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THE DECOMPOSITION OF THE SPACE OF ALGEBRAIC CURVATURE TENSORS

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The Decomposition of the Space of Algebraic Curvature Tensors

A Thesis

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Abstract

We decompose the space of algebraic curvature tensors (ACTs) on a finite dimensional, real inner product space under the action of the orthogonal group into three inequivalent and irreducible subspaces: the real numbers, the space of trace-free symmetric bilinear forms, and the space of Weyl tensors. First, we decompose the space of ACTs using two short exact sequences and a key result, Lemma 3.5, which allows us to express one vector space as the direct sum of the others. This gives us a decomposition of the space of ACTs as the direct sum of three subspaces, which at this point may or may not be inequivalent or irreducible. We then count the number of nonzero, independent ways of contracting an ACT down to a real number and determine there are exactly three ways of doing so. We conclude with a verification that the subspaces in our decomposition are in fact inequivalent and irreducible by applying another key result, Lemma 3.7, a representation theoretic tool used to sense irreducibility. Since the number of terms in our decomposition (three) is equal to the number of nonzero, independent ways to contract an ACT down to the real numbers (three), we conclude that our decomposition is the orthogonal decomposition of the space of ACTs where the subrepresentations are inequivalent and irreducible.

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Table of Contents

Chapter 1

Introduction

In this thesis, we will study representations, which require a group, a vector space, and an action of that group on the vector space. We will also use an inner product on our vector space V. We will fix V to be a real vector space with finite dimension n and nondegenerate inner product $< \cdot, \cdot >$ defined in Chapter 2, Section 2.4. We will fix our group to be the orthogonal group on V, the group of orthogonal linear transformations preserving the inner product.

The goal of this thesis is to decompose the space of algebraic curvature tensors (ACTs) into irreducible modules under the orthogonal group $\mathcal{O}$. We will discover that the space of ACTs (denoted $\mathcal{A}(V)$) decompses into three irreducible and inequivalent subspaces, the real numbers ($\mathbb{R}$), the space of trace-free symmetric bilinear forms (S_0^2), and the space of Weyl tensors ($\mathcal{W}$). Our main result comes from Peter Gilkey's Geometric Realizations of Curvature [VGN12] and is fully verified in Chapter 4 of this thesis:

Theorem 1.1. *Let* $n \geq 4$.

1. *There is an orthogonal module decomposition:*
   (a) $\mathcal{A}(V) \cong \mathbb{R} \oplus S_0^2 \oplus \mathcal{W}$.
2. *The orthogonal modules* $\{\mathbb{R}, S_0^2, \mathcal{W}\}$ *are inequivalent and irreducible.*

Algebraic curvature tensors are interesting because they mimic the behavior of curvature on a manifold at a point. They take in four vectors and produce a real number which describes a manifold's curvature at a specific point. The decomposition of the space of ACTs is not only of interest to representation theorists, who are concerned with the irreducibility of a representation, but it is meaningful from a geometric standpoint as well. Adding to its value, interesting geometric comments can be made about the three subspaces which the space of ACTs decomposes into. When it is stated that a module (or representation) decomposes, it means that the underlying vector space decomposes into vector subspaces which remain invariant under the group action. To complete our goal of decomposing the space of ACTs into irreducible subrepresentations, we will find a decomposition, but then we will verify that the subrepresentations are invariant and irreducible.

A brief outline of the contents of this thesis follows. In Chapter 2, we will discuss preliminary information pertaining to our main goal which includes basic notation, important definitions, and results needed in the subsequent chapters for general representation theory as well as the representation specifically involving the space of ACTs. Section 1 will focus on representation theoretic preliminaries. We will define for the reader important concepts such as that of the representation, symmetric bilinear form, and inner product which turns out to itself be symmetric bilinear form with the additional property of being nondegenerate. Section 2 of this chapter will focus on preliminary information related to ACTs. It will define tensor products and ACTs and then will move into a discussion on type changing and contraction. We will introduce a key theorem, Theorem 2.17, which allows us to determine a covector's unique metric equivalent, enabling us to freely change a covector to its metric equivalent vector. This relies heavily on the use of the nondegenerate inner product. Using this theorem, we will then demonstrate how to contract an ACT down to a symmetric bilinear form, called its *Ricci tensor*, and then contract once more to determine the real number that the ACT contracted down to, called the *scalar curvature*. We will also show that, up to a sign, there is only one way to contract an ACT down to its Ricci tensor. This idea of contraction will play a major role in our verification that the space of ACTs decomposes specifically into three subspaces, and that those three subspaces are inequivalent and irreducible. At the end of the chapter, we will discuss short exact sequences involving vector spaces as well as $\mathcal{O}$-modules. These ideas will be utilized in Chapter 4 when we use a short exact sequence to come up with a decomposition of the space of ACTs whose subspaces may or may not be irreducible.

The focus of Chapter 3 will be to discuss general representation theory results that will later be applied in Chapter 4 to complete our goal of decomposing the space of algebraic curvature tensors. All results in this chapter involve general representations of $\mathcal{O}$ on V and yet can be applied to the representation involving the space of ACTs. One of the most important results of Section 1 in relation to our main goal is Lemma 3.5. It tells us that a representation ξ can be decomposed into the direct sum of finitely many other subrepresentations (since our vector space V is finite) where each subrepresentation is irreducible. Equally as important, the lemma tells us that if our representation fits into a short exact sequence:

$$0 \to \xi_1 \xrightarrow{f} \xi \xrightarrow{\pi} \xi_2 \to 0,$$

with ξ_1 and ξ_2 subrepresentations of ξ, then ξ is isomorphic to the direct sum ξ_1 and ξ_2 as representations. The importance of this result cannot be understated as it will allow us to create a decomposition of the space of ACTs. Section 2 of this chapter will provide some tests for the irreducibility of a representation. The highlight of this section is Lemma 3.7, a tool that can be used to determine the irreducibility of a representation. This lemma will provide the test we will use to determine whether or not the subspaces we find in our decomposition of the space of ACTs are irreducible and inequvalent. Its importance cannot be over-emphasized as it is crucial in verifying our main goal.

In our final chapter, Chapter 4, we will complete our goal of decomposing the space of ACTs into its three inequivalent and irreducible subspaces. In Section 1, we use two short exact sequences, one involving the space of ACTs and the other involving $S^2(V)$ (the space of symmetric bilinear forms on V), in combination with the application of Lemma 3.5 to create a decomposition of the space of ACTs where the subspaces may or may not be irreducible. Next, in Section 2, we will determine the dimension of the space of quadratic invariants $\mathcal{I}_2^G(\xi)$ where the vector space associated with ξ is $\mathcal{A}(V)$. This dimension is calculated by counting the number of independent, nonzero ways to contract the ACTs down to the real numbers. This will give us an estimation of the dimension of our representation. Finally, in Section 3, we will combine the short exact sequence results from Section 1 with the dimension count from Section 2 using Assertion (2) of Lemma 3.7 to confirm that the subspaces in our direct sum decomposition from Section 1 are in fact inequivalent and irreducible. This confirms that the decomposition from our main result, Theorem 4.1, is an orthogonal direct sum decomposition of the space of ACTs, where the orthogonal modules $\mathbb{R}$, S_0^2, and $\mathcal{W}$ are inequivalent and irreducible.

Chapter 2

Preliminary Information

In order to achieve our ultimate goal of decomposing the space of algebraic curvature tensors into irreducible modules under the orthogonal group $\mathcal{O}$, we must first discuss some preliminary information. This chapter will include basic notation, important definitions, and results needed in the subsequent chapters.

2.1 Representation Theoretic Preliminaries

We will first begin with some notational conventions that will prove useful in later results. We will use G to represent a group. Recall that a group G is any set that holds the properties of closure under its defined operation, associativity, contains an identity, and contains inverses. Fixing a real vector space V with finite dimension n and a nondegenerate inner product $< \cdot, \cdot >$ (see 2.4), two specific groups which will be involved later are the general linear group and the orthogonal group. We will use GL to represent the general linear group, the invertible linear maps from V to V, and we will use $\mathcal{O}$ to represent the orthogonal group, the subgroup of GL preserving the inner product, i.e., $< gv, gw > = < v, w >$ for all $v, w \in V$ and $g \in \mathcal{O}$.

Our first definition, *endomorphism*, plays a crucial role in one of the main concepts of this section, the representation.

Definition 2.1. *Let V be a vector space. An* endomorphism *T is a linear map from V back to itself. In other words, $T : V \to V$.*

We will use $End(V)$ to represent the set of all invertible endomorphisms of a vector space V. Recall that a group homomorphism is an operation-preserving map between two groups.

Definition 2.2. *If G is a group and V is a vector space, a* representation σ *of G on V is a homomorphism:*

$$\sigma : G \to End(V).$$

We can see that a representation takes group elements and *represents* them as transformations of a vector space. Note that the terms module and representation can be used interchangeably. We will use $\xi = (G, V, \sigma)$ to denote a representation σ of G on V, where the group G is the orthogonal group of our nondegenerate inner product, V is a real, finite vector space of dimension n, and σ is the homomorphism above.

A representation requires a group, a vector space, and an action as in Definition 2. We will also use an inner product on V. Before defining an inner product, however, we will begin with the definition for a symmetric bilinear form, which takes in two vectors and produces a real number, and follows the properties listed below:

Definition 2.3. *Let V be a vector space. A* symmetric bilinear form ϕ *is a function from $V \times V \to \mathbb{R}$ that satisfies two properties:*

1. *ϕ is symmetric: $\phi(v, w) = \phi(w, v)$ for $v, w \in V$, and*

2. *ϕ is bilinear: $\phi(\alpha_1 v_1 + \alpha_2 v_2, w) = \alpha_1 \phi(v_1, w) + \alpha_2 \phi(v_2, w)$ for $v_1, v_2, w \in V$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, and since ϕ is symmetric, we have linearity in the second slot as well.*

The space of all symmetric bilinear forms is denoted $S^2(V)$. Interestingly, an inner product is a symmetric bilinear form with the additional specification of being nondegenerate.

Definition 2.4. *Let V be a vector space. Define an inner product φ, with $\varphi : V \times V \to \mathbb{R}$. φ is* nondegenerate *if for all nonzero vectors $v \in V$ there exists another vector $w \in V$ with $\varphi(v, w) \neq 0$.*

Further, an inner product is said to be *positive definite* for $< v, v > \geq 0$ and $< v, v > = 0$ if and only if $v = 0$. If an inner product is positive definite, then it is nondegenerate [Sha09].

The inner product is also referred to as the *metric*, and we will see this come into play in Section 2.2 of this chapter when we uniquely convert covectors to vectors by finding the metric dual, which utilizes the inner product.

Thinking back to our main goal, we want to decompose the space of ACTs into some number of irreducible, inequivalent pieces. To understand more about what is meant by the term *irreducible*, we will start by discussing the concept of an invariant subspace.

A vector space is called *invariant* if a linear map acting on the space sends it back to itself, see Definition 2.5 below. Invariant subspaces are important because they form what we call subrepresentations, which themselves are representations.

Definition 2.5. *Let T be a linear map, and W be a subspace of a vector space V. W is said to be T-invariant if T sends everything in W back to W. In other words, $W \subseteq V$ is T-invariant if $T(W) \subseteq W$.*

Suppose V is an arbitrary vector space, G is a group, and ξ is a representation. If W is an invariant subspace of V, meaning the group action on W sends W back to itself, then $\xi_W = (G, W, \sigma)$ is its own representation, and it is *subrepresentation* of ξ.

Definition 2.6. *Let ξ be a representation, V a vector space, and G a group. A representation ξ is* irreducible *if the only subspaces of V which are invariant under the action of G are $\{0\}$ and V itself.*

In other words, there is no way to further decompose such a subspace into the direct sum of two nontrivial subrepresentations.

Our next definition, from [VGN12], will prove very useful in later results.

Definition 2.7. *Let V be a finite dimensional real vector space, and W be a subspace of V. W is* totally isotropic *if the restriction of the inner product $< \cdot, \cdot >$ to W is zero.*

In other words, using polarization, one can show that W is totally isotropic if $< w, w > = 0$ for all $w \in W$.

The following definition for the *orthogonal complement* of a subspace, or *perpendicular space*, is adapted from [MR19]:

Definition 2.8. *Let V be a real vector space with finite dimension n, $< \cdot, \cdot >$ a nondegenerate inner product, and W be a subspace of V. The* orthogonal complement *of W is* *the subspace*

$$W^{\perp} = \{v \in V| < v, w >= 0 \text{ for all } w \in W\}.$$

The following two lemmas will be useful when proving subsequent results later in this thesis, and in the following lemma specifically, it is important that our group G preserves the inner product.

