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Exploring Matroid Minors

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Exploring Matroid Minors

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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Abstract

Matroids are discrete mathematical objects that generalize important concepts of independence arising in other areas of mathematics. There are many different important classes of matroids and a frequent problem in matroid theory is to determine whether or not a given matroid belongs to a certain class of matroids. For special classes of matroids that are minor-closed, this question is commonly answered by determining a complete list of matroids that are not in the class but have the property that each of their proper minors is in the class; that is, minor-minimal matroids that are not in the minor-closed class. These minor-minimal matroids that are not in the minor-closed class are called excluded minors. In this thesis, we construct interesting minor-closed classes of matroids and then characterize them by determining their complete sets of excluded minors.

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Chapter 1

Introduction

1.1 Independent sets

Matroid theory incorporates ideas from geometry, graph theory, abstract and linear algebra, and combinatorics. The first paper in matroid theory was authored by Hassler Whitney in 1935. He defined matroids on a finite collection of elements with the purpose of generalizing a notion of independence on subsets of the collection. As soon as Whitney's paper was introduced, many others started to contribute to matroid theory such as Garrett Birkoff, who studied flats of matroids and their lattices. Also, Saunders MacLane studied the relationship of matroids to projective geometry, and Richard Rado made important connections between transversals of bipartite graphs and matroids. Another important contributor to matroid theory was William Tutte, who characterized binary and regular matroids by excluded minors [GM12].

Before continuing, it is good to know that matroids can be defined in a variety of equivalent ways. The most common axiomatic systems through which matroids can be defined are those pertaining to independent sets, bases, circuits, rank, and flats (closed sets). One can prove that each of these axiomatic systems are *cryptomorphic* to one another; that is, they define the same mathematical object but through different perspectives. For what we will discuss in this thesis, defining a matroid in terms of independence will be the most useful.

Definition 1.1. *[GM12] A matroid M is a pair $(E,\mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a family of subsets of E satisfying:*

(I1) $\mathcal{I} \neq \emptyset$.

(I2) If $J \in \mathcal{I}$ and $I \subseteq J$ then $I \in \mathcal{I}$.

(I3) If $I, J \in \mathcal{I}$ and $|I| < |J|$, then there is some element $\{x\} \in J - I$ such that $I \cup \{x\} \in \mathcal{I}$.

We will refer to the conditions (I1), (I2), and (I3) as the *independence axioms*. The set E is called the *ground set* and consists of the elements of M. The family $\mathcal{I}$ is called the *independent sets* of the matroid M. Sometimes we will denoted $\mathcal{I}$ by $\mathcal{I}(M)$ and E by $E(M)$ when specifying the matroid is required. A subset X of E that is not in $\mathcal{I}$ is called a *dependent set*. Briefly, we will make a notational connection throughout this thesis for sets. The family $\mathcal{I}$ is a set whose elements are also sets notating such a collection that can be quite cumbersome due to the large amount of curly brackets required. Thus we will often leave off curly brackets and commas when writing sets. For instance, instead of $\{a, b, c\}$, we will often write abc.

We will illustrate the properties of the family $\mathcal{I}$ with several examples of matroids arising from a different context. We will look at something familiar, a matrix, and how this relates to a matroid.

Example 1.2. Let A be the following matrix whose entries are from the field $\mathbb{R}$.

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ A = & \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \end{array}$$

We are going to focus on the four column vectors $a = (2, 0)$, $b = (1, 1)$, $c = (-1, 3)$, and $d = (1, 2)$. Recall in linear algebra that a set of vectors is linearly dependent if there exists a nontrivial *linear combination* of the vectors that results in the zero vector; otherwise the set of vectors are linearly independent. To find the rank of the matrix A, we transform the matrix A to its *row echelon form* (ref) and the number of nonzero rows determines the rank of the matrix A. That is, a matrix is in row echelon form (if any) all rows that consist of only zeros are at that the bottom of the matrix and the leading entry (the first nonzero element in each row) is 1 where each leading entry of a row is in a column to the right of the leading entry above it. Then the rank of the matrix A is 2, see Figure 1.1.

$$A_{ref} = \begin{array}{c} \begin{matrix} \text{a} & \text{b} & \text{c} & \text{d} \end{matrix} \\ \begin{bmatrix} 1 & 0 & \text{-2} & \text{-1/2} \\ 0 & 1 & 3 & 2 \end{bmatrix} \end{array}$$

Figure 1.1: The matrix A in row echelon form.

Thus, any set of three or more of the column vectors is linear dependent. For instance, we see the set of column vectors abc is linear dependent because $2 \cdot a - 3 \cdot b + 1 \cdot c = 0$:

$$2(2,0) - 3(1,1) + (-1,3) = (4,0) + (-3,-3) + (-1,3) = (0,0)$$

Now we are going to determine all of the column vectors whose sets are linearly independent. Obviously, each single column vector is linearly independent since there is no zero vector as a column. In order for a pair of column vectors Notice for every pair of column vectors there is only two rows and for each row their entries are relatively prime to one another. Thus, there is no linear combination of any pair of column vectors that will result in the zero vector. Then every pair of column vectors are linearly independent. We know any set of three or more of the column vectors is linear dependent. Now we have a complete list of column vectors whose sets are linearly independent of $\mathcal{I} = \{a, b, c, d, ab, ac, ad, bc, bd, cd\}$. One can check these sets satisfy the independence axioms. We call the matroid whose independent sets are $\mathcal{I}$ *representable matroid* since these independent sets arise from the linearly independent sets of column vectors of a matrix over some field. We denote a matroid M that can be represented by a matrix A in this way by $M[A]$. We will later that not all matroids are representable. One of the most general and useful ways to depict a matroid is through geometry. For representable matroids, we can see how this geometry is related to visualizing and plotting the vectors of the ground set of A. However, we will also see later that this geometric perspective is a way to visualize matroids even when they are not representable.

To see the geometry corresponding to the matroid derived from the matrix A, we first graph the four column vectors in $\mathbb{R}^2$ as in Figure 1.2.

Next, we draw a line in a $free\ position$. Free position here means the line is not parallel to any of the four vectors (refer to Figure 1.3 for the drawing). Then the intersection of the line and the (possibly extended) column vectors produces points, which

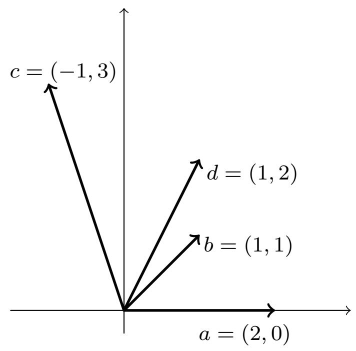


Figure 1.2: Plotting the four vectors

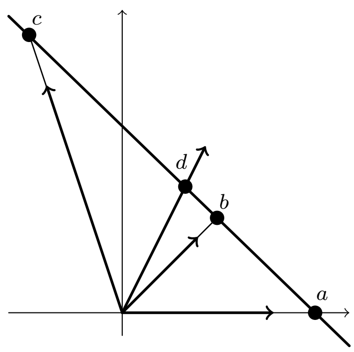


Figure 1.3: Four vectors intersecting the free position line

we label by the corresponding column vectors (elements). The picture of the resulting geometry of the matroid is shown in Figure 1.2. Note the length and direction of a vector does not matter, and replacing any vector by a scalar multiple of the vector would result in the same geometry.

For this example, the matroid geometry just consists of four collinear points. Furthermore, it does not matter how you arrange the points on their common line. The geometry will still correspond to the same matrix (see Figure 1.4). We will call this matroid M. We know the set of points abc in the geometry is dependent since their corresponding vectors are linearly dependent. If we were to only consider the geometry, and disregard the original collection of vectors from which this matroid was constructed, how can we tell which collections of points are independent and which are dependent?

Figure 1.4: A relabeling of the points in the geometry in Figure 1.2

Based on our derivation of the geometry from the vectors, parallel vectors will produce multi-points in the geometry, vectors that all live in a common plane will result in collinear points in the geometry, and vectors that all live in 3-space will result in coplanar points in the corresponding geometry. From this, it follows that two or more points all occurring at the same point (multi-points) in a geometry form a dependent set, three or more points all collinear form a dependent set, and four or more points all coplanar form a dependent set in a geometry. This makes sense! One point spans a point, two points span a line, and three points span a plane. Returning to our example, the implications of all this are that all single points are independent. Also any pair of points are independent. Remember it only takes two points to span a line, and in our example, there is a line containing four points. It follows that the sets abc, abd, acd, bcd, and $abcd$ are dependent sets in the matroid. Referring back to matrix A, we see that the corresponding sets of column vectors are linearly dependent.

In Example 1.2, we showed how the matrix A was associated with a matroid. Generally speaking, every matrix corresponds with a matroid in this way. This is stated in the following theorem:

Theorem 1.3. *[GM12] Let E be the columns of a matrix A with entries from a field $\mathbb{F}$. Let $\mathcal{I}$ be the collection of all subsets of E that are linearly independent. Then $M = (E, \mathcal{I})$ is a matroid. That is, $\mathcal{I}$ satisfies the independence axioms (I1), (I2), and (I3).*

We omit the proof of this theorem. We mention that if all matroids arise in this way (if all matroids are representable), then the study of matroids is little more than linear algebra. Fortunately, there are non-representable matroids which cannot be viewed in terms of matrices over a field. This implies that the independence that is truly more general than linear independence.

Example 1.4. Suppose $E = \{a, b, c, d\}$ and $\mathcal{I} = \{\emptyset, a, b, c, d, ab, ac, ad, bc, cd\}$. We wish to determine if the pair $(E, \mathcal{I})$ is a matroid, independent of knowing whether or not the ground set can be represented by vectors over some field. We first check that $\mathcal{I}$

satisfies (I1), (I2), and (I3). Clearly, $\mathcal{I}$ satisfies (I1) and (I2) since $\mathcal{I} \neq \emptyset$ and every subset of an element of $\mathcal{I}$ is in $\mathcal{I}$. When checking (I3), it helps to look at the larger independent set and choose the smaller independent set to be a set that is not a subset of the larger set. For example, if we take ab to be the larger independent set, then avoid choosing a, b, or $\emptyset$ as the smaller independent set (we are always going to avoid choosing $\emptyset$ since $\emptyset$ is a proper subset of any set). We will choose either c or d as the smaller independent set. Focusing on c first, by (I3), we need either ac or bc to be in $\mathcal{I}$, but actually, both are in $\mathcal{I}$. Now consider the element d. We see that ad is in $\mathcal{I}$ but bd is not. Thus the sets d and ab in $\mathcal{I}$ still satisfy (I3). Continuing this way, one can prove that $\mathcal{I}$ satisfies (I3). So the pair $(E, \mathcal{I})$ is a matroid.

Obviously this is not an efficient way to check if $\mathcal{I}$ satisfies the independence axioms. Especially when $\mathcal{I}$ contains over 20 subsets of E. One can see how tedious this can get.

Example 1.5. Let us look at an example where $\mathcal{I}$ does not satisfy some independence axioms. We will look at a small example. Suppose $E = \{a, b, c\}$ and $\mathcal{I} = \{\emptyset, a, b, ac\}$. Here $\mathcal{I}$ satisfies (I1) but not (I2) or (I3). For example, the subset ac does not satisfy (I2) since c is not an element of $\mathcal{I}$ and (I3) is not satisfies since b and ac both being independent require either ab or bc to be independent. However, neither ab nor bc are in $\mathcal{I}$. Hence the pair $(E, \mathcal{I})$ is not a matroid.

When we are given a pair $(E, \mathcal{I})$, where $\mathcal{I}$ satisfies (I1), (I2), and (I3), the pair $(E, \mathcal{I})$ is a matroid and we treat each element of E as a point in the geometric representation of $(E, \mathcal{I})$. As we continue through this chapter, we will casually reveal how the geometry is determined. First, we need to introduce more terminology in order to explain the geometry of a matroid.

1.2 Properties of a matroid in terms of independence

In this section, we will introduce some of the fundamental terminology in matroid theory. The goal is to become familiar with these new terms and to clearly understand their meaning through examples.

A maximal independent subset of the ground set E of a matroid M is called a *basis* of M. Maximal here means that the set is not properly contained in any other

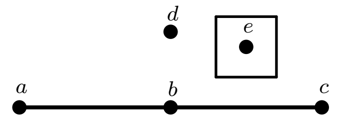


Figure 1.5: Matroid for Example 1.6

independent set. The collection of all the bases of M is denoted by $\mathcal{B}$ or $\mathcal{B}(M)$. In Example 1.4, we were given $E = \{a, b, c, d\}$ and $\mathcal{I} = \{\emptyset, a, b, c, d, ab, ac, ad, bc, cd\}$. We determined that the pair $(E, \mathcal{I})$ is a matroid M. The bases of M are the sets ab, ac, ad, bc, and cd since they are the maximal independent sets in $\mathcal{I}$. A *coloop* is a single element x of the ground set E that is in every basis. A *loop* is a single element x of the ground set E that is in no basis. A subset S of E is called *spanning* if S contains a basis of M. We will illustrate these terms in the next example.

Example 1.6. In this example, we will determine the coloops and the loops of a matroid that is defined in terms of independent sets. Note that not all matroids contain coloops or loops. Let M be a matroid on the ground set $E = \{a, b, c, d, e\}$ and let $\mathcal{I} = \{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd, abd, acd, bcd\}$. It is easily checked that $\mathcal{I}$ satisfies the independence axioms so that $M = (E, \mathcal{I})$ is a matroid. Is there a coloop in M? Observe that this matroid contains a coloop. That is, there is a single element of E that is in every basis. The collection of all the bases of M is $\mathcal{B} = \{abd, acd, bcd\}$. Notice d is in every basis. Therefore, d is a coloop. This matroid also contains a loop. Indeed, e is in no basis, which means that e is a loop. The geometry of M is shown in Figure 1.5, which illustrates how coloops and loops are depicted geometrically. Every element in the box in Figure 1.5 is a loop, and the element d is the only element that is not collinear with the three points abc.

Notice we do not draw line segments connecting two points if they are the only two points on that line. We do not draw two-point lines because every two distinct points determine a line and this will cause more clutter in the picture.

Example 1.7. Let M be the matroid on the ground set $E = \{a, b, c, d\}$ and let $\mathcal{I} = \{\emptyset, a, b, c, d, ab, ac, ad, bc, cd\}$ (see Figure 1.6). Then the dependent sets of M are $\{bd, abc, abd, acd, bcd, abcd\}$ since none of these sets are in $\mathcal{I}$.

The set of *minimal dependent sets*, that is, dependent sets all of whose *proper* subsets are independent sets, of M are $\{bd, abc, acd\}$.

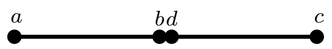


Figure 1.6: Matroid for Example 1.7

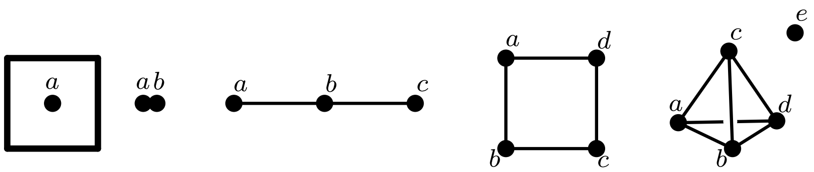


Figure 1.7: Starting on the left is a 1-circuit (loop), 2-circuit (parallel points), 3-circuit (3 collinear points), 4-circuit (4 coplanar points), and 5-circuit (5 points in a 3-space).

A minimal dependent set in a matroid M is called a *circuit* of M. The collection of all circuits of M is denoted by $\mathcal{C}$ or $\mathcal{C}(M)$. A set $C \subseteq E(M)$ is a *spanning circuit* if C is a circuit and is a spanning set. It is necessary to say that circuits do not contain other circuits. A circuit of M having k elements will be called an *k-circuit*. We can characterize loops and coloops in terms of circuits. If x is a loop, then x is a 1-circuit. If x is a coloop, then x is not contained in any circuit. In Figure 1.7, some different sizes of circuits are depicted geometrically. These are the only geometric representation of circuits we can illustrate. A circuit itself is a matroid, the subsets of a circuit satisfies the independence axioms. We will see why this is true in some examples in the next section.

