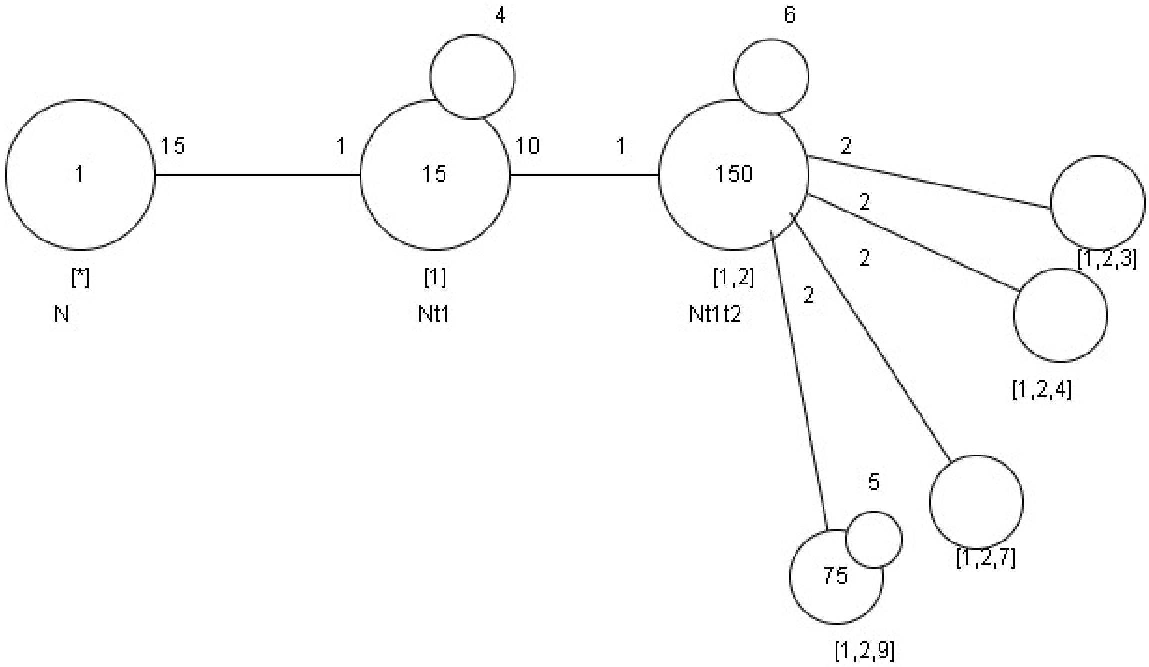

Figure 8.5: Cayley diagram for G over S_{15}

Now look at the generators for [1,2,3] $Nt_1t_2t_9t_5, Nt_1t_2t_9t_{10}, Nt_1t_2t_9t_{11}$ has three symmetric generator goes to [1,2,3].

Cayley Diagram

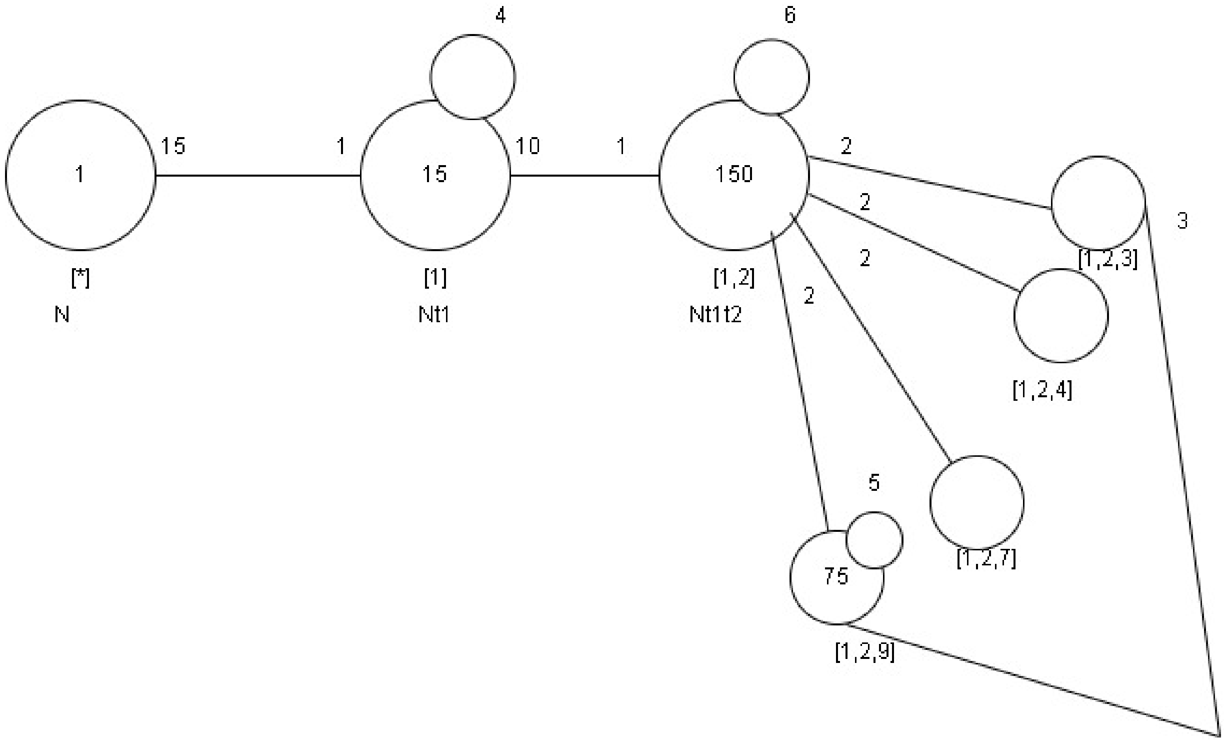

Figure 8.6: Cayley diagram for G over S_{15}

Now look at the generators for [1,2,4] $Nt_1t_2t_9t_1, Nt_1t_2t_9t_3$ has four symmetric generator goes to [1,2,4].

Cayley Diagram

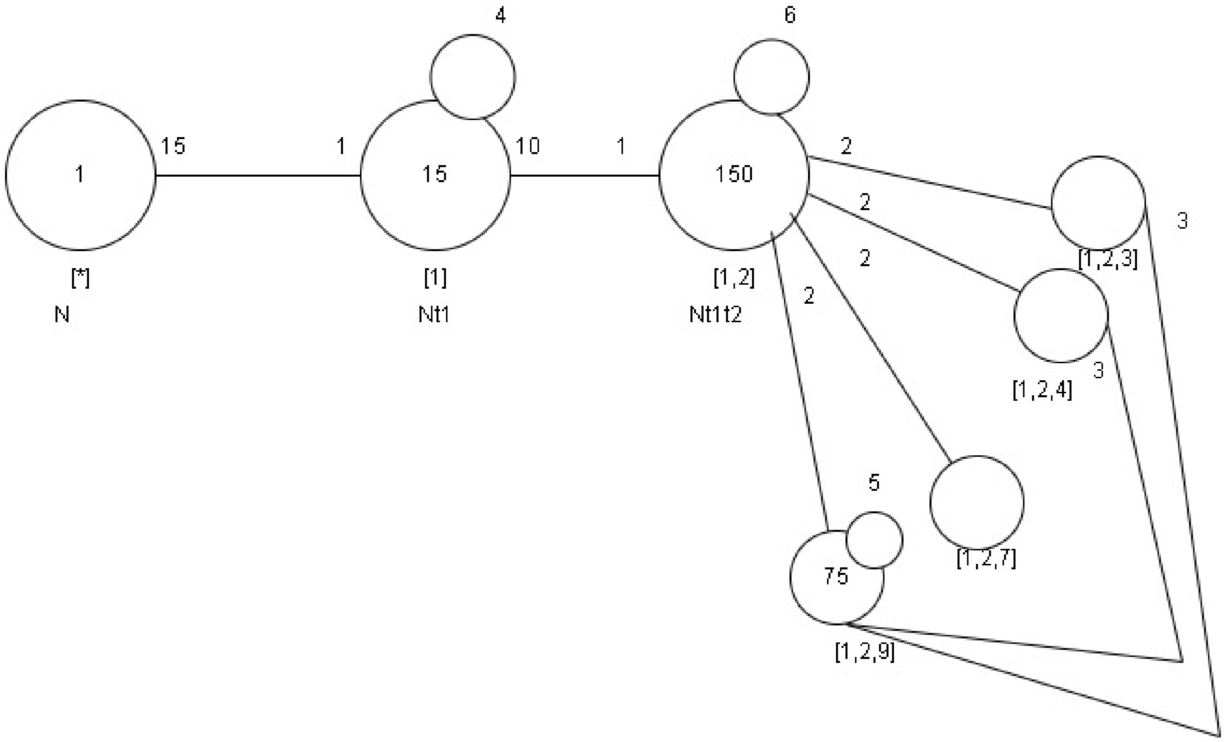

Figure 8.7: Cayley diagram for G over S_{15}

• **Fifth Double Coset** $Nt_1t_2t_7N = [127]$

$Nt_1t_2t_7 = \{N(t_1t_2t_7)^n | n \in N\}$. We now find the Coset Stabilizer $N^{(127)}$. We first find the point stabilizer of 1, 2 and 7 in N.

$N^{127} = \{n \in N | (127)^n = 127\}$

$N^{127} = \langle (1,2,15)(3,6,9)(4,7,11)(5,13,14)(8,12,10) \rangle$.

We have $N(t_1t_2t_7)^{(1,2,15)(3,6,9)(4,7,11)(5,13,14)(8,12,10)}$.

$\implies$ $(1,2,15)(3,6,9)(4,7,11)(5,13,14)(8,12,10) \in N^{(127)}$ Thus $N^{((127)} \geq \langle N^{127}, (1,2,15)$
$(3,6,9)(4,7,11)(5,13,14)(8,12,10) \rangle$

The number of right cosets in $Nt_1t_2t_7N$ is calculated by the formula,

$\frac{|N|}{|N^{(127)}|} = \frac{150}{3} = 50.$

The orbits of $N^{(129)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$ are orbits $\{1,2,15\}, \{3,6,9\}, \{4,7,11\}, \{5,13,14\}, \{8,12,10\}$.

We will determine the double cosetes$Nt_1t_2t_7t_1, Nt_1t_2t_7t_3, Nt_1t_2t_7t_4, Nt_1t_2t_7t_5,$

$Nt_1t_2t_7t_8$ by selecting one representative from this orbit such as,
$Nt_1t_2t_7t_1N$is a new double coset which has three symmetric generators we will denote [1271].

$Nt_1t_2t_7t_3N$is a new double coset which has three symmetric generators we will denote [1273].

$Nt_1t_2t_7t_4N$is a new double coset which has three symmetric generators we will denote [1274].

$Nt_1t_2t_7t_5N$is a new double coset which has three symmetric generators we will denote [1275].

$Nt_1t_2t_7t_8N$is a new double coset which has three symmetric generators we will denote [1278].

Cayley Diagram

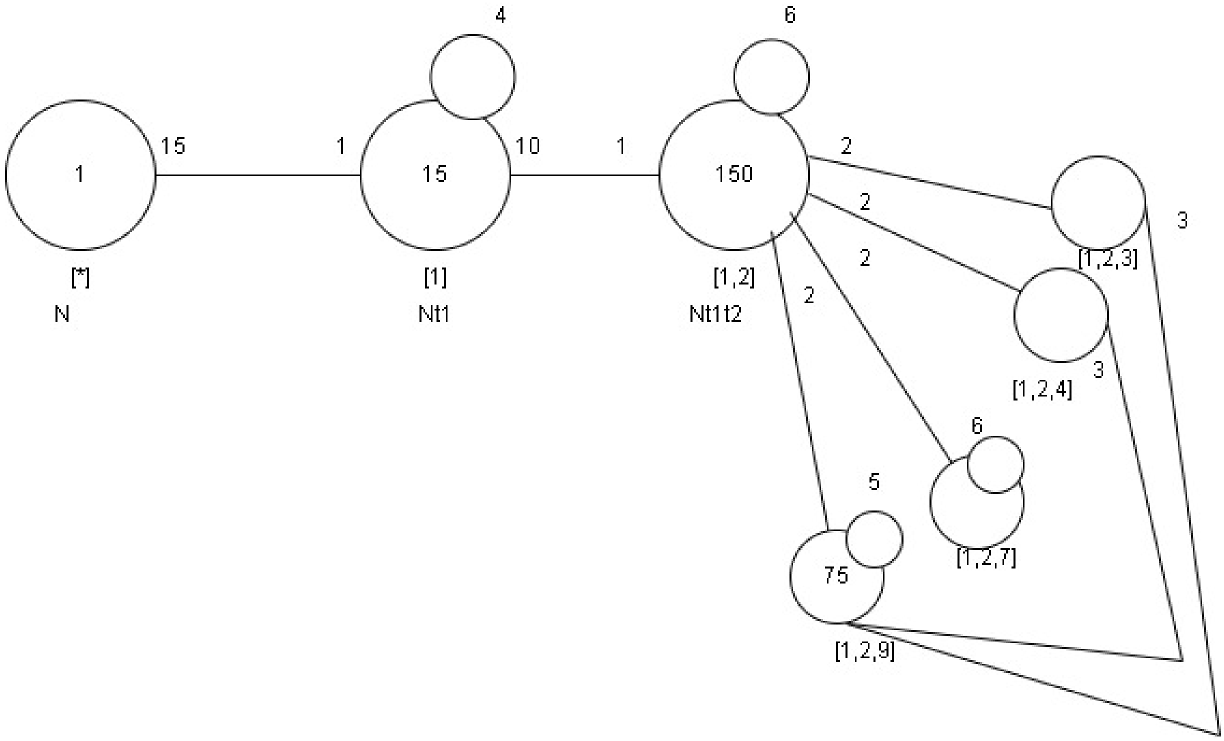

Figure 8.8: Cayley diagram for G over S_{15}

Now look at the generators for [1,2,7] that connect to [124] and three symmetric generator goes from [1,2,7]to [124].

Cayley Diagram

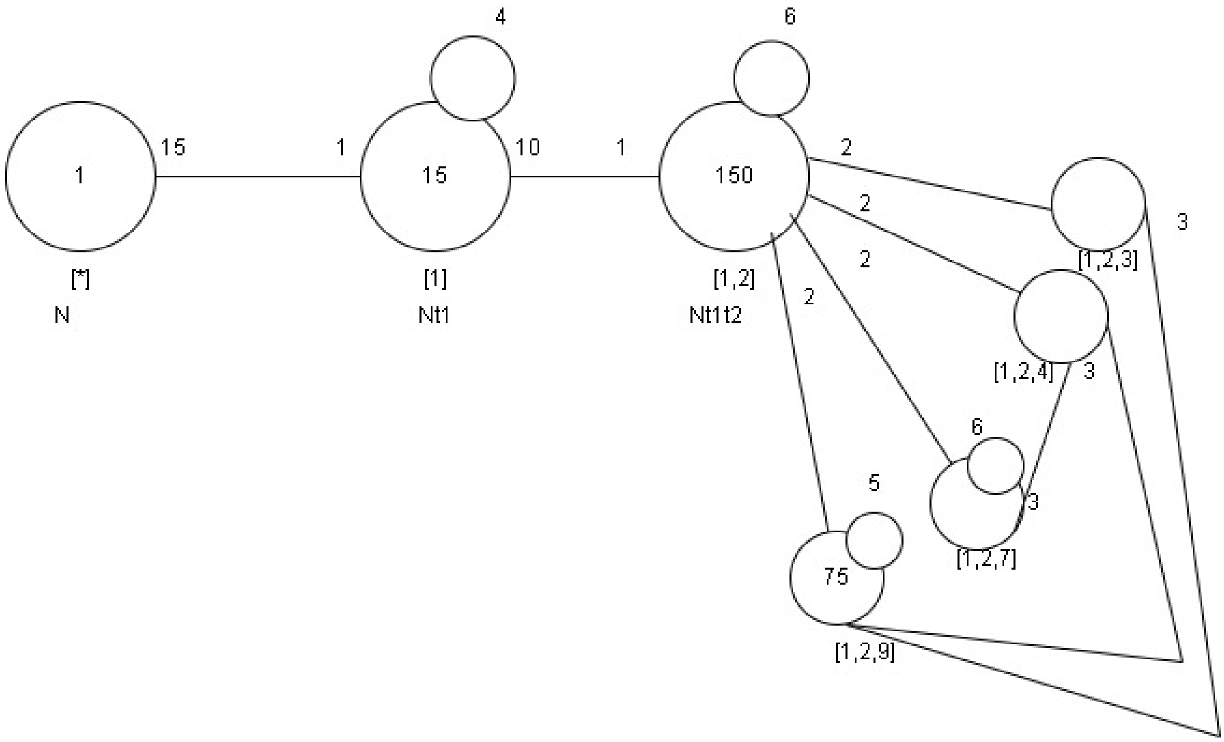

Figure 8.9: Cayley diagram for G over S_{15}

• **Sixth Double Coset** $Nt_1t_2t_4N = [124]$

$Nt_1t_2t_4 = \{N(t_1t_2t_4)^n | n \in N\}$. We now find the Coset Stabilizer $N^{(124)}$. We first find the point stabilizer of 1, 2 and 4 in N.

$N^{124} = \{n \in N | (124)^n = 124\}$

$N^{124} = \langle(1, 6, 15)(2, 13, 14)(3, 11, 9)(4, 7, 10)(5, 8, 12)\rangle$.

We have $N(t_1t_2t_7)^{(1,6,15)(2,13,14)(3,11,9)(4,7,10)(5,8,12)}$.

$\implies (1, 6, 15)(2, 13, 14)(3, 11, 9)(4, 7, 10)(5, 8, 12) \in N^{(124)}$ Thus $N^{((124)} \geq \langle N^{124}, (1, 6, 15)$ $(2, 13, 14)(3, 11, 9)(4, 7, 10)(5, 8, 12)\rangle$

The number of right cosets in $Nt_1t_2t_4N$ is calculated by the formula,

$\frac{|N|}{|N^{(124)}|} = \frac{150}{3} = 50$.

The orbits of $N^{(124)}$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ are orbits $\{1, 6, 15\}, \{2, 13, 14\}, \{3, 11, 9\}, \{4, 7, 10\}, \{5, 8, 12\}$

We will determine the double cosetes$Nt_1t_2t_4t_1, Nt_1t_2t_4t_2, Nt_1t_2t_4t_3, Nt_1t_2t_4t_4, Nt_1t_2t_4t_5$ by selecting one representative from this orbit such as,

$Nt_1t_2t_4t_1N$is a new double coset which has three symmetric generators we will denote [1241].

$Nt_1t_2t_4t_2N$is a new double coset which has three symmetric generators we will denote [1242].

$Nt_1t_2t_4t_3N$is a new double coset which has three symmetric generators we will denote [1243].

$Nt_1t_2t_4t_4N$is a new double coset which has three symmetric generators we will denote [1244].

$Nt_1t_2t_4t_5N$ is a new double coset which has three symmetric generators we will denote [1245].

Cayley Diagram

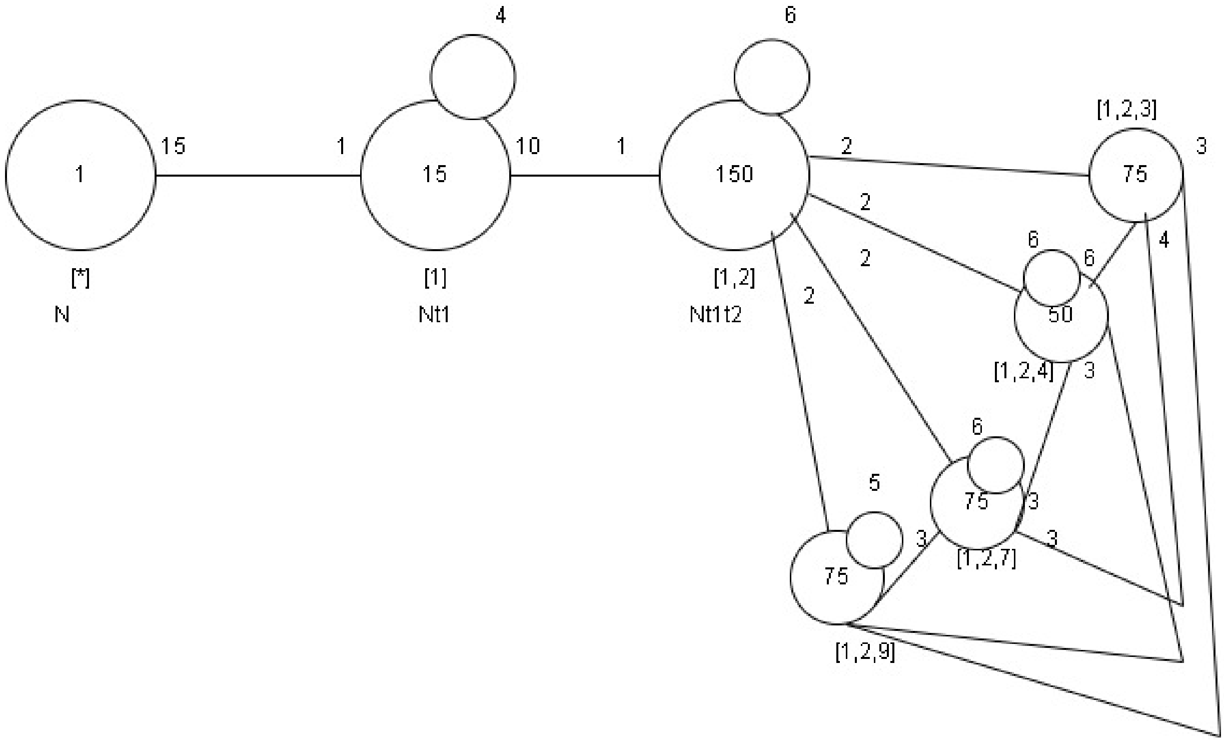

Figure 8.10: Cayley diagram for G over S_{15}

• **Seventh Double Coset** $Nt_1t_2t_3N = [123]$

$Nt_1t_2t_3 = \{N(t_1t_2t_3)^n | n \in N\}$. We now find the Coset Stabilizer $N^{(123)}$. We first find the point stabilizer of 1, 2 and 3 in N.

