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Error Terms for the Trapezoid, Midpoint, and Simpson's Rules

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A Thesis

Presented to the

Faculty of

California State University,

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In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

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Abstract

When it is not possible to integrate a function we resort to Numerical Integration. For example the ubiquitous Normal curve tables are obtained using Numerical Integration. The antiderivative of the defining function for the normal curve involves the formula for antiderivative of e^{-x^2} which can't be expressed in the terms of basic functions.

One of the best known Numerical Integration formula is the so-called Simpson's rule. The rule states that we can replace $\int_a^b f(x)\,dx$ by

$$\frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)].$$

Of course for most functions Simpson's rule is going to give us just an approximation for the true value of the integral, so it is very important to be able to control the error for this approximation. It is known that for four times differentiable functions

$$\int_a^b f(x)\,dx - \frac{b-a}{6}[f(a)+4f(\frac{a+b}{2})+f(b)] = -\frac{1}{90}(\frac{b-a}{2})^5 f^{iv}(\psi) \qquad (0.0.1)$$

where $f^{iv}(\psi)$ is the value of the fourth derivative at some point $\psi \in (a,b)$. From this formula it follows that Simpon's rule can be used to evaluate integrals off all polynomials of degree 3 or less. This is because the fourth derivative for these polynomials is identically zero hence the error term $-\frac{1}{90}(\frac{b-a}{2})^5 f^{iv}(\psi)$ is zero. For other functions we need an estimate on the size of its fourth derivatives.

Simpson's rule is studied in most Calculus books, and in all undergraduate Numerical Analysis books, but proofs of (0.0.1) are not provided. Hence if one is interested in a proof of (0.0.1), either it can be found in advanced Numerical Analysis books as a special case of the so called Newton-Cotes formulas, or in math journals such as American Mathematical Monthly. My thesis adviser Hajrudin Fejzić, has recently published yet another proof in [Fej17]. In this thesis I plan to introduce Numerical Integration formulas such as simpler Composite and Midpoint rules as well as Simpson's rule and I will provide the proofs to these rules using the ideas developed in [Fej17] as well as new proofs based on ideas of Dr. Fejzić that were communicated to me.

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Chapter 1

Introduction

If $f(x)$ is positive over the interval $[a, b]$, then the area between the x axis and the graph of $f(x)$ over the interval $[a, b]$ is $\int_a^b f(x)\,dx$. In order to find this integral we first have to find the antiderivative of $f(x)$ which may be very difficult to do, or as in the case of the Gaussian function, $f(x) = e^{-x^2}$, a function that is used to describe the normal distributions, has no antiderivative (other than $F(x) = \int_{-\infty}^{x} e^{-t^2}\,dt$.) Fortunately, we can approximate this integral to any level of accuracy with the help of Numerical Integration.

The simplest Numerical Integration formulas are Trapezoid rule

$$\int_a^b f(x)\,dx \approx \frac{f(a)+f(b)}{2}(b-a)$$

the Midpoint rule

$$\int_a^b f(x)\,dx \approx f\left(\frac{a+b}{2}\right)(b-a)$$

and Simpson's rule

$$\int_a^b f(x)\,dx \approx \frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}(b-a).$$

These rules are special cases of the so called Newton-Cotes formulas, and they are obtained by approximating the function with the interpolating polynomials. For example Trapezoid rule is obtained by integrating the line connecting $(a, f(a))$ and $(b, f(b))$. This integral represents the area of the corresponding trapezoid, hence the name Trapezoid rule. The Midpoint rule is obtained by integrating the line $y = f(\frac{a+b}{2})$, while the Simpson's rule is obtained by integrating the parabola through the points $(a, f(a))$, $(\frac{a+b}{2}, f(\frac{a+b}{2}))$, and $(b, f(b))$.

It is known that the errors obtained using Newton-Cotes formulas for differentiable functions, depend only on higher order derivatives of $f(x)$ evaluated at some point $\psi \in (a, b)$. For example the errors for Trapezoid, Midpoint, and Simpson's rule are

$$E_T = \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a) = -\frac{(b-a)^3}{12} f''(\psi)$$

$$E_M = \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)(b-a) = \frac{(b-a)^3}{24} f''(\psi)$$

$$E_S = \int_a^b f(x)\,dx - \frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}(b-a) = -\frac{(b-a)^5}{180} f^{iv}(\psi)$$

respectively.

Introductory books on Numerical Integration give only proof of Trapezoid Rule, and no proof for Midpoint and Simpson's rules. We will reproduce the proof for Trapezoid Rule given in [CK13]. Advanced books on Numerical Integration, see [IK66], provide proofs for all Newton-Cotes formulas based on the theory of divided differences. Hence the formulas for the errors E_T, E_M, and E_S can be derived from the statement of this advanced theorem.

The lack of elementary proofs for Midpoint and Simposn's rule in Introductory books on Numerical Integration has been puzzling so much that a number of papers have been published in journals devoted to undergraduate research such as American Mathematical Monthly and Mathematics Magazine. See [Tal06] and [CN03]. Each of these proofs comes with authors claim that they are elementary enough and appropriate for undergraduate text in Numerical Integration. However the authors of Introductory Numerical Analysis have not bought into their claims yet. We will illustrate some of these ideas from [Gor02]. Gordon in [Gor02] provides elementary proofs to these rules through a series of lemmas and exercises with hints. We will provide complete proofs using ideas from Gordon's book.

Finally we will give original proofs to these three rules and to Simpson's $\frac{3}{8}$'s rule that were communicated to me by Dr. Fejzić. Dr. Fejzić's proofs are elementary in nature and only use the basic properties of continuous, differentiable and integrable functions. We point out that Dr. Fejzić's results are stronger than above mentioned results since the requirement for $f(x)$ at the end points is the continuity only, while most other proofs, including the theorem from advanced books on Numerical Integration, require that $f(x)$ is n times continuously differentiable on $[a, b]$. For comparison, this is equivalent to replacing the conditions of Rolle's theorem, f is continuous on $[a, b]$ and differentiable on (a, b) with $f(x)$ is continuously differentiable on $[a, b]$.

1.1 Riemann Integral

In this section we will review basic definitions and properties of Riemann integrals.

Definition 1.1. *Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and P is a partition $x_0, \ldots, x_n$ of $[a, b]$. The lower Riemann sum $L(f, P, [a, b])$ and the upper Riemann sum $U(f, P, [a, b])$ are defined by*

$$L(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1}, x_j]} f$$

and

$$U(f, P, [a, b]) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{[x_{j-1}, x_j]} f.$$

Theorem 1.2. *Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and P, P' are partitions of $[a, b]$ such that the list defining P is a sublist of the list defining P'. Then*

$$L(f, P, [a, b]) \leq L(f, P', [a, b]) \leq U(f, P', [a, b]) \leq U(f, P, [a, b]).$$

Definition 1.3. *Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. The lower Riemann integral $L(f, [a, b])$ and the upper Riemann integral $U(f, [a, b])$ of f are defined by*

$$L(f, [a, b]) = \sup_{P} L(f, P, [a, b])$$

and

$$U(f, [a, b]) = \inf_{P} U(f, P, [a, b]),$$

where the supremum and infimum above are taken over all partitions P of $[a, b]$.

Definition 1.4. *A bounded function on a closed bounded interval is called Riemann integrable if its lower Riemann integral equals its upper Riemann integral.*
If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then the Riemann integral $\int_a^b f$ is defined by

$$\int_a^b f = L(f, [a, b]) = U(f, [a, b]).$$

Theorem 1.5. *Every continuous real-valued function on each closed bounded interval is Riemann integrable.*

It is important to note that the converse is not true. A function that is Riemann integrable could be a function that is not continuous. For example, consider the piecewise function,

$$f(x) = \begin{cases} -1 & -2 \leq x < 0 \\ 0 & x = 0 \\ 1 & 0 < x \leq 2 \end{cases}$$

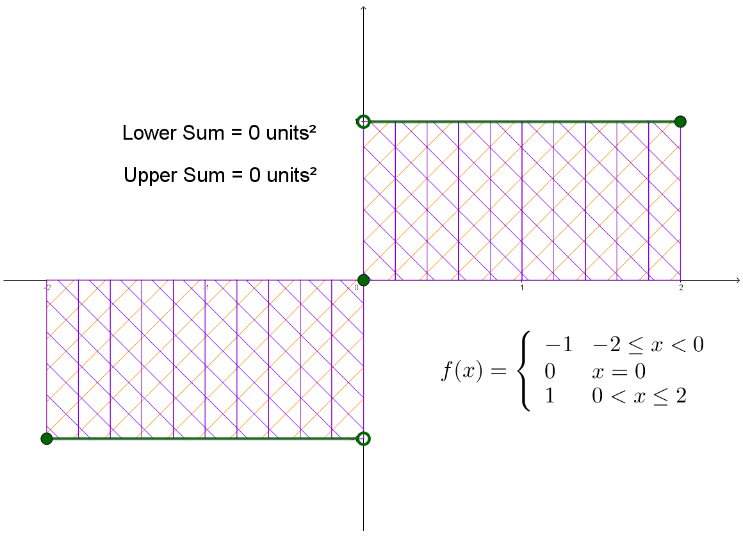

Figure 1.1: Riemann Integrable: Piecewise Function

By definition, Figure 1.1 is a piecewise function that is Riemann integrable since its lower Reimann integral equals its upper Riemann integral. However, despite being Riemann integrable, it is not continuous. Therefore, the converse is not true.

Theorem 1.6 (Mean Value Theorem for Integrals)**.** *If f is continuous on $[a,b]$, then there exists a point $c \in [a,b]$ such that $f(c)(b-a) = \int_a^b f$.*

Theorem 1.7 (Fundamental Theorem of Calculus)**.** *Suppose that f is Riemann integrable on $[a,b]$.*

a) If a function F is defined by $F(x) = \int_a^x f$ for each $x \in [a,b]$, then F is continuous on $[a,b]$ and differentiable at each point $x \in [a,b]$ for which f is continuous. At these points, $F'(x) = f(x)$.

b) If G is an antiderivative of f on $[a,b]$, then $\int_a^b f = G(b) - G(a)$.

Theorem 1.8. *Let $f : [a,b] \to \mathbb{R}$ and let $c \in (a,b)$.*

a) If f is Riemann integrable on $[a,b]$, then f is Riemann integrable on each subinterval of $[a,b]$.

b) If f is Riemann integrable on each of the intervals $[a,c]$ and $[c,b]$, then f is Riemann integrable on $[a,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

1.2 Basic properties of continuous and differentiable functions

In this section we will review basic properties of continuous and differentiable functions.

Theorem 1.9. *Let I be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. If f is differentiable at c, then f is continuous at c. Consequently, if f is differentiable on an interval J, then f is continuous on J.*

Proposition 1.10. *If $f(x) > 0$ for $a < x < b$ then $\int_a^b f(x)\,dx > 0$.*

Theorem 1.11 (Mean Value Theorem)**.** *If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists a point $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{(b-a)}$.*

Theorem 1.12 (Intermediate Value Theorem for continuous functions (and derivatives))**.** *Suppose that $f : [a,b] \to \mathbb{R}$ is continuous (differentiable) on $[a,b]$. If v is a number between $f(a)$ and $f(b)$, ($f'(a)$ and $f'(b)$) then there is a point $c \in (a,b)$ such that $f(c) = v$ ($f'(c) = v$).*

Theorem 1.13 (First Derivative Test). *Suppose f is continuous on an open interval (a, b) and differentiable on (a, b) except possibly at the point $c \in (a, b)$ and assume that c is a critical point of f.*

a) If f' is positive on (a, c) and negative on (c, b), then f has a relative maximum value at c.

b) If f' is negative on (a, c) and positive on (c, b), then f has a relative minimum value at c.

Theorem 1.14 (Second Derivative Test). *Suppose that f is twice differentiable on an open interval (a, b) and that $f'(c) = 0$ for some point $c \in (a, b)$.*

a) If $f''(c) < 0$, then f has a relative maximum value at c.

b) If $f''(c) > 0$, then f has a relative minimum value at c.

1.3 Rolle's Theorem and Fundamental Lemma

Rolle's theorem will play an important role in our treatment of the subject.

Theorem 1.15 (Rolle's Theorem). *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.*

The Rolle's Theorem relies on three conditions. If any of these conditions fail, the Rolle's Theorem does not work. The three conditions are:

1. f is differentiable on an open interval, (a, b)
2. f is continuous on the closed interval, $[a, b]$
3. $f(a) = f(b)$

Counterexample 1: In this example condition (2) is not met. Consider the function

$$f(x) = \begin{cases} 2x - 2 & x < 2 \\ 2x - 4 & x \geq 2 \end{cases}$$

Figure 1.2 shows that this function is differentiable on the open interval, $(1, 2)$ and $f(1) = f(2)$, however it is not continuous at $x = 2$. When we derive this piecewise function, we get $f'(x) = 2$, thus there can't exist a point $c \in (1, 2)$ where $f'(c) = 0$.

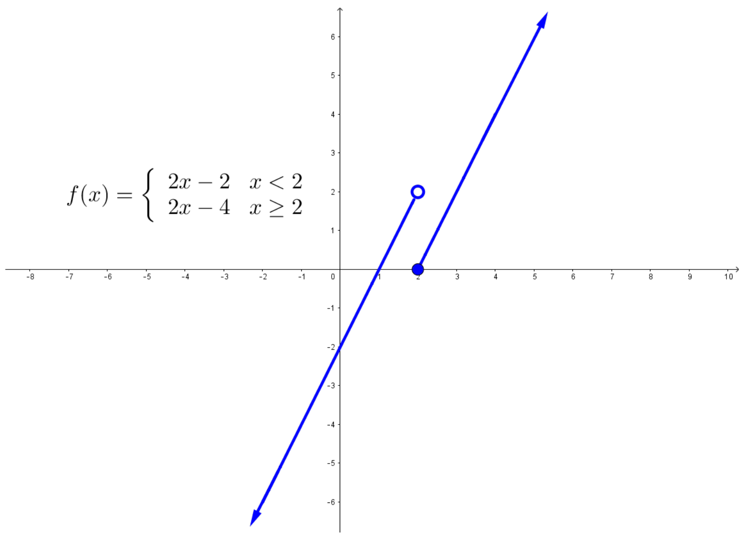

Figure 1.2: Rolle's Theorem's counterexample 1: Piecewise Function

Counterexample 2: In this example, condition (3) is not met. Consider the function $f(x) = x$. Figure 1.3 shows that this function is differentiable on any open interval, (a, b) and it is continuous on any closed interval, $[a, b]$, however $f(a) \neq f(b)$. When we derive this function, we get $f'(x) = 1$, thus there can't exist a point $c \in (a, b)$ where $f'(c) = 0$.

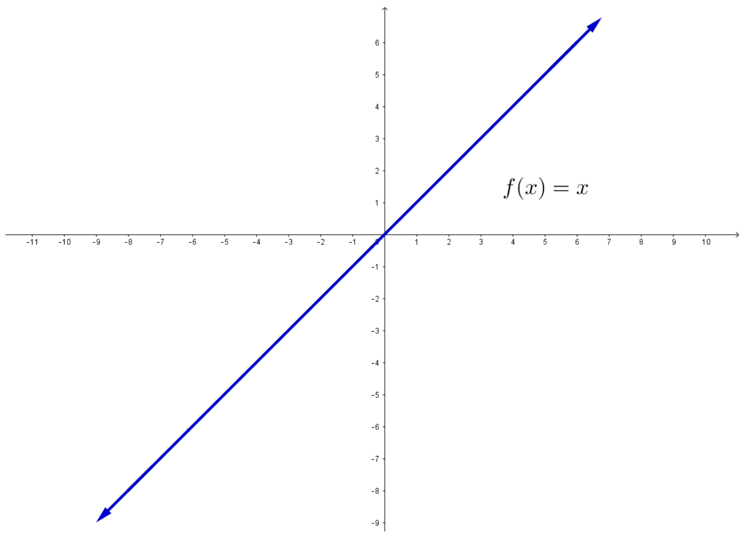

Figure 1.3: Rolle's Theorem's counterexample 2: Linear Function

Counterexample 3: In this example, condition (1) is not met. Consider the function $f(x) = |x| - 2$. Figure 1.4 shows that this function is continuous on the closed interval, $[-2, 2]$ and $f(-2) = f(2)$, however it is not differentiable on the open interval $(-2, 2)$. When we derive this function, we get $f'(x) = \frac{x}{|x|}$ and with this rational equation, there doesn't exist a point $c \in (-2, 2)$ where $f'(c) = 0$.

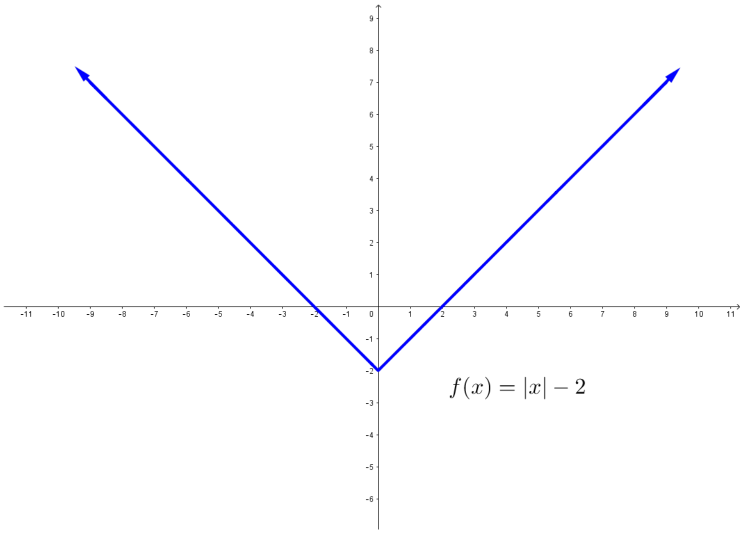

Figure 1.4: Rolle's Theorem's counterexample 3: Absolute Value Function

Together with Rolle's theorem, the following simple lemma will be crux in Dr. Fejzić's proofs of the error terms E_T, E_M and E_S.

