The Examination of the Arithmetic Surface (3, 5) Over Q

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THE EXAMINATION OF THE ARITHMETIC SURFACE (3, 5) OVER \( \mathbb{Q} \)

A Thesis
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Mathematics

by
Rachel Arguelles
May 2022
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Abstract

This thesis is centered around the construction and analysis of the principal arithmetic surface \((3, 5)\) over \(\mathbb{Q}\). By adjoining the two symbols \(i, j\), where \(i^2 = 3, j^2 = 5\), such that \(ij = -ji\), I can produce a quaternion algebra over \(\mathbb{Q}\). I use this quaternion algebra to find a discrete subgroup of \(SL_2(\mathbb{R})\), which I identify with isometries of the hyperbolic plane. From this quaternion algebra, I produce a large list of matrices and apply them via Möbius transformations to the point \((0, 2)\), which is the center of my Dirichlet domain. This list of transformed points is my orbit sampling. The possible walls of the Dirichlet domain are perpendicular bisectors of the hyperbolic geodesic segments between \((0, 2)\) and points from my orbit sampling. Once I produced the list of all perpendicular bisectors, I plot them. From this plot, I refine the collection to only the walls for my Dirichlet domain. I then analyze this surface by finding its hyperbolic area, and the matrices that correspond to gluing one side to the other.

The production of this hyperbolic surface by adjoining \((3, 5)\) over \(\mathbb{Q}\) is an “example problem”. This example problem produces a hyperbolic polygon that could be used to show others the process of finding a Dirichlet domain, or producing a hyperbolic polygon. One reason why one may want to find a hyperbolic surface is for the list of generators that correlates to the edge gluings. This list of generators can be used in other areas of mathematics and science.

This project brought together many areas of math, including topology, hyperbolic geometry, linear algebra, group theory, and noncommutative algebra. Additionally this project involved a significant programming component, in which I wrote and implemented code in Sage, a Python based computer algebra system.
Acknowledgements

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# Table of Contents

Abstract iii

Acknowledgements iv

List of Figures vii

1 Introduction 1

2 Surfaces 5
   2.1 Classification of Surfaces ................................. 5
   2.2 Group Actions and Covering Spaces ....................... 6
   2.3 Hyperbolic Geometry ...................................... 14
   2.4 Genus 2 Surface Coverings ................................. 22
      2.4.1 Sage Coding - Producing Signatures of Surfaces that Cover a Genus 2 Surface ................................................. 31
   2.5 Genus 3 Surface Coverings ................................. 32
      2.5.1 Sage Coding - Producing Signatures of Surfaces that Cover a Genus 3 Surface ................................................. 40

3 Hyperbolic Transformations 42
   3.1 Möbius Transformations ..................................... 42
   3.2 Circle Inversions ............................................ 44
   3.3 Discrete Subgroups .......................................... 47
   3.4 Dirichlet Domains ........................................... 47

4 Arithmetic Constructions 49
   4.1 Quadratic Extensions ........................................ 49
   4.2 Quaternion Algebras ........................................ 50

5 The Research Process 52
   5.1 The Quaternion Algebra ..................................... 52
   5.2 Orbit ....................................................... 53
   5.3 Finding the Dirichlet Domain ............................... 54
   5.4 Analyzing the Polygon ..................................... 67
6 Conclusion

Bibliography
# List of Figures

1.1 Clay figures-1 .................................................. 2
1.2 Clay figures-2 .................................................. 2
2.1 A of genus 2 surface covering genus 1 surface with 2 cone points .......... 7
2.2 Genus 3 surface with two different placements of the axis of rotation ..... 9
2.3 A genus 2\(n + 1\) covers a genus 3 surface ................................. 11
2.4 A genus \(n + 1\) surface covers a genus 2 surface .......................... 13
2.5 Triangle with ideal point ................................................. 15
2.6 Area of triangle with no ideal points ..................................... 16
2.7 strategy 1 ................................................................ 17
2.8 strategy 2 ................................................................ 18
2.9 Area of surface with \(n\) many cone points. ................................. 20
2.10 Area of surface with \(n\) many cone points and one boundary .......... 21
2.11 Area of surface with \(g\) genus and one boundary ......................... 21
2.12 Area of surface with \(g\) many genus and \(n\) many cone points ........ 22
3.1 circle inversion .......................................................... 44
3.2 Similar triangles under circle inversion ..................................... 45
3.3 Circle inversion maps circles to lines ......................................... 45
3.4 Circle inversion maps circle to circles ....................................... 46
3.5 Circle inversion circle maps to itself ......................................... 46
5.1 Orbit sampling ......................................................... 54
5.2 3 points create a triangle .................................................. 55
5.3 Production of the circumcenter ............................................. 56
5.4 Circle through 3 noncollinear points ......................................... 56
5.5 Line through points \(AB\) .................................................. 57
5.6 Circle \(AB\) on boundary ................................................... 58
5.7 segment \(AB\) ............................................................ 59
5.8 Circle \(d\) ................................................................ 60
5.9 Segment \(AD\) ........................................................... 61
5.10 Circle \(e\) ............................................................... 62
5.11 Perpendicular bisector of segment \(AB\) ...................................... 63
5.12 Graph of all perpendicular bisectors between segment $AB$ . . . . . . . . . . . . . . 66
5.13 Hyperbolic polygon . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 67
5.14 Hyperbolic polygon with gluings . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 69
5.15 Hyperbolic Surface . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 70
Chapter 1

Introduction

In Larry Guth’s *Metaphors in Systolic Geometry*, he states “Arithmetic Hyperbolic surfaces are among the strangest and most interesting examples in (Riemannian) geometry”. I decided to study hyperbolic geometry because I thought it was different. What made it different was that this was the first type of non-Euclidean geometry that I have experienced. I remember the first day learning about $\mathbb{H}^2$, the hyperbolic plane. My mind is blown when learning that in Euclidean geometry, we have lines but in hyperbolic geometry we have circles that are geodesics. When learning about surfaces, I found it was very hard for me to visualize the concept of bending and stretching of surfaces. This actually motivates me to build the surfaces I am working on by hand with clay. It gives me a deeper understanding to construct what covers a genus 2 or a genus 3 surface. I use straws as a “literal pole” (see Figure 2.14), to understand how the fundamental domain is being rotated.
Once I had these constructions, it was the quantitative features of these surfaces that intrigue me the most. For example, two surfaces could look the same, but they could
have different cone points (see Section 2). A hyperbolic surface can be understood to be a hyperbolic polygon with gluing maps. A special class of such polygons can be built from arithmetic objects called quaternion algebras. For my research, I create a hyperbolic polygon from the quaternion algebra $(3, 5)$ over $\mathbb{Q}$. This algebra is formed by algebraically adjoining the square roots $\sqrt{3}$ and the $\sqrt{5}$ in a special way. More information on this quaternion algebra can be found in Chapter 4.2.

The groups formed by this type of constructions are a special type of what are called Fuchsian groups. These groups have deep connections through out geometry and algebra. For example, in *Small Generators of Cocompact Arithmetic Fuchsian Groups*, by Michelle Chu and Han Li [Li16], they state there is a special connection of the subclass formed from arithmetic Fuchsian groups to number theory and to automorphic forms. An automorphic form is a function from a topological group $G$ to the complex numbers. The action for the discrete subgroup $\Gamma \subset G$ the topological group is invariant. Automorphic forms were first discovered as a generalization of trionometric and elliptic functions by the French mathematician Henri Poincare. For more information on automorphic forms, you can use the following resource [Dei12].

It is a nontrivial problem to determine a set of finite generators for Fuchsian groups. To determine the set of generators, you can create a special polygon in $\mathbb{H}^2$ called a fundamental domain and the generators then correspond to certain side pairings of the polygon. The polygonal fundamental domain can be found in certain contexts using algorithms by Johansson, Voight and Macasieb, and others have written computer codes to implement this strategy. I do the same. I use Sage, a computer algebra system.

To create this polygonal fundamental domain, I first find the quaternion algebra, and use the quaternion algebra to produce matrices (See Chapter 4.2). I use the matrices to transform the point $(0,2)$, which is the center of my Dirichlet domain (see Chapter 3.4). The list of points which I produce from the transformation of $(0,2)$ is my orbit sampling. Using the points from my orbit sampling, I find the perpendicular bisectors between $(0,2)$ and each point in my orbit sampling. I use the perpendicular bisectors to find my Dirichlet domain (see Figure 5.14).

Once I have the Dirichlet domain, to check that I have the correct walls, I use three different area formulas: Gauss-Bonnet Theorem 2.25, the Arithmetic Area Theorem 4.14, and Riemann-Hurwitz Theorem 2.29, to confirm the correct area of my founded
hyperbolic polygon. The Arithmetic Area Formula is actually being used in many papers like [Mac08]. The Riemann-Hurwitz Theorem is named after the German mathematicians Georg Friedrich Bernhard Riemann and Adolf Hurwitz. The Gauss-Bonnet Theorem (Theorem 2.25) is actually named after two mathematicians Carl Friedrich Gauss, who has developed a version of the theorem, but has never published it, and Pierre Ossian Bonnet who has published a special case of the theorem.

In finding this hyperbolic polygon, I will then have produced a list of finite generators for my Fuchsian group. This list of finite generators, could be used in other areas such as spectral geometry, systolic geometry, and many other fields.
Chapter 2

Surfaces

2.1 Classification of Surfaces

I now begin with some background on the topology of surfaces. For a detailed point-set topology reference, see [Mun18]. A surface is a shape that locally looks like the plane. For example, the Earth is a surface and it is locally flat, and it is for this reason people thought it was flat for so long.

Definition 2.1. A (topological) surface is a Hausdorff topological space with a countable basis, each point of which has a neighborhood homeomorphic to an open subset of $\mathbb{R}^2$.

Definition 2.2. Let $S$ be a compact surface. An orientation on $S$ is a continuous choice of an “outside” to the surface and $S$ is orientable if it admits an orientation.

Definition 2.3. An orbisurface is a Hausdorff topological space with a countable basis $\{U_i\}$ such that to each $U_i$, there exists an open set $\tilde{U}_i$ such that $U_i \cong \tilde{U}_i/\Gamma_i \subset \mathbb{R}^2$ and $\Gamma_i \subset \text{Isom}(\mathbb{R}^2)$ is a finite subgroup. If $\Gamma_i$ is a rotation group, then the image of $(0,0)$ in $U_i$ is a cone point. The order of $\Gamma_i$ is called to degree of cone point, and is denoted $e_i$.

Definition 2.4. A good orbisurface is an orbisurface which is the quotient of a surface under the action of a discrete group of homeomorphisms.

When working with the topology of surfaces, we can bend, fold, and stretch them as long as we keep the orientation of the surface. We can think of surfaces like clay.

Definition 2.5. Let $S$ be a compact orientable surface. The genus of $S$ is the number of “donut holes”. (A more rigorous definition can be found in [Mun18, Chapter 12])
Definition 2.6. A **punctured surface** is a compact surface with \( p \) many “marked points”, which we consider having been deleted. We denote the punctured orientable surface with genus \( g \) and \( p \) punctures as \( S_{g,p} \).