1.3 Other properties of matroid

Given a subset A of E in an arbitrary matroid M, the cardinality of the largest independent subset of A is called the *rank* of A and is denoted $r(A)$. It follows from (I2) and (I3) that if B_1 and B_2 are bases of M, then $|B_1| = |B_2|$. Thus, the *rank of the matroid* M, denoted by $r(M)$, is the size of any basis of M. When specifying the rank of some set A in a particular matroid M is important, we use the notation $r_M(A)$. Given a subset C of E in an arbitrary matroid where C is a k-circuit, then $r(C) = k - 1$ since the largest independent set contained in C has one less element than C.

Example 1.8. This example focuses on $uniform$ matroids. A rank r matroid on a ground set E, where $|E| = n$, and $\mathcal{I} = \{I \subset E : |I| \leq r\}$ is called a uniform matroid, which is denoted $U_{r,n}$. It is easy to check that $\mathcal{I}$ satisfies the independence axioms since

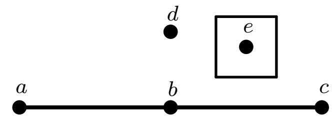


Figure 1.8: The matroid in Example 1.6

any subset of E with r or fewer elements is independent. Notice that the maximal independent sets in $U_{r,n}$ are the subsets of E of cardinality r. Thus, it makes sense that $r(U_{r,n}) = r$. Also, if $U_{r,n}$ contains circuits, then all such circuits must have exactly $r+1$ elements since we cannot have a dependent set with r or fewer elements. Recall from the previous section when suggested that there exist matroids consisting entirely of a circuit. Here, we see that the entire ground set of an uniform matroid of the form $U_{n-1,n}$ is a n-circuit. In fact, in Figure 1.7, we have provided examples of the geometries of uniform matroids of the form $U_{n-1,n}$ for $n = 1, 2, 3, 4, 5$.

If, in a uniform matroid, $r = n$, then we have the matroid $U_{n,n}$, which is called the *Boolean algebra*. In this matroid, every subset of the ground set is independent. In fact, the entire ground set forms the unique basis of $U_{n,n}$.

A matroid of rank r is called a *paving matroid* if all of its circuits are k-circuits, where $k \geq r$. Examples of paving matroids are all uniform matroids. Even $U_{n,n}$ is a paving matroid, since this matroid does not contain any circuits. For these next examples, we will be referring to the matroid in Example 1.6.

Example 1.9. The rank of the matroid in Figure 1.5 is three since the bases have cardinality three. Given any subset A of $E - \{d, e\}$, we notice that $r(A \cup d) = r(A) + 1$ and $r(A \cup e) = r(A)$.These are actually defining properties of coloops and loops. In this matroid, d is a coloop and e is a loop. In general, if x is a coloop in a matroid on E, then for all $A \subseteq E - \{x\}$, $r(A \cup x) = r(A) + 1$. Also, if x is a loop in a matroid on E, then $r(A \cup x) = r(A)$. Now we can organize the subsets of $E(M)$ by their rank. The sets with rank 0 are $\emptyset$ and e (e is a loop and all loops have rank 0). Sets with rank 1 are all 1-element except for the set $\{e\}$, as well as all of the 2-element subsets of $E(M)$ that contain the element e. The sets with rank 2 are found by choosing any of the 2-elements sunsets of $\{a, b, c, d\}$, together with all the 3-element subsets of $\{a, b, c, d, e\}$ that contain the element e. There are other rank 2 sets that we have missed. Namely, *abc* has rank 2 since *abc* is a 3-circuit. Also, *abcde* has rank 2 since e is loop. The largest independent set we can find in this matroid has three elements. Thus, this matroid has rank 3. The 3-element sets of rank 3 are the bases of M. Other rank 3 sets are all spanning sets of M since these sets contains bases of M.

A *flat* in a matroid M is a subset of E that is $rank - maximal$. That is, if F is a flat, then for all $e \in E - F$, $r(F \cup e) > r(F)$. The closure of A is denoted $\overline{A}$ is the unique smallest flat containing A. A hyperplane H is a special type of flat such that $r(H) = r(M) - 1$. The collection of hyperplanes of M is denoted $\mathcal{H}$. If an arbitrary rank r matroid contains an r-circuit C having the property that $r(C \cup e) > r(C)$, for all $e \in E - C$, then C is also a flat. In fact, C is hyperplane since $r(C) = r - 1$. We call such a circuit a *circuit-hyperplane*.

Example 1.10. We will determine all the flats for the matroid in Figure 1.8. The only rank 0 flat is e. All the rank 1 flats are *ae*, *be*, *ce*, and *de*. All the rank 2 flats (hyperplanes) are *abce*, *ade*, *bde*, and *cde*. The only rank 3 flats is the entire ground set E. It is important to notice, is that loops are in every flat, $\emptyset$ is a flat when M has no loops, and E is always the unique and largest flat possible in any matroid.

Chapter 2

Matroid constructions

2.1 Deletion and contraction

In this section, we introduce two important operations on matroids. These operations will give us ways to construct new matroids from existing matroids.

Definition 2.1. *[GM12] Let M be a matroid on the ground set E with independent sets $\mathcal{I}$.*

(i) ***Deletion*** *For $e \in E$ (e not an coloop), the matroid $M\backslash e$ has ground set $E - \{e\}$ and independent sets that are those members of $\mathcal{I}$ that do not contain e.*

(ii) ***Contraction*** *For $e \in E$ (e not an loop), the matroid M/e has ground set $E - \{e\}$ and independent sets that are formed by choosing all those members of $\mathcal{I}$ that contain e, and then removing e from each such set.*

The following proposition states that by performing either of these operations on a matroid, we obtain another matroid.

Proposition 2.2. *[GM12] Let M be a matroid on the ground set E with independent set $\mathcal{I}$. If e is not a coloop, then $M\backslash e$ is a matroid. If e is not a loop, then M/e is a matroid.*

We omit the proof of Proposition 2.2.

Example 2.3. Suppose we are given a matroid M on the ground set $E = \{a, b, c, d, e\}$ with $\mathcal{I} = \{\emptyset, a, b, c, d, e, ab, ac, ad, ae, bc, bd, be, cd, ce, de, abd, abe, acd, ace, bcd, bce, bde, cde\}$ (see Figure 2.1 for matroid M).

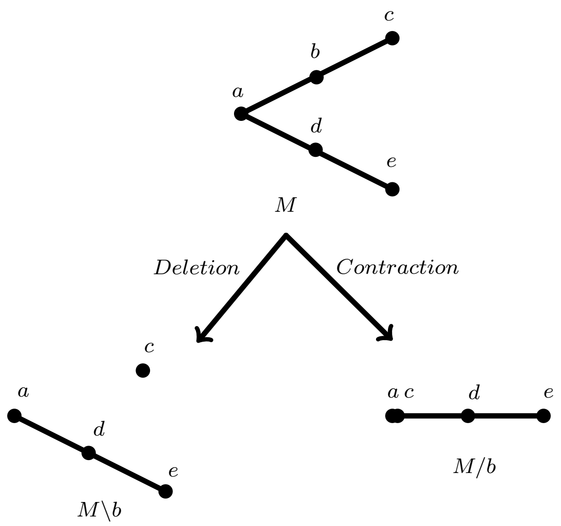


Figure 2.1: Picture of $M\backslash b$ and M/b from M

Let us find all of the independent sets in the matroid $M\backslash b$. By the definition of deletion, all the independent sets of $M\backslash b$ are the independent sets in $\mathcal{I}(M)$ that do not contain the element b. Thus the independent sets of $M\backslash b$ are $\{\emptyset$, a, c, d, e, ac, ad, ae, cd, ce, de, acd, ace, $cde\}$.

Now to find the independent sets of M/b. By the definition of contraction, all the independent sets of M/b are the independent sets in $\mathcal{I}(M)$ that contain the element b with b then removed from each such set. Thus the independent sets of M/b are $\{\emptyset$, a, c, d, e, ad, ae, cd, ce, $de\}$. In Figure 2.1, we show the geometry of matroids $M\backslash b$ and M/b.

2.2 The geometry of deletion and contraction

In this section we show how to derive the geometry of $M\backslash e$ and M/e from the matroid M. In this thesis, we will always assume the element e is an arbitrary element of M and e is not a coloop (when deleting e) or a loop (when contracting e).

Example 2.4. Let $M = P_4$ be the rank 3 paving matroid in Figure 2.2 with 4-elements consisting of 3 collinear points as a, b, and c and a coloop d. Consider the deletion of the element a from M. To obtain the geometry of $M\backslash a$, we simply remove the point a from P_4. Any circuit that contained a will no longer be a circuit. In this example, the

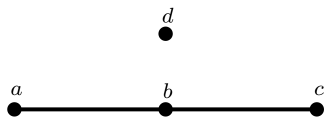


Figure 2.2: A picture of P_4.

Figure 2.3: A picture of $P_4\backslash a$.

only circuit that contains a is the set abc. Every pair of elements are independent sets in $P_4\backslash a$, and we see in Figure 2.3 that $P_4\backslash a$ consist of a 3-element independent set.

The process of drawing the geometry of a matroid after deleting an element is straight forward. In contrast, the geometry of contracting an element is more involved. Geometrically, contraction deals with projecting all points in the geometry through the point that is being contracted. This will be made more precise in the following example.

Example 2.5. Let $M = P_4$ again and consider the contraction of the element a. Geometrically, when we contract a, we remove the point a from the geometry and project the remaining points onto a rank 2 space in general position. To visualize the projection through the point a, draw lines from point a through each of the other points of P_4 and observe where each of these lines intersects the rank 2 space we placed in general position (A rank 2 space is a line). The points b, c, and d will project onto this line as depicted in

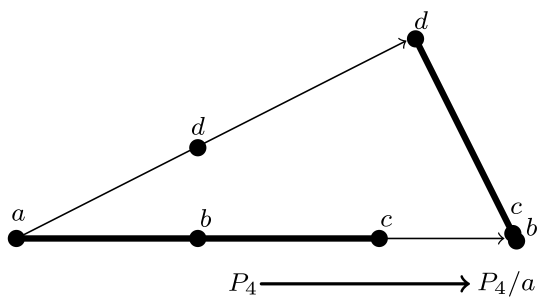


Figure 2.4: Drawing a line from point a through each points of P_4 and projecting the points b, c, and d onto a line in general position.

Figure 2.5: The resulting matroid P_4/a.

Figure 2.4. Note that b and c project onto the same point since they are both collinear with a. After erasing the unnecessary lines in Figure 2.4, we obtained the matroid P_4/a in Figure 2.5.

2.3 Properties of deletion and contraction

In this section, we will discuss how bases, circuits, and the rank function of a matroid M behave under the operations of deletion and contraction. The following proposition describes the bases, circuits, and rank of $M \backslash e$ and M/e in terms of those of M.

Proposition 2.6. *[GM12] Let M be a matroid on the ground set E and let $e \in E$, where e is neither a coloop nor a loop. Then*

(1) ***Bases***

- *(a)* ***Deletion*** *The bases of $M \backslash e$ are those bases of M that do not contain e.*
- *(b)* ***Contraction*** *The bases of M/e are those bases of M that do contain e, with e then removed from each such basis.*

(2) ***Circuits***

- *(a)* ***Deletion*** *C is a circuit of $M \backslash e$ if and only if $e \notin C$ and C is a circuit of M.*
- *(b)* ***Contraction*** *C is a circuit of M/e if and only if*
  - *(i) $C \cup e$ is a circuit of M, or*
  - *(ii) C is a circuit of M, $C \cup e$ contains no circuits except C, and C is nonspanning.*

(3) ***Rank*** *Let $A \subseteq E$ with $e \notin A$. Then*

- *(a)* ***Deletion*** *$r_{M \backslash e}(A) = r_M(A)$.*
- *(b)* ***Contraction*** *$r_{M/e}(A) = r_M(A \cup e) - 1$.*

We omit the proof of this proposition. This next proposition describes how a very important type of flat behaves under the operations of deletion and contraction.

Proposition 2.7. *[Oxl11] Let M be a matroid on the ground set E and let $e \in E$, where e is neither a coloop nor a loop. Then*
(1) ***Hyperplanes***

(a) ***Deletion*** *The hyperplanes of $M \backslash e$ are the collection of maximal proper subsets of $E - e$ of the form $H - \{e\}$ where $H \in \mathcal{H}(M)$.*

(b) ***Contraction*** *The hyperplanes of M/e are those hyperplanes of M that contain e, with e then removed from such each hyperplane.*

We omit the proof of this proposition as well. One may be concerned if the order of deletion and contraction matters. As a matter of fact, this is not the case. This next proposition shows how deletion and contraction commutes.

Proposition 2.8. *Let $a, b \in E$, where E is ground set of the matroid M and a and b are not loops nor coloops. Then we have:*

(1) $(M \backslash a) \backslash b = (M \backslash b) \backslash a$*;*

(2) $(M/a)/b = (M/b)/a$*; and*

(3) $(M/a) \backslash b = (M \backslash b)/a$*.*

We refer the reader to [GM12] for a proof.

2.4 Truncation

We almost have all of the tools we need to discuss the main results of this thesis. In this section we will to explore another way to construct new matroids from existing matroids. We define a new operation on a matroid M in terms of independence.

Definition 2.9. *[GM12] Let M be a rank r matroid on E. Let $\mathcal{I}_k$ be the collection of all independent sets of M of cardinality at most k. Then the truncation of M, denoted $T(M)$, is the matroid whose independent sets are $\mathcal{I}_{r-1}$.*

Clearly $\mathcal{I}_{r-1}$ satisfies the independence axioms. If this were not true, then there would exist $I \in \mathcal{I}_r$ such that $|I| \leq r-1$ and I does not satisfy at least one of the independence axioms since $I \in \mathcal{I}(M)$. This contradicts the fact that M is a matroid. It is possible to preform two truncation of M, which is the same as truncating $T(M)$. We denote this by $T^2(M)$. Each truncation yields another matroid. In a matroid M of rank r, we can to perform a sequence of truncation at most r-times, denoted $T^j(M)$, where $j \leq r$. The matroid $T^r(M)$ is a matroid with independent sets $\mathcal{I}_0$. That is, $\mathcal{I}(T^r(M)) = \{\emptyset\}$ and there does not exist an independent that is smaller than $\emptyset$.

Let M be a matroid of rank r on ground set E. We define the *free extension* of M by the element $x \notin E$, denoted $M + x$, as the matroid whose independent sets are given by:

$$\mathcal{I}(M + x) = \mathcal{I} \cup \{I \cup x : I \in \mathcal{I}(M) \text{ and } |I| \leq r - 1\}$$

This operation does not increase the rank of M. The matroid $(M + x)/x$ has the ground set $E(M+x)-\{x\} = E(M)$, its independent sets are all independent sets of $\mathcal{I}((M+x)/x)$ that contain x, with x then removed from each such set. That is $\mathcal{I}((M + x)/x) = \{I : I \in \mathcal{I}(M) \text{ and } |I| \leq r - 1\}$. The truncation of M, $T(M)$, can be viewed as $(M + x)/x$. The cardinality of the ground set $E(M)$ is preserved since $x \notin E(M)$.

Example 2.10. Let $M = U_{r,n}$ be a rank r, where $0 < r$, uniform matroid on n-elements and let $\mathcal{I}_r$ be the collection of all independent sets of cardinality at most r of $U_{r,n}$. Then, by the definition of truncation, the independent sets of $T(U_{r,n})$ are the collection of all independent sets of cardinality at most $r - 1$ of $U_{r,n}$. We know any set of size r or less is an independent set of $U_{r,n}$. Thus any set of size $r - 1$ or less is an independent set of $T(U_{r,n})$. Therefore, $T(U_{r,n}) = U_{r-1,n}$.