$N^{123} = \{n \in N | (123)^n = 123\}$

$N^{123} = \langle(1,5)(2,3)(6,7)(10,14)(11,12)\rangle$.

We have $N(t_1t_2t_3)^{(1,5)(2,3)(6,7)(10,14)(11,12)}$.

$\implies (1,5)(2,3)(6,7)(10,14)$

$(11,12) \in N^{(123)}$ Thus $N^{((123)} \geq \langle N^{123}, (1,5)(2,3)(6,7)(10,14)(11,12)\rangle$

The number of right cosets in Nt_1t_2t34N is calculated by the formula,

$\frac{|N|}{|N^{(123)}|} = \frac{150}{2} = 75$.

The orbits of $N^{(123)}$ on $X = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$ are orbits $\{8\},\{9\},\{13\},\{15\},\{1,5\},\{2,3\},\{6,7\},\{10,14\},\{11,12\}$.

We will determine the double cosetes$Nt_1t_2t_3t_8, Nt_1t_2t_3t_9, Nt_1t_2t_3t_{13}, Nt_1t_2t_3t_{15}$, $Nt_1t_2t_3t_1, Nt_1t_2t_3t_2, Nt_1t_2t_3t_6, Nt_1t_2t_3t_{10}, Nt_1t_2t_3t_{11}$.

By selecting one representative from this orbit such as,

$Nt_1t_2t_3t_4N$is a new double coset which has one symmetric generators we will denote [1234].

$Nt_1t_2t_3t_8N$is a new double coset which has one symmetric generators we will denote [1238].

$Nt_1t_2t_3t_9N$is a new double coset which has one symmetric generators we will denote [1239].

$Nt_1t_2t_3t_{13}N$is a new double coset which has one symmetric generators we will denote [12313].

$Nt_1t_2t_3t_{15}N$is a new double coset which has one symmetric generators we will denote [12313].

$Nt_1t_2t_3t_1N$is a new double coset which has two symmetric generators we will denote [1231].

$Nt_1t_2t_3t_2N$is a new double coset which has two symmetric generators we will denote [1232].

$Nt_1t_2t_3t_6N$is a new double coset which has two symmetric generators we will denote [1232].

$Nt_1t_2t_3t_{10}N$is a new double coset which has two symmetric generators we will denote [12310].

$Nt_1t_2t_3t_{11}N$is a new double coset which has two symmetric generators we will denote [12311].

Cayley Diagram

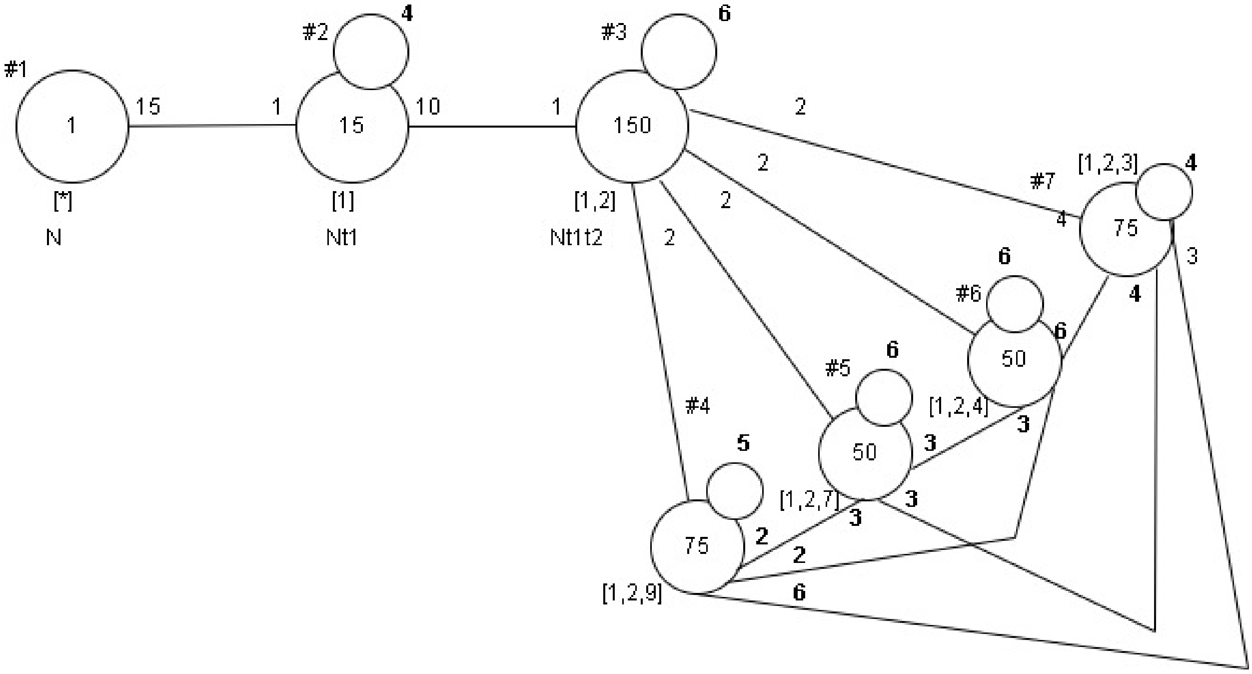

Figure 8.11: Cayley diagram for G over S_{15}

8.2.1 Magma Work $U(3,4):2$ Over $N \sim 5^2 : S_3$

S:=Sym(15);

xx:=S!(1, 15, 12, 8, 3, 9, 14, 13, 7, 4)(2, 11, 5, 6, 10);

yy:=S!(1, 11, 14, 6, 12, 2, 7, 5, 3, 10)(4, 8, 13, 15, 9);

N:=sub$< S|xx, yy >$;

♯ N; /*150*/

G$< x, y, t >$:=Group$< x, y, t|(y^{-1} * x^{-1})^3, (y^{-1} * x)^3, x^{-1} * y^{-1} * x^3 * y^{-1} * x^{-1} * y, x^2 * y * x^2 * y^3, t^2, (t, y^{-1} * x^2 * y^{-1}), (t, x^3 * y^{-1} * x), (x^{(-1)} * y * t)^5, (x^{(-2)} * t^{(y*x^{(-1)})})^3 >$;

♯G;

/*124800*/

f,G1,k:=CosetAction(G,sub$< G|x, y >$);

CompositionFactors(G1);

```
/*
    G

    |  Cyclic(2)

    *

    |  2A(2, 4)                   = U(3, 4)

    1

*/
NL:=NormalLattice(G1);
NL;
/*

Normal subgroup lattice

-----------------------

[3]  Order 124800  Length 1  Maximal Subgroups: 2

---
[2]  Order 62400   Length 1  Maximal Subgroups: 1

---
```

```
[1]  Order 1       Length 1  Maximal Subgroups:

*/
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;
/*
1
*/
/*to get the largest abelain group*/
IsAbelian(NormalLattice(N)[2]);
/*true*/
IsCyclic(NormalLattice(N)[2]);
/*false*/
q,ff:=quo<N|NormalLattice(N)[2]>;
q;
/*
Permutation group q acting on a set of cardinality 3
Order = 6 = 2 * 3
    (2, 3)
    (1, 2)
*/
IsIsomorphic(q,Sym(3));
/*
true Isomorphism of GrpPerm: q, Degree 3, Order 2 * 3
into GrpPerm: $, Degree 3, Order 2 * 3 induced by
    (2, 3) |--> (2, 3)
    (1, 2) |--> (1, 2)
*/
FPGroup(q);
/*
Finitely presented group on 2 generators
Relations
    $.1^2 = Id($)
    $.2^2 = Id($)
    ($.2 * $.1)^3 = Id($)
*/
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
/*12*/
DoubleCosets(G,H, sub<G|x,y>);

H:=sub<G|x,y,x * y * t * x^-2 * t * x^2 * t * y^-1>;
#H;
/*300*/
Index(G,H);
```

```
/*416*/
f,G1,k:=CosetAction(G,H);
IN:=sub<G1|f(x),f(y)>;
IH:=sub<G1|f(x),f(y),f(x * y * t * x^-2 * t * x^2 * t * y^-1)>;
#DoubleCosets(G,H,sub<G|x,y>);
/*7*/
/* Do DCE of G over H and N*/
DoubleCosets(G,H, sub<G|x,y>);
/*{ <GrpFP: H, Id(G), GrpFP>, <GrpFP: H, t * x^-1 * t * y
* t, GrpFP>, <GrpFP: H, t * x * t * x * t, GrpFP>,
<GrpFP: H, t, GrpFP>, <GrpFP: H, t * x * t, GrpFP>,
<GrpFP: H, t * x * t * y^-1 * t, GrpFP>, <GrpFP: H, t *
y * t * y * t, GrpFP> }*/

DC:=[Id(G1),f(t),f(t * x * t),
f(t * x^-1 * t * y * t),f(t * x * t * x * t),
f( t * x * t * y^-1 * t),
f( t * y * t * y * t)  ];
cst := [null : i in [1 .. 416]] where null is [Integers() | ];
prodim := function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
NN<a,b>:=Group<a,b| (b^-1 * a^-1)^3 , (b^-1 * a)^3 ,
a^-1 * b^-1 * a^3 * b^-1 * a^-1 * b  ,a^2 * b * a^2 * b^3 >;
#NN;
/*150*/
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
```

```
end for;
for j in [2..15] do for i in [1..#Sch] do if 1^ArrayP[i] eq j then j,
Sch[i];
break;
end if; end for;
end for;
/*
2 b * a^-1
3 b^-2
4 a^-1
5 b * a
6 b^-1 * a^-1
7 a^-2
8 a^-1 * b
9 a * b
10 b^-1
11 b
12 a^2
13 a * b^-1
14 b^2
15 a
*/

ts := [ Id(G1): i in [1 .. 15] ];
ts[1]:=f(t);
ts[2]:=f(t^(y * x^-1)); ts[3]:=f(t^(y^-2)); ts[4]:=f(t^(x^-1));
ts[5]:=f(t^(y * x));
ts[6]:=f(t^(y^-1 * x^-1));
ts[7]:=f(t^(x^-2));
ts[8]:=f(t^(x^-1 * y)); ts[9]:=f(t^(x*y));
ts[10]:=f(t^(y^-1 )); ts[11]:=f(t^y);
ts[12]:=f(t^(x^2 )); ts[13]:=f(t^(x*y^-1));
ts[14]:=f(t^(y^2)); ts[15]:=f(t^x);

N1:=Stabiliser(N,1);
N1;
/*
Permutation group N1 acting on a set of cardinality 15
Order = 10 = 2 * 5
    (2, 5, 10, 11, 6)(4, 8, 13, 15, 9)
    (2, 13)(4, 10)(5, 8)(6, 15)(9, 11)
*/
for g in IH do for i in [1..15] do if ts[1] eq g*ts[i]
then i; end if; end for; end for;
```

```
S:={[1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IH do if ts[1]
eq g*ts[Rep(SSS[i])[1]]
then print SSS[i];
end if; end for; end for;
/*
{
    [ 1 ]
}

*/
N1:=Stabiliser(N,1);
N1s:=N1;
#N1s;
/*10*/

tr1:=Transversal(N,N1s);
for i:=1 to #tr1 do
ss:=[1]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
/*15*/
m:=0; for i in [1..416] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
Orbits(N1s);
/*
[
    GSet{@ 1 @},
    GSet{@ 3 @},
    GSet{@ 7 @},
    GSet{@ 12 @},
    GSet{@ 14 @},
    GSet{@ 2, 5, 13, 10, 8, 15, 11, 4, 9, 6 @}
]

*/
```

for i in $[1 \cdots 7]$ do

for g in IH do for h in IN do

if $ts[1] * ts[1] eqg * (DC[i])^h$ then i;

break i; break g; break h; end if; end for;end for;end for;

/*1*/

for i in [1···7] do

for g in IH do for h in IN do

if ts[1] $* ts[3] eqg * (DC[i])^h$ then i;

break i; break g; break h; end if; end for;end for;end for;

/*2*/

```
for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[3] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[7] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[12] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[14] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/

S:={[1,2]};
SS:=S^N;
```

```
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IH do if ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
/*
{
    [ 1, 2 ]
}

*/
N12:=Stabiliser(N,[1,2]);
N12;
/*
Permutation group N12 acting on a set of cardinality 15
Order = 1
*/
N12s:=N12;
#N12s;
/*1*/
tr1:=Transversal(N,N12s);
for i:=1 to #tr1 do
ss:=[1,2]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..416] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*165*/
Orbits(N12s);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @},
    GSet{@ 7 @},
    GSet{@ 8 @},
    GSet{@ 9 @},
    GSet{@ 10 @},
    GSet{@ 11 @},
```

```
    GSet{@ 12 @},
    GSet{@ 13 @},
    GSet{@ 14 @},
    GSet{@ 15 @}
]

*/
for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[1] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[2] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/
for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[3] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*7*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[4] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*6*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[5] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[6] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/

for i in [1..7] do
for g in IH do for h in IN do
```

```
if ts[1]*ts[2]*ts[7] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*5
*/
for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[8] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[9] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*4*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[10] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[11] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[12] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*5*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[13] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*6*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[14] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
```

```
/*7*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[15] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*4*/
N129:=Stabiliser(N,[1,2,9]);
N129;
/*
Permutation group N129s acting on a set of cardinality
15
Order = 2
    (1, 13)(3, 15)(4, 12)(7, 9)(8, 14)
*/
N129s:=N129;
for g in N do if [1,2,9]^g eq [13,2,7]
then N129s:=sub<N|N129s,g>; end if ; end for;
[1,2,9]^N129s;
/*
GSet{@
    [ 1, 2, 9 ],
    [ 13, 2, 7 ]
@}
*/
#N/#N129s;
/*75*/

tr1:=Transversal(N,N129s);
for i:=1 to #tr1 do
ss:=[1,2,9]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..416] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*240*/
Orbits(N129s);
/*
[
    GSet{@ 2 @},
    GSet{@ 5 @},
    GSet{@ 6 @},
    GSet{@ 10 @},
```

```
    GSet{@ 11 @},
    GSet{@ 1, 13 @},
    GSet{@ 3, 15 @},
    GSet{@ 4, 12 @},
    GSet{@ 7, 9 @},
    GSet{@ 8, 14 @}
]
*/
```

for i in $[1\cdots 7]$ do

for g in IH do for h in IN do

if $ts[1]*ts[2]*ts[9]*ts[2]eqg*(DC[i])^h$ then i;

break i; break g; break h; end if; end for;end for;end for;

/* 4 */

for i in $[1\cdots 7]$ do

for g in IH do for h in IN do

if $ts[1]*ts[2]*ts[9]*ts[5]eqg*(DC[i])^h$ then i;

break i; break g; break h; end if; end for;end for;end for;

/*7 */

for i in $[1\cdots 7]$ do

for g in IH do for h in IN do

if $ts[1]*ts[2]*ts[9]*ts[6]eqg*(DC[i])^h$ then i;

break i; break g; break h; end if; end for;end for;end for;

/* 4 */

for i in $[1\cdots 7]$ do

for g in IH do for h in IN do

if $ts[1]*ts[2]*ts[9]*ts[10]eqg*(DC[i])^h$ then i;

break i; break g; break h; end if; end for;end for;end for;

/* 7 */

for i in $[1\cdots 7]$ do

for g in IH do for h in IN do

if $ts[1] * ts[2] * ts[9] * ts[11] eqg * (DC[i])^h$ then i;

break i; break g; break h; end if; end for;end for;end for;

/* 7 */

for i in $[1 \cdots 7]$ do

for g in IH do for h in IN do

if $ts[1] * ts[2] * ts[9] * ts[1] eqg * (DC[i])^h$ then i;

break i; break g; break h; end if; end for;end for;end for;

/*6 */

for i in $[1 \cdots 7]$ do

for g in IH do for h in IN do

if $ts[1] * ts[2] * ts[9] * ts[3] eqg * (DC[i])^h$ then i;

break i; break g; break h; end if; end for;end for;end for;

/* 6 */

for i in $[1 \cdots 7]$ do

for g in IH do for h in IN do

if $ts[1] * ts[2] * ts[9] * ts[4] eqg * (DC[i])^h$ then i;

break i; break g; break h; end if; end for;end for;end for;

/*4 */

for i in $[1 \cdots 7]$ do

for g in IH do for h in IN do

if $ts[1] * ts[2] * ts[9] * ts[7] eqg * (DC[i])^h$ then i;

break i; break g; break h; end if; end for;end for;end for;

/* 3 */

for i in $[1 \cdots 7]$ do

for g in IH do for h in IN do

if $ts[1] * ts[2] * ts[9] * ts[8] eqg * (DC[i])^h$ then i;

break i; break g; break h; end if; end for;end for;end for;

/* 3 */

```
S:={[1,2,7]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IH do if ts[1]*ts[2]*ts[7]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;
/*
{
    [ 1, 2, 7 ]
}
{
    [ 2, 15, 11 ]
}
{
    [ 15, 1, 4 ]
}
*/
N127:=Stabiliser(N,[1,2,7]);
N127;
/* Permutation group N127 acting on a set of cardinality 15
Order = 1
*/
N127s:=N127;
for g in N do if [1,2,7]^g eq [2,15,11] then N127s:=sub<N|N127s,g>;
end if ; end for;
[1,2,7]^N127s;
N127s;
/*Permutation group N127s acting on a set of cardinality
15
Order = 3
    (1, 2, 15)(3, 6, 9)(4, 7, 11)(5, 13, 14)(8, 12, 10)
*/
#N/#N127s;
/*50*/

tr1:=Transversal(N,N127s);
for i:=1 to #tr1 do
ss:=[1,2,7]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
```

```
end for;
m:=0; for i in [1..416] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*290*/
Orbits(N127s);
/*
[
    GSet{@ 1, 2, 15 @},
    GSet{@ 3, 6, 9 @},
    GSet{@ 4, 7, 11 @},
    GSet{@ 5, 13, 14 @},
    GSet{@ 8, 12, 10 @}
]

*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[7]*ts[1] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3 */

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[7]*ts[3] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*5*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[7]*ts[4] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[7]*ts[5] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/* 7 */

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[7]*ts[8] eq g*(DC[i])^h then i;
```

```
break i; break g; break h; end if; end for;end for;end for;
/* 7 */

S:={[1,2,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IH do if ts[1]*ts[2]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;
/*
{
    [ 1, 2, 4 ]
}
{
    [ 6, 13, 7 ]
}
{
    [ 15, 14, 10 ]
}

*/
```

N124:=Stabiliser(N,[1,2,4]);

N124s:=N124;

for g in N do if $[1,2,4]^g$ eq [6,13,7] then

N124s:=sub$< N|N124s, g >$; end if ; end for;

$[1,2,4]^{N124s}$;

```
    /*
GSet{@
    [ 1, 2, 4 ],
    [ 6, 13, 7 ],
    [ 15, 14, 10 ]
@}
*/
N124s;
/*
 (1, 6, 15)(2, 13, 14)(3, 11, 9)(4, 7, 10)(5, 8, 12)
*/
#N/#N124s;
```

```
/*50*/

tr1:=Transversal(N,N124s);
for i:=1 to #tr1 do
ss:=[1,2,4]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..416] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*340*/
Orbits(N124s);
/*
[
    GSet{@ 1, 6, 15 @},
    GSet{@ 2, 13, 14 @},
    GSet{@ 3, 11, 9 @},
    GSet{@ 4, 7, 10 @},
    GSet{@ 5, 8, 12 @}
]
*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[4]*ts[1] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/* 4 */

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[4]*ts[2] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*6 */

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[4]*ts[3] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/* 3*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[4]*ts[4] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
```

```
/* 3 */

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[4]*ts[5] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/* 4 */
```

S:=[1,2,3];

$SS := S^N$;