Lemma 2.9. *Let V be finite dimensional, real vector space with subspace V_1 and $V_1^{\perp}$ its associated perpendicular space. If V_1 is invariant under the action of the orthogonal group G, then $V_1^{\perp}$ is invariant under the action of G.*

Proof. We need to show that if $w \in V_1^{\perp}$ and $g \in G$, then $gw \in V_1^{\perp}$. Choose $v \in V_1$. We will show that $< v, gw >= 0$. Now, since G preserves the inner product,

$$\begin{aligned} < v, gw > &= < gg^{-1}v, gw > \\ & < g^{-1}v, w > . \end{aligned}$$

We know that V_1 is invariant under the action of G, and so $hv \in V_1$ for all $h \in G$ and $v \in V_1$. If we set $h = g^{-1}$, then we have $g^{-1}v \in V_1$, since $v \in V_1$. Therefore, since $g^{-1}v \in V_1$ and $w \in V_1^{\perp}$, we have $< g^{-1}v, w >= 0$, and so, $V_1^{\perp}$ is invariant under the action of G. □

Lemma 2.10. *Let V be a finite dimensional, real vector space with subspace V_1 and $V_1^{\perp}$ be its associated perpendicular space. $V_1 \cap V_1^{\perp}$ is a totally isotropic subspace of a vector space V.*

Proof. We want to show that $< v, v >= 0$ for all $v \in V_1 \cap V_1^{\perp}$. Since $v \in V_1 \cap V_1^{\perp}$, we know $v \in V_1$ and $v \in V_1^{\perp}$. Therefore, $< v, v >= 0$. □

2.2 Algebraic Curvature Tensor Preliminaries

This section will discuss all the notation, definitions, and concepts related to the space of algebraic curvature tensors. It will be useful in Chapter 4 when we decompose the space of ACTs.

To begin, we will discuss the tensor product. We will use V to denote a finite dimensional real vector space, and $\{e_i\}$ will be a basis for V. We will use V^* to denote the associated dual vector space, where V^* is the set of all linear maps from V to $\mathbb{R}$, and $\{e^i\}$ will be the dual basis for V^*, that is, e^i is characterized by the relations $e^i(e_j) = \delta_{ij}$.

Definition 2.11. *Let V be a finite dimensional real vector space and V^* the associated dual vector space. Suppose $\theta, \psi \in V^*$ and $v, w \in V$. Then define $\theta \otimes \psi : V \times V \to \mathbb{R}$ to be multilinear, where*

$$(\theta \otimes \psi)(v, w) = \theta(v) \cdot \psi(w).$$

Similarly, $V \otimes V$ is the set of all multilinear maps $V^* \otimes V^*$ to $\mathbb{R}$.

The tensor product of bases is a basis for the tensor product [Lee97]. This is illustrated in the example below:

Example 2.12. *Suppose vector space V has a basis $\{e_1, e_2\}$ and its associated dual space V^* has the dual basis $\{e^1, e^2\}$. Then, $V \otimes V^*$ has basis $\{e_1 \otimes e^1, e_1 \otimes e^2, e_2 \otimes e^1, e_2 \otimes e^2\}$.*

Each tensor has a *type* under which it can be classified. If $T \in V \otimes V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$ where there are r number of V's tensored together and s number of V^*'s tensored together, then T has type (r, s).

Our understanding of the concept of the tensor product will lead us into discussing algebraic curvature tensors, with the following definition found in [DEMH18].

Definition 2.13. *An* algebraic curvature tensor *on a finite dimensional real vector space V is a multilinear function $R \in \otimes^4 V^*$ that satisfies all of the following for $x, y, z, w \in V$:*

1. $R(x, y, z, w) = -R(y, x, z, w)$,
2. $R(x, y, z, w) = R(z, w, x, y)$,
3. $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$ *(this property is known as the* Bianchi Identity*).*

In short, ACTs are objects that take in four vectors and produce a real number, i.e., they are tensors of type $(0, 4)$, and they behave under the three properties listed above. As previously stated, we will use $\mathcal{A}(V)$ to denote the space of all algebraic curvature tensors on a finite dimensional real vector space V. $\mathcal{A}(V)$ is a vector space under the operations of summing of functions and scaling by real numbers [Gil01].

2.2.1 Type Changing and Contraction

Let V be a real, finite dimensional vector space, and V^* be the associated dual space. What would an element R of $\otimes^4 V^*$ look like with basis $\{e^i, e^j, e^k, e^l\}$? An algebraic curvature tensor, $R \in \otimes^4 V^*$, would be a linear combination of some constants, called *components*, times the tensor product of the basis vectors. In other words,

$$R = \sum R_{ijkl} e^i \otimes e^j \otimes e^k \otimes e^l$$

where $R_{ijkl} \in \mathbb{R}$, and $e^i \otimes e^j \otimes e^k \otimes e^l \in \otimes^4 V^*$. Note that if $\{e_i\}$ is a basis for V and $\{e^i\}$ is a basis for V^*, the *component* of R is the output value when basis vectors are fed into R. More specifically, R_{ijkl} is the number we get when we feed the quadruple (e_i, e_j, e_k, e_l) to R. In other words, $R(e_i, e_j, e_k, e_l) = R_{ijkl}$. For an ACT, changing the input order of the vectors affects the sign of the component based on the properties of an ACT. For example, swapping the order of the first two vectors introduces a negative to the component: $R_{ijkl} = -R_{jikl}$, since $R(e_i, e_j, e_k, e_l) = -R(e_j, e_i, e_k, e_l)$.

Interestingly, we can take an ACT of type $(0, 4)$ and produce a symmetric bilinear form of type $(0, 2)$, called the *Ricci tensor* ρ, using a process called *contraction*. We can use the method of contraction on a symmetric bilinear form as well to produce a real number (a tensor of type $(0, 0)$), also called the *scalar curvature* τ. Later in this section, we will demonstrate the contraction an ACT once to get a symmetric bilinear form, and then we will contract that to get real number. The diagram below illustrates the concept:

$$\mathcal{A}(V) \overset{contract}{\longrightarrow} S^2(V) \overset{contract}{\longrightarrow} \mathbb{R}$$

To contract, however, we need to pair one V with one V^*. Since ACTs are of type $(0, 4)$, they consist of four V^*'s, and zero V's with which to pair and contract. To solve this dilemma, we will convert one V^* to a V through a process described in Theorem 2.14 below. Theorem 2.14 guarantees that this process is a unique process. We will see that using the inner product will allow us to determine a covector's *metric equivalent* vector. Converting one V^* to a V will give us a tensor of type $(1, 3)$, and we will then pair one V with one V^* and contract to get a tensor of type $(0, 2)$. We will repeat this process of converting, pairing, and contracting until we get a real number of type $(0, 0)$. To demonstrate more clearly in a diagram, we will be doing the following:

$$(0, 4) \overset{type\ change}{\longrightarrow} (1, 3) \overset{contract}{\longrightarrow} (0, 2) \overset{type\ change}{\longrightarrow} (1, 1) \overset{contract}{\longrightarrow} (0, 0).$$

The following theorem, from [O'N83], defines the process of contraction by using the inner product to account for the difference in length when we convert a covector to a vector. It allows us to determine a covector's unique *metric equivalent*.

Theorem 2.14. *Given a nondegenerate inner product φ, and $\theta \in V^*$, there exists a unique $v \in V$ so that for all $w \in V$,*

$$\theta(w) = \varphi(v, w).$$

Let $\{e_i\}$ be a basis for V, and let $< \cdot, \cdot >$ be an inner product on V. Define $g_{ij} = < e_i, e_j >$, where e_i and e_j are basis vectors for V. We will express the components of g in the matrix $[g_{ij}]$. In our matrix $[g_{ij}]$, we will use g_{ij} to represent the i^{th} row and j^{th} column of the matrix. In order to raise or lower indices and contract, we will use the g matrix and its inverse. The following lemma from [O'N83] gives us valuable insight about the invertibility of our matrix $[g_{ij}]$.

Lemma 2.15. *A symmetric bilinear form is nondegenerate if and only if its matrix relative to one (hence every) basis is invertible.*

Since our inner product is nondegenerate, we know that the matrix $[g_{ij}]$ has an inverse that exists, call it $[g^{ij}]$. We will use g^{ij} to represent the i^{th} row and j^{th} column of the inverse of the matrix $[g_{ij}] = [g^{ij}]^{-1}$. In other words, $[g_{ij}]^{-1} = [g^{ij}]$, where g_{ij} are the entries of $[g_{ij}]$, and g^{ij} are the entries of $[g^{ij}]$.

Using the g^{-1} matrix, the metric equivalent of e^i is $\sum g^{ik} e_k$. A tangible example of determining a covector's metric equivalent vector follows below.

Example 2.16. *Let V be a real vector space with basis $\{e_1, e_2\}$ and V^* be the associated dual space with basis $\{e^1, e^2\}$. Let g be a symmetric bilinear form with $g_{ij} = g(e_i, e_j) = < e_i, e_j >$. Suppose $g = 2e^1 \otimes e^1 - 3e^2 \otimes e^2$. We want to determine the metric equivalent of e^1. We will first construct a 2×2 matrix, the matrix $[g_{ij}]$.*

We know that $g(e_1, e_1) = 2$, and so $g_{11} = 2$. This means that in the first row, first column of our matrix, the entry will be 2. Further, we know $g(e_2, e_2) = -3$, and so $g_{22} = -3$, meaning the second row, second column of our matrix will be -3. The remaining entries of our matrix, g_{12} and g_{21}, will be 0, since $g(e_1, e_2) = g(e_2, e_1) = 0$.

We now have our matrix:

$$[g_{ij}] \quad = \quad \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

The inverse of this matrix is:

$$[g^{ij}] = [g_{ij}]^{-1} \quad = \quad \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}.$$

So, $g^{11} = \frac{1}{2}$*,* $g^{22} = -\frac{1}{3}$*, and* $g^{12} = g^{21} = 0$*. In order to determine the metric equivalent of* e^1*, we will use the fact that the metric equivalent of* e^i *is* $\sum g^{ik} e_k$*. Now,*

$$\begin{aligned} e^1 &= \textstyle\sum_{k=1}^{2} g^{1k} e_k \\ &= g^{11} e_1 + g^{12} e_2 \\ &= \tfrac{1}{2} e_1 + 0 e_2 \\ &= \tfrac{1}{2} e_1. \end{aligned}$$

Therefore, the metric equivalent of e^1 *is* $\frac{1}{2} e_1$*.*

In summary, given a vector space with a nondegenerate inner product and a dual vector, there is a unique way that you can take in a dual vector and produce a vector from it. This is an isomorphism from V^* to V. To summarize this process with an example, if $R \in \otimes^4 V^*$, then we can contract R to an element of $\otimes^2 V^*$ by choosing an index to raise and then choosing another index to contract it with.

Contracting an ACT down to its Ricci Tensor

Let $\xi = (V, G, \sigma)$ be a representation. The vector space of symmetric bilinear forms on V which are G-invariant is known as the space of *quadratic invariants* and denoted as $\mathcal{I}_2^G(\xi)$. Although we will go into more detail about $\mathcal{I}_2^G(\xi)$ in Chapter 3, Section 2, we want to provide the reader with some foreshadowing and context in this section. The space $\mathcal{I}_2^G(\xi)$ is generated by full contractions of indices (see [VGN12], page 49). These contractions are examples of elements in $\mathcal{I}_2^G(\xi)$, where $\xi = \mathcal{A}(V)$, and the dimension of this space is equal to the number of independent ways to fully contract. This dimension will prove to be important when we want to verify that the submodules we have found in the decomposition of the space of ACTs are irreducible and inequivalent, using Lemma 3.7, Assertion (2), and we will determine this dimension in Chapter 4.

In the following example of contraction, we will start with an algebraic curvature tensor and contract the second and third indices with the result being a symmetric bilinear form known as a *Ricci tensor*.

Example 2.17. *This example will demonstrate how exactly to contract an ACT down to its Ricci tensor, and we will carefully explain our method. The final result will be the Ricci tensor. There are a few ways to contract an ACT to its Ricci tensor, but in this example, we will choose to lower the third index and contract against the second. We will* *examine the contraction of an ACT down to its Ricci tensor choosing other indices to lower and contract against later.*

We will start with our definition of an ACT, and then we will lower the third index and contract it against the second.

$$R = \sum R_{ijkl} e^i \otimes e^j \otimes e^k \otimes e^l$$

Change e^k *to its metric equivalent by replacing it with* $\sum_p g^{kp} e_p$*, which becomes:*

$$\sum g^{kp} R_{ijkl} e^i \otimes e^j \otimes e_p \otimes e^l$$

We have now changed R *from a type* $(0,4)$ *tensor to a type* $(1,3)$ *tensor. We will now contract* e^j *and* e_p*:*

$$\sum g^{kp} R_{ijkl} e^i \otimes e^l \cdot e^j(e_p)$$

The only time $e^j(e_p) \neq 0$ *is when* $p = j$*, so we can replace all* p*'s with* j*'s, which leaves us:*

$$\sum g^{kj} R_{ijkl} e^i \otimes e^l = \rho$$

We now have our Ricci tensor. In other words, if we feed the Ricci tensor (e_i, e_l)*, we would get:*

$$\rho(e_i, e_l) = \sum_{k,j} g^{kj} R_{ijkl}$$

The symmetric bilinear form that resulted from our contraction example above is known as the *Ricci Tensor* of the ACT from which it came. A succinct definition of the Ricci tensor follows.