Definition 2.7. The **signature** of a good orbisurface with genus \( g \), \( p \) punctures, and \( r \) many cone points of orders \( e_1, ..., e_r \), is \((g; e_1, ..., e_r; p)\).

Theorem 2.8. Good orientable orbisurfaces are completely characterized by 3 pieces of data: genus \( g \), punctures \( p \), and cone points (with orders) \((e_1, ..., e_r)\). [Mar12, Theorem 1.1]

In other words, if two such orbisurfaces have the same signature, then they are homeomorphic orbifolds.

### 2.2 Group Actions and Covering Spaces

Definition 2.9. A **discrete group** is a group with discrete topology.

Definition 2.10. If \( X \) is a topological space and \( \Gamma \) is a group acting discretely on \( X \), then a **fundamental domain** for the action of \( \Gamma \) is a closed subset \( D \subset X \) such that

1. \( \cup_{\gamma \in \Gamma}(\gamma D) = X \)
2. If \( \gamma_1 \neq \gamma_2 \), then \( \gamma_1 D \cap \gamma_2 D = \phi. \)

Definition 2.11. A **covering space** of a space \( X \) is a space \( \tilde{X} \) together with a map \( p: \tilde{X} \rightarrow X \) satisfying the following condition: There exists an open cover \( \{U_\alpha\} \) of \( X \) such that for each \( \alpha \), \( p^{-1}(U_\alpha) \) is a disjoint union of open sets in \( \tilde{X} \), each of which is mapped by \( p \) homeomorphically onto \( U_\alpha \). We do not require \( p^{-1}(U_\alpha) \) to be nonempty, so \( p \) need not to be surjective.

Definition 2.12. A covering \( p: \tilde{X} \rightarrow X \) is **\( s \)-sheeted coverings** if \( p \) is surjective and there exist an open cover of \( X \), \( \{U_\alpha\} \), where \( p^{-1}(U_\alpha) \) is a disjoint union of \( s \)-many sets homeomorphic to \( U_\alpha \).

The following is a well known result regarding orbifolds. It follows from the work of [Thu97, Chapter 13] and is useful in my research.

Proposition 2.13. If \( \Gamma \) is a finite symmetry group acting on an orbisurface, then this gives a gluing map on the fundamental domain. This newly glued surface is the **quotient** and we get a covering map with \(|\Gamma|\)-sheets.
I start with building an understanding of the topology of surfaces. The topology of surfaces refers to the qualitative characteristics of a surface. When a surface covers another, this brings the opportunity for a surface to have cone points. Imagine taking a literal pole, see Figure 2.14, and intersecting this pole with a surface, a fundamental domain is rotated around this pole at a certain degree, covering the surface. The points of intersection between the pole and the surface are where the cone points are made.

**Example 2.14.** Let’s look at an example of a genus 2 surface.

Figure 2.1: A of genus 2 surface covering genus 1 surface with 2 cone points

*Here we are starting with a genus 2 surface. By placing this pole in the middle of this surface, we have two points of intersection between the pole and the surface. That*
is the top entry point and the bottom exit point. We rotate the surface 180° around this axis. The right side (green) is a fundamental domain. Since our fundamental region is rotated by 180°, our two intersection points in the quotient will be cone points with 180° around them. So in this example, a genus 2 surface covers a genus 1 surface that has two cone points that both have 180° around them. This is a 2-sheeted cover.

Going forward, in these context, I will refer to this pole around which we are rotating the surface as the axis of rotation. Looking at the example above, let’s notice where I place the axis of rotation in the figure. It is directly in the middle between the two genus. You can place the axis in more than one place. The fundamental domain must be symmetric to the surrounding region. Let’s look at an example where I place this axis in two different places.

Example 2.15. Here I look at two distinct axes of rotation for a genus 3 surface, and analyze the resulting quotient orbifolds.
Figure 2.2: Genus 3 surface with two different placements of the axis of rotation

One surface has a fundamental domain that is being rotated by $180^\circ$. This creates four cone points that each have a degree of 2. In the visual, where the pole is placed directly in the middle, the fundamental domain is rotated by $120^\circ$. The two cone points both have a degree of 3. The way to distinguish between surfaces is by the signature. A genus 3 surfaces covers two genus one surfaces with the signatures of $(1; 2, 2, 2, 2)$ and $(1; 3, 3)$. There exist a 2-sheeted cover $\phi_1 : (3; -) \to (1; 2, 2, 2, 2)$ and a 3-sheeted
The following result which is useful in my work is a well known theorem regarding surfaces. It follows from the work of [Mun18, Chapter 12].

**Theorem 2.16.** A genus $g$ surface can be formed from a $4g$-gon with appropriate edge gluings.

**Proposition 2.17.** A genus $2n + 1$ surface, where $n \geq 1$, covers a genus 3 surface.

*Proof.* Start with a genus $2n + 1$ surface. Let’s first start by placing one of the holes in the center and then the others circled around it. Looking at the figure below, we can group together the adjacent pairs of holes around the center hole. The individual pair along with the center hole is the fundamental domain (see Figure 2.3).
Figure 2.3: A genus $2n + 1$ covers a genus 3 surface
By placing an axis of rotation through the center hole, you can rotate the fundamental domain \((\frac{360}{n})^\circ\). Upon identifying the boundaries of the fundamental domain, you get a genus 3 surface.

**Proposition 2.18.** A genus \(n + 1\) surface for all \(n \geq 1\) covers a genus 2 surface.

**Proof.** If we start with a genus, \((n + 1)\) surface, we can place one hole in the center, then the others around it. Notice by putting one hole in the center of our figure, we have \(n\) surrounding it. We can then group together each hole with the center hole. This group of paired up holes is actually the fundamental domain. Now by placing the axis of rotation through the center hole, we can then rotate the fundamental domain by \(\frac{360^\circ}{n}\). So a genus \(n + 1\) surface covers a genus 2 surface.
The geometry of surfaces is the “quantitative” features of shape. To learn about the geometry of surfaces, we dive into the hyperbolic plane, the hyperbolic metric, and hyperbolic surfaces.
2.3 Hyperbolic Geometry

The upper half plane of $\mathbb{R}^2$ is a model for the hyperbolic plane. Listed below are some general definitions that can be found in [Kat92].

**Definition 2.19.** The hyperbolic plane, $\mathbb{H}^2$, is modelled by the upper half-plane $\{(x, y) \in \mathbb{R}^2 | y > 0\}$ endowed with the hyperbolic metric, $ds^2 = \frac{dx^2 + dy^2}{y^2}$.

**Definition 2.20.** If $c : [a, b] \rightarrow \mathbb{H}^2$ and $t \rightarrow (x(t), y(t))$ is a curve, its length is $\ell(c) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2} \, dt$.

**Definition 2.21.** A geodesic is a curve that locally minimized length.

**Theorem 2.22.** The geodesics in $\mathbb{H}^2$ are lines and semicircles which all are perpendicular to the $x$-axis. [Buc11, Theorem 2.7]

**Definition 2.23.** In the upper half-plane model of hyperbolic geometry, the area of a region $R$ described in Cartesian coordinates, denoted as $A(R)$ is given by

$$A(R) = \iint_R \frac{1}{y^2} \, dx \, dy.$$ 

**Definition 2.24.** A hyperbolic polygon is a polygon in $\mathbb{H}^2 \cup \partial \mathbb{H}^2$, where $\partial \mathbb{H}^2$ is the $x$-axis, whose edges are geodesics. If at least one of it vertices lie on the boundary, we say it is an ideal polygon.

In order to find the area of hyperbolic surfaces, we use hyperbolic triangles. Hyperbolic triangles are like Euclidean triangles, instead of three points being connected by line segments, the three points are connected by geodesics. We do this by using the following Gauss-Bonnet theorem:

**Theorem 2.25** (Gauss-Bonnet). If a triangle is a hyperbolic triangle with angles $\alpha, \beta, \gamma$, then the hyperbolic area of the hyperbolic triangle is $\pi - \alpha - \beta - \gamma$.

**Proof.** If a triangle has angles $\alpha, \beta, \gamma$, there are two possible cases. Case 1: If a triangle has the angles $\alpha, \beta, \gamma$ such that $\gamma$ is an ideal point. We can use transformations such as Möbius transformation discussed in Chapter 3.1 to send the ideal point to infinity and send circle $c$ to the unit circle (see Figure 2.5). To find the area of the shaded region $T$
(see Figure 2.5), then we will use integration.

\[
\int\int_T \frac{\cos \beta}{y^2} \, dx \, dy = \int_{-\cos \alpha}^{\cos \beta} \int_{-\infty}^{\infty} \frac{1}{y^2} \, dy \, dx
\]

\[
= \int_{-\cos \alpha}^{\cos \beta} \left[ \frac{-1}{y} \right]_{-\infty}^{\infty} \, dx
\]

\[
= \int_{-\cos \alpha}^{\cos \beta} \left[ \left( \frac{1}{\infty} \right) - \left( \frac{-1}{\sqrt{1-x^2}} \right) \right] \, dx
\]

\[
= \int_{-\cos \alpha}^{\cos \beta} \left( \frac{1}{\sqrt{1-x^2}} \right) \, dx
\]

\[
= \left[ \arcsin \left( \frac{\cos \beta}{\cos(\pi-\alpha)} \right) \right]
\]

\[
= \arcsin(\cos \beta) - \arcsin(\cos(\pi - \alpha))
\]

\[
= \frac{\pi}{2} - \beta(\frac{\pi}{2} - (\pi - \alpha))
\]

\[
= \frac{\pi}{2} - \beta + \frac{\pi}{2} - \alpha
\]

\[
= \pi - \alpha - \beta
\]

Figure 2.5: Triangle with ideal point
Case 2: The intersection of three geodesics gives us a hyperbolic triangle. If the triangle has no ideal points, then it looks like $T_1$ which is the area of they hyperbolic triangle we want to find. By creating another geodesic, the red geodesic in Figure 2.6, we create triangle $T_2$, which is an ideal triangle that is proved in case 1. We then use the area of $T_2$ to help us find the area of $T_1$. To find the area of triangle $T_1$, we subtract the $area(T_1 \cup T_2) - area(T_2)$ shown in Figure 2.6.

\[
area(T_1 \cup T_2) = \pi - \alpha - (\delta + \gamma) - \beta
\]

\[
area(T_2) = \pi - \alpha - \delta(\pi - \alpha)
\]

\[
area(T_1) = area(T_1 \cup T_2) - area(T_2) = \pi - \alpha - \beta - \gamma
\]

Figure 2.6: Area of triangle with no ideal points

From this theorem, we have the following consequence:

**Corollary 2.26.** If $p$ is a hyperbolic $n$-gon with interior angles $\alpha_1, \ldots, \alpha_n$ then $area(p) = (n - 2)\pi - \alpha_1 - \alpha_2 - \cdots - \alpha_n$

**Proof.** To find the area of a hyperbolic polygon, we split it into hyperbolic triangles. We do this because we know how to find the area of a hyperbolic triangle (see Theorem 2.25). We use two different strategies to do so. Say we have a polygon with $n$ number of sides.