SSS:=Setseq(SS);

for i in [1$\cdots\sharp$ SSS] do

for g in IH do if ts[1]*ts[2]*ts[3]

eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]

then print SSS[i];

end if; end for; end for;

```
/*
{
    [ 1, 2, 3 ]
}
{
    [ 5, 3, 2 ]
}
*/

N123:=Stabiliser(N,[1,2,3]);
N123s:=N123;
for g in N do if [1,2,3]^g eq [5,3,2]
then N123s:=sub<N|N123s,g>; end if ; end for;
[1,2,3]^N123s;
/*
GSet{@
    [ 1, 2, 3 ],
    [ 5, 3, 2 ]
@}
*/
N123s;
/*
Permutation group N123s acting on a set of cardinality
15
```

```
Order = 2
    (1, 5)(2, 3)(6, 7)(10, 14)(11, 12)
*/

#N/#N123s;
/*75*/
Orbits(N123s);
/*[
    GSet{@ 4 @},
    GSet{@ 8 @},
    GSet{@ 9 @},
    GSet{@ 13 @},
    GSet{@ 15 @},
    GSet{@ 1, 5 @},
    GSet{@ 2, 3 @},
    GSet{@ 6, 7 @},
    GSet{@ 10, 14 @},
    GSet{@ 11, 12 @}
]

*/

tr1:=Transversal(N,N123s);
for i:=1 to #tr1 do
ss:=[1,2,3]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..416] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*415*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[3]*ts[4] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*7*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[3]*ts[8] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*4*/
```

```
for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[3]*ts[9] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*4*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[3]*ts[13] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*4*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[3]*ts[15] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*7*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[3]*ts[1] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*5*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[3]*ts[2] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[3]*ts[6] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*3*/

for i in [1..7] do
for g in IH do for h in IN do
if ts[1]*ts[2]*ts[3]*ts[10] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*7*/

for i in [1..7] do
for g in IH do for h in IN do
```

```
if ts[1]*ts[2]*ts[3]*ts[11] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*5*/
```

S:=[1,2,3,4];

$SS := S^N$;

SSS:=Setseq(SS);

for i in [1⋯♯ SSS] do

for g in IH do if ts[1]*ts[2]*ts[3]*ts[4]

eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])[4]]

then print SSS[i];

end if; end for; end for;

```
    /*
{
    [ 1, 2, 3, 4]
}
{
    [ 5, 3, 2, 4 ]
}

*/

N1234:=Stabiliser(N,[1,2,3,4]);
N1234s:=N1234;
for g in N do if [1,2,3,4]^g eq [5,3,2,4]
then N1234s:=sub<N|N1234s,g>; end if ; end for;
[1,2,3,4]^N1234s;
#N/#N1234s;
/* 75 */

tr1:=Transversal(N,N123s);
for i:=1 to #tr1 do
ss:=[1,2,3,4]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..416] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*
415
```

```
*/

S:={[1,2,3,8]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IH do if ts[1]*ts[2]*ts[3]*ts[8]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
*ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])[4]]
then print SSS[i];
end if; end for; end for;
/*
{
    [ 1, 2, 3, 8 ]
}
{
    [ 5, 3, 2, 8 ]
}

*/

N1238:=Stabiliser(N,[1,2,3,8]);
N1238s:=N1238;
for g in N do if [1,2,3,8]^g eq [5,3,2,8]
then N1238s:=sub<N|N1238s,g>; end if ; end for;
[1,2,3,8]^N1238s;
#N/#N1238s;
/*
75
*/

tr1:=Transversal(N,N123s);
for i:=1 to #tr1 do
ss:=[1,2,3,8]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..416] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*
415
*/

S:={[1,2,3,1]};
```

```
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IH do if ts[1]*ts[2]*ts[3]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
*ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])[4]]
then print SSS[i];
end if; end for; end for;
/*
{
    [ 1, 2, 3, 1 ]
}
{
    [ 11, 15, 10, 11 ]
}
{
    [ 4, 12, 8, 4 ]
}

*/
```

N1231:=Stabiliser(N,[1,2,3,1]);

N1231s:=N1231;

for g in N do if $[1,2,3,1]^g$ eq [11,15,10,11]then N1231s:=sub$< N|N1231s,g >$; end if ; end for;

$[1,2,3,1]^{N1231s}$;

♯ N/ ♯ N1231s;

```
    /*
50
*/

tr1:=Transversal(N,N123s);
for i:=1 to #tr1 do
ss:=[1,2,3,1]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..416] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/* 415 */
```

8.3 Double Coset Enumeration Of S_6 Over S_5

Consider N = $< x, y >$; where

where x $\sim$ (2, 3)(4, 6)(5, 8)(7, 11)(9, 14)(10, 15)(12, 18)(13, 20)(17, 24)(19, 26)(21,25)(23, 28),

y $\sim$ (1, 2, 4, 7, 12, 19)(3, 5, 9, 15, 22, 27)(6, 10, 16, 23, 18, 25)(8, 13,21)(11, 17, 14)(20, 26, 29, 30, 28, 24).

Our progenitor $2^{*30} : S5$. We prove that

$S_6 \cong$ Group$< x, y, t | x^2, y^6, (y * x * y^{-1} * x)^2, (x * y^{-1})^5, (t, x), (t, y^2 * x * y^{-2} * x * y^2), (y * x * t)^3, (y * x * t^{(y^2 * x)})^4 >$;

We perform manual double coset enumeration of G over N. We need to determine all distinct double coset NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find the index which is the order of G over the order of N.

Hence, $\frac{|G|}{|N|} = \frac{720}{120} = 6$. So, we have total 6 single cosets.

• **First Double Coset [*]**

$NeN = \{Ne^n | n \in N\} = \{N\}$.

The double coset NeN is denoted by [*] which contains 1 right coset.The coset stsbiliser of the coset Ne is N.

The number of right cosets in [*] is equal to $\frac{|N|}{|N|} = \frac{120}{120} = 1$.

Since N is transitive on {1,2,3,4,5,6,8,9,10,11,12,13,14,15,16,17,18,19, 20,21,22,23,24,25, 26,27,28,29,30},

We need to determine the double coset of the right coset Nt_1.

Thus, the Thirty $t_i's$ which are $Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}$, Nt_{11}, Nt_{12}, $Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16}, Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}, Nt_{21}, Nt_{22}, Nt_{23}, Nt_{24}, Nt_{25}$, $Nt_{26}, Nt_{27}, Nt_{28}$, Nt_{29}, Nt_{30} extend the new double coset [1], that mean thirty generators goes forward to Nt_1.

Cayley Diagram

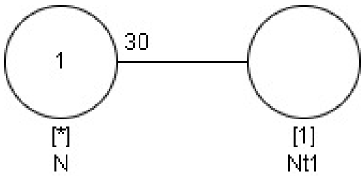

Figure 8.12: Cayley diagram for G over S_{30}

• **Second Double Coset** $Nt_1N = [1]$

$Nt_1N = \{Nt_1^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12},$

$Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16}, Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}, Nt_{21}, Nt_{22}, Nt_{23}, Nt_{24},$

$Nt_{25}, Nt_{26}, Nt_{27}, Nt_{28}, Nt_{29}, Nt_{30}\}$. Firstly, find the point stabilizer of 1 in N.

$N^1 = n \in N | 1^n = 1$

```
N^1 =
(2, 3)(4, 6)(5, 8)(7, 11)(9, 14)(10, 15)(12,18)
(13,20)(17, 24)(19, 26)(21, 25)(23, 28)(2, 26)
(3, 19)(4, 18)(5, 13)(6, 12)(7,11)(8,20)(9,17)
(10, 28)(14, 24)(15, 23)(21, 25)(22,30)(27,29).
```

Thus,$N^{(1)} \geq$

```
(2, 3)(4, 6)(5, 8)(7, 11)(9, 14)(10, 15)(12,18)
(13,20)(17, 24)(19, 26)(21, 25)(23, 28)(2, 26)
(3, 19)(4, 18)(5, 13)(6, 12)(7,11)(8,20)(9,17)
(10, 28)(14, 24)(15, 23)(21, 25)(22,30)(27,29).
```

$\Rightarrow$ (1,2,3,4,5,6,8,9,10,11,12,13,14,15,16,17,18,19, 20,21,22,23,24,

25,26,27,28,29,30) $\in N^{(}1)$.

The orbits of $N^{(1)}$ on

$X = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,$

$20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\}$ are

$\{1, 4, 6, 18, 12, 16\}, \{2, 3, 26, 7, 9, 30, 19, 11, 14, 17, 22, 24\},$

$\{5, 8, 13, 15, 21, 29, 20, 10, 23, 27, 25, 28\}.$

Then $N^{(1)} \leq N_1, (1, 4, 12)(2, 7, 19)(3, 9, 22)(5, 15, 27)(6, 16, 18)(8, 21, 13)(10, 23, 25)$

$(11, 14, 17)(20, 29, 28)(24, 26, 30) > \cong S_4$.

So, Order of$< N_1, (1, 4, 12)(2, 7, 19)(3, 9, 22)(5, 15, 27)(6, 16, 18)(8, 21, 13)(10, 23, 25)$
$(11, 14, 17)(20, 29, 28)(24, 26, 30) >$ is 24.

The number of right cosets in [1] is equal to
$\frac{|N|}{|N|^{(1)}} = \frac{120}{24} = 5$.

Now we choose the represenatative from the first orbit and 5 from the second orbit and determine the double coset of Nt_1t_1, Nt_1t_5.

This shows us the following:

$Nt_1t_1 = Nt_1^2 = N \in [*]$

Since the orbit $\{1\}$ contains one element, then one symmetric generator goes back to the double coset [*].

$Nt_1 \in [1]$

Six symmetric generator will go back to [1].

$Nt_1t_2N$is a new double coset which we will denote [12].

Twelve symmetric generators will go to the new double coset [12].

$Nt_1t_5N$is a new double coset which we will denote [15].

Twelve symmetric generators will go to the new double coset [12].

<u>Cayley Diagram</u>

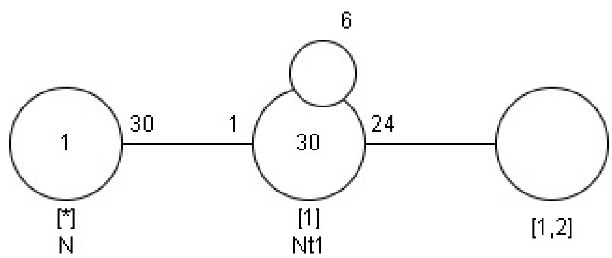

Figure 8.13: Cayley diagram for G over S_{30}

• **Third Double Coset** $Nt_1 2N = [12]$

$Nt_{12}N = \{Nt_{12}^n | n \in N\}$.

$=\{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7, Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12},$

$Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16}, Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}, Nt_{21}, Nt_{22}, Nt_{23},$

$Nt_{24}, Nt_{25}, Nt_{26}, Nt_{27}, Nt_{28}, Nt_{29}, Nt_{30}\}$.

Firstly, find the point stabilizer of 1 in N.

$N^1 = n \in N | 1^n = 1$.

N^1 =1

Thus,$N^{(1)} \geq (1)$

$\Rightarrow$ (1,2,3,4,5,6,8,9,10,11,12,13,14,15,16,17,18,19, 20,21,22,23,24,25,

26,27,28,29,30) $\in N^(1)$

The number of right cosets in [1] is equal to

$\frac{|N|}{|N|^{(1)}} = \frac{120}{1} = 120$.

The orbits of $N^{(1)}$ on

$X = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25,$

$26, 27, 28, 29, 30\}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}, \{15\},$

$\{16\}, \{17\}, \{18\}, \{19\}, \{20\}, \{21\}, \{22\},$

$\{23\}, \{24\}, \{25\}, \{26\}, \{27\}, \{28\}, \{19\}, \{30\}$.

The number of right cosets in [1] is equal to

$\frac{|N|}{|N|^{(1)}} = \frac{120}{1} = 120$.

Now we choose the represenatative from determine the double coset of

$Nt_1t_1, Nt_1t_2, Nt_1t_3, Nt_1t_4, Nt_1t_5, Nt_1t_6,$

$Nt_1t_7, Nt_1t_8, Nt_1t_9, Nt_1t_{10}, Nt_1t_{11}, Nt_1t_{12}, Nt_1t_{13}, Nt_1t_{14}, Nt_1t_{15},$

$Nt_1t_{16}, Nt_1t_{17}, Nt_1t_{18}, Nt_1t_{19}, Nt_1t_{20}, Nt_1t_{21}, Nt_1t_{21}, Nt_1t_{22}, Nt_1t_{23}$

$, Nt_1t_{24}, Nt_1t_{25}, Nt_1t_{26}, Nt_1t_{27}, Nt_1t_{28}, Nt_1t_{29}, Nt_1t_{30}$.

This shows us the following:

Cayley Diagram

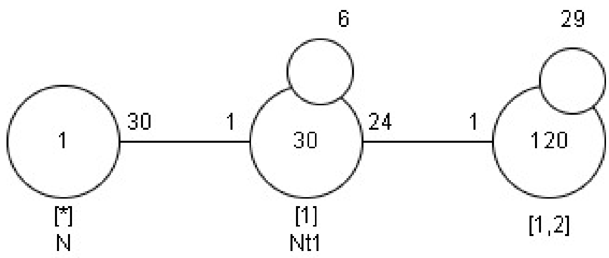

Figure 8.14: Cayley diagram for G over S_{30}

It is possible that the coset stabiliser of $N^{(w)}$ of the coset Nw increases and therefore $\frac{|N|}{|N^{(w)}|}$ decreases. Our cayley diagram shows that
$G = N \cup Nt_1 \cup Nt_1t_2$
$|G| \leq (|N| + \frac{|N|}{|N^1|} + \frac{|N|}{|N^{12}|} X|N|$
$|G| \leq (1 + 30 + 120)X120 \implies |G| \leq 151X120 \implies |G| \leq 18000.$
G acts on 720 cosets that are given in the cayley diagram.
Let X be the set of these 720 cosets.
Now, $f : G \to S_x$ is a homomorphism.
$\frac{G}{Kerf} \cong Imf$ (First Isomorphism Theorem).
$\implies \frac{G}{Kerf} \cong \langle f(x), f(y), f(t) \rangle$
$\implies |\frac{G}{Kerf}| = |\langle f(x), f(y), f(t) \rangle|.$
But $\#\langle f(x), f(y), f(t) \rangle = 8064.$
So, $|\frac{G}{kerf}| = 18000$
This means $|G| \geq 18000$. We know $|G| \leq 18000$ from cayley diagram.
Therefore, $|G| = 18000$.
From $|G| = 18000 \times |Kerf|$ we find $|Kerf| = 1$
$G \cong \langle f(x), f(y), f(t) \rangle$ but, $\langle f(x), f(y), f(t) \rangle \cong (2^6 : S_6)(7 : 3)$
$\implies G \cong (2 * 30 : S_5).$

8.3.1 Magma Work S_6 Over S_5

```
S:=Sym(30);
xx:=S!(2, 3)(4, 6)(5, 8)(7, 11)(9, 14)(10, 15)(12, 18)(13, 20)(17, 24)
(19, 26)(21,25)(23, 28);
yy:=S!(1, 2, 4, 7, 12, 19)(3, 5, 9, 15, 22, 27)(6, 10, 16, 23, 18, 25)
(8, 13,21)(11, 17, 14)(20, 26, 29, 30, 28, 24);
N:=sub<S|xx,yy>;
#N;
/*120*/
G<x,y,t>:=Group<x,y,t|x^2, y^6, (y*x*y^-1*x)^2,(x*y^-1)^5,
(t,x),(t,y^2*x*y^-2*x*y^2),(y*x*t)^3,
(y*x*t^(y^2*x))^4>;
#G;
720
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  Alternating(6)
    1
NL:=NormalLattice(G1);
NL;

Normal subgroup lattice
-----------------------

[3]  Order 720  Length 1  Maximal Subgroups: 2
---
[2]  Order 360  Length 1  Maximal Subgroups: 1
---
[1]  Order 1    Length 1  Maximal Subgroups:

IsAbelian(NormalLattice(N)[1]);
true
q,ff:=quo<N|NormalLattice(N)[2]>;
q;
Permutation group q acting on a set of cardinality 2
Order = 2
    (1, 2)
    (1, 2)
IsIsomorphic(G1,Sym(6));
true Isomorphism of GrpPerm: G1, Degree 6, Order 2^4 *
3^2 * 5 into GrpPerm: $, Degree 6, Order 2^4 * 3^2 * 5
```

```
induced by
    (3, 4) |--> (3, 4)
    (2, 3)(4, 5, 6) |--> (2, 3)(4, 5, 6)
    (1, 2)(3, 4) |--> (1, 2)(3, 4)
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
2
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
{ <GrpFP, Id(G), GrpFP>, <GrpFP, t, GrpFP> }
IN:=sub<G1|f(x),f(y)>;
IN;
Permutation group IN acting on a set of cardinality 6
    (3, 4)
    (2, 3)(4, 5, 6)
#G1/#N;
/*6*/
ts := [ Id(G1): i in [1 .. 30] ];
ts[1]:=f(t);
ts[2]:=f(t^y );
ts[3]:=f(t^(y*x));
ts[4]:=f(t^(y^2));
ts[5]:=f(t^(y * x*y));
ts[6]:=f(t^(y^2 * x));
ts[7]:=f(t^(y^3));
ts[8]:=f(t^(y*x)^2);
ts[9]:=f(t^(y*x*y^2));
ts[10]:=f(t^(y^2*x*y ));
ts[11]:=f(t^(y^3*x));
ts[12]:=f(t^(x^2));
ts[13]:=f(t^(x*y^-1));
ts[14]:=f(t^(y*x*y^2*x));
ts[15]:=f(t^(y*x*y^3));
ts[16]:=f(t^(y^2*x*y^2));
ts[17]:=f(t^(y^3*x*y));
ts[18]:=f(t^(y^-2*x));
ts[19]:=f(t^(y^-1));
ts[20]:=f(t^(y^-1*x*y^-1));
ts[21]:=f(t^(y*x*y*x*y^-1));
ts[22]:=f(t^(y*x*y^-2));
ts[23]:=f(t^(y^-2*x*y^-1)); ts[24]:=f(t^(y^-1*x*y^-2));
ts[25]:=f(t^(y^2*x*y^-1));
ts[26]:=f(t^(y^-1*x));
ts[27]:=f(t^(y*x*y^-1)); ts[28]:=f(t^(y^-1*x*y^3));
ts[29]:=f(t^(x^y));
ts[30]:=f(t^(y^-1*x*y^2));
```

```
DC:=[Id(G1),f(t) ];
cst := [null : i in [1 .. 6]] where null is [Integers() | ];
prodim := function(pt, Q, I)
function> v := pt;
function> for i in I do
function|for> v := v^(Q[i]);
function|for> end for;
function> return v;
function> end function;
N1:=Stabiliser(N,1);
N1;
Permutation group N1 acting on a set of cardinality 30
Order = 4 = 2^2
    (2, 3)(4, 6)(5, 8)(7, 11)(9, 14)(10, 15)
    (12, 18)(13,20)(17, 24)(19, 26)(21, 25)
    (23, 28)(2, 26)(3, 19)(4, 18)(5, 13)
    (6, 12)(7, 11)(8, 20)(9,17)(10, 28)
    (14, 24)(15, 23)(21, 25)(22, 30)(27,29)

S:={[1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]
eq g*ts[Rep(SSS[i])[1]]
then print SSS[i];
end if; end for; end for;
{
    [ 1 ]
}
{
    [ 4 ]
}
{
    [ 6 ]
}
{
    [ 12 ]
}
{
    [ 16 ]
}
{
```

```
    [ 18 ]
}
N1:=Stabiliser(N,1);
N1s:=N1;
#N1s;
4
Orbits(N1s);
[
    GSet{@ 1 @},
    GSet{@ 16 @},
    GSet{@ 7, 11 @},
    GSet{@ 21, 25 @},
    GSet{@ 22, 30 @},
    GSet{@ 27, 29 @},
    GSet{@ 2, 3, 26, 19 @},
    GSet{@ 4, 6, 18, 12 @},
    GSet{@ 5, 8, 13, 20 @},
    GSet{@ 9, 14, 17, 24 @},
    GSet{@ 10, 15, 28, 23 @}
]
N1s:=sub<N|N1,(1, 4, 12)(2, 7, 19)(3, 9, 22)(5,15,27)(6, 16, 18)
(8,21, 13)(10, 23, 25)(11, 14, 17)(20, 29, 28)(24,26, 30)>;
#N1s;
24
Orbits(N1s);
[
    GSet{@ 1, 4, 6, 18, 12, 16 @},
    GSet{@ 2, 3, 26, 7, 19, 9, 30, 11, 14, 17, 22, 24
    @},
    GSet{@ 5, 8, 13, 15, 20, 21, 10, 23, 27, 29, 25, 28
    @}
]
for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[1] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*1*/
for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/
for i in [1..2] do
for g in IN do for h in IN do
```

```
if ts[1]*ts[5] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/
S:={[1,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
{
    [ 1, 2 ]
}
{
    [ 1, 3 ]
}
{
    [ 5, 13 ]
}
{
    [ 7, 15 ]
}
{
    [ 9, 16 ]
}
{
    [ 11, 18 ]
}
{
    [ 9, 21 ]
}
{
    [ 11, 10 ]
}
{
    [ 14, 16 ]
}
{
    [ 14, 25 ]
}
{
    [ 22, 13 ]
}
```

```
{
    [ 22, 12 ]
}
{
    [ 27, 25 ]
}
{
    [ 23, 15 ]
}
{
    [ 6, 30 ]
}
{
    [ 27, 21 ]
}
{
    [ 23, 20 ]
}
{
    [ 6, 3 ]
}
{
    [ 24, 12 ]
}
{
    [ 28, 10 ]
}
{
    [ 4, 30 ]
}
{
    [ 26, 18 ]
}
{
    [ 4, 2 ]
}
{
    [ 19, 20 ]
}
N12:=Stabiliser(N,[1,2]);
N12s:=N12;
#N12s;
1
N12;
```

```
Permutation group N12 acting on a set of cardinality 30
Order = 1
tr1:=Transversal(N,N12s);
for i:=1 to #tr1 do
ss:=[1,2]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..6] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
5
Orbits(N12s);
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @},
    GSet{@ 7 @},
    GSet{@ 8 @},
    GSet{@ 9 @},
    GSet{@ 10 @},
    GSet{@ 11 @},
    GSet{@ 12 @},
    GSet{@ 13 @},
    GSet{@ 14 @},
    GSet{@ 15 @},
    GSet{@ 16 @},
    GSet{@ 17 @},
    GSet{@ 18 @},
    GSet{@ 19 @},
    GSet{@ 20 @},
    GSet{@ 21 @},
    GSet{@ 22 @},
    GSet{@ 23 @},
    GSet{@ 24 @},
    GSet{@ 25 @},
    GSet{@ 26 @},
    GSet{@ 27 @},
    GSet{@ 28 @},
    GSet{@ 29 @},
    GSet{@ 30 @}
]
```

```
for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[1] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[2] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[3] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[4] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[5] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*1*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[6] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[7] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
```

```
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[8] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*1*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[9] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[10] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[11] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[12] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[13] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[14] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[15] eq g*(DC[i])^h then i;
```

```
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[16] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[17] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*1*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[18] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[19] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[20] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[21] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[22] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*1*/
```

```
for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[23] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[24] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*1*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[25] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[26] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[27] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[28] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/

for i in [1..2] do
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[29] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*1*/

for i in [1..2] do
```

```
for g in IN do for h in IN do
if ts[1]*ts[2]*ts[30] eq g*(DC[i])^h then i;
break i; break g; break h; end if; end for;end for;end for;
/*2*/
```

8.4 Double Coset Enumeration Of $3^3 : 2^3$ Over $3^2 : 2$

Let $N \cong (3^2 : 2)$ with $\langle x, y\rangle$ where
$x \sim (1,4)(2,5)(3,6)$, and
$y \sim (1,2,3)$.
Our progenitor $3^3 : 2^3$. We prove that $3^3 : 2^3 \cong$ Group< $x, y, t | x^2, y^3, y^-1 * x * y^-1 * x * y * x * y * x, t^2, (t, y^x), (y * x * t)^4$ >.
We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double cosets NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw. We find our index which is the order of G over the order of N. Hence, $\frac{|G|}{|N|} = \frac{216}{18} = 12$. We have 12 distinct single cosets.

