Partial Representations for Ternary Matroids

Ebony Perez
Partial Representations for Ternary Matroids

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Ebony Perez

August 2021
PARTIAL REPRESENTATIONS FOR TERNARY MATROIDS

A Thesis
Presented to the
Faculty of
California State University,
San Bernardino

by
Ebony Perez
August 2021
Approved by:

Dr. Jeremy Aikin, Committee Chair
Dr. Corrine Johnson, Committee Member
Dr. Davida Fischman, Committee Member
Dr. Madiline Jetter, Chair, Department of Mathematics

Date
Dr. Corey Dunn
Graduate Coordinator, Department of Mathematics
Abstract

In combinatorics, a matroid is a discrete object that generalizes various notions of dependence that arise throughout mathematics. All of the information about some matroids can be encoded (or represented) by a matrix whose entries come from a particular field, while other matroids cannot be represented in this way. However, for any matroid, there exists a matrix, called a partial representation of the matroid, that encodes some of the information about the matroid. In fact, a given matroid usually has many different partial representations, each providing different pieces of information about the matroid. In this thesis, we investigate when a given partial representation actually encodes all of the information about some matroid. In particular, we restrict our attention to the class of ternary matroids, which are those that can be fully represented by a matrix over the Galois field $GF(3)$. We explore some of the conditions under which such a matroid is also uniquely determined by one of its partial representations.
Acknowledgements

This project would not be possible without the guidance and support of my thesis advisor, Dr. Jeremy Aikin. His cultivation of a positive environment has greatly contributed to my growth as a mathematician and will have a lifelong affect.

I would like to thank my research partner Aurora. Her collaboration has been vital to the success of this project. Learning and working with her was an immense honor.

I am forever thankful to my parents. Their sacrifice and dedication have opened many gateways of opportunity in my life.

Thank you Brent, Tonionna, Timothy, Ki-jana, Brianna, Sofia, Oliver, and Bella for being my inspiration and light. If any of you happen upon this thesis, know that the sky is your limit.

Lastly, all the thanks and glory belongs to God.
# Table of Contents

**Abstract** iii

**Acknowledgements** iv

**List of Figures** vi

1 **Introduction** 1

2 **Fundamental Ideas in Matroid Theory** 3
   1 Fundamental Ideas ................................................. 3
   2 Bases and Fundamental Circuits ................................. 11
   3 Matroid Minors, Extensions, and Coextensions ............... 14

3 **Representations and Partial Representations of Matroids** 19
   1 Representations ................................................. 19
   2 Partial Representations ........................................ 22
   3 From Partial Representations to Representations Over GF(3) . 26

4 **Partial Representations for Simple Ternary Non-Binary Matroids** 32
   1 Rank 3 STNB Matroids as Restrictions of PG(2,3) ............... 32
   2 Results Using Sage ............................................ 37
   3 General Result .................................................. 43

5 **Conclusion** 48

**Bibliography** 49
List of Figures

2.1 Geometry for M[A] ............................................. 7
2.2 Geometry for Example 2.6 ..................................... 8
2.3 Non-Fano Plane ($F_7^{-}$): Geometry for Example 2.7 ... 8
2.4 Uniform Matroids ............................................. 9
2.5 Graph G for Cycle Matroid $M(G)$ .......................... 11
2.6 The Matroid $W_3$ ........................................... 12
2.7 Constructions and Deletions of Matroid $M$ ................. 15
2.8 $M/a$. Contract the Element $a$ by Projecting onto a Line 15
2.9 $(M/a) - f$: Contraction of $a$ and Deletion of $f$ from the Matroid $M$ 16
2.10 Lifts of Matroid $N$ ........................................ 17

3.1 Left: $PG(2, 2)$. Right: $PG(2, 3)$ .......................... 20
3.2 The Matroid $P_7$ ........................................... 21
3.3 A Rank 3 Matroid with a Loop ............................... 23
3.4 Nonisomorphic Matroids with Partial Representation $P$ ... 24
3.5 The Matroid Geometry for Example 3.10 ..................... 28
3.6 Size 6 STNB Matroids ...................................... 31

4.1 $PG(2, 3)$ .................................................. 32
Chapter 1

Introduction

Matroid theory is a relatively new branch of mathematics that was introduced by Hassler Whitney in the year 1935. Around this time, many mathematicians were actively studying ideas equivalent to the concepts Whitney described as matroid theory. In his paper “On the Abstract Properties of Linear Dependence”, Whitney introduces a set of axioms defining a matroid. He defined a matroid as a finite set with special subsets called independent sets. His axioms state that the independent sets must obey the following properties [Whi35]:

1. Any subset of an independent set is independent.

2. Independent sets preserve augmentation (We will learn more about the meaning behind this in Definition 2.1).

When studying matroid theory, one can find many connections with other areas of mathematics. Structures that preserve this notion of independence include, vector spaces, graphs, matchings in bipartite graphs, various algebraic structures, and finite geometries. Along with this definition of a matroid in terms of independence, matroids can also be equivalently defined in terms of other related concepts such as rank, bases, circuits, flats, hyperplanes, and closure (the equivalent ways of defining a matroid is not limited to this list). The concepts of rank, bases, and circuits will be discussed in Chapter 2 in detail. Since the initial discovery of matroids in 1935, the field of matroid theory has grown tremendously, and research in matroid theory continues to be very active. [GM12]. In this thesis, we will mostly explore binary and ternary matroids; matroids that are representable over fields $GF(2)$ and $GF(3)$, respectively. Throughout the subsequent chapters
we will define a matroid, give many examples of matroids, and gather the necessary tools to present the main ideas of partial representations of certain classes of matroids.
Chapter 2

Fundamental Ideas in Matroid Theory

1 Fundamental Ideas

Our aim of this chapter is to introduce the colorful world of matroids with fundamental definitions, theorems, and examples. In similar fashion to Whitney, let us begin by defining a matroid in terms of independent sets.

**Definition 2.1.** A matroid $M$ is an ordered pair $(E, \mathcal{I})$ consisting of a finite set $E$ and a collection $\mathcal{I}$ of subsets of $E$ having the following three properties:

(I1) $\emptyset \in \mathcal{I}$. (The empty set is independent.)

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$. (Every subset of an independent set is independent.)

(I3) If $I_1$ and $I_2$ are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element $e$ of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$. (The augmentation property or the independent set exchange property.)

The finite set $E$ is called the ground set of the matroid. This set consists of all of the elements of the matroid. Elements are defined according to the type of matroid being considered. For example, in a representable matroid the elements are the column vectors of a matrix. In a graphic matroid the elements are defined as the edges of the graph. We will see examples of representable and graphic matroids later in this section.
Given any subset \( A \) of the ground set \( E \) of a matroid, the *rank* of \( A \), denoted \( r(A) \), is the size of a largest independent subset of \( A \):

\[
r(A) := \max_{I \subseteq A} \{|I| : I \in \mathcal{I}\}.
\]

We call \( r(E) \) the *rank of the matroid*. Sometimes we write \( r(M) \) instead of \( r(E) \). We define \( r(E) \) as the size of a largest independent set of the matroid. If \( M \) is a matroid with independent sets \( \mathcal{I} \), then \( B \) is a basis of the matroid \( M \) if \( B \) is a *maximal independent set*. We use \( \mathcal{B}(M) \), or just \( \mathcal{B} \), to denote the collection of bases of a matroid.

\[
\mathcal{B}(M) = \{B \in \mathcal{I} | B \subseteq A \in \mathcal{I} \text{ implies } B = A\}
\]

One striking property is that all bases in a matroid have the same cardinality. A proof of this goes as follows: Let \( B_1 \) and \( B_2 \) be maximal independent sets in a matroid \( M \) and suppose that \( |B_1| < |B_2| \). Using (I3) of Definition 2.1, there is some element \( x \in B_2 - B_1 \) such that \( B_1 \cup \{x\} \in \mathcal{I} \). Since \( B_1 \) is a maximal independent set, no independent set \( I \) properly contains \( B_1 \). Thus we arrive at a contradiction, allowing us to conclude \( |B_1| = |B_2| \). With this in mind, the definitions of rank and basis implies that \( r(E) = |B| \), for any \( B \in \mathcal{B} \). The cardinality of a bases \( |B| \) refers to the number of elements in the set \( B \). This notation for cardinality is universal for all subsets of the ground set. Cardinality of a set of is not to be confused with the rank of a set. In fact, the only time the cardinality and rank of a set coincide is when the set itself is independent. That is, for \( X \subseteq E \), \( r(X) = |X| \) precisely when \( X \in \mathcal{I} \). This equality between rank and cardinality does not hold for dependent sets.