1. Using 1 vertex the $n$-gon, we can draw line segments to the nonadjacent vertices. Doing this splits the polygon into $n - 2$ triangles. Recalling Theorem 2.25, we know the area of a triangle is $\pi - \alpha - \beta - \gamma$, where $\alpha$, $\beta$, and $\gamma$, are the angles of the triangle. To find the area of this hyperbolic polygon, we add the areas of the triangles.

\[
area(t_1) + area(t_2) + area(t_3) + area(t_4) + area(t_5) + area(t_6) + \ldots + area(t_{n-2}).
\]
Now the area is \( \text{area}(\text{polygon}) = \pi - N - O - E + \pi - P - \alpha_7 + \pi - A - \alpha_2 - G + \pi - H - I - B + \pi - C - J - K + \pi - D - L - M - \ldots \) But, \( \alpha_1 = A + B + C + D + E + F, \) \( \alpha_3 = G + H, \) \( \alpha_4 = I + J, \) \( \alpha_5 = K + L, \) \( \alpha_6 = O + P, \) \( \alpha_{(n-2)} = M + N. \) So we can rewrite this as \( \text{area}(\text{polygon}) = \pi(n - 2) - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 - \ldots - \alpha_{n-2} \)

Figure 2.7: strategy 1

2. Using 1 point in the interior of the polygon, and drawing line segments from the interior point to each vertex, we split the polygon into \( n \) triangles. Recalling Theorem 2.25, we know the area of a triangle is \( \pi - \alpha - \beta - \gamma, \) where \( \alpha, \beta, \) and \( \gamma \) are the angles of the triangle. Now the \( \text{area}(\text{polygon}) = \text{area}(T_1) + \text{area}(T_2) + \text{area}(T_3) + \text{area}(T_4) + \text{area}(T_5) + \text{area}(T_6) + \text{area}(T_7) + \ldots + \text{area}(T_n) \) So we have \( \text{area}(\text{polygon}) = \pi - S - I - J + \pi - V - K - L + \pi - W - M - N + \pi - O - A - B + \pi - P - C - D + \pi - Q - E - F + \pi - R - G - H + \ldots \) But, \( \alpha_1 = A + N, \) \( \alpha_2 = B + C, \) \( \alpha_3 = D + E, \) \( \alpha_4 = F + G, \) \( \alpha_5 = J + K, \) \( \alpha_6 = M + L, \) \( \alpha_{n} = H + I. \) We can rewrite this as \( \text{area}(\text{polygon}) = \pi(n - 2) - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 - \ldots - \alpha_{n-2} \)
Proposition 2.27. The area for a hyperbolic surface of genus $g$ is $A = 4\pi(g - 1)$.

Proof. Suppose we have a hyperbolic surface of genus $g$ with no cone points. We take surface $g$, and we can map it to its associated polygon. For example, a torus is a genus 1 surface, and it maps to a rectangle. Every genus will be a $4g$-gon. A point has $360^\circ$ or $2\pi$ radians, when opening a surface, points are "split" to "unglue" a surface. When a point is "split" these points will add up to $2\pi$. From Corollary 2.26, $(n - 2)\pi - \alpha_1 - \ldots - \alpha_n$ where $n$ is the number of sides. The polygon will have $4g$ number of sides, so we will replace $n$ with $4g$. So we have $A = (4g - 2)\pi - 2\pi$, and by factoring and adding like terms, we can rewrite this as, $A = 4\pi(g - 1)$.

Proposition 2.28. The LCM of the orders of cone points divide the number of $s$-sheeted covers.

Proof. By Definition 2.4 there exist a smooth and compact surface $S$ with boundary and $H < \text{Sym}(S)$ the symmetry group of $S$, is finite such that $S \to S/H \cong A$. Let $p$ be a preimage of a cone point. We have that $\text{stab}_H p = \{h \in H | h(p) = p\}$. The $\text{stab}_H p < H$ is a subgroup where the size of $H$ is $s$, which is the number of sheets, and the size of $\text{stab}_H p$ is the degree $e_1$ of the cone point associated with $p$. Similarly, we find cone points
with the orders of \( e_1, \ldots, e_n \). By Lagrange’s Theorem (group theory) we know the order of the subgroup \( \text{stab}_H p \) divides the order of \( H \), therefore \( e_1 \mid s \). Since \( e_i \mid s \) for each \( i \), \( \text{LCM}(e_1 \cdots e_n) \mid s \).

Understanding surfaces as polygons with edge gluings, I am now able to compute the area of hyperbolic surfaces.

**Theorem 2.29** (Riemann-Hurwitz Theorem). If \( S \) is a hyperbolic orbisurface with signature \((g; e_1, \ldots, e_n)\), and cuts open to a hyperbolic polygon \( P \), then

\[
\text{area}(S) := \text{area}(P) = 2\pi \left( 2g - 2 + \sum_{i=1}^{n} \frac{e_i - 1}{e_i} \right),
\]

with \( n \)-many cone points \( e_1, \ldots, e_n \), let \( i \) denote indices which range between 1 and \( n \), and \( g \) is the number of genus a surface has.

This formula outputs the area of surface given its genus and cone points.

**Proof.** Let’s first examine the area of a surface with 0 genus, and \( n \) cone points. We can open this surface to a polygon with \( 2n \) many edges. We first start by cutting this surface, and by opening it up we can find its polygon. Now using Corollary 2.26, we split this polygon into triangles.
Using Corollary 2.26 we know that the area of a polygon with \( n \) cone points is 
\[
(n - 2)\pi - \alpha_1 - \cdots - \alpha_n.
\]
Now to find the area of surface with \( n \) many cone points and one boundary. Let’s say we have a surface with 0 genus and cone points \( e_1, e_2, e_3 \) with one boundary. We can create a copy of our surface, and paste them together to have 1 complete surface (see Figure 2.10). We know how to find the area of this surface now that it is whole. By dividing this area by 2 we compute the area of a surface with \( n \) many cone points and one boundary. Therefore, 
\[
area = \frac{2\pi[(2n-2)-2(\frac{1}{e_1} + \cdots + \frac{1}{e_n})]}{2},
\]
which will simplify to 
\[
area = 2\pi[(n - 1) - (\frac{1}{e_1} + \cdots + \frac{1}{e_n})].
\]
Figure 2.10: Area of surface with \( n \) many cone points and one boundary

Now consider a genus \( g \) surface with one boundary component. By Proposition 2.27 the area of a genus \( g \) surface given is \( area(genus) = 4\pi(g - 1) \). Now using the same process above we have the area of a surface with genus \( g \) and one boundary is \( area(genus) = \frac{4\pi(2g-1)}{2} \) and this will simplify to \( area(genus) = 2\pi(2g - 1) \) (see Figure 2.11).

Figure 2.11: Area of surface with \( g \) genus and one boundary.

Therefore the area of a surface with \( g \) genus with \( n \) many cone points is \( Area = 2\pi(2g - 2 + n - (\frac{1}{\epsilon_1} + ... + \frac{1}{\epsilon_n})) \) (see Figure 2.12).
Using this formula, I am able to write a Sage code that gives me my own list of surfaces that could possibly be covered by a genus 2 surface. Creating my own list using Sage and hand drawn constructions gives me motivation to then produce a list of surfaces that are covered by a genus 3 surface. I go over this in the next section.

2.4 Genus 2 Surface Coverings

I begin to study what is covered by a genus 2 surface. This leads me into studying Lemma 3.2 from [Mac08]:

**Lemma 2.30.** [Mac08, Lemma 3.2] Let \( S \) be a hyperbolic orbisurface covered by a genus 2 surface. Then \( S \) has one of the following signatures: \((0; 2, 3, 7), (0; 2, 3, 8), (0; 2, 3, 9), (0; 2, 3, 10), (0; 2, 3, 12), (0; 2, 4, 5), (0; 2, 4, 6), (0; 2, 4, 8), (0; 2, 4, 12), (0; 2, 5, 5), (0; 2, 5, 6), (0; 2, 5, 10), (0; 2, 6, 6), (0; 2, 8, 8), (0; 3, 3, 4), (0; 3, 3, 5), (0; 3, 3, 6), (0; 3, 3, 9), (0; 3, 4, 4), (0; 3, 6, 6), (0; 4, 4, 4), (0; 5, 5, 5), (0; 2, 2, 2, 3), (0; 2, 2, 2, 4), (0; 2, 2, 2, 6), (0; 2, 2, 3, 3), (0; 2, 2, 4, 4), (0; 3, 3, 3, 3), and (0; 2, 2, 2, 2, 2), (0; 2, 2, 2, 2, 2), (1; 2), (1; 3), or (1; 2, 2).

This result claims to produce a list that contains all orbisurfaces which a genus 2 surface covers. While examining [Mac08, Lemma 3.2], I produce constructions of orbisurfaces which the genus 2 surface covers. In doing this, I have found one surface that is not on Macasieb’s list. However, it is unknown to me how to construct all of the surfaces that Macasieb claims are covered by a genus 2 surface.
In the following figures, I show the progression of surface coverings for a genus 2 surface. Here is an example of how read the information in the figures below. Starting with a genus 2 surface, there are three ways to place the axis of rotation. The three ways to place the axis are numbered as 1, 2, and 3. Let’s take a look at 1. If I place the axis of rotation in the middle, then the left and right side of the genus 2 surface is symmetric. Notice above the axis of rotation there is the degree of rotation. Let’s look at the shaded region on the right side of this genus 2 surface, this is the fundamental domain. For this example, the fundamental domain is rotated 180°. On the right hand side of the genus 2 surface we started with you can see the quotient with its associated signature. Notice that this new surface (1;2,2) still has symmetry. For this specific surface, we can place the axis of rotation in two different places. These two different axis of rotation are indexed as 1.a and 1.b. Let’s look at one way to find the covering for the indexed surface 1.a. The image on 1.a, shows the orbisurface with cone points, as well as the rotation transformation of 180°. This rotation transformation produces the signature (0;2,2,2,2,2). The surface corresponding to this signature is still symmetric. I have found 4 ways to place the axis of rotation on this surface, it is indexed as 1.a.1, 1.a.2, 1.a.3, and 1.a.4. I use the indexing this way to show you this was the surface found from the index 1.a.

Let’s look at 1.b. We start with the same signature as 1.a, (1;2,2), but I place the axis of rotation in a different location. 1.b produces the signature (1;2). Notice this surface still has symmetry, and it produces the signature (0;2,2,2,4), but this surface has already been covered above, so I write the indexing of where you can find the corresponding surface coverings for this surface.

If the letters in the indexing have the same “place value” that means it is the same figure with the axis of rotation being placed through the surface in a different location. If that surface has symmetry, then it is continued, and I add the number 1 following the letter to show that it is a continue to the previously used axis of rotation. The indexing is continued in this form.