• **First Double Coset [*]**

NeN = $\{Ne^n | n \in N\}$ = N.
The double coset NeN is denoted by [*] which contains 1 right coset.The coset stsbiliser of the coset Ne is N.
The number of right cosets in [*] is equal to $\frac{|N|}{|N|} = \frac{18}{18} = 1$.
Since N is transitive on {1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,},
We need to determine the double coset of the right coset Nt_1.
Thus, the eighteen $t_i's$ which are extend the double coset [$*$], that mean eighteen generators goes forward to Nt_1.

Cayley Diagram

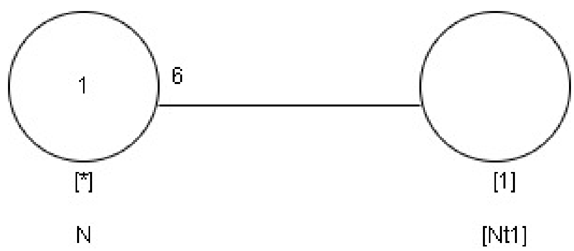

Figure 8.15: Cayley diagram for$3^2 : 2$ over $3^3 : 2^3$

• **Second Double Coset** $Nt_1N = [1]$

$Nt_1N = \{\mathrm{Nt_1}^n \mid n \in N\}$.

$=\{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6\}$.Firstly, find the point stabilizer of 1 in N.

$N^1 = n \in N|1^n = 1$

$N^1 =$ (4,5,6)

Thus, $N^{(1)} \geq N^1 = \langle(4,5,6)\rangle$.

The number of right cosets in Nt_1N is calculated by the formula,

$\frac{|N|}{|N^{(1)}|} = \frac{18}{3} = 6$.

The orbits of $N^{(1)}$ on $X = \{1, 2, 3, 4, 5, 6\}$ are

$\{1\}, \{2\}, \{3\}, \{4, 5, 6\}$.

We will determine the double cosetes by selecting one representative from each orbit such as,

$Nt_1t_1, Nt_1t_2, Nt_1t_3, Nt_1t_4$ belongs.

This shows us the following:

$Nt_1t_1 = Nt_1^2 = N \in [*]$.

Since the orbit $\{1\}$ contains one element, then one symmetric generator goes back to the double coset [*].

$Nt_1 \in [1]$

One symmetric generator will go back to [1].

$Nt_1t_2N$is a new double coset which we will denote [12].

One symmetric generators will go to the new double coset [14].

$Nt_1t_3N$is a new double coset which we will denote [13].

One symmetric generators will go to the new double coset [14].

$Nt_1t_4N$is a new double coset which we will denote [14].

Three symmetric generators will go to the new double coset [13].

Cayley Diagram

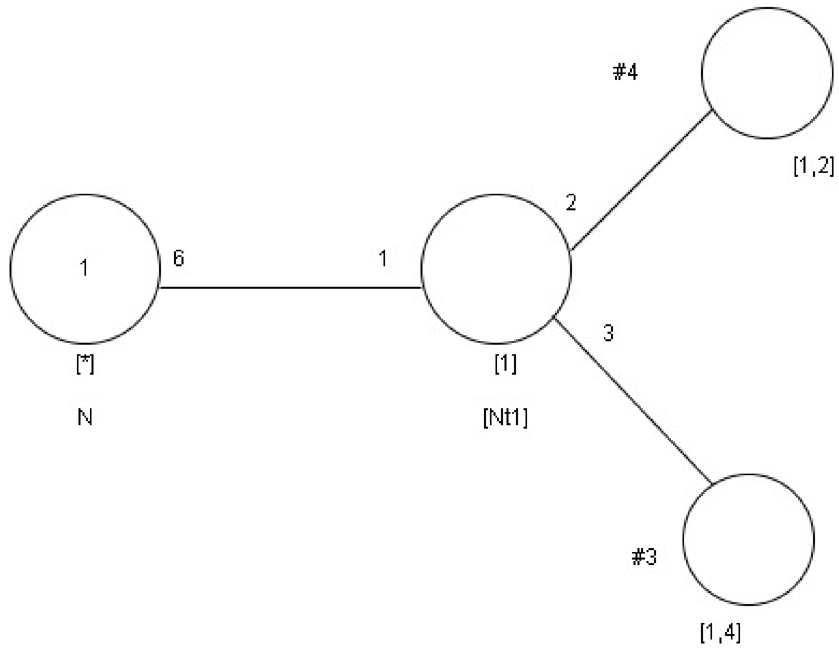

Figure 8.16: Cayley diagram for $3^2 : 2$ over $3^3 : 2^3$

- **Third Double Coset** $Nt_1t_2N = [12]$

$Nt_1\ t_2 = \{N(t_1t_2)^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(12)}$. Firstly, find the point stabilizer of 1 and 2 in N.

$N^{(12)} = n \in N | (12)^n = 12$

$Nt_1\ t_2N$ is denoted by [12]

Thus,$N^{12} = (4, 5, 6)$.

The number of right cosets in [12] is equal to

$\frac{|N|}{|N|^{(12)}} = \frac{18}{3} = 6$.

The orbits of $N^{(12)}$ on X= $\{1, 2, 3, 4, 5, 6\}$ are $\{1\}, \{2\}, \{3\}, \{4, 5, 6\}$.

We take t_1, t_2, t_3, t_4 from from each orbit respectively, and determine to which double coset $Nt_1t_2t_1, Nt_1t_2t_3, Nt_1t_2t_4$ belong.

As $Nt_1t_2t_2 = Nt_1t_2^2 = Nt_1 \in [1]$.

Thus $Nt_1t_2t_1$ is a new double coset which will donate by [121] One symmetric generator will go to [121].

$Nt_1t_2t_3$ is a new double coset which will donate by [123] One symmetric generator will go to [123]

$Nt_1t_2t_4$ is a new double coset which will donate by [124] Three symmetric generator will go back to [124]

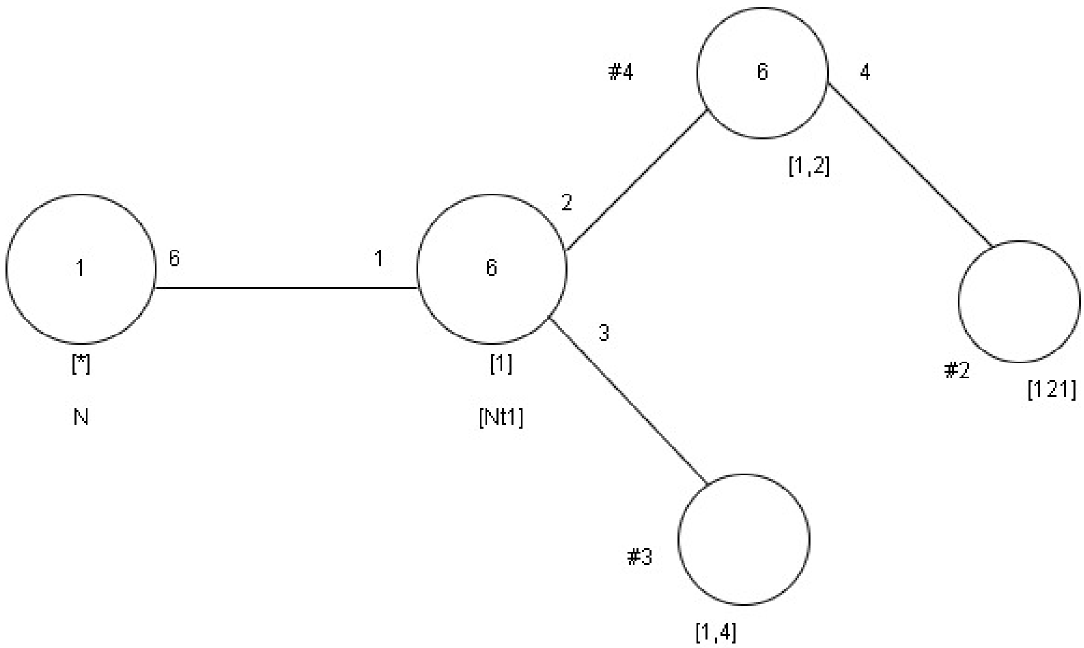

Figure 8.17: Cayley diagram for $3^2 : 2$ over $3^3 : 2^3$

- **Fourth Double Coset** $Nt_1t_4N = [14]$

$Nt_1\ t_4 = \{N(t_1t_4)^n | n \in N\}$.
We now find the Coset Stabilizer $N^{(14)}$. Firstly, find the point stabilizer of 1 and 4 in N.
$N^{(14)} = n \in N|(14)^n = 14$.
$Nt_1\ t_4N$ is denoted by [14].
The number of right cosets in [14] is equal to
$\frac{|N|}{|N|^{(14)}} = \frac{18}{1} = 18$.
The orbits of $N^{(14)}$ on X= $\{1,2,3,4,5,6\}$ are $\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}$.

We take $t_1, t_2, t_3, t_5, t_4, t_6$ from from each orbit respectively, and determine to which double coset $Nt_1t4t_1, Nt_1t_4t_2, Nt_1t_4t_3, Nt_1t_4t_4, Nt_1t_4t_5, Nt_1t_4t_6$ belong.
Thus $Nt_1t_4t_1$ is a new double coset which will donate by [141] One symmetric generator will go to [121].

Thus $Nt_1t_4t_2$ is a new double coset which will donate by [141] One symmetric generator will go to [121].

$Nt_1t_4t_3$ is a new double coset which will donate by [143] One symmetric generator will go to [121]

Thus $Nt_1t_4t_4$ is a new double coset which will donate by [144] One symmetric generator will go to [121].

Thus $Nt_1t_4t_5$ is a new double coset which will donate by [145] One symmetric generator will go to [121].

Thus $Nt_1t_4t_6$ is a new double coset which will donate by [146] One symmetric generator will go to [121].

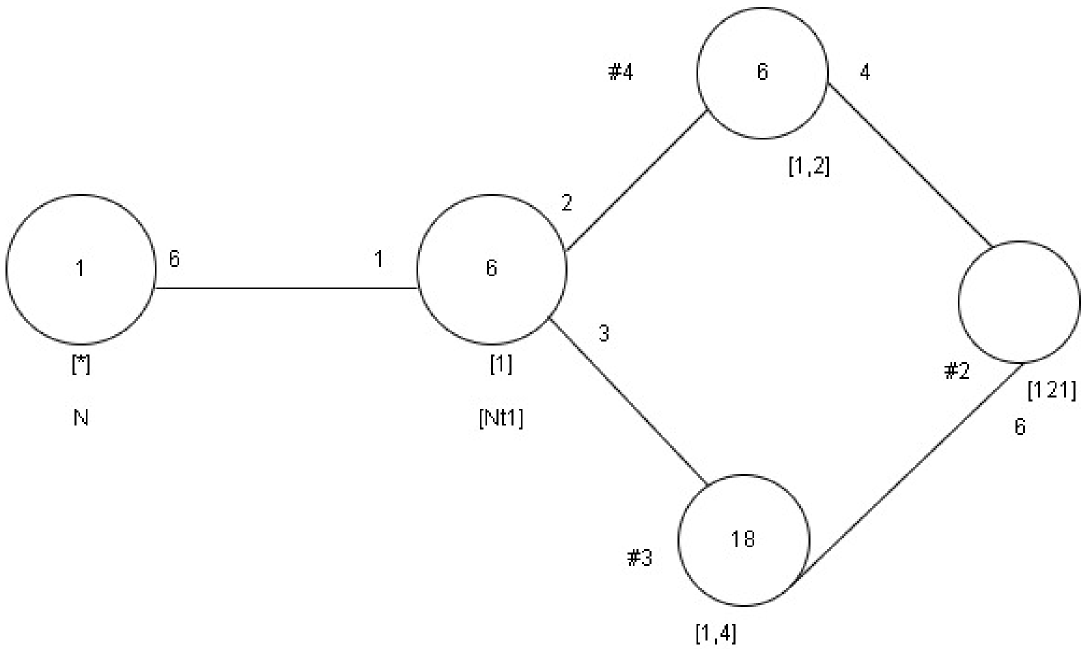

Figure 8.18: Cayley diagram for $3^2 : 2$ over $3^3 : 2^3$

• **Fifth Double Coset** $Nt_1t_2t_1N = [121]$

$Nt_1\ t_2t_1 = \{N(t_1t_2){t_1}^n | n \in N\}$.

We now find the Coset Stabilizer $N^{(121)}$. Firstly, find the point stabilizer of 1,2 and 1 in N.

$N^{(121)} = n \in N | (121)^n = 121$.

$Nt_1\ t_2t_1N$ is denoted by [121].

The number of right cosets in [121] is equal to

$\frac{|N|}{|N|^{(123)}} = \frac{18}{3} = 6$.

The orbits of $N^{(123)}$ on X= $\{1, 2, 3, 4, 5, 6\}$ are $\{1\}, \{2\}, \{3\}, \{4, 5, 6\}$.

We take t_1, t_2, t_3, t_4 from from each orbit respectively, and determine to which double coset $Nt_1t_2t_1t_1, Nt_1t_2t_1t_2, Nt_1t_2t_1t_3, Nt_1t_2t_1t_4$ belong.

Thus $Nt_1t_2t_1t_1$ and $Nt_1t_2t_1t_2$ are the double coset whose two symmetric generators will go to [12].

Thus $Nt_1t_2t_1t_3$ is the double coset which whose one symmetric generators will go back to [1].

Thus $Nt_1t_2t_1t_4$ is the double coset which whose three symmetric generators will go back to [14].