Given an algebraic curvature tensor $R \in \mathcal{A}(V)$, an inner product φ, and an orthonormal basis, where $\varphi(e_i, e_j) = \varepsilon_i \delta_{ij}$ and $\varepsilon_i = \pm 1$, we can use φ to construct the Ricci tensor (denoted ρ). After lowering the third index and contracting the middle two indices against each other in an ACT, we get the Ricci tensor. We prove below that the Ricci tensor ρ is, in fact symmetric. In summary, the Ricci tensor is a symmetric bilinear form, $\rho \in S^2(V)$, where $\rho(x, y) = \sum_{i=1}^{n} \varepsilon_i R(x, e_i, e_i, y)$, with $\{e_i\}$ being the orthonormal basis for V and $x, y \in V$.

Proposition 2.18. *The Ricci tensor* ρ *is a symmetric bilinear form, and so* $\rho \in S^2(V)$*.*

Proof. Let V be a finite dimensional, real vector space with orthonormal basis $\{e_i\}$, with $x, y \in V$, and let $R \in \mathcal{A}(V)$. Using the algebraic curvature tensor properties, we can show that the Ricci tensor ρ is symmetric:

$$\begin{aligned}\rho(x,y) &= \sum R(x,e_i,e_i,y)\\ &= \sum R(e_i,y,x,e_i)\\ &= \sum -R(y,e_i,x,e_i)\\ &= \sum R(y,e_i,e_i,x)\\ &= \rho(y,x).\end{aligned}$$

Thus we can see that the Ricci tensor is symmetric. □

Note that algebraic curvature tensors can be divided into two categories: those that have a zero Ricci tensor (called *the Weyl conformal tensors*, a subspace of $\mathcal{A}(V)$ which we will discuss later) and those that have nonzero Ricci tensors. We will use $\mathcal{W}$ to denote the space of Weyl tensors, where $\mathcal{W}$ is the kernel of the map from the algebraic curvature tensors to the Ricci tensor. We can express the space of Weyl tensors as $\mathcal{W} = \ker(R \mapsto \rho)$.

There are a few ways to do the contraction from an ACT to the Ricci tensor because there are four V^*'s we can choose to convert to a V, and then we have a choice of three V^*'s with which to pair our newly converted V with. For the process of contraction, we will define $e^i : V \to \mathbb{R}$

$$e^i(e_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

where $e^i \in V^*$, and $e_j \in V$. As previously mentioned, we will utilize the process of type changing and contraction in Chapter 4 when we determine the number of independent ways to contract the ACTs down to a real number.

In Example 2.17 we lowered the third index of our ACT and contracted it against the second. Any other choice of contraction from the $(0,4)$ tensor to some $(0,2)$ tensor involves lowering just one of the four indices and contracting against some other one of the remaining three indices. Our next proposition notes that after contracting an ACT, its Ricci tensor, ρ, is the same up to a sign.

Proposition 2.19. *Up to a sign, there is only one nonzero way to contract $R \in \mathcal{A}(V)$ to a $(0,2)$ tensor.*

Proof. In order to change an $(0,4)$ tensor to a $(0,2)$ tensor, we need to pick one slot to type change and another slot to contract it against. Choose any one of the four slots to type change. Up to a sign, we can make our chosen slot be in any position we want. In Example 2.17, we chose slot number three to type change. If we chose it to be the first slot, we could swap indices in groups to now make our chosen slot be slot number three. If we chose it to be slot number two, we can swap the first two indices which introduces a minus sign, and then swap the indices in groups so that our chosen index is in slot number three. Up to a sign, we can always move the index we have chosen to type change to the third slot. It does not matter which vector we want to type change because, up to a sign, we can use the symmetries of the curvature tensor to move it to the third slot.

Once the vector in the third slot is chosen to type change, the vector we contract against must either be the fourth vector (in which case we contract to be zero due to the antisymmetric property of an ACT) or the vector in slot one or two. However, up to a sign, it may as well be the second slot. Thus, up to a sign, there is only one nonzero way to contract a $(0,4)$ tensor to a $(0,2)$ tensor. □

Contracting the Ricci Tensor to the Scalar Curvature

The Ricci tensor is a type $(0,2)$ tensor, so we are only able to lower one of the two indices and contract it against the other. However, since it is symmetric, we will get the same output no matter which index we choose to lower and subsequently contract against the remaining index.

Continuing our contraction from Example 2.17, we will start off with the symmetric bilinear form we ended up with when contracting the ACT. We will contract this once more to get a real number, our scalar curvature τ. Begin with our Ricci tensor from Example 2.17:

$$\rho \;=\; \sum g^{kj} R_{ijkl} e^i \otimes e^l$$

Changing e^l to its metric equivalent $\sum_q g^{lq} e_q$, gives us:

$$\sum g^{kj} g^{lq} R_{ijkl} e^i \otimes e_q$$

Contracting e^i and e_p, we have:

$$\sum g^{kj} g^{lq} R_{ijkl} e^i(e_q)$$

We now have our scalar curvature:

$$\sum g^{kj} g^{lq} R_{ijkl} = \tau.$$

2.2.2 Exact Sequences of Vector Spaces and Representations

As described earlier in this chapter, the space of algebraic curvature tensors gets sent to the space of symmetric bilinear forms by the map defined by the process of contraction, call it ρ. Diagrammatically,

$$\mathcal{A}(V) \xrightarrow{\rho} S^2(V^*).$$

The kernel of the map ρ is all of the ACTs that contract to zero Ricci tensors, which we denoted $\mathcal{W}$ earlier, and can be included in the space of algebraic curvature tensors:

$$\ker(\rho) \longrightarrow \mathcal{A}(V) \xrightarrow{\rho} S^2(V).$$

The above is the beginning of a sequence of vector spaces which will be revisited later in this thesis when we decompose the space of algebraic curvature tensors. We will see that $\mathcal{A}(V) = \mathcal{W} \oplus S^2(V)$.

Let V_1, V_2, and V_3 be arbitrary vector spaces, with f a map from V_1 to V_2 and g a map from V_2 to V_3. Recall that an *exact sequence* is a sequence of vector spaces with linear maps between them satisfying a certain property. The sequence is said to be *exact* at V_2 if the image of the map f is equal to the kernel of the map g. See the diagram below:

$$V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3, \text{ where } im(f) = \ker(g).$$

Further, the sequence is said to be *short exact* if instead of being exact everywhere and of arbitrary length, it looks like the sequence below:

$$0 \longrightarrow V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3 \longrightarrow 0$$

The assumption that the sequence above is short exact allows us to make additional conclusions about the maps f and g. In a short exact sequence, the map $0 \to V_1$ is predetermined because it is a linear function that has only 0 in its domain, and linear maps send 0 to 0. Further, $im(0) = \ker f$ because the sequence is exact, and so $\ker f = 0$, which means f is injective. Similarly, the map from $V_3 \to 0$ is predetermined as well because everything in V_3 gets sent to 0. Futher, $im(g)$ is equal to the kernel of the map from V_3 to 0, but the kernel of this map is everything, which means $im(g)$ is everything, and so g is surjective.

The last category of sequence we will use is an *$\mathcal{O}$-equivariant short exact sequence*, which involves $\mathcal{O}$-representations. Suppose there is a group $\mathcal{O}$ that acts on V_1, V_2, and V_3 via $h \in \mathcal{O}$; V_1, V_2 and V_3 are $\mathcal{O}$-representations. An element in $\mathcal{O}$ acting on an element from V_i would be a function from V_i to V_i, with $i = 1, 2, 3$. For every $h \in \mathcal{O}$, the following diagram commutes, and we call it an $\mathcal{O}$-equivariant short exact sequence:

$$
\begin{array}{c}
0 \rightarrow V_1 \stackrel{f}{\rightarrow} V_2 \stackrel{g}{\rightarrow} V_3 \rightarrow 0 \\
\downarrow_{h\cdot(\ )} \downarrow_{h\cdot(\ )} \downarrow_{h\cdot(\ )} \\
0 \rightarrow V_1 \stackrel{f}{\rightarrow} V_2 \stackrel{g}{\rightarrow} V_3 \rightarrow 0
\end{array}
$$

In short, the action of the group commutes with the functions that are involved. We will use an $\mathcal{O}$-equivariant short exact sequence in Chapter 4 when we decompose the space of ACTs.

Chapter 3

General Representation Theory

This chapter will discuss general representation theory results that will later be applied in our next chapter to complete our goal of decomposing the space of algebraic curvature tensors. Although the following results involve are more generalized, our specific representation involving the ACTs with G the orthogonal group of our nondegenerate inner product, vector space $\mathcal{A}(V) \subseteq \otimes^4 V^*$, and action $(gR)(x, y, z, w) = R(g^{-1}x, g^{-1}y, g^{-1}z, g^{-1}w)$ with $g \in G$, and $R \in \mathcal{A}(V)$, meets the conditions of the hypotheses, and thus, each result can be applied to our representation involving the ACTs. All theorems and lemmas in this chapter follow from Gilkey's Geometric Realizations of Curvature [VGN12]. In Section 1, we will discus some general results about decomposing a representation. Then, in Section 2, we will discuss some basic tests for irreducibility of a representation. Throughout the chapter, we will let V be a finite dimensional, real vector space, G be the orthogonal group of our nondegenerate inner product where $G = O(< \cdot, \cdot >) = \{A \in GL | < Ax, Ay > = < x, y >\}$, and σ be a group homomorphism from G to GL. Then, $\xi := (V, \sigma)$ is a *module*, or *representation of* G, with vector space V and group G as specified above. If $v \in V$ and $g \in G$, then we will define $g \cdot v := \sigma(g)v$.

3.1 Representations for the Group G

The focus of this section is to examine a few general but important results related to decomposing a representation. Our first definition explains how to make the inner product on $\otimes^k V^*$ by restricting it to a subspace, which maintains the inner product's nondegenerate property. Further, it mentions how to use the inner product to identify V

with V^*, which is explained in Theorem 2.14 in our Preliminaries Chapter.

Definition 3.1. *Let V^k denote the Cartesian product $V \times \cdots \times V$. If $\vec{v} = (v_1, ..., v_k)$ and $\vec{w} = (w_1, ..., w_k)$ are elements of V^k, the map*

$$\vec{v} \times \vec{w} \to < v_1, w_1 > \cdots < v_k, w_k >$$

is a bilinear symmetric map from $V^k \times V^k$ to $\mathbb{R}$ which extends to a symmetric inner product that is the extension of $< \cdot, \cdot >$ to $\otimes^k V$. If $\{e_i\}$ is an orthonormal basis for V and if $I = (i_1, ..., i_k)$ is a multi-index, let $e_I := e_{i_1} \otimes \cdots e_{i_k}$. The collection $\{e_I\}_{|I|=K}$ forms a basis for $\otimes^k V$ with

$$< e_I, e_K > = \begin{cases} 0 & \text{if } I \neq K \\ < e_{i_1}, e_{i_1} > \cdots < e_{i_k}, e_{i_k} > & \text{if } I = K. \end{cases}$$

Since $< e_I, e_I > = \pm 1$, $< \cdot, \cdot >$ is non-degenerate on $\otimes^k V$. The orthogonal group $\mathcal{O}$ extends to act naturally on $\otimes^k V$ and preserves this inner product.

If we start with an inner product on V, we can modify it to be an inner product on V^*. We could also modify it to be an inner product on tensor powers of V, and then subsequently modify that to be an inner product on tensor powers of V^*. This is how we end up with the natural extension of the inner product on $\otimes^k V^*$.

Let $G \in \mathcal{O}$. We will define the action of G on the tensor $\otimes^k V^*$ as $(T^*\theta)(v_1, ..., v_k) := \theta(Tv_1, ..., Tv_k)$ [VGN12]. In a representation $\sigma : G \to End(V)$ must be a homomorphism. However the action of the group defined as above is not a representation because it would lead to $\sigma(gh) = \sigma(h)\sigma(g)$. Nothing changes if we use the action defined above or if we precompose with the inverse. So, for convenience, we will define the action of the group on a tensor as shown below:

$$(g \cdot \theta)(v_1, ..., v_k) = \theta(g^{-1}v_1, ..., g^{-1}v_k).$$

Our next lemma plays an important role in this chapter because it allows following results to apply to our defined representation ξ. Its proof (see [VGN12] page 39) is standard and will be omitted for brevity.