A subset of \( E \) that is not found in the collection of independent sets \( \mathcal{I} \) is called a *dependent set*. Let \( M \) be a matroid. If \( C \) is dependent, but every proper subset of \( C \) is independent, then \( C \) is called a *circuit* of the matroid. That is, circuits are minimally dependent sets. We use \( \mathcal{C} \) to denote the collection of all circuits of a matroid (or \( \mathcal{C}(M) \) if we wish to specify the matroid \( M \)). The collection \( \mathcal{C} \) of circuits of a matroid are characterized by the three properties in the following theorem.

**Theorem 2.2.** Let \( M \) be a matroid on ground set \( E \). A collection \( \mathcal{C} \) of subsets of \( E \) is the collection of circuits of \( M \) if and only if \( \mathcal{C} \) satisfies the following three properties:

\[
(C1) \emptyset \notin \mathcal{C}.
\]
(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \neq C_2$, and $x \in C_1 \cap C_2$, then $C_3 \subseteq C_1 \cup C_2 - x$ for some $C_3 \in \mathcal{C}$.

We note that circuits depend on the notion of dependence, which in turn, depends on the notion of independence. Alternatively, we could have begun with a notion of dependence and circuits and defined independent set as those subsets of $E$ that contain no circuits. Thus, with a little work, one can show that a matroid can be determined if we are given $(E, \mathcal{I})$ by checking that $\mathcal{I}$ satisfies $(I1)$, $(I2)$, and $(I3)$, or equivalently, if we are given $(E, \mathcal{C})$ by checking that $\mathcal{C}$ satisfies $(C1)$, $(C2)$, and $(C3)$.

Traditionally, the notation $\{a, b, c\}$ translates as the set consisting of $a, b,$ and $c$. Because we will frequently be working with sets whose elements are also sets, we will translate this notation and instead write $abc$ to mean the set consisting of $a, b,$ and $c$. That is $abc$ should be taken to mean $\{a, b, c\}$ throughout this paper. Thus far, we have discussed many fundamental ideas about matroids and will now illustrate these ideas with the following examples.

**Example 2.3.** Let $E = \{♣, ♦, ♠, ♥\}$ be the ground set of a matroid $M$ and let $\mathcal{I} = \{∅, ♦, ♠, ♠♥, ♦♠\}$ be the collection of independent sets of $M$.

Recall, we write $♠♥$ instead of $\{♠, ♥\}$. We begin by verifying the independence axioms for $\mathcal{I}$. We can see that $∅ \in \mathcal{I}$, so that axiom $(I1)$ is satisfied. The subsets of $\mathcal{I}$ also satisfy $(I3)$; the exchange property. However, notice the subset $♠♥$ violates axiom $(I2)$. Axiom $(I2)$ states that every subset of an independent set is independent however $♥ \subset ♠♥$ and $♥ \not\in \mathcal{I}$. To rectify this violation we might alter $\mathcal{I}$ by adding $♥$ as an element of $\mathcal{I}$. One can check that if we add the set $♥$ to the collection $\mathcal{I}$, then $\mathcal{I}$ will satisfy $(I2)$, as well as $(I1)$ and $(I3)$. Thus the ground set $E$ together with the independent sets $\mathcal{I} \cup ♥$ now describes a matroid $M$. We can see $r(M) = 2$, because the size of the a maximal independent set, or basis, in $E(M)$ is 2. We also see that the element $♣$ is found in every basis of the matroid $M$ and the element $♠$ is found in no basis of $M$. The characteristics of these two elements lead us to the following definitions. Let $M$ be a matroid on the ground set $E$. A **coloop** is an element $x \in E$ that is every basis. A **loop** is an element $x \in E$ that is in no basis. **Simple matroids** are void of loops (dependent singletons) and **pairs of parallel elements** (dependent doubletons). The matroid in Example 2.3 is not a
simple matroid of the ground set is independent and every pair of elements is independent. The element ♠ is a loop and the dependent set ♢♦ is a pair of parallel elements.

**Definition 2.4.** A matroid whose ground set $E$ is the set of column vectors in a matrix $A$ over a fixed field $\mathbb{F}$ is called an $\mathbb{F}$-representable matroid. We denote such a matroid by $M[A]$.

The linearly independent sets (over $\mathbb{F}$) of column vectors are the independent sets of $M[A]$. A set of vectors is *linearly dependent* if a vector in the set can be expressed as a linear combination (over $\mathbb{F}$) of other vectors in the set. Otherwise, a set of vectors is *linearly independent*. We will continue our discussion of representable matroids in Chapter 3.

**Example 2.5.** Let $A$ be the following $3 \times 5$ matrix over $\mathbb{R}$:

$$
A = \begin{pmatrix}
a & b & c & d & e \\
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0
\end{pmatrix}
$$

The elements in the set $E = \{a, b, c, d, e\}$ are the columns of matrix $A$ corresponding to their labels, and $E$ is the ground set of the matroid $M[A]$. Note that the columns of $A$ are vectors in $\mathbb{R}^3$. Let us describe the independent sets, bases, and circuits of $M[A]$. Because there are no columns in $A$ that correspond to all zeros, matroid $M[A]$ has no loop elements and every single element is independent. Every pair of columns are independent, thus there are no pairs of parallel elements. Parallel pairs would be a set of two columns vectors that are scalar multiples of each other. Observe that for all $B \in \mathcal{B}$, we have $|B| = 3$. Therefore, $r(M[A]) = 3$. Because this is a simple rank 3 matroid we can consider all combinations of three columns and sort these combinations by their dependency status; independent combinations will be bases and dependent combinations will be circuits of size 3. We know the dependent sets of size 3 are minimally dependent because every two columns of $A$ are independent. There are $\binom{5}{3} = 10$ combinations of three columns in $A$, which is a relatively small number to consider. If $A$ was a larger matrix, to efficiently find this information, we could use a computer program such as Sage. The complete collection of bases is $\mathcal{B} = \{abc, abd, acd, ace, ade, bcd, bce, bde, cde\}$. The complete list of circuits is $\mathcal{C} = \{abc, abcd, acde, bcede\}$. A spanning circuit $C$ of matroid
Figure 2.1: Geometry for $M[A]$

$M$ is a circuit of size $r + 1$, where $r = r(M)$. For a spanning circuit $C \in \mathcal{C}$, and for all $x \in C$ the set $C - x$ will be a basis of $M$. We see that that $M[A]$ has three spanning circuits.

Geometrically, we can describe the column dependencies of matrix $A$. We can construct the matroid geometry representing the column dependencies by projecting the vectors of matrix $A$ in $\mathbb{R}^3$ onto a hyperplane (a copy of $\mathbb{R}^2$) in $\mathbb{R}^3$. In a rank 3 finite geometry,

- Circuits of size at most four consist of loops (dependent singletons), parallel points (two points occupying the same space), three points on a common line. In rank 3, all sets of size 4 are dependent.

- Independent sets of size at most four are single non-loop points, nonparallel pairs of points, and three non-collinear points, and four non coplanar points.

We state again that the matrix $A$ does not have a loop element. A loop element in a matrix would correspond to a column of all zeros. No two columns of $A$ are scalar multiples of each other. Thus $A$ does not have pairs of parallel elements. We see that $(1,0,0) + -1(0,1,0) = (1,-1,0)$ so that $e$ can be expressed as a linear combination of columns $a$ and $b$. These columns correspond in the geometry to three colinear points. Figure 2.1 describes the the dependent and independent sets of $M[A]$. Each point in the geometry corresponds to the column of $A$ with the same label.