Lemma 1.16. *If g is continuous on $[a, b]$ and $\int_a^b g(x)\,dx = 0$, then there exists a point c, with $a < c < b$ such that $g(c) = 0$.*

Proof. Suppose that the statement of Lemma 1.16 is false; that is suppose that for all $a < x < b$, $g(x) \neq 0$. Since g is continuous, then by the Intermediate Value Property:
(1) $g(x) > 0$ for all $a < x < b$ or
(2) $g(x) < 0$ for all $a < x < b$.
Let $p = a + \frac{b-a}{3}$, and $q = b - \frac{b-a}{3}$. Then the interval $[p, q]$ is the middle third of the interval $[a, b]$. Since g is continuous on the closed interval $[p, q]$, it attains the minimum m and maximum M on $[p, q]$. Thus for all $x \in [p, q]$ we have $g(x) \geq m$ and $g(x) \leq M$.

Now consider the case that $g(x) > 0$ for all $a < x < b$. It follows that $m > 0$ and hence

$$\int_a^b g(x)\ dx = \int_a^p g(x)\ dx + \int_p^q g(x)\ dx + \int_q^b g(x)\ dx \geq$$

$$\int_a^p 0\ dx + \int_p^q m\ dx + \int_q^b 0\ dx = m(q-p) > 0$$

contradicting the assumption that $\int_a^b g(x)\ dx = 0$.

It remains to show that $g(x) < 0$ for all $a < x < b$ also leads to a contradiction.

In this case, $M < 0$ and hence,

$$\int_a^b g(x)\ dx = \int_a^p g(x)\ dx + \int_p^q g(x)\ dx + \int_q^b g(x)\ dx \leq$$

$$\int_a^p 0\ dx + \int_p^q M\ dx + \int_q^b 0\ dx = M(q-p) < 0$$

contradicting the assumption that $\int_a^b g(x)\ dx = 0$.

Therefore, if g is continuous on $[a, b]$ and $\int_a^b g(x)\ dx = 0$, then there exists a point c, with $a < c < b$ such that $g(c) = 0$. $\square$

Chapter 2

Error Term for the Trapezoid Rule

The first rule we will analyze is the error term for the Trapezoid Rule. We will provide two different proofs from [CK13] and [Gor02] respectively. We will finish this section with Dr. Fejzić's original proof.

2.1 Kincaid and Cheney proof for Trapezoid Rule

The first proof is from [CK13]. The version in [CK13] is called Composite Trapezoid Rule. I will comment on the part of the proof that is in bold afterwords.

Theorem 2.1. *If f'' exists and is continuous on the interval $[a, b]$ and if the composite trapezoid rule T with uniform spacing h is used to estimate the integral $I = \int_a^b f(x)\,dx$, then for some ζ in (a, b),*

$$I - T = -\frac{1}{12}(b-a)h^2 f''(\zeta) = \mathcal{O}(h^2)$$

Proof. Prove when $a = 0$, $b = 1$, and $h = 1$.

$$\int_0^1 f(x)\,dx - \frac{1}{2}[f(0) + f(1)] = -\frac{1}{12}f''(\zeta) \tag{2.1.1}$$

We will need the error formula for polynomial interpolation. Let p be the polynomial of degree 1 that interpolates f at 0 and 1. Then p is given by:

$$p(x) = f(0) + [f(1) - f(0)]x$$

Hence we have

$$\int_0^1 p(x)\ dx = \frac{1}{2}[f(0) + f(1)]$$

By the error formula that governs polynomial interpolation, we have (here, of course, $n = 1$, $x_0 = 0$, and $x_1 = 1$)

$$f(x) - p(x) = \frac{1}{2} f''[\xi(x)]x(x-1) \tag{2.1.2}$$

where $\xi(x)$ depends on x in $(0,1)$. It follows that:

$$\int_0^1 f(x)\ dx - \int_0^1 p(x)\ dx = \frac{1}{2}\int_0^1 f''[\xi(x)]x(x-1)\ dx$$

That $f''[\xi(x)]$ is continuous can be proved by solving Equation (2.1.2) for $f''[\xi(x)]$ and verifying continuity. Notice that $x(x-1)$ does not change sign in the interval $[0,1]$. Hence, by the Mean-Value Theorem for Integrals, there is a point $x = s$ for which $\xi = \xi(s)$ and

$$\begin{aligned}\int_0^1 f''[\xi(x)]x(x-1)\ dx &= f''[\xi(s)]\int_0^1 x(x-1)\ dx \\ &= -\frac{1}{6} f''(\zeta)\end{aligned}$$

By putting all these equations together, we obtain Equation (2.1.1). Then, by making a change in variable, we obtain the basic trapezoid rule with its error term:

$$\int_a^b f(x)\ dx = \frac{b-a}{2}[f(a) - f(b)] - \frac{1}{12}(b-a)^3 f''(\xi) \tag{2.1.3}$$

The details of this are as follows: Let $g(t) = f(a + t(b-a))$ and $x = a + (b-a)t$. Thus, as t traverses the interval $[0,1]$, x traverses the interval $[a,b]$. Also, $dx = (b-a)dt$, $g'(t) = f'[a + t(b-a)](b-a)$ and $g''(t) = f''[a + t(b-a)](b-a)^2$. Hence, by Equation (2.1.1), we have

$$\begin{aligned}\int_a^b f(x)\ dx &= (b-a)\int_0^1 f[a + t(b-a)]\ dt \\ &= (b-a)\int_0^1 g(t)\ dt \\ &= (b-a)\left\{\frac{1}{2}[g(0) + g(1)] - \frac{1}{12} g''(\zeta)\right\} \\ &= \frac{b-a}{2}[f(a) - f(b)] - \frac{(b-a)^3}{12} f''(\xi)\end{aligned}$$

This is the trapezoid rule and error term for the interval $[a, b]$ with only one subinterval, which is the entire interval. Thus, the error term is $\mathcal{O}(h)^3$, where $h = b - a$. Here ξ is in (a, b).

Now let the interval $[a, b]$ be divided into n equal subintervals by points $x_0, x_1, ..., x_n$ with spacing h. Applying Equation (2.1.3) to subinterval $[x_i, x_{i+1}]$, we have

$$\int_{x_i}^{x_{i+1}} f(x)\ dx = \frac{h}{2}[f(x_i) + f(x_{i+1})] - \frac{1}{12}h^3 f''(\xi_i) \tag{2.1.4}$$

where $x_i < \xi_i < x_{i+1}$. We use this result over the interval $[a, b]$, obtaining the composite trapezoid rule:

$$\begin{aligned}\int_a^b f(x)\ dx &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} f(x)\ dx \\ &= \frac{h}{2}\sum_{i=0}^{n-1}[f(x_i) - f(x_{i+1}) - \frac{h^3}{12}\sum_{i=0}^{n-1} f''(\xi_i) \end{aligned} \tag{2.1.5}$$

The final term in Equation (2.1.5) is the error term, and it can be simplified in the following way: since $h = \frac{(b-a)}{n}$, the error term for the composite trapezoid rule is

$$\begin{aligned}-\frac{h^3}{12}\sum_{i=0}^{n-1} f''(\xi_i) &= -\frac{(b-a)}{12}h^2\left[\frac{1}{n}\sum_{i=0}^{n-1} f''(\xi_i)\right] \\ &= -\frac{(b-a)}{12}h^2 f''(\zeta) \\ &= \mathcal{O}(h^2)\end{aligned}$$

Here, we have reasoned that the average $\left[\frac{1}{n}\right]\sum_{i=0}^{n-1} f''(\xi_i)$ lies between the least and greatest values of f'' on the interval (a, b). Hence, by the Intermediate Value Theorem for derivatives, it is $f''(\zeta)$ for some point ζ in (a, b). This completes Kincaid's proof of the error formula. □

As promised we will comment on the part of the proof in bold letters. Here, Kincaid and Cheney state that we can find that $f''[\xi(x)]$ is continuous by solving for it in Equation (2.1.2). However, when one does this, it results in the equation:

$$f''[\xi(x)] = \frac{2(f(x) - p(x))}{x(x-1)}$$

This function does not exist at $x = 0$ and $x = 1$. We could use L'Hopital's rule to show that this function is indeed continuous at 0 and at 1, which of course would add to the complexity of their proof.

2.2 Gordon's proof for Trapezoid Rule

In this section we will reproduce the proof in [Gor02]. The proof starts with a lemma that has a specialized single interval where it is symmetric, before applying it to a more general case.

Lemma 2.2. *If g is twice differentiable on an interval $[-r, r]$ for some positive constant r, Then there exists a point z in the interval $(-r, r)$ such that*

$$\int_{-r}^{r} g - r(g(r) + g(-r)) = -\frac{(2r)^3}{12} g''(z).$$

Proof. Let k be the constant that satisfies the equation

$$\int_{-r}^{r} g - r(g(r) + g(-r)) = k(2r)^3$$

(In other words, the number k is the left side of the displayed equation divided by $8r^3$.) We must show that $k = -\frac{g''(z)}{12}$ for some point $z \in (-r, r)$. Define function $G : [0, r] \to \mathbb{R}$ by

$$G(x) = \int_{-x}^{x} g - x(g(x) + g(-x)) - k(2x)^3$$

Since $G(0) = 0 = G(r)$, Rolle's Theorem guarantees the existence of a point $c \in (0, r)$ such that $G'(c) = 0$.

Using the Fundamental Theorem of Calculus to find G', we obtain

$$\begin{aligned} G'(x) &= \frac{d}{dx}\int_{-x}^{x} g + \frac{d}{dx} - x(g(x) + g(-x)) - \frac{d}{dx}k(2x)^3 \\ &= -x(g'(x) - g(-x) + 24kx) \end{aligned}$$

Since $G'(c) = 0$ and $c \neq 0$, it follows that $g'(c) - g'(-c) + 24kc = 0$. Applying the Mean Value Theorem for derivatives to the function g' on the interval $[-c, c]$ yields

$$k = -\frac{1}{12} \cdot \frac{g'(c) - g'(-c)}{2c} = -\frac{1}{12} g''(z)$$

where $z \in (-c, c) \subseteq (-r, r)$. This completes the proof. □

Next, Gordon proves Lemma 2.2 where the symmetric interval $[-r, r]$ is replaced by an arbitrary interval $[c, d]$.

Lemma 2.3. *If f is twice differentiable on an interval $[c, d]$, then there exists a point v in the interval (c, d) such that*

$$\int_c^d f - \frac{d-c}{2}(f(d) + f(c)) = -\frac{(d-c)^3}{12} f''(v)$$

Proof. Let $m = (d + c)/2$, let $r = (d - c)/2$, and define a function g on $[-r, r]$ by $g(x) = f(x + m)$. Notice that

$$\int_c^d f - \frac{d-c}{2}(f(d) + f(c)) = \int_{-r}^r g - r(g(r) + g(-r))$$

By the previous lemma, there exists a point $z \in (-r, r)$ such that

$$\int_c^d f - \frac{d-c}{2}(f(d) + f(c)) = -\frac{(2r)^3}{12} g''(z) = -\frac{(d-c)^3}{12} f''(v),$$

where $v = z + m$ is a point in the interval (c, d). □

Now the Composite Trapezoid Rule follows easily from Lemma 2.3.

Theorem 2.4. *If f is twice differentiable on an interval $[a, b]$ and n is a positive integer, then there exists a point v in the interval (a, b) such that*

$$\int_a^b f - T_n = -\frac{(b-a)^3}{12n^2} f''(v),$$

where T_n is the nth trapezoidal estimate to the integral.

Proof. Fix a positive integer n and let $x_i = a + i\frac{(b-a)}{n}$ for $0 \leq i \leq n$. Using the previous lemma, we obtain

$$\begin{aligned}
\int_a^b f - T_n &= \sum_{i=1}^n \left(\int_{x_{i-1}}^{x_i} f - \frac{x_i - x_{i-1}}{2}(f(x_i) + f(x_{i-1})) \right) \\
&= \sum_{i=1}^n -\frac{(x_i - x_{i-1})^3}{12} f''(v_i) \\
&= -\frac{(b-a)^3}{12n^3} \sum_{i=1}^n f''(v_i)
\end{aligned}$$

where $v_i \in (x_{i-1}, x_i)$ for each i.

By the Intermediate Value Theorem for derivatives, the function f'' has the intermediate value property on the interval $[v_1, v_n]$. Since the average

$$\frac{f''(v_1) + f''(v_2) + ... + f''(v_n)}{n}$$

is between $min\{f''(v_i) : 1 \le i \le n\}$ and $max\{f''(v_i) : 1 \le i \le n\}$, there exists a point $v \in (v_1, v_n) \subseteq (a, b)$ such that

$$\int_a^b f - T_n = -\frac{(b-a)^3}{12n^2} \cdot \frac{1}{n} \sum_{i=1}^{n} f''(v_i) = -\frac{(b-a)^3}{12n^2} f''(v).$$

□

2.3 Dr. Fejzić's proof of the Trapezoid Rule

Theorem 2.5. *Let f be twice differentiable on an open interval (a, b) and continuous on $[a, b]$. Then $\int_a^b f(x)\,dx = \frac{f(a)+f(b)}{2}(b-a) + E_T$, where the error $E_T = -\frac{f''(\psi)}{12}(b-a)^3$ for some $\psi \in (a, b)$.*

Proof. Let

$$g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a) - \frac{E_T}{\int_a^b (x-a)(x-b)\,dx}(x-a)(x-b).$$

It is straight-forward to check that $g(a) = 0$, $g(b) = 0$, $\int_a^b g(x)\,dx = 0$ and that g is twice differentiable. Here are the details: First, let's check to see if $g(a) = 0$.
Let

$$g(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a) - \frac{E_T}{\int_a^b (a-a)(a-b)\,dx}(a-a)(a-b)$$

When simplified,we get:

$$g(a) = f(a) - f(a) - 0 - 0$$

Thus, $g(a) = 0$
Next, let's check to see if $g(b) = 0$.
Let

$$g(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) - \frac{E_T}{\int_a^b (b-a)(b-b)\,dx}(b-a)(b-b)$$

When simplified, we get:

$$g(b) = f(b) - f(a) - (f(b) - f(a)) - 0$$

Thus, $g(b) = 0$.
We will now check to see if $\int_a^b g(x)\,dx = 0$. We will use that $\int_a^b f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\,dx =$

$\frac{f(a)+f(b)}{2}(b-a)$, which can be derived directly or using the formula for the area of a trapezoid.

$$\int_a^b g(x)\,dx = \int_a^b \Big(f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)-\frac{E_T}{\int_a^b (x-a)(x-b)\,dx}(x-a)(x-b)\Big)\,dx$$

Using the additive properties for integrals we get

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b f(a) - \frac{f(b)-f(a)}{b-a}(x-a)\,dx - \frac{E_T\int_a^b (x-a)(x-b)\,dx}{\int_a^b (x-a)(x-b)\,dx} =$$

$$\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}(b-a) - E_T = E_T - E_T = 0.$$

For the last preliminary part, we will check to see if $g(x)$ is twice differentiable.
First derivative:

$$g'(x) = f'(x) - 0 - \frac{f(b)-f(a)}{b-a} - \frac{E_T}{\int_a^b (x-a)(x-b)\,dx}(2x-a-b)$$

Second derivative:

$$g''(x) = f''(x) - 0 - 2\frac{E_T}{\int_a^b (x-a)(x-b)\,dx}$$

It can be stated that $g(x)$ is twice differentiable.

From Fundamental Lemma 1.16 that $a < c < b$ such that $g(c) = 0$. With this information, we will show that there is ψ such that $g''(\psi) = 0$. To do this, we will prove the following lemma:

Lemma 2.6. *If g is continuous on $[a, b]$, where $a < c < b$ and $g(a) = g(c) = g(b) = 0$, and if g is twice differentiable on (a, b), then there is $\psi \in (a, b)$ such that $g''(\psi) = 0$.*

Proof. Since g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < c < b$ and $g(a) = g(c) = g(b)$, then g is continuous on the intervals $[a, c]$ and $[c, b]$, and differentiable at every point in (a, c) and (c, b).
By Rolle's Theorem, since g is continuous on the interval $[a, c]$ and differentiable at every point in (a, c) such that $g(a) = g(c)$ then

$$g'(d_1) = 0$$

for some d_1 with $a \leq d_1 \leq c$.
Similarly, since g is continuous on the interval $[c, b]$ and differentiable at every point in (c, b) such that $g(c) = g(b)$ then

$$g'(d_2) = 0$$

for some d_2 with $c \leq d_2 \leq b$.
Thus, we get $g'(d_1) = g'(d_2) = 0$.
Let $h(x) = g'(x)$. $h(x)$ is differentiable on (a, b) (by the given) and if a function is differentiable on (a, b), then the function must also be continuous on the interval $[a, b]$. Since $d_1, d_2 \in [a, b]$, then $h(x)$ is differentiable on (d_1, d_2) and continuous on $[d_1, d_2]$. It has also been found above that $h(d_1) = h(d_2)$ and by the Rolle's Theorem, $h'(\psi) = 0$ for some ψ with $d_1 \leq \psi \leq d_2$.
Thus, if g is continuous on $[a, b]$, where $a < c < b$ and $g(a) = g(c) = g(b) = 0$, and if g is twice differentiable on (a, b), then there is $\psi \in (a, b)$ such that $g''(\psi) = 0$. □

Now we have the following result: $g(a) = 0$, $g(b) = 0$, $\int_a^b g(x)\,dx = 0$ and g is twice differentiable. We also found that there is a c with $a < c < b$ such that $g(c) = 0$ and finally, that there is ψ with $\psi \in (a, b)$ such that $g''(\psi) = 0$.
We will now combine all of this to find the error term:

$$g''(x) = f''(x) - \frac{E_T}{\int_a^b (x-a)(x-b)\,dx} \cdot 2$$

Substituting x with ψ gives us:

$$g''(\psi) = f''(\psi) - \frac{2E_T}{\int_a^b (x-a)(x-b)\,dx}$$

$$0 = f''(\psi) - \frac{2E_T}{\int_a^b (x-a)(x-b)\,dx}$$

Solving for the error term results in:

$$\begin{aligned} E_T &= \frac{f''(\psi)}{2} \cdot \int_a^b (x-a)(x-b)\,dx \\ &= \frac{f''(\psi)}{2} \cdot \frac{-(b-a)^3}{6} \\ &= -\frac{f''(\psi)}{12}(b-a)^3 \end{aligned}$$

This concludes Dr. Fejzic's proof on the error term for the Trapezoid Rule. □

The error for Composite Trapezoid Rule uses the same standard argument as in [Gor02].

