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Matroids Determinable by Two Partial Representations

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Matroids Determinable by Two Partial Representations

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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Abstract

A matroid is a mathematical object that generalizes and connects notions of independence that arise in various branches of mathematics. Some matroids can be represented by a matrix whose entries are from some field; whereas, other matroids cannot be represented in this way. However, every matroid can be partially represented by a matrix over the field $GF(2)$. In fact, for a given matroid, many different partial representations may exist, each providing a different collection of information about the matroid with which they are associated. Such a partial representation of a matroid usually does not uniquely determine the matroid on its own. That is, if we are given a partial representation P and seek to find a unique matroid having P as one of its partial representations, we may not be able to do so. On the other hand, the more partial representations we are given, the more likely it is that this collection of partial representations uniquely determines a matroid. In this thesis, we investigate matroids that can be uniquely determined by two partial representations. Specifically, we provide a characterization of the rank-3 matroids for which there exist two distinct partial representations that combine to encode all of the matroid information.

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Chapter 1

Introduction

Matroid theory grew out of the common interest among several mathematicians to formalize an abstract notion of dependence that generalizes and connects the algebraic and combinatorial concepts of dependence. One mathematician who gained fame from introducing matroids was Hassler Whitney. Whitney was an American mathematician who wrote the first paper in the field, "On the abstract properties of linear dependence," which was published in the American Journal of Mathematics in 1935 [GM12]. During the same time, matroid theory was also introduced by Takeo Nakasawa, a Japanese mathematician [Oxl18]. In the same year as Whitney, Nakasawa published the first of three papers that also introduced the theory of matroids [Oxl18]. The work of Nakasawa has received little acknowledgement for his contribution to matroid theory. Sadly, Nakasawa died at the young age of 33 just 11 years after his first publication [Oxl18]. Whitney completed his doctoral degree in 1932, writing his dissertation "The Coloring of Graphs," whose primary focus was on graph coloring, and where some of his findings were used to solve the four-color problem [OR]. With his wide interest of mathematics and his graph theory expertise, Whitney laid down the fundamental ideas for matroids using the motivation that graphs and matroids have a close connection. Not very long after the debut of matroids, other mathematicians such as Saunders Mac Lane, B. L. van der Waerden, and Richard Rado, to name a few, also published work describing what Whitney called a "matroid". One of the most recognized mathematicians in matroid theory is W. T. Tutte, who wrote many outstanding papers in matroid theory in the 1950's. The first textbook on matroid theory was written by Dominic Welsh and published in 1976.

Since its debut, this branch of mathematics has become an active area of research, and the effects of this have immensely contributed to the growth of matroid theory. The structure of a matroid generalizes the idea of independence. This idea of independence can be found in various areas of mathematics including linear algebra, abstract algebra, combinatorics, graph theory, and finite geometry, to name a few. We will see that matroid theory "borrows" extensively from the terminology of linear algebra and graph theory. Matroid theory is a perfect example of combining several branches of mathematics and making connections between them. Moving forward, a matroid can be defined in several equivalent ways. Each definition offers a slightly different perspective.

We promise to give a formal definition of a matroid, but first we would like to explore an example. Keep in mind that although this example uses a graph to help aid our understanding of a matroid, we do not depend on graphs to study matroids. Consider the graph G in Figure 1.1, with edge set $E = \{a, b, c, d, e, f\}$. Notice that certain subsets of the set E do not contain cycles in the graph G. For example, the sets $\{a, b, d\}$ and $\{a, b, e\}$ form an acyclic subset of edges in our graph. We let the set $\mathcal{I}$ be the collection of all acyclic subsets of edges in our graph G. Since the set $\mathcal{I}$ contains elements that are also sets, for readability purposes, instead of writing elements of $\mathcal{I}$ as, for example, $\{\{a, b, d\}, \{a, b, e\}, \ldots\}$ we will relax the set notation of the acyclic edges and instead write them in the form $\{abd, abe, \ldots\}$. In this particular example, the collection of acyclic subsets of edges in the graph G are $\mathcal{I} = \{\emptyset, a, b, c, d, e, ab, ac, ad, ae, bc, bd, be, cd, ce, abd, abe, acd, ace, bcd, bce\}$. Notice that the subsets f and de are cycles in our graph G. It is usually the case that an edge joins two distinct vertices, but when we let an edge join a vertex to itself we call this a *loop*. Thus, the edge f is considered a loop in our example. Two (or sometimes more!) distinct edges can join the same two vertices. Such edges are called *parallel edges*. In our particular example, the edges d and e are parallel edges.

From the perspective of a matroid, the acyclic subsets of edges and the sets of edges corresponding to the cycles play important roles. In particular, the cycles of a graph form the building blocks for Whitney's abstract notion of dependence in the context of graphs. Cycles correspond to minimally dependent sets. Hence, any set of edges of a graph that contains a cycle forms a dependent set. Conversely, a set of edges in a graph is called independent if it is acyclic; meaning that it contains no cycles. In

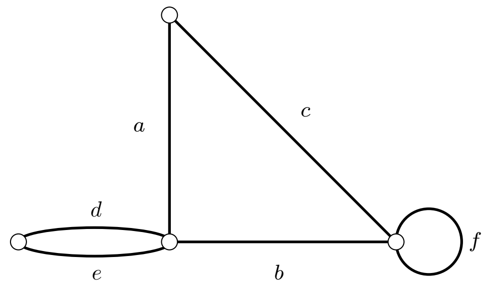


Figure 1.1: The graph G.

broad terms, matroids are defined by first identifying a finite set of objects (edges of a graph, for example) and then declaring what it means for subsets of this finite set to be independent (acyclic in our graph example). In turns out that graphs form a very important and fundamental class of matroids.

1.1 Matroid Defined

Formally, a *matroid* M is a pair $(E, \mathcal{I})$ where E is a finite set called the *ground set* and $\mathcal{I}$ is a family of subsets of E, called *independent sets.* The set $\mathcal{I}$ must satisfy the following three axioms, also known as the *independence axioms*:

(I1) $\mathcal{I} \neq \emptyset$;

(I2) If $J \in \mathcal{I}$ and $I \subseteq J$, then $I \in \mathcal{I}$;

(I3) If $I, J \in \mathcal{I}$ with $|I| < |J|$, then there is some element $x \in J - I$ such that $I \cup \{x\} \in \mathcal{I}$.

As stated previously, the acyclic subsets of edges of a graph G satisfy the independence axioms so that graphs are matroids. This is fairly easily checked. We call such matroids *cycle matroids* of their associated graph G and denote them by $M(G)$. We would like to note that not all matroids have an associated graph, and we will discuss this more later. Also, throughout this thesis, we may encounter a time when we will be discussing two distinct matroids at once. So, for matroid specification purposes, we may occasionally refer to the ground set as $E(M)$ and independent sets of matroid M as $\mathcal{I}(M)$.

Above, we have defined a matroid using independent sets. However, we can also equivalently define a matroid using minimal dependent sets. First we should define a dependent set. A subset of the ground set that in not contained in $\mathcal{I}$ is called a *dependent set*. In short, a set that is not independent is dependent. A *minimal dependent set* is a subset Q of $E(M)$ such that Q is dependent and, for all $x \in Q$, the set $Q - x$ is independent. Minimal dependent sets are also called *circuits*. We denote the collection of all circuits in a matroid by $\mathcal{C}$.

To help solidify our understanding of the previous definitions, return to the graph in Figure 1.1 and determine the collection $\mathcal{C}$ of circuits. We mentioned that the acyclic subsets of edges in G form the independent sets of the cycle matroid $M(G)$. Thus, the cycles in G are dependent sets in $M(G)$. Moreover, for any cycle C, and for any edge $e \in C$, the set $C - e$ is acyclic. Thus, the cycles in G are minimally dependent in $M(G)$, and are therefore the circuits of the cycle matroid $M(G)$. The following theorem characterizes the set of circuits in a matroid.

Theorem 1.1. *Given a finite set E, a collection $\mathcal{C}$ of subsets of E are the circuits of a matroid M on E if and only if $\mathcal{C}$ satisfies the following three properties:*

(C1) $\emptyset \notin \mathcal{C}$*;*

(C2) If C_1,$C_2 \in \mathcal{C}$ *and* $C_1 \subseteq C_2$, *then* $C_1 = C_2$*;*

(C3) If C_1 *and* C_2 *are distinct elements in* $\mathcal{C}$ *and* $x \in C_1 \cap C_2$, *then there exists* $C_3 \in \mathcal{C}$ *such that* $C_3 \subseteq ((C_1 \cup C_2) - \{x\})$.

A matroid can also be viewed equivalently as a function on the power set of a finite set $E(M)$. The *rank function* r is a mapping from the set of all subsets of the ground set $E(M)$ of a matroid M to the non-negative integers. The rank of a subset X of E is the maximum size of an independent set contained in X.

Theorem 1.2. *Let* $M = (E,\mathcal{I})$ *be a matroid and let* $K \subseteq E$*. The rank of* K*, denoted* $r(K)$*, is equal to the size of a largest independent subset of* K*:*

$$r(K) = \max_{I \subseteq K}\{|I| : I \in \mathcal{I}\}.$$

In terms of our graph in Figure 1.1 we will discuss subsets of the edge set $E(M)$ and their ranks. We know that the rank of $\emptyset$ is zero since $\emptyset \in \mathcal{I}$ and $|\emptyset| = 0$. In general, finding the ranks of all possible subsets of our ground set $E(M)$ would be extremely tedious. We would need to observe all $2^{|E(M)|}$ possibilities. In our particular example, in Figure 1.1, this would involve listing all the ranks of $2^6 = 64$ subsets. Below, we have saved you the time and provided all 64 subsets with their respective ranks.