Suppose we are tasked with describing the dependencies of a matroid given a finite geometry, such as the one given in Figure 2.1. We would begin by identifying the independent sets of the matroid. The rank of the matroid would correspond to a maximal independent set in the geometry. In a rank 4 geometry, any set consisting of four non-coplanar points would be maximally independent in the matroid. For a rank 3 geometry, any set consisting of three non-colinear points would be maximally independent, in fact,
a basis in the matroid. In a rank 2 geometry, any set consisting of two non-parallel points would be maximally independent in the matroid. Every nonloop element is independent in a matroid. Similarly, we could describe the dependent sets of a matroid geometry. To illustrate this process, we consider the following example.

**Example 2.6.** Let $M$ be the matroid whose ground set consists of the points $a$, $b$, $c$, $d$, and $e$ shown in Figure 2.2.

The collection $I = \{\emptyset, a, b, c, d, ab, ac, ad, bc, bd\}$ are the independent sets of $M$. We observe that not every single point in $M$ is independent. The cloud-like element $e$ floating towards the top of the figure is how we depict a dependent singleton or loop, in a matroid geometry. This geometric depiction of a loop element was adapted from Gordon and McNulty’s wonderful book *Matroids: A Geometric Introduction*. The pair of dependent points $cd$ are a pair of parallel elements. Any set of three colinear points are dependent. Furthermore, because all bases of $M$ have cardinality 2, we see that $r(M) = 2$. Any set of three or more points is dependent. We continue our exploration with another geometric example.
Example 2.7. Let $M$ be the matroid given by the geometry, shown in Figure 2.3, whose ground set is the set of points \{a, b, c, d, e, f, g\}.

The collection of independent sets includes the empty set, all single points, all pairs of points and all sets of triples excluding the elements of \{abd, acf, aeg, bce, bfg, cdg\}. This collection of dependent triples represents the three-point lines in the geometry of the matroid. We observe that $r(M) = 3$. Thus all sets of four or more points are dependent. This 7-element matroid is the well-known non-Fano plane, and is denoted $F_7^\text{\small \rightarrow}$.

Example 2.8. Let $M$ be a matroid defined on a ground set of $n$ elements such that every $r$-element subset of the ground set, where $0 \leq r \leq n$, is independent.

This is called the uniform matroid of rank $r$ on $n$ points, denoted $U_{r,n}$. If we let $r = 2$ and $n = 4$. The resulting matroid $U_{2,4}$, will have 4 elements and independent sets of size less than or equal to 2. This means that single points are independent and every two points are independent. So the matroid would have no loops or parallel elements. The resulting rank 2 matroid will be four distinct points on a line. Figure 2.4 shows the geometry of the uniform matroids $U_{2,4}$, $U_{2,5}$, and $U_{3,5}$.

Example 2.9. Graphs are matroids.

Let $G$ be a graph with edge set $E$, and let $\mathcal{I}$ be the collection of all subsets of $E$ that do not contain a cycle. Then $\mathcal{I}$ forms the independent sets of a matroid on the ground set $E$, called the cycle matroid $M(G)$ of the graph $G$. This is a fundamental result showing that the collection $\mathcal{I}$ of acyclic subsets of edges satisfies the independent set axioms of a matroid [GM12]. Graph theory was an early application of matroid theory. Whitney’s early work highlighted the close relationship between graphs and matroids.
He stated that although graphs are a small subclass of matroids, many theorems in the already established field of graph theory applied to matroid theory. One important example of this is the generalization of the notion of the planar dual of a planar graph, which consequently, has one of Whitney’s primary motivations for developing the idea of a matroid. Similar to the fact that matrices are matroids, every graph is a matroid. Thus results proven about matroids are results proven about graphs [GM12].

In this section we defined matroids as objects that satisfy axioms (I1),(I2), and (I3). We discussed the freedom of equivalently defining matroids in terms of circuits (minimally dependent sets). There are many other equivalent ways to define a matroid. Among the many examples of matroids, in this section we explored matroids as finite sets, matrices, finite geometries, and graphs.
2 Bases and Fundamental Circuits

In a matroid $M$, we have defined the bases of $M$ as the maximal independent sets and the circuits of $M$ as the minimal dependent sets. This section will highlight an important relationship between bases and circuits.

Example 2.10. Consider the cycle matroid $M(G)$ associated with the graph $G$ in Figure 2.5.

![Figure 2.5: Graph $G$ for Cycle Matroid $M(G)$](image)

The set $bcd$ is independent in $M(G)$ because it does not contain any cycle. The cycle $abc$ is a dependent set in $M(G)$. Let $I = bcd$ and let $x = a$. The set $bcd \cup a$ is a dependent set and there exists a unique circuit $C$ contained in $abcd$. Namely, $C = acd$.

Another efficient and equivalent way to describe a matroid $M$ is by providing the collection of bases $B$ of $M$. Note that by considering all possible subsets of the elements in the collection $B$, we can construct the collection $I$ of $M$.

Theorem 2.11. Let $B$ be a set of subsets of a set $E$. Then $B$ is the collection of bases of a matroid on $E$ if and only if $B$ has the following properties:

(B1) $B$ is nonempty.

(B2) If $B_1$ and $B_2$ are in $B$ and $x \in B_1 - B_2$, then there is an element $y$ of $B_2 - B_1$ such that $(B_1 - x) \cup y \in B$.

Bases are maximal independent sets, therefore the inheritance of the following relationship between circuits and bases is expected.

Theorem 2.12. Let $B$ be a basis of a matroid $M$. Let $x$ be an element of $E(M) - B$. Then the set $B \cup x$ contains a unique circuit.
Proof. We will prove Theorem 2.12 by contradiction. Notice the set $B \cup x$ is a dependent set, and because $B$ is independent, all circuits contained in $B \cup x$ must contain $x$. Let $C$ and $C'$ be distinct circuits such that $C \subset B \cup x$ and $C' \subset B \cup x$. By property (C3) in Theorem 2.2 there is some circuit $C'' \subseteq (C \cup C') - x$. However, the circuit does not contain $x$, implying that $C'' \subseteq B$. Since $B$ is independent, this is a contradiction. Therefore $C$ must be equal to $C'$ and thus the set $B \cup x$ contains a unique circuit.

Given a basis $B$ and an element $x \in E - B$, the unique circuit contained in $B \cup x$ that is guaranteed by Theorem 2.12 is called the fundamental circuit of $x$ with respect to basis $B$. Revisiting the matroid associated to the graph in Figure 2.5 and applying Theorem 2.12 let $B = ace$ and let $x = b$. The fundamental circuit of $b$ with respect to $ace$ is the circuit $abe$.

![Figure 2.6: The Matroid $W_3$](image)

**Example 2.13.** In the matroid $W_3$, whose geometry is pictured in Figure 2.6, we find the fundamental circuits with respect to the bases $abc$, $def$, and $aef$.

- The fundamental circuits of $d$, $e$, and $f$ with respect to the basis $abc$ are the three point lines $abd$, $ace$, and $bcf$, respectively.
- The fundamental circuits of $a$, $b$, and $c$ with respect to the basis $def$ are the spanning circuits $aef$, $bdef$, and $cdef$, respectively.
- The fundamental circuits of $c$, $d$, and $b$ with respect to the basis $aef$ are the circuits $ace$, $aef$, and $abef$, respectively.

There are many results that include the properties of fundamental circuits. In
Chapter 3 we will describe a tool to partially determining a matroid through the use of fundamental circuits.
3 Matroid Minors, Extensions, and Coextensions

In this section, we develop a notion of a structured way to obtain a smaller matroid from an existing matroid. The smaller matroid will be referred to as a minor of the larger matroid. Our motivation is that the presence of certain minors for a given matroid \( M \) provide useful information about \( M \). In particular, in this thesis, we are concerned with matroid minors because of the following two results:

**Theorem 2.14.** A matroid is binary if and only if it does not contain a minor isomorphic to \( U_{2,4} \).

**Theorem 2.15.** A matroid is ternary if and only if it does not contain a minor isomorphic to one of the following matroids: \( U_{2,5}, U_{3,5}, F_7, \) and \( F^* \).

