The edge-isoperimetric problem for the square tessellation of plane

Sunmi Lee

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THE EDGE-ISOPERIMETRIC PROBLEM FOR THE SQUARE TESSELLATION
OF PLANE

A Project
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
Of the Requirements for the Degree
Master of Arts
in
Mathematics

by
Sunmi Lee
June 2000
THE EDGE-ISOPERIMETRIC PROBLEM FOR THE SQUARE TESSELLATION OF PLANE

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ABSTRACT

The solution for the edge-isoperimetric problem (EIP) of the square tessellation of plane is investigated and solved. Summaries of the stabilization theory and previous research dealing with the EIP are stated. These techniques enable us to solve the EIP of the cubical tessellation.
ACKNOWLEDGMENTS

I wish to acknowledge indebtedness to all those who have directly or indirectly contributed to this project. Special tribute is due to my advisor, Dr. Joseph Chavez who primarily has been helping my study. His stimulating discussions, unlimited patience, and time have been of invariable to me. Thanks to my family and friends who have supported me with deep love, otherwise this paper would not have been come to light.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>iv</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>vi</td>
</tr>
<tr>
<td>1  INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2  SUMMARY OF THE LITERATURE</td>
<td>4</td>
</tr>
<tr>
<td>3  THE EIP OF THE SQUARE TESSELLATION OF $R^2$</td>
<td>5</td>
</tr>
<tr>
<td>3.1 Solutions for $k=1,2,3,4$</td>
<td>5</td>
</tr>
<tr>
<td>3.1.1 The stability order of $V_s$</td>
<td>6</td>
</tr>
<tr>
<td>3.2 Solutions for all $k$</td>
<td>8</td>
</tr>
<tr>
<td>4  The EIP of the cubical tessellation of $R^3$</td>
<td>12</td>
</tr>
<tr>
<td>5  Conclusion</td>
<td>16</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>17</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

FIGURE 1 .......................................................... 1
FIGURE 2 .......................................................... 3
FIGURE 3 .......................................................... 5
FIGURE 4 .......................................................... 7
FIGURE 5 .......................................................... 8
FIGURE 6 .......................................................... 9
FIGURE 7 ......................................................... 12
1 Introduction

The edge-isoperimetric problem (EIP) has emerged during the past few decades as an important research area in graph theory. As the name implies, the EIP is coined from classical isoperimetric problems that involve notions of area and the length of boundary. Harper first brought out an analogy between continuous and discrete isoperimetric problems [2]. Many authors have made contributions to the development of the EIP. In this paper, we are particularly interested in the EIP of the square tessellation of $R^2$ and the cubical tessellation of $R^3$.

A graph, $G$, is defined as a structure with a set of vertices, $V$, and a set of edges, $E$, which join pairs of distinct vertices. Given a graph $G = (V, E, \partial)$ having vertex-set, $V$, edge-set, $E$, and boundary function, $\partial : E \to \binom{V}{2}$, which identifies the pair of vertices incident to each edge, we let $\Theta(S) = \{e \in E : \partial(e) = \{v, w\}, v \in S \text{ and } w \notin S\}$. The number of edges that have one vertex in $S$ and the other vertex not in $S$ is counted by $|\Theta(S)|$. Then given $k \in Z^+$, the edge-isoperimetric problem (EIP) is to minimize $|\Theta(S)|$ over all $S \subseteq V$ such that $|S| = k$ [1]. We may think of $|\Theta(S)|$ as the boundary for $S$ and $|S|$ to be the area of $S$. Minimizing $|\Theta(S)|$ is analogous to the classical isoperimetric problem.

Let $G$ be the graph of a square having vertex-set, $V$ (Figure 1). The minimum value of $|\Theta(S)|$ over all $S \subseteq V$ with $|S| = k$ for $k = 1, 2, 3, 4$ is given in Table 1.

