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Excluded minors for nearly-paving matroids

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Excluded Minors for Nearly-Paving Matroids

A Thesis

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in

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Abstract

Matroids capture an abstract notion of independence that generalizes linear independence in linear algebra, edge independence in graph theory, as well as algebraic independence. Given a particular property of matroids, all the matroids that possess that property form a matroid class. A common research theme in matroid theory is to characterize matroid classes so that, given a matroid M, it is possible to determine whether or not M belongs to a given class. An excluded minor of a minor-closed matroid class $\mathcal{M}$ is a matroid N that is in a sense, minimal with respect to not being in $\mathcal{M}$. An attractive way to characterize a minor-closed matroid class $\mathcal{M}$ is to determine the complete list of excluded minors for $\mathcal{M}$. In this thesis, we study several fundamental minor-closed classes of matroids. One such class is the class of paving matroids. We define a new closely related minor-closed class of matroids called nearly-paving matroids, and we provide an excluded minor characterization for the class of nearly-paving matroids.

Acknowledgements

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Table of Contents

List of Figures

Chapter 1

Introduction to matroid theory

Matroid theory combines concepts from graph theory, linear and abstract algebra, combinatorics, and geometry. Matroids were introduced in a paper written by Hassler Whitney in 1935 [Oxl11]. Whitney formalized the definition of a matroid through the properties of dependence from similar graphs and matrices. Early on in the field, mathematicians Garrett Birkhoff, Saunders MacLane, B. L. Van der Waerden, and Richard Rado also studied matroids shortly after Whitney's paper. Birkhoff explored the flats of a matroid through a lattice-theoretic perspective, and MacLane demonstrated the connections between matroids to projective geometry [GM12]. In addition, Van der Waerden recognized commonalities between three properties of linear and algebraic dependence, and Rado's work related matroids to the transversals of a bipartite graph [GM12]. Though these mathematicians made contributions to matroid theory, the work of William Tutte was highly significant. Tutte did quite a bit of work in a variety of areas such as generalizing the notions of graph theory and connectivity to matroids [Oxl11]. Tutte also proved many theorems that characterized certain classes of matroids by their excluded minors [GM12]. Lastly, Gian-Carlo Rota was a mathematician who preferred to call matroids combinatorial pre-geometries and is known for his conjecture (now a theorem) on excluded minors [GM12]. These are just a few of the many important mathematicians who were involved in the devolopement of matroid theory. In fact, matroid theory is a very active area of research today. While many important results in matroid theory were established through 1960s, many of the major results in the field are from the 21st century.

Interestingly, there are many equivalent ways to define a matroid. This is one of the features that attracts many researchers to matroid theory. In this thesis, we will consider characterizing minor-closed classes of matroids through excluded minors. In particular, We will be studying the classes of nearly-uniform matroids and nearly-paving matroids. These are essentially matroids that are in some sense almost in the minor-closed class of uniform and paving matroids. Characterizing fundamental classes of matroids in terms of excluded minors is currently a very attractive area in matroid theory. As we mentioned, William Tutte contributed greatly in this area. Specifically, he proved that binary and regular matroids can be characterized through excluded minors. The work that we will demonstrate in the following chapters extends this line of research by characterizing a certain class of matroids by their excluded minors.

In Chapter 1, we introduce matroids in terms of three axioms and define important key terms. To build a better understanding of what a matroid is, we go through a few examples. These examples illustrate matroids from a geometric perspective and will help us learn how to distinguish the different components that make up a matroid by simply looking at the geometric object. In addition, we introduce two classes of matroids that will be the focus throughout this thesis. Namely, uniform matroids and paving matroids. In Chapter 2, we discuss how to construct new matroids from existing matroids by implementing operations like deletion and contraction. These operations can be performed multiple times to create a new matroid which we will call a minor of the given matroid. Furthermore, we define the concept of a minor-closed class $\mathcal{M}$ and we define what it means to be an excluded minor of a minor-closed class $\mathcal{M}$. In Chapter 3, we explore the idea of almost being in a minor-closed class $\mathcal{M}$. We call these matroids the nearly-$\mathcal{M}$ matroids. We will investigate nearly-$\mathcal{M}$ matroids for several specific matroid classes $\mathcal{M}$. We first consider $\mathcal{M}$ to be uniform, and prove that the class of nearly-uniform matroids is minor-closed. Hence, we investigate the excluded minors for nearly-uniform matroids. Chapter 4 then concludes with our major results. The focus in Chapter 4 is the study of the class of nearly-paving matroids, and we characterize this class by its excluded minors.