- Rank 0 sets: $r(\emptyset) = 0$ is always true, however, a set consisting entirely of loop(s) will also have rank zero. In our particular case, the sets with rank zero are $\emptyset$, and f.

- Rank 1 sets: All singleton subsets of $E(M(G))$, aside from the loop f, have rank one. We can also add elements that create a cycle within our single element subsets. For example, d is an acyclic set with rank one. Adding the element e creates the cycle de. However, the rank of de remains one. We can also add f to the singletons without increasing the rank. With this observation, we complete the list of rank one subsets of $E(M(G))$: $\{a, b, c, d, e, de, af, bf, cf, df, ef, def\}$.

- Rank 2 sets: Ignoring the loop f for a moment, the subsets of $E(M(G))$ of size two that are not multiple edges (acyclic sets in our graph) are independent sets. Among the edges a, b, c, d, and e, choosing sets of size two aside from de will form acyclic sets. These subsets have rank two. Just as before, we can add the loop f to our sets without increasing the rank. We can also add elements that create a cycle within our two element subsets. For example, ab is acyclic and has rank two. Adding c creates a cycle abc. However, the set abc still has rank two. Also, adding f to abc does not change the rank. That is, $abcf$ is also a rank two subset of $E(M(G))$. We would like to also note that, in general, if an acyclic subset S of $E(M)$ contains an element x that is parallel to another element y, then $r(S) = r(S \cup y)$. With this observation, we complete the list of rank two subsets of $E(M(G))$: $\{ab, ac, ad, ae, bc, bd, be, cd, ce, abc, ade, bde, cde, abf, acf, adf, aef, bcf, bdf, bef, cdf, cef, abcf, adef, bdef, cdef\}$.

- Rank 3 sets: Again, ignoring the loop f, the subsets of $E(M(G))$ of size three that are acyclic sets in our graph are subsets of rank three. Among the edges a, b, c, d, and e, choosing sets of size three that do not contain abc, de, or f will form acyclic sets. As we have done before, we can add the loop f to our sets without increasing the rank. We can also add elements that create cycles within our three element subsets. For example, abd is acyclic and has rank three. Adding e creates the cycle de. However, the set $abde$ still has rank three. Also, adding f to $abde$ does not change the rank. That is, $abdef$ is a rank three subset of $E(M(G))$. Notice that we can also add c creating the cycle abc, thus $abcdef$ is also a rank three subset of E. With this observation, we complete the list of rank three subsets of $E(M(G))$: $\{abd, abe, acd, ace, bcd, bce, abdf, abef, acdf, acef, bcdf, bcef, abcd, abce, abde, acde, bcde, abcdf, abcef, abdef, acdef, bcdef, abcde, abcdef\}$.

We can also discuss the *rank of a matroid*. In general, the rank of the matroid M, denoted $r(M)$, is simply the maximum size of an independent subset of $E(M)$. In our particular example, notice that we are unable to find acyclic subsets of E in G of size four. The maximum size of an acyclic set of edges found in G is of size three. Thus, it must be that the rank of the corresponding cycle matroid $M(G)$, denoted $r(M(G))$, is equal to three.

1.2 Bases and Fundamental Circuits

Just as minimal dependent sets play an important role in the study of matroids, maximal independent sets also have great importance. Given a matroid $M(E, \mathcal{I})$, a set $B \subseteq E$ is maximally independent if $B \cup x$ is dependent for all $x \in E - B$. A maximal independent set B in matroid M is called a *basis* of M. We use $\mathcal{B}$ to denote the collection of bases of M.

Proposition 1.3. *Let M be a matroid with $B \in \mathcal{B}(M)$ and $I \in \mathcal{I}(M)$, then $|I| \leq |B|$.*

Proof. Suppose, to the contrary, that $|I| > |B|$. Then, by $(I3)$, there exists an element $x \in I - B$, such that $B \cup x \in \mathcal{I}$. This contradicts the maximality of basis B. □

By Proposition 1.3, it follows that no two bases can differ in cardinality and that $r(M) = |B|$, for any basis $B \in \mathcal{B}(M)$. Referring back to our matroid example in Figure 1.1 our independent sets are $\mathcal{I}(M(G)) = \{\emptyset, a, b, c, d, e, ab, ac, ad, ae, bc, bd, be, cd, ce, abd, abe, acd, ace, bcd, bce\}$. We can deduce the collection of bases $\mathcal{B}$ by analyzing each independent sets and determining whether or not they are maximal independent sets. Consider the set ab and the element d. We know that abd is contained in the collection of independent sets $\mathcal{I}$, thus ab is not a maximal independent set. On the other hand, the set abd is a maximal independent set since for all $x \in E - \{abd\}$, $abd \cup x$ is not contained in the collection of independent sets $\mathcal{I}$. Therefore abd must be a dependent set. Upon investigating the remaining independent sets further, we deduce that the collection of bases is $\mathcal{B} = \{abd, abe, acd, ace, bcd, bce\}$. Observe that all these maximal independent subsets of E have the same cardinality. These maximal subsets are maximal with respect to inclusion. This is not to be mistaken for maximum subsets which refer to the maximum size subsets. While it may be true that a maximum size subset is maximal, the converse may not true.

Theorem 1.4. *Let $M = (E, \mathcal{I})$ be a matroid. A collection $\mathcal{B}$ of subsets of E form the set of bases of M if and only if $\mathcal{B}$ satisfies the following three conditions:*

(B1) $\mathcal{B} \notin \emptyset$;

(B2) If B_1,$B_2 \in \mathcal{B}$ *and* $B_1 \subseteq B_2$*, then* $B_1 = B_2$;

(B3) If $B_1, B_2 \in \mathcal{B}$ *and* $x \in B_1 - B_2$*, then there is an element* $y \in B_2 - B_1$ *so that* $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$.

The following theorem is foundational to our particular focus in our study of matroids throughout this thesis.

Theorem 1.5. *Suppose that B is a basis of a matroid M and let $x \in E(M) - B$. Then there is a unique circuit C of M that is contained in $B \cup \{x\}$.*

Proof. Consider a basis B of the matroid M and let $x \in E(M) - B$. Since B is a basis, it must be that B is a maximal independent set. Thus, $B \cup \{x\}$ is not independent, so it must contain a circuit. Since B itself is not a circuit it must be that x is contained in any circuit that is contained in $B \cup \{x\}$. Suppose, toward a contradiction, that there are two distinct circuits C_1 and C_2 that are both contain in $B \cup \{x\}$. As previously mentioned, x is contained in every circuit in $B \cup \{x\}$, so it must be that $x \in C_1 \cap C_2$. Applying property (C3), we obtain a circuit $C_3 \subseteq C_1 \cup C_2 - \{x\}$. Notice that C_3 does not contain x, so it must be that $C_3 \subseteq B$, a contradiction to the assumption that B is an independent set, which cannot contain a circuit. Thus, it must be that $B \cup \{x\}$ contains a unique circuit. □

We will call the unique circuit obtained by adding a single element to a basis a *x-fundamental circuit with respect to basis B.* This will be important later in Chapter 2.

Consider the rank 3 matroid in Figure 1.1. Also, recall the collection of bases $\mathcal{B}$ that was stated earlier: $\mathcal{B} = \{abd, abe, acd, ace, bcd, bce\}$. If we consider, for example, the basis *abe* we see that the elements of E that are not in *abe* are c, d, and f. According to Theorem 1.5, it must be that *abec* contains a unique circuit, which is true. We know from our collection of circuits from the previous section that *abc* is minimally dependent. We can also quickly see that in our graph G, the set *abc* forms a cycle. Thus, our c-fundamental circuit is *abc*.

1.3 Matroid Geometry

After this section, we should be able to understand that the finite geometric representation of a matroid is not dependent on graphs. We should also be able to determine independents sets, bases, and circuits by analyzing the finite geometry of our matroid. Our main point is, that while it may be true that some matroids come from graphs it does not complete the entire spectrum of matroids. We know that there are matroids that are not derived from graphs. This is a major motivation to understanding how to derive $\mathcal{I}$, $\mathcal{B}$, and $\mathcal{C}$ if we are given a matroid that in not the cycle matroid of a graph.

As mentioned previously, the field of matroids incorporates ideas that build connections between branches of mathematics. We know that matroids are used to formalize the notion of dependence. The column dependences of a matrix also formalize the notice of dependence of matroids. We will begin by looking at two examples, which will help with our understanding of this notion. Recall that the structure of a matroid generalizes the idea of independence. Thus, the following examples will use matrices to construct the finite geometry of matroids. Consider the following matrix G:

$$G = \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

It is easily seen that the collection $\mathcal{I}$ of all linearly independent subsets of column vectors of a matrix satisfies the three independence axioms.