![Figure 1: Square](image)

The wirelength problem is an application of the EIP. We may consider a given
graph as an electronic circuit with the vertices representing components and the edges representing wires connecting them. Suppose that we place the components on a linear chassis, then the wirelength problem is to minimize the total length of all the wires. Let $w_l$ be the total length of all the wires. A numbering of $G$ is a function, $\phi : V \rightarrow \{1, 2, \ldots, n\}$, where $|V| = n$. The following statement shows the relationship between the wirelength problem and the EIP.

$$w_l(\phi) = \sum_{k=0}^{n} |\Theta(S_k(\phi))|,$$

where $S_k(\phi) = \{v \in V : \phi(v) \leq k\}$.

Let $G$ be the graph of the square. Suppose we want to place all vertices of $G$ on a linear chassis, each a unit distance from the next, and in doing so minimize the total length of all wires. The square has $4! = 24$ numberings, but also 8 symmetries. Any two numberings symmetric to each other have the same wirelengths. The three numberings in Figure 2 below represent the $24/8 = 3$ equivalence classes of numberings. The first two numberings minimize the wirelength with $w_l = 6$ while the third one maximizes it with $w_l = 8$.

\[
\begin{array}{c|c}
 k & \min|\Theta(S)| \\
\hline
 1 & 2 \\
 2 & 2 \\
 3 & 2 \\
 4 & 0 \\
\end{array}
\]

Table 1: Min $|\Theta(S)|$
Figure 2: Three numberings of the square

We state a variation of $\Theta(S)$ that is very useful in finding solution sets for the EIP of the square tessellation and the cubical tessellation. For $S \subseteq V$, let $I(S) = \{e \in E : \partial(e) = \{v, w\}, v \in S$ and $w \in S\}$. The members of $I(S)$ are called internal edges of $S$. If $G$ is regular of degree $\delta$, then $|\Theta(S)| = \delta|S| - 2|I(S)|$, so $|I(S)| = 1/2(\delta|S| - |\Theta(S)|)$, and for $|S| = k$, fixed, minimizing $|\Theta(S)|$ is equivalent to maximizing $|I(S)|$. If we define $\iota(v) = \{w \in V; \exists e \in E, \partial(e) = \{v, w\}$ and $w <_{so} v\}$ then $|I(S)| = \sum_{v \in S} |\iota(v)|$.

In order to understand the techniques we will use, a summary of the literature is given in the next section. In Section 3, solutions of the EIP of the square tessellation of $R^2$ will be given. In Section 4, the EIP of the cubical tessellation of $R^3$ will be solved by tools similar to those used for the square tessellation of $R^2$. 


2 Summary of the literature

This section is devoted to explaining the stabilization theory which plays an important part in solving the EIP for both the square and cubical tessellation.

Let \( G = (V, E, \partial) \) be a finite graph embedded in \( \mathbb{R}^d \), \( d \)-dimensional Euclidean space, and let \( R \) be a reflection which acts as a symmetry of \( G \).

**Definition 1** \( R \) is called stabilizing if for all \( e \in E, \partial(e) = \{v, w\} \), when \( v \) and \( w \) are on opposite sides of the fixed hyperplane of \( R \), then \( R(v) = w \).

**Definition 2** If \( R \) is stabilizing for \( G \), \( p \in \mathbb{R}^d \) is not fixed by \( R \) and \( S \subseteq V \) with

\[
\sum = \{ v \in S : ||v - p|| > ||R(v) - p|| \text{ and } R(v) \notin S \},
\]

then \( \text{Stab}_{R,p}(S) = S - \sum + R(\sum) \).

**Theorem 1** \( |\text{Stab}_{R,p}(S)| = |S| \) and \( |\Theta(\text{Stab}_{R,p}(S))| \leq |\Theta(S)| \). Also if \( S \subseteq S' \subseteq V \) then \( \text{Stab}_{R,p}(S) \subseteq \text{Stab}_{R,p}(S') \).

**Proof.** See [2].

**Definition 3** Let \( R = \{R_0, R_1, R_2, \ldots R_{k-1}\} \) be the set of stabilizing symmetries of \( G \). Then, a set \( S \subseteq V \) such that \( \text{Stab}_{R_i,p}(S) = S \) for \( i = 0, 1, \ldots, k - 1 \) is called stable.