Lemma 3.2. *Use Definition 3.1 to extend the given inner product $< \cdot, \cdot >$ to $\otimes^k V^*$. Let W be a non-trivial subspace of $\otimes^k V^*$ which is invariant under the action of the group*

G where G belongs to $\mathcal{O}$. Then the restriction of $< \cdot, \cdot >$ to W is non-degenerate. In particular, W is not totally isotropic.

In short, under the specified conditions of the hypothesis, any invariant subspace is not going to be totally isotropic. This means that even under the specified conditions, we can still split off subspaces from the vector space when we decompose, and our direct sum decomposition will not have an overlap between vector spaces.

Let ξ_1 and ξ_2 be modules with structure group G, g an element of the group G, and V_1, V_2 vector spaces associated with ξ_1, ξ_2, respectively. We will define

$$\begin{aligned} Hom(\xi_1, \xi_2) &:= Hom_G(V_1, V_2) \\ &:= \{T \in Hom(V_1, V_2) : T(g \cdot v_1) = g \cdot Tv_1\} \end{aligned}$$

as the linear space of *intertwining operators.*

Definition 3.3. *Let ξ_1 and ξ_2 be modules with structure group G, g an element of the group G, and V_1, V_2 vector spaces associated with ξ_1, ξ_2, respectively. ξ_1 and ξ_2 are* isomorphic modules with structure group G *if there exists a $T \in Hom(\xi_1, \xi_2)$ that is a vector space isomorphism.*

Our next lemma notes that two irreducible modules are isomorphic if and only if the dimensions of the homomorphism between them is positive. Restated, if the dimension of the homomorphisms between two irreducible modules ξ_1 and ξ_2 is positive, there is a nonzero map between the modules that is an isomorphism, that is g-intertwining. The proof is well-known and can be found in [VGN12] on page 41. This lemma will play a crucial role in the proof of Lemma 3.6.

Lemma 3.4. *Two irreducible modules ξ_1 and ξ_2 with structure group G are isomorphic if and only if* $\dim\{Hom(\xi_1, \xi_2)\} > 0$.

The following lemma has two main parts. Part one tells us that the vector space associated with representation ξ decomposes into the direct sum of finitely many other subrepresentations (since the vector space is finite), and each subrepresentation is irreducible. Further, the vector space corresponding to one of them is perpendicular to the vector space corresponding to any other separate one. In this lemma, the fact that no non-trivial submodule is totally isotropic is crucial because otherwise the direct sum decomposition may have an overlap. Keeping in mind our main goal of decomposing the space of ACTs, the lemma tells us that we can decompose the vector space associated with our representation into irreducible subspaces.

Part two of the lemma tells us that when the sequence involved is short exact, then the middle term is the direct sum of the two on either side of it. This will play a key role in our next chapter when decomposing the space of ACTs. The importance of this lemma cannot be understated when it comes to the decomposition of the space of ACTs in our next chapter. The lemma states that every representation is completely reducible into orthogonal subrepresentations.

Lemma 3.5. *Let $\xi = (V, \sigma)$ be a module with structure group G admitting a non-degenerate inner product $< \cdot, \cdot >$ which is invariant under the action of G so that no non-trivial submodule of V is totally isotropic. Then:*

1. *There is an orthogonal direct sum decomposition $\xi = \eta_1 \oplus \cdots \oplus \eta_l$ where the η_i are irreducible and where $V_{\eta_i} \perp V_{\eta_j}$ for $i \neq j$.*
2. *Let $0 \to \xi_1 \to \xi \to \xi_2 \to 0$ be a short exact sequence of modules for the group G. Then ξ is isomorphic to $\xi_1 \oplus \xi_2$.*

Proof. If ξ is irreducible, then there is no further decomposition, and Assertion (1) has been proven. So, suppose ξ is reducible. Then, by definition, there exists some nontrivial, invariant subspace, call it V_1. If V_1 were not to be irreducible, we could look inside of V_1 to find a nontrivial, proper subspace of V_1 which may or may not be irreducible. If it is not irreducible, we could look inside of that space for a nontrivial, proper subspace that may or may not be irreducible. Eventually, this search will result in a nontrivial, proper, and irreducible subspace since the dimension of V is finite, and each time we find a proper subspace, the dimension decreases until we eventually end up with a subspace that is irreducible.

Choose V_1 to be a proper, irreducible subspace of V with $V_1 \neq 0$. Let the action on V_1 be $\sigma_1 = \sigma|_{V_1}$. Restricting σ to V_1 makes ξ_{V_1} its own representation since V_1 was chosen to be invariant, and so σ_1, which is the action on V_1 from the same group, sends V_1 back to V_1. Now, $V_1^\perp$ is also a subspace of V, and it is invariant under the action of G. Further, since $V_1 \cap V_1^\perp$ is a totally isotropic subspace of V, and by assumption, there is no non-trivial submodule of V which is totally isotropic, we have $V_1 \cap V_1^\perp = \{0\}$. Thus, we can split V into an orthogonal direct sum of our two nontrivial subpaces V_1 and $V_1^\perp$, both of which are invariant under the action of G. In other words, $V = V_1 \oplus V_1^{\perp}$. Now, if $V_1^{\perp}$ is irreducible, then we are done. But, if $V_1^{\perp}$ is reducible, then we continue on with this same process until V has been decomposed into the orthogonal direct sum of only irreducible, nontrivial subspaces, all of which are invariant under the action of G. Since each subspace that we split V into is nontrivial, the dimension of V reduces with each subspace that splits off. So, we have $\dim(V) > \dim(V_1^{\perp}) > ...$, all of which are positive integers, and the process does not continue indefinitely. Therefore, Assertion (1) now follows.

We now begin the proof of Assertion (2). In our short exact sequence $0 \to \xi_1 \to \xi \to \xi_2 \to 0$, let us define the maps $f : V_{\xi_1} \to V_{\xi}$ and $\pi : V_{\xi} \to V_{\xi_2}$. Because our sequence is short exact, we know that the function f is injective, and the function π is surjective. We claim that $im(f) \cong V_{\xi_1}$. We know $f : V_{\xi_1} \to V_{\xi}$, and if we redefine our function f to be $f : V_{\xi_1} \to im(f)$, we now have our function f as a surjective function. Since our function f is both injective and surjective, f is an isomorphism from V_{ξ_1} to the image of f, and so $im(f) \cong V_{\xi_1}$. Now, $V_{\xi_1} \cong im(f)$. The image of a representation is itself a representation, and so $f(\xi_1) \cong \xi_1$.

We must now determine whether or not $f(V_{\xi_1})$ is invariant under the action of G, meaning we want to know if $g \cdot f(v_1) \in f(V_{\xi_1})$ if $v_1 \in V_{\xi_1}$. By definition, f is a module homomorphism from V_{ξ_1} to V_{ξ} that is intertwining. Therefore, $g \cdot f(v_1) = f \cdot g(v_1)$. Since V_{ξ_1} is invariant under the action of G, $gv_1 \in V_{\xi_1}$, and now applying the function f, we have $f(g(v_1)) \in f(V_{\xi_1})$. So, $f(V_{\xi_1})$ is invariant under the action of G. Because $f(\xi_1)$ is a submodule of ξ, so, too, is $f(\xi_1)^{\perp}$.

Since f is intertwining, the representation associated with the vector space of $im(f)$ is isomorphic to the vector space associated with V_{ξ_1}, or ξ_1. Further, since $f(\xi_1) \cap f(\xi_1)^{\perp}$ is a totally isotropic submodule of ξ, and by assumption, there are no non-trivial submodules of ξ which are totally isotropic, we have $f(V_{\xi_1}) \cap f(V_{\xi_1})^{\perp} = \{0\}$. So, we have $V_{\xi} = f(V_{\xi_1}) \oplus f(V_{\xi_1})^{\perp}$. Now, $im(f) = f(V_{\xi_1}) = \ker(\pi)$ since the sequence is exact. Consider, the map $\widetilde{\pi} : f(V_{\xi_1})^{\perp} \to V_{\xi_2}$, it is the map π with the domain restricted to $f(V_{\xi_1})^{\perp}$. We know that the map π is onto, and we want to show that the map $\widetilde{\pi}$ is onto as well. Let $w \in V_{\xi_2}$. We need to find a $\widetilde{v} \in im(f)^{\perp}$ such that $\widetilde{\pi}(\widetilde{v}) = w$. Since $\pi : V_{\xi} \to V_{\xi_2}$ is onto, and $w \in V_{\xi_2}$, we know there exists a $v \in V_{\xi}$ such that $\pi(v) = w$. Recall that $V_{\xi} = f(V_{\xi_1}) \oplus f(V_{\xi_1})^{\perp}$, and so vector $v \in V_{\xi} = f(V_{\xi_1})$ can be expressed as

$v = v_1 + \widetilde{v}$, with $v_1 \in f(V_{\xi_1})$ and $\widetilde{v} \in im(f)^{\perp}$. So, we have:

$$\begin{aligned} w &= \pi(v) \\ &= \pi(v_1 + \widetilde{v}) \\ &= \pi(v_1) + \pi(\widetilde{v}) \text{ but since } v_1 \text{ comes from } f(V_{\xi_1}) \cong im(f) = ker(\pi), \text{ we know } \pi(v_1) = 0. \\ &= \pi(\widetilde{v}) \end{aligned}$$

Therefore, the map $\widetilde{\pi}$ is onto.

Now, since the map $\widetilde{\pi}$ is both intertwining (since it's a module homomorphism) and onto, we claim that if $\widetilde{\pi}$ is injective, then $\widetilde{\pi}$ is an isomorphism of modules. In order to show that the map $\widetilde{\pi}$ is injective, we will show that $\ker \widetilde{\pi} = \{0\}$. Let $u \in ker(\widetilde{\pi})$. So, $u \in f(V_{\xi_1})^{\perp}$, and $\widetilde{\pi}(u) = \pi(u)$. Suppose that $\widetilde{\pi}(u) = 0$. Then, $\pi(u) = 0$, and so $u \in ker(\pi)$. But, $ker(\pi) = im(f)$ since the sequence is exact, and we know that $im(f) = f(V_{\xi_1})$, and so $u \in f(V_{\xi_1})$ and $u \in f(V_{\xi_1})^{\perp}$. Thus, $u \in f(V_{\xi_1}) \cap f(V_{\xi_1})^{\perp}$. Recall that $f(V_{\xi_1}) \cap f(V_{\xi_1})^{\perp} = \{0\}$ since it's a totally isotropic subspace, and so $u = 0$. So, $\widetilde{\pi}$ is injective!

Now, putting all the pieces together, we have the representation $\xi = f(\xi_1) \oplus f(\xi_1)^{\perp}$, with $f(\xi_1) \cong \xi_1$ via the isomorphism f, and $f(\xi_1)^{\perp} \cong \xi_2$ via the isomorphism $\widetilde{\pi}$. Thus, $\xi = \xi_1 \oplus \xi_2$. □

3.2 Tests for Irreducibility

This section will focus on some tests for the irreducibility of a representation. Our representation $\xi = (G, V, \sigma)$ remains the same as before where the group G is the orthogonal group of our nondegenerate inner product, V is a real, finite vector space of dimension n, and σ is the action of the group G on the vector space V. We will express the decomposition of ξ as $\xi = \sum n_i \xi_i$ where the ξ_i's are invariant subrepresentations of ξ and there are n_i isomorphic copies of each ξ_i. Further, each ξ_i has a corresponding subspace, call it V_i or V_{ξ_i}, with $V_i \subset V$. Another way of expressing this is $\xi = \sum \eta_i$ where $\eta_i = n_i \xi_i$.

It is important to discuss a few key concepts that are crucial in the following lemma. Let g be an element of the group G, and V_1, V_2 be finite, real dimensional vector spaces, with $v_1 \in V_1$ and $v_2 \in V_2$. We will call $Hom^{sa}(\xi, \xi)$ the subspace of self-adjoint maps from ξ to ξ that are equivariant under the action of G:

$$Hom^{sa}(\xi,\xi) \quad := \quad \{T \in Hom(\xi,\xi) :< Tv_1, v_2 >=< v_1, Tv_2 >\}.$$

We will use $\mathcal{I}(\xi) = \mathcal{I}^G(\xi)$ to represent the set of linear maps from V to $\mathbb{R}$ that are G-equivariant. This is the space of *linear invariants.* In other words, it is a linear function $\theta : V \to \mathbb{R}$ so that:

$$\theta(v) \quad = \quad \theta(gv).$$

$\mathcal{I}_2^G(\xi)$, the space of *quadratic invariants*, will represent the vector space of symmetric bilinear forms on V which are G-invariant. Note that $\mathcal{I}_2^G(\xi)$ is nonempty because our inner product is a symmetric bilinear form that is invariant under the action of the group G, and so $\dim\{\mathcal{I}_2^G(\xi)\} > 0$.