Step-by-step drawing procedure of a rank two matroid from a matrix

- Step 1: Draw the column vectors in the plane as shown in Figure 1.2.
- Step 2: After we draw the vectors in a two dimensional space we would like to draw the line l that intersects each subspace spanned by vector, but be careful to not draw l such that l is parallel to any vectors. If needed, shrink or extend vectors in order to observe where the span of each vector will intersect with l. See Figure 1.3.
- Step 3: To reveal our matroid we would need to discard the original vectors we drew and keep only the line l along with the intersections of l with our scaled vectors. In Figure 1.4 we have given the matroid after removing our vectors. This matroid just consists of four collinear points.

Figure 1.2: The column vectors of G.

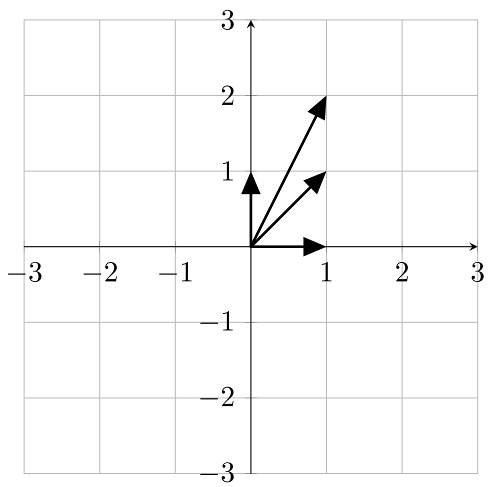


Figure 1.3: The column vectors of G with line l.

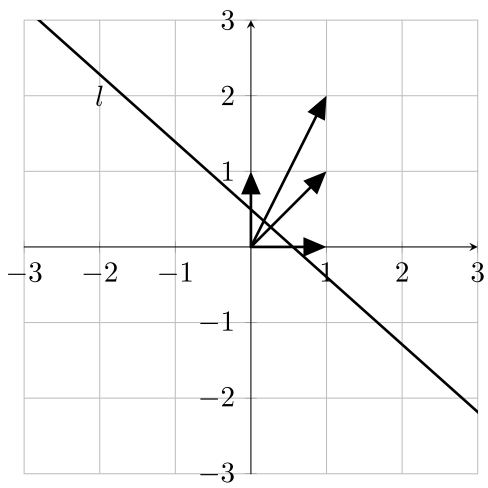


Figure 1.4: The matroid produced from the matrix G.

We would like to note that the size or length of our vector is not significant when constructing our matroid. For example, in Figure 1.2 we could have exchanged $\binom{1}{2}$ with $\binom{2}{4}$. Also, we can change the direction of vectors without altering our matroid. For example, in Figure 1.2 we could have exchanged $\binom{1}{1}$ with $\binom{-1}{-1}$. This would not change the dependency of our vectors.

Briefly, we would like to define the difference between a *simple matroid* and a *non-simple matroid*. We believe the best way to do so is to show an example. An example of a *non-simple matroid* is seen in Figure 1.1. We say a *non-simple matroid* is a matroid that contains an element that is a loop or contains a pair of elements that appear as multiple edges in the graph. Subsequently, a matroid that does not contain any loops or multiple edges is referred to as a *simple matroid*.

Now let us explore how we can recover the independent sets, bases, and circuits solely using the matroid geometry. Consider the matroid in Figure 1.4. Since our matroid in question has rank two and is simple, we use the following guideline to construct the independent sets:

Determining independent sets of a rank two simple matroid

- The empty set is contained in $\mathcal{I}$.
- Every single point is contained in $\mathcal{I}$.
- Every pair of points are independent.
- No set with more than two points is independent.

If the matroid is not simple, we must take the following into consideration:

- An element that is contained in a cloud is a single element dependent set.
- Any pair of elements of a multiple point is a two-element dependent set.

With respect to our example, we know that the empty set is independent. We can quickly see that every point is also independent. We have no pair of points drawn as multiple points, thus every pair of points are independent. Hence, the collection of independent sets are: $\{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd\}$. Recall from the previous section how we defined a basis. A basis is a maximal independent set. In our example, these sets would be $\{ab, ac, ad, bc, bd, cd\}$.

Since we do not have any multiple points in our matroid, we know that dependent sets are those subsets of $E = \{a, b, c, d\}$ of size greater than two. We also know that, since bases have cardinality two, any subsets of E of size greater than or equal to three will be dependent. Thus, the dependent sets are $\{abc, abd, acd, bcd, abcd\}$. However, to derive the collection of circuits, we should remove the sets from this list that are minimally dependent. It follows that, $abcd$ would not be a circuit since if we remove a, we are left with the set bcd, but we know that bcd is still dependent. The remaining dependent sets in this list are, in fact, minimally dependent. Thus, the collection of circuits are $\{abc, abd, acd, bcd\}$.

Collections of $\mathcal{I}$, $\mathcal{B}$, $\mathcal{C}$ of M

- The collection of independent sets $\mathcal{I}$ are: $\{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd\}$.
- The collection of bases $\mathcal{B}$ are: $\{ab, ac, ad, bc, bd, cd\}$.
- The collection of circuits $\mathcal{C}$ are: $\{abc, abd, acd, bcd\}$.

Notice that we did not need to use the vectors in matrix G to determine our independent sets, bases, and circuits. This is very useful in terms of analyzing a matroid. Given a matroid, we can deduce a lot of information about it without having to find a matrix associated with the matroid, which can be time consuming.

Let's do another example in a higher rank. This time, consider the matrix H:

$$H = \begin{array}{c} \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Note that we will be following a similar procedure as before, but instead of attempting to draw a line l, we will be drawing a plane p.

Step-by-step drawing procedure of a rank three matroid from a matrix

- Step 1: Draw the column vectors in a three dimensional space. See Figure 1.5.
- Step 2: Next we will draw a plane p that does not contain any of our vectors, however, each vector should span a space that intersects this plane. See Figure 1.6. If needed, shrink or extend vectors in order to observe where each rescaled vector will intersect with p.
- Step 3: To reveal our matroid geometry, we need to discard the original vectors we drew and keep only the plane p along with the points of intersection of p with our rescaled vectors. Removing the vectors and keeping the points in the plane p produces the matroid geometry in Figure 1.7.

Figure 1.5: The vectors of H.

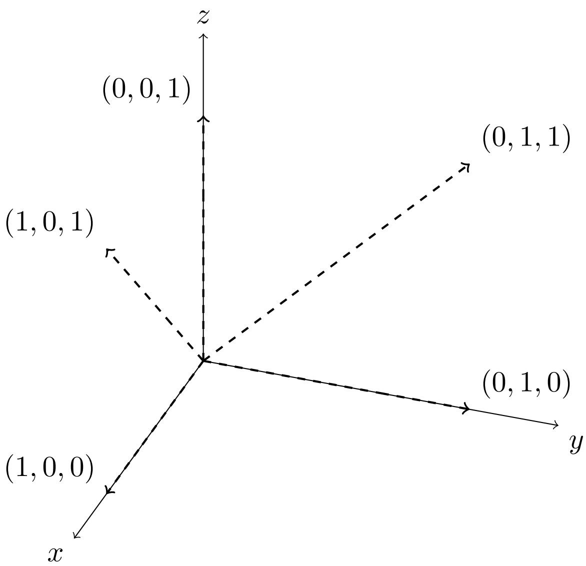


Figure 1.6: The vectors of H with plane p.

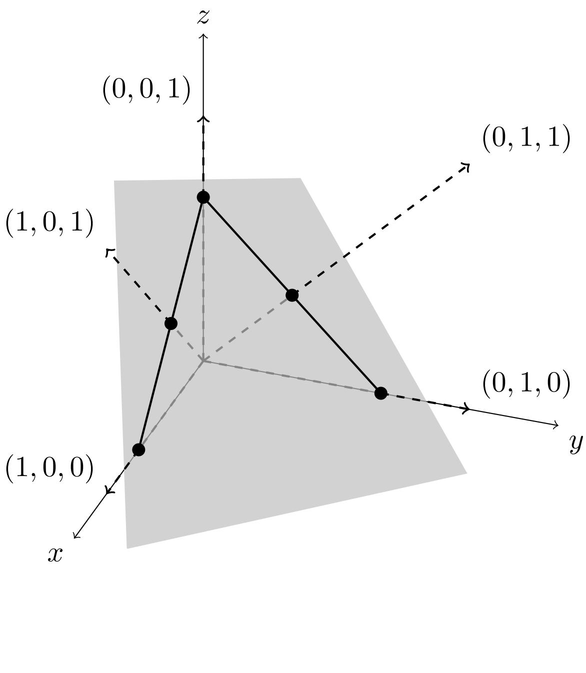
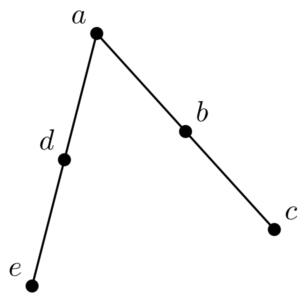


Figure 1.7: The matroid produced from the matrix H.



We note that in the Figure 1.7 we do not draw line segments connecting two points in the plane p if they are the only two points on that line. For instance, the points c and e form a two-point line, but we omit this line to reduce the clutter it could create in our picture. In this particular example, the clutter would not be as problematic, but in other cases where our ground set is larger, it could be more problematic. Though we will not be drawing these lines they are still considered to be in our finite geometry.