Note, we are not able to truncate a rank 0 matroid ($U_{0,n}$), since freely adding element x to the ground set of $E(U_{0,n})$ is a loop. We cannot contract loops. An important thing to know is that $T(M)$ reduces the rank of M by one. It follows from (I2) that the bases of $T(M)$ are formed by taking any basis of M and removing an element.

Example 2.11. What if we are only given the geometric representation of a matroid and we wish to truncate this matroid. Let M be the matroid in Figure 2.6.

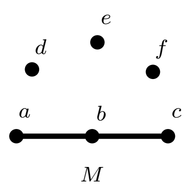


Figure 2.6: The matroid M in Example 2.11.

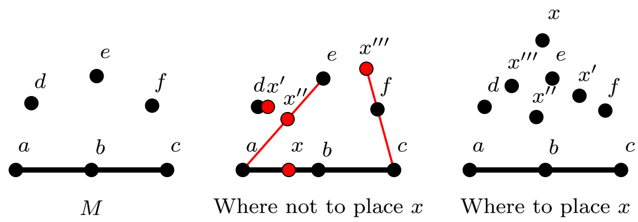


Figure 2.7: Examples where x is not freely placed and where x is freely placed in $M + x$.

First, we freely add an element x to the ground set $E(M)$. In general, if N is a rank r matroid, then freely adding x to the ground set $E(N)$ means that for all $I \in \mathcal{I}(N)$ with $|I| \leq r-1$. We have $I \cup \{x\} \in \mathcal{I}(N+x)$. Notice M is a rank 3 matroid and every 2 points in M is independent. Thus every 3 points that contain x must be an independent set of $M+x$. In Figure 2.7, x', x'', and x''' all represent different positions for x where x is not freely placed and where x is freely place of $M+x$.

After x is freely placed, we contract x. Note, it does not matter where we position x. When contracting x from $M+x$, the operation $(M+x)/x$ will always result in the same matroid, as long as x is freely placed. We freely place x in the most convenient position for us to contract x in Figure 2.8. Our resulting matroid, $T(M) = (M+x)/x$, consists of 6 collinear points, since, for all $I \in \mathcal{I}(M)$ such that $|I| = 2$, are the bases of $T(M)$. Recall, every 2 points of M is an independent set. Thus $T(M)$ is the rank 2 uniform matroid $U_{2,6}$ (see Figure 2.9).

What are the bases, the circuits, and the rank of M after truncation? The next proposition tells us about the bases, circuits, and the rank of $T(M)$. The following proposition describes the bases, circuits, and rank of $T(M)$ in terms of those of M.

Proposition 2.12. *[Bry86] Let M be a rank r matroid on E and $\mathcal{I}_r$ be the collection of all independent sets of cardinality at most r. Then,*
(1) $\mathcal{B}(T(M)) = \{I \colon I \in \mathcal{I}(M)$ and $|I| = r-1\}$.

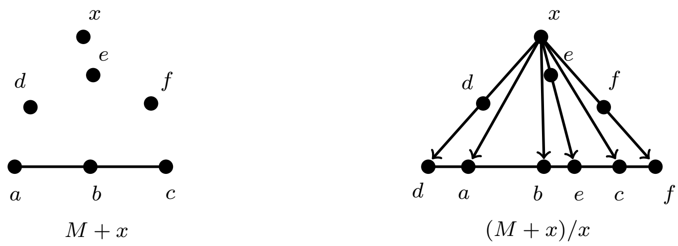


Figure 2.8: Contracting x from $(M+x)$, $(M+x)/x$.

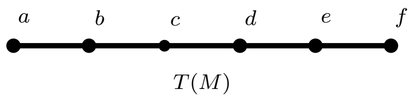


Figure 2.9: $(M+x)/x = T(M) \cong U_{2,4}$.

(2) $\mathcal{C}(T(M))$= $\{C$*: C is a nonspanning circuits of* $M\} \cup \{B$*: B is a basis of* $M\}$.

(3) $r(T(M)) = r(M) - 1 = r - 1$

Recall that in Section 2.3, we showed that the operations of deletion and contraction commute. Do these the operations commute with operation of truncation? The next proposition states that deletion and contraction commutes with truncation.

Proposition 2.13. *Let M be a matroid on ground set E, and let $e \in E$. Assuming everything is well-defined, we have:*

(1) $T(M\backslash e) = T(M)\backslash e$*; and*

(2) $T(M/e) = T(M)/e$.

Before we begin the proof, well-defined here means that the element e is not a coloop in (1) and e is not a loop in (2).

Proof. ***(1)*** We will prove this statement by showing that the set $S = \{I \in \mathcal{I}(M) : |I| = r(M) - 1 \text{ and } e \notin I\}$ is the collection of bases for both matroids $T(M\backslash e)$ and $T(M)\backslash e$.

Let $B \in \mathcal{B}(T(M\backslash e))$. Then, by Proposition 2.6, B is an independent set of $M\backslash e$ of size $r(M\backslash e) - 1$. That is, $B \in \mathcal{I}(M)$, such that $|B| = r(M) - 1$ and $e \notin B$. Hence, $B \in S$. Now let $I \in S$. We need to show $I \in \mathcal{B}(T(M\backslash e))$. Since $I \in S$, we know $I \in \mathcal{I}(M)$, $|I| = r(M) - 1$, and $e \notin I$. Thus, $I \in \mathcal{I}(M\backslash e)$ and $|I| = r(M\backslash e) - 1 = r(M) - 1$. By Proposition 2.12, this implies $I \in \mathcal{B}(T(M\backslash e))$. We have now shown that $S = \mathcal{B}(T(M\backslash e))$.

Next, we will show that $\mathcal{B}(T(M)\backslash e) = S$. Let $B' \in \mathcal{B}(T(M)\backslash e)$. By Proposition 2.12, the bases of $T(M)$ are the independent sets of M of size $r(M) - 1$. When we delete the element e from the ground set $E(T(M))$, by Proposition 2.6, the bases of the resulting matroid are the bases of $T(M)$ that do not contain the element e, which describes B'. That is $B' \in \mathcal{I}(M)$ and $|B'| = r(M) - 1$. Moreover, $e \notin B'$. Therefore, $B' \in S$. Now let $I \in S$. We need to show $I \in \mathcal{B}(T(M)\backslash e)$. Since $I \in S$, we know $I \in \mathcal{I}(M)$ and $|I| = r(M) - 1$ such that $e \notin I$. Thus, $I \in \mathcal{I}(T(M))$, since $|I| = r(M) - 1 = r(T(M))$. By Proposition 2.6, this implies $I \in \mathcal{B}(T(M)\backslash e)$ since $e \notin I$. Now that we have shown $\mathcal{B}(T(M\backslash e)) = S$ and $\mathcal{B}(T(M)\backslash e) = S$, this shows that $\mathcal{B}(T(M\backslash e)) = \mathcal{B}(T(M)\backslash e)$. Therefore, $T(M\backslash e) = T(M)\backslash e$.

(2) We will prove this statement by showing that the set $S' = \{I' - \{e\} : I' \in \mathcal{I}(M), |I'| = r(M) - 1 \text{ and } e \in I'\}$ is the collection of bases for both matroids $T(M/e)$ and $T(M)/e$.

Let $\hat{B} \in \mathcal{B}(T(M/e))$. Then, by Proposition 2.12, $\hat{B} \in \mathcal{I}(M/e)$ and $|\hat{B}| = r(M/e) - 1 = r(M) - 2$. Thus, by Proposition 2.6, $\hat{B} \cup \{e\} \in \mathcal{I}(M)$ and $|\hat{B} \cup \{e\}| = r(M) - 1$. It follows that $\hat{B} \in S'$. Now let $X \in S'$. We need to show $X \in \mathcal{B}(T(M/e))$. But $X = I' - \{e\}$, for some $I' \in \mathcal{I}(M)$ with $|I'| = r(M) - 1$ such that $e \in I'$. Thus, by Proposition 2.6, $X \in \mathcal{I}(M/e)$ and $|X| = r(M/e) - 1 = r(M) - 2$, since $X \cup e \in \mathcal{I}(M)$. By Proposition 2.12, this implies $X \in \mathcal{B}(T(M/e))$, since $X \in \mathcal{I}(M/e)$ and $|X| = r(M/e) - 1 = r(M) - 2$. Therefore, $S' = \mathcal{B}(T(M/e)$.

Next, we will show that $\mathcal{B}(T(M)/e) = S'$. Let $\hat{B}' \in \mathcal{B}(T(M)/e)$. By Proposition 2.12, the bases of $T(M)$ are the independent sets of M of size $r(M) - 1 = r(T(M))$. When we contract the element e from the ground set $E(T(M))$, by Proposition 2.6, the bases of $T(M)/e$ are the bases of $T(M)$ that contain the element e, with e then removed from each such basis, which describes $\hat{B}'$. That is $\hat{B}' \cup \{e\} \in \mathcal{I}(M)$, such that $|\hat{B}' \cup \{e\}| = r(M) - 1 = r(T(M))$. Hence, $\hat{B}' \in S'$. Now, let $X' \in S'$. Then, $X' = I' - \{e\}$, for some $I' \in \mathcal{I}(M)$ with $|I'| = r(M) - 1$, such that $e \in I'$. We need to show $X' \in \mathcal{B}(T(M)/e)$. By Proposition 2.12, we know $X' \cup \{e\} \in \mathcal{B}(T(M))$ since $X' \cup \{e\} \in \mathcal{I}(M)$ and $|X' \cup \{e\}| = r(M) - 1 = r(T(M))$. By Proposition 2.6, this implies $X' \in \mathcal{B}(T(M)/e)$ since $X' \cup \{e\} \in \mathcal{B}(T(M))$. Now that we have shown $\mathcal{B}(T(M/e)) = S'$ and $\mathcal{B}(T(M)/e) = S'$, this shows that $\mathcal{B}(T(M/e)) = \mathcal{B}(T(M)/e)$. Therefore, $T(M/e) = T(M)/e$. □

Chapter 3

Matroid minors

3.1 Minor-closed classes

Recall that the operations of deletion and contraction of a matroid results to a new matroid. We call the new matroid a *minor* of the original matroid. That is, given a matroid M, a *minor* of M is any matroid N that can be obtained from M by performing any sequence of deletions and contractions.

Example 3.1. Let the matroid P_4 be a rank 3 paving matroid with 4-elements consisting of 3 collinear points and a coloop (see Figure 3.1). We will show P_4 has $U_{2,2}$ as a minor. The matroid $U_{2,2}$ is a rank 2 matroid consisting of 2-elements, which are two coloops. We will determine which operations of deletion and contraction is needed to obtain $U_{2,2}$ as a minor. We know $r(P_4) = 3$ and $|E(P_4)| = 4$. We need to operate at most two operations of either deletion or contraction on P_4 since preforming either operations reduces the size of the ground set of P_4 to obtain a minor with 2-elements. One of the two operations must be contraction since contracting a point reduces the size and the rank of P_4 resulting to some rank 2 matroid with 3-elements. Then the other operation must be deletion since deleting a point reduces the size of P_4 but do not reduce the rank of P_4. Thus the two operations needed to obtain a rank 2 matroid with 2-elements is one contraction and one deletion. Note that we can contract any point of P_4 but we are able to delete any other point except for d since d is a coloop of P_4.

Consider contracting d, by Proposition 2.6, the set abc is preserved since abc is a 3-circuit and d is not contained in this 3-circuit. Then P_4/d consist of only a 3 collinear

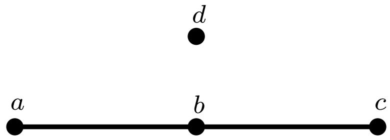


Figure 3.1: The rank 3 paving matroid with 4-elements consisting of 3 collinear points and a coloop P_4.

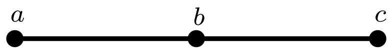


Figure 3.2: P_4/d.

points (see Figure 3.2).

Deleting any of the three points a, b, or c will result to $U_{2,2}$, for instance, deleting a gives us $P_4/d\backslash a \cong U_{2,2}$. Note, Proposition 2.8 allows to delete and contract in any order. So this is true as well $P_4\backslash a/d \cong U_{2,2}$. The geometry of this matroid is given in Figure 3.3.

This is one possible way for us to achieve $U_{2,2}$ as a minor from P_4. For instance, we can contract the point a of P_4. By Proposition 2.6, the set $abc - \{a\}$ is a 2-circuit contain in P_4/a. So P_4/a is a rank 2 matroid consisting of parallel points bc and a coloop d, see Figure 3.4. Then we are able to delete either point b or c, since neither points are coloops, to obtain the matroid $U_{2,2}$.

Now that we know what a minor of a matroid is, we can go over the central idea of this thesis. A class of matroids $\mathcal{M}$ is called a *minor-closed class* if $\mathcal{M}$ is closed under the operations of deletion and contraction. That is, let $\mathcal{M}$ be an arbitrary class of matroids and let $M \in \mathcal{M}$. The class $\mathcal{M}$ is minor-closed if for all $e \in E(M)$, then $M\backslash e$ and M/e are in the class $\mathcal{M}$. Furthermore, all possible minors of M must be in the class $\mathcal{M}$. We will go over this concept in the following example.

Example 3.2. For this example we are going to show a class of matroids $\mathcal{M}$ that is not minor-closed and the class $\mathcal{C}_k$ that is minor-closed.

Figure 3.3: $P_4/d\backslash a = P_4\backslash a/d \cong U_{2,2}$

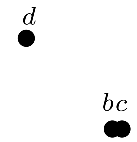


Figure 3.4: A picture of P_4/a.

Let $\mathcal{M}$ be the class of matroids that contains at least 10-elements and let $M \in \mathcal{M}$ on the ground set E. By definition of deletion and contraction, both operations decreases the ground set E of M by one for each operation that is done to a matroid. If there is exist a matroid M such that $|E(M)| = 10$, then, for all $e \in E(M)$, $|E(M\backslash e)| = |E(M/e)| = 9$. Then every possible minor of M is not in the class $\mathcal{M}$. That is, for all $e \in E(M)$, $M\backslash e, M/e \notin \mathcal{M}$.Thus he class $\mathcal{M}$ is not closed under the operations of deletion or contraction. Therefore, $\mathcal{M}$ is not minor-closed.

Let $\mathcal{C}_k$ be the class of matroids whose circuits are at most k-circuits and let $M \in \mathcal{C}_k$. Assuming everything is well-defined, then by Proposition 2.6, both operations of deletion and contraction do not increase the size of circuits of M. Then all possible minors of M contains circuits at most k-circuits. Therefore, $\mathcal{C}_k$ is minor-closed.

There are many possible classes of matroids that are minor-closed. We are going to introduce some minor-closed classes that will be used throughout the rest of this thesis.

Theorem 3.3. *The classes of uniform and paving matroids are minor-closed.*

Proof. Let $\mathcal{U}$ be the class of uniform matroids and let $U_{r,n} \in \mathcal{U}$. Let $U_{r,n}$ be a uniform matroid on the ground set E and $e \in E$, then

$$U_{r,n}\backslash e \cong \{U_{r,n-1} \text{ if } 0 \leq r < n;$$

and

$$U_{r,n}/e \cong \{U_{r-1,n-1} \text{ if } 0 < r \leq n.$$

Thus the class $\mathcal{U}$ is minor-closed. Notice when $r = 0$, then every element of $U_{0,n}$ is a loop, so every proper minor of $U_{0,n}$ is only done through the operation of deletion. Also when $r = n$, then every element of $U_{n,n}$ is a coloop, so every minor of $U_{n,n}$ is only done through the operation of contraction.

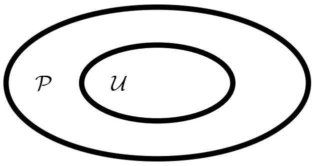


Figure 3.5: The class $\mathcal{P}$ contain the class $\mathcal{U}$.