1.1 Definition of a matroid and terminology

In this section, we will define a matroid, introduce some terminology, and give examples of matroids. Much of the terminology in matroid theory is inherited from graph theory and linear alegbra. A *matroid* M is a pair $(E, \mathcal{I})$ where E is a finite set, called the *ground set*, and $\mathcal{I}$ is a collection of subsets of E, called *independent sets*, such that $\mathcal{I}$ satisfies the following axioms:

(I1) The empty set is in $\mathcal{I}$;

(I2) Every subset of an independent set is independent;

(I3) If I and J are independent sets with $|I| < |J|$, then there is some element $x \in J - I$ such that $I \cup \{x\}$ belongs to $\mathcal{I}$.

Throughout this thesis, we often denote a set as ab instead of $\{a, b\}$. We avoid using commas and curly brackets in our notation for the sake of cleanliness, since we will often be working with sets whose elements are also sets.

Example 1.1. Let $E = \{a, b, c\}$ and $\mathcal{I} = \{\emptyset, a, b, c, ab\}$. We determine whether or not the pair $(E, \mathcal{I})$ is a matroid.

By the definition of a matroid, we need to determine if $\mathcal{I}$ satisfies axioms (I1), (I2), and (I3). Clearly, axiom (I1) holds since $\emptyset \in \mathcal{I}$. However, the pair $(E, \mathcal{I})$ is not a matroid since axiom (I3) fails. Axiom (I3) is not satisfied since sets c and ab are independent, but neither sets ac or bc are in $\mathcal{I}$. However, if we add the set abc to $\mathcal{I}$, it will satisfy axiom (I3), but the pair $(E, \mathcal{I})$ will still not be a matroid since every proper subset of abc is not in $\mathcal{I}$, and therefore axiom (I2) is not satisfied. Now if we add every proper subset of abc to $\mathcal{I}$, then axiom (I2) holds. We leave it to the reader to check that axioms (I1), (I2), and (I3) are satisfied when $\mathcal{I} = \{\emptyset, a, b, c, ab, ac, bc, abc\}$. Therefore, adding subsets ac, bc, and abc to $\mathcal{I}$ will make the pair $(E, \mathcal{I})$ a matroid M.

With this example it is easy to determine what a non-matroid looks like and how we can simply add new sets to $\mathcal{I}$ to make the pair $(E, \mathcal{I})$ a matroid M. We will now introduce some terminology and properties that describe a matroid M in terms of independence.

By the independence axiom (I2), we know that subsets of independent sets of M remain independent. However, instead of listing all of the independent sets in a matroid M, we will often focus on the *maximal independent sets*. The maximal sets of $\mathcal{I}$ are the sets that are not contained in any other sets in $\mathcal{I}$. We define a *basis* B of M to be a maximal independent set. Let $\mathcal{B}(M)$, or $\mathcal{B}$ for short, denote the collection of all bases of M. It follows from axioms (I2) and (I3) that every basis of a matroid has the same number of elements. We state this formally in the following lemma.

Lemma 1.2. *If B_1 and B_2 are in $\mathcal{B}(M)$, then $|B_1| = |B_2|$.*

Proof. Suppose, to the contrary, that there exist two bases B_1 and B_2 such that $|B_1| < |B_2|$. By axiom (I3), there exists an element $x \in B_2 - B_1$, such that $B_1 \cup \{x\}$ is independent. So B_1 is not a maximal independent set, which is a contradiction. Thus, $|B_1| \geq |B_2|$. A similar argument shows that if $|B_2| < |B_1|$, a contradiction also arises. Hence, we conclude that $|B_2| \geq |B_1|$. Therefore, if B_1 and B_2 are in $\mathcal{B}(M)$, then $|B_1| = |B_2|$. □