Proposition 2.31. For a genus 2 surface, I have constructed all of the following signatures: (0;2,3,8), (0;2,3,9), (0;2,3,12), (2,3,18), (0;2,4,5), (0;2,4,6), (0;2,4,8), (0;2,5,10), (0;2,6,6), (0;2,8,8), (0;3,3,4), (0;3,4,4), (0;3,6,6), (0;4,4,4), (0;2,2,2,3), (0;2,2,2,4), (0;2,2,3,3), (0;2,2,4,4), (0;3,3,3), (0;2,2,2,2,2), (0;2,2,2,2,2,2), (1;2), (1;2,2).
Proof. Proof by construction.
2. b

2. c

2. d

3.
3 a

3 a 1

3 a 1 a

3 a 1 b

3 a 2

3 a 3
While constructing these signatures, I have found the signature \((0; 2, 3, 18)\) that is not listed in Macasieb’s Lemma 3.2. There are signatures that I cannot construct on Macasieb’s list, so I further my research to see if either I can rule out any signatures.
or add to my list of constructions. In doing this I also want to confirm my findings of the signature (0; 2, 3, 18). To do this, I use Sage coding to help in my research and calculations.

2.4.1 Sage Coding - Producing Signatures of Surfaces that Cover a Genus 2 Surface

Sage is a Python-based programming language. I use Sage coding throughout my research. Using the Riemann-Hurwitz Theorem (Theorem 2.29) and Proposition 2.28, I create a Sage code that produces possible signatures for a genus two surface. Since we start with a genus 2 surface, the surfaces that cover a genus 2 are a genus 1 and genus 0. This constrains possible number of cone points and degrees. From this, I produce a finite list of signatures that may be realizable. In this code, we find a range of values for cone points. From Proposition 3.23, we know the area of the quotient orbisurface divides $4\pi(g - 1)$. For a genus 2 surface, the quotient must divide $4\pi$. To find the maximum number of cone points reasoning about the fact that when covering a surface, the quotient often has higher cone points but a smaller amount of cone points. The greatest number of possible cone points comes when the surface has cone points of smallest degree which is order 2. When covering a surface, there has to be symmetry to rotate the surface around the axis of rotation. The smallest order is 2 which is $180^\circ$. We could use the function $4\pi = (\text{integer} > 1) \cdot 2\pi(-2 + r(1/2))$, where $r$ is the number of cone points. In solving we get there are 6 possible cone points for order 2. This is the process I go through to find possible signatures.

```python
for a in range(2,3):
    for b in range(a,7):
        M=[2,a,b]
        L=lcm(M)
        temp=0
        for m in M:
            temp += (m-1)/m
        if temp<=(2+2/L):
            print((2,a,b))

for a in range(2,3):
    for b in range(a,7):
        for c in range(b,100):
            M=[2,a,b,c]
```


L=lcm(M)
temp=0
for m in M:
    temp += (m-1)/m
if temp<=(2+2/L):
    print((2,a,b,c))
for a in range(2,10):
    for b in range(a,10):
        for c in range(b,100):
            for d in range(c,100):
                for e in range(d,100):
                    M=[a,b,c,d,e]
                    L=lcm(M)
                    temp=-2
                    for m in M:
                        temp += (m-1)/m
                    if temp<=(4/L) and temp>0:
                        print((a,b,c,d,e))

These codes are examples of how possible signatures can be found. Shown above is a genus zero surface with 3, 4, or 5 cone points. Macasieb’s paper [Mac08, Lemma 3.2] has a smaller list. It is an open question as to why she excludes certain signatures. With this code, I notice that all of the signatures listed on Macasieb’s paper are on mine except the signature (0; 2,5,6). I am able to exclude this signature from my list using the Riemann-Hurwitz Theorem (Theorem 2.29). Macasieb does not list the signature (0; 2,3,18), and I construct it in Proposition 2.31, and also produce it using Sage.

2.5 Genus 3 Surface Coverings

Finding the constructions of quotients of a genus 2 surface motivates me to find the constructions of a genus 3 surface. In the figures below, I show the progression of surface coverings for a genus 3 surface. Each image shows the orbisurface with cone points, as well as the degree of the rotation transformation. Each surface on the right hand side of the figures shows an arrow to the right, that arrow points to the quotient as well as the associated signature. The indexing is the same as Proposition 2.31 constructions. Notice that a genus 2 surface covers a genus 3 surface, and the constructions for the genus 2 surface is shown in Proposition 2.31. I list them here, but I do not reconstruct them.
Proposition 2.32. For a genus 3 surface, I construct all of the following signatures:

\( (0;2,3,8) \), \( (0;2,3,9) \), \( (0;2,3,12) \), \( (2,3,18) \), \( (0;2,4,5) \), \( (0;2,4,6) \), \( (0;2,4,8) \), \( (0;2,4,12) \), \( (0;2,5,5) \), \( (0;2,5,10) \), \( (0;2,6,6) \), \( (0;2,6,9) \), \( (0;2,7,14) \), \( (0;2,8,8) \), \( (0;2,12,12) \), \( (0;3,3,4) \), \( (0;3,4,4) \), \( (0;3,6,6) \), \( (0;4,4,4) \), \( (0;4,4,6) \), \( (0;2,2,2,3) \), \( (0;2,2,2,4) \), \( (0;2,2,2,6) \), \( (0;2,2,3,3) \), \( (0;2,2,4,4) \), \( (0;2,2,6,6) \), \( (0;3,3,3,3) \), \( (0;2,2,2,2,3) \), \( (0;2,2,2,2,2) \), \( (0;2,2,2,2,4) \), \( (0;2,2,2,2,6) \), \( (0;2,2,2,2,9) \), \( (0;2,2,2,2,10) \), \( (0;2,2,2,2,12) \), \( (1;2) \), \( (1;3) \), \( (1;2,2) \), \( (1;2,2,2) \), and \( (2;0) \).

Proof. Proof by construction.
2 a 1

2 a 1 a

2 a 1 b

2 a 2

2 a 3

\[(0, 2, 2.2, b)\]

\[(0, 2.4, 12)\]

\[(0, 2.3.18)\]

\[(0, 2.4, 12)\]

\[(0, 2.6, q)\]
2c. 

2c.1 

2d. 

3. 

3a. 

3b. 

\((1, 3)\) 

\((0, 2, 2, 2, 2)\) 

\((1, 1, 2, 2, 2)\) 

\((1, 2, 2)\) 

\((0, 2, 2, 2, 4, 4)\)
The above constructions show what a genus 3 surface covers. Notice that a genus 2 surface covers a genus 3 surface, see Proposition 2.31 for genus 2 constructions. I will now use Sage coding to see if there could be any signatures that I could not construct.
2.5.1 Sage Coding - Producing Signatures of Surfaces that Cover a Genus 3 Surface

Using Sage, I created code that will produce a list of signatures which can be covered by a genus 3 surface. This code is created using the same reasoning as Section 2.4.1, but a genus 3 surface can be covered by a genus 2, genus 1, and a genus 0.

for a in range(2,10):
    for b in range(a,10):
        for c in range(b,100):
            M=[a,b,c]
            L=lcm(M)
            temp=-2
            for m in M:
                temp += (m-1)/m
            if temp<=(4/L) and temp>0:
                print((a,b,c))

for a in range(2,10):
    for b in range(a,10):
        for c in range(b,100):
            for d in range(c,100):
                M=[a,b,c,d]
                L=lcm(M)
                temp=-2
                for m in M:
                    temp += (m-1)/m
                if temp<=(4/L) and temp>0:
                    print((a,b,c,d))

for a in range(2,10):
    for b in range(a,10):
        for c in range(b,100):
            for d in range(c,100):
                for e in range(d,100):
                    M=[a,b,c,d,e]
                    L=lcm(M)
                    temp=-2
                    for m in M:
                        temp += (m-1)/m
                    if temp<=(4/L) and temp>0:
                        print((a,b,c,d,e))

This Sage code produces the possible signatures of a genus 0 surface that has 3, 4, or 5 cone points.
Proposition 2.33. Using the Riemann-Hurwitz Theorem, for a genus 3 surface I produce the following list of possible surface coverings: (0; 2, 3, 7), (0; 2, 3, 8), (0; 2, 3, 9), (0; 2, 3, 10), (0; 2, 3, 12), (0; 2, 3, 14), (0; 2, 3, 15), (0; 2, 3, 18), (0; 2, 3, 24), (0; 2, 3, 30), (0; 2, 4, 5), (0; 2, 4, 6), (0; 2, 4, 7), (0; 2, 4, 8), (0; 2, 4, 10), (0; 2, 4, 12), (0; 2, 4, 16), (0; 2, 4, 20), (0; 2, 5, 5), (0; 2, 5, 6), (0; 2, 5, 10), (0; 2, 6, 6), (0; 2, 6, 9), (0; 2, 6, 12), (0; 2, 7, 7), (0; 2, 7, 14), (0; 2, 8, 8), (0; 3, 3, 4), (0; 3, 3, 5), (0; 3, 3, 6), (0; 3, 3, 7), (0; 3, 3, 9), (0; 3, 3, 12), (0; 3, 3, 15), (0; 3, 4, 4), (0; 3, 4, 6), (0; 3, 4, 12), (0; 3, 5, 5), (0; 3, 6, 6), (0; 3, 9, 9), (0; 4, 4, 4), (0; 4, 4, 6), (0; 4, 4, 8), (0; 4, 8, 8), (0; 5, 5, 5), (0; 6, 6, 6), (0; 7, 7, 7), (0; 2, 2, 2, 3), (0; 2, 2, 2, 4), (0; 2, 2, 2, 5), (0; 2, 2, 6, 6), (0; 2, 3, 3, 3), (0; 2, 3, 3, 6), (0; 2, 4, 4, 4), (0; 3, 3, 3, 3), (0; 4, 4, 4, 4), (0; 2, 2, 2, 2, 2), (0; 2, 2, 2, 2, 3), (0; 2, 2, 2, 2, 4), (0; 2, 2, 2, 4, 4), (0; 3, 3, 3, 3, 3).

I produce this list using the Riemann-Hurwitz Theorem and Sage, but there maybe signatures that cannot be realized. The signatures that are constructed in Proposition 2.32 are proven to be actual signatures for surfaces that are covered by a genus 3 surface. It is an open question as to how to certify that the remaining ones on Proposition 2.33 that I do not list in Proposition 2.32.