Theorems 2.14 and 2.15 are examples of excluded minor results. One familiar such result in graph theory is Wagner’s Theorem, which states that a graph \( G \) is planar if and only if \( G \) has no minor isomorphic to \( K_5 \) or \( K_{3,3} \). Thus by examining the minors of a matroid, we can gain important information about the representability of the matroid. Representability over is preserved when performing contractions and deletions. We will discuss with more detail, additional characteristics of ternary and binary matroids, in Chapter 2 and 3. In this section, we are gathering the tools necessary for constructing matroid minors. Consider the following definitions of deletion and contraction. Let \( M \) be a matroid on the ground set \( E \) with independent sets \( I \).

1. **Deletion** For \( x \in E \) where \( x \) is not a coloop, the matroid \( M \) delete \( x \), denoted, \( M - x \) has ground set \( E - \{x\} \) and independent sets that are those members of \( I \) that do not contain \( x \). That is, \( I \) is independent in \( M - x \) if and only if \( e \notin I \) and \( I \) is independent in \( M \). Applying this to maximal independent sets and noting that coloops are contained in all such sets, we observe that since \( x \) is not a coloop, \( r(M - e) = r(M) \).

2. **Contraction** For \( x \in E \), where \( x \) is not a loop, the matroid \( M \) contract \( x \), denoted \( M/x \), has ground set \( E - \{x\} \) and independent sets that are formed by choosing all those members of \( I \) that contain \( x \), and then removing \( x \) from each such set. That is, \( I - x \) is independent in \( M/x \) if and only if \( e \in I \) and \( I \) is independent in \( M \). We
see that under contraction, the cardinality of maximal independent sets decreases. Hence, if $x$ is not a loop, then $r(M/x) = r(M) - 1$.

**Example 2.16.** Let $M$ be the matroid on $E = \{a, b, c, d, e, f\}$ with the collection $\mathcal{I}$ consisting of $\emptyset$, all singletons, all pairs of points, and all triples except for the sets $abf, bde, bdc, and dec$.

The geometries of $M$, $M - a, M/a, M - b, M/b$ are shown in Figure 2.7. Both deletion
and contraction removes an element from the matroid and thus reduces the size of the
matroid. To delete a point in $M$ we simply remove the point from the geometry. To
contract a point in $M$ we project from the point onto a line. If we construct a pencil of
lines through the point to be contracted, we can project the rest of the points of the the
matroid onto the line. This process is illustrated in Figure 2.8.

Formally a minor $N$ of a matroid $M$ is a matroid obtained from $M$ by performing
any (possibly empty) sequence of deletions an contractions of elements in $M$.

Example 2.17. Revisiting the matroid $M$ in Example 2.16 the deletion of element $f$
followed by the contraction of element $a$, denoted $(M/a)−f$ is shown in Figure 2.9. The
result is that the uniform matroid $U_{2,4}$ is a minor of $M$.

![Figure 2.9: $(M/a)−f$. Contraction of $a$ and Deletion of $f$ from the Matroid $M$.](image)

We now consider inverse operations of deletion and contraction. If there exists
an element $x$ in a matroid $M$ such that $M−x = N$, then we say that $N$ is a single-element
extension of $M$, and we write $M = N + x$. Similarly, if there exists an element $y$ in a
matroid $M$ such that $M/y = N$, then we say that $M$ is a single-element lift of $M$ by
the element $y$, and we write $M = N * y$. We note that these operations are not entirely
well-defined. Given a matroid $N$, there are usually many non-isomorphic matroids that
could result from single-element extensions or single-element lifts of $N$.

Example 2.18. Let $N$ be the matroid in Figure 2.10. The single-element lift of element
$f$ is not unique.

The lift $N * f$ allows us freedoms to place $f$ in different circuits and independent
sets. In the lift, we must either place the point $f$ on a line with points $a$ and $e$ or place
the points $a$ and $e$ in parallel. The placement of points is not completely free. When we
contract the point $f$ in the matroid $N * f$, the result of $(N * f)/f$ should be the original
matroid $N$. 
Example 2.19. Let $A$ be the following $2 \times 4$ matrix over $\mathbb{R}$:

$$
A = \begin{bmatrix}
a & b & c & d \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & -1
\end{bmatrix}
$$

Then we have a matroid $M = M[A]$ on the ground set $E = \{a, b, c, d\}$. We can extend and lift the matroid $M[A]$ by element $e = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. For the extension of $e$, we add a column vector to the matrix.

$$
A + e = \begin{bmatrix}
a & b & c & d & e \\
1 & 0 & 1 & 1 & -1 \\
0 & 1 & 1 & -1 & 0
\end{bmatrix}
$$

To lift the matrix by element $e$, we add a row and column to the matrix.

$$
A * e = \begin{bmatrix}
a & b & c & d & e \\
1 & 0 & 1 & 1 & -1 \\
0 & 1 & 1 & -1 & 0 \\
0 & 0 & 1 & 1 & 1
\end{bmatrix}
$$

There are many possible rows to add to $M$ with entries in $\mathcal{R}$. The result of $(M[A] * e)/e$ must be $M[A]$ Thus we must add $e$ such that when row reduce the matrix $A * e$ there is
exactly one nonzero entry in the column corresponding to $e$ and the row that was added in lift. If this condition is satisfied then the contraction $e$ from $(M[A] * e)$ will produce $M[A]$. There are many matrices associated with the lift of the matrix $A$ by the element $e$.

In more generality, we say matroid $M$ is an extension of matroid $N$ if $M$ can be obtained from $N$ by performing a sequence of single-element extensions of $N$. We say matroid $M$ is a lift of matroid $N$ if $M$ can be obtained from $N$ by performing a sequence of single-element lifts of $N$. Note that if $M$ is an extension or lift of $N$ then $N$ is a minor of $M$. 
Chapter 3

Representations and Partial Representations of Matroids

1 Representations

Matroids arise from various mathematical objects and we will be studying the highly motivating example of representable matroids. In particular, we will be exploring representable matroids of low rank over the unique fields $GF(2)$ and $GF(3)$; the Galois fields of two elements and three elements, respectively. Matroids that are representable over $GF(2)$ are called binary matroids and matroids that are representable over $GF(3)$ are called ternary matroids. To demonstrate our results later in this chapter, we sometimes simultaneously construct the matrix representation and the geometries of the matroids.

In this thesis we are exploring rank 3 projective geometries representable over $GF(2)$ and $GF(3)$. These geometries are considered projective planes will satisfy the following two properties:

1. Every pair of points determines a unique line, and
2. every pair of lines intersect in a unique point.

Definition 3.1. The rank $r$ projective geometry is denoted $PG(r – 1, p)$ and is representable over the field $\mathbb{F}_p$.

Theorem 3.2. Let $M$ be a simple rank $r$ matroid representable over the field $\mathbb{F}_p$. Then $M$ is a submatroid of the projective geometry $PG(r – 1, p)$. 
Proof. We assume a simple matroid $M$ is representable over some field $\mathbb{F}$. Then we can represent the elements of $M$ as vectors in $\mathbb{F}^n$. A simple matroid is void of loops or parallel points, thus each element in $M$ corresponds to a point in $PG(n, \mathbb{F})$.

Every rank 3 binary matroid is representable over $GF(2)$ and is contained in the projective geometry $PG(2, 2)$. The projective geometry $PG(2, 2)$ is also known as the Fano plane. Let $A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$ be a $3 \times 7$ matrix, and consider the column dependences over $GF(2)$, The matroid $M = M[A]$ on the ground set $E = \{a, b, c, d, e, f, g\}$ is the matrix representation of the Fano plane as shown in Figure 3.1. In the Fano plane every point is in exactly three three-point lines and every line has exactly three points. Projective planes have this symmetry between points and lines.

Every rank 3 ternary matroid is representable over $GF(3)$ and is contained in the projective geometry $PG(2, 3)$. This matroid has 13 elements and 13 lines. We see symmetry between point and lines again. In $PG(2, 3)$ every point is in exactly four four-point lines and every line has exactly four points. Let $B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 & 0 & 2 & 1 & 2 & 0 & 1 & 1 \end{bmatrix}$ be a $3 \times 13$ matrix, and consider the column dependences over $GF(3)$. The matroid $M = M[B]$ on the ground set $E = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$ is the matrix representation of $PG(2, 3)$ as shown in Figure 3.1. The entries in the matrix are elements in the field $GF(3)$. 