To continue our fun, we will look only at our rank three matroid geometry in Figure 1.7 and describe the collections of independent sets, bases, and circuits. We will once again determine the independent sets first.

Determining independent sets of a rank three simple matroid

- As before, the empty set is contained in $\mathcal{I}$.
- Every point is independent.
- Every pair of points are independent.
- A triple of points is independent if and only if the triple of points are not collinear.
- A set of more than three points is not independent.

When the matroid in consideration is not simple:

- An element that is contained in a cloud is a single element dependent set.
- Any pair of elements of a multiple point is a two-element dependent set.

Since our matroid in Figure 1.7 does not have any loops or multiple points we can say that all sets of points of size at most two are independent. Now since our matroid has collinear subsets of size three, we know that these sets are not independent. Thus, all triples of E aside from abc and ade are independent. Therefore, the collection $\mathcal{I}$ of independent sets is: $\{\emptyset, a, b, c, d, e, ab, ac, ad, ae, bc, bd, be, cd, ce, de, abd, abe, acd, ace, bcd, bce, bde, cde\}$. As before, from our collection $\mathcal{I}$, we will take the maximal independent sets and identify those as our collection of bases $\mathcal{B}$. Thus, we see that $\mathcal{B} = \{abd, abe, acd, ace, bcd, bce, bde, cde\}$.

Finding the dependent sets is a little more tricky as compared with our previous example. In this example, we already know the collinear triple of points are abc and ade. However, we also have subsets of size four that are dependent. These would be $abcd, abce, abde, acde$, and $bcde$. We now have the collection of dependent sets to be $\{abc, ade, abcd, abce, abde, acde, bcde\}$. To find the circuits, we choose the minimal dependent sets. Again, we first give an example of a non-minimal dependent set. For example, $abcd$ is not a minimal dependent set since if we remove the element d, the remaining set abc is also dependent. Thus, $abcd$ is not an element of $\mathcal{C}$. Upon investigating the remaining dependent sets, we deduce that the collection of circuits are $\{abc, ade, bcde\}$.

Collections of $\mathcal{I}$, $\mathcal{B}$, $\mathcal{C}$ of M

- The collection of independent sets $\mathcal{I}$ is $\{\emptyset, a, b, c, d, e, ab, ac, ad, ae, bc, bd, be, cd, ce, de, abd, abe, acd, ace, bcd, bce, bde, cde\}$.
- The collection of bases $\mathcal{B}$ is $\{abd, abe, acd, ace, bcd, bce, bde, cde\}$
- The collection of circuits $\mathcal{C}$ is $\{abc, ade, bcde\}$.

Returning to our graph example of a matroid in Figure 1.1, we would like to give the finite geometry of the matroid $M(G)$. However, since G is a graph we will provide the matrix A such that A carries the same linear dependence and independence according to the cycles and acyclic sets of G. The matrix A that encodes this information is called the incidence matrix of G. The columns of A are labeled by the edges of G and the rows of A are labeled by the vertices of G. An entry a_{ij} in A is 1 precisely when the corresponding edge j is incident with vertex i. Here is the matrix A (we suppress the row labels):

$$A = \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

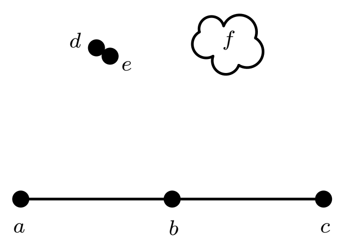
Since we determined that our matroid $M(G)$ is of rank three we use the same process as we did for the matroid in Figure 1.7 to construct the finite geometry of the matroid $M(G)$. The only additional guidelines we would need to address are the following:

Additional notes for drawing procedure of a rank three matroid from a matrix

- If our matrix contains a column vector of zeros, then the element associated with this column vector is a loop in the matroid. Thus, it must belong in a cloud.
- If our matrix contains two (or sometimes more!) column vectors that are identical to each other, these elements are multiple points in our matroid that are slightly stacked on top of one another, indicating that they occupy the same rank 1 subspace in the geometry. We refer to points of this form in a matroid geometry as *parallel points*.

After much discussion, we would like to introduce the matroid $M(G)$:

Figure 1.8: The matroid $M(G)$.



With practice, it is relatively easy to determine independent sets of a matroid by only using the geometry. We hope to have convinced the reader that we can study matroids entirely in the context of finite geometry without depending on matrices or graphs to identify the collections $\mathcal{I}$, $\mathcal{B}$, and $\mathcal{C}$. In fact, as we will see, some matroids cannot be associated with a graph or a matrix. In the next chapter we will see matroids of this form and use their finite geometries in our investigation of them.

Chapter 2

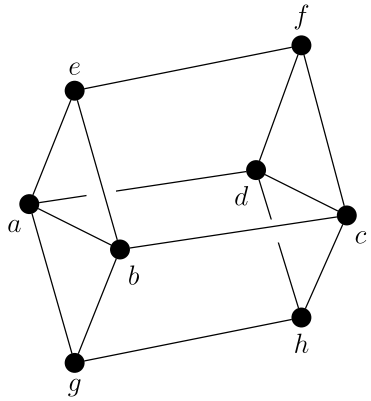
Matroid Representations

2.1 $\mathbb{F}$-Representable Matroids

In the previous chapter, we hinted at the fact that not all matroids are derived from graphs or matrices. However, all of the matroids we have investigated thus far were derived in this manner. We refer to the matroids explored in previous sections as *representable matroids.* In general, if M is isomorphic to a vector matroid associated with matrix A over some field $\mathbb{F}$, then M is *representable over* $\mathbb{F}$, where linear dependence of column vectors of A match the dependencies of M. Thus, the columns of A form the ground set of M and the linearly independent sets of A form the collection of independent sets $\mathcal{I}$ of M. We denote a matroid of this form by $M_{\mathbb{F}}[A]$ (if we want to specify the field), or when we wish to not emphasize the field, $M[A]$. To give a specific example of a representable matroid, refer to the matroid in Figure 1.7. This matroid is representable by matrix H over the field $GF(2)$. Now we question, *is every matroid representable?* We hope you have answered no to this question (since we have answered this before). If all matroids were representable, the theory of matroids would just boil down to linear algebra.

If a matroid M is not representable over any field $\mathbb{F}$, we say that M is a non-representable matroid. We love examples here, so we would like to introduce the smallest non-representable matroid, which has a ground set of size eight and is known as the Vámos matroid, denoted V_8. See Figure 2.1. Note that the elements e, f, g, and h are not coplanar.

Figure 2.1: The Vámos matroid V_8.



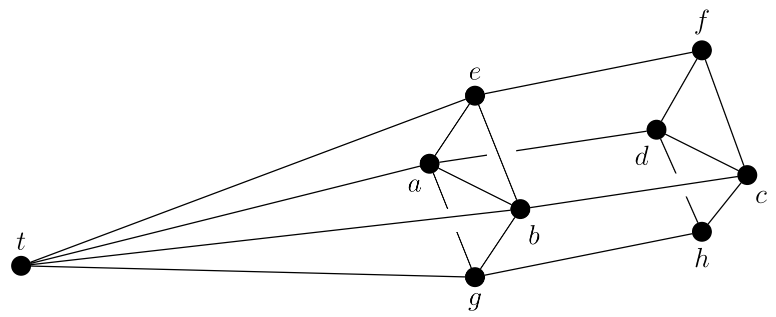
We would like to prove the following:

Theorem 2.1. *The Vámos matroid* V_8 *is not representable over any field* $\mathbb{F}$*.*

Proof. Suppose, toward a contradiction, that V_8 is representable over some field $\mathbb{F}$. This implies that V_8 is contained in a rank four projective geometry. Since V_8 is of rank four, there exists a 4 by 8 matrix A over $\mathbb{F}$ where the column vectors of A are labeled by the elements in the set $\{a, b, c, d, e, f, g, h\}$, such that $V_8 = M[A]$. Then we know that the columns of A are in the four dimensional vector space $V(4, \mathbb{F})$ over $\mathbb{F}$. Suppose that $X \subseteq E(V_8)$. Let A' be a submatrix of A such that the columns of A' are labeled by the set X. We know that the rank of X is equal to the rank of the submatrix A'. Consider $X = \{x_1, x_2, \ldots, x_n\} \subseteq E(V_8)$ and let $W(x_1, x_2, \ldots, x_n)$ denote the subspace of $V(4, \mathbb{F})$ spanned by X. Notice that by the dimension identity for vector spaces, $\dim(W(ef)) \cap \dim(W(abcd)) = \dim(W(ef)) + \dim(W(abcd)) - \dim(W(ef) + W(abcd)) = 2 + 3 - 4 = 1$. Let $< i >= \dim(W(ef)) \cap \dim(W(abcd))$ be the subspace of $V(4, \mathbb{F})$ generated by some non-zero vector i. Upon computing the dimension of the intersection of the subspaces spanned by the planes $adef$ and $abcd$, we obtain $\dim(W(adef)) \cap \dim(W(abcd)) = \dim(W(adef)) + \dim(W(abcd)) - \dim(W(ef) + W(abcd)) = 3 + 3 - 4 = 2$. We know that $W(ad)$ is a rank two subspace of $W(adef) \cap W(abcd)$. Thus, $< i >= W(ef) \cap W(abcd) \subseteq W(adef) \cap W(abcd) = W(ad)$. This implies that $< i > \subseteq W(ad)$. By a similar argument, $< i > \subseteq W(bc)$. It follows that the dimension of the intersection of the subspaces spanned by ad and bc

is $\dim(W(ad)) \cap \dim(W(bc)) = \dim(W(ad)) + \dim(W(bc)) - \dim(W(ad) + W(bc)) = 2 + 2 - 3 = 1$. This implies that $< i >= W(ad) \cap W(bc)$, since $< i >$ is a subspace of $\dim(W(ad)) \cap \dim(W(bc))$. If we are assuming that V_8 is representable, then geometrically we can add the point t, where t belongs to the subspace $< i >$. See Figure 2.2.