2.4 Comparison of the three proofs

The proof given in [CK13] requires that f is continuously twice differentiable on $[a, b]$. The conditions in [Gor02] are relaxed by requiring that f is twice differentiable on $[a, b]$, and finally in Dr. Fejzić's proof the conditions are further relaxed to f is twice differentiable on (a, b) and continuous on $[a, b]$. Hence Dr. Fejzić's proof is the most general and in my opinion the simplest of the three.

Chapter 3

Error Term for the Midpoint Rule

The second error term we will analyze is for the Midpoint Rule. We will follow the outline given by Gordon in [Gor02]. Next we will present two original proofs due to Dr.Fejzić.

3.1 Gordon's proof for Midpoint Rule

Similar to the Trapezoid Rule, Gordon starts with a lemma that has a specialized single interval where it is symmetric.

Lemma 3.1. *If g is twice differentiable on an interval $[-r, r]$ for some positive constant r, then there exists a point z in the interval $(-r, r)$ such that*

$$\int_{-r}^{r} g - 2rg(0) = \frac{(2r)^3}{24} g''(z).$$

Proof. Let k be the constant that satisfies the equation:

$$\int_{-r}^{r} g - 2rg(0) = k(2r)^3$$

We must show that $k = \frac{g''(z)}{24}$ for some point $z \in (-r, r)$. Define function $G : [0, r] \to \mathbb{R}$ by

$$G(x) = \int_{-x}^{x} g - 2xg(0) - k(2x)^3$$

To do this, we will show that $G(0) = 0$ and $G(r) = 0$.

$$\begin{aligned} G(0) &= \int_0^0 g - 2(0)g(0) - k(2(0))^3 \\ &= 0 - 0 - 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} G(r) &= \int_{-r}^r g - 2(r)g(0) - k(2(r))^3 \\ &= k(2r)^3 - k(2r)^3 \\ &= 0 \end{aligned}$$

Since $G(0) = G(r) = 0$, Rolle's Theorem guarantees the existence of a point $d \in (0, r)$ such that $G'(d) = 0$. This has been demonstrated in earlier proofs for the trapezoid rule. The derivative of G is

$$G'(x) = g(x) + g(-x) - 2g(0) - 24kx^2$$

Substituting 0 for x, we have

$$\begin{aligned} G'(0) &= g(0) - g(0) - 2g(0) - 24k(0)^2 \\ &= 2g(0) - 2g(0) - 0 \\ &= 0 \end{aligned}$$

Since $G'(d) = G'(0) = 0$, Rolle's Theorem guarantees the existence of a point $c \in (0, d)$ such that $G''(c) = 0$. Using the Fundamental Theorem of Calculus to find G'', we obtain:

$$G''(x) = g'(x) - g'(-x) - 48kx$$

And since we know from Rolle's Theorem that $G''(c) = 0$, we can rewrite the equation as:

$$0 = g'(c) - g'(-c) - 48kc$$

Solving for k, we get:

$$k = \frac{g'(c) - g'(-c)}{2c} \cdot \frac{1}{24}$$

We see that we can apply the Mean Value Theorem to the function g' on the interval $[-c, c]$. By the Mean Value Theorem, there exists a point $z \in (-c, c) \subseteq (-r, r)$ such that $g''(z) = \frac{g'(c)-g'(-c)}{2c}$ and this yields:

$$\begin{aligned} k &= \frac{g'(c) - g'(-c)}{2c} \cdot \frac{1}{24} \\ &= \frac{g'(c) - g'(-c)}{c - (-c)} \cdot \frac{1}{24} \\ &= g''(z) \cdot \frac{1}{24} \\ &= \frac{g''(z)}{24} \end{aligned}$$

□

Now the general case is obtained in a similar way as in Gordon's proof for Trapezoid Rule.

3.2 Dr. Fejzić's first proof of the Midpoint Rule

This proof is similar to Dr. Fejzić's proof of Trapezoid Rule.

Theorem 3.2. *Let f be twice differentiable on an open interval (a, b) and continuous on $[a, b]$. Let $c = \frac{a+b}{2}$. Then $\int_a^b f(x)\,dx = f(c)(b-a) + E$, where the error $E = \frac{f''(\psi)}{24}(b-a)^3$ for some $\psi \in (a, b)$.*

Proof. Let

$$g(x) = f(x) - f(c) - k(x-c) - \frac{E}{\int_a^b x(x-c)\,dx} x(x-c)$$

with $E = \int_a^b f(x)\,dx - f(c)(b-a)$.

We will first show $g(c) = 0$.

$$g(c) = f(c) - f(c) - k(c-c) - \frac{E}{\int_a^b x(x-c)\,dx} c(c-c) = 0.$$

We will now show that g is twice differentiable:

$$g'(x) = f'(x) - k - \frac{2xE}{\int_a^b x(x-c)\,dx}$$

$$g''(x) = f''(x) - \frac{2E}{\int_a^b x(x-c)\,dx}$$

Next, we will show that $\int_a^b g(x)\,dx = 0$.

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx - f(c)(b-a) - 0 - \frac{E}{\int_a^b x(x-c)\,dx}\int_a^b x(x-c)\,dx = 0.$$

Since $\int_a^b g(x)\,dx = \int_a^c g(x)\,dx + \int_c^b g(x)\,dx$ and $\int_a^b g(x)\,dx = 0$, then we will find the k values that makes $\int_a^c g(x)\,dx = 0$ and $\int_c^b g(x)\,dx = 0$. We will first find the k value that makes $\int_a^c g(x)\,dx = 0$.

$$\int_a^c g(x)\,dx = \int_a^c f(x)\,dx - \int_a^c f(c)(b-a)\,dx - \int_a^c k(x-c)\,dx - \frac{E\int_a^c x(x-c)\,dx}{\int_a^b x(x-c)\,dx}$$

$$= \int_a^c f(x)\,dx - f(c)(b-a)(c-a) + k\frac{(b-a)^2}{8} + \frac{E}{\int_a^b x(x-c)\,dx}\cdot\frac{(5a+b)(b-a)^2}{48}$$

We will now solve for k that makes $\int_a^c g(x)\,dx = 0$ true. To do so, We will make

$$R_1 = \int_a^c f(x)\,dx - f(c)(b-a)(c-a) + \frac{E}{\int_a^b x(x-c)\,dx}\cdot\frac{(5a+b)(b-a)^2}{48}$$

Thus, we get

$$\int_a^c g(x)\,dx = R_1 + \frac{k(b-a)^2}{8}$$

$$0 = R_1 + \frac{k(b-a)^2}{8}$$

Solving for k results in:

$$k = -\frac{8R_1}{(b-a)^2}$$

This means that in order for $\int_a^c g(x)\,dx = 0$, $k = -\frac{8R_1}{(b-a)^2}$. Now we will find the k value that makes $\int_c^b g(x)\,dx = 0$.

$$\int_c^b g(x)\,dx = \int_c^b f(x)\,dx - \int_c^b f(c)(b-a)\,dx - \int_c^b k(x-c)\,dx - \frac{E\int_c^b x(x-c)\,dx}{\int_a^b x(x-c)\,dx}$$

$$= \int_c^b f(x)\,dx - f(c)(b-a)(b-c) - k\frac{(b-a)^2}{8} - \frac{E}{\int_a^b x(x-c)\,dx}\cdot\frac{(a+5b)(b-a)^2}{48}$$

We will now solve for k that makes $\int_c^b g(x)\,dx = 0$ true. To do so, We will make

$$R_2 = \int_c^b f(x)\,dx - f(c)(b-a)(b-c) - \frac{E}{\int_a^b x(x-c)\,dx} \cdot \frac{(a+5b)(b-a)^2}{48}$$

Thus, we get

$$\int_c^b g(x)\,dx = R_2 - \frac{k(b-a)^2}{8}$$

$$0 = R_2 - \frac{k(b-a)^2}{8}$$

Solving for k results in:

$$k = \frac{8R_2}{(b-a)^2}$$

This means that in order for $\int_c^b g(x)\,dx = 0$ to be true, $k = \frac{8R_2}{(b-a)^2}$.
Since $\int_a^b g(x)\,dx = 0$ for all possible values of k, then let $k = -\frac{8R_1}{(b-a)^2}$. Applying the basic property of definite integrals we can rewrite $\int_a^b g(x)\,dx$ as the following:

$$\int_a^b g(x)\,dx = \int_a^c g(x)\,dx + \int_c^b g(x)\,dx$$

And when $k = -\frac{8R_1}{(b-a)^2}$, we have:

$$0 = 0 + \int_c^b g(x)\,dx$$

Which could only mean that $\int_c^b g(x)\,dx = 0$ when $k = -\frac{8R_1}{(b-a)^2}$ also. This results with the conclusion that:

$$R_1 = -R_2$$

Thus, $\int_a^c g(x)\,dx = 0$ and $\int_c^b g(x)\,dx = 0$ when $k = -\frac{8R_1}{(b-a)^2}$.

From Lemma 1.16, we know that there is a d_1 with $a < d_1 < c$ such that $g(d_1) = 0$ and a d_2 with $c < d_2 < b$ such that $g(d_2) = 0$. With this, we will now show that there is a ψ with $\psi \in (a,b)$ such that $g''(\psi) = 0$. To do this, we will prove the following lemma:

Lemma 3.3. *If g is continuous on $[a,b]$, where $a < d_1 < c < d_2 < b$ and $g(d_1) = g(c) = g(d_2) = 0$, and if g is twice differentiable on (a,b), then there is a a ψ with $\psi \in (a,b)$ such that $g''(\psi) = 0$*

Proof. Since g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < d_1 < c < d_2 < b$ and $g(d_1) = g(c) = g(d_2)$, then g is continuous on the intervals $[d_1, c]$ and $[c, d_2]$, and differentiable at every point in (d_1, c) and (c, d_2).
By Rolle's Theorem, since g is continuous on the interval $[d_1, c]$ and differentiable at every point in (d_1, c) such that $g(d_1) = g(c)$ then

$$g'(\xi_1) = 0$$

for some ξ_1 with $d_1 \leq \xi_1 \leq c$.

Similarly, using Rolle's Theorem, we also find that $g'(\xi_2) = 0$ for some ξ_2 with $d_1 \leq \xi_2 \leq c$. Thus, we get $g'(\xi_1) = g'(\xi_2) = 0$.

Let $h(x) = g'(x)$. We know from the given that $h(x)$ is differentiable on (a, b) and if a function is differentiable on (a, b), then the function must also be continuous on the interval $[a, b]$. Since $\xi_1, \xi_2 \in [a, b]$, then $h(x)$ is differentiable on (ξ_1, ξ_2) and continuous on $[\xi_1, \xi_2]$. It has also been found above that $h(\xi_1) = h(\xi_2) = 0$ and by the Rolle's Theorem, $h'(\psi) = 0$ for some ψ with $\xi_1 \leq \psi \leq \xi_2$.

Thus, if g is continuous on $[a, b]$, where $a < d_1 < c < d_2 < b$ and $g(d_1) = g(c) = g(d_2) = 0$, and if g is twice differntiable on (a, b), then there is a ψ with $\psi \in (a, b)$ such that $g''(\psi) = 0$. □

Now we have the following result: $g(c) = 0$, g is twice differentiable, $\int_a^b g(x)\,dx = 0$, $\int_a^c g(x)\,dx = 0$, and $\int_c^b g(x)\,dx = 0$. We also found that there is a ψ with $\psi \in (a, b)$ such that $g''(\psi) = 0$.
We will now combine all of this to find the error term:

$$g''(x) = f''(x) - 0 - 0 - \frac{E}{\int_a^b x(x-c)\,dx} \cdot 2$$

Substituting x with ψ gives us:

$$g''(\psi) = f''(\psi) - \frac{2E}{\int_a^b x(x-c)\,dx}$$

$$0 = f''(\psi) - \frac{2E}{\int_a^b x(x-c)\,dx}$$

Solving for the error term results in:

$$\begin{aligned} E &= \frac{f''(\psi)}{2} \cdot \int_a^b x(x-c)\ dx \\ &= \frac{f''(\psi)}{2} \cdot \frac{(b-a)^3}{12} \\ &= \frac{f''(\psi)}{24}(b-a)^3 \end{aligned}$$

□

3.3 Dr. Fejzić's second proof for Midpoint Rule

Here we present a different proof for Midpoint Rule.

Theorem 3.4. *Let f be twice differentiable on an open interval (a, b) and continuous on $[a, b]$. Let $c = \frac{a+b}{2}$. Then $\int_a^b f(x)\ dx = f(c)(b-a) + E$, where the error $E = \frac{f''(\psi)}{24}(b-a)^3$ for some $\psi \in (a, b)$.*

Proof. Let

$$g(x) = f(x) - f(c) - k(x-c) - \frac{E(x-2)^2}{\int_a^b (x-c)^2\ dx}$$

where $E = \int_a^b \Big(f(x) - f(c)\Big)\ dx$ and $c = \frac{a+b}{2}$.
We will first need to show that $g(c) = 0$, $\int_a^b g(x)\ dx = 0$, and that g is twice differentiable. Substituting c into $g(c)$ gives us the following:

$$\begin{aligned} g(c) &= f(c) - f(c) - k(c-c) - \frac{E(c-c)^2}{\int_a^b (x-c)^2\ dx} \\ &= 0 - 0 - 0 \\ &= 0 \end{aligned}$$

Thus, as we can see above, $g(c) = 0$. Now we will integrate $g(x)$ on the interval $[a, b]$:

$$\begin{aligned}
\int_a^b g(x)\,dx &= \int_a^b \left(f(x) - f(c) - k(x-c) - \frac{E(x-c)^2}{\int_a^b (x-c)^2\,dx} \right) dx \\
&= \int_a^b \Big(f(x) - f(c) \Big)\,dx - \int_a^b k(x-c)\,dx - \frac{E\int_a^b (x-c)^2\,dx}{\int_a^b (x-c)^2\,dx} \\
&= \int_a^b \Big(f(x) - f(c) \Big)\,dx - 0 - E \\
&= \int_a^b \Big(f(x) - f(c) \Big)\,dx - \int_a^b \Big(f(x) - f(c) \Big)\,dx \\
&= 0
\end{aligned}$$

And upon integrating, we have $\int_a^b g(x) = 0$. Lastly, we will differentiate g twice to show that it is twice differentiable:

$$\begin{aligned}
g'(x) &= f'(x) - 0 - k - \frac{2E(x-c)}{\int_a^b (x-c)^2\,dx} \\
&= f'(x) - k - \frac{2E(x-c)}{\int_a^b (x-c)^2\,dx}
\end{aligned}$$

$$\begin{aligned}
g''(x) &= f''(x) - 0 - \frac{2E}{\int_a^b (x-c)^2\,dx} \\
&= f''(x) - \frac{2E}{\int_a^b (x-c)^2\,dx}
\end{aligned}$$

As demonstrated above, we can state that g is twice differentiable.
Now we will pick k such that $g'(c) = 0$.

$$g'(x) = f'(x) - k - \frac{2E(x-c)}{\int_a^b (x-c)^2\,dx}$$

$$g'(c) = f'(c) - k - \frac{2E(c-c)}{\int_a^b (x-c)^2\,dx}$$

$$0 = f'(c) - k - 0$$

$$k = f'(c)$$

Rewriting $g(x)$ with $k = f'(c)$ gives us the following function:

$$g(x) = f(x) - f(c) - f'(c)(x-c) - \frac{E(x-c)^2}{\int_a^b (x-c)^2\, dx}$$

We will let this be our $g(x)$ for the remainder of this proof.

Since $\int_a^b g(x)\, dx = 0$, then there must exist at least one point that makes $g(x) = 0$, besides $g(a)$, $g(b)$, and $g(c)$. This is because we know that $g'(c) = 0$, which indicates that c is a critical point, a relative minimum or maximum, thus the function doesn't cross the x-axis at c. We consider the following three cases.

Case 1: $g''(c) = 0$. We will explore the case when $g''(c) = 0$, which makes c an inflection point, where the function changes from concave up/down or vice versa. If $g''(c) = 0$, then we can solve for E:

$$g''(c) = f''(c) - \frac{2E}{\int_a^b (x-c)^2\, dx}$$

$$0 = f''(c) - \frac{2E}{\int_a^b (x-c)^2\, dx}$$

$$E = \frac{f''(c)}{2} \int_a^b (x-c)^2\, dx$$

$$E = \frac{f''(c)}{2} \cdot \frac{(b-a)^3}{12}$$

$$E = \frac{f''(c)}{24}(b-a)^3$$

Case 2: $g''(c) > 0$. We will explore the case when $g''(c) > 0$, which makes c a relative minimum and the function doesn't cross the x-axis at c. Since $\int_a^b g(x)\, dx = 0$, this means that there must be another point, d such that $g(d) = 0$. Point d can either exist within the interval (a, c) or in (c, b).

Suppose $d < c$. Since $g(d) = g(c) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < d < c < b$, then by Rolle's Theorem, there exists a point, $\xi \in (d, c)$ such that $g'(\xi) = 0$. And since $g'(\xi) = g'(c) = 0$, then by Rolle's Theorem again, there exists a point $\psi \in (\xi, c)$ such that $g''(\psi) = 0$. If $g''(\psi) = 0$, then we can solve for E:

$$g''(\psi) = f''(\psi) - \frac{2E}{\int_a^b (x-c)^2 \, dx}$$

$$0 = f''(\psi) - \frac{2E}{\int_a^b (x-c)^2 \, dx}$$

$$E = \frac{f''(\psi)}{2} \int_a^b (x-c)^2 \, dx$$

$$E = \frac{f''(\psi)}{2} \cdot \frac{(b-a)^3}{12}$$

$$E = \frac{f''(\psi)}{24} (b-a)^3$$

Suppose $d > c$. Since $g(d) = g(c) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < c < d < b$, then by Rolle's Theorem, there exists a point, $\xi \in (c, d)$ such that $g'(\xi) = 0$. And since $g'(\xi) = g'(c) = 0$, then by Rolle's Theorem again, there exists a point $\psi \in (c, \xi)$ such that $g''(\psi) = 0$. If $g''(\psi) = 0$, then we can solve for E:

$$g''(\psi) = f''(\psi) - \frac{2E}{\int_a^b (x-c)^2 \, dx}$$

$$0 = f''(\psi) - \frac{2E}{\int_a^b (x-c)^2 \, dx}$$

$$E = \frac{f''(\psi)}{2} \int_a^b (x-c)^2 \, dx$$

$$E = \frac{f''(\psi)}{2} \cdot \frac{(b-a)^3}{12}$$

$$E = \frac{f''(\psi)}{24} (b-a)^3$$

Case 3: $g''(c) < 0$. We will explore the case when $g''(c) < 0$, which makes c a relative maximum and the function doesn't cross the x-axis at c. Since $\int_a^b g(x)\ dx = 0$, this means that there must be another point, d such that $g(d) = 0$. Point d can either exist within the interval (a, c) or in (c, b).