Figure 3.1: Left: $PG(2, 2)$. Right: $PG(2, 3)$.
Example 3.3. Let $M$ be the matroid in Figure 3.2. Construct a matrix $C$ whose column dependencies corresponds to the geometric dependencies of the matroid $P_7$.

We gather the following details about the matroid $P_7$ from the geometry

- Notice $r(M) = 3$ and $|E| = 7$, so we know $C$ must be a 3 by 7 matrix.

- To make this construction a bit more interesting, we can choose the basis $def$. We use the standard basis vectors for the columns corresponding to these elements.

- The point $a$ is not special, but together with the basis shows four points in a plane.
  We can let the column $a$ be $[1, 1, 1]^T$. So far, matrix $C$ has the following form:

$$
C = \begin{pmatrix}
d & e & f & a & b & c & g \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & \ldots \\
0 & 0 & 1 & 1
\end{pmatrix}
$$

We now find the columns of the remaining points. Since $c$ is on a line with $a$ and $e$, then $c$ is a linear combination of $[0, 1, 0]^T$ and $[1, 1, 1]^T$. Notice $c$ is also on a line with $b$ and $f$. So we can let column $c$ be $[1, 2, 1]^T$ and column $b$ be $[2, 1, 1]^T$. 

![Figure 3.2: The Matroid $P_7$](image)
\[
C = \begin{bmatrix}
\begin{array}{cccccc}
d & e & f & a & b & c & g \\
1 & 0 & 0 & 1 & 2 & 1 \\
0 & 1 & 0 & 1 & 1 & 2 \\
0 & 0 & 1 & 1 & 1 & 1
\end{array}
\end{bmatrix}
\]

For the element \( g \) we consider the lines \( ef \) and \( cdg \). We deduce that the vector \([0, 2, 1]^T\) will suffice for the column corresponding to element \( g \).

\[
C = \begin{bmatrix}
\begin{array}{cccccc}
d & e & f & a & b & c & g \\
1 & 0 & 0 & 1 & 2 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 2 & 2 \\
0 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}
\end{bmatrix}
\]

The matroid \( M = M[C] \) describes the dependencies of the matroid geometry \( P_7 \).

When a matroid is representable over some field, constructing its matrix representation is not always easily done. It is important to mention that every matroid does not arise from a matrix, nor is every matroid representable over some field. In fact, Rónyai et al. proved a result which implies that almost all matroids are not representable over some field. [Nel18]

### 2 Partial Representations

A motivating aspect of partial representations is that every matroid has a partial representation. Our study of partial representations describes the conditions under which the information about a matroid is adequately encoded from a partial representation. Recall Theorem 2.12 from Chapter 2. Let \( B \) be a basis of a matroid \( M \). Let \( x \) be an element of \( E(M) - B \). Then the set \( B \cup x \) contains a unique circuit. Called the fundamental circuit of \( x \) with respect to basis \( B \).

**Definition 3.4.** A partial representation \( P_B(M) \) with respect to a basis \( B = \{e_1, e_2, \ldots, e_r\} \) is a matrix \([a_{i,j}]\) such that \( a_{i,j} \in \{0,1\} \), row \( i \) is labeled by the basis element \( e_i \), and column \( j \) is labeled by and element \( x_j \in E(M) - B \), where \( a_{i,j} = 1 \) precisely when element \( e_i \) is in the \( x_j \)-fundamental circuit with respect to \( B \).
A partial representation of a matroid $M$ is sometimes called a $B$–fundamental–circuit incidence matrix of $M$. In the following example, we illustrate how to construct a partial representation for a matroid when we are given the geometry of the matroid.

**Example 3.5.** Construct the partial representation of the matroid shown below in Figure 3.3.

![Figure 3.3: A Rank 3 Matroid with a Loop](image)

We begin by choosing a basis $B$. Let $B = abc$, then the complement of the basis $E - B$, is the set $c\ d\ f\ g\ j$. We label the rows by elements in $B$ and the columns by elements in $E - B$.

$$P_{\{abe\}} = \begin{bmatrix} a & c & d & f & g & j \\ b & 1 & 0 & 1 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We now find the fundamental circuits with respect to $B$. Consider the element $c$. The $c$-fundamental circuit with respect to $B$ is the circuit $abc$. Thus the column labeled $c$ should have a 1 corresponding to the basis elements $a$ and $b$.

$$P_{\{abe\}} = \begin{bmatrix} a & c & d & f & g & j \\ b & 1 & 0 & 1 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Consider the remaining elements in the complement of the basis, $d, e, f, g, j$. The $d$-fundamental circuit with respect to $B$ is the circuit $abd$. The $f$-fundamental circuit with
respect to $B$ is the circuit $ef$. This is the pair of parallel points in Figure 3.3. The $g$-fundamental circuit with respect to $B$ is the circuit $abeg$. This is a spanning circuit in the matroid. Notice $j$ is a loop element thus the column corresponding to $j$ should consist of all 0’s. The resulting partial representation is as follows,

$$
P_{\{abe\}} = \begin{bmatrix}
    c & d & f & g & j \\
    a & 1 & 1 & 0 & 1 & 0 \\
    b & 1 & 1 & 0 & 1 & 0 \\
    e & 0 & 0 & 1 & 1 & 0
\end{bmatrix}.
$$

As shown in Example 3.5, partial representations provide important information about a matroid. Given a partial representation, we can determine the rank of the matroid, a basis of the matroid, some of the independent sets, as well as fundamental circuits with respect to the chosen basis. Partial representations share many properties with representations of matroids. However unlike representations, partial representations do not always tell us all of the independent and dependents sets of a matroid.

**Lemma 3.6.** Let $P = \begin{bmatrix} c & d \\ a & 1 & 1 \\ b & 1 & 1 \end{bmatrix}$ be a partial representation for a matroid, then there are two nonisomorphic matroids with this partial representation.

![Figure 3.4: Nonisomorphic Matroids with Partial Representation P](image)

**Proof.** The matroid with partial representation $P$ has a basis $B = ab$ and $r(M) = 2$. The partial representation $P$ shows that $abc$ and $abd$ are spanning circuits in the matroid. If $M$ be a ternary non-binary matroid that has the partial representation with partial representation $P = \begin{bmatrix} c & d \\ a & 1 & 1 \\ b & 1 & 1 \end{bmatrix}$, then the elements $a$, $b$, $c$, and $d$ lie distinctly on a line and $M$ be $U_{2,4}$. 
If $M'$ is a binary matroid that has the partial representation $P = \begin{pmatrix} a & c & d \\ b & 1 & 1 \\ 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ then the matroid does not contain a $U_{2,4}$ minor and the elements $e$ and $f$ must be contained in a circuit of size 2. Thus $M'$ is three points on a line with one pair of parallel points; a binary matroid. Therefore $P$ produces two nonisomorphic matroids $M$ and $M'$.

An important property of partial representations that differs from full representations is that two identical columns in a partial representation do not always correspond to parallel elements. However in both a partial representation and a full representation, two columns that are not scalar multiples of each other correspond to two distinct nonparallel points in the matroid.

One of the questions that we ask is, “What additional information about a matroid’s independent and dependent sets is inherited from a partial representation if we are given the representability of the given matroid?”. Beginning with the field $GF(2)$, binary matroids together with their partial representations, produces a seemingly trivial result that is stated in Theorem 3.7. Increasing the size of the field by one, lands us in the realm of ternary matroids that are represented over the field $GF(3)$. To reduce redundancies, we specifically explored matroids that are ternary and not binary.

**Theorem 3.7.** If $M$ is binary, then every partial representations of $M$ uniquely determines $M$ or a matroid isomorphic to $M$.

**Proof.** Let $P$ be any partial Representation of $M$. Since the entries in $P$ are over $GF(2)$, we see that $P$ is a complete representation of $M$ over $GF(2)$. Any other of partial representation of $M$ is also a complete representation by a matrix that is equivalent to $P$.