Let $\mathcal{P}$ be the class of paving matroids and let $M \in \mathcal{P}$. Let $\mathcal{C}(M)$ be the collection of all circuits of M and let $C \in \mathcal{C}(M)$. Then we know for every matroid M in $\mathcal{P}$ is a rank r matroid and all of circuits C are at least r-circuits. We are going to show, for any paving matroid M, the circuits of $M \backslash e$ are at least r-circuits as $M \backslash e$ is a rank r matroid. Similarly, we are also going show the circuits of M/e are at least $(r-1)$-circuits as M/e is a rank $(r-1)$ matroid. For all $e \in E(M)$, by Proposition 2.6, $r(M \backslash e) = r(M)$ and the circuits of $M \backslash e$ are all $C \in \mathcal{C}(M)$ such that $e \notin C$. We know all circuits C of M are at least r-circuits. Thus $M \backslash e$ is a paving matroid since $M \backslash e$ is a rank r matroid with circuits C such that $e \notin C$ and $|C| \geq r$. Note, a matroid that does not contain circuits is still consider paving. For all $e \in E(M)$, by Proposition 2.6, $r(M/e) = r(M) - 1$ and the circuits of M/e are all nonspanning circuits C of M with $e \notin C$, that is $|C| = r$. Also all circuits C of M with $e \in C$ such that the set $C - \{e\}$ is a circuit of M/e of size r or $r-1$ since the circuits of M are either r-circuits or $(r+1)$-circuits. Thus M/e is paving since M/e is a rank $(r-1)$ matroid having at least $(r-1)$-circuits. Therefore, paving matroids is a minor-close class.

□

All uniform matroids are paving matroids. If uniform matroids contains circuits, then the circuits are spanning circuits. Thus the class of paving matroid $\mathcal{P}$ contain the class of uniform matroids $\mathcal{U}$, see Figure 3.5.

Another minor-closed class is binary matroids. We omit the proof of binary matroids is a minor-closed class. Binary matroids are representable matroids over $GF(2)$, the field of two elements $\{0, 1\}$. For instance, the matroid in Figure 3.6 can be represented by the following matrix A over the field $GF(2)$.

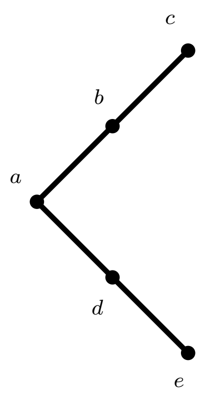


Figure 3.6: One of many binary matroids.

$$A = \begin{array}{c} \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

One can check that the column dependences of matrix A corresponds to the matroid in Figure 3.6.

3.2 New minor-closed classes from old minor-closed classes

This section is all about creating new minor-closed classes from existing minor-closed classes. The following theorem is one way we can create a new minor-closed class from an existing minor-closed class. We will not focus too much on this way to create new minor-closed classes. We will see in Chapter 4 what we mean by not focusing too much.

Theorem 3.4. *Let $\mathcal{M}$ be a minor-closed class of matroids. Let $\mathcal{Z}_{\mathcal{M}}$ be the collection of matroids such that for all $M \in \mathcal{Z}_{\mathcal{M}}$, whenever $e \in E(M)$, either $M \backslash e$ or M/e is in $\mathcal{M}$. Then $\mathcal{Z}_{\mathcal{M}}$ is a minor-closed class of matroids that contains $\mathcal{M}$.*

Before we begin the proof, the class $\mathcal{Z}_{\mathcal{M}}$ is called the class of *nearly-$\mathcal{M}$* matroids.

Proof. Let $M \in \mathcal{Z}_{\mathcal{M}}$, whenever $e \in E(M)$, either $M \backslash e$ or M/e is in $\mathcal{M}$. Let y be any element in our ground set $E(M)$. Let $\mathcal{M}$ be a minor-closed class. Suppose $M \in \mathcal{M}$, then $M\backslash y$ and M/y are in $\mathcal{M}$ since $\mathcal{M}$ is minor-closed. Thus all matroids in $\mathcal{M}$ are all nearly-$\mathcal{M}$ matroids.

Now suppose $M \notin \mathcal{M}$. Then there exist $y \in E(M)$ such that $M\backslash y \notin \mathcal{M}$. Is $M\backslash y$ still nearly-$\mathcal{M}$ matroid? That is, whenever $e \in E(M\backslash y)$, either $(M\backslash y)\backslash e$ or $(M\backslash y)/e$ is in $\mathcal{M}$ then $M\backslash y \in \mathcal{Z}_{\mathcal{M}}$. We know, whenever $e \in E(M)$, either $M\backslash e$ or M/e is in $\mathcal{M}$. If $M\backslash e \in \mathcal{M}$, then $(M\backslash e)\backslash y \in \mathcal{M}$ since $\mathcal{M}$ is minor-closed. By Proposition 2.8, $(M\backslash e)\backslash y = (M\backslash y)\backslash e$. This implies $(M\backslash y)\backslash e \in \mathcal{M}$. If $M/e \in \mathcal{M}$, then $(M/e)\backslash y \in \mathcal{M}$ since $\mathcal{M}$ is minor-closed. Again, by Proposition 2.8, $(M/e)\backslash y = (M\backslash y)/e$. Hence $(M\backslash y)/e \in \mathcal{M}$. Thus $M\backslash y \in \mathcal{Z}_{\mathcal{M}}$.

Finally suppose $M \notin \mathcal{M}$. Then there exist $y \in E(M)$ such that $M/y \notin \mathcal{M}$. Is M/y still nearly-$\mathcal{M}$ matroid? That is, whenever $e \in E(M/y)$, either $(M/y)\backslash e$ or $(M/y)/e$ is in $\mathcal{M}$ then $M/y \in \mathcal{Z}_{\mathcal{M}}$. We know, whenever $e \in E(M)$, either $M\backslash e$ or M/e is in $\mathcal{M}$. If $M\backslash e \in \mathcal{M}$, then $(M\backslash e)/y \in \mathcal{M}$ since $\mathcal{M}$ is minor-closed. By Proposition 2.8, $(M\backslash e)/y = (M/y)\backslash e$. For this reason, $(M/y)\backslash e \in \mathcal{M}$. If $M/e \in \mathcal{M}$, then $(M/e)/y \in \mathcal{M}$ since $\mathcal{M}$ is minor-closed. By Proposition 2.8, $(M/e)/y = (M/y)/e$. As a result, $(M/y)/e \in \mathcal{M}$. Thus $M/y \in \mathcal{Z}_{\mathcal{M}}$. Therefore $\mathcal{Z}_{\mathcal{M}}$ is a minor-closed class of matroids that contains $\mathcal{M}$. □

This theorem allows us to expand existing minor-closed classes. For instance, the class $\mathcal{Z}_{\mathcal{M}}$ contains the class $\mathcal{M}$ and some matroids not in $\mathcal{M}$. Note that all matroids in $\mathcal{M}$ are already nearly-$\mathcal{M}$ matroids since $\mathcal{M}$ is a minor-closed class. Let us use Theorem 3.4 on the class of uniform matroids.

Let $\mathcal{U}$ be the class of uniform matroids and let $\mathcal{Z}_{\mathcal{U}}$ be the class of nearly-uniform matroids. By Theorem 3.4, $\mathcal{Z}_{\mathcal{U}}$ is a minor-closed class. We are going to describe all matroids in the class $\mathcal{Z}_{\mathcal{U}}$ in the next theorem. Before begin with this theorem, we will need the following lemma to help us prove some types of matroids that are nearly-uniform.

Lemma 3.5. *If a matroid M of rank r has exactly one circuit C and $|C| = r$, then M has exactly one other element $x \in E - C$ and x is a coloop.*

Proof. Will prove this by contradiction. We know M has exactly one circuit C. Also, $|C| = r$, which means $r(C) = r - 1$. We need to show that C is a flat. That is, $\overline{C} = C$. Suppose, towards a contradiction, C is not a flat. Then $\overline{C} = C \cup X$, where $X \subseteq E(M) - C$. The set $C \cup X$ must contain another circuit of size r. Since $X \subset cl(C)$, for some $e \in C$ and for some $x \in X$, $r(C - \{e\}) = r - 1$ where $|C - \{e\}| = r - 1$ and $r(C - \{e\} \cup x) = r - 1$

where $|C-\{e\}\cup x| = r$. This implies $C \notin C-\{e\}\cup x$ and $C-\{e\}\cup x$ is dependent. Thus, $C-\{e\}\cup x$ contains a r-circuit that is not C. This contradicts that M has one circuit C. Thus, there exists an element $x \in E(M)-C$ such that $r(C\cup x) = r$ and for all $e \in C$, $(C-\{e\}\cup x) \in \mathcal{B}$. Suppose there also exists an element $y \in E(M)-C$, where $y \neq x$, and for all $e \in C$, $|(C-\{e\})\cup y| = r$. Hence, $|(C-\{e\})\cup\{x,y\}| = r+1$, so it must contain a circuit that is not C. But M only has one circuit, which is C; a contradiction. □

Recall that a rank r matroid M is uniform if its independence sets are precisely those subsets of $E(M)$ having at most r-elements. As a consequence of this, a matroid is uniform if and only if all of its circuits are spanning. That is all circuits of M have $(r+1)$-elements. The following theorem provides a characterization of nearly-uniform matroids:

Theorem 3.6. *A rank r matroid M is nearly-uniform if and only if M satisfies the following conditions:*

- *M is a paving matroid.*
- *M has at most one circuit C of size r.*
- *If M has a circuit C of size r, then C is also a hyperplane.*

Proof. Let M be a nearly-uniform matroid of rank r on the ground set E. We need to show M satisfies the following conditions: M is paving, M has at most one circuit of size r, and if it has a circuit of size r, then this circuit is a hyperplane. To see that M is paving, suppose, towards a contradiction, that M has a circuit C such that $|C| \leq r-1$. Then M is not paving. Since $r(C) < r(E)$, then there exists $y \in E(M)-C$. If y is a loop, then we cannot contract y, but we are able to delete y. By Proposition 2.6, $M\backslash y$ is a rank r matroid and contains the circuit C with $|C| \leq r-1$. Thus $M\backslash y$ is not uniform. If y is a coloop, then we cannot delete y, but we are able to contract y. By Proposition 2.6, we know $r(M/y) = r-1$ and M/y is also containing the circuit C with $|C| \leq r-1$. Hence M/y is not uniform. This implies, if y is neither a loop or a coloop, then, by Proposition 2.6, both matroids $N\backslash y$ and N/y contains the circuit C with $|C| \leq r-1$, which are not uniform matroids. This contradicts our assumption that M is nearly-uniform. It follows that M must be a paving matroid.

Now, suppose M has two distinct r-circuits, C_1 and C_2. Then there exists an element $e \in C_2 - C_1$. By Proposition 2.6, $M \backslash e$ is a rank r matroid that contains the circuit C_1 with $|C_1| = r$. Thus, $M \backslash e$ is not uniform. Similarly, by Proposition 2.6, M/e is a rank $(r-1)$ matroid that contains the circuit $C_2 - \{e\}$ where $|C_2 - \{e\}| = r - 1$. Thus, M/e is not uniform. It follows that if M has at least two r-circuits, then M is not nearly-uniform, which is a contradiction.

Now suppose M has a single r-circuit C. Then $r(C) = r - 1$. Take the closure of C, then $\overline{C} = C$. We need to show that C is a flat. Suppose, towards a contradiction, C is not a flat. Then $\overline{C} = C \cup X$, where $X \subseteq E(M) - C$. The set $C \cup X$ must contain another circuit of size r. Since $X \subset cl(C)$, for some $e \in C$ and for some $x \in X$, $r(C - \{e\}) = r - 1$ where $|C - \{e\}| = r - 1$ and $r(C - \{e\} \cup x) = r - 1$ where $|C - \{e\} \cup x| = r$. This implies $C \notin C - \{e\} \cup x$ and $C - \{e\} \cup x$ is dependent. Thus, $C - \{e\} \cup x$ contains a r-circuit that is not C. This contradicts that M has at most one circuit of size r. Thus C is a circuit-hyperplane. Therefore, if M is nearly-uniform, then M satisfies the three given conditions.

Let N be a rank r paving matroid with at most one r-circuit, such that any r-circuit is a hyperplane. Note, if N has no r-circuits and is paving, then N is a uniform matroid, which is nearly-uniform. We shall show N is nearly-uniform. To do so, we will show that the deletion of any element in C will result in a uniform matroid and that the contraction of any element not in C will also result in a uniform matroid.

Since coloops are not contained in any circuits, for any $e \in C$, $N \backslash e$ is well-defined. By Proposition 2.6 and Theorem 3.3, $N \backslash e$ is a rank r paving matroid. Moreover, all circuits of $N \backslash e$ are either $(r+1)$-circuits or $N \backslash e$ has no circuits, exclusively. Since the set $C - \{e\}$ is not a circuit of $N \backslash e$ and C is the only r-circuit of N, it follows that $N \backslash e$ is uniform.

Since C is the only r-circuit contain in N, the smallest possible circuit C can be is a 1-circuit (a loop). For any $y \in E - C$, N/y is well-defined. By Proposition 2.6, we know N/y is a rank $(r-1)$ matroid. The element y is contain either in a $(r+1)$-circuit or y is contain in no circuit, exclusively. If y is contain in a $(r+1)$-circuit C', then by Proposition 2.6, N/y contains the set $C' - \{e\}$ as a r-circuit. Any circuit of N that does not contain y must be C and $(r+1)$-circuits. Then C will remain as a r-circuit in N/y. Also any $(r+1)$-circuit of N that did not contain y will no longer be a circuit of N/y.

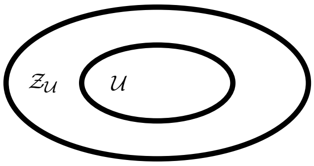


Figure 3.7: The class $\mathcal{Z}_{\mathcal{U}}$ contain the class $\mathcal{U}$.

In this case, N/y is uniform since $r(N/y) = r - 1$ and all circuits of N/y are r-circuits. Now if y is contained in no circuit then y is a coloop by Lemma 3.5. Hence $N/y = C$. In Chapter 1, we stated if the entire matroid is a circuit, then the matroid is uniform. For this case, $N/y \cong U_{r-1,r}$ since $|C| = r$. It follows that N is nearly-uniform. $\square$

Then the class of nearly-uniform matroids $\mathcal{Z}_{\mathcal{U}}$ contain the class of uniform matroids $\mathcal{U}$, see Figure 3.7.

There is one more theorem that can produce a new minor-closed class of matroids from an existing minor-closed class of matroids. This will be our main focus of minor-closed classes for Section 2 of Chapter 4. This theorem involves truncation and to prove this theorem we will need Proposition 2.13 which state that truncation commutes with deletion and contraction.