**Definition 4** Let \( \text{SO}(V; R; p) = \{(v, w) \in V \times V : R(v) = w \text{ and } ||v - p|| < ||w - p||\} \). Then the stability order, \( \text{SO}(V; R_0, R_1, \ldots, R_{k-1}; p) \), is defined to be the transitive closure of \( \bigcup_{i=0}^{k-1} \text{SO}(V; R_i; p) \).

**Theorem 2** A set \( S \subseteq V \) is stable if and only if it is a lower set in \( \text{SO}(V; R_0, R_1, \ldots, R_{k-1}; p) \).

**Proof.** See [2].
3 The EIP of the square tessellation of $R^2$

The main goal of this section is to prove the solution for the EIP for the square tessellation of $R^2$. The square tessellation of $R^2$ is an infinite set of square tiles fitting together to cover the whole plane. Let $S$ be the graph of the square tessellation of $R^2$ (see Figure 3). Denote by $V_S$ the vertex set of $S$. In order to gain some insight into the solution sets for the square tessellation, we begin by looking at solution sets and $\min|\Theta(S)|$ for small $k \in Z^+$.

![Figure 3: The square tessellation of $R^2$](image)

3.1 Solutions for $k = 1, 2, 3, 4$

We are to investigate solution sets of the EIP for small values of $k$.

$k = 1$: Every vertex in the square tessellation can be the solution set of size 1 and $\min_{|S|=1} |\Theta(S)| = 4$.

$k = 2$: We have two cases for $|S| = 2$. If two vertices are connected by an edge, $|\Theta(S)| = 6$. If they are not, $|\Theta(S)| = 8$. Therefore $\min_{|S|=1} |\Theta(S)| = 6$.

$k = 3$: There are two types of set with $|S| = 3$ and $|I(S)| = 2$ and none with
$|I(S)| > 2$. However, either of the two types has the same value of $|\Theta(S)|$. Thus $\min_{|S|-1}|\Theta(S)| = 8$.

$k = 4$: There is only one type of set with $|S| = 4$ and $|I(S)| = 4$, the vertices of a square. There is no set with $|I(S)| > 4$. Therefore $\min_{|S|-1}|\Theta(S)| = 4 \cdot 4 - 2 \cdot 4 = 8$.

For larger $k$ if we continue in the same manner, we could find solution sets, but it would get very complicated. We need a more powerful method to find solution sets for any $k$. This is when we use the theory of stabilization.

### 3.1.1 The stability order of $V_S$

There are some connections among solution sets of the EIP for any $k$. We start with how to build the stability order of $V_S$. In Figure 4, we are given a point, $p$. The darkened lines represent the three basic reflections of $S$. These three reflections are called basic since they bound the smallest region containing $p$ with the minimum number of reflections. Also they generate the Coxeter group of $S$, $R_3$. The small triangle bounded by the basic reflections is called the fundamental region of $S$ (for more information see [2]).

According to Definition 4 in Section 2 we can build the stability order of $V_S$. Note that $a$ is the only vertex in the fundamental region. Therefore $a$ is the closest point to $p$ so we start ordering with $a$. Since $a$ is on two of the reflections, $a$ is reflected to $b$ by the remaining reflection. Similarly $b$ is reflected to $c$. From $c$ we have two choices to make. $c$ is reflected to $d$ by the vertical reflection and to $e$ by the horizontal reflection.
By continuing in this manner we have the stability order of $V_S$ for $k \leq 25$ in Figure 5. Denote the stability order of $V_S$ as $SO$. From inspection of the stability order, we determine the solutions for $k \leq 25$ given in Table 2.