We will use the following equation in the proof of Lemma 3.6. It uses the idea of type changing described previously in Section 2.1 to identify bilinear forms with linear maps. We can start with a linear map and produce a bilinear form from it, and the process is reversible; we can start with a bilinear form and from that produce a linear map. Let θ be a bilinear form on a finite dimensional, real vector space V, and let T_θ be a linear map of V_ξ uniquely determined by this equation:

$$\theta(v_1, v_2) = < v_1, T_\theta v_2 > . \tag{3.1}$$

For our next lemma, it may help the reader to point out the contrast that $\mathcal{I}(\xi \otimes \xi)$ is the vector space of all bilinear forms which are invariant under the action of G, whereas $\mathcal{I}_2^G(\xi)$ is the vector space of all symmetric bilinear forms on V which are invariant under the action of G. In Lemma 3.6, Assertion (1) notes that given a bilinear form, we can produce a function that takes us from $\mathcal{I}(\xi \otimes \xi)$ to $Hom(\xi,\xi)$, or from $\mathcal{I}_2^G(\xi)$ to $Hom^{sa}(\xi,\xi)$. Assertion (2) provides more detail about $Hom^{sa}(\xi,\xi)$, and Assertion (3) tells us the dimension of $\mathcal{I}_2^G(\xi)$ in terms of the dimensions of these Hom spaces.

Lemma 3.6. *Let ξ be a module for the group G admitting a non-degenerate inner product $< \cdot, \cdot >$ which is invariant under the action of G with no non-trivial totally isotropic submodules.*

1. *The map $\theta \to T_\theta$ identifies $\mathcal{I}(\xi \otimes \xi)$ with $Hom(\xi,\xi)$ and $\mathcal{I}_2^G(\xi)$ with $Hom^{sa}(\xi,\xi)$.*

2. *Suppose $\xi = \eta_1 \oplus \cdots \oplus \eta_l$ where the modules η_i are not necessarily irreducible. Then $Hom^{sa}(\xi,\xi) = \oplus_i Hom^{sa}(\eta_i,\eta_i) \oplus_{i<j} Hom(\eta_i,\eta_j)$.*

3. *Suppose that each* η_i *decomposes as* n_i *copies of irreducible modules* ξ_i *where* ξ_i *is not isomorphic to* ξ_j *for* $i \neq j$*. Then* $\dim\{Hom(\eta_i, \eta_j)\} = 0$ *for* $i \neq j$ *and* $\dim\{\mathcal{I}_2^G(\xi)\} = \sum_i n_i \dim\{Hom^{sa}(\xi_i, \xi_i)\} + \frac{1}{2}n_i(n_i - 1)\dim\{Hom(\xi_i, \xi_i)\}$.

Proof. The map $\theta \to T_\theta$ takes in a bilinear form θ, and produces a function T_θ. By Equation 3.1, we can take a bilinear form, an element of $V^* \otimes V^*$ and associate it with a linear map.

Now, $\mathcal{I}(\xi \otimes \xi)$ are multilinear functions that take in pairs of vectors and produce a real number, in other words, $\mathcal{I}(\xi \otimes \xi)$ are bilinear forms. So, $\theta \in \mathcal{I}(\xi \otimes \xi)$ is a bilinear function that is G-equivariant. In other words, $\theta(v_1, v_2) = \theta(gv_1, gv_2)$. Given such a θ, we can produce a linear map $T_\theta : V \to V$, and so θ is a homomorphism from $V \to V$. In order for θ to be a module homomorphism $Hom(\xi, \xi)$, we need to check that it is intertwining. We need $T_\theta(gv) = gT_\theta(v)$. Now, the function that takes in θ and produces T_θ is a vector space isomorphism from θ to T_θ. And,

$$\begin{aligned} \theta(gv_1, gv_2) &= < gv_1, T_\theta g v_2 > \text{ (by Equation 3.1)} \\ &= < g^{-1}gv_1, g^{-1}T_\theta g v_2 > \\ &= < v_1, g^{-1}T_\theta g v_2 > . \end{aligned}$$

θ is a linear invariant, meaning:

$$\begin{aligned} \theta(gv_1, gv_2) &= \theta(v_1, v_2) \\ &= < v_1, T_\theta v_2 > \text{ (by Equation 3.1).} \end{aligned}$$

Since $< \cdot, \cdot >$ is nondegenerate, $< v_1, T_\theta v_2 > = < v_1, g^{-1}T_\theta g v_2 >$.

Finally, multiplying both sides of $g^{-1}T_\theta g = T_\theta$ by g on the left, we have $T_\theta g = gT_\theta$, and we have that T_θ is equivariant under the action of the group G.

We have:

$$< v_1, T_\theta v_2 > - < v_2, T_\theta, v_1 > = \theta(v_1, v_2) - \theta(v_2, v_1).$$

We can see in the above equation that if T_θ is self-adjoint, θ must be symmetric, and similarly, if θ is symmetric, T_θ must be self-adjoint. Thus, θ is symmetric if and only if T_θ is self-adjoint with respect to the inner product. Therefore, $\mathcal{I}_2^G(\xi)$ can be identified with $Hom^{sa}(\xi, \xi)$. This concludes the proof of Assertion (1).

Suppose $\xi = \eta_1 \oplus \cdots \oplus \eta_l$ where the modules η_i are not necessarily irreducible, and let $V_\xi = V_{\eta_1} \oplus \cdots \oplus V_{\eta_l}$ be the vector spaces associated with the respective modules.

Suppose $\dim(V_{\eta_i}) = d_i$, and notate the basis for V_{η_i} as $V_{\eta_i} = \{e^i_1, e^i_2, \cdots, e^i_{d_i}\}$. Let θ be a bilinear form on V which is invariant under the action of G. Our bilinear form θ is determined by its values on pairs of basis vectors. If $v = \sum v_k$ and $w = \sum w_k$, where $v_k, w_k \in V_{\eta_k}$, then define $\theta_{ij}(v,w) = \theta(v_i, w_j) = < v_i, T_\theta w_j >$. Now, because θ is symmetric, each θ_{ii} is symmetric, however, each θ_{ij} with $i \neq j$ is not necessarily symmetric.

Define $\theta^t(x,y) = \theta(y,x)$ with $x, y \in V_\xi$. If $i = j$, then $\theta^t_{ii} = \theta_{ii}$ since θ_{ii} is symmetric. We know that symmetric bilinear forms θ_{ii} belong to $\mathcal{I}^G_2(\xi)$, and by Assertion (1), $\mathcal{I}^G_2(\xi)$ corresponds to $Hom^{sa}(\xi,\xi)$. So, θ_{ii} corresponds to $T_{\theta_{ii}} \in Hom^{sa}(\xi,\xi)$.

Now, for $i \neq j$, we have $\theta^t_{ji}(x,y) = \theta_{ji}(y,x)$ with $x, y \in V_\xi$. These bilinear forms need not be symmetric but can be expressed as θ_{ij} with $i < j$. Because these bilinear forms are invariant under the action of G but are not necessarily symmetric, they belong to $\mathcal{I}(\xi \otimes \xi)$, and by Assertion (1), $\mathcal{I}(\xi \otimes \xi)$ corresponds to $Hom(\xi,\xi)$.

Let π_i be the orthogonal projection onto V_i. A short digression to demonstrate that π_i is self-adjoint, we claim that $< \pi_i x, y > = < x, \pi_i y >$. We can write x as $x = x_i + x_i^\perp$, and y as $y = y_i + y_i^\perp$. Then, since projecting onto the i^{th} factor eliminates everything that is perpendicular to that factor, we have:

$$< \pi_i x, y > \quad = \quad < x_i, y_i + y_i^\perp > .$$

But, $y_i^\perp$ is perpendicular to anything that is in V_i, and $x_i \in V_i$, and so we have:

$$= \quad < x_i, y_i > + < x_i, y_i^\perp > .$$

However, x_i and $y_i^\perp$ are orthogonal since $x_i, y_i \in V_i$, and $y_i^\perp \in V_i^\perp$, so their inner product is zero. This leaves us with:

$$= \quad < x_i, y_i > .$$

We can apply the same line of reasoning starting with $< x, \pi_i y >$ and ending up with $< x_i, y_i >$. In short, π_i is self-adjoint, and this ends our digression.

Now, since π_i is self-adjoint, we have:

$$\begin{aligned} \theta_{ij}(v,w) = \theta(v_i, w_j) &= < v_i, T_\theta w_j > \\ &= < \pi_i v, (T_\theta \circ \pi_j) w > \\ &= < v, (\pi_i \circ T_\theta \circ \pi_j) w > . \end{aligned}$$

We want to conclude that $\theta_{ij}(v,w) = < v, (\pi_i \circ T_\theta \circ \pi_j)w >$, so that θ_{ij} corresponds to $Hom(V_j, V_i)$; if we feed elements from V_j to $\pi_i \circ T_\theta \circ \pi_j$, we get elements from V_i out.

We now need to check to see if $\pi_i \circ T_\theta \circ \pi_j \in Hom(V_j, V_i)$. In other words, we need to check if the map $\pi_i \circ T_\theta \circ \pi_j$ G-intertwining. We know that T_θ is G-intertwining since $gT_\theta = T_\theta g$. This is because we started off with θ being a symmetric bilinear form, and since $T_\theta \in Hom(V,V)$, it is intertwining.

To check that π_i and π_j are G-intertwining, we need to show that $\pi_i gv = g\pi_i v$. But, we know that $v = \sum v_i$, and so $\pi_i v = v_i$. So it follows that $g\pi_i v = gv_i$. We know that $gv = \sum gv_i$, and so $\pi_i gv = gvi$. Thus, π_i is G-intertwining, and π_j is G-intertwining as well.

So, to check to see if the composition $\pi_i \circ T_\theta \circ \pi_j$ is G-intertwining, we need to see if:

$$(\pi_i \circ T_\theta \circ \pi_j)g = g(\pi_i \circ T_\theta \circ \pi_j).$$

Since each map is G-interwining on its own, we can move g across the composition:

$$\begin{aligned}(\pi_i \circ T_\theta \circ \pi_j)g &= (\pi_i \circ T_\theta)g\pi_j \\ &= \pi_i g T_\theta \pi_j \\ &= g(\pi_i \circ T_\theta \circ \pi_j).\end{aligned}$$

Therefore, because all of the functions involved are G-intertwining, their composition must be as well. Thus, $\theta_{ij} \longleftrightarrow \pi_i \circ T_\theta \circ \pi_j \in Hom(V_j, V_i)$. So, θ_{ij} corresponds to $T_{\theta ij} \in Hom(\xi_j, \xi_i)$ with $i < j$.

We can express θ as the sum $\theta = \sum_{i=1}^n \theta_{ii} + \sum_{i<j} \theta_{ij}$. Applying the connection from Assertion (1), we can identify θ with T_θ, where $T_\theta = \sum_{i=1}^n T_{\theta_{ii}} + \sum_{i<j} T_{\theta_{ij}}$, with $\sum_{i=1}^n T_{\theta_{ii}} \in Hom^{sa}(\eta_i, \eta_i)$ and $\sum_{i<j} T_{\theta_{ij}} \in Hom(\eta_j, \eta_i)$. So, $\theta \in \mathcal{I}_2^G(\xi)$ breaks up into the sum of elements from $Hom^{sa}(\eta_i, \eta_i)$ and $Hom(\eta_j, \eta_i)$. Then, $\mathcal{I}_2^G(\xi)$ breaks up into the direct sum of $\sum_{i=1}^n \theta_{ii}$ and $\sum_{i<j} \theta_{ij}$. However, $\mathcal{I}_2^G(\xi) \cong Hom^{sa}(\xi, \xi)$. Therefore, $\mathcal{I}_2^G(\xi) \cong \oplus_{i=1}^n Hom^{sa}(\eta_i, \eta_i) \oplus_{i<j} Hom(\eta_j, \eta_i)$.