Theorem 3.7. *Let $\mathcal{M}$ be a minor-closed class of matroids and let $\mathcal{T}_{\mathcal{M}}$ be the class of matroids whose truncation is in $\mathcal{M}$. Then $\mathcal{T}_{\mathcal{M}}$ is minor-closed.*

Proof. Let $M \in \mathcal{T}_{\mathcal{M}}$. Then $T(M) \in \mathcal{M}$. We need to show that $M\backslash e$ and M/e are in $\mathcal{T}_{\mathcal{M}}$, for all $e \in E(M)$. But, $T(M) \in \mathcal{M}$ and $\mathcal{M}$ is minor-closed. Hence, $T(M)\backslash e$ and $T(M)/e$ are in $\mathcal{M}$, for all $e \in E(T(M))$. By Proposition 2.13, we know that $T(M\backslash e) = T(M)\backslash e$ and $T(M/e) = T(M)/e$. So $T(M\backslash e)$ and $T(M/e)$ are in $\mathcal{M}$. Thus $M\backslash e$ and M/e are in $\mathcal{T}_{\mathcal{M}}$. Therefore, $\mathcal{T}_{\mathcal{M}}$ is minor-closed. $\square$

Paving matroids are closed under truncation. That is, let $\mathcal{P}$ be the class of paving matroids and let $M \in \mathcal{P}$, where M is a rank r matroid. Then $T(M) \in \mathcal{P}$. This can be proven by Proposition 2.12, which state that truncation preserve nonspanning circuits of M and the bases of M are circuits of $T(M)$. Since M has at least r-circuits and the bases of M have cardinality r. Hence $T(M)$ has at least r-circuits and $r(T(M)) = r(M) - 1 = r - 1$. Therefore, $T(M)$ is paving. Moreover, $T(M)$ is uniform. We just proved all truncation of paving matroids is uniform. We also proved uniform matroids are closed under truncation in Example 2.10. We showed that $T(U_{r,n}) = U_{r-1,n}$. So uniform matroids are also closed under truncation. We have the following theorem:

Theorem 3.8. *Let $\mathcal{U}$ be the class of uniform matroids and let $\mathcal{P}$ be the class of paving matroids. Let $\mathcal{T}_{\mathcal{U}}$ be a class of matroids whose truncation is in $\mathcal{U}$. For all $M \in \mathcal{P}$, then $M \in \mathcal{T}_{\mathcal{U}}$.*

This theorem describes all matroids whose truncation is uniform and the class $\mathcal{T}_{\mathcal{U}} = \mathcal{P}$. If a matroid M has rank r and is not paving, then $T(M)$ is not uniform. The reason for this, for M to not be paving M must have an k-circuit, where $k \leq r-1$, which implies this k-circuit is nonspanning. By Proposition 2.12, the circuits of $T(M)$ are all nonspanning circuits of M and the bases of M, which all bases of M has cardinality r. Thus the k-circuit is preserved in $T(M)$ and $r(T(M)) = r-1$. For $T(M)$ to be uniform, its independence sets are all subsets of $E(T(M))$ having at most $(r-1)$-elements but $T(M)$ has an k-circuit, where $k \leq r-1$. Therefore, $T(M)$ is not uniform when M is not paving.

What about all matroids whose truncation is paving? This can be easily found if we think about how we found all matroids whose truncation is uniform. We know what a uniform matroid is. The matroid M is a rank r uniform matroid if and only if every circuit of M is a $(r+1)$-circuit. We showed that matroids with circuits having at least r-elements (paving matroids), their truncation is uniform. Now suppose M is a rank r paving matroid, then every circuit of M has at least r-elements. We can prove that for every rank r matroid with circuits having at least $(r-1)$-elements, their truncation results to a paving matroid. In a similar fashion how we proved all truncation of paving matroids are uniform matroids, simply by using Proposition 2.12, which states every nonspanning circuits of M and the bases of M are the complete set of circuits of $T(M)$. Now consider M to be a rank r matroid having at least $(r-1)$-circuits. By Proposition 2.12, $T(M)$ has at least $(r-1)$-circuits and $r(T(M)) = r-1$. Thus $T(M)$ is paving. We have the following theorem.

Theorem 3.9. *Let $M \in \mathcal{N}$ and $\mathcal{N}$ be the class of matroids M with circuits having at least $(r(M)-1)$-elements. Let $\mathcal{P}$ be the class of paving matroids. Let $\mathcal{T}_{\mathcal{P}}$ be a class of matroids whose truncation is in $\mathcal{P}$. For all $M \in \mathcal{N}$, then $M \in \mathcal{T}_{\mathcal{P}}$.*

So the class $\mathcal{T}_{\mathcal{P}}$ contains the classes $\mathcal{U}$, $\mathcal{T}_{\mathcal{U}}$, and $\mathcal{P}$ (see Figure 3.8). By Theorem 3.7, the

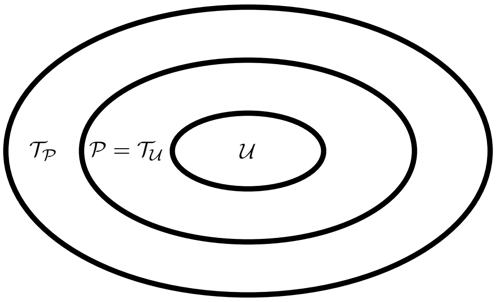


Figure 3.8: The class $\mathcal{T}_{\mathcal{P}}$ contains both $\mathcal{P}$ and $\mathcal{U}$.

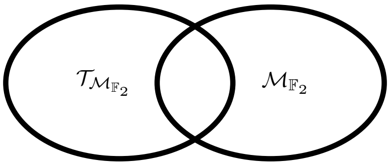


Figure 3.9: The class $\mathcal{T}_{\mathcal{M}_{\mathbb{F}_2}}$ does not contain the class $\mathcal{M}_{\mathbb{F}_2}$.

classes $\mathcal{T}_{\mathcal{U}}$ and $\mathcal{T}_{\mathcal{P}}$ are minor-closed classes.

Note, not all classes of matroids are closed under truncation. For example, the class of binary matroids is not closed under truncation. Let $\mathcal{M}_{\mathbb{F}_2}$ be the class of binary matroids and let $\mathcal{T}_{\mathcal{M}_{\mathbb{F}_2}}$ be the class of matroids whose truncation are binary. Then, by Theorem 3.7, $\mathcal{T}_{\mathcal{M}_{\mathbb{F}_2}}$ is a minor-closed class. Thus the class $\mathcal{T}_{\mathcal{M}_{\mathbb{F}_2}}$ contains some binary matroids and some non-binary matroids, see Figure 3.9. In Chapter 4 we will explain why binary matroids are not closed under truncation.

Chapter 4

Characterizing minor-closed classes of matroids

At the end of Chapter 3, we claim that not all classes of matroids are closed under truncation. In particular, we said $\mathcal{M}_{\mathbb{F}_2}$ is not closed under truncation. That is, there exist a matroid $M \in \mathcal{M}_{\mathbb{F}_2}$ such that $T(M) \notin \mathcal{M}_{\mathbb{F}_2}$. An important question arises from this example how do we know when a matroid is in the class of binary matroids? In general, how do we know if a matroid is in a particular minor-closed class $\mathcal{M}$? In the following section we will answer these questions.

4.1 Excluded minors

In this section we will introduce a particular method for characterizing minor-closed classes of matroids. Let $\mathcal{M}$ be a minor-closed class of matroids. Then a matroid N is an *excluded minor* for the class $\mathcal{M}$ if $N \notin \mathcal{M}$, but $N\backslash e$ and N/e are in $\mathcal{M}$, for all $e \in E(N)$. That is, N is a *minor-minimal* matroid that is not in the minor-closed class $\mathcal{M}$. The set of all excluded minors for $\mathcal{M}$ is denoted $\mathcal{EX}(\mathcal{M})$. It may be difficult to determine all the members of $\mathcal{EX}(\mathcal{M})$ and to determine whether $\mathcal{EX}(\mathcal{M})$ is finite or infinite.

We can characterize a class $\mathcal{M}$ by determining the set of $\mathcal{EX}(\mathcal{M})$. An example of this is Tutte's characterization of binary matroids: A matroid is binary if and only if it has no $U_{2,4}$ as a minor. Here, $\mathcal{M}$ is the class binary matroids and $\mathcal{EX}(\mathcal{M}) = \{U_{2,4}\}$.

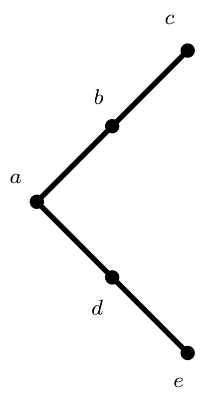


Figure 4.1: Binary matroid M.

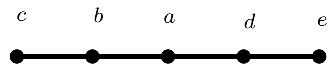


Figure 4.2: The truncation of the binary matroid M, $T(M) \cong U_{2,5}$.

In general, if we are given a minor-closed class of matroids $\mathcal{M}$, then we may wish to characterize $\mathcal{M}$ by determining the set of excluded minors $\mathcal{EX}(\mathcal{M})$. We state Tutte's theorem here for completeness.

Theorem 4.1. *[Oxl11] A matroid M is binary if and only if M has no $U_{2,4}$ as a minor.*

Now we can finally discuss why binary matroids are not closed under truncation. For example, let M be the matroid in Figure 4.1. We can see M is a rank 3 paving matroid. One can check that M does not have $U_{2,4}$ as a minor. Then, by Theorem 4.1, M is binary. By Theorem 3.8 and Proposition 2.12, $T(M)$ is a rank 2 uniform matroid with 5-elements. Thus, $T(M) \cong U_{2,5}$, as shown in Figure 4.2. We can delete any one point in $U_{2,5}$ to see that $U_{2,4}$ is a minor. therefore, $T(M)$ is not binary, and so in general, binary matroids are not closed under truncation.

One may ask, what are the excluded minors for the other minor-closed classes we discussed in Chapter 3?

Example 4.2. Let us find all excluded minors for the class $\mathcal{C}_k$ of matroids whose circuits are at most k-circuits. First we are going to show $U_{k,k+1}$ is an excluded minor for $\mathcal{C}_k$. The matroid $U_{k,k+1}$ is certainly not in $\mathcal{C}_k$ since the entire matroid is a $(k+1)$-circuit. We know if $e \in E(U_{k,k+1})$, then $U_{k,k+1} \backslash e = U_{k,k}$ and $U_{k,k+1}/e = U_{k-1,k}$ (note, for contraction, $k > 0$). Both matroids $U_{k,k}$ and $U_{k-1,k}$ have at most k-circuits. Thus $U_{k,k+1}$ is an excluded minor for $\mathcal{C}_k$.

Let N be an arbitrary excluded minor for $\mathcal{C}_k$. Then N must have circuits C of size at least $(k+1)$ elements to not be in $\mathcal{C}_k$. But, for all $e \in E(N)$, $N\backslash e$ and N/e are both in $\mathcal{C}_k$. Since, by Proposition 2.6, if $e \in C$, then the set $C - e$ is not a circuit in $N\backslash e$. If $e \notin C$, then the set C is a circuit in $N\backslash e$. Then every $e \in E(N)$ must be contained in all circuits C since we need to eliminate these circuits C for $N\backslash e \in \mathcal{C}_k$. This implies N has at most one circuit C. What can the size of circuit C be? Now suppose we contract an element of N. By Proposition 2.6, if $e \in C$ and recall $|C| \leq k+1$, then the set $C - e$ is a circuit in N/e such that $|C - e| \leq k$. Also,if $e \notin C$ and C is nonspanning such, then the set C is a circuit in N/e. Thus the circuit of N must be a $(k+1)$-circuit and every $e \in E(N)$ must be contained in the $(k+1)$-circuit to reduce the circuit size to a k-circuit for $N/e \in \mathcal{C}_k$. Thus the entire matroid N is a $(k+1)$-circuit. Therefore N is the uniform matroid $U_{k,k+1}$. Notice every $e \in E(N)$ is contained the $(k+1)$-circuit. Then we know e is not a coloop, since coloops are not contained in circuits. Also e is not a loop contained in N for the case where $k \geq 1$, since N is a $(k+1)$-circuit. For the case where $\mathcal{C}_k$ is a class of matroids whose circuits are at most k-circuits, where $k = 0$. That is, a class of matroids that contain no circuits. Then N is $U_{0,1}$. We can only delete in $U_{0,1}$ since $U_{0,1}$ is just a loop.

Now for the class $\mathcal{U}$ and $\mathcal{P}$, describing their excluded minors will require the following proposition.

Proposition 4.3. *[Oxl11] Let M_1 and M_2 be the matroids $(E_1, \mathcal{I}_1)$ and $(E_2, \mathcal{I}_2)$ where $E_1 \neq E_2$. Let*

$$M_1 \oplus M_2 = (E_1 \cup E_2, \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}).$$

Then $M_1 \oplus M_2$ is a matroid.

We omit the proof of this proposition. The matroid $M_1 \oplus M_2$ is called the *direct sum* of M_1 and M_2. The rank of $M_1 \oplus M_2$ is the sum of the rank of M_1 and M_2 since the size of the bases determines the rank of a matroid. The bases $\mathcal{B}(M_1 \oplus M_2) = \{B_1 \cup B_2 : B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2)$ since B_1 and B_2 are maximal independent sets of M_1 and M_2, respectively. Thus $r(M_1 \oplus M_2) = r(M_1) + r(M_2) = |B_1 \cup B_2|$. What are the collection of circuits in $M_1 \oplus M_2$. We have the following result.

Proposition 4.4. *[Oxl11]* $\mathcal{C}(M_1 \oplus M_2) = \mathcal{C}(M_1) \cup \mathcal{C}(M_2)$

We omit the proof of Proposition 4.4. Consequently, uniform matroids are not closed under direct sum. Which leads us to our next results.

Theorem 4.5. *The unique excluded minor for the class of uniform matroids $\mathcal{U}$ is $U_{0,1} \oplus U_{1,1}$.*

Proof. First, we are going to show the matroid $U_{0,1} \oplus U_{1,1}$ (this matroid just consist of a loop and a coloop, see Figure 4.3) is an excluded minor for the class $\mathcal{U}$. The matroid $U_{0,1} \oplus U_{1,1}$ is not uniform since this matroid has rank 1 but not every 1-element is independent. Consider if we delete an element in the matroid $U_{0,1} \oplus U_{1,1}$, by the definition of deletion, we are only able to delete the loop in $U_{0,1} \oplus U_{1,1}$. This results to the matroid $U_{1,1}$, which is uniform. Now consider if we contract an element in the matroid $U_{0,1} \oplus U_{1,1}$, by the definition of contraction, we are only able to contract the coloop in $U_{0,1} \oplus U_{1,1}$, resulting to the matroid $U_{0,1}$, which is uniform. Thus $U_{0,1} \oplus U_{1,1}$ is an excluded minor for $\mathcal{U}$.

Suppose that N is an excluded minor for the class $\mathcal{U}$ and let N be a rank r matroid. We shall show that $N \cong U_{0,1} \oplus U_{1,1}$. We know $N \notin \mathcal{U}$, but $N \backslash e$ and N/e are in $\mathcal{U}$, for all $e \in E(N)$. This means N must have circuits C, where $|C| \leq r$, for N not to be a uniform. If we delete any element in N, then we must delete an element contain in all C of N to get rid of all the circuits C to be uniform. Hence N does not have multiple disjoint circuits C. Keep in mind, we cannot contract an element in these circuits C, since by Proposition 2.6, contracting element $e \in C$ of N the set $C - \{e\}$ is a circuit with $|C - \{e\}| \leq r - 1$ contained in N/e and $r(N/e) = r - 1$. Resulting N/e to be a nonuniform matroid. The only circuit we know we cannot contract in is a loop. Thus the circuits C of N are loops. Recall N cannot contain multiple disjoint circuits. Then N has one circuit C since multiple loops are disjoint circuits. Then $r(N) = 1$, since contracting reduces the rank of N by one and N contains a 1-circuit. Again, when deleting an element in N we must delete an element contain in C, but we cannot delete any other element not contain in C. If we are able to delete an element $e \notin C$, by Proposition 2.6, C is a 1-circuit contain in $N \backslash e$ and $r(N \backslash e) = 1$. Resulting $N \backslash e$ to be a nonuniform matroid. We know we cannot delete coloops. Then the remaining elements not contain in C of N are coloops. So we are forced to delete an element contain in C. Now we know N is a

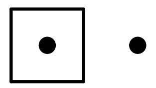


Figure 4.3: A loop and a coloop.

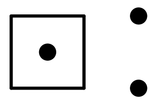
rank 1 matroid that contains a loop and the only elements besides the loop are coloops. This implies N has one coloop to be a rank 1 matroid. Therefore, N only consists of a loop and a coloop, which is the matroid $U_{0,1} \oplus U_{1,1}$.