Proposition 2.34. The list of genus 3 surface coverings found in Proposition 2.33, but can not be constructed in Proposition 2.32 is: (0; 2, 3, 7), (0; 2, 3, 10), (0; 2, 3, 14), (0; 2, 3, 15), (0; 2, 3, 24), (0; 2, 3, 30), (0; 2, 4, 7), (0; 2, 4, 10), (0; 2, 4, 12), (0; 2, 4, 16), (0; 2, 4, 20), (0; 2, 5, 5), (0; 2, 5, 6), (0; 2, 5, 10), (0; 2, 6, 6), (0; 2, 6, 9), (0; 2, 6, 12), (0; 2, 7, 7), (0; 2, 7, 14), (0; 2, 8, 8), (0; 3, 3, 4), (0; 3, 3, 5), (0; 3, 3, 6), (0; 3, 3, 7), (0; 3, 3, 9), (0; 3, 3, 12), (0; 3, 3, 15), (0; 3, 4, 4), (0; 3, 4, 6), (0; 3, 4, 12), (0; 3, 5, 5), (0; 3, 6, 6), (0; 3, 9, 9), (0; 4, 4, 4), (0; 4, 4, 6), (0; 4, 4, 8), (0; 4, 8, 8), (0; 5, 5, 5), (0; 6, 6, 6), (0; 7, 7, 7), (0; 2, 2, 2, 3), (0; 2, 2, 2, 4), (0; 2, 2, 2, 5), (0; 2, 2, 6, 6), (0; 2, 3, 3, 3), (0; 2, 3, 3, 6), (0; 2, 4, 4, 4), (0; 3, 3, 3, 3), (0; 4, 4, 4, 4), (0; 2, 2, 2, 2, 2), (0; 2, 2, 2, 2, 3), (0; 2, 2, 2, 2, 4), (0; 2, 2, 2, 4, 4), (0; 3, 3, 3, 3, 3).
Chapter 3

Hyperbolic Transformations

3.1 Möbius Transformations

A question I ask myself is: what is the difference between a surface and a hyperbolic surface? Surfaces are built from polygons along with their associated gluing maps. A hyperbolic surface is built from hyperbolic polygons along with their associated gluing maps. A hyperbolic polygon has edges that are geodesics. The gluing maps for hyperbolic polygons are isometries of $\mathbb{H}^2$. Listed below are general definitions that can be found in [Hat01].

**Definition 3.1.** An *isometry* of $\mathbb{H}^2$ is a transformation $T : \mathbb{H}^2 \to \mathbb{H}^2$, that preserves the hyperbolic metric.

In my research, I use Möbius transformations of $\mathbb{H}^2$, denoted as Möb($\mathbb{H}^2$).

**Definition 3.2.** A *Möbius transformation* is a transformation $T : \mathbb{H}^2 \to \mathbb{H}^2$, $z \to \frac{az+b}{cz+d}$, for any real numbers $a, b, c, d$ such that $ad - bc \neq 0$.

**Theorem 3.3.** [Leh14, Proposition 5] Möbius transformations preserve angles.

**Theorem 3.4.** [Kat92, Theorem 1.3.1] Möb($\mathbb{H}^2$) $\cong$ PSL$_2(\mathbb{R})$

**Theorem 3.5.** [Bon55, Proposition 2.18] Möbius transformations send circles and lines to circles and lines.

Let $Isom(\mathbb{H}^2)$ be the set of all isometries, and $Isom^+(\mathbb{H}^2)$ be the subset of all orientation preserving isometries.
**Theorem 3.6.** [Bon55, Theorem 2.11] If $T \in \text{Mob}(\mathbb{H}^2)$ then $T \in \text{Isom}(\mathbb{H}^2)$. Furthermore, $\text{Isom}^+(\mathbb{H}^2) \cong \text{Mob}(\mathbb{R}^2)$.

An example of an isometry is the identity map, translations, rotations, and reflections. In my research, I use Möbius transformations for the edge gluings of my hyperbolic polygon. For convenience, I work in $SL_2(\mathbb{R})$ and I keep in mind, $A^u = -A$ of $\text{Mob}(\mathbb{H}^2)$.

**Example 3.7.** Möbius transformations are

1. $z \rightarrow z + b$ given by matrix \[
\begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix},
\] which is a parabolic transformation;

2. $z \rightarrow \left(\frac{1}{z}\right)$ given by matrix \[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix},
\] which is an elliptic transformation;

3. $z \rightarrow a^2z$ given by matrix \[
\begin{pmatrix}
a & 0 \\
0 & a^{-1}
\end{pmatrix},
\] this is a hyperbolic transformation.

Let $A \in SL_2(\mathbb{R})$.

**Definition 3.8.** The **discriminate** of $A$ is $\text{disc}(A) = b^2 - 4ac$.

**Definition 3.9.** The **trace** of $A$ is $\text{tr}(A) = a + d$.

**Proposition 3.10.** [Kat92, Section 2] There are three fundamentally distinct types of Möbius transformations. Let $A \in SL_2(\mathbb{R})$ and let $T_A$ denote the associated Möbius transformation.

1. $A$ is hyperbolic:
   - $\text{disc}(A) > 0$
   - $|\text{tr}(A)| > 2$
   - $A$ is diagonalizable
   - $T_A$ has precisely two distinct fixed points in $\mathbb{H}^2$, both of which lie on the boundary.

2. $A$ is parabolic:
• $\text{disc}(A) = 0$
• $|\text{tr}(A)| < 2$
• $A$ is triagonalizable
• $T_A$ has precisely one fixed point in $\mathbb{H}^2$, and it lies on the boundary.

3. $A$ is elliptic:
• $\text{disc}(A) < 0$
• $|\text{tr}(A)| < 2$
• $A$ is rotationalizable
• $T_A$ has precisely one fixed point in $\mathbb{H}^2$, and it lies in $\mathbb{H}^2$

3.2 Circle Inversions

For my research, I use Möbius transformations to find the perpendicular bisector between two points. In this process, I use two types of inversions, inverting a circle over another circle, and inverting a inscribed circle over the circle it is inscribed in. Let’s first start with what is an inversion. Below I list general definitions from [Nee97].

Definition 3.11. A circular inversion is a transformation where point $A$ in the Cartesian plane is transformed based on a circle $\alpha$ with radius $k$ and center $O$ such that $OA \cdot OA' = k^2$.

Figure 3.1: circle inversion
Proposition 3.12. [Nee97, Theorem 3.2 (5)] If inversion in a circle is centered at $q$ maps two points $a$ and $b$ to $a'$ and $b'$, then the triangles $aqb$ and $b'q'a'$ are similar.

Figure 3.2: Similar triangles under circle inversion

Theorem 3.13. [Nee97, Theorem 3.2 (7)] If a line $L$ does not pass through the centre $q$ of $K$, then inversion in $K$ maps $L$ to a circle that passes through $q$.

Figure 3.3: Circle inversion maps circles to lines

Theorem 3.14. [Nee97, Theorem 3.2 (9)] If a circle $C$ does not pass through the centre
$q$ of $K$, then inversion in $K$ maps $C$ to another circle not passing through $q$.

Figure 3.4: Circle inversion maps circle to circles

**Theorem 3.15.** [Nee97, Theorem 3.2 (10)] Under inversion in $K$, every circle orthogonal to $K$ is mapped to itself.

Figure 3.5: Circle inversion circle maps to itself
Theorem 3.16. [Kat92, Theorem 1.3.1] Circle inversion centered on the $x$-axis are orientation-reversing hyperbolic isometries.

3.3 Discrete Subgroups

We want to find the Dirichlet domain (see Section 3.4) because this gives us my end goal of finding a hyperbolic polygon to build a hyperbolic surface. In order to find a Dirichlet domain, we use discrete subgroups of $SL_2\mathbb{R}$, which we are thinking of as gluing isometries of $\mathbb{H}^2$. In topology, a discrete subset of a space is one which you can separate all points with disjoint open balls. Below I list general definitions from [Kat92].

Definition 3.17. If $\Gamma \subset SL_2\mathbb{R}$ is a subgroup which is discrete relative to the subspace topology, it is a **discrete subgroup**.

Definition 3.18. If $\Gamma$ is a group acting on a set $X$, and $x \in X$, then the **orbit** of $x$ is the set $\Gamma x = \{g \cdot x | g \in \Gamma\}$, the full set of objects $x$ is sent to under the action of $\Gamma x$.

Definition 3.19. The action of a subgroup, $\Gamma \subset SL_2(\mathbb{R})$ is **properly discontinuous**, if for any $x \in \mathbb{H}^2$, and any closed and bounded $K \subset \mathbb{H}^2$, $K \cap \Gamma x$ is finite.

Definition 3.20. A **Fuchsian group** is a discrete subgroup of $PSL_2(\mathbb{R})$.

Proposition 3.21. [Kat92, Corollary 8.7] $\Gamma$ is discrete if and only if its action on $\mathbb{H}^2$ is properly discontinuous.

Proposition 3.22. [Buc11, Theorem 3.2] Fuchsian groups act properly discontinuously on the hyperbolic plane $\mathbb{H}^2$.

Proposition 3.23. [Kat92, Section 3.6] Let $\Gamma$ be a Fuchsian group, and $F$ be a fundamental domain, then the quotient $\mathbb{H}^2/\Gamma \cong F/\Gamma$ and the area($\mathbb{H}^2/\Gamma$) := area($F$).

Discrete subgroups will help us to find the associated Dirichlet domain.

3.4 Dirichlet Domains

In order to find a hyperbolic polygon, I construct a Dirichlet domain. Dirichlet domains are an explicit process for producing fundamental domains for the action of discrete subgroups on $\mathbb{H}^2$. The following result, which is useful in my work, is a well known theorem on Dirichlet domains [Kat92, Section 3.2].
Theorem 3.24. The process of finding a Dirichlet domain is:

Step 1: Fix a point $x_0 \in \mathbb{H}^2$ to be the center of the domain;

Step 2: Compute the orbit $\Gamma x_0$;

Step 3: For each $x_1 \in \Gamma x_0$, find the perpendicular bisector between $x_0x_1$; and

Step 4: The domain, $D$, is the intersection of all half-planes containing $x_0$.

For Fuchsian groups, a Dirichlet domain only has finite sides. Therefore, I do not need the full orbit, just a well-chosen finite subset. This is fortunate, as it is impossible to construct the infinite orbit, but I can construct a well-chosen subset, which I will call my orbit sampling.

Theorem 3.25. [Kat92, Theorem 11.8] Let $\{T_i\}$ be the subset of $\Gamma$ consisting of those elements which pair the sides of some fixed Dirichlet region $F$. Then $\{T_i\}$ is a set of generators for $\Gamma$. 
Chapter 4

Arithmetic Constructions

4.1 Quadratic Extensions

For my research, I produce a Dirichlet domain. To do this, we use a discrete subgroup. A discrete subgroup tessellates the plane with the Dirichlet domain, i.e. translates of out domain will not overlap. In order to get a discrete subgroup, we use quaternion algebras. To help in my understanding of quaternion algebras I have to gain an understanding of quadratic extensions. Listed below are general definitions that can be found in [Voi97].

Definition 4.1. A quadratic extension of \( \mathbb{Q} \) is the algebra \( A = \mathbb{Q}[a] \) where \( a^2 = n \), \( n \) is square-free integer.

Quadratic field extensions have embeddings into \( \mathbb{R} \) and \( \mathbb{C} \). In order to know where the quadratic field extension is embedded, we look at the mapping. Does the polynomial have distinct roots, complex conjugate roots. For example, let \( \mathbb{Q}[a] = \mathbb{Q}[x]/(x^2 - 2) \), where \( a \to \sqrt{2} \) and \( a \to -\sqrt{2} \). \( \mathbb{Q}[a] \) has two real roots, both of which are embedded in \( \mathbb{R} \). Since we are working with quadratic field extensions, then we have two embeddings.