We now argue as follows to prove Assertion (3). Suppose that $\xi = \eta_1 \oplus \cdots \oplus \eta_l$ with $\eta_i = ni\xi_i$. Applying Lemma 3.4, since ξ_i is not isomorphic to ξ_j, we know that $dim\{Hom(\eta_i, \eta_j)\} = \oplus_{i,j} n_i n_j Hom(\xi_i, \xi_j) = 0$. Suppose that η_i is n_i copies of ξ_i where ξ_i is irreducible. In other words, let $\eta_i = n_i \xi_i$. We know from Assertion (1) that $\mathcal{I}_2^G(\xi)$ is identified with $Hom^{sa}(\xi, \xi)$. Our goal is to compute the dimension of $\mathcal{I}_2^G(\xi)$. We will instead compute the dimension of $Hom^{sa}(\xi, \xi)$, and this will give us the dimension of

$\mathcal{I}_2^G(\xi)$, as desired, since $Hom^{sa}(\xi,\xi)$ is isomorphic to $\mathcal{I}_2^G(\xi)$. Assertion (2) tells us that $Hom^{sa}(\xi,\xi)$ decomposes into $\oplus_i Hom^{sa}(\eta_i,\eta_i) \oplus_{i<j} Hom(\eta_i,\eta_j)$. But, as stated above, we know that $dim\{Hom(\eta_i,\eta_j)\} = 0$. So, we are left with $Hom^{sa}(\xi,\xi) = \oplus_i Hom^{sa}(\eta_i,\eta_i)$. We need to determine $\dim\{Hom(\eta_i,\eta_i)\}$, where η_i is n_i copies of ξ_i. Since $\xi = n_i\xi_i$, we have $\xi = \xi_i \oplus \xi_i \oplus \cdots \oplus \xi_i$, with n_i number of ξ_i. This results in n_i repeated summands of $\dim\{Hom^{sa}(\xi_i,\xi_i)\}$, which can be expressed as $\sum_i n_i \dim\{Hom^{sa}(\xi_i,\xi_i)\}$. This leaves us to determine the dimension of the Hom of any ξ_i with any other ξ_i. We have already accounted for the first set of summands of $\oplus_i Hom^{sa}(\eta_i,\eta_i) \oplus_{i<j} Hom(\eta_i,\eta_j)$, and this leaves us with $i < j$ up to n_i copies of ξ_i with any other ξ_i, or $\binom{n_i}{2}$ copies. So, we have $\frac{1}{2}n_i(n_i-1)\dim\{Hom(\xi_i,\xi_i)\}$. Counting up the total dimension, we have $\dim\{\mathcal{I}_2^G(\xi)\} = \sum_i n_i \dim\{Hom^{sa}(\xi_i,\xi_i)\} + \frac{1}{2}n_i(n_i-1)\dim\{Hom(\xi_i,\xi_i)\}$. □

Our final result for the section provides an estimation test for the decomposition of a representation. In short, it states that if we apply the following estimation to a particular decomposition of a representation and find equality, then we have found the correct decomposition, and each subrepresentation in the decomposition is irreducible and not isomorphic to any of the others. The assumption is, however, that we know *how* the representation decomposes into these invariant subspaces. If we find equality, then the groupings of subspaces has been done correctly in the decomposition, and the subrepresentations do not decompose any further, meaning there are no more subrepresentations to find. In the next chapter, we will apply this lemma to our decomposition of the space of ACTs.

Lemma 3.7. *Let $\xi = (V,\sigma)$ be a module for the group G admitting a non-degenerate inner product which is invariant under the action of G with no non-trivial totally isotropic submodules.*

1. *Suppose $\xi = \sum_i n_i\xi_i$ decomposes as a sum of modules ξ_i for the group G. Then*
$$\dim\{\mathcal{I}_2^G(\xi)\} \geq \sum_i \frac{1}{2}n_i(n_i+1).$$
2. *If equality holds in Assertion (1), then each ξ_i is irreducible and ξ_i is not isomorphic to ξ_j for $i \neq j$.*

Proof. Recall our equation from Lemma 3.6, Assertion (3):

$$\dim\{\mathcal{I}_2^G(\xi)\} = \sum_i n_i \dim\{Hom^{sa}(\xi_i,\xi_i)\} + \frac{1}{2}n_i(n_i-1)\dim\{Hom(\xi_i,\xi_i)\}.$$

We know that restricting the inner product to any non-trivial submodule is non-trivial, and our inner product restricted to ξ_i lives in $\mathcal{I}_2^G(\xi_i)$. So, we know that $\dim\{\mathcal{I}_2^G(\xi_i)\} \geq 1$. However, $\mathcal{I}_2^G(\xi_i)$ corresponds to $Hom^{sa}(\xi_i,\xi_i)$, and so we know that $\dim\{Hom^{sa}(\xi_i,\xi_i)\} \geq 1$. Now, $\mathcal{I}(\xi\otimes\xi)$ is the set of linear maps from $\xi\otimes\xi$ to $\mathbb{R}$. In other words, they are G-invariant bilinear forms, and so our inner product restricted to ξ_i lives in $\mathcal{I}(\xi\otimes\xi)$. Thus, $\dim\{\mathcal{I}(\xi\otimes\xi)\} \geq 1$. However, $\mathcal{I}(\xi\otimes\xi)$ is identified with $Hom(\xi_i,\xi_i)$, and therefore $\dim\{Hom(\xi_i,\xi_i)\} \geq 1$. Now, if we replace both $\dim\{Hom^{sa}(\xi_i,\xi_i)\}$ and $\dim\{Hom(\xi_i,\xi_i)\}$ with 1 in our above equation, we can underestimate $\dim\{\mathcal{I}_2^G(\xi)\}$, as shown below:

$$\begin{aligned}
\dim\{\mathcal{I}_2^G(\xi)\} &= \textstyle\sum_i n_i \dim\{Hom^{sa}(\xi_i,\xi_i)\} + \frac{1}{2}n_i(n_i-1)\dim\{Hom(\xi_i,\xi_i)\} \\
&\geq \textstyle\sum_i n_i + \frac{1}{2}n_i(n_i-1) \\
&= \textstyle\sum_i (n_i + \frac{1}{2}n_i^2 - \frac{1}{2}n_i) \\
&= \textstyle\sum_i (\frac{1}{2}n_i^2 + \frac{1}{2}n_i) \\
&= \textstyle\sum_i \frac{1}{2}n_i(n_i+1).
\end{aligned}$$

Therefore, we have $\dim\{\mathcal{I}_2^G(\xi)\} \geq \sum_i \frac{1}{2}n_i(n_i+1)$.

We now begin our proof of Assertion (2). Let ξ be a representation which can be expressed as the direct sum $\xi = n_1\xi_1 \oplus \cdots \oplus n_k\xi_k$, where there are n_i copies of each subrepresentation ξ_i, with not all subrepresentations ξ_i irreducible. Suppose we have equality in the estimation from Assertion (1), $\dim\{\mathcal{I}_2^G(\xi)\} = \sum_i \frac{n_i(n_i+1)}{2}$. Choose one subrepresentation that is not irreducible and re-index the direct sum decomposition so that the first subrepresentation is not irreducible. So, we have:

$$\xi_1 = a_1\alpha_1 \oplus \cdots \oplus a_l\alpha_l.$$

Fitting this back into our original decomposition, there are n_1 copies of ξ_1, and we can express our representation as:

$$\xi = n_1a_1\alpha_1 \oplus n_1a_2\alpha_2 \oplus \cdots \oplus n_1a_l\alpha_l \oplus n_2\xi_2 \oplus \cdots \oplus n_k\xi_k.$$

Now, we have:

$$\dim\{\mathcal{I}_2^G(\xi)\} = \textstyle\sum_i \frac{n_i(n_i+1)}{2} \quad = \quad \sum_{i=1}^{l} \frac{(n_1a_i)(n_1a_i+1)}{2} + \sum_{i=2}^{k} \frac{n_i(n_i+1)}{2}.$$

Subtract $\sum_{i=2}^{k} \frac{n_i(n_i+1)}{2}$ to get:

$$\frac{n_1(n_1+1)}{2} \quad = \quad \sum_{i=1}^{l} \frac{(n_1 a_i)(n_1 a_i+1)}{2},$$

and the only way the two sides our our equation could be equal is if one of the a_i's is equal to one, and the rest are equal to zero. Choose $a_1 = 1$, with all remaining a_i's equal to zero. If this is the case, then we can eliminate all $n_1 a_i$'s from the decomposition below, except $n_1 a_1$:

$$\xi = n_1 a_1 \alpha_1 \oplus n_1 a_2 \alpha_2 \oplus \cdots \oplus n_1 a_l \alpha_l \oplus n_2 \xi_2 \oplus \cdots \oplus n_k \xi_k,$$

which results in:

$$\xi = n_1 a_1 \alpha_1 \oplus n_2 \xi_2 \oplus \cdots \oplus n_k \xi_k.$$

Further, if $a_1 = 1$, then we have:

$$\xi = n_1 \alpha_1 \oplus n_2 \xi_2 \oplus \cdots \oplus n_k \xi_k,$$

which is just another way of labelling our original decomposition:

$$\xi = n_1 \xi_1 \oplus \cdots \oplus n_k \xi_k.$$

Therefore, we have a contradiction, and $n_1 \xi_1$ is an irreducible submodule. Thus, it is the case that all of the subrepresentations in our original decomposition are irreducible. $\square$

Chapter 4

The Decomposition of the Space of ACTs

In this chapter, we will decompose the space of algebraic curvature tensors into three irreducible and inequivalent subspaces. We will verify Theorem 4.1 found in Chapter 4 of Gilkey's Geometric Realizations of Curvature [VGN12] which states that the space of ACTs decomposes into the real numbers ($\mathbb{R}$), the space of trace-free symmetric bilinear forms (S_0^2), and the space of Weyl tensors ($\mathcal{W}$), and these are inequivalent and irreducible:

Theorem 4.1. *Let $n \geq 4$.*

1. *There is an orthogonal module decomposition:*
   (a) $\mathcal{A}(V) \cong \mathbb{R} \oplus S_0^2 \oplus \mathcal{W}$.
2. *The orthogonal modules $\{\mathbb{R}, S_0^2, \mathcal{W}\}$ are inequivalent and irreducible.*

In Section 1 we will use two short exact sequences in combination with the application of Lemma 3.5 to create a decomposition of the space of ACTs where the subspaces may or may not be irreducible. Then, in Section 2, we will determine the dimension of $\mathcal{I}_2^G(\xi)$ where the vector space associated with ξ is $\mathcal{A}(V)$. We will determine this dimension by counting the number of independent, nonzero ways to contract the ACTs down to a real number. This will give us an estimation of the dimension of our representation. Finally, in Section 3, we will combine the short exact sequence results from Section 1 with the dimension count from Section 2 using Assertion (2) of Lemma

3.7 to confirm that Theorem 4.1 is an orthogonal direct sum decomposition of the space of ACTs.

The representation ξ we will be decomposing has vector space $\mathcal{A}(V)$, group $\mathcal{O}$, the orthogonal group of our nondegenerate inner product $G = O(< \cdot, \cdot >) = \{A : V \to V| < Ax, Ay > = < x, y >\}$ with A invertible, and group action σ. We will use g to represent the group action. $g \in \mathcal{O}$ acts on $R \in \mathcal{A}(V)$ by scrambling each of the inputs through precomposition so that $(gR)(x, y, z, w) = R(gx, gy, gz, gw)$.

4.1 A Decomposition of the Space of ACTs

The following section will provide a decomposition of the space of ACTs using two short exact sequences combined with the splitting result from Lemma 3.5.

Let $\mathcal{A}(V)$ be the space of algebraic curvature tensors, $R \in \mathcal{A}(V)$, $S^2(V)$ be the space of all symmetric bilinear forms, ρ be the Ricci function that takes in a curvature tensor and produces a symmetric bilinear form, and let $\mathcal{W} = \ker \rho$ be the set of ACTs with a Ricci tensor equal to zero. Define $\rho(R)$ to be the Ricci tensor of R, and $i(\mathcal{A}(V)) = \mathcal{A}(V)$, where i stands for inclusion.

Theorem 4.2. *The sequence* $0 \longrightarrow \mathcal{W} \xrightarrow{i} \mathcal{A}(V) \xrightarrow{\rho} S^2(V) \longrightarrow 0$ *is short exact.*

Proof. In order to verify that the above sequence is short exact, we need to verify three things: $im(i) = \ker \rho$, the map i is injective, and the map ρ is surjective. In order to show that ρ is surjective, we will demonstrate that the sequence splits at σ, where $\sigma : S^2(V) \to \mathcal{A}(V)$. We will show that starting at $S^2(V)$ and applying the map σ will bring us into $\mathcal{A}(V)$. Then, applying the map ρ will bring us back into $S^2(V)$. Thus, the composition of these functions is equal to the identity function of $S^2(V)$. Because the identity function is surjective, with the identity being equal to the composition $\rho \circ \sigma$, we know it is the case that ρ is surjective as well.

The image of the map i is defined to be the kernel of ρ. More specifically, $\mathcal{W}$, or the image of the map i, is the set of all set of all curvature tensors that give have a zero Ricci tensor. In other words, it is the kernel of the map ρ.

The inclusion map i takes elements from $\mathcal{W}$ and transfers them into the larger set of ACTs. In short, the function includes $\mathcal{W}$ into $\mathcal{A}(V)$, and so i is injective.