□

Theorem 4.6. *The unique excluded minor for the class of paving matroids $\mathcal{P}$ is $U_{0,1} \oplus U_{2,2}$.*

Proof. The matroid $U_{0,1} \oplus U_{2,2}$ (this matroid just consist of a loop and two coloops, see Figure 4.4) is not paving since $U_{0,1} \oplus U_{2,2}$ is a rank 2 matroid and not every circuit of $U_{0,1} \oplus U_{2,2}$ has at least 2-elements. If we delete an element in $U_{0,1} \oplus U_{2,2}$, then we are only able to delete the loop contain in $U_{0,1} \oplus U_{2,2}$. This results to the matroid $U_{2,2}$, which is paving. If we contract an element contain in $U_{0,1} \oplus U_{2,2}$, then we are only able to contract either one of the two coloops contain in $U_{0,1} \oplus U_{2,2}$. This results to the matroid $U_{0,1} \oplus U_{1,1}$, which is paving, since $U_{0,1} \oplus U_{1,1}$ is a rank 1 matroid and every circuits in $U_{0,1} \oplus U_{1,1}$ has at least 1-element. Thus $U_{0,1} \oplus U_{2,2}$ is an excluded minor for the class $\mathcal{P}$.

Suppose that N is an excluded minor for the class $\mathcal{P}$ and let N be a rank r matroid on the ground set E. We shall show $N \cong U_{0,1} \oplus U_{2,2}$. Then we know $N \notin \mathcal{P}$, but $N \backslash e$ and N/e are in $\mathcal{P}$, for all $e \in E(N)$. Then N is not paving, which implies N must have circuits C such that $|C| \leq r-1$. Due to contraction, the smallest size circuit N can possibly have is a circuit C containing $(r-1)$-elements. Then there exists $x \in E - C$ that is not a loop since $r(M) = r$ and $r(C) = r - 2$ meaning there are independent subsets of $E - C$. By Proposition 2.6, N/x is a rank $r - 1$ matroid and contains the circuits C where $|C| = r - 1$, which is now paving. If we would had contracted an element $y \in C$, then Proposition 2.6, the set $C - \{y\}$ will be a $(r - 2)$-circuit contain in N/y and $r(N/y) = r - 1$, which is not paving. For this given reason, we must be force to not contract in these circuits C. We cannot contract loops and loops are 1-circuits. Thus the circuits C of N are loops and C is a $(r - 1)$-circuit with the respect to the rank of



$U_{0,1} \oplus U_{2,2}$

Figure 4.4: A loop and two coloops.

N. Hence, N is a rank 2 matroid. We gained all of this information by contracting all possible elements of N. Due to deletion, we must be force to delete an element in the circuits C, since by Proposition 2.6, deleting an element $e \in C$ of N, the set $C - \{e\}$ is no longer a circuit contain in $N \backslash e$. We are not able to delete coloops. Then the remaining elements not contain in the circuits C are coloops. Thus N has two coloops to be a rank 2 matroid. How many loops does N have? The matroid N does not contain more than one loop since for all $e \in E(N)$, $N \backslash e$ must result to a paving matroid. For all $e \in E(N)$, if N has more than one loop and we delete the element e of N, then there is a loop that is not e, by Proposition 2.6, $N \backslash e$ is a rank 2 matroid that has 1-circuits, which is not paving. Then N has one loop as circuit C. Now we know N is a rank 2 matroid with one loop and two coloops. Therefore, $N \cong U_{0,1} \oplus U_{2,2}$. □

4.2 Main results

In this section, we devote to characterize the new minor-closed classes in Section 3.2 by their excluded minors. Then we will attempt to find all excluded minors for the new minor-closed classes by a direct connection from their original minor-closed class's excluded minors.

We will begin with a theorem that describes all of the sets of excluded minors of $\mathcal{EX}(\mathcal{Z}_{\mathcal{U}})$, where $\mathcal{Z}_{\mathcal{U}}$ is the class of nearly-uniform.

Theorem 4.7. *A matroid N with $r(M) \geq r$ is nearly-uniform if and only if it does not have any of the following matroids as a minor:*

- $U_{0,2} \oplus U_{1,1}$
- $U_{0,1} \oplus U_{2,2}$

- $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$, *where* $k \geq 1$.

Proof. ($\Leftarrow$) Let N be a rank r matroid. So, if N has either $U_{0,2} \oplus U_{1,1}$, $U_{1,2} \oplus U_{1,2}$, $U_{0,1} \oplus U_{2,2}$, or $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$ as a minor, then N is not nearly-uniform. We are going to prove that these matroids are not nearly-uniform by showing that they do not satisfy Theorem 3.6.

Suppose N has $U_{0,2} \oplus U_{1,1}$ as a minor. The matroid $U_{0,2} \oplus U_{1,1}$ is paving since $U_{0,2} \oplus U_{1,1}$ is a rank 1 matroid having at least 1-circuits. But $U_{0,2} \oplus U_{1,1}$ has more than one 1-circuits. So $U_{0,2} \oplus U_{1,1}$ fails the second condition of Theorem 3.6. Thus, $U_{0,2} \oplus U_{1,1}$ is not nearly-uniform.

Now, suppose N has $U_{1,2} \oplus U_{1,2}$ as a minor. The matroid $U_{1,2} \oplus U_{1,2}$ is paving since $U_{1,2} \oplus U_{1,2}$ is a rank 2 matroid having at least 2-circuits. But $U_{1,2} \oplus U_{1,2}$ has more than one 2-circuits. So $U_{1,2} \oplus U_{1,2}$ fails the second condition of Theorem 3.6. Thus, $U_{0,1} \oplus U_{2,2}$ is not nearly-uniform.

Next, suppose N has $U_{0,1} \oplus U_{2,2}$ as a minor. The matroid $U_{0,1} \oplus U_{2,2}$ is not paving since this matroid has rank 2 containing a 1-circuit. So $U_{0,1} \oplus U_{2,2}$ fails the first condition of Theorem 3.6. Thus, $U_{0,1} \oplus U_{2,2}$ is not nearly-uniform.

Lastly, suppose N has $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$, where $k \geq 3$, as a minor. By Proposition 2.12, we know $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$ is a rank $k+1$ matroid consisting of only two disjoint $(k+1)$-circuits. So $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$ fails the second condition of Theorem 3.6 since $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$ has more than one $(k+1)$-circuits. Thus, $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$ is not nearly-uniform. Therefore N is not nearly-uniform if N has either $U_{0,2} \oplus U_{1,1}$, $U_{1,2} \oplus U_{1,2}$, $U_{0,1} \oplus U_{2,2}$, or $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$ as a minor.

($\Rightarrow$) Let $\mathcal{Z}_{\mathcal{U}}$ be the class of nearly-uniform matroids. If $N \notin \mathcal{Z}_{\mathcal{U}}$, then N has either $U_{0,2} \oplus U_{1,1}$, $U_{1,2} \oplus U_{1,2}$, $U_{0,1} \oplus U_{2,2}$, or $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$ as a minor. We are going to prove this by exhaustion through cases.

Let N be a rank r matroid. Suppose N is not a paving matroid. Then N has circuits C such that $|C| \leq r-1$. We are going to check every nonpaving matroids through each rank. (Note, it is trivial to check a rank 0 matroid since every rank 0 matroid is uniform and all rank 1 matroids are paving). Suppose N is a rank 2 matroid. Then N contains 1-circuits for N not to be a paving matroid. Since N has rank 2, the bases B of N are size 2. Let N only has one circuit C and C is a 1-circuit (loop). Given that N only has one circuit C such that $|C| = 1$, the 2-elements of B are coloops since each element of B is not contained in C. Then $N \cong U_{0,1} \oplus U_{2,2}$, a loop and two coloops. For all $e \in E(N)$, the matroid $N\backslash e$, we are only able to delete the loop of N since we cannot delete coloops. Resulting $N\backslash e \cong U_{2,2}$, which is already nearly-uniform. Now, for all $e \in E(N)$, the matroid N/e, we are only able to contract one of the two coloops of N since we cannot contract loops. Resulting $N/e \cong U_{0,1} \oplus U_{1,1}$. The matroid $U_{0,1} \oplus U_{1,1}$ is paving since $U_{0,1} \oplus U_{1,1}$ has rank 1 and all circuits of $U_{0,1} \oplus U_{1,1}$ is a 1-circuit. Notice $U_{0,1} \oplus U_{1,1}$ contains exactly one 1-circuit and the 1-circuit is a hyperplane since the only other element of $U_{0,1} \oplus U_{1,1}$ is a coloop, which satisfy Theorem 3.6. Hence, $U_{0,1} \oplus U_{1,1}$ is nearly-uniform. Therefore, $U_{0,1} \oplus U_{2,2}$ is an excluded minor for $\mathcal{Z}_{\mathcal{U}}$.

Now, suppose N is still a rank 2 matroid only containing more than one 1-circuit. If f is any 1-circuit of N, then, by Proposition 2.6, $N\backslash f$ will remain as a non-paving matroid since $N\backslash f$ will have 1-circuits and $r(N\backslash f) = 2$, which fails the first condition of Theorem 3.6. Hence $N\backslash f$ is not nearly-uniform. Thus N cannot have more than one 1-circuit to be nearly-uniform. Now, let N have multiple circuits of different sizes with only one 1-circuit since N cannot have more than one 1-circuit. Recall N is a rank 2 matroid, so N can only have circuits at most 3-circuits. If N has any other circuits other than the 1-circuit, then we are able to delete in those circuits of N that is not the 1-circuit since coloops are not contained in circuits. That is there exist an element e where e is contain in the circuits of N except the 1-circuit. By Proposition 2.6, $N\backslash e$ is a non-paving matroid since $r(N\backslash e) = 2$ and $N\backslash e$ has a 1-circuit because e is not contain in the 1-circuit of N. Therefore, a rank 2 matroid N cannot have any other circuits involved expect for exactly one 1-circuit to be nearly-uniform. This exhausts all possible cases for rank 2.

This time, suppose N is rank n, where $n \geq 3$, non-paving matroid. If N has circuits C such that $|C| < n$ and we contract an element contain in a circuit C that is not a 1-circuit, then, by Proposition 2.6, contracting decreases the rank and circuit by one. Thus, for all $e \in C$, N/e result to a non-paving matroid. For any rank n non-paving matroid that only has 1-circuits, will still result to a non-paving matroid since we cannot contract loops. For all $e \in E(N)$, the matroid N/e needs to be paving to satisfy one of the conditions of Theorem 3.6. We must somehow contract once to match the rank of N/e with the size of a 1-circuit. This is impossible through one contraction with a matroid that has at least rank 3. These matroids cannot be an excluded minor for the class $\mathcal{Z}_{\mathcal{U}}$ since N for this case is not closed under contraction (Note if N fails one operation, then

N cannot be an excluded minor). So $U_{0,1} \oplus U_{2,2}$ is the only non-paving matroid that is an excluded minor for the class $\mathcal{Z}_{\mathcal{U}}$.

Now consider the case where N is a paving matroid and $N \notin \mathcal{Z}_{\mathcal{U}}$. Then N must have not satisfied at least one of the conditions from Theorem 3.6. Given that N is paving, which leaves us with a few options. Our matroid N has more than one r-circuit or N has one r-circuit C but C is not a hyperplane.

Suppose N is a matroid on the ground set E and has one r-circuit C but C is not a hyperplane. If C is not a hyperplane, then there exist a subset $X \in E - C$ such that the set $C \cup X$ is a hyperplane. Let N be a rank 1 matroid with a 1-circuit as C and C is not a flat, precisely a hyperplane. Since C has rank 0 then $C \cup X$ resulting the set X being loops. Then there exists more than one loop. This is a contradiction since we know that N has one loop (1-circuit).

Similarly, consider N has rank m, where $m \geq 2$, and only has one circuit C such that C is a m-circuit. The set $C \cup X$ is a hyperplane by assumption, where the set X contains no loops since N is paving and $r(N) = m$. Then N has more than one m-circuit. Since $X \subset cl(C) = C \cup X$, for some $e \in C$ and for some $x \in X$, $r(C - \{e\}) = m - 1$ such that $|C - \{e\}| = m - 1$ and $r(C - \{e\} \cup x) = m - 1$ such that $|C - \{e\} \cup x| = m$. This implies $C \notin C - \{e\} \cup x$ and $C - \{e\} \cup x$ is dependent since the cardinality of the set is larger that the rank of the set. Given that N is paving, the set $C - \{e\} \cup x$ is a m-circuit that is not C. This is a contradiction since we know that N has one m-circuit as C. Thus, the case where N has one circuit C but C is not a hyperplane is impossible for paving matroids.

Now let us suppose N is a rank r paving matroid and let N have more than one r-circuit. Consider the case where N has two r-circuits. Suppose N is a rank 1 matroid, then N has two 1-circuits. Two things can occur, our matroid N may have a coloop joined with two loops, $U_{0,2} \oplus U_{1,1}$, or a class of parallel points joined with two loops, $U_{0,2} \oplus U_{1,n}$, where $n \geq 2$ (see Figure 4.5). If $N \cong U_{0,2} \oplus U_{1,1}$, then for all element $e \in E(N)$, $N \backslash e \cong U_{0,1} \oplus U_{1,1}$ since we cannot delete the coloop. The matroid $U_{0,1} \oplus U_{1,1}$ is nearly-uniform since $U_{0,1} \oplus U_{1,1}$ satisfy Theorem 3.6. For all element $e \in N$, $N/e \cong U_{0,2}$ since we cannot contract the loops. The matroid $U_{0,2}$ also satisfy Theorem 3.6. Thus the matroid $U_{0,2} \oplus U_{1,1}$ is an excluded minor for the class $\mathcal{Z}_{\mathcal{U}}$. Now suppose N has a pair of parallel points with two loops, which is the matroid $U_{0,2} \oplus U_{1,2}$. We can delete

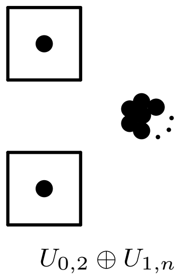


Figure 4.5: The matroid $U_{0,2} \oplus U_{1,n}$

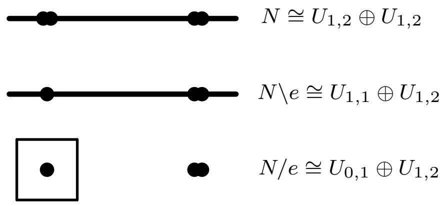


Figure 4.6: Supposing N is a rank 2 matroid consisting of two disjoint 2-circuits, for any one deletion of N ($N \backslash e$), and for any one contraction of N (N/e).

any point in the parallel points since a coloop is not contain the parallel points to obtain $U_{0,2} \oplus U_{1,1}$ as a minor, which is not nearly-uniform. We just established $U_{0,2} \oplus U_{1,1}$ is an excluded for $\mathcal{Z}_{\mathcal{U}}$. So N cannot have a parallel class of points to be a nearly-uniform rank 1 matroid. Next, let us assume N is a rank 1 matroid with three loops and with just a coloop, $U_{0,3} \oplus U_{1,1}$, since N cannot contain a parallel class. We can obtain $U_{0,2} \oplus U_{1,1}$ as a minor by deleting one of the three loops. Notice that if we keep adding more loops in our matroid N, we can always delete some loops to obtain $U_{0,2} \oplus U_{1,1}$ as a minor. Thus $U_{0,2} \oplus U_{1,1}$ is the only excluded minor as a rank 1 paving matroid having more than one 1-circuit.

Now suppose N is a rank 2 matroid consisting of two 2-circuits. Assuming for this case these circuits do not intersect and are the only two circuits of N. Thus we can assume N is a rank 2 matroid only consisting of two disjoint 2-circuits, that is $N \cong U_{1,2} \oplus U_{1,2}$ (see Figure 4.6).