Proposition 4.2. An element in \( \mathbb{Q}[a] \) is of the form \( x + ya \) where \( x, y \in \mathbb{Q} \). \( \mathbb{Q}(a) \simeq \left\{ \begin{pmatrix} x & ny \\ y & x \end{pmatrix} \right\} \subset \text{Mat}_{2 \times 2}(\mathbb{Q}) \)

Proof. Let \( x, y \in \mathbb{Q} \), where \( x + ya \in \mathbb{Q}[a] \) is an element. Let \( A = \begin{pmatrix} 0 & n \\ 1 & 0 \end{pmatrix} \) and \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).
So \( \begin{pmatrix} x & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & n \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} x & ny \\ y & x \end{pmatrix} \mid x, y \in \mathbb{Q} \) \( \subset \text{Mat}_{2 \times 2}(\mathbb{Q}) \). \hfill \Box

**Theorem 4.3.** If \( n \) is square free and \( x, y \in \mathbb{Z} \) such that they are primitive solution to Pell’s equation, i.e. \( x^2 - ny^2 = 1 \), then:

1. The matrix corresponding to Pell’s equation \( \begin{pmatrix} x & ny \\ y & x \end{pmatrix} \) is a hyperbolic matrix.

2. The fixed points under a Möbius transformation are \(-\sqrt{n}, \sqrt{n}\).

**Proof.** Proof of 1: Let \( n \) be square free and \( x, y \in \mathbb{Z} \). The trace of \( \begin{pmatrix} x & ny \\ y & x \end{pmatrix} \) is \( 2x \). This matrix corresponds to Pell’s equation, where \( x = 1 \) is not a possible solution because if \( x = 1 \), then we would have \(-ny^2 = 0 \) and \( n \) must be a square free number. So \( x \geq 2 \), and \( \text{tr} \left( \begin{pmatrix} x & ny \\ y & x \end{pmatrix} \right) \geq 2 \). By Proposition 3.10, this is a hyperbolic matrix.

Proof of 2: Let \( n \) square free and \( x, y \in \mathbb{Z} \) then \( x^2 - ny^2 = 1 \). We have \( x^2 = 1 - ny^2 \). Then \( \begin{pmatrix} x & ny \\ y & x \end{pmatrix} \cdot r = r \). Then it can be written as a Möbius transformation (See Definition 3.2). So we have \( \frac{xr + ny}{yr + x} = r \). Now we will solve for the roots, so \( xr + ny = yr^2 + xr \). Now we will add like terms and have \( yr^2 - ny = 0 \). Now \( yr^2 = ny \). So \( r = \sqrt{n} \) and \( r = -\sqrt{n} \). \hfill \Box

**Definition 4.4.** The **ring of integers** of a \( \mathbb{Q}[a] \) is the subset \( R \) of \( \mathbb{Q}[a] \) consisting of elements that satisfy monic polynomials over \( \mathbb{Z} \).

### 4.2 Quaternion Algebras

Listed below are general definitions that can be found in [Voi97].

**Definition 4.5.** A **quaternion algebra** over \( \mathbb{Q} \) is the algebra \( A = \mathbb{Q}[i, j, k] \), where \( a, b \in \mathbb{Q} \) \( i^2 = a, j^2 = b, \) and \( ij = -ji \). This is denoted with the symbol \( A = (a, b) \)

For this project, restrict to the case \( a, b > 0 \). Quaternion algebras are like quadratic field extensions, it is the adjoining of two symbols, but they anti-commute. The maximal orders that come from quaternion algebras give us the discrete subgroup needed to find the Dirichlet domain.
Definition 4.6. The standard embedding I will use is \( \rho : A \to SL_2(\mathbb{Q}[\sqrt{a}]) \) such that
\[
\rho : 1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\quad j \mapsto \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix},
\quad i \mapsto \begin{bmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{bmatrix},
\quad k \mapsto \begin{bmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{bmatrix} \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix},
\]
and extend linearly.

Theorem 4.7. [Kat92, Theorem 5.2.7] If \( O \subset A \) is an order, and \( \rho : A \to \text{Mat}_{2 \times 2}(\mathbb{R}) \), then \( \rho(O) \) is discrete.

We use the maximal order of \( O \) to then find the norm 1 elements, which will help us to find the quaternion algebras. Below is a list of definitions from [Voi97].

Definition 4.8. An order \( O \subset A \) is maximal if it is not properly contained in another order.

Theorem 4.9. [Voi97, Proposition 2.3.1] A quaternion algebra \( (a, b) \) is either a division algebra or isomorphic to the matrix algebra of \( 2 \times 2 \) matrices over \( \mathbb{R} \).

Definition 4.10. If \( t + xi + yj + zk \in A \), then the norm is:
\[
Nrm(t + xi + yj + zk) = t^2 - ax^2 - by^2 + abz^2.
\]

Theorem 4.11. [Kat92, Equation 5.2.11] If we have the embedding \( \rho : A \to \text{Mat}_{2 \times 2}(\mathbb{Q}[\sqrt{a}]) \), and \( a \in A \), then \( Nrm(a) = \det\rho(a) \).

Theorem 4.12. [Kat92, Theorem 5.2.7] If \( A \) is a quaternion algebra \((a, b)\) over \( \mathbb{Q} \), and \( O \subset A \) is a maximal order, and \( O^{(1)} \subset O \) is the set of norm one elements, then \( \Gamma := \rho(O^{(1)}) \subset SL_2(\mathbb{R}) \) is a discrete subgroup.

Definition 4.13. A group of this form is a principal arithmetic Fuchsian subgroup.

Theorem 4.14. [Bor81, Arithmetic Area Formula Theorem 7.3] If \( \Gamma \) is as in Theorem 4.12, then the area of \( \mathbb{H}^2/\Gamma \) is:
\[
\text{area}(\mathbb{H}^2/\Gamma) = \frac{\pi}{3} \prod_{p \in \triangle A} (p - 1),
\]
where \( \triangle A \) is the set of primes for which \( A \) ramifies.

Theorem 4.14 is very technical and powerful. The proof of this theorem is beyond the scope of the thesis. The Riemann-Hurwitz Theorem (Theorem 2.29) can be constructed by finding the area of a surface by breaking it into sections (See Figure 2.12). Unlike the Riemann-Hurwitz Theorem, the Arithmetic Area Formula is an arithmetic formula that does not require the qualitative aspects of a surface to find its area. These three area functions are from completely different perspectives.
Chapter 5

The Research Process

5.1 The Quaternion Algebra

The goal of my research is to produce a hyperbolic surface associated with the maximal order in the quaternion division algebra $(3,5)$ over $\mathbb{Q}$. To do this, I find the Dirichlet domain. I center my Dirichlet domain at the point $(0,2)$. I find the quaternion algebra, which helps me to find matrices that transform the point $(0,2)$. A list of sufficiently many transformed points is my orbit sampling. In order to find the orbit sampling, I first find the quaternion algebra by adjoining $(3,5)$. I use Sage to help me find the maximal order, and the norm 1 elements. This produces a list of matrices. These matrices transform the point $(0,2)$.

```python
A.<i,j,k> = QuaternionAlgebra(3,5)
O = A.maximal_order()

#brute force (Not Optimal)
norm_one_elements = []
r = 10
for a in range(-r,r):
    for b in range(-r,r):
        for c in range(-r,r):
            for d in range(-r,r):
                o = a*w+b*x+c*y+d*z
                if o.reduced_norm()==1:
                    norm_one_elements.append(o)
```

From the norm 1 elements, we produce matrices.
# turns norm one elements to a matrix

def turn_order_to_matrix(o):
    R.<x> = PolynomialRing(QQ)
    K.<d> = NumberField(x^2-3)
    Id = matrix(K, 2, 2, [1, 0, 0, 1])
    I = matrix(K, 2, 2, [d, 0, 0, -d])
    J = matrix(K, 2, 2, [0, 5, 1, 0])
    K = I*J
    Matrix_Gens = [Id, I, J, K]
    M = sum([list(o)[n]*Matrix_Gens[n] for n in range(4)])
    return M

def matrix_rounded(U):
    U_rounded = Matrix(2, 2, [sigma(u) for u in U.list()])
    return U_rounded

Listed above, I have exact and rounded matrices. The exact matrices include the $\sqrt{3}$, $\sqrt{5}$. The rounded matrices, rounds the $\sqrt{3}$, $\sqrt{5}$, and matrix with 4 rounded decimal entries.

Once we have the matrices produced from the function above, we use these matrices to create M"obius transformations.

# take the founded matrices, and using the point (0, 2) or complex number 2i to find where matrix takes 2i.

def mobius(B, P0=2*sqrt(-1)):
    N1 = (B[0][0]*P0+B[0][1])/(B[1][0]*P0+B[1][1])
    return N1

5.2 Orbit

Using the M"obius transformations found from the quaternion algebra we can now create an orbit sampling by applying these transformations to the point (0, 2)

orbit = []
for o in norm_one_elements:
    B = turn_order_to_matrix(o)
    C = matrix_rounded(B)
    P1 = mobius(C)
    orbit.append(CC(P1))
5.3 Finding the Dirichlet Domain

To find the Dirichlet domain, I have to find the perpendicular bisectors of the geodesics that go through the point (0, 2) and the points in my orbit sampling.

Proposition 5.1. Given 3 noncolinear points, I can construct a circle through them.

Proof. Let $A = (x_0, y_0)$, $B = (x_1, y_1)$, and $A' = (x_0, -y_0)$ be 3 noncolinear points, by drawing line segments between them we can construct a triangle.
The intersection of the Euclidean perpendicular bisectors for each edge is the circumcenter. To algebraically find the circumcenter, we create two of the euclidean perpendicular bisector lines and solve for the intersection. Let’s find the euclidean perpendicular bisector between segment $AB$, let’s call the line that goes through points $AB$ $f(x)$, and let’s call the euclidean perpendicular bisector of $AB$ $g(x)$. To find $g(x)$, we find the average the midpoint between points $A, B$. To do this, we take the average between the $x$ and $y$ values for points $A, B$, so we have $a = \frac{x_1 + x_0}{2}$ and $b = \frac{y_1 + y_0}{2}$. $g(x)$ has a slope that is the opposite reciprocal of $g(x)$. So the slope of $f(x)$ is $M = \frac{y_1 - y_0}{x_1 - x_0}$. Now the slope of $g(x)$ is $M_{g(x)} = \frac{-1}{M}$. so $f(x) = m(x - a) + b$ and $g(x) = \frac{-1}{m}(x - a) + b$. Now we need another line to intersect $g(x)$ with to solve for the circumcenter. Let’s find the Euclidean perpendicular bisector for the line that goes through the points $AA'$. Since line $AA'$ is a vertical line, the Euclidean midpoint is just the average of the $y$ values, but since $A' = (x_0, -y_0)$, the midpoint is the $x - axis$. The Euclidean perpendicular bisector is $y = 0$. We set it equal to $g(x)$ and solve for the $x$ value of the intersection, which I call $x_2$. In doing this, we have $x_2 = mb + a$. 