Figure 2.2: The matroid V_8 with point t, where t is collinear with $\{ef, ad, bc, gh\}$.



Using symmetry to our advantage, we know since $\dim(W(ad)) \cap \dim(W(bc)) = \dim(W(ef)) \cap \dim(W(abcd))$, then it must be that $\dim(W(ad)) \cap \dim(W(bc)) = \dim(W(gh)) \cap \dim(W(abcd))$. Since $< i >\subseteq W(gh)$ and $< i >\subseteq W(ef)$, this implies $< i >\subseteq W(gh) \cap W(ef)$. This along, with the fact that we assumed V_8 to be contained in a projective geometry of rank four forces geometrically that t is both on the same line as the pairs ef and gh. This implies that $efgh$ is a dependent set. So, $\dim(W(efgh)) = \dim(W(ef) + W(gh)) = \dim(W(ef)) + \dim(W(gh)) - \dim(W(ef)) \cap \dim(W(gh)) = 2 + 2 - \dim(W(ef)) \cap \dim(W(gh)) \leq 3$, since we know that $\dim(W(ef)) \cap \dim(W(gh))$ is at least equal to 1. However, this contradicts the geometry of V_8 where we know that the rank of $efgh$ is equal to four. We conclude that we are unable to find a vector space with eight vectors such that its linear independencies are the same as those of Vámos matroid V_8. Therefore, the Vámos matroid is a non-representable matroid. □

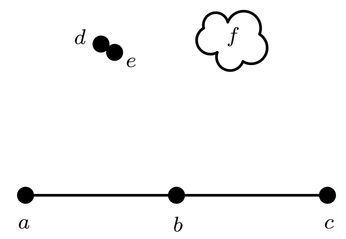
This example illustrates that not all matroids are representable. In general, we know that proving something is impossible is very difficult. However, this example demonstrates that not every matroid is derived from the linear dependencies of a collection of vectors. In fact, it has been shown that asymptotically, most matroids are not representable [Nel16]. This is a major motivation for the next section of this chapter.

2.2 Partial Representations

In the previous section, we provided an example of a non-representable matroid. Since such matroids exist (in fact, many such matroids exists), we rely greatly on studying matroids through their finite geometries. However, perhaps there is a way to encode a large amount of information from a matroid by some sort of matrix, even if the matroid is not representable. This thought motivates us to develop the technique of constructing *partial representations* of matroids. This method is used to represent some or occasionally all information about a matroid. Interestingly, even if a matroid M, such as the Vámos matroid V_8, is not representable we can still construct a *partial representation* for M. In this section, we will define *partial representations* and provide some examples that illustrate how to construct them. Also, we will illustrate how *partial representations* sometimes entirely encode a given matroid, while often times they do not. To formally define a *partial representation*, recall the following property of a basis of a matroid M. Theorem 1.5 states that for $x \notin B$, the set $B \cup x$ is dependent, and that the set $B \cup x$ contains a unique circuit. We refer to this unique circuit as the *x-fundamental circuit with respect to basis B.*

Definition 2.2. *A partial representation $P_B(M)$ with respect to a basis $B = \{b_1, b_2, \ldots, b_r\}$ is a matrix $[a_{i,j}]$ such that $a_{i,j} \in \{0, 1\}$, row i is labeled by the basis element b_i, and column j is labeled by an element $f_j \in E(M) - B$, where $a_{i,j} = 1$ precisely when element b_i is in the f_j-fundamental circuit with respect to B.*

Note that a partial representation of a matroid M tells us about a particular basis of M as well as information about some of the circuits of the matroid. To aid us in solidifying our understanding of this definition, consider the matroid M in Figure 1.8. We illustrate how to construct a partial representation of M with respect to the basis $B = abe$. We will provide the geometry of Figure 1.8:



- Step 1: The basis elements will label the rows in our partial representation and the columns will be labeled by the remaining elements of $E(M)$. So far, our partial representation has the following form:

$$\begin{array}{cc} & \begin{array}{ccc} c & d & f \end{array} \\ P_{abe}(M) = \begin{array}{c} a \\ b \\ e \end{array} & \left[\begin{array}{ccc} \ & \ & \ \\ \ & \ & \ \\ \ & \ & \ \end{array}\right] \end{array}$$

- Step 2: Now we use the finite geometry of M in Figure 1.8 to determine the x-fundamental circuits of M, which will be used to complete $P_{abe}(M)$.

  1. Consider the element $c \in E - abe$. We know from Theorem 1.5 that $abe \cup c$ contains a unique circuit which is considered a c-fundamental circuit. Specifically we observe in the finite geometry that abc is a circuit. Thus, in the partial representation we will place 1's in positions $a_{1,1}$, $a_{2,1}$ and 0 in the $a_{3,1}$ position to show that the basis elements a and b are in this circuit, while e is not.
  2. Next consider the element $d \in E - \{abe\}$. We deduce that the d-fundamental circuit with respect to abe is de. Thus, in the partial representation we will place a 1 in the position $a_{3,2}$. The remaining positions $a_{1,2}$, and $a_{2,2}$ will be set to 0.
  3. Last, we consider the element $f \in E - abe$. Since f itself is a circuit, there are no basis elements in the unique f-fundamental circuit with respect to the the basis abe. Thus, we place a zero in each of the positions $a_{1,3}$, $a_{2,3}$, and $a_{3,3}$.

- Step 3: We combine our findings from Step 2 to complete the partial representation

$P_{abe}(M)$:

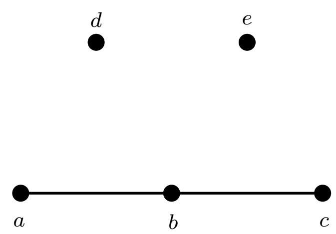
$$P_{abe}(M) = \begin{array}{c} \\ a \\ b \\ e \end{array} \begin{array}{c} \begin{array}{ccc} c & d & f \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \end{array}$$

We would like to note that our notation for partial representations is actually shorthand for (and equivalent to) the following notation in which we include the basis *abe* as columns of the identity matrix instead of labeling the rows by the basis elements:

$$P_{abe}(M) = \begin{array}{c} \begin{array}{cccccc} a & b & e & c & d & f \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \end{array}$$

In this particular example, the partial representation $P_{abe}(M)$ determines all the circuits of the matroid M. That is, if we were given solely the partial representation $P_{abe}(M)$, we could build the matroid shown in Figure 1.8 with no ambiguity. What is interesting about the matroid M in Figure 1.8 is that no matter what basis we choose to construct a partial representation we will be able to determine the collection of circuits of M. When this occurs, we say that M is *uniquely determined by every single partial representation.* Although the ability to determine the entire collection of circuits is quite nice, it may not be the case every time. For example, if we construct the partial representation $P_{bce}(M)$ for the matroid M in Figure 2.3, we observe that not all circuits are determined.

Figure 2.3: The matroid M not uniquely determined by any single partial representation.



The partial representation $P_{bce}(M)$ has the following form:

$$P_{bce}(M) = \begin{array}{c} \\ b \\ c \\ e \end{array} \begin{array}{c} \begin{array}{cc} a & d \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$

Notice that column d is composed of all 1's. This is because the unique circuit contained in *bcde* is the entire set itself. We know from the finite geometry of M that *ade* is not circuit. However, just by looking at our partial representation $P_{bce}(M)$, we would not be able to make the conclusion that *ade* is not a circuit. What if we choose another basis to construct a partial representation? Let us try with the basis *cde*.

The partial representation $P_{cde}(M)$ has the following form:

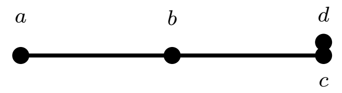
$$P_{cde}(M) = \begin{array}{c} \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

If we observed $P_{cde}(M)$, it is not apparent that *abc* is a circuit of M. It is possible to see that *acde* and *bcde* are 4-element circuits. However, it is not possible through other methods to determine whether *abc* is a circuit. Since it is not possible, then this is an example of a partial representation that does not uniquely determine M. As it turns out in this example, no matter which basis we choose for a partial representation we will not be able to determine the entire collection of circuits of M. This makes the matroid M in Figure 2.3 not uniquely determined by any single partial representation.

In the first example, we explored a case that demonstrate there exist matroids in which any basis could be used to construct a partial representation that uniquely determines a matroid M. In the second example, we explored a case that demonstrates there exist matroids in which none of the bases could be used to construct a partial representation that uniquely determines a matroid M. Interestingly, there also exist matroids that are uniquely determined by a single partial representation, but only if we carefully select a certain basis.

Consider the following matroid N:

Figure 2.4: The geometry of matroid N.