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Table 2: $max_{|S|=k}|I(S)|$
3.2 Solutions for all $k$

Our work in this section will focus on the proof of solutions for any positive integer. For $r \in \mathbb{Z}^+$, we define $B_r$ to be the set of vertices within the square with radius $r$, that is a $2r \times 2r$ square ball centered at $v_0$ (Figure 6). This is true that $|B_r| = (2r+1)^2$. 
All the vertices of the square tessellation lie on two families of parallel lines, denoted \( L \). \( R_3 \) is the group of symmetries of \( S \) (see the Table IV of [3]). The stabilization theory summarized in Section 2 may apply to geometric objects such as lines, as well as to vertices, so that we can define the stability order of lines. If \( R \in R_3 \), then we let \( SO(L; R; p) = \{ (L, R(L)); \|L - p\| < \|R(L) - p\| \} \). Then the stability order of \( L \) with respect to \( R_3 \) and \( p \) is the transitive closure of \( \cup_{R \in R_3} SO(L; R; p) \).

Denote the stability order of lines as \( LO \). The symmetries of \( B_r \) constitute a dihedral group \( D_4 \), which acts transitively on the four lines bounding \( B_r \). The stability order of any \( D_n \) acting on the sides of a regular \( n \)-gon is total so the relative order in \( LO \) of the four lines bounding \( B_r \) is total. Thus we may denote them as \( L_{r,i}, 0 \leq i \leq 3 \) with \( L_{r,i} <_{LO} L_{r,j} \) if \( i < j \).

**Lemma 1** The stability order of lines, \( LO \), is total.

**Proof.** \( L_{r,i} <_{LO} L_{r+1,0} \) is true for \( 0 \leq i \leq 3 \) since there is \( R \in R_3 \) such that
\[
\|L_{r,3} - p\| < \|L_{r+1,0} - p\| = \|R(L_{r,3}) - p\|. \]

\( \square \)
We have \( V_s = \{v_0\} \cup \bigcup_{i=0}^{\infty} \bigcup_{r_i=1}^{2^r} (V_s \cap L_{r_i}) \). Each vertex, except \( v_0 \), is contained in multiple \( L_{r_i} \)'s. However if we let 
\[
L'_{r,i} = V_s \cap L_{r,i} - \bigcup_{j=i+1}^{2^r} (V_s \cap L_{r,j}) \cup \bigcup_{i=0}^{\infty} (V_s \cap L_{s,i}) \],
then \( \{v_0\} \) and the \( L'_{r,i} \) form a partition of \( V_s \). Also, \( B_r = \{v_0\} \cup \bigcup_{s=1}^{2^r} \bigcup_{i=0}^{2^{r+s}} (V_s \cap L'_{s,i}) \) and 
\[
|L'_{r,i}| = \begin{cases} 
2r - 1 & \text{if } i = 0 \\
2r & \text{if } i = 1 \text{ or } 2 \\
2r + 1 & \text{if } i = 3
\end{cases} \tag{1}
\]

Note that \( V_s \cap L'_{r,i} \) is totally ordered by \( SO \), the vertex nearest \( p \) being its least element and this lies at the midpoint of \( B_r \cap L_{r,i} \). The others follow in increasing order of their distance from \( p \) so that they alternate from side to side. \( L'_{r,i} \) is an initial segment in this order. Note that for \( v \in L'_{r,i}, \)
\[
|\iota(v)| = \begin{cases} 
0 & \text{if } v = v_0 \\
1 & \text{if } v \neq v_0, \text{ } v \text{ is minimal in } L'_{r,i} \\
2 & \text{otherwise}
\end{cases} \tag{2}
\]

**Theorem 3** There is a total order, \( TO \), on \( V_s \) such that for all \( k \in \mathbb{Z}^+ \) the initial \( k \)-set of \( TO \) minimizes \( |\Theta(S)| \) over all \( S \subseteq V_s \) with \( |S| = k \).