**Conjecture 3.8.** If every partial representation of $M$ uniquely determines $M$, then $M$ is binary.

Conjecture 3.8 is the converse statement of Theorem 3.7. In order to prove this theorem we would consider any matroid despite its representability or lack thereof. We would also consider matroids that are either simple or not simple. In turn, we would have to consider all classes of matroids. A counter example of this statement would involve
a matroid that is not binary whose every partial representation with respect to every basis of the said matroid uniquely determines the matroid. Although this is the last time we consider this conjecture in this thesis. The validity of this conjecture will enlighten us on the structure of matroids that are completely determined by their fundamental circuits with respect to a single basis. It will also help determine the usefulness of partial representations in a general sense.

3 From Partial Representations to Representations Over GF(3)

While exploring partial representations of simple ternary nonbinary (STNB) matroids (matroids representable over the field GF(3) and not GF(2)), we used a process of constructing full representations of ternary matroids given a partial representation. By changing the value of some of the 1’s in the partial representation to −1’s we can create a ternary representation of a matroid. Note we can also interchange the 1’s with the value 2 instead of −1, we see that 2 ≡ −1 mod(3) in the field GF(3).

**Theorem 3.9.** $M$ is representable over a field $F$ if and only if we can extend a partial representation into a representation matrix by interchanging the entries of 1’s in the partial representation by appropriate nonzero entries of $F$.

In light of Theorem 3.9, given a partial representation $P$ for an $F$-representable matroid $M$ it is possible to extend $P$ to a full representation for $M$ by strategically replacing some of the non-zero entries in $P$ with other non-zero entries in $F$. In this way, depending on the non-zero entries we change, $P$ can be extended to representations for a variety of nonisomorphic $F$-representable matroids. Moreover if we aim to extend $P$ to a full representation for a specific $F$-representable matroid $M$, there is often more than one way to do this by replacing non-zero entries of $P$ with non-zero entries from $F$. In this chapter we use this process to determine which partial representations uniquely determine a matroid $M$ with the assumption the matroid is a STNB matroid. When considering STNB matroids there are a few structures in a full representation that we must avoid in order to ensure the matroid is simple. Recall simple matroids are void of loops and parallel elements. Thus the following matrix highlights structures that will not
be allowed in partial representations representing simple ternary nonbinay matroids:

\[
\begin{array}{ccc}
    & d & e & f \\
    a & 1 & 0 & 0 \\
    \text{P} = b & 1 & 0 & 0 \\
    c & 1 & 0 & 1 \\
\end{array}
\]

The column that corresponds to the element \( f \) represents an element in parallel with the basis element \( c \). Because we are only considering simple matroids the partial representations will not have columns with a single 1. The column corresponding to \( e \) is a column of all 0’s, which is a loop in the matroid and thus not a structure in a simple matroid.

In a full representation of a simple matroid, the following structures in matrix \( A \) will not appear in its associated matrix.

\[
A = 
\begin{bmatrix}
    a & b & c & d & e & f \\
    \begin{array}{cccc}
        1 & 0 & 0 & 1 & -1 \\
        0 & 1 & 0 & -1 & 1 \\
        0 & 0 & 1 & 0 & -1 & 1 \\
    \end{array}
\end{bmatrix}
\]

- The columns corresponding to the elements \( e \) and \( f \) are scalar multiples of each other. This means that the elements \( e \) and \( f \) are linearly dependent; which are parallel elements in the matroid.
- The column corresponding to \( d \) is a column of all 0’s which is a loop in the matroid and thus not a structure in a simple matroid.

Let us construct a full representation of a rank 3 STNB matroids given one of its partial representations.

**Example 3.10.** Let \( M \) be a STNB matroid. We find corresponding representations of \( M \) over \( GF(3) \) from the partial representation

\[
\begin{array}{ccc}
    & d & e & f \\
    a & 1 & 1 & 1 \\
    \text{P} = b & 1 & 0 & 1 \\
    c & 0 & 1 & 1 \\
\end{array}
\]
Consider all of the possible ways of replacing the 1’s with \(-1\) while keeping the 0’s in the same positions. Notice the bases are now linearly independent column vectors. This partial representation yielded three distinct matrices with entries over \(GF(3)\); there are no other possible combinations of 1’s and \(-1\)’s up to replacing column vectors with parallel vectors.

\[
\begin{bmatrix}
a & b & c & d & e & f \\
1 & 0 & 0 & -1 & -1 & -1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
\end{bmatrix}
\quad
\begin{bmatrix}
a & b & c & d & e & f \\
1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 1 \\
\end{bmatrix}
\quad
\begin{bmatrix}
a & b & c & d & e & f \\
1 & 0 & 0 & -1 & -1 & -1 \\
0 & 1 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 1 \\
\end{bmatrix}
\]

As it turns out, the matroids associated with these matrices are all isomorphic. We can see that each of these representable matroids describe the dependencies of a single matroid shown in Figure 3.5, where the labeling of the points depends on which matrix we use. Since these matrices all correspond to the same STNB matroid \(M\), we say that these are \(GF(3)\)-equivalent representations of \(M\).

Consider the following lemma which describes the conditions that determine equivalent \(\mathbb{F}\)-representations.

**Theorem 3.11.** Let \(\mathbb{F}\) be a field and \(A_1\) and \(A_2\) be equivalent \(\mathbb{F}\)-representations of a matroid of positive rank. Then either

1. \(A_1\) and \(A_2\) are positively equivalent, meaning \(A_1 = A_2\); or

2. \(A_1\) and \(A_2\) are not positively equivalent and \(A_2\) can be obtained from \(A_1\) by a sequence of operations a-g (list shown below) that involves exactly one operation of type g. Moreover, this operation can be done either first or last in the sequence.

   (a) Interchange two rows.
(b) Multiply a row by a non-zero element of \( F \).

(c) Replace a row by the sum of that row and another row.

(d) Adjoin or remove a zero row.

(e) Interchange two columns (moving their labels with the column).

(f) Multiply a column by a non-zero element of \( F \).

(g) Replace each matrix entry by its entry by its image under some automorphism of \( F \).

Example 3.12. Let \( P \) represent a STNB matroid. Find all of the corresponding representations over \( GF(3) \) from the partial representation

\[
P = \begin{bmatrix}
d & e & f \\
a & 1 & 1 & 1 \\
b & 1 & 1 & 1 \\
c & 1 & 1 & 0 \\
\end{bmatrix}
\]

We begin by considering the combinations of all the possible columns of 1’s and -1’s, and with these combinations, we ensure our resulting representation is describing a STNB matroid. Notice the columns \([-1, 1, 1]^T\) and \([1, -1, -1]^T\) are scalar multiples of each other. The columns \([-1, 1, 0]^T\) and \([1, -1, 0]^T\) are scalar multiples of each other. Because we are representing simple matroids, we avoid columns that are scalar multiples of other columns. The following matrices \( B_i \) for \( 1 \leq i \leq 13 \), are the result of all possible substitutions of non-zero elements in \( P \) by non-zero entries from \( GF(3) \), up to replacing a column by a scalar multiple of that column.
We compare the dependencies of elements in each of the representable matroids $M[B_i]$. After careful analysis, we find that:
Figure 3.6: Size 6 STNB Matroids

- $M[B_1] \cong M[B_2] \cong M[B_4] \cong M[B_5] \cong M[B_8] \cong M[B_9] \cong M[B_{11}] \cong M[B_{12}]$,
- $M[B_3] \cong M[B_{10}]$, and
- $M[B_6] \cong M[B_7]$

Thus, we conclude that there are three distinct matroids associated with the partial representation $P = \begin{bmatrix} d & e & f \\ a & 1 & 1 & 1 \\ b & 1 & 1 & 1 \\ c & 1 & 1 & 0 \end{bmatrix}$. The corresponding geometries that result from the matroids are shown in Figure 3.6. The isomorphisms between the representable matroids means that the representations can be shown to be equivalent found by applying Theorem 3.11. Notice if we were considering matroids that were not simple, the number of possible nonisomorphic matroids would grow substantially. Example 3.12 was quite the exercise, and for future analysis, we will use Sage to determine these equivalent representations.
Chapter 4

Partial Representations for Simple Ternary Non-Binary Matroids

1 Rank 3 STNB Matroids as Restrictions of PG(2,3)

Recall the matroid $PG(2,3)$ shown in Figure 4.1. Let $P$ be the partial representation of PG(2,3), then

$$P = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}.$$

With respect to any basis the matroid PG(2,3) will have a partial representation similar to the partial representation $P$. Two partial representation are positively
equivalent or one can be obtained from the other by interchanging columns with their column labels or rows.