![Figure 5.2: 3 points create a triangle](image)
To find the radius, we just find the distance from $(x_2, 0)$ and point $A$. So $r = \sqrt{(x_0 - x_2)^2 + (y_0)^2}$.

**Proposition 5.2.** Given two points $A, B \in \mathbb{H}^2$, we can construct a circle that goes through these two points.

**Proof.** Let $A, B \in \mathbb{H}^2$ such that $A = (x_0, y_0)$ and $B = (x_1, y_1)$. There are four possible
cases we encounter in the creation of the circle, \( c \), that goes through points \( A \) and \( B \).

Case 1: \( x_0 = x_1 \) and \( y_0 = y_1 \). This is the case where the points are the same. So there is no circle that goes through the two points. Case 2: \( x_0 = x_1 \). This is the case where the points are vertical. The geodesic that goes through these points is a vertical line that is centered at \((x_0, 0)\) with a radius of \(\infty\).

Case 3: \( y_0 = y_1 \) and \( y_0 = 0 \). This is the case where the points are on the boundary. When this is the case, to find the center of the circle between points \( A, B \) we take the average between the \( x \)-values, I call this \( x \)-value \( a \). The radius is the distance from the center of the circle to one of \( x \)-values. So the center of circle \( c \) is found from \(|(a - x_0)|\).
Case 4: This is the case where point $A = (x_0, y_0)$ and $B = (x_1, y_1)$ are two distinct points. If two points are not equal, then by reflecting one of the two points over the $x$-axis, we can construct a unique circle through these three noncolinear points. By Proposition 5.1 we create a circle centered at $(x_2, 0)$ with a radius of $r = \sqrt{(x_0 - x_2)^2 + (y_0)^2}$ (see Figure 5.4 for visual).

This is shown in the code below. This code is classical and well known.

```python
def hyperbolic_geodesic_circle(P0, P1):
    # given two points, this function finds the geodesic
    x_0 = P0[0]
    y_0 = P0[1]
    x_1 = P1[0]
    y_1 = P1[1]
    if y_0 == y_1 and x_0 == x_1:
        print("same point")
        # single equals is an assignment
        return((0,0),0)
    elif x_0 == x_1:
        print("same x’s but different y’s")
        r = "infinity"
        return ((x_0,0), r)
    elif y_0 == y_1 and y_0 == 0:
        print("both points on boundary")
        a = (x_1+x_0)/2
        center=(a,0)
        radius=abs(a-x_0)
        return(center,radius)
```
else:
    print("this is the generic case")
a=(x_1+x_0)/2
b=(y_1+y_0)/2
m = (y_1-y_0)/(x_1-x_0)
#f(x) = m*(x-a) + b #equation of line through starting points
#g(x) = -1/m*(x-a) + b #
x_2=m*b+a
r = sqrt((x_0-x_2)^2+(y_0)^2)
return ((x_2,0),r)

Figure 5.7 Below shows an example of the geodesic between the point (0, 2), and (3, 3).

![Figure 5.7: segment AB](image)

We will find the perpendicular bisector of segment $AB$. I create a function that uses a circle inversion to take this geodesic curve to a vertical line. To do this, I start by creating a circle, $d$, that goes through point $A$ and has a center that is the left endpoint of circle $C$. See the Figure 5.8 below.
To create circle $d$, we first find the left endpoints for circle $c$. To do this, I create a function that inputs the center and radius of the circle, and outputs the endpoints. Since a geodesic circle is perpendicular to the $x$-axis, our endpoints will be on the hyperbolic boundary.

```python
def circle_endpoints(P0,r):
    x0 = P0[0]
    y0 = P0[1]
    L0 = x0 - r
    R0 = x0 + r
    return ((L0,0),(R0,0),r)
```

I use the left endpoint of circle $c$.

```python
def radius_of_circle_d(P0,P1): #radius of circle d
    x0 = P0[0]
    y0 = P0[1]
    x1 = P1[0]
    y1 = P1[1]
    r1 = sqrt((x0 - x1)^2 + (y0 - y1)^2)
    return (P0,r1)
```

The purpose in the creation of circle $d$ is when I compute a circle inversion transformation of circle $c$ being reflected over circle $d$, this moves circle $c$ to the $y$-axis.
Proposition 5.3. If there is a hyperbolic circle inversion fixing $A$ that takes circle $c$ to the vertical line $\ell$ going through $A$, then point $B$ on circle $c$ goes to point $D$, where $D$ is the intersection of $\ell$ with the vertical axis.

Proof. By Proposition 5.2, I compute the center and radius of $c$ which I denote $center = (P[0], P[1])$, radius $= r$. To find where point $B$ has moved, we create a line that goes through the center of circle $d$, and point $B$. The intersection of this line along with the $y$-axis gives us the location of the image of $B$ under this transformation. In Figure 5.9 below, this is point $D$. Notice that segment $AB$ has moved to segment $AD$. Using the left endpoint of circle $c$ which I denote as $L$ where $L = P[0] - r$, we create a circle, $d$, that goes through point $A$. The inversion of circle $c$ over circle $d$ will move circle $c$ to a vertical line that goes through points $AA'$ (See Theorem 3.5). By Definition 3.11 to find where $B$ has moved, we can draw a line from $L$ to point $B$. The intersection of line through $LB$ which I will call $f(x)$, and the line through $AA'$ which I will call $g(x)$, is the transformed $B$, called point $D$ in Figure 5.9. The slope for $f(x)$ is $m = \frac{(0-y_1)}{(L-x_1)}$, and using point $B$, we use the point slope formula to create a line. So $f(x) = m(x - x_1) + y_1$, now $g(x)$ is the line through $AA'$, but lets recall from Proposition 5.2 that $A$ and $A'$ both have the $x$-value of $x_0$. so $g(x) = x_0$, and the intersection is $y_2 = m(x_0 - x_1) + y_1$. We have $D = (x_0, y_2)$.

Figure 5.9: Segment AD

I create a Sage function, that computes Proposition 5.3, the circle inversion that transforms point $B$ to point $D$ shown in Figure 5.9.
def circle_inversion1(P0,P1):
    (P,r) = hyperbolic_geodesic_circle(P0,P1)
    x_0 = P0[0]
    y_0 = P0[1]
    x_1 = P1[0]
    y_1 = P1[1]
    if r == "infinity":
        return ((x_0,0),r)
    else:
        L = P[0]-r
        m = (0-y_1)/(L-x_1)
        #f(x)=m(x-x_1)+y_1
        #g(x)=x_0
        y_2 = m*(x_0-x_1)+y_1
        return(x_0,y_2)

We now have the transformed segment $AD$, and the next step is to find the perpendicular bisector of this vertical segment.

**Proposition 5.4.** If there exist a circle $e$ with center $F = (x_0,0)$, and radius $r$, then $r^2 = FD \cdot FA$.

*Proof.* Let $A = (x_0,y_0)$, $D = (x_1,y_1)$, and $F = (0,0)$ using the Definition 3.11 we create a circle centered at the origin where the length of the radius takes point $D$, which in Figure 5.10 is inside of circle $e$ to the point $A$, which is outside of circle $e$. Similarly for $A$ being the inversion of $D$. So $r = \sqrt{|y_0 \cdot y_1 - y_0|}$.
In Sage, I create a function for the circle inversion that sends $FD$ to $FA$ by creating circle $e$.

```python
def perp_bisectors_segment_h(P_0, P_1):
    x0 = P_0[0]
    y0 = P_0[1]
    x1 = P_1[0]
    y1 = P_1[1]
    radius_F = sqrt(abs(y0*y1-y0))
    return((0,0),radius_F)
```

**Proposition 5.5.** Inversion of circle $e$ over circle $d$, takes $e$ to circle $e'$

**Proof.** Let circle $d$ have the center $(x_0, y_0)$ with radius $r$ and circle $e$ has the left endpoint $(x_1, y_1)$ and the right endpoint $(x_2, y_2)$. By Definition 3.11, we know the left endpoint goes to the transformed right endpoint, $TR$ (See Theorem 3.14). If $x_0 < x_1$, then $TR = \left(\frac{r^2}{|x_0-x_1|}\right) + x_0$, and if $x_0 \geq x_1$, then $TR = -\left(\frac{r^2}{|x_0-x_1|}\right) + x_0$. The right endpoint goes to the transformed left endpoint, $TL$. If $x_0 < x_2$, then $TL = \left(\frac{r^2}{|x_0-x_2|}\right) + x_0$, and if $x_0 \geq x_2$, then $TL = -\left(\frac{r^2}{|x_0-x_2|}\right) + x_0$. Now once you have the transformed left and right points, using Proposition 5.2 we find the geodesic between two points, this gives us $e'$ (See Figure 5.11).

![Figure 5.11: Perpendicular bisector of segment AB](image)

Using Proposition 5.5, I create a Sage function that finds the endpoints of circle $e'$. 
def perpbisector_e_transformed(P0, P1, P2, r):
    # P0 is the center of the circle d
    x0 = P0[0]
y0 = P0[1]
    # P1 is the left endpoint of circle e
    x1 = P1[0]
y1 = P1[1]
    # P2 is the right endpoint of circle e
    x2 = P2[0]
y2 = P2[1]
    # TL is transformed left point from inversion 2
    if x0 < x1:
        TR = (r^2/(abs(x0-x1)))+x0
    else:
        TR = -(r^2/(abs(x0-x1)))+x0
    # TR is transformed right point from inversion 2
    if x0 < x2:
        TL = (r^2/(abs(x0-x2)))+x0
    else:
        TL = -(r^2/(abs(x0-x2)))+x0
    return ((TL, 0), (TR, 0))

The created function above requires lots of information: circle d’s center and radius, and circle e’s endpoints. I loop it all together to keep track of the orbit point with its associated perpendicular bisector, this list is called data3. Recall from Proposition 5.2, that if A, B have the same x-values, but different y-values, then the geodesic that goes through both points is centered at (x0, 0) and has a radius of infinity. If this is the case, then I do not need to move point b to the vertical line that goes through AA’. I use Proposition 5.4, to find circle e. If points A and B have the same y-values, then the code uses Proposition 5.4 to find the perpendicular bisector of the vertical segment.

data3 = []
for w in range(1, len(complex_to_real_even_i_points_list)):
    try:
        original_point = (complex_to_real_even_i_points_list[w][0],
                         complex_to_real_even_i_points_list[w][1])
        # pt from orbit
        if original_point[0] == 0:
            perp_bisectors = perp_bisectors_segment_h(original_point,
                                                       (0, 2))
            data3.append([original_point, perp_bisectors])
        else:
transformed_point = circle_inversion1((0,2), original_point)
circle_C = hyperbolic_geodesic_circle((0,2), original_point)
#starting circle
circle_C_endpoints = circle_endpoints(circle_C[0], circle_C[1])
#starting circle endpoints
left_endpoint_for_circle_C = circle_C_endpoints[0]
circle_D = radius_of_circle_d(left_endpoint_for_circle_C, (0,2))
#circle D
circle_D_radius = circle_D[1]
circle_D_endpoints = circle_endpoints(circle_D[0], circle_D[1])
segment_H = perp_bisectors_segment_h((0,2),
transformed_point)
#perpendicular bisectors of a and b'
circle_e_endpoints =
circle_endpoints(segment_H[0], segment_H[1])
circle_eprime_endpoints =
perpbisector_e_transformed(circle_D[0],
circle_e_endpoints[0],
circle_e_endpoints[1], circle_D_radius)
perp_bisectors = hyperbolic_geodesic_circle
(circle_eprime_endpoints[0], circle_eprime_endpoints[1])
data3.append([original_point, perp_bisectors])
except:
    print("line")

This list produces all of the perpendicular bisectors of segments AB. We now have enough information to build the Dirichlet domain. I start by graphing the perpendicular bisectors.
Figure 5.12: Graph of all perpendicular bisectors between segment $AB$.