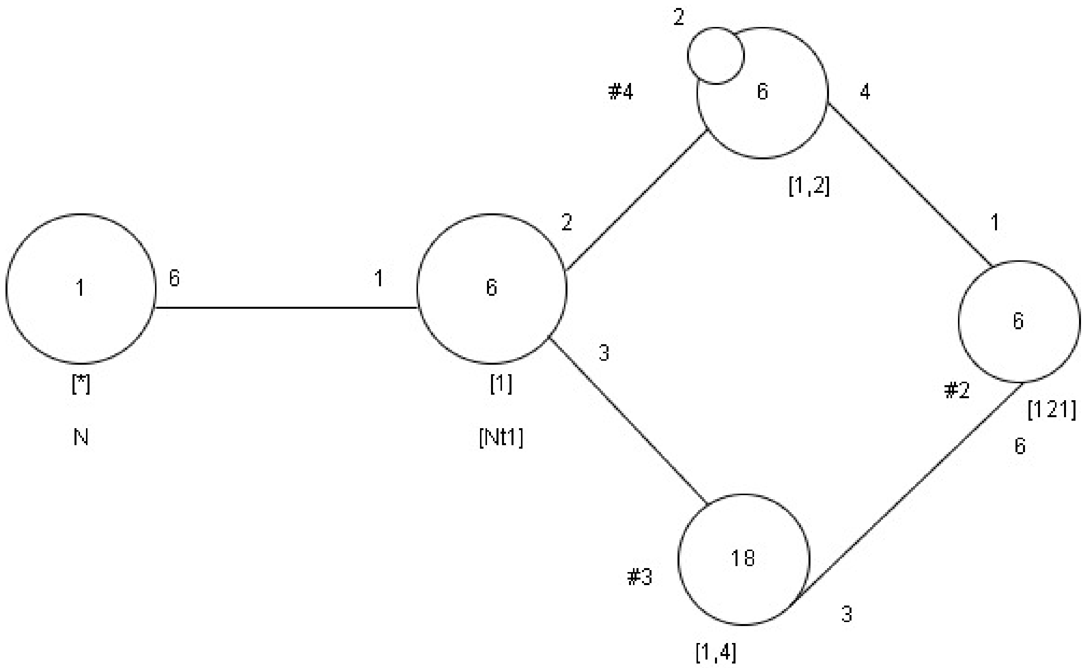

Figure 8.19: Cayley diagram for $3^2 : 2$ over $3^3 : 2^3$

8.4.1 Magma Work $3^2 : 2$ Over $3^3 : 2^3$

```
S:=Sym(6);
xx:=S!(1, 4)(2, 5)(3, 6) ;
yy:=S!(1,2,3);
N:=sub<S|xx,yy>;
#N;
/* 18 */
 CompositionFactors(N);
   /*
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1
*/
G<x,y,t>:=Group<x,y,t| x^2,y^3,y^-1 * x * y^-1 * x *y * x * y * x,
t^2,
(t,y^x),(y*x*t)^4>;
#G;
CompositionFactors(G);
/* 216 */
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
/*
 G
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1
*/
NL:=NormalLattice(G1);
NL;
/*
Normal subgroup lattice
```

```
-----------------------

[11]  Order 216  Length 1  Maximal Subgroups: 8 9 10
---
[10]  Order 108  Length 1  Maximal Subgroups: 7
[ 9]  Order 108  Length 1  Maximal Subgroups: 6 7
[ 8]  Order 108  Length 1  Maximal Subgroups: 7
---
[ 7]  Order 54   Length 1  Maximal Subgroups: 4 5
[ 6]  Order 36   Length 1  Maximal Subgroups: 4
---
[ 5]  Order 27   Length 1  Maximal Subgroups: 2 3
[ 4]  Order 18   Length 1  Maximal Subgroups: 3
---
[ 3]  Order 9    Length 1  Maximal Subgroups: 1
---
[ 2]  Order 3    Length 1  Maximal Subgroups: 1
---
[ 1]  Order 1    Length 1  Maximal Subgroups:
*/
IsAbelian(NormalLattice(N)[2]);
/* true */
q,ff:=quo<N|NormalLattice(N)[2]>;
q;
/*
Permutation group q acting on a set of cardinality 3
Order = 6 = 2 * 3
    (2, 3)
    (1, 2, 3)
*/
FPGroup(q);
/* Finitely presented group on 2 generators
Relations
    $.1^2 = Id($)
    $.2^-3 = Id($)
    ($.2^-1 * $.1)^2 = Id($)
*/
IsIsomorphic(q,Sym(3));
/*
true Isomorphism of GrpPerm: q, Degree 3, Order 2 * 3
into GrpPerm: $, Degree 3, Order 2 * 3 induced by
    (2, 3) |--> (2, 3)
    (1, 2, 3) |--> (1, 2, 3)
*/
```

```
H:=sub<G|x,y,t * y * t * y^-1>;
#H;
/* 54 */
IN:=sub<G1|f(x),f(y)>;
IH:=sub<G1|f(x),f(y),f(t * y * t * y^-1)>;
#DoubleCosets(G,H,sub<G|x,y>);
/*3*/
DoubleCosets(G,H, sub<G|x,y>);
/* { <GrpFP: H, Id(G), GrpFP>, <GrpFP: H, t * x * t,
GrpFP>, <GrpFP: H, t, GrpFP> }
*/
NN<a,b>:=Group<a,b| a^2,b^3,b^-1 * a * b^-1 * a * b * a * b * a>;
#NN;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..#N]];
for i in [2..#N] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for j in [2..6] do for i in [1..#Sch] do
if 1^ArrayP[i] eq j then j,Sch[i];
break;
end if; end for;
end for;

/*
2 b
3 b^-1
4 a
5 b * a
6 b^-1 * a
*/

DC:=[Id(G1),f(t),f(t * x * t) ];
IN:=sub<G1|f(x),f(y)>;
ts := [ Id(G1): i in [1 .. 6] ];
```

```
ts[1]:=f(t);
ts[2]:=f(t^(y));
ts[3]:=f(t^(y^-1));
ts[4]:=f(t^(x));
ts[5]:=f(t^(y * x));
ts[6]:=f(t^(y^-1 * x));
#DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
/*4*/
DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
/*{ <GrpFP, Id(G), GrpFP>, <GrpFP, t * y * t, GrpFP>,
<GrpFP, t, GrpFP>, <GrpFP, t * x * t, GrpFP> }
*/

DC:=[Id(G1),f(t),f(t * x * t),f(t*y*t) ];
Index(G1,IN);
/*12*/
cst := [null : i in [1 .. Index(G1,IN)]] where null is [Integers() | ];
prodim := function(pt, Q, I)
v := pt;
for i in I do
     v := v^(Q[i]);
    end for;
return v;
end function;
for i := 1 to 6 do
      cst[prodim(1, ts, [i])] := [i];
     end for;
m:=0; for i in [1..12] do if cst[i] ne [] then m:=m+1;
 end if; end for;m;
/*6*/
Orbits(N);
/*[
    GSet{@ 1, 4, 2, 5, 3, 6 @}
]
*/
N1:=Stabiliser(N,1);
Orbits(N1);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4, 6, 5 @}
]
```

```
*/
#N/#N1;
/*6*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[1] eq m*(DC[i])^n then i;
break; end if; end for;end for;
/*1*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2] eq m*(DC[i])^n then i;
break; end if; end for;end for;
/*4*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3] eq m*(DC[i])^n then i;
break; end if; end for;end for;
/*4*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[4] eq m*(DC[i])^n then i;
break; end if; end for;end for;
/*3*/
S:={[1,2]};
SS:=S^N;SS;
/*
GSet{@
    {
        [ 1, 2 ]
    },
    {
        [ 4, 5 ]
    },
    {
        [ 2, 3 ]
    },
    {
        [ 5, 6 ]
    },
    {
        [ 3, 1 ]
    },
    {
        [ 6, 4 ]
    }
@}
*/
```

```
SSS:=Setseq(SS);
for i in [1..#SSS] do
 for g in IN do if ts[1]*ts[2]
 eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
 then print SSS[i];
 end if; end for; end for;
/*
{
    [ 1, 2 ]
}
{
    [ 2, 3 ]
}
{
    [ 3, 1 ]
}
*/
N12:=Stabiliser(N,[1,2]);
#N12;
/*3*/
N12;
/*Permutation group N12 acting on a set of cardinality 6
Order = 3
    (4, 6, 5)
*/
N12s:=N12;
#N12s;
/*3*/
tr1:=Transversal(N,N12s);
for i:=1 to #tr1 do
ss:=[1,2]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..12] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*8*/


Orbits(N12s);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
```

```
    GSet{@ 3 @},
    GSet{@ 4, 6, 5 @}
]
*/
#N/#N12s;

 /*6*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/*2*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[2] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/* 2*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/*2*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4] eq m*(DC[i])^n
then i; break; end if; end for;end for;
/*2*/
S:={[1,4]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
/*
    [ 1, 4 ]
}
{
    [ 4, 1 ]
}
{
    [ 5, 2 ]
}
{
    [ 2, 5 ]
}
{
    [ 6, 3 ]
```

```
}
{
    [ 3, 6 ]
}1818
*/
N14:=Stabiliser(N,[1,4]);
N14;
/*Permutation group N14 acting on a set of cardinality 6
Order = 1
*/

N14s:=N14;
#N14s;
/*1*/


tr1:=Transversal(N,N14s);
for i:=1 to #tr1 do
ss:=[1,4]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..12] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*11*/

Orbits(N14s);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4 @},
    GSet{@ 5 @},
    GSet{@ 6 @}
]
*/

for i in [1..#DC] do for m,n in IN do
if ts[1]*ts[4]*ts[1] eq m*(DC[i])^n then i;
break; end if; end for;end for;
/*2*/

for i in [1..#DC] do for m,n in IN do
```

```
if ts[1]*ts[4]*ts[2] eq m*(DC[i])^n then i;
break; end if; end for;end for;
/*2*/

for i in [1..#DC] do for m,n in IN do
if ts[1]*ts[4]*ts[3] eq m*(DC[i])^n then i;
break; end if; end for;end for;

for i in [1..#DC] do for m,n in IN do
if ts[1]*ts[4]*ts[4] eq m*(DC[i])^n then i;
break; end if; end for;end for;

for i in [1..#DC] do for m,n in IN do
if ts[1]*ts[4]*ts[5] eq m*(DC[i])^n then i;
break; end if; end for;end for;

for i in [1..#DC] do for m,n in IN do
if ts[1]*ts[4]*ts[6] eq m*(DC[i])^n then i;
break; end if; end for;end for;
/*2*/

S:={[1,2,1]};
 SS:=S^N;SS;
/*
GSet{@
    {
        [ 1, 2, 1 ]
    },
    {
        [ 4, 5, 4 ]
    },
    {
        [ 2, 3, 2 ]
    },
    {
        [ 5, 6, 5 ]
    },
    {
        [ 3, 1, 3 ]
    },
    {
        [ 6, 4, 6 ]
    }
@}
*/
```

```
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
 then print SSS[i];
end if; end for; end for;
N121:=Stabiliser(N,[1,2,1]);
#N121;
/*3*/

N121s:=N121;
[1,2,1]^N121s;
/*
GSet{@
    [ 1, 2, 1 ]
@}
*/
#N/#N121s;
/*6*/

tr1:=Transversal(N,N121s);
for i:=1 to #tr1 do
ss:=[1,2,1]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..12] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
/*11*/
 Orbits(N121s);
/*
[
    GSet{@ 1 @},
    GSet{@ 2 @},
    GSet{@ 3 @},
    GSet{@ 4, 6, 5 @}
]
*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[1] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*4*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[2] eq
```

```
m*(DC[i])^n then i; break; end if; end for;end for;
/*4*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[3] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*1*/

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[4] eq
m*(DC[i])^n then i; break; end if; end for;end for;
/*3*/
```

8.5 Construction Of $2{\times}(A_5 \times A_5)$ Over $D_5 \times D_5$

Consider N = $< x, y >$, where
x = (1, 19, 11, 17, 2)(3, 16, 15, 5, 6, 9, 24, 8, 21, 22)(4, 18, 13, 25, 7, 14, 20, 10, 23, 12)
and
y = (1, 16)(2, 8)(3, 20)(4, 24)(5, 17)(6, 13)(7, 15)(9, 19)(10, 18)(11, 22)(14, 25)(21, 23).
Our progenitor $D_5 \times D_5$.We prove that $D_5 \times D_5 \cong$ Group $< x, y, t|y^2, (x*y*x)^2, x^10, x^-1*y*x^-1*y*x^-1*y*x^-1*y*x^-1*y*x*y*x*y*x*y*x*y*x^-1*y, t^2, (t, y*x*y*x*y*x*y*x^-1*y), (t, x*y*x*y*x*y*x^-1*y*x*y), (y*x^(2)*t^(x^(-1)))^3, (y*x*y*t^(x*y*x^(-1)))^3 >$.
We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double coset NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find the index which is the order of G over the order of N.
Hence,$\frac{|G|}{|N|} = \frac{7200}{100} = 72$. So, we have 72 single cosets.

• **First Double Coset [*]**

$NeN = \{Ne^n|n \in N\} = \{N\}$.
The double coset $NeN = [*]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.
The number of right coset in $*$ is equal to $\frac{|N|}{|N|} = \frac{100}{100} = 1$.
Since N is transitive on
$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25\}$.
We need only determine the double coset of the right coset Nt_1.
Thus twenty-five cosets extend to the new double coset [1], that mean the twenty-five generators go forward to Nt_1.

• **Second Double Coset $Nt_1N = [1]$**

$Nt_1N = \{Nt_1^n|n \in N\}$.
$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7,$
$Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12}, Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16},$
$Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}, Nt_{21}, Nt_{22}, Nt_{23}, Nt_{24}, Nt_{25}\}$.
Firstly, the point stabilizer of 1 in N.

$N^1 = \{n \in N | 1^n = 1\}$.

$N^1 = <(2,19)(3,16)(4,14)(5,21)(6,8)(7,18)(9,24)(10,23)(11,17)(12,20)(13,25)$
$(15,22)(3,9)(4,14)(5,21)(6,22)(7,12)(8,15)(10,13)(16,24)(18,20)(23,25)>$.

The number of right cosets in [1] is equal to
$\frac{|N|}{|N|^{(1)}} = \frac{100}{2} = 50$.

The orbits of $N^{(1)}$ on X= $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,$
$19,20,21,22,23,24,25\}$ are $\{1\},\{2,19\},\{4,14\},\{5,21\},\{11,17\},\{3,16,9,24\},$
$\{6,8,22,15\},\{7,18,12,20\},\{10,23,13,25\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_1, Nt_1t_2, Nt_1t_4, Nt_1t_5, Nt_1t_{11}, Nt_1t_3, Nt_1t_6, Nt_1t_7, Nt_1t_{10}$ belongs.

This shows us the following:

$Nt_1t_1 = Nt_1^2 = N \in [*]$.

Since the orbit $\{1\}$ contains one element, then one symmetric generator goes back to the double coset [*].

One symmetric generator will go back to [1].

$Nt_1t_2 \in Nt_1t_2N$ which is a new double coset. We denote this double coset by [12].

Two symmetric generators will go to the new double coset [12].

$Nt_1t_4 \in Nt_1t_3N$ which is a new double coset. We denote this double coset by [14].

Two symmetric generators will go to the new double coset [14].

$Nt_1t_5 \in Nt_1t_5N$is a new double coset which we will denote [14].

Two symmetric generators will go to the new double coset [15].

$Nt_1t_11 \in Nt_1t_11N$is a new double coset which we will denote [111].

Two symmetric generators will go to the new double coset [111].

$Nt_1t_3 \in Nt_1t_3N$is a new double coset which we will denote [13].

Four symmetric generators will go to the new double coset [13].

$Nt_1t_6 \in Nt_1t_6N$is a new double coset which we will denote [16].

Four symmetric generators will go to the new double coset [16].

$Nt_1t_7 \in Nt_1t_7N$is a new double coset which we will denote [17].
Four symmetric generators will go to the new double coset [17].

$Nt_1t_{10} \in Nt_1t_{10}N$is a new double coset which we will denote [110].
Four symmetric generators will go to the new double coset [110].

• **Third Double Coset** $Nt_1t_2N = [12]$

$Nt_1t_2N = \{Nt_1t_2^n | n \in N\}$
Firstly, the point stabilizer of 1 and 2 in N,
$N^{12} = \{n \in N | 12^n = 12\}$
$N^{12} = <(3,9)(4,14)(5,21)(6,22)(7,12)(8,15)(10,13)(16,24)(18,20)(23,25)>$.
The number of right cosets in [1] is equal to
$\frac{|N|}{|N^{(12)}|} = \frac{100}{2} = 50$.
The orbits of $N^{(12)}$ on X= $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20, 21,22,23,24,25\}$ are $\{1\},\{2\},\{11\},\{17\},\{19\},\{3,9\},\{4,14\},\{5,21\},\{6,22\},\{7,12\}, \{8,15\},\{10,13\},\{16,24\},\{18,20\},\{23,25\}$.
Now we select a representative from each orbit and determine to which double coset $Nt_1t_1, Nt_1t_2, Nt_1t_{11}, Nt_1t_{17}, Nt_1t_{19}, Nt_1t_3, Nt_1t_4, Nt_1t_5, Nt_1t_6, Nt_1t_7, Nt_1t_8, Nt_1t_{10}, Nt_1t_{16}, Nt_1t_{18}, Nt_1t_{23}$ belongs.
This shows us the following:
$Nt_1t_2t_1 \in Nt_1t_2t_1N$ which is a new double coset. We denote this double coset by [121].
One symmetric generators will go to the new double coset [121].

$Nt_1t_2t_2 \in Nt_1t_2t_2N$ which is a new double coset. We denote this double coset by [122].
One symmetric generators will go to the new double coset [122].

$Nt_1t_2t_{11} \in Nt_1t_2t_{11}N$ which is a new double coset. We denote this double coset by [1211].
One symmetric generators will go to the new double coset [1211].

$Nt_1t_2t_{17} \in Nt_1t_2t_{17}N$ which is a new double coset. We denote this double coset by [1217].
One symmetric generators will go to the new double coset [1217].

$Nt_1t_2t_{19} \in Nt_1t_2t_{19}N$ which is a new double coset. We denote this double coset by [1219].
One symmetric generators will go to the new double coset [1219].

$Nt_1t_2t_3 \in Nt_1t_2t_3N$ which is a new double coset. We denote this double coset by [123].
Two symmetric generators will go to the new double coset [123].

$Nt_1t_2t_4 \in Nt_1t_2t_4N$ which is a new double coset. We denote this double coset by [124].
Two symmetric generators will go to the new double coset [124].

$Nt_1t_2t_5 \in Nt_1t_2t_5N$ which is a new double coset. We denote this double coset by [125].
Two symmetric generators will go to the new double coset [121].

$Nt_1t_2t_6 \in Nt_1t_2t_6N$ which is a new double coset. We denote this double coset by [126].
Two symmetric generators will go to the new double coset [126].

$Nt_1t_2t_7 \in Nt_1t_2t_7N$ which is a new double coset. We denote this double coset by [127].
Two symmetric generators will go to the new double coset [127].

$Nt_1t_2t_8 \in Nt_1t_2t_8N$ which is a new double coset. We denote this double coset by [128].
Two symmetric generators will go to the new double coset [128].

$Nt_1t_2t_{10} \in Nt_1t_2t_{10}N$ which is a new double coset. We denote this double coset by [1210].
Two symmetric generators will go to the new double coset [1210].

$Nt_1t_2t_{16} \in Nt_1t_2t_{16}N$ which is a new double coset. We denote this double coset by [1216].

Two symmetric generators will go to the new double coset [1216].

$Nt_1t_2t_{18} \in Nt_1t_2t_{18}N$ which is a new double coset. We denote this double coset by [1218].
Two symmetric generators will go to the new double coset [1218].

$Nt_1t_2t_{23} \in Nt_1t_2t_{23}N$ which is a new double coset. We denote this double coset by [1223].
Two symmetric generators will go to the new double coset [1223].

• **Fifth Double Coset** $Nt_1t_5N = [15]$
$Nt_1t_5N = \{Nt_1t_5^n | n \in N\}$
Firstly, the point stabilizer of 1 and 5 in N,
$N^{15} = \{n \in N | 15^n = 15\}$
$N^{15} = <(2,19)(3,24)(6,15)(7,20)(8,22)(9,16)(10,25)(11,17)(12,18)(13,23)>$.
The number of right cosets in [1] is equal to
$\frac{|N|}{|N|^{(15)}} = \frac{100}{2} = 50$.
The orbits of $N^{(15)}$ on X= $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,$ $22,23,24,25\}$ are $\{1\},\{4\},\{5\},\{14\},\{21\},2,19\},\{3,24\},\{6,15\},\{7,20\},\{8,22\},\{9,16\},$ $\{10,25\},\{11,17\},\{12,18\},\{13,23\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_5t_1, Nt_1t_5t_4, Nt_1t_5t_5, Nt_1t_5t_14, Nt_1t_5t_{21}, Nt_1t_5t_2, Nt_1t_5t_3, Nt_1t_5t_6, Nt_1t_5t_7,$ $Nt_1t_5t_8, Nt_1t_5t_9, Nt_1t_5t_{10}, Nt_1t_5t_{11}, Nt_1t_5t_{12}, Nt_1t_5t_{13}$ This shows us the following:
$Nt_1t_5t_1 \in Nt_1t_5t_1N$ which is a new double coset. We denote this double coset by [151].
One symmetric generators will go to the new double coset [151].

$Nt_1t_5t_4 \in Nt_1t_5t_4N$ which is a new double coset. We denote this double coset by [154].
One symmetric generators will go to the new double coset [154].

$Nt_1t_5t_5 \in Nt_1t_5t_5N$ which is a new double coset. We denote this double coset by [155].
One symmetric generators will go to the new double coset [155].

• **Sixth Double Coset** $Nt_1t_6N = [16]$

$Nt_1t_6N = \{Nt_1t_6^n | n \in N\}$

Firstly, the point stabilizer of 1 and 6 in N,

$N^{16} = \{n \in N | 16^n = 16\}$

$N^{16} = < Id >$.

The number of right cosets in [1] is equal to

$\frac{|N|}{|N^{(16)}|} = \frac{100}{1} = 100$.

The orbits of $N^{(16)}$ on X= $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25\}$ are $\{1\}, \{4\}, \{5\}, \{6\}, \{7\}, \{7\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}, \{15\}, \{16\}, \{17\}, \{18\}, \{19\}, \{20\}, \{21\}, \{22\}, \{23\}, \{24\}, \{25\}$.

Now we select a representative from each orbit and determine to which double coset belongs to which coset. This shows us the following:

$Nt_1t_6t_1 \in Nt_1t_6t_1N$ which is a new double coset. We denote this double coset by [161]. One symmetric generators will go to the new double coset [161].

$Nt_1t_6t_4 \in Nt_1t_6t_4N$ which is a new double coset. We denote this double coset by [164]. One symmetric generators will go to the new double coset [164].

$Nt_1t64t_5 \in Nt_1t_6t_5N$ which is a new double coset. We denote this double coset by [165].

One symmetric generators will go to the new double coset [165].

• **Seventh Double Coset** $Nt_1t_7N = [17]$

$Nt_1t_7N = \{Nt_1t_7^n | n \in N\}$

Firstly, the point stabilizer of 1 and 7 in N,

$N^{17} = \{n \in N | 17^n = 17\}$

$N^{17} = < (1) >$.

The number of right cosets in [1] is equal to

$\frac{|N|}{|N^{(17)}|} = \frac{100}{1} = 100$.

The orbits of $N^{(17)}$ on X= $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21,$

$22, 23, 24, 25\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}$, $\{15\}, \{16\}, \{17\}, \{18\}, \{19\}, \{20\}, \{21\}, \{22\}, \{23\}, \{24\}, \{25\}$.
Now we select a representative from each orbit and determine to which double coset $Nt_1t7t_1, Nt_1t7t_2, Nt_1t7t_3, Nt_1t_7t_4, Nt_1t_7t_5, Nt_1t_7t_6, Nt_1t_7t_7, Nt_1t_7t_8, Nt_1t_7t_9$, Nt_1t_{710}, $Nt_1t_7t_{11}, Nt_1t_7t_{12}, Nt_1t_7t_{13}, Nt_1t_7t_{14}, Nt_1t_7t_{15}, Nt_1t_7t_{16}, Nt_1t_7t_{17}, Nt_1t_7t_{18}$, $Nt_1t_7t_{19}$, $Nt_1t_7t_{20}, Nt_1t_7t_{21}, Nt_1t_7t_{22}, Nt_1t_7t_{23}, Nt_1t_7t_{24}, Nt_1t_7t_{25}$.
This shows us the following:

$Nt_1t_7t_1 \in Nt_1t_7t_1N$ which is a new double coset. We denote this double coset by [171]. One symmetric generators will go back to double coset [171].

$Nt_1t_7t_2 \in Nt_1t_7t_2N$ which is a new double coset. We denote this double coset by [172]. One symmetric generators will go back to double coset [172].

$Nt_1t_7t_3 \in Nt_1t_7t_3N$ which is a new double coset. We denote this double coset by [173].

One symmetric generators will go back to double coset [173].
$Nt_1t_7t_4 \in Nt_1t_7t_4N$ which is a new double coset. We denote this double coset by [174]. One symmetric generators will go back to double coset [174].

$Nt_1t_7t_5 \in Nt_1t_7t_5N$ which is a new double coset. We denote this double coset by [175]. One symmetric generators will go back to double coset [175].

$Nt_1t_7t_6 \in Nt_1t_7t_6N$ which is a new double coset. We denote this double coset by [176]. One symmetric generators will go back to double coset [176].

$Nt_1t_7t_8 \in Nt_1t_7t_8N$ which is a new double coset. We denote this double coset by [178]. One symmetric generators will go back to double coset [178].

$Nt_1t_7t_9 \in Nt_1t_7t_9N$ which is a new double coset. We denote this double coset by [179]. One symmetric generators will go back to double coset [179].