For all element $e \in E(N)$, the matroid $N \backslash e$ will result to a rank 2 paving matroid with one 2-circuit, that is $N \backslash e \cong U_{1,1} \oplus U_{1,2}$, which satisfies Theorem 3.6. For all element $e \in E(N)$, the matroid N/e will result to a rank 1 paving matroid with one 1-circuit, that

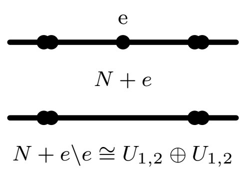


Figure 4.7:

is $N/e \cong U_{0,1} \oplus U_{1,2}$, which satisfies Theorem 3.6. Thus $U_{1,2} \oplus U_{1,2}$ is an excluded minor for the class $\mathcal{Z}_{\mathcal{U}}$. Now suppose N is a rank 2 matroid with two parallel points that are disjoint and has one additional point e that is not contain in any of two parallel points, see Figure 4.7. Obviously e is not a coloop since e is contain in a 3-circuit. Then we are able to delete the point e that results to the matroid $U_{1,2} \oplus U_{1,2}$ as a minor. Previously, we establish $U_{1,2} \oplus U_{1,2}$ is an excluded for $\mathcal{Z}_{\mathcal{U}}$. Hence for all rank 2 paving matroids with two parallel points that are disjoint and has n additional points that are not contain in any of two parallel points has $U_{1,2} \oplus U_{1,2}$ as a minor since we are able to delete all n additional points because each point is contain in some 3-circuit.

Notice a rank r matroid consisting of two disjoint r-circuits is not nearly-uniform since this matroid fails the second condition of Theorem 3.6, but every proper minor of this matroid will satisfy Theorem 3.6. This implies, these matroids are excluded minors for $\mathcal{Z}_{\mathcal{U}}$. We already showed a rank 1 matroid consisting of two disjoint 1-circuits ($U_{0,2} \oplus U_{1,1}$) and rank 2 matroid consisting of two disjoint 2-circuits ($U_{1,2} \oplus U_{1,2}$) are indeed excluded minors for $\mathcal{Z}_{\mathcal{U}}$.

Now to describe a rank 3 or greater matroid consisting of two disjoint 3-circuits or greater, respectively, is more complicated. This matroid $U_{1,2} \oplus U_{1,2} \cong T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$ only when $k = 1$ since $T^{1-1}(U_{1,1+1} \oplus U_{1,k+1}) = T^{0}(U_{1,2} \oplus U_{1,2})$ and we truncate $U_{1,2} \oplus U_{1,2}$ 0-times, meaning we are not truncating $U_{1,2} \oplus U_{1,2}$. In general, $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$, where $k \geq 1$, is a direct sum of two uniform matroids, where both entire uniform matroids are $(k+1)$-circuits, such that we truncate $U_{k,k+1} \oplus U_{k,k+1}$ $(k-1)$-times. Recall the rank of the direct sum of two matroids is the sum of the rank of both matroids and truncating $(k-1)$-times decreases the rank by $k-1$. We know $r(U_{k,k+1} \oplus U_{k,k+1}) = 2k$ and , by Proposition 4.4, the two $(k+1)$-circuits are the only circuits of $U_{k,k+1} \oplus U_{k,k+1}$. Then, by Proposition 2.12, $r(T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})) = 2k-(k-1) = 2k-k+1 = k+1$ and the two $(k+1)$-circuits are preserved since the $(k+1)$-circuits size is the same as the rank of $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$, which implies the $(k+1)$-circuits were nonspanning circuits for each truncation. Thus $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$ is a rank $k+1$ paving matroid consisting of two disjoint $(k+1)$-circuits, where half of the elements of $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$ is contain in one of the $(k+1)$-circuit and the other half is contain in the other $(k+1)$-circuit. The fact that the $k+1$-circuit contains half of the elements and the other half of the elements are disjointed implies every element not contain in one the $k+1$-circuit will increase the rank of the $k+1$-circuit. Hence these $(k+1)$-circuits are flats and the rank of these $(k+1)$-circuits are k. Thus these $(k+1)$-circuits are hyperplanes.

Suppose $N \cong T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$. We will show every deletion of N is nearly-uniform by satisfying Theorem 3.6. We are able to delete all $e \in E(N)$ since every element of N is contain in some $(k+1)$-circuit, which implies e is not a coloop. For all $e \in E(N)$, by Proposition 3.3, $N \backslash e$ is a paving since N is paving. By Proposition 2.6, deletion of an element e in one of the two $(k+1)$-circuits do not create new dependences and preserves the other $(k+1)$-circuit. Then $N \backslash e$ has at most one $(k+1)$-circuit. By Proposition 2.7, the $k+1$-circuit of $N \backslash e$ is a hyperplane since e is not contain this $k+1$-circuit. Thus $N \backslash e$ is nearly-uniform. Now we are going show for every contraction of N is nearly-uniform by satisfying Theorem 3.6. We are able to contract all $e \in E(N)$ since N is a least a rank 2 paving matroid, meaning N will never have a loop as an element. For all $e \in E(N)$, by Proposition 2.6, contracting the element e in one of the $k+1$-circuits of N will decrease to a (k)-circuit and preserves the other $k+1$-circuit as the rank of N/e is k. Then N/e has at most one k-circuit and N/e is paving since paving matroid is minor-closed. By Proposition 2.7, we know that the (k)-circuit of N/e is a hyperplane since k-circuit union e is a hyperplane of N. Therefore if $N \cong T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$, then there are infinitely many excluded minors of this form for the class $\mathcal{Z}_{\mathcal{U}}$. Now suppose N is a rank r matroid and consist of more than two disjoint r-circuit. We can eliminate as many r-circuit by the operation of deletion until we obtain $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$ as a minor since coloops are not contained in circuits.

Consider N is a rank r matroid with more than one r-circuit such that these circuits are intersecting. If e is subset of $E(N)$ and e is contained in the intersection of the r-circuits, then, for all $e \in E(N)$, contracting e will decrease all r-circuits that shared e by one as well as the rank of the matroid N. However, the matroid N/e will have more than one $r-1$-circuits, but we can only have at most one $r-1$-circuit to satisfy one the condition of Theorem 3.6. So $U_{0,2} \oplus U_{1,1}$ and $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$ are the only paving matroids that are excluded minors for the class $\mathcal{Z}_{\mathcal{U}}$. Therefore $U_{0,1} \oplus U_{2,2}$, $U_{0,2} \oplus U_{1,1}$ and $T^{k-1}(U_{k,k+1} \oplus U_{k,k+1})$ are excluded minors for the class $\mathcal{Z}_{\mathcal{U}}$. □

Recall back in Chapter 3 we stated we will not focus too much on these class of nearly-$\mathcal{M}$. What we really meant is that we will not be seeking a theorem to find all excluded minor for nearly-$\mathcal{M}$. But we are willing to discuss some thoughts that what we found interesting while investing $\mathcal{Z}_{\mathcal{U}}$. Notice we were able to find these excluded minors for $\mathcal{Z}_{\mathcal{U}}$ by using Theorem 3.6, which introduce all matroids contained in $\mathcal{Z}_{\mathcal{U}}$. We were able to come up with Theorem 3.6 because we notice the excluded minor $(U_{0,1} \oplus U_{1,1})$ for $\mathcal{U}$ was a nearly-uniform matroid. We investigated what conditions made the matroid $U_{0,1} \oplus U_{1,1}$ nearly-uniform. We noticed $U_{0,1} \oplus U_{1,1}$ was a paving matroid, we suggested if all paving matroids are nearly-uniform. We found counter examples, that is there exist paving matroids that are not nearly-uniform. Looking back at $U_{0,1} \oplus U_{1,1}$, we notice this matroid only has one circuit, which was a 1-circuit and $r(U_{0,1} \oplus U_{1,1}) = 1$. This is where we looked at a rank r paving matroid M that contains one r-circuit and we were able prove M is nearly-uniform in Theorem 3.6. Also M can contain as many $(r+1)$-circuits as long this matroid only contain one or no r-circuit to be nearly-uniform. By claiming if M has a r-circuit as C and C is a circuit-hyperplanes suggests that taking the closure of C, then $\overline{C}$ does not contain any other elements. This was need to help prove Theorem 3.6. Any other matroids we tried that did not satisfy Theorem 3.6 was not nearly-uniform.

This is where we were able to consider matroids to be an excluded minor for $\mathcal{Z}_{\mathcal{U}}$ by investigating the matroids that did not satisfy Theorem 3.6. In search for the complete set of excluded minor $\mathcal{EX}(\mathcal{Z}_{\mathcal{U}})$ we divided into two section such that we looked at every nonpaving matroids and every paving matroids with more than one r-circuit. This is how we where able to identify the matroids in $\mathcal{EX}(\mathcal{Z}_{\mathcal{U}})$ through exhaustion. Again, we were able to gather all of this information by studying the excluded minor $U_{0,1} \oplus U_{1,1}$ for $\mathcal{U}$. Which leads us to this claim.

Conjecture 1. *Let $\mathcal{M}$ be a minor-closed class and let $\mathcal{Z}_{\mathcal{M}}$ be the class of nearly-$\mathcal{M}$. We are able to find all $\mathcal{EX}(\mathcal{Z}_{\mathcal{M}})$ by exploring the excluded minors of $\mathcal{M}$. That is, for all $N \in \mathcal{EX}(\mathcal{M})$ is always a nearly-$\mathcal{M}$ matroid since N is minor minimal. Then describe the condition what makes N nearly-$\mathcal{M}$. To find all $\mathcal{EX}(\mathcal{Z}_{\mathcal{M}})$, we will consider all matroids that do not satisfy the same conditions as N.*

Now for our main focus, we going to introduce the excluded minors for the class whose truncation is in the class $\mathcal{U}$, $\mathcal{P}$, and $\mathcal{M}_{\mathbb{F}_2}$ as the following theorems.

Theorem 4.8. *The unique excluded minor for* $\mathcal{T}_{\mathcal{U}}$ *is* $U_{0,1} \oplus U_{2,2}$.

Proof. First, we are going to show $U_{0,1} \oplus U_{2,2}$ is an excluded minor for the class $\mathcal{T}_{\mathcal{U}}$. Notice $T(U_{0,1} \oplus U_{2,2}) \notin \mathcal{T}_{\mathcal{U}}$, since $T(U_{0,1} \oplus U_{2,2}) \cong U_{0,1} \oplus U_{1,2}$ has $U_{0,1} \oplus U_{1,1}$ as a minor. Thus, by Theorem 4.5, $U_{0,1} \oplus U_{1,2}$ is not uniform. If we delete an arbitrary element e in the ground set $E(U_{0,1} \oplus U_{2,2})$, then we are only able to delete the loop. Then $U_{0,1} \oplus U_{2,2} \backslash e \cong U_{2,2}$, which is uniform, and the class of uniform matroids is closed under truncation. That is $T(U_{0,1} \oplus U_{2,2} \backslash e) = T(U_{2,2}) = U_{1,2}$. Thus $T(U_{0,1} \oplus U_{2,2} \backslash e) \in \mathcal{T}_{\mathcal{U}}$, for all $e \in U_{0,1} \oplus U_{2,2}$. Moreover, if we contract an arbitrary element e in the ground set $E(U_{0,1} \oplus U_{2,2})$, we are only able to contract one of the two coloops. Then $U_{0,1} \oplus U_{2,2}/e \cong U_{0,1} \oplus U_{1,1}$. Since $U_{0,1} \oplus U_{1,1}$ is a rank 1 matroid consisting of 2-elements and, by Proposition 2.12, truncating any rank 1 matroid results to a rank 0 matroid. All rank 0 matroids are uniform matroids of the form $U_{0,n}$, where $n = \{1, 2, 3, ...\}$. Hence $T(U_{0,1} \oplus U_{2,2}/e) = T(U_{0,1} \oplus U_{1,1}) \cong U_{0,2}$, which is uniform. Thus $T(U_{0,1} \oplus U_{2,2}/e) \in \mathcal{T}_{\mathcal{U}}$, for all $e \in U_{0,1} \oplus U_{2,2}$. Therefore, $U_{0,1} \oplus U_{2,2}$ is an excluded minor for $\mathcal{T}_{\mathcal{U}}$.

Now suppose that N is an excluded minor for the class $\mathcal{T}_{\mathcal{U}}$. We shall show that $N \cong U_{0,1} \oplus U_{2,2}$. We know $T(N) \notin \mathcal{T}_{\mathcal{U}}$, but $T(N \backslash e)$ and $T(N/e)$ are in $\mathcal{T}_{\mathcal{U}}$. By Theorem 3.8, we know all the matroids in the class $\mathcal{T}_{\mathcal{U}}$ are paving matroids. Then N is not a paving matroid. However, for all $e \in E(N)$, $N \backslash e$ and N/e must be paving such that we can truncate this paving matroid to be uniform. That is $T(N \backslash e)$ and $T(N/e)$ are both in $\mathcal{T}_{\mathcal{U}}$. This implies any proper minor of N is paving. By Theorem 4.6, the only matroid that is not paving but very proper minor is paving is the matroid $U_{0,1} \oplus U_{2,2}$. Therefore, $N \cong U_{0,1} \oplus U_{2,2}$. □

Theorem 4.9. *The unique excluded minor for* $\mathcal{T}_{\mathcal{P}}$ *is* $U_{0,1} \oplus U_{3,3}$.

Proof. First, we are going to show $U_{0,1} \oplus U_{3,3}$ (see Figure 4.8) is an excluded minor for the class $\mathcal{T}_{\mathcal{P}}$. Notice $T(U_{0,1} \oplus U_{3,3}) \notin \mathcal{T}_{\mathcal{P}}$, since $T(U_{0,1} \oplus U_{3,3}) \cong U_{0,1} \oplus U_{2,3}$ has $U_{0,1} \oplus U_{2,2}$ as a minor. Thus, by Theorem 4.6, $U_{0,1} \oplus U_{2,3}$ is not paving. If we delete an arbitrary element e in the ground set $E(U_{0,1} \oplus U_{3,3})$, then we are only able to delete the loop. Then $U_{0,1} \oplus U_{3,3} \backslash e \cong U_{3,3}$, which is uniform, and the class of uniform matroids is closed under truncation. That is $T(U_{0,1} \oplus U_{3,3} \backslash e) = T(U_{3,3}) \cong U_{2,3}$, which is paving since all

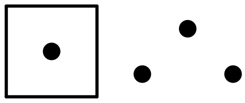


Figure 4.8: The excluded minor for $\mathcal{T}_{\mathcal{P}}$.

uniform matroids are paving. Thus $T(U_{0,1} \oplus U_{3,3} \backslash e) \in \mathcal{T}_{\mathcal{P}}$, for all $e \in E(U_{0,1} \oplus U_{3,3})$. Moreover, if we contract an arbitrary element e in the ground set $E(U_{0,1} \oplus U_{3,3})$, we are only able to contract one of the three coloops. Then $U_{0,1} \oplus U_{3,3}/e \cong U_{0,1} \oplus U_{2,2}$. By Proposition 2.12, $T(U_{0,1} \oplus U_{3,3}/e) \cong T(U_{0,1} \oplus U_{2,2}) \cong U_{0,1} \oplus U_{1,2}$, which is paving. Thus $T(U_{0,1} \oplus U_{3,3}/e) \in \mathcal{T}_{\mathcal{P}}$, for all $e \in E(U_{0,1} \oplus U_{3,3})$. Therefore, $U_{0,1} \oplus U_{3,3}$ is an excluded minor for $\mathcal{T}_{\mathcal{P}}$.