Suppose $d < c$. Since $g(d) = g(c) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < d < c < b$, then by Rolle's Theorem, there exists a point, $\xi \in (d, c)$ such that $g'(\xi) = 0$. And since $g'(\xi) = g'(c) = 0$, then by Rolle's Theorem again, there exists a point $\psi \in (\xi, c)$ such that $g''(\psi) = 0$. If $g''(\psi) = 0$, then we can solve for E:

$$\begin{aligned}
g''(\psi) &= f''(\psi) - \frac{2E}{\int_a^b (x-c)^2\ dx} \\
0 &= f''(\psi) - \frac{2E}{\int_a^b (x-c)^2\ dx} \\
E &= \frac{f''(\psi)}{2} \int_a^b (x-c)^2\ dx \\
E &= \frac{f''(\psi)}{2} \cdot \frac{(b-a)^3}{12} \\
E &= \frac{f''(\psi)}{24}(b-a)^3
\end{aligned}$$

Suppose $d > c$. Since $g(d) = g(c) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < c < d < b$, then by Rolle's Theorem, there exists a point, $\xi \in (c, d)$ such that $g'(\xi) = 0$. And since $g'(\xi) = g'(c) = 0$, then by Rolle's Theorem again, there exists a point $\psi \in (c, \xi)$ such that $g''(\psi) = 0$. If $g''(\psi) = 0$, then we can solve for E:

$$\begin{aligned}
g''(\psi) &= f''(\psi) - \frac{2E}{\int_a^b (x-c)^2\ dx} \\
0 &= f''(\psi) - \frac{2E}{\int_a^b (x-c)^2\ dx} \\
E &= \frac{f''(\psi)}{2} \int_a^b (x-c)^2\ dx \\
E &= \frac{f''(\psi)}{2} \cdot \frac{(b-a)^3}{12}
\end{aligned}$$

$$E = \frac{f''(\psi)}{24}(b-a)^3$$

From these three possible cases, we can see that no matter if $g''(c) = 0$, $g''(c) > 0$ or $g''(c) < 0$, the error term came out the same way: $E = \frac{f''(\psi)}{24}(b-a)^3$. This concludes the proof. $\square$

3.4 Some remarks on the three proofs of Midpoint Rule

For this paper, we explored three proofs on the error term for the Midpoint Rule. The first proof was Gordon's where similar to his approach on the Trapezoid Rule's error term, he began by proving a symmetric domain. I did find that his proof was easy to follow, but I didn't like that his proof was a specialized case and not a generalized case. Unlike the Trapezoid Rule's error term, Gordon did not provide a generalized case. The second proof we followed was Dr. Fejzić's first method. Dr. Fejzić's first method is similar to how the proof of Trapezoid Rule's error term went. The main difference came from the inclusion of the k value that we had to find in order to make $\int_a^c g(x)\ dx = 0$ and $\int_c^b g(x)\ dx = 0$. Nonetheless, the skills needed to understand and complete this proof are the basics of analysis, relying on algebra, basic differentiation rules, basic integration rules, Riemann integral theorems, Lemma 1.16, and Rolle's Theorem. This was a very straightforward proof that an entry-level analysis student would have little to no problems understanding and following. The last proof was Dr. Fejzić's second method. This proof relied on the same ideas as the first method with the addition of the First and Second Derivative Tests while exploring three cases. I found Dr. Fejzić's second method to be eloquent and effortless, the most ideal for students in an entry-level analysis course.

Chapter 4

Error Term for the Simpson's Rule

The third rule we will analyze is for the Simpson's Rule.

4.1 Gordon's proof of Simpson's Rule

Similar to the Trapezoid Rule, Gordon starts with a lemma that has a specialized single interval where it is symmetric and then he applies that to the general case, making the final theorem.

Lemma 4.1. *If g has a fourth derivative on an interval $[-r, r]$ for some positive constant r, then there exists a point z in the interval $(-r, r)$ such that*

$$\int_{-r}^{r} g - \frac{r}{3}\Big(g(-r) + 4g(0) + g(r)\Big) = -\frac{r^5}{90} g''''(z).$$

Proof. Let k be the constant that satisfies the equation

$$\int_{-r}^{r} g - \frac{r}{3}\Big(g(-r) + 4g(0) + g(r)\Big) = kr^5.$$

Define a function G on the interval $[0, r]$ by

$$G(x) = \int_{-x}^{x} g - \frac{x}{3}\Big(g(-x) + 4g(0) + g(x)\Big) - kx^5.$$

In order to make Gordon's next statement in his proof, we will need to show and find several things. The first thing that we need to show is that $G(0) = 0$ and $G(r) = 0$.

$$\begin{aligned} G(0) &= \int_0^0 g - \frac{0}{g}\Big(g(0) + 4g(0) + g(0)\Big) - k(0)^5 \\ &= 0 - 0 - 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} G(r) &= \int_{-r}^r g - \frac{r}{3}\Big(g(-r) + 4g(0) + g(r)\Big) - k(r)^5 \\ &= kr^5 - kr^5 \\ &= 0 \end{aligned}$$

Next, we will need to find $G'''(x)$ and show that G is fourth differentiable.

$$\begin{aligned} G'(x) &= g(x) + g(-x) + \frac{x}{3}g'(-x) - \frac{1}{3}g(-x) - \frac{4}{3}g(0) - \frac{x}{3}g'(x) - \frac{1}{3}g(x) - 5kx^4 \\ &= \frac{2}{3}g(x) + \frac{2}{3}g(-x) + \frac{x}{3}g'(-x) - \frac{x}{3}g'(x) - \frac{4}{3}g(0) - 5kx^4 \end{aligned}$$

$$\begin{aligned} G''(x) &= \frac{2}{3}g'(x) - \frac{2}{3}g'(-x) - \frac{x}{3}g''(-x) + \frac{1}{3}g'(-x) - \frac{x}{3}g''(x) - \frac{1}{3}g'(x) - 20kx^3 \\ &= \frac{1}{3}g'(x) - \frac{1}{3}g'(-x) - \frac{x}{3}g''(-x) - \frac{x}{3}g''(x) - 20kx^3 \end{aligned}$$

$$\begin{aligned} G'''(x) &= \frac{1}{3}g''(x) + \frac{1}{3}g''(-x) + \frac{x}{3}g'''(-x) - \frac{1}{3}g''(-x) - \frac{x}{3}g'''(x) - \frac{1}{3}g''(x) - 60kx^2 \\ &= \frac{x}{3}g'''(-x) - \frac{x}{3}g'''(x) - 60kx^2 \\ &= -\frac{x}{3}\Big(g'''(x) - g'''(-x) + 180kx\Big) \end{aligned}$$

$$G''''(x) = -\frac{x}{3}g''''(-x) + \frac{1}{3}g'''(-x) - \frac{x}{3}g''''(x) - \frac{1}{3}g'''(x) - 120kx$$

With the equations we found above, we will show that $G'(0) = 0$ and $G''(0) = 0$.

$$\begin{aligned} G'(0) &= \frac{2}{3}g(0) + \frac{2}{3}g(0) + \frac{x}{3}g'(0) - \frac{x}{3}g'(0) - \frac{4}{3}g(0) - 5k(0)^4 \\ &= 0 \end{aligned}$$

$$\begin{aligned}G''(0) &= \frac{1}{3}g'(0) - \frac{1}{3}g'(0) - \frac{0}{3}g''(0) - \frac{0}{3}g''(0) - 20k(0)^3\\ &= 0\end{aligned}$$

Since G is continuous on $[0,r]$ and differentiable on $(0,r)$ and $G(0) = G(r) = 0$, then by Rolle's Theorem, there exists a point $e \in (0,r)$ such that $G'(e) = 0$. Likewise, since G' is continuous on $[0,r]$ and differentiable on $(0,r)$ and $G'(0) = G'(e) = 0$, then by Rolle's Theorem, there exists a point $d \in (0,e)$ such that $G''(d) = 0$. Once again, since G'' is continuous on $[0,r]$ and differentiable on $(0,r)$ and $G''(0) = G''(d) = 0$, then by Rolle's Theorem, there exists a point $c \in (0,d)$ such that $G'''(c) = 0$. Now that we have this information, we can continue where Gordon left off in his proof:

Then $G'''(x) = -\frac{x}{3}\Big(g'''(x) - g'''(-x) + 180kx\Big)$ and it can be shown that there is a point $c \in (0,r)$ for which $G'''(c) = 0$.

Gordon then points out that since $c \neq 0$, it follows that

$$k = \frac{g'''(c) - g'''(-c)}{-180c} = -\frac{g''''(z)}{90}$$

for some $z \in (-c,c)$. Gordon obtained this from the substituting in c into $G'''(x)$ and then solving for k.

$$\begin{aligned}G'''(x) &= -\frac{x}{3}\Big(g'''(x) - g'''(-x) + 180kx\Big)\\ G'''(c) &= -\frac{c}{3}\Big(g'''(c) - g'''(-c) + 180kc\Big)\\ 0 &= -\frac{c}{3}\Big(g'''(c) - g'''(-c) + 180kc\Big)\end{aligned}$$

$$\begin{aligned}k &= \frac{g'''(c) - g'''(-c)}{-180c}\\ &= \frac{g'''(c) - g'''(-c)}{2c} \cdot -\frac{1}{90}\\ &= \frac{g'''(c) - g'''(-c)}{c - (-c)} \cdot -\frac{1}{90}\end{aligned}$$

Applying the Mean Value Theorem, we draw the same conclusion as Gordon with

$$k = -\frac{g''''(z)}{90}$$

where the existence of the point $z \in (-c,c) \subseteq (-r,r)$ is guaranteed by the Mean Value Theorem. And this concludes Gordon's specialized case. $\square$

We will now explore Gordon's general case for the Simpson's Rule Error Term.

Lemma 4.2. *If f has a fourth derivative on an interval $[c, d]$, then there exists a point v in the interval (c, d) such that*

$$\int_c^d f - \frac{d-c}{6}\Big(f(c) + 4f(m) + f(d)\Big) = -\frac{(d-c)^5}{32 \cdot 90} f''''(v)$$

where $m = \frac{(c+d)}{2}$ is the midpoint of the interval $[c, d]$.

Theorem 4.3. *If f has a fourth derivative on an interval $[a, b]$ and n is an even positive integer, then there exists a point v in the interval (a, b) such that*

$$\int_a^b f - S_n = -\frac{(b-a)^5}{180n^4} f''''(v),$$

where S_n is the nth Simpson's rule estimate to the integral.

Proof. For this theorem, Gordon starts by letting $m = \frac{d+c}{2}$ and $r = \frac{d-c}{2}$ and define a function g on $[-r, r]$ by $g(x) = f(x + m)$. He then points out that

$$\int_c^d f - \frac{d-c}{6}\Big(f(c) + 4f(m) + f(d)\Big) = \int_{-r}^r g - \frac{r}{3}\Big(g(-r) + 4g(0) + g(r)\Big)$$

Now this may not be immediately apparent or obvious. To help clarify this, I will show the following:

$$g(-r) = f(-r + m) = f(\frac{-d+c}{2} + \frac{d+c}{2}) = f(c)$$

$$g(r) = f(r + m) = f(\frac{d-c}{2} + \frac{d+c}{2}) = f(d)$$

$$g(0) = f(0 + m) = f(m)$$

$$\frac{r}{3} = \frac{d-c}{2 \cdot 3} = \frac{d-c}{6}$$

Gordon finishes his proof by stating that by the previous lemma (for the specialized case), there exists a point $z \in (-r, r)$ such that

$$\int_c^d f - \frac{d-c}{6}\Big(f(c) + 4f(m) + f(d)\Big) = -\frac{r^5}{90} g''''(z) = -\frac{(d-c)^5}{32 \cdot 90} g''''(z)$$

□

4.2 Dr. Fejzić's first proof of Simpson's Rule

The following proof we will follow is the Simpson's Rule for the error term using one of Dr. Fejzic's proofs and methods. Similar to the Midpoint Rule above, we will compose the function, $g(x)$ with Langrange's Interpolation Error Formula and introduce an unknown, k. The more concise version is posted on https://arxiv.org/abs/1708.07727.

Theorem 4.4. *Let f be fourth differentiable on an open interval (a, b) and continuous on $[a, b]$. Then $\int_a^b f(x)\ dx = \frac{(b-a)}{6}\big(f(a) + 4f(c) + f(b)\big) + E$, where the error $E = -\frac{f^{(iv)}(\psi)}{2880}(b-a)^5$ for some $\psi \in (a, b)$.*

Proof. Let

$$g(x) = f(x) - p(x) - k(x-a)(x-b)(x-c) - \frac{E(x-a)(x-b)(x-c)x}{\int_a^b (x-a)(x-b)(x-c)x\ dx}$$

with $E = \int_a^b \Big(f(x) - p(x)\Big)\ dx$ and

$$p(x) = f(a)\frac{(x-b)(x-c)}{(a-b)(a-c)} + f(b)\frac{(x-a)(x-c)}{(b-a)(b-c)} + f(c)\frac{(x-a)(x-b)}{(c-a)(c-b)}$$

For this proof, we will begin by checking that $g(a) = 0$, $g(b) = 0$, $g(c) = 0$, g is fourth differentiable, $\int_a^b g(x)\ dx = 0$, $\int_a^c g(x)\ dx = 0$, and $\int_c^b g(x)\ dx = 0$.
First, let's check to see if $g(a) = 0$.
Let

$$g(a) = f(a) - f(a)\frac{(a-b)(a-c)}{(a-b)(a-c)} - f(b)\frac{(a-a)(a-c)}{(b-a)(b-c)} - f(c)\frac{(a-a)(a-b)}{(c-a)(c-b)}$$
$$-k(a-a)(a-b)(a-c) - \frac{E(a-a)(a-b)(a-c)a}{\int_a^b (x-a)(x-b)(x-c)x\ dx}$$

When simplified, we get:

$$g(a) = f(a) - f(a) - 0 - 0 - 0 - 0$$

Thus, $g(a) = 0$. Next, let's check to see if $g(b) = 0$.

$$g(b) = f(b) - f(a)\frac{(b-b)(b-c)}{(a-b)(a-c)} - f(b)\frac{(b-a)(b-c)}{(b-a)(b-c)} - f(c)\frac{(b-a)(b-b)}{(c-a)(c-b)}$$
$$-k(b-a)(b-b)(b-c) - \frac{E(b-a)(b-b)(b-c)b}{\int_a^b (x-a)(x-b)(x-c)x\ dx}$$

When simplified, we get:

$$g(b) = f(b) - 0 - f(b) - 0 - 0 - 0$$

Thus, $g(b) = 0$. We will next check to see if $g(c) = 0$.

$$g(c) = f(c) - f(a)\frac{(c-b)(c-c)}{(a-b)(a-c)} - f(b)\frac{(c-a)(c-c)}{(b-a)(b-c)} - f(c)\frac{(c-a)(c-b)}{(c-a)(c-b)}$$
$$-k(c-a)(c-b)(c-c) - \frac{E(c-a)(c-b)(c-c)c}{\int_a^b (x-a)(x-b)(x-c)x\,dx}$$

When simplified, we get:

$$g(c) = f(c) - 0 - 0 - f(c) - 0 - 0$$

Thus, $g(c) = 0$. We will now check to see if g is fourth differentiable.

$$g^{(iv)}(x) = f^{(iv)}(x) - 0 - 0 - 0 - 0 - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)x\,dx}$$

Thus, g is fourth differentiable. Next, we will check if $\int_a^b g(x)\,dx = 0$.

$$\int_a^b g(x)\,dx = \int_a^b \Big(f(x) - p(x) - k(x-a)(x-b)(x-c) - \frac{E(x-a)(x-b)(x-c)x}{\int_a^b (x-a)(x-b)(x-c)x\,dx}\Big)\,dx$$

Using the additive properties for integrals we get

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx - \int_a^b p(x)\,dx - k\int_a^b (x-a)(x-b)(x-c)\,dx$$
$$-\frac{E}{\int_a^b (x-a)(x-b)(x-c)x\,dx}\int_a^b (x-a)(x-b)(x-c)x\,dx$$

Integrating yields the following:

$$\int_a^b g(x)\,dx = \int_a^b f(x) - \frac{(b-a)}{6}\Big(f(a) + 4f(c) + f(b)\Big) - \int_a^b f(x)\,dx$$
$$-0 + \frac{(b-a)}{6}\Big(f(a) + 4f(c) + f(b)\Big)$$

Thus resulting with $\int_a^b g(x)\,dx = 0$ for any $k \in \mathbb{R}$.
Now we will show that there is k such that $\int_a^c g(x)\,dx = 0$ and $\int_c^b g(x)\,dx = 0$.

$$\int_a^c g(x)\,dx = \int_a^c \Big(f(x) - p(x) - k(x-a)(x-b)(x-c) - \frac{E(x-a)(x-b)(x-c)x}{\int_a^b (x-a)(x-b)(x-c)x\,dx}\Big)\,dx$$

Integrating this function gives us the following result:

$$\int_a^c g(x)\,dx = \int_a^c f(x)\,dx - \frac{(b-a)}{24}\Big(5f(a)+8f(c)-f(b)\Big) - k\frac{(b-a)^4}{64} - E\frac{(23a+7b)}{16(b-a)}$$

Let

$$R_1 = \int_a^c f(x)\,dx - \frac{(b-a)}{24}\Big(5f(a)+8f(c)-f(b)\Big) - E\frac{(23a+7b)}{16(b-a)}$$

Then we have:

$$\int_a^c g(x)\,dx = R_1 - k\frac{(b-a)^4}{64}$$

This means that in order for $\int_a^c g(x)\,dx = 0$ to be true, $k = \frac{64R_1}{(b-a)^4}$. Now let's check the other interval:

$$\int_c^b g(x)\,dx = \int_c^b \Big(f(x) - p(x) - k(x-a)(x-b)(x-c) - \frac{E(x-a)(x-b)(x-c)x}{\int_a^b (x-a)(x-b)(x-c)x\,dx}\Big)\,dx$$

Integrating this function gives us the following result:

$$\int_c^b g(x)\,dx = \int_c^b f(x)\,dx - \frac{(b-a)}{24}\Big(-f(a)+8f(c)+5f(b)\Big) + k\frac{(b-a)^4}{64} + E\frac{(7a+23b)}{16(b-a)}$$

Let

$$R_2 = \int_c^b f(x)\,dx - \frac{(b-a)}{24}\Big(-f(a)+8f(c)+5f(b)\Big) + E\frac{(7a+23b)}{16(b-a)}$$

Then we have:

$$\int_c^b g(x)\,dx = R_2 + k\frac{(b-a)^4}{64}$$

This means that in order for $\int_c^b g(x)\,dx = 0$ to be true, $k = -\frac{64R_2}{(b-a)^4}$.
Since $\int_a^b g(x)\,dx = 0$ for all possible values of k, then let $k = \frac{64R_1}{(b-a)^4}$. Applying the basic property of definite integrals we can rewrite $\int_a^b g(x)\,dx$ as the following:

$$\int_a^b g(x)\,dx = \int_a^c g(x)\,dx + \int_c^b g(x)\,dx$$

And when $k = \frac{64R_1}{(b-a)^4}$, we have:

$$0 = 0 + \int_c^b g(x)\,dx$$

Which could only mean that $\int_c^b g(x)\,dx = 0$ when $k = \frac{64R_1}{(b-a)^4}$ also. This results with the conclusion that:

$$R_1 = -R_2$$

Thus, $\int_a^c g(x)\ dx = 0$ and $\int_c^b g(x)\ dx = 0$ when $k = \frac{64R_1}{(b-a)^4}$. From Lemma 1.16, we know that there is a d_1 with $a < d_1 < c$ such that $g(d_1) = 0$ and there is a d_2 with $c < d_2 < b$ such that $g(d_2) = 0$.