Choosing the basis ab and constructing the partial representation $P_{ab}(N)$, we obtain the following:

$$P_{ab}(N) = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} c & d \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array}.$$

The issue we see with our choice of basis is that the partial representation, $P_{ab}(N)$, does not tell us whether or not $cd \subseteq E(M)$ is dependent. In the geometry, we can see that cd is a minimally dependent set. However, this is not determined by $P_{ab}(N)$. In fact, the partial representation $P_{ab}(N)$ also represents the matroid in Figure 1.4. We conclude that $P_{ab}(N)$ does not uniquely determine N since there is two distinct matroids that have the same partial representation $P_{ab}(N)$. The underlying ambiguity stems from the fact that c and d label the same column vectors in $P_{ab}(N)$ for both matroids in Figure 2.4 and Figure 1.4. This motivates the following lemma.

Lemma 2.3. *Let $M = (E, \mathcal{I})$ be a matroid having a partial representation $P_B(M)$. Let a and b be elements of $E(M) - B$. If the column vectors in $P_B(M)$ corresponding to a and b are distinct, then a and b are not parallel elements of $E(M)$.*

Proof. Suppose that M is a matroid on the ground set $E(M)$. Let $B = \{e_1, e_2, \ldots, e_r\}$ be the basis of M used in the partial representation $P_B(M)$. Let a and b be elements of $E(M) - B$. Let S_a be the subset of $\{e_1, e_2, \ldots, e_r\}$ that forms a fundamental circuit with a. That is, $S_a \cup a \in \mathcal{C}$. Let S_b be the subset of $\{e_1, e_2, \ldots, e_r\}$ that forms a circuit with b. Similarly, $S_b \cup b \in \mathcal{C}$. We know that S_a is not equal to or contained in S_b, so it must be that there exists $e_i \in S_a - S_b$. If it is the case that a and b are parallel elements, then we know that $S_a \cup b \in \mathcal{C}$ since $S_a \cup a \in \mathcal{C}$. Similarly, $S_b \cup a \in \mathcal{C}$. We now know that $S_a \cup a, S_b \cup b, S_a \cup b, S_b \cup a \in \mathcal{C}$. Recall circuit property (C3) in Theorem 1.1, which states: If C_1 and C_2 are distinct elements in $\mathcal{C}$ and $x \in C_1 \cap C_2$, then there exists

$C_3 \subseteq (C_1 \cup C_2) - x$. Consider circuits $S_b \cup b$ and $S_a \cup b$. Notice that $b \in (S_b \cup b) \cap (S_a \cup b)$, so by circuit property (C3) in Theorem 1.1, there exists a circuit C_3 that is a subset of $(S_b \cup b) \cup (S_a \cup b) - b$. Note, $(S_b \cup b) \cup (S_a \cup b) - b = S_b \cup S_a$ However, we know that $S_a \cup S_b \subseteq B$. This is a contradiction, since a circuit cannot be contained in a basis. A similar argument can be made for $S_a \cup a$ and $S_b \cup a$ that are also in $\mathcal{C}$. □

Now using the Lemma 2.3, we prove that the matroid N in Figure 2.4 is uniquely determined. Choosing the basis element bd, the partial representation $P_{bd}(M)$ takes the following form:

$$P_{bd}(M) = \begin{matrix} & a & c \\ b & 1 & 0 \\ d & 1 & 1 \end{matrix}.$$

In this partial representation we can determine the entire collection of circuits of M. Since we know that abd is a three element circuit and cd are parallel points, it must be that abc is also a three element circuit in N. In fact, we observe that all subsets of size one or two are determined by $P_{bd}(N)$ to either be independent or dependent. Since no column in $P_{bd}(N)$ is a column of only 0's, all subsets of $E(M)$ of size one are independent. The subsets of size two, $\{ab, ac, ad, bc, bd\}$, are independent. We are given that bd is a basis, thus it is independent. The sets ab, and ad are contained in the circuit abd, thus they are independent. The set bc is independent since in our partial representation, we see that in the location $a_{1,2}$, we have a zero. Thus, bc is not contained in a circuit and must be independent. Lastly, by Lemma 2.3, we can say that ac is also an independent set. Since we know all circuits of size at most two in N and since N has rank 2, we conclude that using the partial representation $P_{bd}(N)$, we can uniquely determine the matroid N in Figure 2.4.

As we have seen, we can occasionally uniquely determine matroids with a single partial representation using any basis or we can uniquely determine matroids with a single partial representation if we carefully choose a basis. These interesting observations raise questions. Can we classify matroids uniquely determined by any single partial representation? Can we classify the matroids M for which there exists a single partial representation that uniquely determines M? For those matroids that are not uniquely determined by a single partial representation, can we find all matroids in which there exist two distinct bases B_1 and B_2 such that, together, $P_{B_1}(M)$ and $P_{B_2}(M)$ uniquely determines the matroid?

Chapter 3

2-PR Determinable Matroids

3.1 Determinability

In the previous chapter, we introduced the idea of partial representations of matroids. We described how a partial representation is constructed and how sometimes partial representations do or do not uniquely determine a matroid M. The main focus of this chapter is to classify those matroids that are uniquely determined given two distinct partial representations of the matroid. When we question whether or not a matroid is uniquely determined by a given pair of partial representations, this entails using the partial representations to deduce which subsets of the ground set are bases and circuits. Sometimes, this provides us with enough information to uniquely construct the matroid geometry. When we are able to uniquely determine the matroid given two distinct partial representations, we say the matroid is *2-PR determinable.* The process of determining if a matroid is 2-PR determinable can be rather difficult especially when we are not given any other information about a matroid aside from two partial representations. As we previously showed, partial representations may not directly provide all the information about a matroid M, thus making the matroid a little trickier to construct. However, instead of only using the matrices of partial representations we also explore determinability in a more geometric way.

We would like to give a few additional definitions that will aid us in describing this new approach. Suppose M is a matroid with the ground set E. We say a set $X \subseteq E$ is *determinable* if we can determine from two given partial representations whether the set X is independent or dependent. Otherwise, we say X is *indeterminable* by the given pair of partial representations. Additionally, let M be a matroid with bases B_r and B_b. We refer to the elements of B_r as *red elements* and the elements of B_b as *blue elements*. We refer to the elements of $E-(B_r \cup B_b)$ as *gray elements*. If matroid M is simple, every two elements in M determine a line. Every line in M contains at least two elements of M. We call a line *red-determined* if it contains at least two red elements. We call a line *blue-determined* if it contains at least two blue elements. We say a line l is *determined* if l is either red-determined or it is blue-determined. Note that it is possible for a line to be simultaneously red-determined and blue-determined. An *undetermined line* is one that is not determined.

Before we dive into examples we would like to note that we are working with simple matroids. We do so since there exists examples of matroids in rank 3 that would be 2PR-determinable if it were not for parallel elements. It turns out that by adding elements in parallel, we can increase at will the ambiguity in the information provided by partial representations. We will demonstrate an example of when this issue arises in the next section. Now that we have covered some fundamental ideas we will explore some examples to help aid the notion of determined lines. Note that we will approach the concept of partial representations from a geometric point of view. First, consider the example is shown in Figure 3.1. This is a rank 3 simple matroid that is 2-PR determinable with two distinct bases $B_r = \{r_1, r_2, r_3\}$ and $B_b = \{b_1, b_2, r_2\}$.

In Figure 3.2, we have colored the elements r_1 and r_3 of B_r red, and have also colored the red-determined lines. Note that the red-determined lines are also dashed. We also colored the basis elements in B_b blue, and we have colored the blue-determined lines. Note that the blue-determined lines are solid. We would also like to note that the single element r_2 in $B_r \cap B_b$ has been colored purple to designate it is both a red and blue element. All other elements not in $B_r \cup B_b$ are colored gray and are labeled g_1, g_2, g_3, g_4, and g_5.

Figure 3.1: A rank 3 simple matroid that is 2-PR determinable.

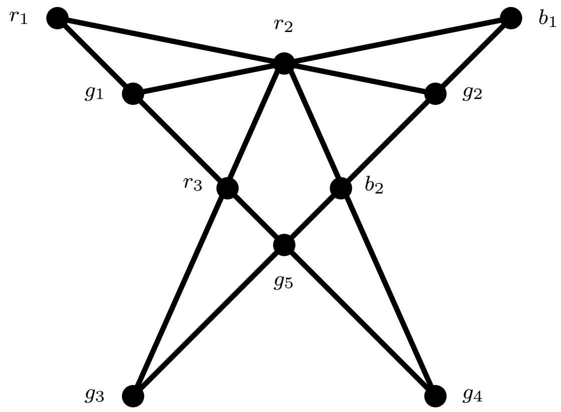
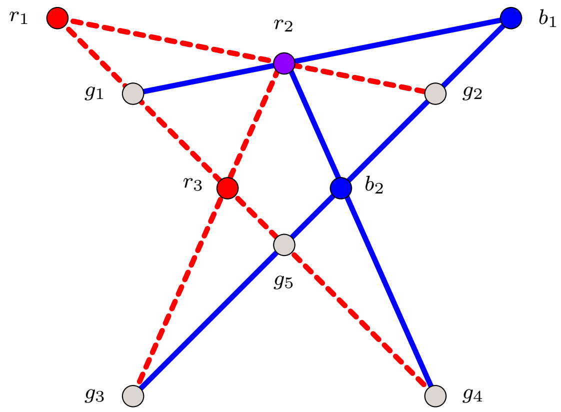


Figure 3.2: Figure 3.1 colored with determined lines.