**Theorem 4.1.** Any rank 3 simple nonbinary ternary matroid with $|E| < 13$ has a partial representation that is a submatrix contained in $P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$.

**Proof.** Suppose there exists a rank 3 simple nonbinary ternary matroid with $|E| < 13$ with a partial representation that strictly contains a partial representation equivalent to $T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ or $V = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$. It suffices to show that the partial representations $T$, $U$ and $V$ cannot be properly contained in a partial representation that represents a simple nonbinary ternary matroid. Notice the partial representation that contains $T$ represents a matroid that has an element in parallel to a basis element. Thus any matroid that strictly contains $T$ will not be a simple matroid.

Now consider a matroid that has a partial representation that strictly contains $U$ or $V$. If we consider the full representation extended from $U$ and $V$ over $GF(3)$. The corresponding full representations will have at least one pair of parallel elements to ensure $GF(3)$ representability. Thus any rank 3 simple nonbinary ternary matroid with $|E| < 13$ has a partial representation that is a submatrix contained in $P$. \qed

**Lemma 4.2.** Let $P$ be a partial representation for a rank 3 simple ternary non-binary matroid, then $P$ has the following structure:

- $P$ has no columns consisting of a single one and two zeros.
- $P$ has at most four columns consisting of all ones.
- $P$ has at most six columns consisting of two ones and one zero.
- $P$ has at most two identical columns consisting of two ones and one zero.

This lemma is a direct consequence of Theorem 4.1. A complete classification of partial representations that arise from submatrices of $P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$ is given below:
- **Size 5**
  Partial representations of size 5 STNB matroids

\[
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
0 & 0 \\
\end{bmatrix}
\]

Partial representation that does not represent size 5 STNB matroids

\[
\begin{bmatrix}
1 & 1 \\
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

- **Size 6** Partial representations of size 6 STNB matroids

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}
\]

- **Size 7** Partial representations of size 7 STNB matroids

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
\end{bmatrix}
\]

- **Size 8** Partial representations of size 8 STNB matroids
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

- **Size 9** Partial representations of size 9 STNB matroids

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

- **Size 10** Partial representations of size 10 STNB matroids

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\]
• **Size 11** Partial representations of size 11 STNB matroids

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}
\]

• **Size 12** Partial representations of size 12 STNB matroids

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
\]

• **Size 13** Partial representations of size 13 STNB matroids

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

We want to determine which of these partial representations encodes all of the information about a STNB matroid. We have observed that some partial representations describe multiple distinct matroids. We are interested in the partial representations that uniquely determine a single matroid. We begin by finding all the corresponding representations from a given partial representation. We then determine if the representations are
equivalent and generate isomorphic matroids, or if the representations describe distinct matroids. As shown in Example 3.12 this process is extensive. In the next section, we use the computational software Sage to explore the ternary representations we obtain from the partial representations.

2 Results Using Sage

In the previous section, we produced the possible partial representations for rank 3 STNB matroids. In this section, we use Sage to study this class and study the relationships between the partial representations and equivalence of representations over \( GF(3) \). We use Sage to define a matroid given the matroid’s matrix representation. We then ask Sage to verify that the matrix is not binary and is simple. Once two matroids are defined, we ask Sage to determine if these matroids are isomorphic. Once we collect the information Sage provides, we interpret the results.

Example 4.3.

Using Sage [Sag21], we find the corresponding STNB matroids that have the partial representation

\[
P = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}.
\]
With the partial representation $P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, there are two possible matrix representations. In the Sage code these representations are matrices $A$ and $B$. Sage shows that the matroids, $M$ and $M_1$, associated with these matrices are isomorphic. This shows that the partial representation $P$ uniquely represents a single STNB matroid, which is shown in Line 9 of the code.

**Example 4.4.**

Using Sage [Sag21], we find the corresponding STNB matroids that have the partial representation

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$
In [1]: # Rank 3 Size 8 Simple Ternary Matroids Zero

In [2]: R = Matrix(GF(3), [[1, 0, 0, -1, 1, 1, 1, 1], [0, 1, 0, 1, -1, 1, 0, 1],
                   [0, 0, 1, 1, 1, 1, 1, 0]])

  N = Matroid(R)
  N.is_binary()

Out[2]: False

In [3]: N.is_simple()

Out[3]: True

In [4]: Matrix(R)

Out[4]:
\[
\begin{pmatrix}
1 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 2 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix}
\]

In [5]: S = Matrix(GF(3), [[1, 0, 0, -1, 1, 1, 1, 1], [0, 1, 0, 0, 1, -1, 0, 1],
                   [0, 0, 1, 1, 1, 1, 1, 0]])

  N1 = Matroid(S)
  N1.is_binary()

Out[5]: False

In [6]: N1.is_simple()

Out[6]: True

In [7]: Matrix(S)

Out[7]:
\[
\begin{pmatrix}
1 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 2 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix}
\]

In [8]: T = Matrix(GF(3), [[1, 0, 0, -1, 1, 1, 1, 1], [0, 1, 0, 0, 1, -1, 0, 1],
                   [0, 0, 1, 1, 1, 1, 1, 0]])

  N2 = Matroid(T)
  N2.is_binary()

Out[8]: False

In [9]: N2.is_simple()

Out[9]: True

In [10]: Matrix(T)

Out[10]:
\[
\begin{pmatrix}
1 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 2 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix}
\]

In [11]: U = Matrix(GF(3), [[1, 0, 0, -1, 1, 1, 1, 1], [0, 1, 0, 0, 1, -1, 0, 1],
                   [0, 0, 1, 1, 1, 1, 1, 0]])

  N3 = Matroid(U)
  N3.is_binary()

Out[11]: False

In [12]: N3.is_simple()

Out[12]: True

In [13]: Matrix(U)

Out[13]:
\[
\begin{pmatrix}
1 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 2 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix}
\]
In [14]:
V = Matrix(GF3), [[1, 0, 0, -1, 1, -1, 1], [0, 1, 0, -1, 1, -1, 1],
[0, 0, 1, 1, 1, 1, 1]]
N4 = Matroid(V)
N4_is_binary()

Out[14]: False

In [15]: N4_is_simple()

Out[15]: True

In [16]: Matrix(V)

Out[16]:
[1 0 0 2 1 0 0]
[0 1 0 1 2 0 0]
[0 0 1 1 1 0 0]

In [17]: W = Matrix(GF3), [[1, 0, 0, -1, 1, -1, 1], [0, 1, 0, -1, 1, -1, 1],
[0, 0, 1, 1, 1, 1, 1]]
NS = Matroid(W)
NS_is_binary()

Out[17]: False

In [18]: NS_is_simple()

Out[18]: True

In [19]: Matrix(W)

Out[19]:
[1 0 0 2 1 2 2]
[0 1 0 1 2 0 0]
[0 0 1 1 2 0 0]

In [20]: X = Matrix(GF3), [[1, 0, 0, -1, 1, -1, 1], [0, 1, 0, -1, 1, -1, 1],
[0, 0, 1, 1, 1, 1, 1]]
N6 = Matroid(X)
N6_is_binary()

Out[20]: False

In [21]: N6_is_simple()

Out[21]: True

In [22]: Matrix(X)

Out[22]:
[1 0 0 2 1 2 1]
[0 1 0 1 2 0 1]
[0 0 1 1 2 0 0]

In [23]: Y = Matrix(GF3), [[1, 0, 0, -1, 1, -1, 1], [0, 1, 0, -1, 1, -1, 1],
[0, 0, 1, 1, 1, 1, 1]]
NT = Matroid(Y)
NT_is_binary()

Out[23]: False

In [24]: NT_is_simple()

Out[24]: True

In [25]: Matrix(Y)