We will now show that there is a ψ with $\psi \in (a, b)$ such that $g^{(iv)}(\psi) = 0$. To do this, we will prove the following lemma:

Lemma 4.5. *If g is continuous on $[a, b]$, where $a < d_1 < c < d_2 < b$ and $g(a) = g(d_1) = g(c) = g(d_2) = g(b) = 0$, and if g is fourth differentiable on (a, b), then there is a ψ with $\psi \in (a, b)$ such that $g^{(iv)}(\psi) = 0$*

Proof. Since g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < d_1 < c < d_2 < b$ and $g(a) = g(d_1) = g(c) = g(d_2) = g(b)$, then g is continuous on the intervals $[a, d_1]$, $[d_1, c]$, $[c, d_2]$, and $[d_2, b]$, and differentiable at every point in (a, d_1), (d_1, c), (c, d_2), and (d_2, b).

By Rolle's Theorem, since g is continuous on the interval $[a, d_1]$ and differentiable at every point in (a, d_1) such that $g(a) = g(d_1)$ then

$$g'(\xi_1) = 0$$

for some ξ_1 with $a \leq \xi_1 \leq d_1$.

Similarly, using Rolle's Theorem, we also find that $g'(\xi_2) = 0$, $g'(\xi_3) = 0$, and $g'(\xi_4) = 0$ for some ξ_2 with $d_1 \leq \xi_2 \leq c$, ξ_3 with $c \leq \xi_3 \leq d_2$, and ξ_4 with $d_2 \leq \xi_4 \leq b$. Thus, we get $g'(\xi_1) = g'(\xi_2) = g'(\xi_3) = g'(\xi_4) = 0$.

Let $h(x) = g'(x)$. We know from the given that $h(x)$ is differentiable on (a, b) and if a function is differentiable on (a, b), then the function must also be continuous on the interval $[a, b]$. Since $\xi_1, \xi_2, \xi_3, \xi_4 \in [a, b]$, then $h(x)$ is differentiable on (ξ_1, ξ_2), (ξ_2, ξ_3), and (ξ_3, ξ_4) and continuous on $[\xi_1, \xi_2]$, $[\xi_2, \xi_3]$, and $[\xi_3, \xi_4]$. It has also been found above that $h(\xi_1) = h(\xi_2) = h(\xi_3) = h(\xi_4)$ and by the Rolle's Theorem, $h'(\rho_1) = h'(\rho_2) = h'(\rho_3) = 0$ for some ρ_1 with $\xi_1 \leq \rho_1 \leq \xi_2$, a ρ_2 with $\xi_2 \leq \rho_2 \leq \xi_3$, and a ρ_3 with $\xi_3 \leq \rho_3 \leq \xi_4$. Thus, we have $h'(\rho_1) = h'(\rho_2) = h'(\rho_3) = 0$.

Let $j(x) = h'(x) = g''(x)$. We know from the given that $j(x)$ is differentiable on (a, b) and if a function is differentiable on (a, b), then the function must also be continuous on the interval $[a,b]$. Since $\rho_1, \rho_2, \rho_3 \in [a,b]$, then $j(x)$ is differentiable on (ρ_1, ρ_2) and (ρ_2, ρ_3) as well as being continuous on $[\rho_1, \rho_2]$ and $[\rho_2, \rho_3]$. It has also been found above that $j(\rho_1) = j(\rho_2) = j(\rho_3)$ and by the Rolle's Theorem, $j'(\zeta_1) = j'(\zeta_2) = 0$ for some ζ_1 with $\rho_1 \le \zeta_1 \le \rho_2$ and a ζ_2 with $\rho_2 \le \zeta_2 \le \rho_3$. Thus, we have $j'(\zeta_1) = j'(\zeta_2) = 0$.

Now let $k(x) = j'(x) = h''(x) = g'''(x)$. We know from the given that $k(x)$ is differentiable on (a,b) and if a function is differentiable on (a,b), then the function must also be continuous on the interval $[a,b]$. Since $\zeta_1, \zeta_2 \in [a,b]$, then $k(x)$ is differentiable on (ζ_1, ζ_2) and continuous on $[\zeta_1, \zeta_2]$. It has also been found above that $k(\zeta_1) = k(\zeta_2)$ and by the Rolle's Theorem, $k'(\psi) = 0$ for some ψ with $\zeta_1 \le \psi \le \zeta_2$.

Thus, if g is continuous on $[a,b]$, where $a < d_1 < c < d_2 < b$ and $g(a) = g(d_1) = g(c) = g(d_2) = g(b) = 0$, and if g is fourth differntiable on (a,b), then there is a ψ with $\psi \in (a,b)$ such that $g^{(iv)}(\psi) = 0$. □

Now we have the following result: $g(a) = 0$, $g(b) = 0$, $g(c) = 0$, g is fourth differentiable, $\int_a^b g(x)\,dx = 0$, $\int_a^c g(x)\,dx = 0$, and $\int_c^b g(x)\,dx = 0$. We also found that there is a ψ with $\psi \in (a,b)$ such that $g^{(iv)}(\psi) = 0$.
We will now combine all of this to find the error term:

$$g^{(iv)}(x) = f^{(iv)}(x) - 0 - \frac{E}{\int_a^b (x-a)(x-b)(x-c)x\,dx} \cdot 24 - 0$$

Substituting x with ψ gives us:

$$g^{(iv)}(\psi) = f^{(iv)}(\psi) - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)x\,dx}$$

$$0 = f^{(iv)}(\psi) - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)x\,dx}$$

Solving for the error term results in:

$$\begin{aligned} E &= \frac{f^{(iv)}(\psi)}{24} \cdot \int_a^b (x-a)(x-b)(x-c)x\,dx \\ &= \frac{f^{(iv)}(\psi)}{24} \cdot -\frac{(b-a)^5}{120} \\ &= -\frac{f^{(iv)}(\psi)}{2880}(b-a)^5 \end{aligned}$$

□

4.3 Dr. Fejzić's second proof of Simpson's Rule

The second proof of Simpson's Rule is similar to the Dr. Fejzić's second proof of the Midpoint Rule.

Theorem 4.6. *Let f be fourth differentiable on an open interval (a, b) and continuous on $[a, b]$. Then $\int_a^b f(x)\ dx = \frac{(b-a)}{6}\big(f(a) + 4f(c) + f(b)\big) + E$, where the error $E = -\frac{f^{(iv)}(\psi)}{2880}(b-a)^5$ for some $\psi \in (a, b)$.*

Proof. Let

$$g(x) = f(x) - l(x) - kp(x) - \frac{E}{\int_a^b q(x)\ dx} q(x)$$

where

$$E = \int_a^b \Big(f(x) - l(x)\Big)\ dx,$$

and we define $l(x)$, $p(x)$, and $q(x)$ as:

$$l(x) = f(a)\frac{(x-b)(x-c)}{(a-b)(a-c)} + f(c)\frac{(x-a)(x-b)}{(c-a)(c-b)} + f(b)\frac{(x-a)(x-c)}{(b-a)(b-c)},$$

$p(x) = (x-a)(x-b)(x-c)$, and $q(x) = (x-a)(x-b)(x-c)^2$ with $c = \frac{a+b}{2}$. We will first show $g(a) = 0$, $g(b) = 0$ and $g(c) = 0$. Starting with $g(a)$, we have:

$$\begin{aligned} g(a) &= f(a) - l(a) - kp(a) - \frac{E}{\int_a^b q(x)\ dx} q(a) \\ &= f(a) - f(a) - k \cdot 0 - \frac{E}{\int_a^b q(x)\ dx} \cdot 0 \\ &= f(a) - f(a) - 0 - 0 \\ &= 0 \end{aligned}$$

Thus, $g(a) = 0$. Now for $g(b)$:

$$\begin{aligned} g(b) &= f(b) - l(b) - kp(b) - \frac{E}{\int_a^b q(x)\ dx} q(b) \\ &= f(b) - f(b) - k \cdot 0 - \frac{E}{\int_a^b q(x)\ dx} \cdot 0 \\ &= f(b) - f(b) - 0 - 0 \\ &= 0 \end{aligned}$$

Thus, $g(b) = 0$. Finally for $g(c) = 0$:

$$\begin{aligned} g(c) &= f(c) - l(c) - kp(c) - \frac{E}{\int_a^b q(x)\ dx} q(c) \\ &= f(c) - f(c) - k \cdot 0 - \frac{E}{\int_a^b q(x)\ dx} \cdot 0 \\ &= f(c) - f(c) - 0 - 0 \\ &= 0 \end{aligned}$$

Thus, $g(c) = 0$.

We will also show that g is fourth differentiable and $\int_a^b g(x)\ dx = 0$. We will start with showing that g is fourth differentiable:

$$g'(x) = f'(x) - f(a)\frac{(-c+2x-b)}{(a-b)(a-c)} - f(b)\frac{(-c+2x-a)}{(b-a)(b-c)} - f(c)\frac{(-b+2x-a)}{(c-a)(c-b)}$$

$$-k(3x^2-2cx-2bx-2ax+bc+ac+ab) - \frac{E(x-c)(-4x^2-2ab-ac+3ax-bc+3bx+2cx)}{\int_a^b (x-a)(x-b)(x-c)^2\ dx}$$

$$g''(x) = f''(x) - \frac{2f(a)}{(a-b)(a-c)} - \frac{2f(b)}{(b-a)(b-c)} - \frac{2f(c)}{(c-a)(c-b)}$$
$$+2k(a+b+c-3x) - \frac{2E(c^2+6x^2+ab+2ac-3ax+2bc-3bx-6cx)}{\int_a^b (x-a)(x-b)(x-c)^2\ dx}$$

$$g'''(x) = f'''(x) - 6k - \frac{E(24x-6a-6b-12c)}{\int_a^b (x-a)(x-b)(x-c)^2\ dx}$$

$$g^{(iv)}(x) = f^{(iv)}(x) - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)^2\ dx}$$

Now we will show $\int_a^b g(x)\,dx = 0$.

$$\begin{aligned}\int_a^b g(x)\,dx &= \int_a^b \Big(f(x) - l(x) - kp(x) - \frac{E}{\int_a^b q(x)\,dx} q(x)\Big)\,dx \\ &= \int_a^b \Big(f(x) - l(x)\Big)\,dx - k(0) - \frac{E}{\int_a^b q(x)\,dx}\int_a^b q(x)\,dx \\ &= \int_a^b \Big(f(x) - l(x)\Big)\,dx - E \\ &= \int_a^b \Big(f(x) - l(x)\Big)\,dx - \int_a^b \Big(f(x) - l(x)\Big)\,dx \\ &= 0\end{aligned}$$

We will then pick a k such that $g'(c) = 0$, ensuring that c is a critical point.

$$g'(c) = f'(c) - f(a)\frac{(c-b)}{(a-b)(a-c)} - f(b)\frac{(c-a)}{(b-a)(b-c)} -$$

$$f(c)\frac{(-b+2c-a)}{(c-a)(c-b)}$$

$$-k(c^2 - bc - ac + ab) - \frac{E(c-c)(-2c^2 - 2ab + 2ac + 2bc)}{\int_a^b (x-a)(x-b)(x-c)^2\,dx}$$

$$= f'(c) - \frac{f(b) - f(a)}{(b-a)} + k\frac{(b-a)^2}{4}$$

$$k = -\frac{4\Big(f'(c)(b-a) - f(b) + f(a)\Big)}{(b-a)^3}$$

Rewriting $g(x)$ with the k value we found gives us the following function:

$$g(x) = f(x) - l(x) + \frac{4\Big(f'(c)(b-a) - f(b) + f(a)\Big)}{(b-a)^3}p(x) - \frac{E}{\int_a^b q(x)\,dx}q(x).$$

We will let this be our $g(x)$ for the remainder of this proof.
Since $\int_a^b g(x)\,dx = 0$, we will show that there must exist at least one point that makes $g(x) = 0$, besides $g(a)$, $g(b)$, and $g(c)$. To that end we will explore three cases.

Case 1: $g''(c) = 0$. We will explore the case when $g''(c) = 0$, which makes c an inflection point, where the function changes from concave up/down or vice versa. Since $g(a) = g(c) = g(b) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < c < b$, then by Rolle's Theorem, there exists points $\zeta_1 \in (a, c)$ and $\zeta_2 \in (c, b)$ such that $g'(\zeta_1) = 0$ and $g'(\zeta_2) = 0$. And since $g'(\zeta_1) = g'(c) = 0 = g'(\zeta_2) = 0$, then by Rolle's Theorem again, there exists points $\rho_1 \in (\zeta_1, c)$ and $\rho \in (c, \zeta_2)$ such that $g''(\rho_1) = 0$ and $g''(\rho_2) = 0$. But since $g''(\rho_1) = g''(c) = g''(\rho_2) = 0$, then by Rolle's Theorem, there exists points $\xi_1 \in (\rho_1, c)$ and $\xi_2 \in (c, \rho_2)$ such that $g'''(\xi_1) = 0$ and $g'''(\xi_2) = 0$. Since $g'''(\xi_1) == g'''(\xi_2) = 0$, then by Rolle's Theorem one last time, we know that there exists a point $\psi \in (\xi_1, \xi_2)$ such that $g^{(iv)}(\psi) = 0$. If $g^{(iv)}(\psi) = 0$, then we can solve for E:

$$\begin{aligned}
g^{(iv)}(\psi) &= f^{(iv)}(\psi) - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)^2\,dx} \\
0 &= f^{(iv)}(\psi) - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)^2\,dx} \\
E &= \frac{f^{(iv)}(\psi)}{24}\int_a^b (x-a)(x-b)(x-c)^2\,dx \\
E &= \frac{f^{(iv)}(\psi)}{24} \cdot \frac{-(b-a)^5}{120} \\
E &= -\frac{f^{(iv)}(\psi)}{2880}(b-a)^5
\end{aligned}$$

Case 2: $g''(c) > 0$. We will explore the case when $g''(c) > 0$, which makes c a minimum and the function doesn't cross the x-axis at c. The function g has to take on negative values on (a, c) or on (c, b) for otherwise if $g \geq 0$ on (a, b), the integral $\int_a^b g(x)\,dx$, can't be equal to zero. Hence by Intermediate Value Theorem for g, there must be another point, d such that $g(d) = 0$. Point d can either exist within the interval (a, c) or in (c, b). Suppose $d < c$. Since $g(a) = g(d) = g(c) = g(b) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < d < c < b$, then by Rolle's Theorem, there exists points, $\zeta_1 \in (a, d)$, $\zeta_2 \in (d, c)$, and $\zeta_3 \in (c, b)$ such that $g'(\zeta_1) = 0$, $g'(\zeta_2) = 0$, and $g'(\zeta_3) = 0$. And since $g'(\zeta_1) = g'(\zeta_2) = g'(c) = g'(\zeta_3) = 0$, then by Rolle's Theorem again, there exists points $\rho_1 \in (\zeta_1, \zeta_2)$, $\rho_2 \in (\zeta_2, c)$, and $\rho_3 \in (c, \zeta_3)$ such that $g''(\rho_1) = 0$, $g''(\rho_2) = 0$, and $g''(\rho_3) = 0$. Since $g''(\rho_1) = g''(\rho_2) = g''(\rho_3) = 0$, then we know from Rolle's Theorem, there exists points $\xi_1 \in (\rho_1, \rho_2)$ and $\xi_2 \in (\rho_2, \rho_3)$ such that

$g'''(\xi_1) = 0$ and $g'''(\xi_2) = 0$. Since $g'''(\xi_1) = g'''(\xi_2) = 0$, then by Rolle's Theorem one last time, we know that there exists a point $\psi \in (\xi_1, \xi_2)$ such that $g^{(iv)}(\psi) = 0$. If $g^{(iv)}(\psi) = 0$, then we can solve for E:

$$g^{(iv)}(\psi) = f^{(iv)}(\psi) - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)^2\, dx}$$

$$0 = f^{(iv)}(\psi) - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)^2\, dx}$$

$$E = \frac{f^{(iv)}(\psi)}{24} \int_a^b (x-a)(x-b)(x-c)^2\, dx$$

$$E = \frac{f^{(iv)}(\psi)}{24} \cdot \frac{-(b-a)^5}{120}$$

$$E = -\frac{f^{(iv)}(\psi)}{2880}(b-a)^5$$

Suppose $d > c$. Since $g(a) = g(c) = g(d) = g(b) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < c < d < b$, then by Rolle's Theorem, there exists points, $\zeta_1 \in (a, c)$, $\zeta_2 \in (c, d)$, and $\zeta_3 \in (d, b)$ such that $g'(\zeta_1) = 0$, $g'(\zeta_2) = 0$, and $g'(\zeta_3) = 0$. And since $g'(\zeta_1) = g'(c) = g'(\zeta_2) = g'(\zeta_3) = 0$, then by Rolle's Theorem again, there exists points $\rho_1 \in (\zeta_1, c)$, $\rho_2 \in (c, \zeta_2)$, and $\rho_3 \in (\zeta_2, \zeta_3)$ such that $g''(\rho_1) = 0$, $g''(\rho_2) = 0$, and $g''(\rho_3) = 0$. Since $g''(\rho_1) = g''(\rho_2) = g''(\rho_3) = 0$, then we know from Rolle's Theorem, there exists points $\xi_1 \in (\rho_1, \rho_2)$ and $\xi_2 \in (\rho_2, \rho_3)$ such that $g'''(\xi_1) = 0$ and $g'''(\xi_2) = 0$. Since $g'''(\xi_1) = g'''(\xi_2) = 0$, then by Rolle's Theorem one last time, we know that there exists a point $\psi \in (\xi_1, \xi_2)$ such that $g^{(iv)}(\psi) = 0$. If $g^{(iv)}(\psi) = 0$, then we can solve for E:

$$g^{(iv)}(\psi) = f^{(iv)}(\psi) - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)^2\, dx}$$

$$0 = f^{(iv)}(\psi) - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)^2\, dx}$$

$$E = \frac{f^{(iv)}(\psi)}{24} \int_a^b (x-a)(x-b)(x-c)^2\, dx$$

$$E = \frac{f^{(iv)}(\psi)}{24} \cdot \frac{-(b-a)^5}{120}$$

$$E = -\frac{f^{(iv)}(\psi)}{2880}(b-a)^5$$

Case 3: $g''(c) < 0$. We will explore the case when $g''(c) < 0$, which makes c a maximum and the function doesn't cross the x-axis at c. Since $\int_a^b g(x)\,dx = 0$, then as in the proof of the case $g''(c) > 0$, by the Intermediate Value Theorem, there must be another point, d such that $g(d) = 0$. Point d can either exist within the interval (a, c) or in (c, b).