We argue that the matroid in Figure 3.2 is 2-PR determinable, and we do so by observing the coloring of M. We analyze all subsets of our ground set and decide whether or not each subset is determined; that is, we state whether or not the subsets are independent or dependent. Since we are given that the matroid in Figure 3.2 is a simple matroid we can quickly conclude that all subsets of size one and two are independent sets.

It remains to consider subsets of $E(M)$ of size three that are not bases. Once we determine whether sets of size three are either dependent or independent we can easily decipher the dependent subsets of size four. Note that in the colored matroid, in Figure 3.2, we can collect all fundamental circuits with respect to B_r and B_b; for example, $r_1r_3g_1$ and $b_1b_2g_2$, and so on. We can also deduce those subsets of size three that are not fundamental circuits. Let Y be a subset of the ground set E such that $|Y| = 3$, then Y must have one of the following forms:

1. $Y = efg$, and without loss of generality, $e, f \in B_r$ and $g \in B_b$;
2. $Y = efg$, and without loss of generality, $e \in B_r$ and $f, g \in E - (B_r \cup B_b)$;
3. $Y = efg$, and without loss of generality, $e \in B_r$, $f \in B_b$ and $g \in E - (B_r \cup B_b)$; or
4. $Y \subseteq E - (B_r \cup B_b)$.

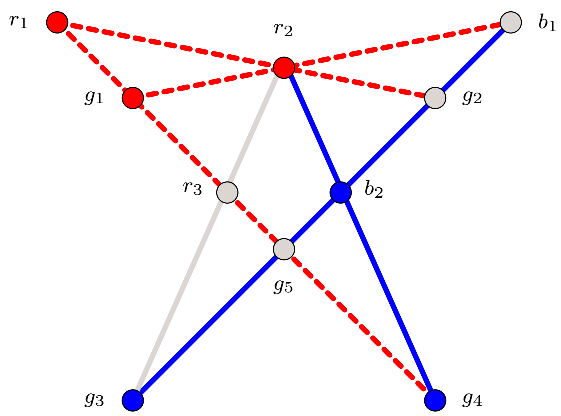
We argue that if Y has any of the four forms we have listed above, we will be able to deduce whether or not Y is an independent set. We would like to direct attention to Figure 3.2. Again, notice that each element in $E-(B_r \cup B_b)$ is both on a red-determined line and a blue-determined line. This is a vital observation in that since we know that each element in $E - (B_r \cup B_b)$ is both on a red-determined line and a blue-determined line, then in each of the four cases, the set Y contains at least two elements that are on a determined line of the same color. We utilize this knowledge of at least two elements of the set Y being on the same-colored determined line to collect the two basis elements that determine this line to argue whether or not the remaining element in Y is also on the same-colored determined line. Hence, we are able to determine these sets as either dependent or independent.

We would like to demonstrate examples of this argument. Consider the set $Y = r_1b_1g_3$. This example takes the form of the third case we have listed above. We would like to argue that the set Y is determinable. First, observe that in the set Y, there is a 2-element subset that is contained in a common blue colored line. Since b_1g_3 is contained on the blue-determined line that is determined by the basis elements b_1 and b_2, we use these basis elements to argue that the last element in Y is in fact not on this blue colored line determined by b_1 and b_2. Hence, the set Y is composed of three non-collinear points, that is, an independent set. Let us look at an example that demonstrates we can determine circuits of M. This time, let the set $Y = r_1g_1g_5$. This example takes the form of the second case we have listed above. In the set Y any 2-element subset is found on the common colored red line. We will discuss using the 2-element subset g_1g_5. Now, since g_1g_5 is contained in a common red colored line that is determined by the basis elements r_1 and r_3 we can collect the fundamental circuits $r_1r_3g_1$ and $r_1r_3g_5$. As a result, the set Y is on the same red-determined line and we know that the set g_1g_5 is not a parallel class since we are given that the matroid M is simple. We can also justify this claim by observing in Figure 3.2 the elements g_1 and g_5 are on the same red line, however, they are on distinct blue lines. Thus, in the partial representation corresponding to the basis $b_1b_2r_2$ the columns g_1 and g_5 will appear as two distinct column vectors. Hence, we can apply Lemma 2.3 to also justify that g_1g_5 is not a parallel class. Similar arguments can be applied to the remaining cases concerning the form of the set Y.

Since we now know all sets Y of size three that are independent, we know all dependent sets of size three as well. Since $r(M) = 3$, then we know every set of size four is dependent. Those sets of size four that do not properly contain a dependent set are circuits. In fact, the minimally dependent sets of size four are special circuits called *spanning circuits*. Their name references the property that if you delete an element from a spanning circuit, you obtain a basis of M. Since we have found the sets Y, with $|Y| = 3$, that are dependent, gathering the spanning circuits is a relatively easy task. Thus, we have determined all the subsets of E of size at most four and collected the circuits of M. Therefore, the matroid in Figure 3.2 is 2PR-determinable.

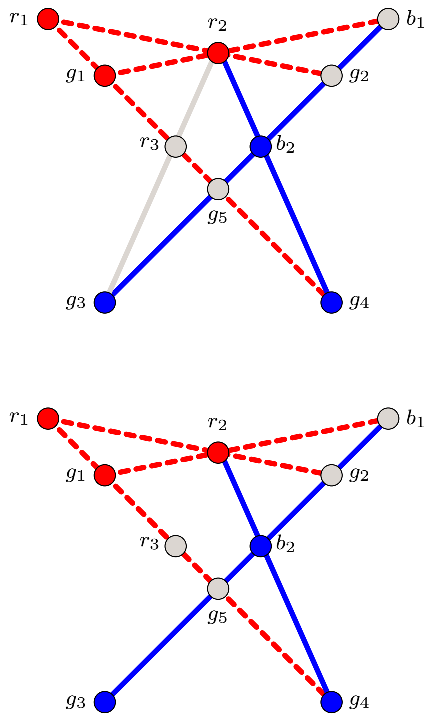
The bases are chosen very carefully. In fact, there exists a pair of bases such that after coloring their determine lines, we are unable to uniquely determine M. For example, consider the matroid in Figure 3.1 colored with a different pair of bases. See Figure 3.3.

Figure 3.3: Figure 3.1 colored with different bases.



Choosing bases $r_1r_2g_2$ and $b_2g_3g_4$ allows us to color five of the six lines (with at least three points) in our matroid in Figure 3.1. However, we are unable to determine the sixth line $r_2r_3g_3$. If we construct the two partial representations $P_{r_1r_2g_2}$ and $P_{b_2g_3g_4}$ we would not be able to determine if the set $r_2r_3g_3$ is dependent or independent. A main reason as to why we are unable to determine the sixth line is because the element r_3 lives exclusively on a red-determined line, but not on any blue-determined lines. Hence, we are unable to find a 2-element subset in $r_2r_3g_3$ that is on a common colored line. Then we are left to question, do the elements $r_2r_3g_3$ live on a line or not? Since we are unable to verify this question, we can conclude the matroid in Figure 3.3 is indeterminable with the bases $r_1r_2g_2$ and $b_2g_3g_4$. In Figure 3.4, we have drawn the two matroids that would both have same two partial representations with respect to bases $r_1r_2g_2$ and $b_2g_3g_4$. Additionally, in one matroid, the elements $r_2r_3g_3$ are collinear while in the other matroid they are not.

Figure 3.4: Two colored matroids with the same partial representations.



3.2 In Rank 3

As we saw in the examples of the previous section, we can use determined lines to prove whether or not a matroid is 2PR-determinable. The way we choose the pair of bases is very specific. In fact, we noticed that the way we choose the pair of bases would need to leave the remaining elements of M on both a red-determined line and a blue-determined line. Thus, allowing us to determine all subsets of M of size $r(M)$. It is also necessary for us to assume the matroids we are considering are simple in order to justify why subsets of $E(M)$ of size one and two are independent, in general. All of this analysis from the previous section culminates in the following theorem. It is the main result of this thesis.

Theorem 3.1. *Let M be a rank 3 simple matroid and let B_r (red) and B_b (blue) be two bases of M. Then M is 2PR-determinable by the pair (B_r, B_b) if and only if, for all $X \subseteq E$ with $|X| = 3$, there exists a 2-element subset of X that is on a common blue-determined line or a common red-determined line.*

Proof. First, suppose M is 2PR-determinable by the pair (B_r, B_b). Suppose, to the contrary, that X is a 3-element subset of E such that no 2-element subset of X is on a red line and no 2-element subset of X is on a blue line. Since we assume that X has no 2-element subset of B_r or B_b we know that no two elements contained in either basis can be found in X. Thus, consider the following cases:

1. $X \subseteq E - (B_r \cup B_b)$.
2. For $X = efg$, without loss of generality, $e \in B_b$ and $f, g \in E - (B_r \cup B_b)$.
3. For $X = efg$, without loss of generality, $e \in B_r$ and $f, g \in E - (B_r \cup B_b)$.
4. For $X = efg$, without loss of generality, $e \in B_b$, $f \in B_r$ and $g \in E - (B_r \cup B_b)$.