Now suppose that N is an excluded minor for the class $\mathcal{T}_{\mathcal{P}}$ and let N be a rank r matroid. We shall show that $N \cong U_{0,1} \oplus U_{3,3}$. We know all the matroids M in the class $\mathcal{T}_{\mathcal{P}}$ are all rank m matroids and every circuit of M has at least $(m-1)$-elements. Thus N must contain circuits C such that $|C| \leq r-2$ for $N \notin \mathcal{T}_{\mathcal{P}}$. Moreover, for any proper minor of N and then truncating the minor of N is in the class $\mathcal{T}_{\mathcal{P}}$. This implies any proper minor of N results to a matroid with every circuit having at least $(n-1)$-elements, where n is the rank of any proper minor of N. This means, for all $e \in E(N)$, $N \backslash e$ results to a matroid with every circuit having at least $(r-1)$-elements since $r(N \backslash e) = r$. Similarly, for all $e \in E(N)$, N/e results to a matroid with every circuit having at least $(r-2)$-elements since $r(N/e)$. Such that the matroids $T(N \backslash e)$ and $T(N/e)$ are in $\mathcal{T}_{\mathcal{P}}$. Recall that N must contain circuits C such that $|C| \leq r-2$. Then N cannot contain multiple disjoint circuits C. Since deleting an element $e \in C$ of N must eliminate all the circuits C in the matroid $N \backslash e$, so that $N \backslash e$ contains circuits having at least $(r-1)$-elements. Due to contraction, we cannot contract an element $e \in C$ of N, by Proposition 2.6, contracting element $e \in C$ the set $C - \{e\}$ is a circuit in N/e such that $|C - \{e\}| \leq r-3$ as $r(N/e) = r-1$. We want the circuits of N/e to have at least $(r-2)$-elements, since $r(T(N/e)) = r-2$. Resulting $T(N/e)$ to be a paving matroid. Knowing we cannot contract an element in C. Thus the circuits C are loops in N and N is a rank 3 matroid. Keep in mind, N cannot have multiple disjoint circuits C, then N contains one 1-circuit as circuit C. We must be forced to delete an element contained in C, since deleting an element in a matroid do not

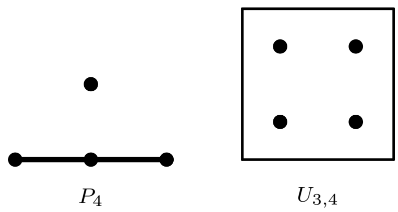


Figure 4.9: Rank 3 paving matroids with 4-elements are only P_4 and $U_{3,4}$.

decrease the rank. We know we cannot delete coloops. Then the remaining elements not contained in C are coloops. This implies N has three coloops to be a rank 3 matroid. Therefore, N is the matroid $U_{0,1} \oplus U_{3,3}$. $\square$

Theorem 4.10. *The excluded minors for the class of matroids whose truncation is binary $\mathcal{T}_{\mathcal{M}_{\mathbb{F}_2}}$ is a rank 3 paving matroid with 4-elements.*

Proof. A rank 3 paving matroid with 4-elements is not binary after truncation since the resulting matroids is $U_{2,4}$. By Theorem 4.1, any matroid that has $U_{2,4}$ as a minor is not binary. Let M be a rank 3 paving matroid with 4-elements (Note, there are only two possible matroids, which are P_4 and $U_{3,4}$, see Figure 4.9). The matroid M is either P_4 or $U_{3,4}$. Then, by Proposition 2.12, $T(M) \cong U_{2,4}$, since every 2-element or less subsets of the ground set $E(P_4)$ and $E(U_{3,4})$ are independent sets. But, for all $e \in E(T(M))$, $T(M)\backslash e$ and $T(M)/e$ are both binary matroids. But, by Proposition 2.13, we know $T(M)\backslash e = T(M\backslash e)$ and $T(M)/e = T(M/e)$. Hence $T(M\backslash e)$ and $T(M/e)$ are in $\mathcal{M}_{\mathbb{F}_2}$, which implies $M\backslash e$ and M/e are in $\mathcal{T}_{\mathcal{M}_{\mathbb{F}_2}}$, for all $e \in M$. Therefore, M is an excluded minor for $\mathcal{M}_{\mathbb{F}_2}$.

Now suppose that N is an excluded minor for the class $\mathcal{T}_{\mathcal{M}_{\mathbb{F}_2}}$. We shall show N is a rank 3 paving matroid with 4-elements. Then $T(N)$ is not binary, but, for all $e \in E(N)$, $T(N\backslash e)$ and $T(N/e)$ are both binary matroids. We know that $T(N\backslash e) = T(N)\backslash e$ and $T(N/e) = T(N)/e$. So $T(N)$ is not binary, but $T(N)\backslash e$ and $T(N)/e$ are binary, for all $e \in E(N)$. Hence, $T(N)$ is an excluded minor for $\mathcal{M}_{\mathbb{F}_2}$. So $T(N) \cong U_{2,4}$. The operation of truncation decreases the rank of N by one and preserve the cardinality of the ground set $E(N)$. This implies N is a rank 3 matroid with 4-elements. If N had any circuits of size at most two, then $T(N)$ would also have these circuits but $T(N) \cong U_{2,4}$. So all circuits of N have size at least 3-elements. Therefore, N is a rank 3 paving matroid with 4-elements. $\square$

Finding the set of excluded minors in $\mathcal{EX}(\mathcal{T_U})$, $\mathcal{EX}(\mathcal{T_P})$, and $\mathcal{EX}(\mathcal{T}_{\mathcal{M}_{\mathbb{F}_2}})$ helped us create the following theorem:

Theorem 4.11. *Let $\mathcal{M}$ be a minor-closed class of matroids whose set of excluded minors is $\mathcal{EX}(\mathcal{M})$. Let $\mathcal{T_M}$ be the collection of matroids M such that $T(M) \in \mathcal{M}$. If there exist a matroid N whose truncation $T(N) \in \mathcal{EX}(\mathcal{M})$, then $N \in \mathcal{EX}(\mathcal{T_M})$.*

Proof. Suppose there exist a matroid N whose truncation $T(N) \in \mathcal{EX}(\mathcal{M})$. Since $T(N)$ in $\mathcal{EX}(\mathcal{M})$, then $N \notin \mathcal{T_M}$, but $T(N)\backslash e$ and $T(N)/e$ are in $\mathcal{M}$, for all $e \in E(T(N)) = E(N)$. By Proposition 2.13, we know that $T(N)\backslash e = T(N\backslash e)$ and $T(N)/e = T(N/e)$. Hence both matroids $T(N\backslash e)$ and $T(N/e)$ are in $\mathcal{T_M}$. Therefore $N \in \mathcal{EX}(\mathcal{T_M})$. □

Theorem 4.11 will assist us by finding some excluded minor for each such classes $\mathcal{T_M}$, where $\mathcal{M}$ is an arbitrary minor-closed class. The reason why this theorem is only capable to find some excluded minors and not all the sets of excluded minors for $\mathcal{EX}(\mathcal{T_M})$ because there may exist a matroid in $\mathcal{EX}(\mathcal{M})$ such that there is no matroid whose truncation is in $\mathcal{EX}(\mathcal{M})$. For instance, there is no matroid whose truncation is in $\mathcal{EX}(\mathcal{U})$ and $\mathcal{EX}(\mathcal{P})$. The excluded minor for $\mathcal{U}$ is $U_{0,1} \oplus U_{1,1}$, which consist of 2-elements where one of the elements is a loop (1-circuit) and the other element is a coloop. If there does exist a matroid N whose truncation $T(N) \cong U_{0,1} \oplus U_{1,1}$, then $|E(N)| = 2$ and N must be a rank 2 matroid with circuits having at least 1-element. We know truncation decreases the rank by one from our original matroid and preserves the cardinality of the ground set $E(N)$ and nonspanning circuits. Notice $U_{0,1} \oplus U_{1,1}$ does not have any 2-circuits. Then N has at most a 1-circuit, since this 2-circuit is a nonspanning set in N. Also the matroid N must have a basis B such that $|B| = 2$ since N has rank 2. This requires the ground set $E(N)$ to have 3-elements, since we need an 2-element set to be a basis and 1-element set to be a 1-circuit of N. But $E(N)$ can only have 2-elements. Thus there does not exist a matroid N whose truncation $T(N) \cong U_{0,1} \oplus U_{1,1}$.

Similarly, the excluded minor for $\mathcal{P}$ is $U_{0,1} \oplus U_{2,2}$, which consist of 3-elements where one of the elements is a loop and the other 2-elements are coloops. If there does exist a matroid N whose truncation $T(N) \cong U_{0,1} \oplus U_{2,2}$, then $|E(N)| = 3$ and N must be a rank 3 matroid with circuits having at least 1-element. Again, truncation decreases the rank by one of our original matroid and preserves the cardinality of the ground set $E(N)$ and nonspanning circuits. Notice $U_{0,1} \oplus U_{2,2}$ does not have any 2-circuits or 3-circuits.

Then N has at most a 1-circuit, since 2-circuits or 3-circuits are nonspanning sets in N. The matroid N must have a basis B such that $|B| = 3$ since N has rank 3. This requires the ground set $E(N)$ to have 4-elements, since we need an 3-element set to be a basis and 1-element set to be a 1-circuit of N. But $E(N)$ can only have 3-elements. Thus there does not exist a matroid N whose truncation $T(N) \cong U_{0,1} \oplus U_{2,2}$. We were still able to find the excluded minors for $\mathcal{T}_{\mathcal{U}}$ and $\mathcal{T}_{\mathcal{P}}$. We noticed the set of excluded minors in $\mathcal{EX}(\mathcal{T}_{\mathcal{U}})$ and $\mathcal{EX}(\mathcal{T}_{\mathcal{P}})$ has the set of excluded minors in $\mathcal{EX}(\mathcal{U})$ and $\mathcal{EX}(\mathcal{P})$ as minors, respectively. So we have the following claim.

Theorem 4.12. *Let $\mathcal{M}$ be a minor-closed class of matroids whose set of excluded minors is $\mathcal{EX}(\mathcal{M})$. Let $\mathcal{T}_{\mathcal{M}}$ be the collection of matroids M such that $T(M) \in \mathcal{M}$. If there does not exist a matroid whose truncation is in $\mathcal{EX}(\mathcal{M})$, then, for all $N \in \mathcal{EX}(\mathcal{M})$, $N \oplus U_{1,1} \in \mathcal{EX}(\mathcal{T}_{\mathcal{M}})$.*

Proof. Suppose there does not exist a matroid whose truncation is in $\mathcal{EX}(\mathcal{M})$. Let N be a rank r excluded minor for $\mathcal{M}$. If there does not exist a matroid whose truncation is N, then, for all $N \in \mathcal{EX}(\mathcal{M})$, $N \oplus U_{1,1} \in \mathcal{EX}(\mathcal{T}_{\mathcal{M}})$. The matroid $U_{1,1,}$ is an 1-element independent set and we will label this 1-element as x. By Proposition 4.3, we know $|E(N \oplus U_{1,1})| = |E(N)| + |E(U_{1,1})| = |E(N)| + 1$, $\mathcal{I}(N \oplus U_{1,1}) = \{I \cup x : I \in \mathcal{I}(N),$ $x \in U_{1,1}\}$, $\mathcal{B}(N \oplus U_{1,1}) = \{B \cup x : B \in \mathcal{B}(N), x \in U_{1,1}\}$ and $r(N \oplus U_{1,1}) = r(N) + r(U_{1,1}) = r(N) + 1$. This implies the collection of all independent sets of $N \oplus U_{1,1}$ has cardinality at most $r + 1$ and the single element x is a coloop since x is contained in every bases of $N \oplus U_{1,1}$. We need to show $N \oplus U_{1,1}$ is indeed an excluded minor for $\mathcal{T}_{\mathcal{M}}$.

The matroid $T(N \oplus U_{1,1}) \notin \mathcal{M}$ since, by the definition of truncation, the collection of all independent sets of $T(N \oplus U_{1,1})$ has cardinality at most r. That is $\mathcal{I}(T(N \oplus U_{1,1})) = \mathcal{I}(N) \cup \{I \cup x : I \in \mathcal{I}(N) \text{ and } |I| \leq r - 1, x \in U_{1,1}\}$ since all the independent sets of N has at most r, by Proposition 2.12, the set $\{B \cup x : B \in \mathcal{B}(N),$ $x \in U_{1,1}\}$ are circuits (every proper subset of $B \cup x$ is independent in $T(N \oplus U_{1,1})$), and it follows from (I2) the set $\{B \cup x : B \in \mathcal{B}(N), x \in U_{1,1}\}$ contains all the independent sets of $T(N \oplus U_{1,1})$. The element x is not a coloop in $T(N \oplus U_{1,1})$ after all x is contained in circuits. Consider deleting x in $T(N \oplus U_{1,1})$, by the definition of deletion, all the independent sets of $T(N \oplus U_{1,1})\backslash x$ are all the independent sets of $T(N \oplus U_{1,1})$ that do not contain the element x. That is $\mathcal{I}(T(N \oplus U_{1,1})\backslash x) = \mathcal{I}(N)$. Thus $T(N \oplus U_{1,1})\backslash x \cong N$. Hence $T(N \oplus U_{1,1})$ has N as a minor. Therefore $N \oplus U_{1,1} \notin \mathcal{T}_{\mathcal{M}}$.

Consider contracting an element in $N \oplus U_{1,1}$. Assuming everything is well-defined, for all $e \in N \oplus U_{1,1}$, the matroid $N \oplus U_{1,1}/e \in \mathcal{T}_{\mathcal{M}}$ since by Proposition 2.6 and Proposition 2.12, $r(T(N \oplus U_{1,1}/e)) = r - 1$ because $r(N \oplus U_{1,1}/e) = r$. Hence $T(N \oplus U_{1,1}/e)$ does not have N as a minor since $r(N) = r$. Now consider deleting an element in $N \oplus U_{1,1}$. Again, assuming everything is well-defined, for all $e \in N \oplus U_{1,1}$, we will only be able to delete an element of the ground $E(N)$ since the remaining element not contained in $E(N)$ is the coloop x of $N \oplus U_{1,1}$. (Note, if N consist of only coloops, the Boolean algebra, then we will only be able to contract the elements of $N \oplus U_{1,1}$). Then, by the definition of deletion, the independent sets of $N \oplus U_{1,1} \backslash e$ are all independent sets of $N \oplus U_{1,1}$ that do not contain e. That is, $\mathcal{I}(N \oplus U_{1,1} \backslash e) = \{I \cup x : I \in \mathcal{I}(N)$ where $e \notin I$, $x \in U_{1,1}\}$. Then the independent sets of $T(N \oplus U_{1,1} \backslash e)$ are the independent sets of $N \oplus U_{1,1} \backslash e$ that has cardinality at most r since, by Proposition 2.6, $r(N \oplus U_{1,1} \backslash e) = r + 1$. That is, $\mathcal{I}(T(N \oplus U_{1,1} \backslash e)) = \{I \cup x : I \in \mathcal{I}(N)$ where $e \notin I$ and $|I| = r - 1$, $x \in U_{1,1}\}$. Thus $T(N \oplus U_{1,1} \backslash e)$ does not has N as a minor since $\mathcal{I}(T(N \oplus U_{1,1} \backslash e)) \neq \mathcal{I}(N)$ and $|E(T(N \oplus U_{1,1} \backslash e))| = |E(N)|$. Then $N \oplus U_{1,1} \backslash e$ and $T(N \oplus U_{1,1}/e)$ are in $\mathcal{M}$. Hence $N \oplus U_{1,1} \backslash e$ and $N \oplus U_{1,1}/e$ are in $\mathcal{T}_{\mathcal{M}}$, for all $e \in E(N \oplus U_{1,1})$. Therefore, $N \oplus U_{1,1} \in \mathcal{EX}(\mathcal{T}_{\mathcal{M}})$.

□

Proving Theorem 4.12, we have the following conjecture.

Conjecture 2. *Let $\mathcal{M}$ be a minor-closed class of matroids whose set of excluded minors is $\mathcal{EX}(\mathcal{M})$. Let $N \in \mathcal{EX}(\mathcal{M})$. Let $\mathcal{T}_{\mathcal{M}}$ be the collection of matroids M such that $T(M) \in \mathcal{M}$. There exist a matroid M' whose truncation $T(M') \cong N$ if and only if $N \oplus U_{1,1} \notin \mathcal{EX}(\mathcal{T}_{\mathcal{M}})$.*

Since there exist only two matroids P_4 and $U_{3,4}$ (see Figure 4.9) whose truncation $T(P_4), T(U_{3,4}) \in \mathcal{EX}(\mathcal{M}_{\mathbb{F}_2})$. Then Theorem 4.11 proves that the matroids P_4 and $U_{3,4}$ are indeed the only sets of excluded minors in $\mathcal{EX}(\mathcal{T}_{\mathcal{M}_{\mathbb{F}_2}})$.

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