Suppose $d < c$. Since $g(a) = g(d) = g(c) = g(b) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < d < c < b$, then by Rolle's Theorem, there exists points, $\zeta_1 \in (a, d)$, $\zeta_2 \in (d, c)$, and $\zeta_3 \in (c, b)$ such that $g'(\zeta_1) = 0$, $g'(\zeta_2) = 0$, and $g'(\zeta_3) = 0$. And since $g'(\zeta_1) = g'(\zeta_2) = g'(c) = g'(\zeta_3) = 0$, then by Rolle's Theorem again, there exists points $\rho_1 \in (\zeta_1, \zeta_2)$, $\rho_2 \in (\zeta_2, c)$, and $\rho_3 \in (c, \zeta_3)$ such that $g''(\rho_1) = 0$, $g''(\rho_2) = 0$, and $g''(\rho_3) = 0$. Since $g''(\rho_1) = g''(\rho_2) = g''(\rho_3) = 0$, then we know from Rolle's Theorem, there exists points $\xi_1 \in (\rho_1, \rho_2)$ and $\xi_2 \in (\rho_2, \rho_3)$ such that $g'''(\xi_1) = 0$ and $g'''(\xi_2) = 0$. Since $g'''(\xi_1) = g'''(\xi_2) = 0$, then by Rolle's Theorem one last time, we know that there exists a point $\psi \in (\xi_1, \xi_2)$ such that $g^{(iv)}(\psi) = 0$. If $g^{(iv)}(\psi) = 0$, then we can solve for E:

$$g^{(iv)}(\psi) = f^{(iv)}(\psi) - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)^2\,dx}$$

$$0 = f^{(iv)}(\psi) - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)^2\,dx}$$

$$E = \frac{f^{(iv)}(\psi)}{24} \int_a^b (x-a)(x-b)(x-c)^2\,dx$$

$$E = \frac{f^{(iv)}(\psi)}{24} \cdot \frac{-(b-a)^5}{120}$$

$$E = -\frac{f^{(iv)}(\psi)}{2880}(b-a)^5$$

Suppose $d > c$. Since $g(a) = g(c) = g(d) = g(b) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < c < d < b$, then by Rolle's Theorem, there exists points, $\zeta_1 \in (a, c)$, $\zeta_2 \in (c, d)$, and $\zeta_3 \in (d, b)$ such that $g'(\zeta_1) = 0$, $g'(\zeta_2) = 0$, and $g'(\zeta_3) = 0$. And since $g'(\zeta_1) = g'(c) = g'(\zeta_2) = g'(\zeta_3) = 0$, then by Rolle's Theorem again, there exists points $\rho_1 \in (\zeta_1, c)$, $\rho_2 \in (c, \zeta_2)$, and $\rho_3 \in (\zeta_2, \zeta_3)$ such that $g''(\rho_1) = 0$, $g''(\rho_2) = 0$, and $g''(\rho_3) = 0$. Since $g''(\rho_1) = g''(\rho_2) = g''(\rho_3) = 0$, then we know from Rolle's Theorem, there exists points $\xi_1 \in (\rho_1, \rho_2)$ and $\xi_2 \in (\rho_2, \rho_3)$ such that $g'''(\xi_1) = 0$ and $g'''(\xi_2) = 0$. Since $g'''(\xi_1) = g'''(\xi_2) = 0$, then by Rolle's Theorem one last time, we know that there exists a point $\psi \in (\xi_1, \xi_2)$ such that $g^{(iv)}(\psi) = 0$. If $g^{(iv)}(\psi) = 0$, then we can solve for E:

$$g^{(iv)}(\psi) = f^{(iv)}(\psi) - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)^2 \, dx}$$

$$0 = f^{(iv)}(\psi) - \frac{24E}{\int_a^b (x-a)(x-b)(x-c)^2 \, dx}$$

$$E = \frac{f^{(iv)}(\psi)}{24} \int_a^b (x-a)(x-b)(x-c)^2 \, dx$$

$$E = \frac{f^{(iv)}(\psi)}{24} \cdot \frac{-(b-a)^5}{120}$$

$$E = -\frac{f^{(iv)}(\psi)}{2880}(b-a)^5$$

From these three possible cases, we can see that no matter if $g''(c) = 0$, $g''(c) > 0$ or $g''(c) < 0$, the error term came out the same way: $E = -\frac{f^{(iv)}(\psi)}{2880}(b-a)^5$. And this concludes the proof. □

4.4 Remarks on the three proofs of Simpson's Rule

We explored three proofs on the error term for the Simpson's Rule. The first proof was Gordon's which was similar to his approach on the Trapezoid Rule's error term, he began by proving a symmetric domain. It is important to note that in this paper I did provide thorough explanations and steps that was not explicitly provided in his proof. I did find that his proof was easy to follow, but just like with his other proofs, I didn't like that his proof relied on a specific situation, being symmetric, in order to formulate the proof for the generalized case. The second proof we followed was Dr. Fejzić's first method. Dr. Fejzić's first method is similar to how the proof of the Midpoint Rule's error term went with the inclusion of the k value that we had to find in order to make $\int_a^c g(x)\, dx = 0$ and $\int_c^b g(x)\, dx = 0$. Unlike the Midpoint Rule's proof, we had to apply the Rolle's Theorem multiple times in order to get our necessary result, $g^{(iv)}(\psi) = 0$. Nonetheless, the skills needed to understand and complete this proof are the basics of analysis, relying on algebra, basic differentiation rules, basic integration rules, Riemann integral theorems, Lemma 1.16, and Rolle's Theorem. This was a very straightforward proof that an entry-level analysis student would have little to no problems understanding and following. The last proof was Dr. Fejzić's second method. This proof follows the exact same methods as the second method for his Midpoint Rule's error term. I appreciated the consistency the second method provided when comparing it to the Midpoint Rule's proof. It was the most simple and direct method that is ideal for undergraduate students.

Chapter 5

Error Term for the Simpson's $\frac{3}{8}$ Rule

The last rule we will analyze is the Simpson's $\frac{3}{8}$ Rule. We will follow two proofs by Dr. Fejzić and analyze each method.

5.1 Dr. Fejzić's first proof of Simpson's $\frac{3}{8}$ Rule

Theorem 5.1. *Let f be fourth differentiable on an open interval (a, b) and continuous on $[a, b]$. Then $\int_a^b f(x)\ dx = \frac{3(b-a)}{8}(f(a) + 3f(c) + 3f(d) + f(b)) + E$, where the error $E = -\frac{f^{(iv)}(\psi)}{6480}(b-a)^5$ for some $\psi \in (a, b)$.*

Proof. Since this rule involves four points, we will first split the function into two separate pieces. For this rule, points a and b are the x-values for the endpoints of this function's interval. We will set $c = \frac{2a+b}{3}$ and $d = \frac{2b+a}{3}$.

The first interval we will evaluate is $[a, d]$. Let

$$g(x) = f(x) - p(x) - k(x-a)(x-c)(x-d) - \frac{E_1(x-a)(x-c)(x-d)(x-b)}{\int_a^d (x-a)(x-c)(x-d)(x-b)\ dx}$$

with

$$p(x) = f(a)\frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)} + f(b)\frac{(x-a)(x-c)(x-d)}{(b-a)(b-c)(b-d)} + f(c)\frac{(x-a)(x-b)(x-d)}{(c-a)(c-b)(c-d)} + f(d)\frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)}$$

and

$$E_1 = \int_a^d \Big(f(x) - p(x)\Big)\, dx$$

We will show that $g(a) = 0$, $g(c) = 0$ and $g(d) = 0$.

$$g(a) = f(a) - p(a) - k(a-a)(a-c)(a-d) - \frac{E_1(a-a)(a-c)(a-d)(a-b)}{\int_a^d (x-a)(x-c)(x-d)(x-b)\, dx}$$

$$= f(a) - f(a) - 0 - 0$$

$$= 0$$

$$g(c) = f(c) - p(c) - k(c-a)(c-c)(c-d) - \frac{E_1(c-a)(c-c)(c-d)(c-b)}{\int_a^d (x-a)(x-c)(x-d)(x-b)\, dx}$$

$$= f(c) - f(c) - 0 - 0$$

$$= 0$$

$$g(d) = f(d) - p(d) - k(d-a)(d-c)(d-d) - \frac{E_1(d-a)(d-c)(d-d)(d-b)}{\int_a^d (x-a)(x-c)(x-d)(x-b)\, dx}$$

$$= f(d) - f(d) - 0 - 0$$

$$= 0$$

Next, we will show that g is fourth differentiable. Indeed

$$g^{(iv)}(x) = f^{(iv)}(x) - 0 - 0 - \frac{24E_1}{\int_a^d (x-a)(x-c)(x-d)(x-b)\, dx}$$

Thus, g is fourth differentiable. We will also show that $\int_a^d g(x)\, dx = 0$.

$$\int_a^d g(x)\, dx = \int_a^d \Big(f(x) - p(x)\Big)\, dx - 0 - E_1 \frac{\int_a^d (x-a)(x-c)(x-d)(x-b)\, dx}{\int_a^d (x-a)(x-c)(x-d)(x-b)\, dx}$$

$$= \int_a^d \Big(f(x) - p(x)\Big)\, dx - 0 - E_1$$

$$= \int_a^d \Big(f(x) - p(x)\Big)\, dx - \int_a^d \Big(f(x) - p(x)\Big)\, dx$$

$$= 0$$

Using the additive properties for integrals we get

$$\int_a^d g(x)\, dx = \int_a^c g(x)\, dx + \int_c^d g(x)\, dx$$

and we will find k such that $\int_a^c g(x)\ dx = 0$ and $\int_c^d g(x)\ dx = 0$.

$$\int_a^c g(x)\ dx = \int_a^c \Big(f(x) - p(x) - \frac{E_1(x-a)(x-c)(x-d)(x-b)}{\int_a^d (x-a)(x-c)(x-d)(x-b)\ dx} -$$

$$k(x-a)(x-c)(x-d)\Big)\ dx$$

Integrating this function gives us the following result:

$$\int_a^c g(x)\ dx = \int_a^c f(x)\ dx - \frac{(b-a)}{72}\Big(9f(a) + 19f(c) - 5f(d) + f(b)\Big) +$$

$$E_1\frac{(38a+7b)}{16(b-a)} - k\frac{(b-a)^4}{324}$$

Let

$$R_1 = \int_a^c f(x)\ dx - \frac{(b-a)}{72}\Big(9f(a) + 19f(c) - 5f(d) + f(b)\Big) + E_1\frac{(38a+7b)}{16(b-a)}$$

Then we have:

$$\int_a^c g(x)\ dx = R_1 - k\frac{(b-a)^4}{324}$$

This means that in order for $\int_a^c g(x)\ dx = 0$ to be true, $k = \frac{324R_1}{(b-a)^4}$. Now let's check the other interval:

$$\int_c^d g(x)\ dx = \int_c^d \Big(f(x) - p(x) - \frac{E_1(x-a)(x-c)(x-d)(x-b)}{\int_a^d (x-a)(x-c)(x-d)(x-b)\ dx} -$$

$$k(x-a)(x-c)(x-d)\Big)\ dx$$

Integrating this function gives us the following result:

$$\int_c^d g(x)\ dx = \int_c^d f(x)\ dx - \frac{(b-a)}{72}\Big(-f(a) + 13f(c) + 13f(d) - f(b)\Big) -$$

$$E_1\frac{(22a+23b)}{16(b-a)} + k\frac{(b-a)^4}{324}$$

Let

$$R_2 = \int_c^d f(x)\ dx - \frac{(b-a)}{72}\Big(-f(a) + 13f(c) + 13f(d) - f(b)\Big) - E_1\frac{(22a+23b)}{16(b-a)}$$

Then we have:

$$\int_c^d g(x)\ dx = R_2 + k\frac{(b-a)^4}{324}$$

This means that in order for $\int_c^b g(x)\ dx = 0$ to be true, $k = -\frac{324R_2}{(b-a)^4}$.
Since $\int_a^d g(x)\ dx = 0$ for all possible values of k, then let $k = \frac{324R_1}{(b-a)^4}$. Applying the basic property of definite integrals we can rewrite $\int_a^d g(x)\ dx$ as the following:

$$\int_a^d g(x)\ dx = \int_a^c g(x)\ dx + \int_c^d g(x)\ dx$$

And when $k = \frac{324R_1}{(b-a)^4}$, we have:

$$0 = 0 + \int_c^d g(x)\ dx$$

Which could only mean that $\int_c^d g(x)\ dx = 0$ when $k = \frac{324R_1}{(b-a)^4}$ also. This results with the conclusion that:

$$R_1 = -R_2$$

Thus, $\int_a^c g(x)\ dx = 0$ and $\int_c^d g(x)\ dx = 0$ when $k = \frac{324R_1}{(b-a)^4}$. From Lemma 1.16, we know that there is an e_1 with $a < e_1 < c$ such that $g(e_1) = 0$ and there is a e_2 with $c < e_2 < d$ such that $g(e_2) = 0$. That leaves us with showing that there is a ψ with $\psi \in (a, b)$ such that $g^{(iv)}(\psi) = 0$. To do this, we will prove the following lemma:

Lemma 5.2. *If g is continuous on $[a, b]$, where $a < e_1 < c < e_2 < d < b$ and $g(a) = g(e_1) = g(c) = g(e_2) = g(d) = 0$, and if g is fourth differentiable on (a, b), then there is a ψ with $\psi \in (a, b)$ such that $g^{(iv)}(\psi) = 0$*

Proof. Since g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < e_1 < c < e_2 < d < b$ and $g(a) = g(e_1) = g(c) = g(e_2) = g(d)$, then g is continuous on the intervals $[a, e_1]$, $[e_1, c]$, $[c, e_2]$, and $[e_2, d]$, and differentiable at every point in (a, e_1), (e_1, c), (c, e_2), and (e_2, d).
By Rolle's Theorem, since g is continuous on the interval $[a, e_1]$ and differentiable at every point in (a, e_1) such that $g(a) = g(e_1)$ then

$$g'(\xi_1) = 0$$

for some ξ_1 with $a \leq \xi_1 \leq e_1$.

Similarly, using Rolle's Theorem, we also find that $g'(\xi_2) = 0$, $g'(\xi_3) = 0$, and $g'(\xi_4) = 0$ for some ξ_2 with $e_1 \leq \xi_2 \leq c$, ξ_3 with $c \leq \xi_3 \leq e_2$, and ξ_4 with $e_2 \leq \xi_4 \leq d$. Thus, we get

$g'(\xi_1) = g'(\xi_2) = g'(\xi_3) = g'(\xi_4) = 0.$

Let $h(x) = g'(x)$. We know from the given that $h(x)$ is differentiable on (a, b) and if a function is differentiable on (a, b), then the function must also be continuous on the interval $[a, b]$. Since $\xi_1, \xi_2, \xi_3, \xi_4 \in [a, b]$, then $h(x)$ is differentiable on (ξ_1, ξ_2), (ξ_2, ξ_3), and (ξ_3, ξ_4) and continuous on $[\xi_1, \xi_2]$, $[\xi_2, \xi_3]$, and $[\xi_3, \xi_4]$. It has also been found above that $h(\xi_1) = h(\xi_2) = h(\xi_3) = h(\xi_4)$ and by the Rolle's Theorem, $h'(\rho_1) = h'(\rho_2) = h'(\rho_3) = 0$ for some ρ_1 with $\xi_1 \leq \rho_1 \leq \xi_2$, a ρ_2 with $\xi_2 \leq \rho_2 \leq \xi_3$, and a ρ_3 with $\xi_3 \leq \rho_3 \leq \xi_4$. Thus, we have $h'(\rho_1) = h'(\rho_2) = h'(\rho_3) = 0$.