Consider the first case. If we assume that $X \subseteq E - (B_r \cup B_b)$, then it must be that the set X is disjoint from the two bases. That is, $|X - B_r| = 3$. Similarly, $|X - B_b| = 3$. This implies that all elements of X will appear in the columns of both partial representations. This makes it impossible to determine whether or not X is a circuit or an independent set since the cardinality of the difference of the sets X and the bases B_1 and B_2 is greater than or equal to two. Additionally, both partial representations would retain their structure whether or not X is a dependent or an independent set.

Consider the second case. Assume that $X = efg$ and, without loss of generality, $e \in B_b$ and $f, g \in E - (B_r \cup B_b)$. Similar to the first case, we argue that X is not determinable. We do so by observing the cardinality of the difference between X and each basis. We can easily see that $|X - B_r| = 3$ and $|X - B_b| = 2$, since $e \in B_b$. Again, since the cardinality of the difference of sets is greater than or equal to two, we are unable to determine whether or not X is a circuit or an independent set. This would imply that in the partial representation corresponding to B_r, the set X will appear in the columns which would make it impossible to determine whether or not X is a circuit or not. In the partial representation corresponding to B_b, a 2-element subset of X will appear in the columns of the partial representations which also makes it impossible to determine the set X. Also, similar to our first case, both partial representations would retain their structure regardless of X being a dependent or an independent set.

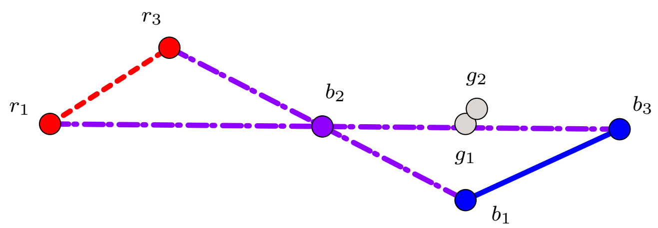
Consider the third case. The argument for this case is symmetric to the argument made for the second case. Assume that $X = efg$ and, without loss of generality, $e \in B_r$ and $f, g \in E - (B_r \cup B_b)$. Similar to the second case, we argue that X is not determinable. We do so by observing the cardinality of the difference between X and each basis. We can easily see that $|X - B_r| = 2$, since $e \in B_r$ and $|X - B_b| = 3$. Again, since the cardinality of the difference of sets is greater than or equal to two, we are unable to determine whether or not X is a circuit or an independent set. This would imply that in the partial representation corresponding to B_b, the set X will appear in the columns which would make it impossible to determine whether or not X is a circuit or not. In the partial representation corresponding to B_r, a 2-element subset of X will appear in the columns of the partial representations which also makes it impossible to determine the set X. Also, similar to previous cases, both partial representations would retain their structure regardless of X being a dependent or an independent set.

Finally, consider the fourth case. Assume that $X = efg$ and, without loss of generality, $e \in B_b$, $f \in B_r$, and $g \in E - (B_r \cup B_b)$. Similar to previous cases, we will argue that if X has this form, there is ambiguity. However, first we would like to clarify again that since we assumed X has no 2-element subset of B_r or B_b, we know that for this particular case, it is impossible for $e = f$. Now, for the last time, let us observe the cardinality of the difference between X and each basis. We see that $|X - B_r| = 2$, since $f \in B_r$ and $|X - B_b| = 2$, since $e \in B_b$. The cardinality of the difference of the sets X and the bases is greater than or equal to two. Thus, in both partial representations corresponding to B_r and B_b, a 2-element subset of X will appear in the columns of the partial representations, which would not allow us to determine the set X. Also, similar to our previous cases, both partial representations would retain their structure regardless of whether X is a dependent or an independent set. Therefore, we have shown that in each case, M is not 2PR-determinable; a contradiction to our initial assumption that M is in fact 2PR-determinable.

Next, we are given that for all $X \subseteq E$ with $|X| = 3$, there exists a 2-element subset of X that is on a common blue-determined line or a common red-determined line. We would like to argue that M is 2PR-determinable. It suffices to show that we can determine whether or not any 3-element subset of E is a circuit. Let $X \subseteq E$ with $|X| = 3$. Then there exist elements $h, i \in X$ such that h and i are on a common line

l that is either red or blue. Without loss of generality, assume l is blue. Let j be the third element of X. If $j \in l$, then X is a circuit determined by B_b. If $j \notin l$, then X must be independent since h, i and j are 3 noncollinear points. Now that we now know all sets X of size three that are independent, and we know all subsets of size at most two are independent we are able to collect the circuits of size four. Since $r(M) = 3$, then we know every subset of E of size four is dependent. Since we know the 3-element subsets of that are independent we also know the 3-element circuits. The sets of size four that do not properly contain a dependent set are the spanning circuits of M. Thus, gathering the spanning circuits is a relatively easy task. Hence, we have determined all 3-element subsets of E that are circuits. Also, we have collected the spanning circuits of M. Now we have collected all the circuits of M, and as a result we are able to construct the matroid M uniquely. Therefore, M is 2PR-determinable. □

Figure 3.5: A non-simple matroid O colored with determined lines.



As we mentioned in the previous section, it is necessary to assume that the matroids in rank 3 are simple. We use the argument of determined lines to verify whether or not the third element of a 3-element subset of $E(M)$ is collinear with the 2-element subset that is on a same-colored determined line. This helps us conclude if the 3-element subset of $E(M)$ is a dependent set, thus it is critical to assume that every 2-element subset of $E(M)$ determines a line. Otherwise, we can encounter an issue where a rank one set, that is, a parallel class, would need to span a line which is impossible to do so. If we relaxed the assumption of our matroid being simple in Theorem 3.1, then we will attain the following conjecture:

Conjecture 3.2. *Let M be a rank 3 matroid and let B_r (red) and B_b (blue) be two bases of M. Then M is 2PR-determinable by the pair (B_r, B_b) if and only if, for all $X \subseteq E$ with $|X| = 3$, there exists a 2-element subset of X that is on a common blue-determined line or a common red-determined line.*

We provide a counterexample to this conjecture, which motivates our need to include the hypothesis that the matroids we consider are simple. Refer to Figure 3.5. The matroid O with ground set $E(O) = \{r_1, r_3, b_1, b_2, b_3, g_1, g_2\}$ shown in Figure 3.5 is colored with red basis $r_1r_3b_2$ and blue basis $b_1b_2b_3$. Note that the element b_2 is in both bases, thus it has been colored purple to designate this. The dashed line is the line determined by the red elements r_1 and r_3. The solid line is the line determined by the blue elements b_1 and b_3. As we mentioned before, determined lines can be simultaneously red and blue colored. The lines $r_3b_1b_2$ and $r_1b_2b_3$ are examples of lines that are simultaneously red and blue. Thus, they are also colored purple and are drawn in a dash dot manner. The two elements g_1 and g_2 are colored gray since they are in the complement of the union of the two bases $r_1r_3b_2$ and $b_1b_2b_3$ and are a parallel class of O. Now, while all of the 3-element subsets of the matroid O have a 2-element subset that are on the same colored line we can find two problematic 3-element subsets. The two problematic 3-element subsets in O are $r_3g_1g_2$ and $b_1g_1g_2$. These two 3-element subsets are problematic since, while they do contain a 2-element subset g_1g_2 that are on a purple colored line, the set g_1g_2 does not determine a line because it is a rank one subset of $E(O)$. As a direct consequence, we would not be able to utilize the purple colored line determined by basis elements r_1, b_2 and b_3 to justify whether or not the sets $r_3g_1g_2$ and $b_1g_1g_2$ are either dependent or independent. For example, consider $r_3g_1g_2$. As we have mentioned, the set g_1g_2 is on a common colored purple line. However, in both partial representations corresponding to bases $r_1r_3b_2$ and $b_1b_2b_3$, the columns g_1 and g_2 would appear as identical column vectors. Thus, we would not be able to justify that the elements g_1 and g_2 form an independent set in order to use fundamental circuits $r_1b_2g_1$ and $r_1b_2g_2$ to reason that the elements g_1 and g_2 are two distinct points on a line and argue that r_3 is not on same the line as g_1 and g_2. If this was the case, then we could conclude that $r_3g_1g_2$ is three non-collinear points; that is, an independent set. A similar argument would follow for $b_1g_1g_2$. However, we are unable to make these conclusions about the sets $r_3g_1g_2$ and $b_1g_1g_2$ and thus, they are indeterminable. We can see in Figure 3.5 that they are dependent since they contain a parallel class, but we would not be able to utilize determined lines to prove this fact. We cannot argue that since they have identical column vectors in both partial representations, they must be a parallel class because if we were to draw the matroid O with g_1 and g_2 as two distinct points, the structures of the partial representations would not change. It would still be impossible to determine whether or not $g_1 g_2$ is a parallel class. Thus, the matroid O is a counterexample to Conjecture 3.2.

Ideally, we would like to generalize Theorem 3.1 for all possible ranks. We conjecture that in higher ranks, we would need would need to assume that subsets of $E(M)$ of cardinality less than $r(M)$ would need to be given as determined sets. The remaining sets of size $r(M)$ would need more reasoning to determine whether or not these sets are independent or dependent. An additional question is whether or not we can classify all the matroids for which all pairs of partial representations uniquely determine M.

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