**Proof.** We define a total order, \( TO \), on \( V_s \) by \( v <_{TO} w \) if \( v \in L'_{r,i}, w \in L'_{s,j} \) with \( r < s \), or \( r = s \) and \( i < j \), or \( r = s \) and \( i = j \) and \( v <_{SO} w \). Note that \( TO \) is an extension of \( SO \), the stability order on \( V_s \). We need to show that if \( S \subseteq V_s \) is a stable set in \( SO \) with \( |S| = k \) and \( S_k \) is the initial segment of \( TO \) of the same cardinality, then \(|I(S_k)| \geq |I(S)|\). If \( S \neq S_k \), then there exists a minimal element, \( a \), with respect to \( TO \), in \( S_k - S \) and a maximal element, \( b \), with respect to \( TO \), in \( S - S_k \). Note that \( a <_{TO} b \) but they must be incomparable with respect to \( SO \). If \( k \leq 3 \), we have a linear order. Thus, we may assume \( k > 3 \), so \(|\iota(a)|, |\iota(b)| = 1 \text{ or } 2.\)
If $|\nu(a)| < |\nu(b)|$, i.e. $a$ is minimal in $L'_{r,i}$ and $b$ is not minimal in $L'_{s,j}$, $s \geq r$ and $j \geq i$, then $a <_{SO} b$, i.e. $a \in S$, which contradicts to the fact $a \in S_k - S$. Therefore, the only case we can have is $|\nu(a)| \geq |\nu(b)|$. If $|\nu(a)| \geq |\nu(b)|$, then $|S + \{a\} - \{b\}| = k$ and $|I(S + \{a\} - \{b\}| \geq |I(S)|$ and a finite series of such switches would give us $|I(S_k)| \geq |I(S)|$. □

**Corollary 1** If $k = (2r + 1)^2$ then the only stable solution is $B_r$. 


4 The EIP of the cubical tessellation of $R^3$

Let $C$ be the graph of the cubical tessellation of $R^3$. Denote by $V_C$ the vertex set of $C$. Our work in this section will focus on the proof of theorem stating that there is a total order on $V_C$. In $C$ we are given a point, $p$, and we have four basic reflection planes of $C$ (see Figure 7; a plane through $a$, $b$, $c$, and $d$, a plane through $a$, $c$, $e$, and $g$, a plane through $a$, $d$, $f$, and $g$, and a perpendicular bisector plane of edge $fg$). The small tetrahedra bounded by the four basic reflections is the fundamental region of $C$ [3]. We can build the stability order of $V_C$, $SOC$, with respect to the four reflections and the point, $p$. The procedure which we used to find and prove the solution for $S$ also suffices for the cubical tessellation. By observation of the stability order of $V_C$, we just list solutions for $k \leq 27$ (see Table 3).

For $r \in Z^+$, let $B_r$ be the set of vertices within the cube with radius $r$ centered at $v_0$. It is easy to see that $|B_r| = (2r + 1)^3$. Notice that there are three families
of parallel planes in which all edges and vertices of $C$ lie. Denote by $P$ these three families of parallel planes. The symmetry group of $C$ is Coxeter [3]. We denote the group of symmetries of $C$ as $R_4$ (See Table 4 of [3]). As we did in Chapter 3, we construct the stability order of planes. If $R \in R_4$, then we let $SOP(P; R; p) = \{(P, R(P)); ||P - p|| < ||R(P) - p||\}$. Then the stability order of $P$ with respect to $R_4$ and $p$ is the transitive closure of $\cup_{R \in R_4} SOP(P; R; p)$.

We denote the stability order of planes as $PO$. The symmetries of $B_r$ act transitively on the six planes bounding $B_r$ [5]. The relative order in $PO$ of the six planes bounding $B_r$ is total. Thus we may denote them as $P_{r,i}, 0 \leq i \leq 5$ with $P_{r,i} < PO P_{r,j}$ if $i < j$.

**Lemma 2** The stability order of planes, $PO$, is total.

**Proof.** $P_{r,i} < PO P_{r+1,0}$ is true for $0 \leq i \leq 5$ since there is $R \in R_4$ such that $||P_{r,5} - p|| < ||P_{r+1,0} - p|| = ||R(P_{r,5}) - p||$. 