Let $j(x) = h'(x) = g''(x)$. We know from the given that $j(x)$ is differentiable on (a, b) and if a function is differentiable on (a, b), then the function must also be continuous on the interval $[a, b]$. Since $\rho_1, \rho_2, \rho_3 \in [a, b]$, then $j(x)$ is differentiable on (ρ_1, ρ_2) and (ρ_2, ρ_3) as well as being continuous on $[\rho_1, \rho_2]$ and $[\rho_2, \rho_3]$. It has also been found above that $j(\rho_1) = j(\rho_2) = j(\rho_3)$ and by the Rolle's Theorem, $j'(\zeta_1) = j'(\zeta_2) = 0$ for some ζ_1 with $\rho_1 \leq \zeta_1 \leq \rho_2$ and a ζ_2 with $\rho_2 \leq \zeta_2 \leq \rho_3$. Thus, we have $j'(\zeta_1) = j'(\zeta_2) = 0$.

Now let $k(x) = j'(x) = h''(x) = g'''(x)$. We know from the given that $k(x)$ is differentiable on (a, b) and if a function is differentiable on (a, b), then the function must also be continuous on the interval $[a, b]$. Since $\zeta_1, \zeta_2 \in [a, b]$, then $k(x)$ is differentiable on (ζ_1, ζ_2) and continuous on $[\zeta_1, \zeta_2]$. It has also been found above that $k(\zeta_1) = k(\zeta_2)$ and by the Rolle's Theorem, $k'(\psi) = 0$ for some ψ with $\zeta_1 \leq \psi \leq \zeta_2$.

Thus, if g is continuous on $[a, b]$, where $a < e_1 < c < e_2 < d < b$ and $g(a) = g(e_1) = g(c) = g(e_2) = g(d) = 0$, and if g is fourth differntiable on (a, b), then there is a ψ with $\psi \in (a, b)$ such that $g^{(iv)}(\psi) = 0$. $\square$

Now we have the following result: $g(a) = 0$, $g(c) = 0$, $g(d) = 0$, g is fourth differentiable, $\int_a^d g(x)\, dx = 0$, $\int_a^c g(x)\, dx = 0$, and $\int_c^d g(x)\, dx = 0$. We also found that there is a ψ with $\psi \in (a, d)$ such that $g^{(iv)}(\psi) = 0$.

We will now combine all of this to find the first section's error term:

$$g^{(iv)}(x) = f^{(iv)}(x) - 0 - 0 - \frac{E_1}{\int_a^d (x-a)(x-c)(x-d)(x-b)\, dx} \cdot 24$$

Substituting x with ψ gives us:

$$g^{(iv)}(\psi) = f^{(iv)}(\psi) - \frac{24E_1}{\int_a^d (x-a)(x-c)(x-d)(x-b)\ dx}$$

$$0 = f^{(iv)}(\psi) - \frac{24E_1}{\int_a^d (x-a)(x-c)(x-d)(x-b)\ dx}$$

Solving for the first section's error term results in:

$$\begin{aligned} E_1 &= \frac{f^{(iv)}(\psi)}{24} \cdot \int_a^d (x-a)(x-c)(x-d)(x-b)\ dx \\ &= \frac{f^{(iv)}(\psi)}{24} \cdot \frac{-4(b-a)^5}{3645} \\ &= -\frac{f^{(iv)}(\psi)}{21870}(b-a)^5 \end{aligned}$$

Now we will evaluate the second interval $[d, b]$. Let

$$g(x) = f(x) - p(x) - \frac{E_2(x-a)(x-c)(x-d)(x-b)}{\int_d^b (x-a)(x-c)(x-d)(x-b)\ dx}$$

with

$$E_2 = \int_d^b \Big(f(x) - p(x)\Big)\ dx$$

We will first show that $g(a) = 0$, $g(c) = 0$, $g(d) = 0$, and $g(b) = 0$.

$$\begin{aligned} g(a) &= f(a) - p(a) - \frac{E_2(a-a)(a-c)(a-d)(a-b)}{\int_d^b (x-a)(x-c)(x-d)(x-b)\ dx} \\ &= f(a) - f(a) - 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} g(c) &= f(c) - p(c) - \frac{E_2(c-a)(c-c)(c-d)(c-b)}{\int_d^b (x-a)(x-c)(x-d)(x-b)\ dx} \\ &= f(c) - f(c) - 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} g(d) &= f(d) - p(d) - \frac{E_2(d-a)(d-c)(d-d)(d-b)}{\int_d^b (x-a)(x-c)(x-d)(x-b)\ dx} \\ &= f(d) - f(d) - 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} g(b) &= f(b) - p(b) - \frac{E_2(b-a)(b-c)(b-d)(b-b)}{\int_d^b (x-a)(x-c)(x-d)(x-b)\ dx} \\ &= f(b) - f(b) - 0 \\ &= 0 \end{aligned}$$

Next, we will show that g is fourth differentiable.

$$g^{(iv)} = f^{(iv)}(x) - 0 - \frac{24E_2}{\int_d^b (x-a)(x-c)(x-d)(x-b)\ dx}$$

Thus, g is fourth differentiable. We will also show that $\int_d^b g(x)\ dx = 0$.

$$\begin{aligned} \int_d^b g(x)\ dx &= \int_d^b \Big(f(x) - p(x)\Big) - E_2 \frac{\int_d^b (x-a)(x-c)(x-d)(x-b)\ dx}{\int_d^b (x-a)(x-c)(x-d)(x-b)\ dx} \\ &= \int_d^b \Big(f(x) - p(x)\Big) - E_2 \\ &= \int_d^b \Big(f(x) - p(x)\Big) - \int_d^b \Big(f(x) - p(x)\Big) \\ &= 0 \end{aligned}$$

We know from Lemma 1.16 that there is a e with $d < e < b$ such that $g(e) = 0$. We will now show that there is a ψ with $\psi \in (a, b)$ such that $g^{(iv)}(\psi) = 0$. To do this, we will prove the following lemma:

Lemma 5.3. *If g is continuous on $[a, b]$, where $a < c < d < e < b$ and $g(a) = g(c) = g(d) = g(e) = g(b) = 0$, and if g is fourth differentiable on (a, b), then there is a ψ with $\psi \in (a, b)$ such that $g^{(iv)}(\psi) = 0$*

Proof. Since g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < c < d < e < b$ and $g(a) = g(c) = g(d) = g(e) = g(b) = 0$, then g is continuous on the intervals $[a, c]$, $[c, d]$, $[d, e]$, and $[e, b]$, and differentiable at every point in (a, c), (c, d), (d, e), and (e, b).
By Rolle's Theorem, since g is continuous on the interval $[a, c]$ and differentiable at every point in (a, c) such that $g(a) = g(c)$ then

$$g'(\xi_1) = 0$$

for some ξ_1 with $a \leq \xi_1 \leq c$.

Similarly, using Rolle's Theorem, we also find that $g'(\xi_2) = 0$, $g'(\xi_3) = 0$, and $g'(\xi_4) = 0$ for some ξ_2 with $c \leq \xi_2 \leq d$, ξ_3 with $d \leq \xi_3 \leq e$, and ξ_4 with $e \leq \xi_4 \leq b$. Thus, we get $g'(\xi_1) = g'(\xi_2) = g'(\xi_3) = g'(\xi_4) = 0$.

Let $h(x) = g'(x)$. We know from the given that $h(x)$ is differentiable on (a, b) and if a function is differentiable on (a, b), then the function must also be continuous on the interval $[a, b]$. Since $\xi_1, \xi_2, \xi_3, \xi_4 \in [a, b]$, then $h(x)$ is differentiable on $(\xi_1, \xi_2), (\xi_2, \xi_3)$, and (ξ_3, ξ_4) and continuous on $[\xi_1, \xi_2]$, $[\xi_2, \xi_3]$, and $[\xi_3, \xi_4]$. It has also been found above that $h(\xi_1) = h(\xi_2) = h(\xi_3) = h(\xi_4)$ and by the Rolle's Theorem, $h'(\rho_1) = h'(\rho_2) = h'(\rho_3) = 0$ for some ρ_1 with $\xi_1 \leq \rho_1 \leq \xi_2$, a ρ_2 with $\xi_2 \leq \rho_2 \leq \xi_3$, and a ρ_3 with $\xi_3 \leq \rho_3 \leq \xi_4$. Thus, we have $h'(\rho_1) = h'(\rho_2) = h'(\rho_3) = 0$.

Let $j(x) = h'(x) = g''(x)$. We know from the given that $j(x)$ is differentiable on (a, b) and if a function is differentiable on (a, b), then the function must also be continuous on the interval $[a, b]$. Since $\rho_1, \rho_2, \rho_3 \in [a, b]$, then $j(x)$ is differentiable on (ρ_1, ρ_2) and (ρ_2, ρ_3) as well as being continuous on $[\rho_1, \rho_2]$ and $[\rho_2, \rho_3]$. It has also been found above that $j(\rho_1) = j(\rho_2) = j(\rho_3)$ and by the Rolle's Theorem, $j'(\zeta_1) = j'(\zeta_2) = 0$ for some ζ_1 with $\rho_1 \leq \zeta_1 \leq \rho_2$ and a ζ_2 with $\rho_2 \leq \zeta_2 \leq \rho_3$. Thus, we have $j'(\zeta_1) = j'(\zeta_2) = 0$.

Now let $k(x) = j'(x) = h''(x) = g'''(x)$. We know from the given that $k(x)$ is differentiable on (a, b) and if a function is differentiable on (a, d), then the function must also be continuous on the interval $[a, d]$. Since $\zeta_1, \zeta_2 \in [a, b]$, then $k(x)$ is differentiable on (ζ_1, ζ_2) and continuous on $[\zeta_1, \zeta_2]$. It has also been found above that $k(\zeta_1) = k(\zeta_2)$ and by the Rolle's Theorem, $k'(\psi) = 0$ for some ψ with $\zeta_1 \leq \psi \leq \zeta_2$.

Thus, if g is continuous on $[a, b]$, where $a < c < d < e < b$ and $g(a) = g(c) = g(d) = g(e) = g(b) = 0$, and if g is fourth differntiable on (a, b), then there is a ψ with $\psi \in (a, b)$ such that $g^{(iv)}(\psi) = 0$. □

Now we have the following result: $g(a) = 0$, $g(c) = 0$, $g(d) = 0$, $g(b) = 0$, g is fourth differentiable, and $\int_d^b g(x)\, dx = 0$. We also found that there is a ψ with $\psi \in (a, b)$ such that $g^{(iv)}(\psi) = 0$.

We will now combine all of this to find the second section's error term:

$$g^{(iv)}(x) = f^{(iv)}(x) - 0 - \frac{E_2}{\int_d^b (x-a)(x-c)(x-d)(x-b)\ dx} \cdot 24$$

Substituting x with ψ gives us:

$$g^{(iv)}(\psi) = f^{(iv)}(\psi) - \frac{24E_2}{\int_d^b (x-a)(x-c)(x-d)(x-b)\ dx}$$

$$0 = f^{(iv)}(\psi) - \frac{24E_2}{\int_d^b (x-a)(x-c)(x-d)(x-b)\ dx}$$

Solving for the second section's error term results in:

$$\begin{aligned} E_2 &= \frac{f^{(iv)}(\psi)}{24} \cdot \int_d^b (x-a)(x-c)(x-d)(x-b)\ dx \\ &= \frac{f^{(iv)}(\psi)}{24} \cdot \frac{-19(b-a)^5}{7290} \\ &= -\frac{19f^{(iv)}(\psi)}{174960}(b-a)^5 \end{aligned}$$

Combining the error terms from the two sections gives us the following:

$$\begin{aligned} E &= E_1 + E_2 \\ &= -\frac{f^{(iv)}(\psi)}{21870}(b-a)^5 - \frac{19f^{(iv)}(\psi)}{174960}(b-a)^5 \\ &= -\frac{f^{(iv)}(\psi)}{6480}(b-a)^5 \end{aligned}$$

□

This concludes the first proof.

5.2 Dr. Fejzić's second proof of Simpson's $\frac{3}{8}$ Rule

Theorem 5.4. *Let f be fourth differentiable on an open interval (a,b) and continuous on $[a,b]$. Then $\int_a^b f(x)\ dx = \frac{3(b-a)}{8}(f(a) + 3f(c) + 3f(d) + f(b)) + E$, where the error $E = -\frac{f^{(iv)}(\psi)}{6480}(b-a)^5$ for some $\psi \in (a,b)$.*

Proof. Similar to the first method, we will split the function apart into two pieces, $[a, d]$ and $[d, b]$.

The first interval we will evaluate is $[a, d]$. Let

$$g(x) = f(x) - l(x) - kp(x) - \frac{E_1}{\int_a^d q_1(x)\,dx} q_1(x)$$

where

$$E_1 = \int_a^d \Big(f(x) - l(x)\Big)\,dx$$

and we define $l(x)$, $p(x)$, and $q_1(x)$ as:

$$l(x) = f(a)\frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)} + f(c)\frac{(x-a)(x-b)(x-d)}{(c-a)(c-b)(c-d)} +$$

$$f(d)\frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)} + f(b)\frac{(x-a)(x-c)(x-d)}{(b-a)(b-c)(b-d)},$$

$p(x) = (x-a)(x-c)(x-d)$, and $q_1(x) = (x-a)(x-c)(x-d)(x-b)$ with $c = \frac{2a+b}{3}$ and $d = \frac{2b+a}{3}$.

We will show that $g(a) = 0$, $g(c) = 0$, and $g(d) = 0$.

$$\begin{aligned}
g(a) &= f(a) - l(a) - kp(a) - \frac{E_1}{\int_a^d q_1(x)\,dx} q_1(a) \\
&= f(a) - f(a) - k(0) - \frac{E_1}{\int_a^d q_1(x)\,dx}(0) \\
&= f(a) - f(a) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
g(c) &= f(c) - l(c) - kp(c) - \frac{E_1}{\int_a^d q_1(x)\,dx} q_1(c) \\
&= f(c) - f(c) - k(0) - \frac{E_1}{\int_a^d q_1(x)\,dx}(0) \\
&= f(c) - f(c) \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
g(d) &= f(d) - l(d) - kp(d) - \frac{E_1}{\int_a^d q_1(x)\ dx} q_1(d) \\
&= f(d) - f(d) - k(0) - \frac{E_1}{\int_a^d q_1(x)\ dx}(0) \\
&= f(d) - f(d) \\
&= 0
\end{aligned}
$$

Next, we will show that g is fourth differentiable.

$$
\begin{aligned}
g^{(iv)}(x) &= f^{(iv)}(x) - 0 - k(0) - \frac{24E_1}{\int_a^d q_1(x)\ dx} \\
&= f^{(iv)}(x) - \frac{24E_1}{\int_a^d q_1(x)\ dx}
\end{aligned}
$$

Thus, g is fourth differentiable. We will then show that $\int_a^d g(x)\ dx = 0$.

$$
\begin{aligned}
\int_a^d g(x)\ dx &= \int_a^d \Big(f(x) - l(x) - kp(x) - \frac{E_1}{\int_a^d q_1(x)\ dx} q_1(x)\Big)\ dx \\
&= \int_a^d \Big(f(x) - l(x)\Big)\ dx - k\int_a^d p(x)\ dx - \frac{E_1}{\int_a^d q_1(x)\ dx}\int_a^d q_1(x)\ dx \\
&= \int_a^d \Big(f(x) - l(x)\Big)\ dx - k(0) - E_1 \\
&= \int_a^d \Big(f(x) - l(x)\Big)\ dx - \int_a^d \Big(f(x) - l(x)\Big)\ dx \\
&= 0
\end{aligned}
$$

Thus, $\int_a^d g(x)\ dx = 0$. We will now pick k such that $g'(c) = 0$. Recall that $g(x) = f(x) - l(x) - kp(x) - \frac{E_1}{\int_a^d q_1(x)\ dx} q_1(x)$ so that

$$
g'(x) = f'(x) - l'(x) - kp'(x) - \frac{E_1}{\int_a^d q_1(x)\ dx} q_1'(x).
$$

Since $p'(c) \neq 0$ we can solve the last equality for k to obtain

$$
k = \frac{g'(c) - f'(c) + l'(c) + \frac{E_1}{\int_a^d q_1(x)\ dx} q_1'(c)}{p'(c)}.
$$

For the rest of the paper we will work with this value of k; so that in addition to the previous properties of $g(x)$ that we have established and that were true for any value of k, we also have that $g'(c) = 0$ for this value of k.

Case 1: $g''(c) = 0$. We will explore the case when $g''(c) = 0$, which makes c an inflection point, where the function changes from concave up/down or vice versa. Since $g(a) = g(c) = g(d) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < c < d < b$, then by Rolle's Theorem, there exists points $\zeta_1 \in (a, c)$ and $\zeta_2 \in (c, d)$ such that $g'(\zeta_1) = 0$ and $g'(\zeta_2) = 0$. And since $g'(\zeta_1) = g'(c) = 0 = g'(\zeta_2) = 0$, then by Rolle's Theorem again, there exists points $\rho_1 \in (\zeta_1, c)$ and $\rho \in (c, \zeta_2)$ such that $g''(\rho_1) = 0$ and $g''(\rho_2) = 0$. But since $g''(\rho_1) = g''(c) = g''(\rho_2) = 0$, then by Rolle's Theorem, there exists points $\xi_1 \in (\rho_1, c)$ and $\xi_2 \in (c, \rho_2)$ such that $g'''(\xi_1) = 0$ and $g'''(\xi_2) = 0$. Since $g'''(\xi_1) == g'''(\xi_2) = 0$, then by Rolle's Theorem one last time, we know that there exists a point $\psi \in (\xi_1, \xi_2)$ such that $g^{(iv)}(\psi) = 0$. If $g^{(iv)}(\psi) = 0$, then we can solve for E_1:

$$
\begin{aligned}
g^{(iv)}(\psi) &= f^{(iv)}(\psi) - \frac{24E_1}{\int_a^d (x-a)(x-d)(x-c)^2\, dx} \\
0 &= f^{(iv)}(\psi) - \frac{24E_1}{\int_a^d (x-a)(x-d)(x-c)^2\, dx} \\
E_1 &= \frac{f^{(iv)}(\psi)}{24} \int_a^d (x-a)(x-b)(x-c)^2\, dx \\
E_1 &= \frac{f^{(iv)}(\psi)}{24} \cdot \frac{-4(b-a)^5}{3645} \\
E_1 &= -\frac{f^{(iv)}(\psi)}{21870}(b-a)^5
\end{aligned}
$$

Case 2: $g''(c) > 0$. We will explore the case when $g''(c) > 0$, which makes c a minimum and the function doesn't cross the x-axis at c. Since $\int_a^d g(x)\, dx = 0$ then by the Intermediate Value Theorem, there must be another point, e such that $g(e) = 0$. Point e can either exist within the interval (a, c) or in (c, d).