Shown in Figure 5.12, there are more walls than necessary, so I clean up and remove most of them to get Figure 5.14.
5.4 Analyzing the Polygon

I start analyzing this surface by finding the area. The Sage code, data3 produces the center and radius of the perpendicular bisectors. Using this information, I find the angle of intersection between each circle using Darboux’s Theorem.

**Theorem 5.6.** [Wei99, Circle-circle intersection] The angle of intersection between two circles is given by $\cos(\theta) = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}$, where $d$ is the distance from the center of the two circles, and $r_1, r_2$ are the radii of the two circles.

This computation produces the external angle of intersection between the two circles. There are two intersections where it is the internal angles I actually need to find the area. In order to get this internal angle, I compute $\pi - \cos \theta$. Using Theorem 5.6, I
produce angles with radian measures of:

- $1.69612415796296$,
- $1.69515132134168$,
- $2.09439510239319$,
- $1.44644133224814$,
- $1.44546849562683$,
- $1.44546849562680$,
- $2.09439510239319$,
- $1.69515132134167$.

Using the Gauss-Bonnet Theorem, Theorem 2.25, I compute an area of $8.37758040957283$ radians. Using the Arithmetic Area Formula 4.14, I compute an area of $8\pi \approx 8.3775804096$. Using the two different area formulas is a way to check that this hyperbolic polygon is the 10-gon that I say it is, with the correct walls (i.e. I am not missing any).

**Proposition 5.7.** If $\gamma$ is a mapping that moves a Dirichlet domain, $F$, to an adjacent domain, then $F$ to an adjacent domain Dirichlet domain with walls $W = \{w_1, ..., w_r\}$, then $\gamma^{-1}w_i \in W$.

**Proof.** If $F$ is a Dirichlet domain with walls $W = \{w_1, ..., w_r\}$ and $\gamma$ moves $F$ to an adjacent domain, then $\gamma F$ has walls $\{\gamma w_1, ..., \gamma w_r\}$. If $F \cap \gamma F$ is $w_i$, then there exist $j \in \{1, ..., r\}$ such that $w_i = \gamma w_j$. Therefore, $\gamma^{-1}w_i \in W$. \qed

**Example 5.8.** Let’s look at the complex point $(-2.8922386553849+0.442678727629470i)$, which is a point I produce from the orbit sample. The perpendicular bisector that correlates to this point is shown in Figure 5.14 below, this is the pink geodesic that has the center $(-3.71437635490060, 0)$. The generator that corresponds to this point is $1/2 - i - 1/2k$. 
The inverse of this generator is $1/2 + i + 1/2k$. Looking at Figure 5.14, the perpendicular bisector that has this generator is also in pink to the left, and has the center $(-1.99554730922155, 0)$. The orbit point that corresponds to this perpendicular bisector is $(-1.74554297357772 + 0.250562173583702i)$. Similarly, if I start with the orbit point $(-1.74554297357772 + 0.250562173583702i)$ the inverse of the generator corresponds to the point $(-2.8922386553849 + 0.442678727629470i)$. It actually does this for every wall in the polygon. Using the symmetry of generator mappings, I produce a list and image of which walls were glued together, see Figure 5.14 below.

Figure 5.14: Hyperbolic polygon with gluings

Notice that the pink walls and the orange walls, are the only walls that are adjacent. This is due to them corresponding to elliptic elements, which means that Möbius transformation is a rotation (see Proposition 3.10). To confirm these are elliptic elements, I computed the trace, and it is 1 for both the pink and orange mappings. A rotation produces cone points in the quotient. All of the other mappings are hyperbolic elements, which means they have a trace that is greater than 2. These transformations are translations. Using these edge gluings, I create a hyperbolic surface.
Figure 5.15: Hyperbolic Surface

In Figure 5.15, we can see that topologically I produce a genus 1 surface with two cone points, i.e. it has signature $(1; 3, 3)$. To see what the cone points are, I use the Riemann-Hurwitz Theorem to verify the area (See Theorem 2.29). By plugging in 0 punctures, 1 genus, and two cone points of $3, 3$ we have $2\pi(2(1) - 2 + 2(3 - 1)) = \frac{8\pi}{3}$. This confirms that this hyperbolic surface has two cone points of order 3.

Theorem 5.9. The hyperbolic orbisurface created from the quaternion algebra $(3, 5)$ over $\mathbb{Q}$ has signature $(1; 3, 3; 0)$, and can be cut into a hyperbolic 10-gon, depicted in Figure 5.15 and Figure 5.14. Furthermore, the following order elements both glue together the walls of the polygon and generate the group $\mathcal{O}^{(1)}$:
1. $1/2 + i + 1/2k$,

2. $1/2 - i - 1/2k$,

3. $-1/2 + i - 1/2k$,

4. $1/2 + i - 1/2k$,

5. $2 + i$,

6. $2 - i$,

7. $3/2 - j + 1/2k$,

8. $3/2 + j - 1/2k$,

9. $3/2 - j - 1/2k$,

10. $3/2 + j + 1/2k$

**Corollary 5.10.** With the notation as above, if $\Gamma$ is the image of $O^{(1)}$ in $PSL_2(Q[d])$, $d = \sqrt{3}$, under the standard embedding (Definition 4.6), then $\Gamma$ is generated by the matrices:

1. $\begin{bmatrix} \frac{1}{2} + d & \frac{-5}{2}d \\ \frac{1}{2}d & \frac{1}{2} - d \end{bmatrix}$

2. $\begin{bmatrix} \frac{1}{2} - d & \frac{5}{2}d \\ \frac{-1}{2}d & \frac{1}{2} + d \end{bmatrix}$

3. $\begin{bmatrix} \frac{-1}{2} + d & \frac{5}{2}d \\ \frac{-1}{2}d & \frac{1}{2} - d \end{bmatrix}$

4. $\begin{bmatrix} \frac{1}{2} + d & \frac{5}{2}d \\ \frac{-1}{2}d & \frac{1}{2} - d \end{bmatrix}$

5. $\begin{bmatrix} 2 + d & 0 \\ 0 & 2 - d \end{bmatrix}$

6. $\begin{bmatrix} 2 - d & 0 \\ 0 & 2 + d \end{bmatrix}$
7. \[
\begin{bmatrix}
\frac{3}{2} & \frac{5}{2}d - 5 \\
-\frac{1}{2}d - 1 & \frac{3}{2}
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
\frac{3}{2} & -\frac{5}{2}d + 5 \\
\frac{1}{2}d + 1 & \frac{3}{2}
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
\frac{3}{2} & -\frac{5}{2}d - 5 \\
\frac{1}{2}d - 1 & \frac{3}{2}
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
\frac{3}{2} & \frac{5}{2}d + 5 \\
-\frac{1}{2}d + 1 & \frac{3}{2}
\end{bmatrix}
\]
Chapter 6

Conclusion

The polygon I produce is created by performing a careful analysis of the principal arithmetic surface \((3, 5)\) over \(\mathbb{Q}\). To find this hyperbolic polygon, I use Sage to adjoin the two symbols \(i^2 = 3, j^2 = 5\), and produce the quaternion algebra over \(\mathbb{Q}\). The quaternion algebra produces a list of matrices that I use to transform the point \((0, 2)\), and find the orbit sampling. Taking each point from the orbit sampling, I create segments between them and \((0, 2)\). I then find the Dirichlet domain by finding the perpendicular bisectors of the segments and look at finitely many, but enough walls. Once I have the Dirichlet domain, I carefully analyze this surface by finding the hyperbolic area, the associated gluing maps, and the matrices that correspond to gluing one side to the other.

Creating a hyperbolic polygon is more than just an example problem. From the quaternion algebra, we produce a list of generators, and these generators are used to find the matrices used to transform the point \((0, 2)\). When you find the Dirichlet domain, you have a specific list of generators that correspond to side pairings of the walls of the hyperbolic polygon. Theorem 5.9 is valuable to other researchers for many reasons. This list of finite generators has uses in other areas of math. Researchers use the generators in systolic geometry, which is the study of invariants of manifolds and polyhedra to find the systole of the surface and its covers. It can be used to find the Cheeger constant of the surface, which is a numerical measure of whether or not the surface has a “bottle neck”. This data is useful for checking future results and hypotheses.

The process leading up to finding my hyperbolic polygon leaves me with a few unanswered questions. It is an open question as to how Macasieb excludes certain sig-
natures from her list of genus 2 surface coverings. I am unable to construct a few of the signatures on her list. It is an open question as to how they can be constructed or if they cannot and should be excluded from the list as well. Generalizing this to the genus 3 surface, is it possible to construct the other signatures Proposition 2.34 for a genus 3 surface covering. Once I have these coverings how do I confirm that this list is correct. Next I want to reevaluate [Mac08] and surface coverings. I really enjoy making the constructions of these surface coverings because I feel like it pushes me outside of my comfort zone. I have never considered myself as a creative person, but in order to make these constructions, I have to be creative in the bending and stretching of these surfaces, as well as where to place the axis of rotation. I want to continue to challenge myself in this area and advance myself as a student and person.

This thesis brought together many areas of math, including topology, hyperbolic geometry, linear algebra, group theory, and noncommutative algebra. Additionally this project involves a significant programming component, in which I write and implement code in the Sage. My project has expanded my mathematical thinking by challenging me to expand my understanding of the algebra that is behind the Euclidean and hyperbolic geometry of these surfaces and figures, like GeoGebra images or surface constructions, and put it into a Python-based Sage code. With surfaces, I have to challenge my mind to see the bending, stretching, and gluings of surfaces. For example, taking my hyperbolic polygon, and finding its corresponding surface (see Figure 5.15). I have never consider myself as a creative person or visual learner, but I feel this thesis challenges me the most in this aspect and make me a better learner. Going into this thesis, I have minimal experience with Sage and I have never created my own functions. I found it extremely difficult to think like a “coder”. I often use GeoGebra to create the images for which I need to create functions and piece together how to get Sage to output the image I am seeing. This is a process. By the end of my thesis, I am able to create my own functions, pull out the data that I need from certain list, and create images (see Figure 5.14). This thesis has expanded my learning by helping me to step outside of my comfort zone, and change my way of thinking as well as challenging me to bring together multiple topics such as topology, hyperbolic geometry, linear algebra, group theory, and noncommutative algebra.
Bibliography