• **Eighth Double Coset** $Nt_1t_8N = [18]$

$Nt_1t_8N = \{Nt_1t_8^n | n \in N\}$

Firstly, the point stabilizer of 1 and 8 in N,

$N^{18} = \{n \in N | 18^n = 18\}$

$N^{18} = <(1)>$.

The number of right cosets in [1] is equal to

$\frac{|N|}{|N^{(18)}|} = \frac{100}{1} = 100$.

The orbits of $N^{(18)}$ on X= $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}, \{15\}, \{16\}, \{17\}, \{18\}, \{19\}, \{20\}, \{21\}, \{22\}, \{23\}, \{24\}, \{25\}$. Now we select a representative from each orbit and determine to which double coset $Nt_1t8t_1, Nt_1t8t_2, Nt_1t8t_3,$
$Nt_1t_8t_4, Nt_1t_8t_5, Nt_1t_8t_6, Nt_1t_8t_7, Nt_1t_8t_8, Nt_1t_8t_9, Nt_1t_{810},$
$Nt_1t_8t_{11}, Nt_1t_8t_{12}, Nt_1t_8t_{13}, Nt_1t_8t_{14}, Nt_1t_8t_{15}, Nt_1t_8t_{16}, Nt_1t_8t_{17}, Nt_1t8t_{18}, Nt_1t8t_{19},$
$Nt_1t_8t_{20}, Nt_1t_8t_{21}, Nt_1t_8t_{22}, Nt_1t_87t_{23}, Nt_1t_8t_{24}, Nt_1t_8t_{25}$.

This shows us the following:

$Nt_1t_8t_1 \in Nt_1t_8t_1N$ which is a new double coset. We denote this double coset by [181]. One symmetric generators will go back to double coset [181].

$Nt_1t_8t_2 \in Nt_1t_8t_2N$ which is a new double coset. We denote this double coset by [182]. One symmetric generators will go back to double coset [182].

$Nt_1t_8t_3 \in Nt_1t_8t_3N$ which is a new double coset. We denote this double coset by [183].

One symmetric generators will go back to double coset [183].

$Nt_1t_8t_4 \in Nt_1t_8t_4N$ which is a new double coset. We denote this double coset by [184]. One symmetric generators will go back to double coset [184].

• **Seventh Double Coset** $Nt_1t_7t_3N = [173]$

$Nt_1t_7t_3N = \{Nt_1t_7t_3^n | n \in N\}$

Firstly, the point stabilizer of 1 and 7 and 3 in N,

$N^{173} = \{n \in N | 173^n = 173\}$

$N^{173} = <(Id)>$

The number of right cosets in [1] is equal to
$\frac{|N|}{|N^{(173)}|} = \frac{100}{1} = 100$.
The orbits of $N^{(173)}$ on X= $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}, \{15\}, \{16\}, \{17\}, \{18\}, \{19\}, \{20\}, \{21\}, \{22\}, \{23\}, \{24\}, \{25\}$. Now we select a representative from each orbit and determine to which double coset $Nt_1t_7t_3t_1, Nt_1t_7t_3t_2, Nt_1t_7t_3t_3, Nt_1t_7t_3t_4, Nt_1t_7t_3t_5, Nt_1t_7t_3t_6, Nt_1t_7t_3t_7, Nt_1t_7t_3t_8, Nt_1t_7t_3t_9, Nt_1t_7t_3t_{10}, Nt_1t_7t_3t_{11}, Nt_1t_7t_3t_{12}, Nt_1t_7t_3t_{13}, Nt_1t_7t_3t_{13}, Nt_1t_7t_3t_{14}, Nt_1t_7t_3t_{15}, Nt_1t_7t_3t_{16}, Nt_1t_7t_3t_{17}, Nt_1t_7t_3t_{18}, Nt_1t_7t_3t_{19}, Nt_1t_7t_3t_{20}, Nt_1t_7t_3t_{21}, Nt_1t_7t_3t_{22}, Nt_1t_7t_3t_{23}, Nt_1t_7t_3t_{24}, Nt_1t_7t_3t_{25}$. This shows us the following:

$Nt_1t_7t_3t_2 \in NNt_1t_7t_3t_2N$ which is a new double coset. We denote this double coset by [1732].
One symmetric generators will go back to double coset [1732].

$Nt_1t_7t_3t_4 \in NNt_1t_7t_3t_4N$ which is a new double coset. We denote this double coset by [1734].
One symmetric generators will go back to double coset [1734]

$Nt_1t_7t_3t_{18} \in Nt_1t_7t_3t_{18}N$ which is a new double coset. We denote this double coset by [17318].
One symmetric generators will go back to double coset [17318].

• **Eighth Double Coset** $Nt_1t_7t_4N = [174]$
$Nt_1t_7t_4N = \{Nt_1t_7t_4^n | n \in N\}$
Firstly, the point stabilizer of 1 and 7 and 4 in N,
$N^{174} = \{n \in N | 174^n = 174\}$
$N^{174} = <(Id)>$

The number of right cosets in [1] is equal to

$\frac{|N|}{|N^{(174)}|} = \frac{100}{1} = 100$.
The orbits of $N^{(174)}$ on X= $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}, \{15\}, \{16\}, \{17\}, \{18\}, \{19\}, \{20\}, \{21\}, \{22\}, \{23\}, \{24\}, \{25\}$. Now we select a representative from each orbit and determine to which double coset $Nt_1t_7t_4t_1, Nt_1t_7t_4t_2, Nt_1t_7t_4t_3, Nt_1t_7t_4t_4, Nt_1t_7t_4t_5, Nt_1t_7t_4t_6, Nt_1t_7t_4t_7, Nt_1t_7t_4t_8, Nt_1t_7t_4t_9, Nt_1t_7t_4t_{10}, Nt_1t_7t_4t_{11}, Nt_1t_7t_4t_{12}, Nt_1t_7t_4t_{13}, Nt_1t_7t_4t_{13}, Nt_1t_7t_4t_{14}, Nt_1t_7t_4t_{15}, Nt_1t_7t_4t_{16}, Nt_1t_7t_4t_{17}, Nt_1t_7t_4t_{18}, Nt_1t_7t_4t_{19}, Nt_1t_7t_4t_{20}, Nt_1t_7t_4t_{21}, Nt_1t_7t_4t_{22}, Nt_1t_7t_4t_{23}, Nt_1t_7t_4t_{24}, Nt_1t_7t_4t_{25}$. This shows us the following:

$Nt_1t_7t_4t_2 \in NNt_1t_7t_4t_2N$ which is a new double coset. We denote this double coset by [1742].
One symmetric generators will go back to double coset [1742].

$Nt_1t_7t_4t_3 \in NNt_1t_7t_4t_3N$ which is a new double coset. We denote this double coset by [1743].
One symmetric generators will go back to double coset [1743]

$Nt_1t_7t_4t_7 \in NNt_1t_7t_4t_7N$ which is a new double coset. We denote this double coset by [1747].
One symmetric generators will go back to double coset [1747]

$Nt_1t_7t_4t_{23} \in Nt_1t_7t_4t_{23}N$ which is a new double coset. We denote this double coset by [17423].
One symmetric generators will go back to double coset [17423].

Cayley Diagram

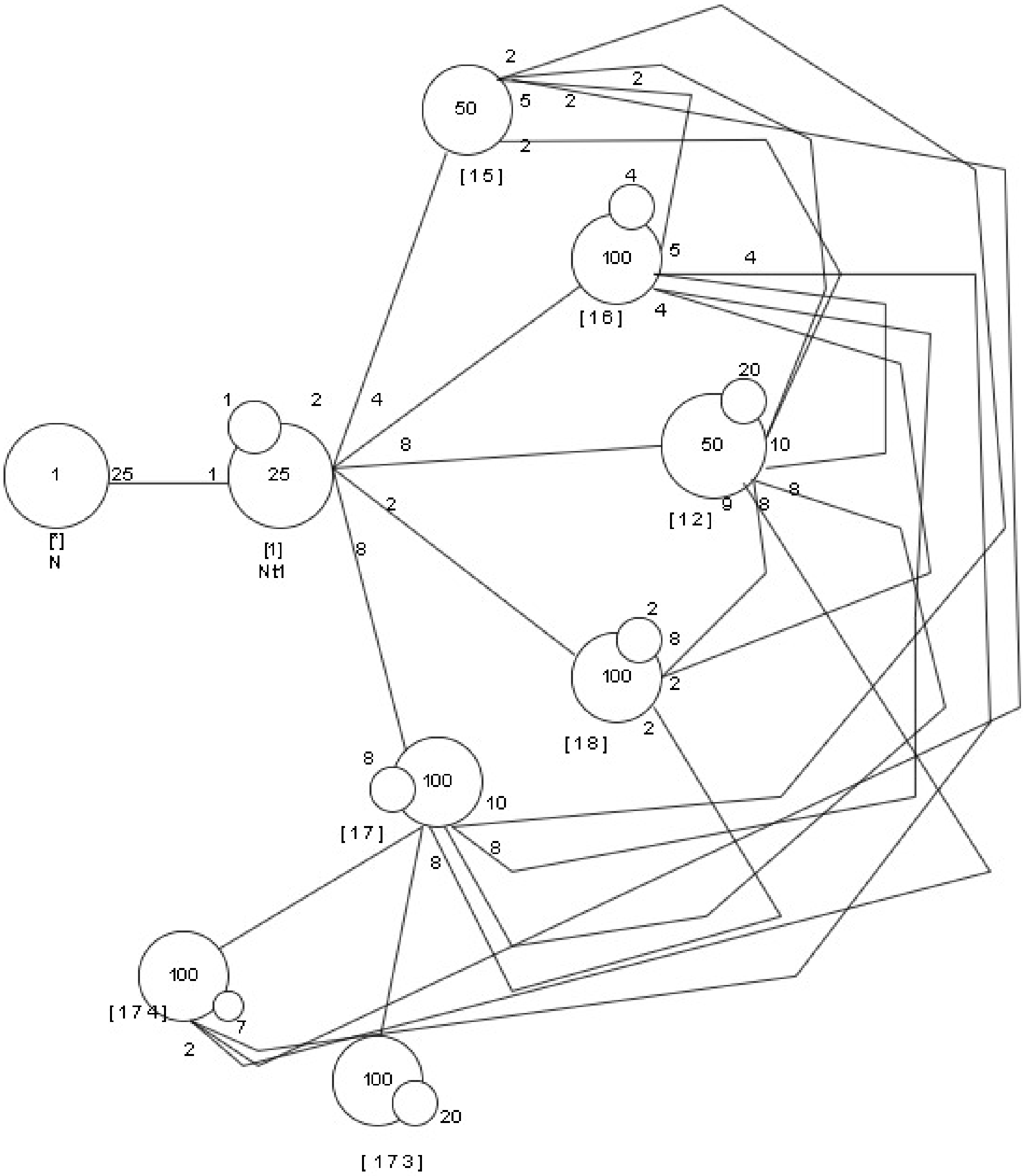

Figure 8.20: Cayley diagram for G over S_{25}

8.6 Construction Of $2^5 : S_5$ Over S_5

Consider N = $< x, y >$, where

x $\sim$ (2, 3)(4, 6)(5, 8)(7, 11)(9, 14)(10, 15)(12, 18)(13, 20)(17, 24)(19, 26)(21,25) (23, 28) and

y $\sim$ (1, 2, 4, 7, 12, 19)(3, 5, 9, 15, 22, 27)(6, 10, 16, 23, 18, 25)(8, 13,21)(11, 17, 14) (20, 26, 29, 30, 28, 24).

Our progenitor S_5. We prove that $S_5 \cong$ G¡x,y,t¿:= Group $< x, y, t|x^2, y^6, (y * x * y^{-1} * x)^2, (x * y^{-1^5}), t^2, (t, x), (t, y^2 * x * y^{-2} * x * y^2), (y * x * t^{(y^3 * x)})^3, (y * x * t^{(y^- 1 * x * y^- 1)})^4 >$.

We will determine the order of G. We perform manual double coset enumeration (DCE) of G over N. We need to determine all distinct double coset NwN and find the number of right cosets in each double coset. It suffices to find the double coset of Nwt_i for one representative t_i from each orbit of the coset stabiliser $N^{(w)}$ of the right coset Nw, so we find the index which is the order of G over the order of N.

Hence, $\frac{|G|}{|N|} = \frac{3840}{120} = 32$. So, we have 32 single cosets.

- **First Double Coset [*]**

$NeN = \{Ne^n | n \in N\} = \{N\}$.

The double coset $NeN = [*]$ contains 1 right coset. The coset stabiliser of the coset Ne is N.

The number of right coset in $*$ is equal to $\frac{|N|}{|N|} = \frac{120}{120} = 1$.

Since N is transitive on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25.26.27.28.29.30\}$.

We need only determine the double coset of the right coset Nt_1.

Thus thirty cosets extend to the new double coset [1], that mean the thirty generators go forward to Nt_1.

- **Second Double Coset** $Nt_1N = [1]$

$Nt_1N = \{Nt_1^n | n \in N\}$.

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7,$

$Nt_8, Nt_9, Nt_10, Nt_11, Nt_12, Nt_13, Nt_14, Nt_15, Nt_16,$

$Nt_17, Nt_18, Nt_19, Nt_20, Nt_21, Nt_22, Nt_23, Nt_24, Nt_25, Nt_26, Nt_27, Nt_28, Nt_29, Nt_30\}$.

Firstly, the point stabilizer of 1 in N,

$N^1 = \{n \in N | 1^n = 1\}$

$N^1 = <(2,3)(4,6)(5,8)(7,11)(9,14)(10,15)(12,18)(13,20)(17,24)(19,26)(21,25)$
$(23,28)(2,26)(3,19)(4,18)(5,13)(6,12)(7,11)(8,20)(9,17)(10,28)(14,24)(15,23)(21,25)$
$(22,30)(27,29)>$.

The number of right cosets in [1] is equal to
$\frac{|N|}{|N^{(1)}|} = \frac{120}{4} = 30$.

The orbits of $N^{(1)}$ on X= $\{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,$
$19,20,21,22,23,24,25,26,27,28,29,30\}$ are $\{1\},\{16\},\{7,11\},\{21,25\},\{22,30\},\{27,29\},$
$\{2,3,26,19\},\{4,6,18,12\},$
$\{5,8,13,20\},\{9,14,17,24\},\{10,15,28,23\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_1, Nt_1t_{16}, Nt_1t_7, Nt_1t_{21}, Nt_1t_{22}, Nt_1t_{27}, Nt_1t_2, Nt_1t_4, Nt_1t_5, Nt_1t_9, Nt_1t_{10}$ belongs.

This shows us the following:

$Nt_1t_1 = Nt_1^2 = N \in [*]$

Since the orbit $\{1\}$ contains one element, then one symmetric generator goes back to the double coset [*].

One symmetric generator will go to the new double coset [1].

$Nt_1t_{16} \in Nt_1t_{16}N$ which is a new double coset. We denote this double coset by [116].
One symmetric generators will go back to double coset [116].

$Nt_1t_{17} \in Nt_1t_{17}N$ which is a new double coset. We denote this double coset by [117].
Two symmetric generators will go back to double coset [117].

$Nt_1t_{21} \in Nt_1t_{21}N$ which is a new double coset. We denote this double coset by [121].
Two symmetric generators will go back to double coset [121].

$Nt_1t_{122} \in Nt_1t_{122}N$ which is a new double coset. We denote this double coset by

.

Two symmetric generators will go back to double coset [122].

$Nt_1t_{27} \in Nt_1t_{27}N$ which is a new double coset. We denote this double coset by [127]. Two symmetric generators will go back to double coset [127].

$Nt_1t_2 \in Nt_1t_2N$ which is a new double coset. We denote this double coset by [12]. Two symmetric generators will go back to double coset [12].

$Nt_1t_4 \in Nt_1t_4N$ which is a new double coset. We denote this double coset by [14]. Two symmetric generators will go back to double coset [14].

$Nt_1t_5 \in Nt_1t_5N$ which is a new double coset. We denote this double coset by [15]. Two symmetric generators will go back to double coset [15].

$Nt_1t_9 \in Nt_1t_9N$ which is a new double coset. We denote this double coset by [19]. Two symmetric generators will go back to double coset [19].

$Nt_1t_{10} \in Nt_1t_{10}N$ which is a new double coset. We denote this double coset by [110]. Two symmetric generators will go back to double coset [110].

• **Third Double Coset** $Nt_{12}N = [12]$

$Nt_{12}N = \{Nt_{12}^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7,$

$Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12}, Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16},$

$Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}, Nt_{21}, Nt_{22}, Nt_{23}, Nt_{24}, Nt_{25}, Nt_{26}, Nt_{27}, Nt_{28}, Nt_{29}, Nt_{30}\}$.

Firstly, the point stabilizer of 1 and 2 in N,

$N^{17} = \{n \in N | 1^n = 1\}$

$N^{17} = < (Id) >$.

The number of right cosets in [1] is equal to

$\frac{|N|}{|N|^{(1)}} = \frac{120}{1} = 120$.

The orbits of $N^{(12)}$ on X= $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18,$

$19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}, \{15\}, \{16\}, \{17\}, \{18\}, \{19\}, \{20\}, \{21\}, \{22\}, \{23\}, \{24\}, \{25\}, \{26\}, \{27\}, \{28\}, \{29\}, \{30\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_2t_1, Nt_1t_2t_2, Nt_1t_2t_3, Nt_1t_2t_3, Nt_1t_2t_4, Nt_1t_2t_5, Nt_1t_2t_6, Nt_1t_2t_7, Nt_1t_2t_8, Nt_1t_2t_9$, Nt$_1t_2t_{10}, Nt_1t_2t_{11}, Nt_1t_2t_{12}, Nt_1t_2t_{13}, Nt_1t_2t_{14}, Nt_1t_2t_{15}, Nt_1t_2t_{16}, Nt_1t_7t_{17}Nt_1t_2t_{18}$, Nt$_1t_2t_{19}, Nt_1t_2t_{20}, Nt_1t_2t_{21}, Nt_1t_2t_{22}, Nt_1t_2t_{23}, Nt_1t_2t_{24}, Nt_1t_2t_{25}, Nt_1t_2t_{26}, Nt_1t_2t_{27}, Nt_1t_2t_{28}, Nt_1t_2t_{29}, Nt_1t_2t_{30}$ belongs.

This shows us the following:

$Nt_1t_2t_1 \in Nt_1t_2t_1N$ which is a new double coset. We denote this double coset by [121]. One symmetric generators will go back to double coset [121].

$Nt_1t_2t_3 \in Nt_1t_2t_3N$ which is a new double coset. We denote this double coset by [123]. One symmetric generators will go back to double coset [123].

- **Fourth Double Coset** $Nt_{14}N = [14]$

$Nt_{14}N = \{Nt_{14}^n | n \in N\}$

$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7,$

$Nt_8, Nt_9, Nt_{10}, Nt_{11}, Nt_{12}, Nt_{13}, Nt_{14}, Nt_{15}, Nt_{16},$

$Nt_{17}, Nt_{18}, Nt_{19}, Nt_{20}, Nt_{21}, Nt_{22}, Nt_{23}, Nt_{24}, Nt_{25}, Nt_{26}, Nt_{27}, Nt_{28}, Nt_{29}, Nt_30\}$.

Firstly, the point stabilizer of 1 and 4 in N,

$N^{14} = \{n \in N | 1^n = 1\}$

$N^{14} = < (Id) >$. The number of right cosets in [1] is equal to

$\frac{|N|}{|N|^{(1)}} = \frac{120}{1} = 120$.

The orbits of $N^{(14)}$ on X= $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}, \{15\}, \{16\}, \{17\}, \{18\}, \{19\}, \{20\}, \{21\}, \{22\}, \{23\}, \{24\}, \{25\}, \{26\}, \{27\}, \{28\}, \{29\}, \{30\}$.

Now we select a representative from each orbit and determine to which double coset $Nt_1t_4t_1, Nt_1t_4t_2, Nt_1t_4t_3, Nt_1t_4t_3, Nt_1t_4t_4, Nt_1t_4t_5, Nt_1t_4t_6, Nt_1t_4t_7, Nt_1t_4t_8, Nt_1t_4t_9$, Nt$_1t_4t_{10}, Nt_1t_4t_{11}, Nt_1t_4t_{12}, Nt_1t_4t_{13}, Nt_1t_4t_{14}, Nt_1t_4t_{15}, Nt_1t_4t_{16}, Nt_1t_4t_{17}Nt_1t_4t_{18}$,

Nt$_1t_4t_{19}, Nt_1t_4t_{20}, Nt_1t_4t_{21}, Nt_1t_4t_{22}, Nt_1t_4t_{23}, Nt_1t_4t_{24}, Nt_1t_4t_{25}, Nt_1t_4t_{26}, Nt_1t_4t_{27}$, $Nt_1t_4t_{28}, Nt_1t_4t_{29}, Nt_1t_4t_{30}$ belongs.
This shows us the following:

$Nt_1t_4t_1 \in Nt_1t_4t_1N$ which is a new double coset. We denote this double coset by [141]. One symmetric generators will go back to double coset [141].