Suppose $e < c$. Since $g(a) = g(d) = g(c) = g(e) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < e < c < d < b$, then by Rolle's Theorem, there exists points, $\zeta_1 \in (a, e)$, $\zeta_2 \in (e, c)$, and $\zeta_3 \in (c, d)$ such that $g'(\zeta_1) = 0$, $g'(\zeta_2) = 0$, and $g'(\zeta_3) = 0$. And since $g'(\zeta_1) = g'(\zeta_2) = g'(c) = g'(\zeta_3) = 0$, then by Rolle's Theorem again, there exists points $\rho_1 \in (\zeta_1, \zeta_2)$, $\rho_2 \in (\zeta_2, c)$, and $\rho_3 \in (c, \zeta_3)$ such that $g''(\rho_1) = 0$, $g''(\rho_2) = 0$, and $g''(\rho_3) = 0$. Since $g''(\rho_1) = g''(\rho_2) = g''(\rho_3) = 0$, then we know from Rolle's Theorem, there exists points $\xi_1 \in (\rho_1, \rho_2)$ and $\xi_2 \in (\rho_2, \rho_3)$ such that $g'''(\xi_1) = 0$ and $g'''(\xi_2) = 0$. Since $g'''(\xi_1) = g'''(\xi_2) = 0$, then by Rolle's Theorem one last time, we know that there exists a point $\psi \in (\xi_1, \xi_2)$ such that $g^{(iv)}(\psi) = 0$. If $g^{(iv)}(\psi) = 0$, then we can solve for E_1:

$$g^{(iv)}(\psi) = f^{(iv)}(\psi) - \frac{24E_1}{\int_a^d (x-a)(x-d)(x-c)^2 \, dx}$$

$$0 = f^{(iv)}(\psi) - \frac{24E_1}{\int_a^d (x-a)(x-d)(x-c)^2 \, dx}$$

$$E_1 = \frac{f^{(iv)}(\psi)}{24} \int_a^d (x-a)(x-b)(x-c)^2 \, dx$$

$$E_1 = \frac{f^{(iv)}(\psi)}{24} \cdot \frac{-4(b-a)^5}{3645}$$

$$E_1 = -\frac{f^{(iv)}(\psi)}{21870}(b-a)^5$$

Suppose $e > c$. Since $g(a) = g(c) = g(d) = g(e) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < c < e < d < b$, then by Rolle's Theorem, there exists points, $\zeta_1 \in (a, c)$, $\zeta_2 \in (c, e)$, and $\zeta_3 \in (e, d)$ such that $g'(\zeta_1) = 0$, $g'(\zeta_2) = 0$, and $g'(\zeta_3) = 0$. And since $g'(\zeta_1) = g'(c) = g'(\zeta_2) = g'(\zeta_3) = 0$, then by Rolle's Theorem again, there exists points $\rho_1 \in (\zeta_1, c)$, $\rho_2 \in (c, \zeta_2)$, and $\rho_3 \in (\zeta_2, \zeta_3)$ such that $g''(\rho_1) = 0$, $g''(\rho_2) = 0$, and $g''(\rho_3) = 0$. Since $g''(\rho_1) = g''(\rho_2) = g''(\rho_3) = 0$, then we know from Rolle's Theorem, there exists points $\xi_1 \in (\rho_1, \rho_2)$ and $\xi_2 \in (\rho_2, \rho_3)$ such that $g'''(\xi_1) = 0$ and $g'''(\xi_2) = 0$. Since $g'''(\xi_1) = g'''(\xi_2) = 0$, then by Rolle's Theorem one last time, we know that there exists a point $\psi \in (\xi_1, \xi_2)$ such that $g^{(iv)}(\psi) = 0$. If $g^{(iv)}(\psi) = 0$, then we can solve for E_1:

$$g^{(iv)}(\psi) = f^{(iv)}(\psi) - \frac{24E_1}{\int_a^d (x-a)(x-d)(x-c)^2 \, dx}$$

$$0 = f^{(iv)}(\psi) - \frac{24E_1}{\int_a^d (x-a)(x-d)(x-c)^2 \, dx}$$

$$E_1 = \frac{f^{(iv)}(\psi)}{24} \int_a^d (x-a)(x-b)(x-c)^2 \, dx$$

$$E_1 = \frac{f^{(iv)}(\psi)}{24} \cdot \frac{-4(b-a)^5}{3645}$$

$$E_1 = -\frac{f^{(iv)}(\psi)}{21870}(b-a)^5$$

Case 3: $g''(c) < 0$. We will explore the case when $g''(c) < 0$, which makes c a maximum and the function doesn't cross the x-axis at c. Since $\int_a^d g(x)\,dx = 0$, then by the Intermediate Value Theorem there must be another point, e such that $g(e) = 0$. Point e can either exist within the interval (a, c) or in (c, d).

Suppose $e < c$. Since $g(a) = g(d) = g(c) = g(e) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < e < c < d < b$, then by Rolle's Theorem, there exists points, $\zeta_1 \in (a, e)$, $\zeta_2 \in (e, c)$, and $\zeta_3 \in (c, d)$ such that $g'(\zeta_1) = 0$, $g'(\zeta_2) = 0$, and $g'(\zeta_3) = 0$. And since $g'(\zeta_1) = g'(\zeta_2) = g'(c) = g'(\zeta_3) = 0$, then by Rolle's Theorem again, there exists points $\rho_1 \in (\zeta_1, \zeta_2)$, $\rho_2 \in (\zeta_2, c)$, and $\rho_3 \in (c, \zeta_3)$ such that $g''(\rho_1) = 0$, $g''(\rho_2) = 0$, and $g''(\rho_3) = 0$. Since $g''(\rho_1) = g''(\rho_2) = g''(\rho_3) = 0$, then we know from Rolle's Theorem, there exists points $\xi_1 \in (\rho_1, \rho_2)$ and $\xi_2 \in (\rho_2, \rho_3)$ such that $g'''(\xi_1) = 0$ and $g'''(\xi_2) = 0$. Since $g'''(\xi_1) = g'''(\xi_2) = 0$, then by Rolle's Theorem one last time, we know that there exists a point $\psi \in (\xi_1, \xi_2)$ such that $g^{(iv)}(\psi) = 0$. If $g^{(iv)}(\psi) = 0$, then we can solve for E_1:

$$g^{(iv)}(\psi) = f^{(iv)}(\psi) - \frac{24E_1}{\int_a^d (x-a)(x-d)(x-c)^2\,dx}$$

$$0 = f^{(iv)}(\psi) - \frac{24E_1}{\int_a^d (x-a)(x-d)(x-c)^2\,dx}$$

$$E_1 = \frac{f^{(iv)}(\psi)}{24}\int_a^d (x-a)(x-b)(x-c)^2\,dx$$

$$E_1 = \frac{f^{(iv)}(\psi)}{24}\cdot\frac{-4(b-a)^5}{3645}$$

$$E_1 = -\frac{f^{(iv)}(\psi)}{21870}(b-a)^5$$

Suppose $e > c$. Since $g(a) = g(c) = g(d) = g(e) = 0$ and g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < c < e < d < b$, then by Rolle's Theorem, there exists points, $\zeta_1 \in (a, c)$, $\zeta_2 \in (c, e)$, and $\zeta_3 \in (e, d)$ such that $g'(\zeta_1) = 0$, $g'(\zeta_2) = 0$, and $g'(\zeta_3) = 0$. And since $g'(\zeta_1) = g'(c) = g'(\zeta_2) = g'(\zeta_3) = 0$, then by Rolle's Theorem again, there exists points $\rho_1 \in (\zeta_1, c)$, $\rho_2 \in (c, \zeta_2)$, and $\rho_3 \in (\zeta_2, \zeta_3)$ such that $g''(\rho_1) = 0$, $g''(\rho_2) = 0$, and $g''(\rho_3) = 0$. Since $g''(\rho_1) = g''(\rho_2) = g''(\rho_3) = 0$, then we know from Rolle's Theorem, there exists points $\xi_1 \in (\rho_1, \rho_2)$ and $\xi_2 \in (\rho_2, \rho_3)$ such that

$g'''(\xi_1) = 0$ and $g'''(\xi_2) = 0$. Since $g'''(\xi_1) = g'''(\xi_2) = 0$, then by Rolle's Theorem one last time, we know that there exists a point $\psi \in (\xi_1, \xi_2)$ such that $g^{(iv)}(\psi) = 0$. If $g^{(iv)}(\psi) = 0$, then we can solve for E_1:

$$g^{(iv)}(\psi) = f^{(iv)}(\psi) - \frac{24E_1}{\int_a^d (x-a)(x-d)(x-c)^2\, dx}$$

$$0 = f^{(iv)}(\psi) - \frac{24E_1}{\int_a^d (x-a)(x-d)(x-c)^2\, dx}$$

$$E_1 = \frac{f^{(iv)}(\psi)}{24} \int_a^d (x-a)(x-b)(x-c)^2\, dx$$

$$E_1 = \frac{f^{(iv)}(\psi)}{24} \cdot \frac{-4(b-a)^5}{3645}$$

$$E_1 = -\frac{f^{(iv)}(\psi)}{21870}(b-a)^5$$

From these three possible cases, we can see that no matter if $g''(c) = 0$, $g''(c) > 0$ or $g''(c) < 0$, the error term came out the same way: $E_1 = -\frac{f^{(iv)}(\psi)}{21870}(b-a)^5$ for the interval $[a, d]$.

We will now evaluate the interval $[d, b]$. For this interval, we will let

$$g(x) = f(x) - l(x) - \frac{E_2 q_2(x)}{\int_d^b q_2(x)\, dx}$$

with $E_2 = \int_d^b \Big(f(x) - l(x)\Big)\, dx$ and $q_2(x) = (x-a)(x-b)(x-c)(x-d)$.

We will begin by showing $g(a) = 0$, $g(b) = 0$, $g(c) = 0$, and $g(d) = 0$.

$$\begin{aligned} g(a) &= f(a) - l(a) - \frac{E_2 q_2(a)}{\int_d^b q_2(x)\, dx} \\ &= f(a) - f(a) - 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} g(b) &= f(b) - l(b) - \frac{E_2 q_2(b)}{\int_d^b q_2(x)\, dx} \\ &= f(b) - f(b) - 0 \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
g(c) &= f(c) - l(c) - \frac{E_2 q_2(c)}{\int_d^b q_2(x)\,dx} \\
&= f(c) - f(c) - 0 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
g(d) &= f(d) - l(d) - \frac{E_2 q_2(d)}{\int_d^b q_2(x)\,dx} \\
&= f(d) - f(d) - 0 \\
&= 0
\end{aligned}
$$

Next, we will show that g is fourth time differentiable.

$$
\begin{aligned}
g^{(iv)} &= f^{(iv)}(x) - 0 - \frac{24E_2}{\int_d^b q_2(x)\,dx} \\
&= f^{(iv)}(x) - \frac{24E_2}{\int_d^b q_2(x)\,dx}
\end{aligned}
$$

Thus, g is fourth differentiable. We will then show that $\int_d^b g(x)\,dx = 0$.

$$
\begin{aligned}
\int_d^b g(x)\,dx &= \int_d^b \left(f(x) - l(x) - \frac{E_2 q_2(x)}{\int_d^b q_2(x)\,dx}\right) dx \\
&= \int_d^b \Big(f(x) - l(x)\Big)\,dx - E_2 \frac{\int_d^b q_2(x)\,dx}{\int_d^b q_2(x)\,dx} \\
&= \int_d^b \Big(f(x) - l(x)\Big)\,dx - E_2 \\
&= \int_d^b \Big(f(x) - l(x)\Big)\,dx - \int_d^b \Big(f(x) - l(x)\Big)\,dx \\
&= 0
\end{aligned}
$$

Since $\int_d^b g(x)\,dx = 0$ and we know that $g(d) = g(b) = 0$, then by the Intermediate Value Theorem, there exists a point $e \in (d, b)$ such that $g(e) = 0$. This means that we have $g(a) = g(c) = g(d) = g(e) = g(b) = 0$. Since g is continuous on the interval $[a, b]$ and differentiable at every point in (a, b), where $a < c < d < e < b$, then by Rolle's Theorem, there exists points $\zeta_1 \in (a, c)$, $\zeta_2 \in (c, d)$, $\zeta_3 \in (d, e)$, and $\zeta_4 \in (e, b)$ such that $g'(\zeta_1) = 0$, $g'(\zeta_2) = 0$, $g'(\zeta_3) = 0$, and $g'(\zeta_4) = 0$. Since $g'(\zeta_1) = g'(\zeta_2) = g'(\zeta_3) = g'(\zeta_4) = 0$, then by Rolle's Theorem, there exists $\xi_1 \in (\zeta_1, \zeta_2)$, $\xi_2 \in (\zeta_2, \zeta_3)$, and $\xi_3 \in (\zeta_3, \zeta_4)$ such that $g''(\xi_1) = 0$, $g''(\xi_2) = 0$, and $g''(\xi_3) = 0$. Since $g''(\xi_1) = g''(\xi_2) = g''(\xi_3) = 0$, then by Rolle's Theorem, there exists $\rho_1 \in (\xi_1, \xi_2)$ and $\rho_2 \in (\xi_2, \xi_3)$ such that $g'''(\rho_1) = 0$ and $g'''(\rho_2) = 0$. Since $g'''(\rho_1) = g'''(\rho_2) = 0$, then by Rolle's Theorem, there exists $\psi \in (\rho_1, \rho_2)$ such that $g^{(iv)}(\psi) = 0$. If $g^{(iv)}(\psi) = 0$, then we can find E_2:

$$g^{(iv)}(\psi) = f^{(iv)}(\psi) - \frac{24E_2}{\int_d^b q_2(x)\,dx}$$

$$0 = f^{(iv)}(\psi) - \frac{24E_2}{\int_d^b q_2(x)\,dx}$$

$$E_2 = -\frac{19f^{(iv)}(\psi)}{174960}(b-a)^5$$

Combining the error terms from the two sections gives us the following:

$$\begin{aligned} E &= E_1 + E_2 \\ &= -\frac{f^{(iv)}(\psi)}{21870}(b-a)^5 - \frac{19f^{(iv)}(\psi)}{174960}(b-a)^5 \\ &= -\frac{f^{(iv)}(\psi)}{6480}(b-a)^5 \end{aligned}$$

□

5.3 Remarks on the two proofs of Simpson's $\frac{3}{8}$ Rule

We explored two proofs on the error term for the Simpson's $\frac{3}{8}$ Rule. Unfortunately, Gordon and Kincaid did not provide a proof for the error term of Simpson's $\frac{3}{8}$ Rule. The first proof we followed was Dr. Fejzić's first method. Dr. Fejzić's first method is similar to how the proofs of the Simpson's Rule and Trapezoid Rule were conducted. Since the Simpson's $\frac{3}{8}$ Rule requires an additional point, we had to split the domain's interval into two parts, $[a, d]$ and $[d, b]$. This resulted with two error terms that needed to be added together to get the final result. Nonetheless, the skills needed to understand and complete this proof are the basics of analysis, relying on algebra, basic differentiation rules, basic integration rules, Riemann integral theorems, Lemma 1.16, and Rolle's Theorem. This was a very straightforward proof that an entry-level analysis student would have little to no problems understanding and following. The second proof was Dr. Fejzić's second method. This proof follows the exact same methods as the second methods for his Simpson's Rule and Midpoint Rule's error terms. Similar to the first method, we had to split the domain interval into the two parts, $[a, d]$ and $[d, b]$. Again, I appreciated the consistency the second method provided when comparing it to the Midpoint Rule's and Simpson's Rule's proof of the error term. The repetitive, straightforward, and elementary approach the second method provides for the proofs of the error terms for the Midpoint, Simpson's and Simpson's $\frac{3}{8}$ Rules makes it the most ideal candidate for being adopted into elementary analysis books.

Chapter 6

Conclusion

After replicating the available proofs of the Trapezoid Rule, Midpoint Rule, Simpson's Rule and Simpson's $\frac{3}{8}$ Rule provided by Kincaid and Cheney, Gordon and Dr. Fejzić, one can see that there is an elementary proof for these rules that should be published in introductory books on Numerical Integration. Dr. Fejzić's proofs were the most elementary and simple to follow, requiring only the basic properties of continuous, differentiable and integrable functions. Not only that, he was able to produce a proof for Simpson's $\frac{3}{8}$ Rule, which is uncommon. Hence, Dr. Fejzić's proofs are the most ideal candidates for finally introducing the proofs of the Midpoint and Simpson's Rules into introductory Numerical Analysis books.

Bibliography

[Axl20] Axler, S., *Measure, Integration, & Real Analysis*, Springer Open, Cham, Switzerland, 2020.

[CK13] Cheney, W., Kincaid, D., *Numerical Mathematics and Computing*, Brooks/Cole, Boston, MA, 2013.

[CN03] Cruz-Uribe, D., Neugebauer, C.J., *"An elementary proof of error estimates for the trapezoidal rule," Mathematics Magazine* Vol. 76, pp. 303-306, 2003.

[DR84] Davis, P.J., Rabinowitz, P., *Methods of Numerical Integration*, Academic Press, Inc., San Diego, CA, 1984.

[Fej17] Fejzić, H. *"A short proof of the error term in Simpson's rule," arXiv*, 2017. Retrieved on July 2020 from http://arxiv.org/abs/1708.07727.

[Gor02] Gordon, R.A., *Real Analysis: A First Course*, Addison-Wesley, Reading, MA, 2002.

[IK66] Isaacson, E., Keller, H.B., *Analysis of numerical methods*, Wiley, New York, 1966.

[Str91] Strang, G., *Calculus*, Wellesley-Cambridge Press, Wellesley, MA, 1991.

[Tal06] Talman, L., *"Simpson's rule is exact for quintics," American Mathematical Monthly*, Vol. 113, pp. 144-155, 2006.