We have $V_C = \{v_0\} \cup \bigcup_{r=1}^{\infty} \bigcup_{i=0}^{5} (V_C \cap P_{r,i})$. Each vertex, except $v_0$, is contained in multiple $P_{r,i}$'s but if we let $P'_{r,i} = V_C \cap P_{r,i} - \bigcup_{j=i+1}^{\infty} (V_C \cap P_{r,j}) \cup \bigcup_{s=r+1}^{\infty} \bigcup_{i=0}^{5} (V_C \cap P_{s,i})$ then $\{v_0\}$ and the $P'_{r,i}$

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Table 3: $max_{|S|=k}|I(S)|$
form a partition of $V_C$. Also, $B_r = \{v_0\} \cup \bigcup_{s=1}^r \bigcup_{t=0}^5 (V_C \cap P_{r,t})$ and

$$|P'_{r,i}| = \begin{cases} (2r - 1)^2 & \text{if } i = 0 \\ (2r - 1)^2 + 2r - 1 & \text{if } i = 1 \\ (2r - 1)^2 + 2r - 1 + 2r & \text{if } i = 2 \\ (2r - 1)^2 + 2r - 1 + 2r + 2r & \text{if } i = 3 \\ (2r - 1)^2 + 2r - 1 + 2r + 2r + 2r + 1 & \text{if } i = 4 \\ +2r^2 - 1 & \text{if } i = 5 \end{cases}$$ (3)

Note that $V_C \cap P'_{r,i}$ is totally ordered by $TO$ on $S$, the vertex $p$ being its least element and this lies at the center of $B_r \cap P_{r,i}$. The others follow according to their order given by $TO$. $P'_{r,i}$ is an initial segment in this order. For $k = 0, 1, 2$, let $P_{0,k}$ be the plane through $v_0$ and parallel to $P_{r,k}$. For $v \in P'_{r,i}$, $v_0$,

$$|\iota(v)| = \begin{cases} 0 & \text{if } v = v_0 \\ 1 & \text{if } v \neq v_0, v \text{ is minimal in } P'_{r,i} \\ 2 & \text{if } v \in P_{0,k} \text{ and } k = 1,3 \\ 3 & \text{if otherwise} \end{cases}$$ (4)

**Theorem 4** There is a total order, $TOC$, on $V_C$ such that for all $k \in \mathbb{Z}^+$ the initial $k$-set of $TOC$ minimizes $|\Theta(S)|$ over all $S \subseteq V_C$ with $|S| = k$.

**Proof.** We define a total order, $TOC$, on $V_C$ by $v <_{TOC} w$ if $v \in P'_{r,i}, w \in P'_{s,j}$ with $r < s$, or $r = s$ and $i < j$, or $r = s$ and $i = j$ and $v <_{TO} w$. We need to show that if $S \subseteq V_C$ is a stable set in $SOC$ with $|S| = k$ and $S_k$ is the initial segment of $TOC$ of the same cardinality, then $|I(S_k)| \geq |I(S)|$. If $S \neq S_k$, then there exists a minimal element, $a$, with respect to $TOC$, in $S_k - S$ and a maximal element, $b$, with respect to $TOC$, in $S - S_k$. Note that $a <_{TOC} b$ but they must be incomparable with respect to $SOC$. For $k > 1$, $|\iota(a)|, |\iota(b)| = 1, 2$, or 3. If $|\iota(a)| < |\iota(b)|$, then we have $|\iota(a)| = 1$ and $|\iota(b)| = 2$ or 3, or $|\iota(a)| = 2$ and $|\iota(b)| = 3$. In first case, $a$ is minimal in $P'_{r,i}$ and $b$ is not minimal in $P'_{s,t}$ where $r \leq s$ and $i \leq t$. In second case, $a$ is in $P'_{0,k}$ and
b is not minimal or in $P_{0,k}$. Either of the two cases leads to $a <_{SOC} b$, that is $a \in S$, which contradicts to the fact $a \in S_k - S$. Therefore, the only case we can have is $|\nu(a)| \geq |\nu(b)|$. If $|\nu(a)| \geq |\nu(b)|$, then $|S + \{a\} - \{b\}| = k$ and $|I(S + \{a\} - \{b\}| \geq |I(S)|$ and a finite series of such switches would give us $|I(S_k)| \geq |I(S)|$. □

**Corollary 2** If $k = (2r + 1)^3$ then the only stable solution is $B_r$. 
References