In order to verify that ρ is a surjective function, we will define a splitting σ which splits ρ, where $\sigma : S^2(V) \to \mathcal{A}(V)$. We will first need to verify that $\sigma(\theta)$ is an ACT, with $\theta \in S^2(V)$. Next, we will check to see if we apply the map ρ to $\sigma(\theta)$, we get back to $S^2(V)$. In summary, we need to check that $\rho \circ \sigma(\theta)$ is the identity map from $S^2(V)$ to $S^2(V)$. This will prove that ρ is surjective because the identity map is surjective, and since $\rho \circ \sigma = Id_{S^2(V)}$, this implies that the second map in the sequence ρ is surjective as well. □

We will follow Gilkey's path found in Geometric Realizations of Curvature [VGN12] to demonstrate that the map $\sigma : \mathcal{A}(V) \to S^2(V)$ splits the map ρ, and that ρ is surjective.

Let $\theta \in S^2(V)$, and let $\tau = \tau(\theta) := g^{ij}\theta_{ij}$ be the scalar curvature, the only way to contract a symmetric bilinear form to the real numbers. We will define:

$$\begin{aligned} \sigma(\theta)(x,y,z,w) \quad := \quad & \tfrac{1}{n-2}\{\theta(x,w) < y,z > + < x,w > \theta(y,z)\} \\ & -\tfrac{1}{n-2}\{\theta(x,z) < y,w > + < x,z > \theta(y,w)\} \\ & -\tfrac{\tau}{(n-1)(n-2)}\{< x,w >< y,z > - < x,z >< y,w >\}. \end{aligned} \tag{4.1}$$

Since $\sigma(\theta)(x,y,z,w) = \sigma(\theta)(z,w,x,y)$, $\sigma \in \mathcal{A}(V)$.

The following equation will be used in our verification that σ splits ρ.

$$g^{ij} < e_i, e_j > = n. \tag{4.2}$$

Equation 4.1 demonstrated that starting with a symmetric bilinear form $\theta \in S^2(V)$, we can apply the map σ with the result being $\sigma(\theta) \in \mathcal{A}(V)$. Now, in the computations below, we will see that applying the map ρ to $\sigma(\theta)$ will bring us back into $S^2(V)$, with $\rho \circ \sigma = Id_{S^2(V)}$. The computation demonstrates that σ splits the map ρ, and since we get the same Ricci tensor if we contract the outer two indices, we have:

$$\begin{aligned} \rho(\sigma(\theta))(y,z) \quad = \quad & \varepsilon^{ij}\tfrac{1}{n-2}\{\theta(e_i,e_j) < y,z > + < e_i,e_j > \theta(y,z)\} \\ & -\varepsilon^{ij}\tfrac{1}{n-2}\{\theta(e_i,z) < y,e_j > + < e_i,z > \theta(y,e_j)\} \\ & -\varepsilon^{ij}\tfrac{\tau}{(n-1)(n-2)}\{< e_i,e_j >< y,z > - < e_i,z >< y,e_j >\} \\ = \quad & \tfrac{1}{n-2}\{< y,z > \tau(\theta) + n\theta(y,z)\} - \tfrac{1}{n-2}\{\theta(y,z) + \theta(y,z)\} \\ & -\tfrac{\tau}{(n-1)(n-2)}\{n < y,z > - < y,z >\} \\ = \quad & \theta(y,z). \end{aligned}$$

We know that the identity map is bijective, and so $\rho \circ \sigma$ is bijective. Further, since $\rho \circ \sigma$ is a composition of functions that is surjective, ρ is surjective.

Because the sequence

$$0 \longrightarrow \mathcal{W} \xrightarrow{i} \mathcal{A}(V) \xrightarrow{\rho} S^2(V) \longrightarrow 0$$

is short exact, we can apply Lemma 3.5, resulting in

$$\mathcal{A}(V) \cong \mathcal{W} \oplus S^2(V).$$

4.1.1 Short Exact Sequence Involving $\mathcal{A}(V)$ is $\mathcal{O}$-Equivariant

Theorem 4.3. *Let $g \in \mathcal{O}$. The sequence is $\mathcal{O}$-equivariant:*

$$\begin{array}{c} 0 \to \mathcal{W} \xrightarrow{i} \mathcal{A}(V) \xrightarrow{\rho} S^2(V) \to 0 \\ \quad \downarrow_{g\cdot()} \qquad \downarrow_{g\cdot()} \qquad \downarrow_{g\cdot()} \\ 0 \to \mathcal{W} \xrightarrow{i} \mathcal{A}(V) \xrightarrow{\rho} S^2(V) \to 0. \end{array}$$

Proof. For the sequence to be $\mathcal{O}$-equivariant, we need to show that starting at $\mathcal{W}$, then applying the group action to get to $\mathcal{W}$ again, and then applying the map i to get to $\mathcal{A}(V)$ will yield the same result as if we started at $\mathcal{W}$, applied the map i to get to $\mathcal{A}(V)$, and then applied the group action to get back to $\mathcal{A}(V)$. The same process would need to be true starting at $\mathcal{A}(V)$ and ending up in $S^2(V)$.

Suppose $A \in \mathcal{W}$. We need to include it, and then apply the group action to it.

$$\begin{aligned} g(i(A))(v_1, v_2, v_3, v_4) &= (g \cdot A)(v_1, v_2, v_3, v_4) \\ &= A(gv_1, gv_2, gv_3, gv_4) \end{aligned}$$

Now, we are going to apply the group action to it first, and then include it:

$$\begin{aligned} i(g \cdot A)(v_1, v_2, v_3, v_4) &= g \cdot A(v_1, v_2, v_3, v_4) \\ &= A(gv_1, gv_2, gv_3, gv_4) \end{aligned}$$

So, the sequence above is equivariant from $\mathcal{W}$ to $\mathcal{A}(V)$. Next, we will check to see if it is equivariant from $\mathcal{A}(V)$ to $S^2(V)$.

Start with $R \in \mathcal{A}(V)$. First we will apply the map ρ, and then we will apply the group action g. Then, we will do the opposite order and compare the results. Now, it it crucial that we are working with the orthogonal group, because our vector spaces are invariant under the action of the orthogonal group. First, we will contract indices, and do so on an orthonormal basis

$$\rho(R)(v, w) = \sum \varepsilon_i R(v, e_i, e_i, w), \text{ where } \varepsilon_i = \pm 1.$$

Next, we will apply the group action, and this is precomposition by g:

$$\begin{aligned} g \cdot \rho(R)(v,w) &= \rho(R)(gv, gw) \\ &= \sum \varepsilon_i R(gv, e_i, e_i, gw) \end{aligned}$$

Now, we will compare this to the mapping starting by applying the group action first:

$$(g \cdot R)(v_1, v_2, v_3, v_4) = R(gv_1, gv_2, gv_3, gv_4)$$

We will find the Ricci tensor of the above using an orthonormal basis:

$$\begin{aligned} \rho(g \cdot R)(v,w) &= \sum \varepsilon_i R(gv, ge_i, ge_i, gw) \\ &= \rho(R)(gv, gw) \end{aligned}$$

But, since the group G is the orthogonal group, and if the basis $\{e_i\}$ is orthonormal, then $\{ge_i\}$ is orthonormal, too. And, with the Ricci tensor, it does not matter which basis we choose to contract indices. The Ricci tensor of (v, w) is:

$$\begin{aligned} \rho(v,w) &= \sum \varepsilon_i R(v, e_i, e_i, w) \\ &= \sum \varepsilon_i R(v, ge_i, ge_i, w), \end{aligned}$$

where $\sum \varepsilon_i R(v, e_i, e_i, w)$ and $\sum \varepsilon_i R(v, ge_i, ge_i, w)$ are the same tensor, just using different orthonormal bases.

In summary, the reason the sequence is equivariant is because it does not matter which orthonormal basis we use to produce the Ricci tensor of a curvature tensor. Using the orthogonal group, when we precompose by the action of g, we are giving ourselves another orthonormal basis, and either one will do. That is what makes the whole sequence $\mathcal{O}$-equivariant. Because the sequence is $\mathcal{O}$-equivariant, the group sends $\mathcal{W}$ back to $\mathcal{W}$, $\mathcal{A}(V)$ back to $\mathcal{A}(V)$, and $S^2(V)$ back to $S^2(V)$. So, $\mathcal{W}$, $\mathcal{A}(V)$, and $S^2(V)$ are all invariant under the action of the orthogonal group. $\square$

4.1.2 A Decomposition of $S^2(V)$

There is an onto map τ that produces the scalar curvature given a symmetric bilinear form. We can contract an element from $S^2(V)$ and get a real number $\mathbb{R}$, and the kernel of this map is $S_0^2(V)$. Define τ to be the only contraction from the space of symmetric bilinear forms to the real numbers, and define $\ker \tau = S_0^2(V)$, where $S_0^2(V)$ are the traceless bilinear forms (which contract to zero).

Theorem 4.4. *The sequence* $0 \longrightarrow S_0^2(V) \overset{i}{\hookrightarrow} S^2(V) \overset{\tau}{\longrightarrow} \mathbb{R} \longrightarrow 0$ *is short exact.*

Proof. The image of the map i is defined to be the kernel of τ. More specifically, $S_0^2(V)$, or the image of the map i, is the set of all symmetric bilinear forms which contract to zero. In other words, it is the kernel of the map τ.

The inclusion map i takes elements from $S_0^2(V)$ and transfers them into the larger set of symmetric bilinear forms. In short, the function includes $S_0^2(V)$ into $S^2(V)$, and so i is injective.

Given a real number, we can determine which symmetric bilinear form gave us that real number. Rephrased, we can trace every real number back to the symmetric bilinear form from which it came. Consider $\frac{\alpha}{n} \cdot < \cdot, \cdot > \overset{\tau}{\mapsto} \alpha$. Therefore, τ is a surjective map. □

Because the sequence in Theorem 4.4 above is short exact, we can apply Lemma 3.5 to get:

$$S^2(V) \cong S_0^2(V) \oplus \mathbb{R}$$

Combining our result from Section 1:

$$\mathcal{A}(V) \cong \mathcal{W} \oplus S^2(V),$$

with our result from above:

$$S^2(V) \cong S_0^2(V) \oplus \mathbb{R},$$

we have a decomposition of the space of ACTs:

$$\mathcal{A}(V) \cong \mathcal{W} \oplus S_0^2(V) \oplus \mathbb{R}.$$

4.1.3 Dimension of $\mathcal{W}$

Using the decomposition of the space of ACTs from our previous section, we can resolve why it is necessary that $n \geq 4$ in Theorem 4.1.

Corollary 4.5. *In the orthogonal module decomposition* $\mathcal{A}(V) \cong \mathbb{R} \oplus S_0^2 \oplus \mathcal{W}$*, it is necessary that* $n \geq 4$.

Proof. We determined earlier in this section that $\mathcal{A}(V) \cong S^2(V) \oplus \mathcal{W}$, with $S^2(V) \cong \mathbb{R} \oplus S_0^2(V)$. It is well-known that $\dim\{S^2(V)\} = \frac{n(n+1)}{2}$, and it is easy to determine that $\dim\{\mathcal{A}(V)\} = \frac{n^2(n^2-1)}{12}$ [VGN12]. We can use the dimensions of $\mathcal{A}(V)$ and $S^2(V)$ to determine the dimension of $\mathcal{W}$. Now,

$$\mathcal{A}(V) \cong S^2(V) \oplus \mathcal{W}.$$

Replacing each vector space in the above with its dimension, we have:

$$\begin{aligned}\dim\{\mathcal{A}(V)\} &= \dim\{S^2(V)\} + \dim\{\mathcal{W}\}\\ \tfrac{n^2(n^2-1)}{12} &= \tfrac{n(n+1)}{2} + \dim\{\mathcal{W}\}\\ \Rightarrow \dim\{\mathcal{W}\} &= \tfrac{n^2(n^2-1)}{12} - \tfrac{n(n+1)}{2}.\end{aligned}$$

Now that we know the dimension of $\mathcal{W}$, we can see that in dimension three, $\mathcal{W}$ is zero-dimensional. If we choose n to be anything less than four, $\mathcal{W}$ will cease to exist. Thus, it is necessary that $n \geq 4$. $\square$

4.2 Number of Nonzero, Independent Ways to Fully Contract ACTs Down to Real Numbers

This section will focus on computing the number of independent ways to fully contract an algebraic curvature tensor down to the real numbers. Recall that we use $\mathcal{A}(V)$ to denote the space of ACTs, and we use R to denote an ACT in $\mathcal{A}(V)$. Also recall that an ACT is a multilinear function $R \in \otimes^4 V^*$ that satisfies all of the following for $x, y, z, w \in V$:

1. $R(x, y, z, w) = -R(y, x, z, w)$,
2. $R(x, y, z, w) = R(z, w, x, y)$,
3. $R(x, y, z, w) + R(z, x, y, w) + R(y, z, x, w) = 0$ (the Bianchi Identity).