$Nt_1t_4t_2 \in Nt_1t_4t_2N$ which is a new double coset. We denote this double coset by [142]. One symmetric generators will go back to double coset [142].

$Nt_1t_4t_3 \in Nt_1t_4t_3N$which is a new double coset. We denote this double coset by [143]. One symmetric generators will go back to double coset [143].

$Nt_1t_4t_5 \in Nt_1t_4t_5N$ which is a new double coset. We denote this double coset by [145]. One symmetric generators will go back to double coset [145].

$Nt_1t_4t_6 \in Nt_1t_4t_6N$ which is a new double coset. We denote this double coset by [146]. One symmetric generators will go to the new double coset [146].

• **Fifth Double Coset** $Nt_{143}N = [143]$

$Nt_{143}N = \{Nt_{143}^n | n \in N\}$
$= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6, Nt_7,$
$Nt_8, Nt_9, Nt_10, Nt_11, Nt_12, Nt_13, Nt_14, Nt_15, Nt_16,$
$Nt_17, Nt_18, Nt_19, Nt_20, Nt_21, Nt_22, Nt_23, Nt_24, Nt_25, Nt_26, Nt_27, Nt_28, Nt_29, Nt_30\}$.
Firstly, the point stabilizer of 1, 4 and 3 in N,
$N^{143} = \{n \in N | 1^n = 1\}$
$N^{143} = < (Id) >$.
The number of right cosets in [1] is equal to
$\frac{|N|}{|N^{(1)}|} = \frac{120}{1} = 120$.
The orbits of $N^{(143)}$ on X= $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}, \{15\}, \{16\}, \{17\}, \{18\}, \{19\}, \{20\}, \{21\}, \{22\}, \{23\}, \{24\}$,

$\{25\}, \{26\}, \{27\}, \{28\}, \{29\}, \{30\}$.
Now we select a representative from each orbit and determine to which double coset $Nt_1t_4t_3t_1, Nt_1t_4t_3t_2, Nt_1t_4t_3t_3, Nt_1t_4t_3t_3, Nt_1t_4t_3t_4, Nt_1t_4t_3t_5, Nt_1t_4t_3t_6, Nt_1t_4t_3t_7,$

$Nt_1t_4t_3t_8, Nt_1t_4t_3t_9$,Nt$_1t_4t_3t_{10}, Nt_1t_4t_3t_{11}, Nt_1t_4t_3t_{12}, Nt_1t_4t_3t_{13}, Nt_1t_4t_3t_{14}, Nt_1t_4t_3t_{15},$
$Nt_1t_4t_3t_{16}, Nt_1t_4t_3t_{17}Nt_1t_4t_3t_{18}, Nt_1t_4t_3t_{19}, Nt_1t_4t_3t_{20}, Nt_1t_4t_3t_{21}, Nt_1t_4tt_{322}, Nt_1t_4t_3t_{23},$

Nt$_1t_4t_3t_{24}, Nt_1t_4t_3t_{25}, Nt_1t_4t_3t_{26}, Nt_1t_4t_3t_{27}, Nt_1t_4t_3t_{28}, Nt_1t_4t_3t_{29}, Nt_1t_4t_3t_{30}$ belongs.
This shows us the following:

$Nt_1t_4t_3t_1 \in Nt_1t_4t_3t_1N$ which is a new double coset. We denote this double coset by [1431].
One symmetric generators will go back to double coset [1431].

$Nt_1t_4t_3t_2 \in Nt_1t_4t_2t_1N$ which is a new double coset. We denote this double coset by [1432].
One symmetric generators will go back to double coset [1432].

$Nt_1t_4t_3t_5 \in Nt_1t_4t_3t_5N$ which is a new double coset. We denote this double coset by [1435].
One symmetric generators will go back to double coset [1435].

$Nt_1t_4t_3t_6 \in Nt_1t_4t_3t_6N$ which is a new double coset. We denote this double coset by [1436].
One symmetric generators will go back to double coset [1436].

$Nt_1t_4t_3t_7 \in Nt_1t_4t_3t_7N$ which is a new double coset. We denote this double coset by [1437].
One symmetric generators will go back to double coset [1437].

$Nt_1t_4t_3t_8 \in Nt_1t_4t_3t_8N$ which is a new double coset. We denote this double coset by

[1438].

One symmetric generators will go back to double coset [1438].

$Nt_1t_4t_3t_9 \in Nt_1t_4t_3t_9N$ which is a new double coset. We denote this double coset by [1439].

One symmetric generators will go back to double coset [1439].

$Nt_1t_4t_3t_{10} \in Nt_1t_4t_3t_{10}N$ which is a new double coset. We denote this double coset by [14310].

One symmetric generators will go back to double coset [14310].

$Nt_1t_4t_3t_{11} \in Nt_1t_4t_3t_{11}N$ which is a new double coset. We denote this double coset by [14311].

One symmetric generators will go back to double coset [14311].

$Nt_1t_4t_3t_{12} \in Nt_1t_4t_3t_{12}N$ which is a new double coset. We denote this double coset by [14312].

One symmetric generators will go back to double coset [14312].

$Nt_1t_4t_3t_{13} \in Nt_1t_4t_3t_{13}N$ which is a new double coset. We denote this double coset by [14313].

One symmetric generators will go back to double coset [14313].

$Nt_1t_4t_3t_{14} \in Nt_1t_4t_3t_{14}N$ which is a new double coset. We denote this double coset by [14314].

One symmetric generators will go back to double coset [14314].

$Nt_1t_4t_3t_{16} \in Nt_1t_4t_3t_{16}N$ which is a new double coset. We denote this double coset by [14316].

One symmetric generators will go back to double coset [14316].

$Nt_1t_4t_3t_{17} \in Nt_1t_4t_3t_{17}N$ which is a new double coset. We denote this double coset by [14317].

One symmetric generators will go back to double coset [14317].

$Nt_1t_4t_3t_{18} \in Nt_1t_4t_3t_{18}N$ which is a new double coset. We denote this double coset by [14318].

One symmetric generators will go back to double coset [14318].

$Nt_1t_4t_3t_{19} \in Nt_1t_4t_3t_{19}N$ which is a new double coset. We denote this double coset by [14319].

One symmetric generators will go back to double coset [14319].

$Nt_1t_4t_3t_{20} \in Nt_1t_4t_3t_{20}N$ which is a new double coset. We denote this double coset by [14320].

One symmetric generators will go back to double coset [14320].

$Nt_1t_4t_3t_{21} \in Nt_1t_4t_3t_{21}N$ which is a new double coset. We denote this double coset by [14321].

One symmetric generators will go back to double coset [14321].

$Nt_1t_4t_3t_{23} \in Nt_1t_4t_3t_{23}N$ which is a new double coset. We denote this double coset by [14323].

One symmetric generators will go back to double coset [14323].

$Nt_1t_4t_3t_{24} \in Nt_1t_4t_3t_{24}N$ which is a new double coset. We denote this double coset by [14324].

One symmetric generators will go back to double coset [14324].

$Nt_1t_4t_3t_{26} \in Nt_1t_4t_3t_{26}N$ which is a new double coset. We denote this double coset by [14326].

One symmetric generators will go back to double coset [14326].

$Nt_1t_4t_3t_{27} \in Nt_1t_4t_3t_{27}N$ which is a new double coset. We denote this double coset by [14327].

One symmetric generators will go back to double coset [14327].

$Nt_1t_4t_3t_{28} \in Nt_1t_4t_3t_{28}N$ which is a new double coset. We denote this double coset by [14328].
One symmetric generators will go back to double coset [14328].

$Nt_1t_4t_3t_{29} \in Nt_1t_4t_3t_{29}N$ which is a new double coset. We denote this double coset by [14329].
One symmetric generators will go back to double coset [14329].

$Nt_1t_4t_3t_{30} \in Nt_1t_4t_3t_{30}N$ which is a new double coset. We denote this double coset by [14330].
One symmetric generators will go back to double coset [14330].

Cayley Diagram

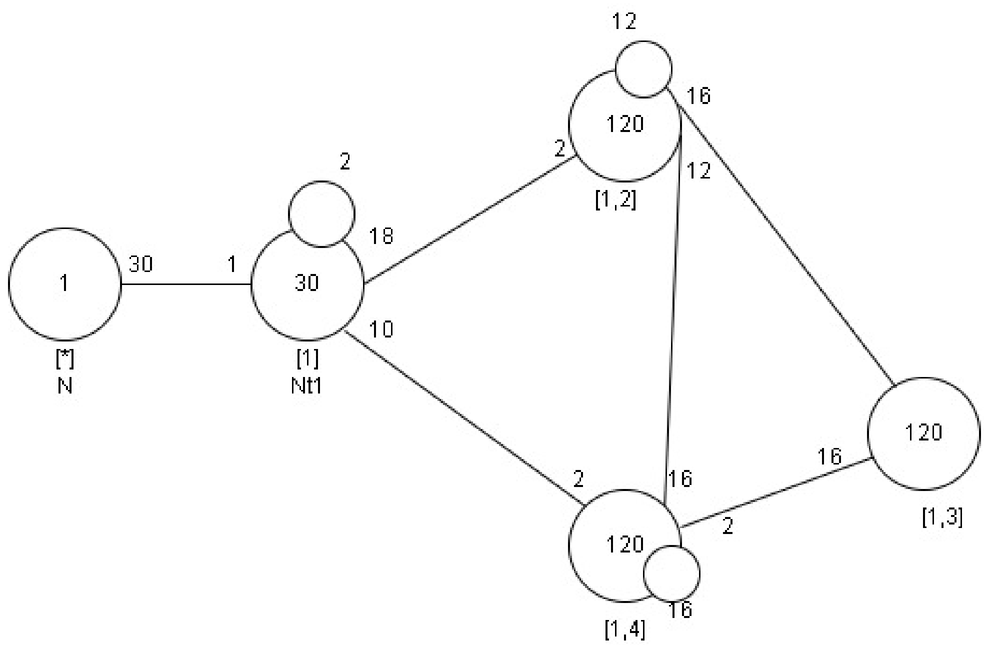

Figure 8.21: Cayley diagram for G over S_{25}

Chapter 9

Isomorphism Types of Transitive Groups

First, we find the number of transitive groups on n letters. Second, we choose a group i from the sequence to investigate further. The group is stored as $N := TransitiveGroup(n, i)$. We give isomorphism types of several such Ns.

9.1 Transitive Groups T(8,14)

T:=TransitiveGroup(8,14); We are given G, a transitive group on 8 letters. Since we have more than two generators we can use the command in Magma to reduce the number of generators to two.

```
for g,h in T[14] do if sub< T[14]|g, h >eq T[14]
then xx:=g; yy:=h;
end if; end for;
xx,yy;
xx;
/*(1, 6)(2, 5)(3, 7)(4, 8)*/
yy;
/*(1, 3, 8)(4, 5, 7)*/
```

Now will check the N by putting below code in magma.

N:=sub< $S|xx, yy$ >;

/*true*/

Now we get two generators xx and yy, so we identify the isomorphizm type of G.

S:=Sym(8);

xx:=$S!(1,6)(2,5)(3,7)(4,8)$;

yy:=$S!(1,3,8)(4,5,7)$;

N:=sub< $S|xx, yy$ >;

/*true*/

N eq T[14];

/* true */

$\sharp N$;

/*24*/

From composition Factors we will find the isomorphism type and from Normal Lattice we will determine if we have a direct product or semi direct.

```
CompositionFactors(N);
/*
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
*/
NL:=NormalLattice(N);
NL;
/*
Normal subgroup lattice
-----------------------

[4]  Order 24  Length 1  Maximal Subgroups: 3
---
[3]  Order 12  Length 1  Maximal Subgroups: 2
```

```
---
[2]  Order 4   Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:

*/
```

We then look to the Largest Abelian subGroup using the code.

$for i in [1..\sharp NL] do if IsAbelian(NL[i]) then i; end if; end for;$

/*

1

2

*/

The largest abelian subgroup in NL[2] and $G/NL[4] \cong q$.

The NL[2] produces two generators which named by A and B.

```
 Generators(NL[2]);
/*
{
    (1, 3)(2, 8)(4, 6)(5, 7),
    (1, 8)(2, 3)(4, 5)(6, 7)
}
*/
```

$A := N!(1,3)(2,8)(4,6)(5,7);$

$B := N!(1,8)(2,3)(4,5)(6,7);$

NL2:=sub$< N|A, B >$;

NL2 eq NL[2];

/*we want to find isomorphisum type of NL2*/

X:=[2,2];

/*

true Mapping from: GrpPerm: NL2 to GrpPerm: , Degree 4, Order 2^2

Composition of Mapping from: GrpPerm: NL2 to GrpPC and

Mapping from: GrpPC to GrpPC and

Mapping from: GrpPC to GrpPerm: , Degree 4, Order 2^2

*/

```
 IsIsomorphic(NL2,AbelianGroup(GrpPerm,X));
q,ff:=quo<N|NL2>;
q;
/*
Permutation group q acting on a set of cardinality 3
Order = 6 = 2 * 3
    (2, 3)
    (1, 2, 3)
*/
IsIsomorphic(q,Sym(3));
/*
true Isomorphism of GrpPerm: q, Degree 3, Order 2 * 3
into GrpPerm: $, Degree 3, Order 2 * 3 induced by
    (2, 3) |--> (2, 3)
    (1, 2, 3) |--> (1, 2, 3)
*/
T:=Transversal(N,NL2);
ff(T[2])eq q.1;
/* true */
ff(T[2])eq q.2;
/* false */
FPGroup(NL[2]);
/*
Finitely presented group on 2 generators
Relations
    $.1^2 = Id($)
    $.2^2 = Id($)
    ($.1 * $.2)^2 = Id($)
*/
FPGroup(q);
/*
Finitely presented group on 2 generators
Relations
    $.1^2 = Id($)
    $.2^-3 = Id($)
    ($.2^-1 * $.1)^2 = Id($)
*/

    for i,j in[1..2] do if A^T[2] eq A^i * B^j then i,j;
end if;
end for;
/* 2 1 */
for i,j in[1..2] do if A^T[3] eq A^i * B^j then i,j;
end if;
end for;
```

```
/* 1 1 */
for i,j in[1..2] do if B^T[2] eq A^i *B^j then i,j;
end if;
end for;
/* 1 2 */
for i,j in[1..2] do if B^T[3] eq A^i * B^j then i,j;
end if;
end for;
/* 1 2 */
G< a, b, c, d >:= Group < a, b, c, d|a^2, b^2,
(a, b), c^2, d^-3, (d^-1* c)^2, a^c = b, a^d =b,
b^c = a,b^d = a * b >;
#G;
/* 24 */
```

f,G,K:=CosetAction($G, sub < G|Id(G) >$);

♯G1;

IsIsomorphic(N,G1);

/*

true Mapping from: GrpPerm: N to GrpPerm: G1

Composition of Mapping from: GrpPerm: N to GrpPC and

Mapping from: GrpPC to GrpPC and

Mapping from: GrpPC to GrpPerm: G1

*/

9.2 Transitive Groups T(35,10)

T := TransitiveGroup(35,10);

We are given G, a transitive group on 35 letters. Since we have more than two generators we can use the command in Magma to reduce the number of generators to two.

T := TransitiveGroups(35);

T[10];

/*

Permutation group acting on a set of cardinality 35

Order = 210 = 2 * 3 * 5 * 7

(1, 16, 21, 11, 31, 26)(2, 17, 22, 12, 32, 27)(3, 18, 23, 13, 33, 28)(4, 19, 24, 14, 34, 29)(5,20, 25, 15, 35, 30)

(1, 28, 25, 2, 29, 21, 3, 30, 22, 4, 26, 23, 5, 27,24)(6, 13, 35, 7, 14, 31, 8, 15, 32, 9, 11, 33, 10, 12, 34)(16, 18, 20, 17, 19)

*/

Now we get two generators xx and yy, so we identity the isomorphism type of G.

S:= Sym(35);

$xx := (1, 16, 21, 11, 31, 26)(2, 17, 22, 12, 32, 27)(3, 18, 23, 13, 33, 28)$
$(4, 19, 24, 14, 34, 29)(5, 20, 25, 15, 35, 30);$

$yy := (1, 28, 25, 2, 29, 21, 3, 30, 22, 4, 26, 23, 5, 27, 24)$
$(6, 13, 35, 7, 14, 31, 8, 15, 32, 9, 11, 33, 10, 12, 34)(16, 18, 20, 17, 19);$

Now will check the N by putting below code in magma.

$N := sub < S|xx, yy >;$

$\sharp N;$

/* 210 */

/* For finding the isomorphism type look for a minimial faithful Perm Rep */

SL:=Subgroups(N);

T:=X‘ subgroup: X in SL;

$\sharp T$;
/*210*/
$TrivCore := H : HinT|\sharp Core(N, H)eq1$;
mdeg := Min (Index(N,H):H in TrivCore);
Good := H:H in TrivCore — Index(N,H) eq mdeg;
$\sharp Good$;
/*1*/
H:= Rep(Good);
$\sharp H$;
/* 6*/
f2,N1,K2:= CosetAction(N,H);
N1;
/*
Permutation group N1 acting on a set of cardinality 35
Order = 210 = 2 * 3 * 5 * 7
(2, 3, 5, 9, 15, 24)(4, 7, 12, 20, 10, 16)(11, 18,21, 17, 25, 13)(14, 19, 26, 22, 29, 28)(23, 30, 34, 35, 32, 33)
(1, 2, 4, 8, 14, 23, 31, 16, 18, 27, 33, 3, 6, 11,19)(5, 10, 17, 26, 32, 9, 7, 13, 22, 30, 24, 12, 21, 28, 34)(15, 20, 25, 29, 35)
*/
N;
/*
Permutation group N acting on a set of cardinality 35
Order = 210 = 2 * 3 * 5 * 7
(1, 16, 21, 11, 31, 26)(2, 17, 22, 12, 32, 27)(3, 18, 23, 13, 33, 28)(4, 19, 24, 14, 34, 29)(5, 20, 25, 15, 35, 30)
(1, 28, 25, 2, 29, 21, 3, 30, 22, 4, 26, 23, 5, 27,24)(6, 13, 35, 7, 14, 31, 8, 15, 32, 9, 11, 33, 10, 12, 34)(16, 18, 20, 17, 19)
*/
Order(xx);
/*6*/

From Composition Factors we will find the isomorphism type and Normal Lattice we will determine if we have a direct product and not.

```
   CompositionFactors(N);
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(7)
    *
    |  Cyclic(5)
    1

  NL:= NormalLattice(N);
 NL;
/*
Normal subgroup lattice
-----------------------

[10]  Order 210  Length 1  Maximal Subgroups: 7 8 9
---
[ 9]  Order 105  Length 1  Maximal Subgroups: 5 6
[ 8]  Order 70   Length 1  Maximal Subgroups: 4 6
[ 7]  Order 42   Length 1  Maximal Subgroups: 4 5
---
[ 6]  Order 35   Length 1  Maximal Subgroups: 2 3
[ 5]  Order 21   Length 1  Maximal Subgroups: 3
[ 4]  Order 14   Length 1  Maximal Subgroups: 3
---
[ 3]  Order 7    Length 1  Maximal Subgroups: 1
[ 2]  Order 5    Length 1  Maximal Subgroups: 1
---
[ 1]  Order 1    Length 1  Maximal Subgroups:
*/
```

We now look to the Largest Abelian sub Group using the code.

for n in [1..$\sharp NL$] do if IsAbelian (NL[i]) then i; end if ; end for;

/*

1

2

3

6

*/

The Largest Abelian subgroup in NL[6] and G/NL[10]≅ q.