Out[25]:
[1 0 0 2 1 2 1]
[0 1 0 1 2 0 1]
[0 0 1 1 2 0 0]
In [36]:
\[ S = \text{Matrix}(\text{GF}(3), [[1, 0, 0, -1, 1, 1, -1, -1], [0, 1, 0, -1, 1, -1, 1, 0], [0, 0, 1, 1, 1, 1, 0])] \]
\[ \text{N} = \text{Matrix}(S) \]
\[ \text{is_binary}() \]
Out[36]:
False

In [37]:
\[ \text{N}.\text{is_simple}() \]
Out[37]:
True

In [38]:
\[ \text{Matrix}(S) \]
Out[38]:
\[
\begin{bmatrix}
1 & 0 & 0 & 2 & 1 & 2 & 2 & 2 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}
\]

In [39]:
\[ \text{N}.\text{is_isomorphic}(\text{M1}) \]
Out[39]:
True

In [40]:
\[ \text{N}.\text{is_isomorphic}(\text{M2}) \]
Out[40]:
False

In [41]:
\[ \text{N}.\text{is_isomorphic}(\text{M3}) \]
Out[41]:
False

In [42]:
\[ \text{N2}.\text{is_isomorphic}(\text{M3}) \]
Out[42]:
False

In [43]:
\[ \text{N}.\text{is_isomorphic}(\text{M4}) \]
Out[43]:
False

In [44]:
\[ \text{N2}.\text{is_isomorphic}(\text{M4}) \]
Out[44]:
True

In [45]:
\[ \text{N}.\text{is_isomorphic}(\text{M5}) \]
Out[45]:
False

In [46]:
\[ \text{N2}.\text{is_isomorphic}(\text{M5}) \]
Out[46]:
False

In [47]:
\[ \text{M3}.\text{is_isomorphic}(\text{M5}) \]
Out[47]:
True

In [48]:
\[ \text{N}.\text{is_isomorphic}(\text{M6}) \]
Out[48]:
True

In [49]:
\[ \text{N}.\text{is_isomorphic}(\text{M7}) \]
Out[49]:
True

In [50]:
\[ \text{N}.\text{is_isomorphic}(\text{M8}) \]
Out[50]:
True

In [51]:
\[ \text{N}.\text{show}() \]
With the partial representation $Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ there are nine possible matrix representations over $GF(3)$. In the Sage code, the representable matroids $N$ through $N_8$ are associated to the matrices label $R$ through $Z$ respectively. Using the isomorphism test, we see that

- $N \cong N_1 \cong N_6 \cong N_7 \cong N_8$
- $N_2 \cong N_4$
- $N_3 \cong N_5$

This shows that the partial representation $Q$ corresponds to three STNB matroids. Sage gave the geometries of these matroids which are shown in Lines 42, 43, and 44 of the code.
3 General Result

**Theorem 4.5.** Let $P$ be a $\{0,1\}$-matrix that can be obtained from one of the following matrices via row permutations, column permutations, or the deletion of columns. Then there is at most one simple, ternary, non-binary matroid having partial representation $P$.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
, \quad
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

**Proof.** Let $P$ be a $\{0,1\}$-matrix that can be obtained via row permutations, column permutations, or the deletion of columns. The possible matrices for $P$ are

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 1 & 1
\end{bmatrix}.
\]

Let $P = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}$. Then \[
\begin{bmatrix}
1 & 1 & 0 & -1 & 1 & 1 \\
1 & 0 & 1 & 1 & -1 & 1 \\
0 & 1 & 1 & 1 & 1 & -1
\end{bmatrix}
\]

is the only possible full matrix representation of a STNB matroid partially represented by $P$ over $GF(3)$. The matroid associated with this representation has the following geometry.

\[\text{Diagram}\]

Let $P = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}$. Then \[
\begin{bmatrix}
1 & 1 & 0 & -1 & 1 & 1 \\
1 & 0 & 1 & 1 & -1 & 1 \\
0 & 1 & 1 & 1 & 1 & -1
\end{bmatrix}
\]

and \[
\begin{bmatrix}
1 & 1 & 0 & -1 & 1 & 1 \\
1 & 0 & 1 & 1 & -1 & 1 \\
0 & 1 & 1 & 1 & 1 & -1
\end{bmatrix}
\]

are the possible full matrix representations of STNB matroids partially represented by $P$ over $GF(3)$. The matroids associated with these representations are isomorphic and have
the following geometry.

Let \( P = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix} \). Then \( \begin{bmatrix}
1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1
\end{bmatrix} \) and \( \begin{bmatrix}
1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1
\end{bmatrix} \) are the possible full matrix representations of STNB matroids partially represented by \( P \) over \( GF(3) \). The matroids associated with these representations are isomorphic and have the following geometry.

Let \( P \) be a \( \{0, 1\} \)-matrix that can be obtained \( \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix} \) via row permutations, column permutations, or the deletion of columns. The possible matrices for \( P \) are

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 1 & 0 \\
1 & 1 \\
0 & 1 \\
\end{bmatrix}
\]

Let \( P = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
\end{bmatrix} \). Then \( \begin{bmatrix}
1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1
\end{bmatrix} \) is the only possible full matrix representation of a STNB matroid partially represented by \( P \).
over $GF(3)$. The matroid associated with this representation has the following geometry.

Let $P = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 & 0 & 1 & -1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ is the only possible full matrix representation of a STNB matroid partially represented by $P$ over $GF(3)$. The matroid associated with this representation has the following geometry.

Let $P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 & 0 & 1 & -1 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$ is the only possible full matrix representation of a STNB matroid partially represented by $P$ over $GF(3)$. The matroid associated with this representation has the following geometry.

Let $P = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Then $\begin{bmatrix} 1 & 1 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$,
and \[
\begin{bmatrix}
1 & 1 & 0 & 1 & -1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 \\
\end{bmatrix}
\]
are the possible full matrix representations of STNB matroids partially represented by \( P \) over \( GF(3) \). The matroid associated with this representation has the following geometry.

Let \( P = \begin{bmatrix}
1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
\end{bmatrix} \). Then \[
\begin{bmatrix}
1 & 1 & 0 & 1 & -1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 \\
\end{bmatrix}
\]
and \[
\begin{bmatrix}
1 & 1 & 0 & 1 & -1 & -1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]
are the possible full matrix representations of STNB matroids partially represented by \( P \) over \( GF(3) \). The matroids associated with these representations are isomorphic and have the following geometry.

Let \( P = \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \). Then \[
\begin{bmatrix}
1 & 1 & 0 & 1 & -1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]
and \[
\begin{bmatrix}
1 & 1 & 0 & 1 & -1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]
are the possible full matrix representations of STNB matroids partially represented by \( P \) over \( GF(3) \). The matroids associated with these representations are isomorphic and have the
following geometry.

Let \( P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \). Then \( \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \), \( \begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \), and \( \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \) are the possible full matrix representations of STNB matroids partially represented by \( P \) over \( GF(3) \). The matroids associated with these representations are isomorphic and have the following geometry.

Let \( P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \). Then \( \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \), is the only possible full matrix representation of a STNB matroid partially represented by \( P \) over \( GF(3) \). The matroid associated with this representation has the following geometry.
Chapter 5

Conclusion

Via the motivation of partial representations, we have constructed the complete list of rank 3 simple ternary nonbinary matroids. We constructed these matroids using technology and took advantage of the ambiguities that arise from partial representations. These ambiguities gave us choices when constructing the relationships between elements of the matroid. We considered matroids of low rank and representatable over $GF(3)$. We found that 14 of the 30 possible partial representations uniquely determined a simple rank 3 ternary non-binary matroids. This is a very small class of matroids and we would like to extended the study of partial representations to other classes of matroids. Some questions to further the exploration of partial representations of matroids would be:

1. In a general sense, what are the minimum assumptions that are needed to completely determine a matroid given a single partial representation.

2. Knowing that partial representations give us the fundamental circuits of a matroid with respects to a single basis and that sometimes this is not enough information to determine all of the independent sets of a matroid. What application of matroids would this ambiguity be useful?

3. Overall, are partial representation mostly helpful when trying to determine the collection of independent sets of a matroid?

The results of this paper involved simple ternary nonbinary matroids of rank 3. These are all reasonable assumptions except the restraint of the matroid having rank 3. We would like to be able to relax our rank restrictions and state similar results in higher rank.
Bibliography