In the previous section, we determined that $\mathcal{A}(V) \cong \mathcal{W} \oplus S_0^2(V) \oplus \mathbb{R}$. In order to determine if each subspace in this decomposition is irreducible, we will apply the estimation test provided in Lemma 3.7. To use the test, we must calculate $\dim\{\mathcal{I}_2^G(\xi)\}$, where $\mathcal{A}(V)$ is the vector space associated with the representation ξ. To do this we will compute $\dim\{\mathcal{I}_2^G(\xi)\}$ which represents number of nonzero, independent ways to fully contract an ACT down to the real numbers. Recall that $\mathcal{I}_2^G(\xi)$ is the set of all symmetric bilinear forms where the vector space is the algebraic curvature tensors, and the way we compute this dimension is through contraction of indices. So, $\dim\{\mathcal{I}_2^G(\xi)\}$ represents the total number of nonzero, independent ways of taking $R \otimes R$ (giving us eight spaces to fill) and contracting them fully in pairs (four total pairs). We will start with $R \otimes R$ and contract to get a real number.

Our goal will be to compute the number of nonzero, independent ways to fully contract $R \otimes R \in \otimes^8 V^*$, only using contraction of indices. We will count the dimension of $\mathcal{I}_2^G(\xi)$, where ξ is the representation with $\mathcal{A}(V)$ as the vector space. So, we are working with the ACTs and their specific properties. Each R has four indices, and so $R \otimes R$ will have eight spots to pair and contract. See the visual below:

$$\underbrace{\_\ \_\ \_\ \_}_{R} \;\Big|\; \underbrace{\_\ \_\ \_\ \_}_{R}$$

In order to determine all of the nonzero, independent ways to contract four pairs of indices in $R \otimes R$ in a methodical fashion, we will ask ourselves how many different indices could appear in each quadruple, and we will start with breaking down how many different indices can occur in the first four slots. Each index occurs twice in the total of eight slots. There are four possibilities: only one index in the first four slots, exactly two different indices appearing in the first four slots with each repeated twice, three different indices appearing in the first four slots with one index repeated, or four different indices appearing in the first four slots. We will examine each case and count up the total number of nonzero, independent results at the end. This number will represent the largest size that $\dim\{\mathcal{I}_2^G(\xi)\}$ could be.

<u>One Index:</u> It is not possible for there to be only index occurring in the first four slots of $R \otimes R$ since we would have to repeat that index four times, and we can only repeat each index twice.

<u>Two Indices:</u> We have an i and j to place in the first four slots, and since there are four pairs of indices, that means i and j both appear twice in the first four slots and will not appear in the second set of four slots. This means that in the second set of slots, only k and l will occur, and twice each.

Because of the first property of ACTs, repeating the same index in either the first two or last two slots will result in 0. Notice that placing i in the first two slots and then swapping would result in $R(e_i, e_i, e_j, e_j) = -R(e_i, e_i, e_j, e_j)$, which implies that $R(e_i, e_i, e_j, e_j) = 0$ since the only number that is the negative of itself is zero. In short, whenever the first two entries or last two entries are repeated, the result is 0. We argue that there is only one independent way to distribute two indices in the first four slots of a curvature tensor. The i's cannot be next to each other in either the first two or last two slots, and the same with the j's. This leaves us with four options for the i's and j's. We can have $ijji$, $ijij$, $jiji$, and $jiij$. However, up to the introduction of a negative, we can rearrange all to be $ijji$, and so the only independent way to fill in the first four spots with two indices would be $ijji$.

After filling in the first four slots, this leaves us $kllk$ as the only unique way to fill in the last four up to a sign, for the same reason we just described. In summary, if there are only two indices that occur in the first four spots, say i and j, then the only nonzero, independent way to place the indices is:

$$\underline{i}\ \ \underline{j}\ \ \underline{j}\ \ \underline{i}\ \ |\ \ \underline{k}\ \ \underline{l}\ \ \underline{l}\ \ \underline{k}$$

Instead of using diagrams, we will introduce a simplified notation. We will set $i = 1$, $j = 2$, $k = 3$, and $l = 4$, and so the above case will be recorded as $I(1221)(3443)$, where I stands for invariant.

<u>Three Indices:</u> We have i, j, k and l as our indices, and we need to fit three of those indices in the four slots, so one index will need to be repeated, say i. Up to a negative sign, we can always maneuver the i to be in the first spot. So, the second place where i can be repeated has to be in spot three or four (otherwise, the result will be zero). See the diagram below:

$$\underline{i}\ \ \_\ \ \underline{i}\ \ \_\ \ |\ \ \_\ \ \_\ \ \_\ \ \_$$

$$\underline{i}\ \ \_\ \ \_\ \ \underline{i}\ \ |\ \ \_\ \ \_\ \ \_\ \ \_$$

Up to a negative, putting i in the third or fourth spot would be the same. So, we will put it in the fourth spot:

$$\underline{i}\ \ \_\ \ \_\ \ \underline{i}\ \ |\ \ \_\ \ \_\ \ \_\ \ \_$$

With the i's placed in the outer two slots, we are left with j and k to fill in to the middle two slots. Notice that if we choose the order to be $ijki$, we can swap the first two indices to get $-jiki$. Swapping the last two indices gives us $jiik$, and finally swapping the first two indices and last two indices in pairs gives us $ikji$. The result is that if the i's stay on the outside, changing the order of the j and k in the middle does not create a new independent contraction. So, we will choose the order to be jk:

$$\underline{i}\quad \underline{j}\quad \underline{k}\quad \underline{i}\quad \Big|\quad \_\quad \_\quad \_\quad \_$$

Now, we have left to fill in j, k, l and l in the second set of four slots. Without loss of generality, the l's will be placed in the outer two slots, and it does not matter the order of j and k:

$$\underline{i}\quad \underline{j}\quad \underline{k}\quad \underline{i}\quad \Big|\quad \underline{l}\quad \underline{j}\quad \underline{k}\quad \underline{l}$$

We will call this $I(1231)(4234)$, and this is the only nonzero, independent way to contract with three different indices in the first four slots.

Four Indices: If all four indices have to occupy all four slots, we may as well just list them in alphabetical order:

$$\underline{i}\quad \underline{j}\quad \underline{k}\quad \underline{l}\quad \Big|\quad \_\quad \_\quad \_\quad \_$$

There will be one i in one of the four slots on the right side. Wherever we place i, we can always maneuver it to the first slot, up to a factor of negative one. For example, if i appears in the second slot, we can move it to the first with the introduction of a minus sign. If the i appears in the third slot, we can move it to the first slot by swapping the first and second slots with the third and fourth. Finally, if i appears in the fourth slot, we can swap it into the third slot with the introduction of a minus sign, and then swap the first and second with the third and fourth, with the result being i appearing in the first slot. So, under any circumstance, we can move the i to be in the first slot at the possible cost of a negative sign. This now leaves us with two possibilities of where to place j. It can either appear in the second slot or in the third or fourth slot (which are the same up to a minus sign). However, the placement of j in the third or fourth slot are the same up to a negative, so we will choose j to be in the fourth slot. This will eliminate the option of having j in the third slot, leaving us with two options:

$$\underline{i}\quad \underline{j}\quad \underline{k}\quad \underline{l}\quad \Big|\quad \underline{i}\quad \underline{j}\quad \_\quad \_$$

$$\underline{i}\quad \underline{j}\quad \underline{k}\quad \underline{l}\quad \Big|\quad \underline{i}\quad \_\quad \_\quad \underline{j}$$

Let us examine the first of the two above options, with j placed in the second slot. We need to place k and l:

$$\underline{i}\quad \underline{j}\quad \underline{k}\quad \underline{l}\quad \Big|\quad \underline{i}\quad \underline{j}\quad \underline{k}\quad \underline{l}$$

$$\underline{i} \quad \underline{j} \quad \underline{k} \quad \underline{l} \quad | \quad \underline{i} \quad \underline{j} \quad \underline{l} \quad \underline{k}$$

Swapping the position of k and l above would be the same up to a negative, so we will choose one way:

$$\underline{i} \quad \underline{j} \quad \underline{k} \quad \underline{l} \quad | \quad \underline{i} \quad \underline{j} \quad \underline{k} \quad \underline{l}$$

We will call this $I(1234)(1234)$.

Now, examining the possibilities with j in the fourth slot, we have another two options of where to place k and l. However, we have no curvature idenities left to distinguish between k being in the second slot and l is in the third slot, or if they are reversed:

$$\underline{i} \quad \underline{j} \quad \underline{k} \quad \underline{l} \quad | \quad \underline{i} \quad \underline{l} \quad \underline{k} \quad \underline{j}$$

$$\underline{i} \quad \underline{j} \quad \underline{k} \quad \underline{l} \quad | \quad \underline{i} \quad \underline{k} \quad \underline{l} \quad \underline{j}$$

So, we are left with three possibilities of placing four different indices in the four slots:

$$\underline{i} \quad \underline{j} \quad \underline{k} \quad \underline{l} \quad | \quad \underline{i} \quad \underline{j} \quad \underline{k} \quad \underline{l} \qquad \text{(a)}$$

$$\underline{i} \quad \underline{j} \quad \underline{k} \quad \underline{l} \quad | \quad \underline{i} \quad \underline{l} \quad \underline{k} \quad \underline{j} \qquad \text{(b)}$$

$$\underline{i} \quad \underline{j} \quad \underline{k} \quad \underline{l} \quad | \quad \underline{i} \quad \underline{k} \quad \underline{l} \quad \underline{j} \qquad \text{(c)}$$

We know that because of the Bianchi Identity, (a) + (b) + (c) = 0. Now, (b) represents $R_{ijkl}R_{ilkj}$ and is a sum of things, where each summand is a curvature term. However, switching the order of multiplication results in $R_{ijkl}R_{iljk} = R_{iljk}R_{ijkl}$. Now, we can reindex this sum. We are summing over all i, j, k, and l, and we will replace l with j, j with k, and k with l. This gives us $\sum R_{iljk}R_{ijkl} = \sum R_{ijkl}R_{iklj}$. This proves that (b) = (c).

Combining the fact that (a) + (b) + (c) = 0, and (b) = (c), we know that (a) + 2(b) = 0, or, (a) = −2(b). This proves that (a) = −2(b) = (c), and so (a), (b), and (c) are all multiples of each other. Therefore, we will only count (a) of the group of (a), (b), and (c) as being linearly independent of the others.

To conclude, we ended up with three independent, nonzero ways to contract the ACTs down to the real numbers:

1. $I(1221)(3443)$

2. $I(1231)(4234)$

3. $I(1234)(1234)$,

and so $\dim\{\mathcal{I}_2^G(\xi)\} \leq 3$, where ξ is the representation with $\mathcal{A}(V)$ as the vector space.

4.3 The Orthogonal Direct Sum Decomposition of the Space of ACTs

In Section 4.1.2, we determined the following decomposition of the space of ACTs: $\mathcal{A}(V) \cong \mathbb{R} \oplus S_0^2 \oplus \mathcal{W}$. We want to verify that this is an orthogonal direct sum decomposition where $\mathbb{R}$, S_0^2, and $\mathcal{W}$ are inequivalent and irreducible. Applying Assertion (1) of Lemma 3.7 to our decomposition above, we have $\xi = n_1\xi_1 \oplus n_2\xi_2 \oplus n_3\xi_3$, where $n_1, n_2, n_3 = 1$, and $\xi_1 = \mathbb{R}$, $\xi_2 = S_0^2$, and $\xi_3 = \mathcal{W}$. These three submodules may or may not be irreducible. We can apply Lemma 3.7, Assertion (1), to determine:

$$\begin{aligned} \dim\{\mathcal{I}_2^G(\xi)\} &\geq \textstyle\sum_{i=1}^{3} \frac{n_i(n_i+1)}{2} \\ &= \tfrac{1(1+1)}{2} + \tfrac{1(1+1)}{2} + \tfrac{1(1+1)}{2} \\ &= 1+1+1 \\ &= 3, \end{aligned}$$

where $\mathcal{A}(V)$ is the vector space associated with representation ξ. However, the dimension counting argument done in Section 2 of this chapter resulted in:

$$\dim\{\mathcal{I}_2^G(\xi)\} \leq 3,$$

since we determined there were three independent, nonzero ways to contract an ACT down to the real numbers.

Combining our above result that $\dim\{\mathcal{I}_2^G(\xi)\} \leq 3$ with our result from Section 2 that $\dim\{\mathcal{I}_2^G(\xi)\} \geq 3$, we have found equality; $\dim\{\mathcal{I}_2^G(\xi)\} = 3$, where $\mathcal{A}(V)$ is the vector space associated with ξ. We can now apply Assertion (2) of Lemma 3.7 to determine that $\mathcal{A}(V) \cong \mathbb{R} \oplus S_0^2(V) \oplus \mathcal{W}$ is a module decomposition where the orthogonal modules $\mathbb{R}$, $S_0^2(V)$, and $\mathcal{W}$ are inequivalent and irreducible!

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