The NL[6] produce the two generators which named by A and B.

```
Generators (NL[6]);
 /*
{
    (1, 16, 31, 11, 26, 6, 21)(2, 17, 32, 12, 27, 7, 22)
    (3, 18, 33,  13, 28, 8, 23)(4, 19, 34, 14, 29, 9, 24)
    (5, 20, 35, 15, 30, 10, 25)
    (1, 3, 5, 2, 4)(6, 8, 10, 7, 9)(11, 13, 15, 12, 14)
    (16, 18, 2, 17, 19)(21, 23, 25, 22, 24)(26, 28, 30, 27, 29)
    (31, 33, 35, 32, 34)
}
*/
  A:=N!((1, 16, 31, 11, 26, 6, 21)(2, 17, 32, 12, 27, 7, 22)
  (3, 18, 33, 13, 28, 8, 23)(4,19, 34, 14, 29, 9, 24)
  (5, 20, 35, 15, 30, 10, 25));

  B:=N!((1, 3, 5, 2, 4)(6, 8, 10, 7, 9)(11, 13, 15, 12, 14)
  (16, 18, 20, 17, 19)(21, 23, 25,22, 24)(26, 28, 30, 27, 29)
  (31, 33, 35, 32, 34));

 /* A and B is the generator  the largest abelian */
 NL6:=sub<N|A,B>;
 /* Check that NL[4] = NL4*/
 NL[6] eq NL6;
 /* True

 N is not an extension of NL4 (normal) by N/NL6
 Can this extnesion be a direct product meaning N =NL6 x N/NL6
 */
 Order (NL6);
 /* 35 */
```

```
 Order (N);
 /* 210 */


 Does NL have a subgroup (normal) of order 6
 It does so it is a direct product (check normal lattice)

Now N = <xx,yy> and NL6=<A,B>. Then N/NL6 = <NL6xx,NL6yy>*/

q,ff: =quo<N|NL6>;
 /* q is the isom type of N/NL6; that is, q N/NL6 */

 T:=Transversal(N,NL6);
 /* T gives right cosets of NL6 in N
 Thus, N/NL6=<NL6T[2]> */
 T[2] eq xx;
 /* true */
  #T;
/* 6*/
```

$/ * T = T[1], T[2], T[3], T[4], T[5]$

```
 N/NL4={NL4, NL4T[2],NL4^T[3]}
 q=<q.1,q.2>, where ff(T[2])=q.1 ff(T[3])=q.2*/
 ff(T[2]) eq q.1;
 /* true */
  q;
 /* true */
 ff(T[2])eq q.1;
 Order(T[2]);
/*6*/
 Order(q.1);
/*6*/



for i in [1..7] do for j in [1..5] do
 if A^T[2] eq A^i*B^j then i,j; end if; end for; end for;
 /*5  5 */
 for i in [1..7] do for j in [1..5] do
 if B^T[2] eq A^i*B^j then i,j; end if; end for; end for;
/* 7 1*/
```

$G < a, b, c >:= Group < a, b, c | a^7, b^5, (a, b), c^6, a^c = a^5 * b^5, b^c = b >;$

$\sharp G;$

/*210*/

$f, G1, k := CosetAction(G, sub < G | Id(G) >);$

$\sharp G1;$

/*210*/

s:=IsIsomorphic(N,G1);

s;

/*true*/

9.3 Transitive Groups T(35,17)

T:=TransitiveGroup(35,17);

We are given G, a transitive group on 35 letters. Since we have more than two generators we can use the command in Magma to reduce the number of generators to two.

T:=TransitiveGroups(35);
T[17];
/*
Permutation group acting on a set of cardinality 35
Order = 840 = 2^3 * 3 * 5 * 7
(1, 21, 9, 30, 14, 34, 19, 5, 22, 10, 29, 15, 31, 16, 3, 23, 6,
27, 12, 32, 17, 4, 24, 8, 26, 13, 33, 20, 2, 25, 7, 28, 11, 35, 18)

(1, 26, 20, 7, 32, 24, 13, 3, 27, 17, 10, 31, 23, 14, 5, 29, 18,
9, 34, 25, 11)(2, 28, 16, 6, 35, 21, 12, 4, 30, 19, 8, 33, 22, 15)
*/

Now we get two generators xx and yy, so we identity the isomorphism type of G.

S:=Sym(35);
xx:=S!(1, 21, 9, 30, 14, 34, 19, 5, 22, 10, 29, 15, 31, 16, 3, 23, 6,
27, 12, 32, 17, 4, 24, 8, 26, 13, 33, 20, 2, 25, 7, 28, 11, 35, 18);

yy:=S!(1, 26, 20, 7, 32, 24, 13, 3, 27, 17, 10, 31, 23, 14, 5, 29, 18, 9,
34, 25, 11)(2, 28, 16, 6, 35, 21, 12, 4, 30, 19, 8, 33, 22, 15);

$NowwillchecktheNbyputtingbelowcodeinmagma.$
N:=sub¡S—xx,yy¿;
$\sharp N$;
/* 840 */

/* For finding the isomorphism type look for a minimial faithful Perm Rep */

SL:=Subgroups(N);

T:=X‘ subgroup: X in SL;

$\sharp T$;

/* 38 */

```
TrivCore := {H:H in T| #Core(N,H) eq 1};
mdeg := Min ({Index(N,H):H in TrivCore});
Good := {H:H in TrivCore | Index(N,H) eq mdeg};
#Good;
/* 1 */
H:= Rep(Good);
#H;
/* 24 */
f2,N1,K2:= CosetAction(N,H);

N1;
/*
Permutation group N1 acting on a set of cardinality 35
Order = 840 = 2^3 * 3 * 5 * 7
    (1, 2, 4, 8, 16, 9, 17, 25, 21, 28, 32, 29, 34, 31, 13, 22, 30,
        23, 5, 10, 19, 11, 20, 26, 3, 6, 12, 7, 14, 18, 15, 24, 27,
        33, 35)
    (1, 3, 7, 15, 10, 20, 6, 13, 23, 19, 28, 34, 22, 16, 25, 32, 35,
        4, 9, 18, 27)(2, 5, 11, 8, 17, 26, 12, 21, 29, 14, 24, 31, 30,
        33)
*/

N;
/*
Permutation group N acting on a set of cardinality 35
Order = 840 = 2^3 * 3 * 5 * 7
    (1, 21, 9, 30, 14, 34, 19, 5, 22, 10, 29, 15, 31, 16, 3, 23, 6,
        27, 12, 32, 17, 4, 24, 8, 26, 13, 33, 20, 2, 25, 7, 28, 11,
        35, 18)
    (1, 26, 20, 7, 32, 24, 13, 3, 27, 17, 10, 31, 23, 14, 5, 29, 18,
        9, 34, 25, 11)(2, 28, 16, 6, 35, 21, 12, 4, 30, 19, 8, 33, 22,
        15)
*/

Order(xx);
/*35*/
```

From Composition Factor we will find the isomorphism type and Normal Lattice we will determine if we have a direct product and not.

```
CompositionFactors(N);

    G
    |  Cyclic(2)
    *
    |  Alternating(5)
    *
    |  Cyclic(7)
    1

NL:= NormalLattice(N);
NL;

Normal subgroup lattice
-----------------------

[6]  Order 840  Length 1  Maximal Subgroups: 4 5
---
[5]  Order 420  Length 1  Maximal Subgroups: 2 3
[4]  Order 120  Length 1  Maximal Subgroups: 3
---
[3]  Order 60   Length 1  Maximal Subgroups: 1
---
[2]  Order 7    Length 1  Maximal Subgroups: 1
---
[1]  Order 1    Length 1  Maximal Subgroups:

Now we look for the Largest Abelian sub group using the code.
for n in [1 #NL] do if IsAbelian (NL[i]) then i; end if ; end for;

/*
1
2
*/
```

The Largest Abelian subgroup in $NL[2] and G/NL[6] \cong q$.

The NL[2] produce the one generator which named by A.

$A := N!(1, 34, 29, 23, 17, 13, 7)(2, 35, 30, 22, 16, 12, 8)$

(3, 32, 26, 25, 18, 14,10)(4, 33, 28, 21, 19, 15, 6)
(5, 31, 27, 24, 20, 11, 9) ;

```
NL2:=sub<N|NL[2]>;
q,ff:=quo<N|NL2>;
q;
Permutation group q acting on a set of cardinality 6
Order = 120 = 2^3 * 3 * 5
    (2, 3, 5, 6, 4)
    (1, 2, 4, 3, 6, 5)

FPGroup(NL[2]);
Permutation group q acting on a set of cardinality 6
Order = 120 = 2^3 * 3 * 5
    (2, 3, 5, 6, 4)
    (1, 2, 4, 3, 6, 5)

T:=Transversal(N,NL2);
ff(T[2])eq q.1;
/* true */

ff(T[3])eq q.2;
/* true */

for i in [1..7] do if A^T[2] eq A^i then i; end if ; end for;
/* 1 */
A^T[2] eq A;
/* true */
for i in [1..7] do if A^T[3] eq A^i then i; end if ; end for;
/* 1 */
A^T[3] eq A;
/* true */
```

$G < a, b, c >:= Group < a, b, c | a^7, b^5, c^6, (b * c * b*)^2, (b * c^{-2})^2 >;$

$\sharp G;$

$f, G1, K := CosetAction(G, sub < G | Id(G) >);$

$\sharp G1;$

IsIsomorphic(G1,N);

/* true */

9.4 Transitive Groups T(35,12)

T:=TransitiveGroup(35,12);

We are given G, a transitive group on 35 letters. Since we have more than two generators we can use the command in Magma to reduce the number of generators to two.

T:=TransitiveGroups(35);
T[12];
/*
Permutation group acting on a set of cardinality 35
Order = 280 = 2^3 * 5 * 7
(1, 12, 5, 14)(2, 15, 4, 11)(3, 13)(6, 7, 10, 9)(16, 32, 20, 34)
(17, 35, 19, 31)(18, 33)(21, 27, 25, 29)(22, 30, 24, 26)(23, 28)
(1, 21)(2, 25)(3, 24)(4, 23)(5, 22)(6, 16)(7, 20)(8, 19)
(9,18)(10, 17)(12, 15)(13, 14)(26, 31)(27, 35)(28, 34)(29, 33)(30, 32)
*/
Now we get two generators xx and yy, so we identity the isomorphism type of G.
S:=Sym(35);
xx:=S! (1, 12, 5, 14)(2, 15, 4, 11)(3, 13)(6, 7, 10, 9)(16, 32, 20, 34)
(17, 35, 19, 31)(18, 33)(21, 27, 25, 29)(22, 30, 24, 26)(23, 28);

yy:=S!(1, 21)(2, 25)(3, 24)(4, 23)(5, 22)(6, 16)(7, 20)(8, 19)(9, 18)
(10, 17)(12, 15)(13, 14)(26, 31)(27, 35)(28, 34)(29, 33)(30, 32);

Now will check the N by putting below code in magma.
$N := sub < S|xx, yy >$;
$\sharp N$;
/*280*/
* For finding the isomorphism type look for a minimial faithful Perm Rep */
SL:=Subgroups(N);
T:=X‘ subgroup: X in SL;
$\sharp T$;
/*32*/

$TrivCore := H : HinT|\sharp Core(N, H)eq1;$

$mdeg := Min(Index(N, H) : HinTrivCore);$

$Good := H : HinTrivCore|Index(N, H)eqmdeg;$

$\sharp Good;$

/* 1 */

H:= Rep(Good);

$\sharp H;$

/* 8 */

f2,N1,K2:= CosetAction(N,H);

N1;

Permutation group N1 acting on a set of cardinality 35

$Order = 280 = 2^3 * 5 * 7$

$(2, 3, 4, 6)(5, 8, 11, 16)(7, 10, 14, 21)(9, 13, 19, 28)$

$(12, 18, 26, 31)(15, 23)(17, 25, 30, 22)(20, 29, 32, 34)(24, 27)(33, 35)$

$(1, 2)(3, 5)(4, 7)(6, 9)(8, 12)(10, 15)(11, 17)(13, 20)(14, 22)$

$(16, 24)(18, 27)(19, 23)(21, 25)(26, 28)(29, 31)(30, 33)(32, 35)$

$N;$

$PermutationgroupNactingonasetofcardinality35$

$Order = 280 = 2^3 * 5 * 7$

$(1, 12, 5, 14)(2, 15, 4, 11)(3, 13)(6, 7, 10, 9)(16, 32, 20, 34)$

$(17, 35, 19, 31)(18, 33)(21, 27, 25, 29)(22, 30, 24, 26)(23, 28)$

$(1, 21)(2, 25)(3, 24)(4, 23)(5, 22)(6, 16)(7, 20)(8, 19)(9, 18)$

$(10, 17)(12, 15)(13, 14)(26, 31)(27, 35)(28, 34)(29, 33)(30, 32)$

$Order(xx);$

$/ * 4 * /$

From Composition Factors we will find the isomorphism type and Normal Lattice we will determine if we have a direct product and not.

```
CompositionFactors(N);
```

```
    G
    |  Cyclic(2)
    *
    |  Cyclic(7)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(5)
    1

NL:= NormalLattice(N);
NL;

Normal subgroup lattice
-----------------------

[14]  Order 280  Length 1  Maximal Subgroups: 11 12 13
---
[13]  Order 140  Length 1  Maximal Subgroups: 7 10
[12]  Order 140  Length 1  Maximal Subgroups: 8 9 10
[11]  Order 140  Length 1  Maximal Subgroups: 10
---
[10]  Order 70   Length 1  Maximal Subgroups: 4 6
[ 9]  Order 70   Length 1  Maximal Subgroups: 6
[ 8]  Order 70   Length 1  Maximal Subgroups: 5 6
[ 7]  Order 20   Length 1  Maximal Subgroups: 4
---
[ 6]  Order 35   Length 1  Maximal Subgroups: 2 3
[ 5]  Order 14   Length 1  Maximal Subgroups: 3
[ 4]  Order 10   Length 1  Maximal Subgroups: 2
---
[ 3]  Order 7    Length 1  Maximal Subgroups: 1
[ 2]  Order 5    Length 1  Maximal Subgroups: 1
---
[ 1]  Order 1    Length 1  Maximal Subgroups:

We now look to the Largest Abelian sub Group using the code.
for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if; end for;

/*
1
2
```

```
3
6
*/
```

The NL[6] is the Largest Abelian sub group and produce the two generators which named A and B.

Chapter 10

Images of Progenitors

In this chapter, we will show the isomorphic images for most of the composition factors we discovered.

10.1 $(5^2 : (3 : 2))$

We have the following information.

S:=Sym(15),

$x \sim (1, 15, 12, 8, 3, 9, 14, 13, 7, 4)(2, 11, 5, 6, 10)$;

$y \sim (1, 11, 14, 6, 12, 2, 7, 5, 3, 10)(4, 8, 13, 15, 9)$;

The order of $|N| = 150$.

Images of 2*15 : $(5^2 : (3 : 2)$

$(5^3 : (3 : 2))$																				
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	u	v
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$2^{*12} : (2 \times A_5)$						
w	z	$a1$	$b1$	$c1$	$d1$	G
0	0	0	3	0	0	$2^3 : 3$
0	0	4	0	2	0	$2^4 : 3$
0	0	5	0	2	0	$2 : Alt_5$
0	0	0	5	3	0	$U(3,4) : 2$

```
G<x,y,t>:=Group<x,y,t| (y^-1 * x^-1)^3 ,
(y^-1 * x)^3,
x^-1 * y^-1 * x^3 * y^-1 * x^-1 * y  ,x^2 * y * x^2 *y^3,t^2,
(t,y^-1 * x^2 * y^-1),
(t,x^3 * y^-1 * x),
(x*t^(y*x^-1))^a,(x*t)^b, (y*t)^c,
(y*t^(y*x^-1))^d,
(y*t^(y^-2))^e,
(x^(-1)*t^(y*x^-1))^f,(x^(-1)*t)^g,
(y^(-1)*t^(y*x^-1))^h,(y^(-1)*t)^i,
(x^(2)*t^(y*x^-1))^j,(x^(2)*t)^k,
(x*y*t^(y*x^-1))^l,
(x*y*t)^m,(x*y^(-1)*t)^n,
(x*y^(-1)*t^(y*x^(-1)))^o,
(x*y^(-1)*t^x^(-1))^p,(y*x*t)^q,
(y*x*t^(y*x^(-1)))^r,(y*x*t^x^(-1))^s,
(y^(2)*t^(y*x^(-1)))^u,(y^(2)*t)^v,
(y*x^(-1)*t^(y*x^(-1)))^w,(y*x^(-1)*t)^z,
(x^(-1)*y*t^(y*x^(-1)))^a1, (x^(-1)*y*t)^b1,(x^(-2)*t^(y*x^(-1)))^c1,
(x^(-2)*t)^d1
>;
```

10.2 $(2^2 : 5^2)$

We have the following information.

S:=Sym(25), we are working with 25 letters.

$x \sim (1,19,11,17,2)(3,16,15,5,6,9,24,8,21,22)(4,18,13,25,7,14,20,10,23,12);$

$y \sim (1,16)(2,8)(3,20)(4,24)(5,17)(6,13)(7,15)(9,19)(10,18)(11,22)(14,25)(21,23);$

The order of $|N| = 100$.

$(5^3 : (3 : 2))$																				
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	u	v
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$(5^3 : (3 : 2))$														
w	z	$a1$	$b1$	$c1$	$d1$	$e1$	$f1$	$g1$	$h1$	$i1$	$j1$	$k1$	$l1$	G
0	0	0	0	0	0	0	0	0	0	0	3	0	0	$2^2 : A_5$
0	0	0	0	0	0	0	0	0	0	3	0	0	0	$2^2 : A_5$
0	0	0	0	0	0	0	0	0	0	3	3	0	0	$(2 : A_5) : a_5$
0	0	0	0	0	0	0	0	0	0	4	4	4	0	$(2^7 \times 5) : A_5$

```
G<x,y,t>:=Group<x,y,t| y^2,
(x * y * x)^2,x^10,
x^-1 * y * x^-1 * y * x^-1 * y * x^-1 * y *
x^-1 * y * x * y * x * y * x * y * x * y * x^-1* y ,
t^2,
(t,y * x * y * x * y * x * y * x^-1 * y),
(t,x * y * x * y * x * y * x^-1 * y * x * y),
(x*t)^a,(x*t^(y*x^-1))^b,
(x*t^(x*y^2))^c,(y*t^(x*y*x^-1))^d,
(y*t)^e,
(y*t^(x^-1))^f,(x^(-1)*t^(y*x*y))^g,
(x^(-1)*t^x^(-1))^h,
(x^(-1)*t^x*y^2)^i,
(x^(-1)*t^(x^(-1)*y))^j,
(x^(-1)*t^(x*y*x^(-1)*y))^k,
(x^(-1)*t)^l,(x^(-1)*t^(y*x^(-1)))^m,
(x^(-1)*t^(y*x^(2)))^n,
(x^(-1)*t^(x*y))^o,(x^2*t)^p,(x*y*t)^q,
(y*x*t)^r,(y*x^(-1)*t)^s,(x^(-1)*y*t)^u,
(x^(-2)*t)^v,(x^3*t)^w,(x^2*y*t)^z,
(x*y*x*t)^a1,(x*y*x*t^(y*x^(-1)))^b1,
(x*y*x*t^((x*y)^2))^c1,
(x*y*x^(-1)*t)^d1,(x*y*x^(-1)*t^(y*x^(-1)))^e1,
(x*y*x^(-1)*t^(x*y^(2)))^f1,
(y*x^(2)*t^(y*x))^g1,(y*x^(2)*t)^h1,
(y*x^(2)*t^(x^(-1)))^i1,
(y*x*y*t^(x*y*x^(-1)))^j1,
(y*x*y*t)^k1,(y*x*y*t^(x^-1))^l1>;
```

10.3 $A_5 * 2$

We have the following information.

S:=Sym(30), we are working with 25 letters.

$x \sim (2,3)(4,6)(5,8)(7,11)(9,14)(10,15)(12,18)(13,20)(17,24)(19,26)(21,25)(23,28);$

$y \sim (1,2,4,7,12,19)(3,5,9,15,22,27)(6,10,16,23,18,25)(8,13,21)(11,17,14)(20,26,29,30,28,24);$

The order of $|N| = 120$.

$A_5 * 2$																				
a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	u	v
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

$(5^3 : (3 : 2))$																
w	z	$a1$	$b1$	$c1$	$d1$	$e1$	$f1$	$g1$	$h1$	$i1$	$j1$	$k1$	$l1$	$m1$	$n1$	G
0	0	0	0	0	0	0	0	0	0	3	0	0	3	4	0	$A_6 : 2$
0	0	0	0	0	0	0	0	0	0	3	0	3	0	4	0	$A_6 : 2$
0	0	0	0	0	0	0	0	0	0	0	3	0	0	0	4	$2^6 : A_5$
0	0	0	0	0	0	0	0	0	0	0	4	4	8	4	0	2×2
0	0	0	0	0	0	0	0	0	0	0	5	0	0	4	3	$A_6 : 2$

```
G<x,y,t>:=Group<x,y,t|x^2,y^6,
(y*x*y^-1*x)^2,
(x*y^-1)^5,
(t,x),(t,y^2*x*y^-2*x*y^2),
(x*t^(y * x^2))^a ,
(x*t^(y^3 * x))^b ,
(x*t)^c , (x*t^(y^2 * x))^d ,
(x*t^( y * x))^e,
(y*t^(y^3))^f, (y*t)^g ,
(y*t^(b))^h,
(y*t^(y * x * y^2))^i,
(y*t^(y*x))^j,
(y*t^(y^2))^k,
(y^-1*t^((y * x)^2))^l,
(y^-1*t^( y^3 * x))^m,
(y^-1*t)^n,
(y^-1*t^(y*x))^o,
(y^-1*t^(y^2*x))^p,
(y*x*t^(y^-1 * x * y^-1))^q,
(x*y*t^(y^3))^r, (x*y*t)^s,
(x*y*t^(y))^u,
(x*y*t^(y*x))^v,
(x*y*t^(y^2))^w,
(x*y*t^(y*x)^2)^z,
(x*y*t^(y*x*y^2))^a1,
(x*y*t^(y^-2))^b1,
(x*y^-1*t)^c1,
(x*y^-1*t^(y))^d1,
(x*y^-1*t^(y^2))^e1,
(x*y^-1*t^(y*x*y))^f1,
(x*y^-1*t^(y*x*y^2))^g1,
(x*y^-1*t^(y^-2))^h1,
(y*x*t^((y*x)^2))^i1,
(y*x*t^(y^3*x))^j1,(y*x*t)^k1,
(y*x*t^(y*x))^l1,(y*x*t^(y^2*x))^m1,
(y*x*t^(y^-1*x*y^-1))^n1 >;
```

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