An element that is in every basis of M is called a *coloop* of M. If x is a coloop in M, then for every independent set $I \subseteq E - \{x\}$, the set $I \cup \{x\}$ is independent. This is just one of several important properties of coloops. We will mention a few others later. It should be noted that some matroids contain coloops and some matroids do not. A set that is not independent is called *dependent*. A minimal dependent set in a matroid is called a *circuit* of M. That is, a circuit C is a dependent set such that any proper subset of C is independent. Hence, any circuit cannot properly contain another circuit since they are minimally dependent. We call a circuit of M having k elements a *$k-$circuit.* Let $\mathcal{C}(M)$, or $\mathcal{C}$ for short, be the collection of all circuits of M. Circuits of a matroid can be of different sizes. For example, a *loop* is a 1-element circuit. Note that a loop is in no basis of M since a loop is a dependent singleton. We must also mention that a coloop is in no circuit. A 2-element circuit is a *parallel class* of size 2. There are also 3-element circuits and 4-element circuits. The largest circuit that we can have in a rank r matroid is an $r + 1$-element circuits, also called a *spanning circuit.* We will encounter some examples illustrating how we depict these k-circuits geometrically very soon. For now, let's continue to discuss a few more important properties about matroids.

Given any subset A of the ground set E of a matroid M, the cardinality of the largest independent subset of A is the *rank* of A, denoted $r(A)$. So the *rank of the matroid* M, denoted $r(M)$ or r, is $r(E)$. That is, by Lemma 1.2, the rank of M is equal to the cardinality of any basis of M. It follows that r is a function from the subsets of E to the non-negative integers. It follows from what we have said that the rank function of a matroid is an increasing function. That is, if $A \subseteq B \subseteq E$, then $r(A) \leq r(B)$.

Loops and coloops have special properties with respect to the rank function of a matroid, as we see in the following proposition.

Proposition 1.3. *[GM12] Let M be a matroid on the ground set E with rank function r.*

(1) An element $x \in E$ is a loop if and only if for all $A \subseteq E$ with $x \notin A$, we have $r(A \cup \{x\}) = r(A)$.

(2) An element $x \in E$ is a coloop if and only if for all $A \subseteq E$ with $x \notin A$, we have $r(A \cup \{x\}) = r(A) + 1$.

We omit the proof of Proposition 1.3. The following example demonstrates how we can determine the bases, circuits, and rank function of a matroid given the independent sets.

Example 1.4. Let M be a matroid on $E = \{a, b, c\}$ with $\mathcal{I} = \{\emptyset, a, b, c, ab, ac, bc, abc\}$. We determine the bases, circuits, and rank function of M.

Notice the set $abc \in \mathcal{I}$ is a maximal independent set since the set abc is not properly contained in any other set in $\mathcal{I}$. So, abc is a basis of M. In fact, abc is the only basis of M. The elements a, b, and c are also in every basis of M, so a, b, and c are all coloops of M and are all independent sets of M. Thus, M does not contain any dependent sets. Furthermore, since all the elements in the ground set E are independent, M contains no circuits. Lastly, since $|abc| = 3$, we know the rank of M is 3.

One final matroid concept we need to introduce is a flat. Let M be a matroid on the ground set E. A *flat* F is a subset of E that is *rank-maximal.* In other words, if we take a flat F and add an element to F such that that element is not in F, the rank of F increases. We define this formally as the following: A subset $F \subseteq E$ is a *flat* if $r(F \cup \{x\}) > r(F)$ for any $x \notin F$ [GM12]. For all $X \subseteq E$, the *closure* of X in M, denoted $cl(X)$, is defined by $cl(X) = \{x \in E : r(X \cup \{x\}) = r(X)\}$. This concept of flats comes from matroid geometry. Some examples of flats are points, lines, planes, and higher-dimensional (hyper)planes of a geometry [GM12]. With that being said, we have flats of various ranks. For example, there is exactly one rank 0 flat of M. That is, if M contains loops, then the rank 0 flat is the set of all loops. If M contains no loops, then the rank 0 flat is $\emptyset$. Recall that loops do not increase the rank of a set since the set of loops has rank 0. Thus, loops are in every flat. A flat of rank $r(M) - 1$ in a matroid of rank $r(M)$ is called a *hyperplane* H. We state this formally as follows: A subset $H \subseteq E$

is a *hyperplane* if H is a flat of M and if $r(H) = r(M) - 1$ [GM12]. Hyperplanes are particularly significant flats in a matroid. We would also like to note the ground set E is always the unique flat of any matroid.

At this point, we have finally covered the basic components of matroids and terminology. While we have defined matroids in terms of their independent sets, it is also possible to equivalently define matroids in terms of their bases $\mathcal{B}$ or their circuits $\mathcal{C}$. These multiple perspectives of the same object is what makes matroids mathematically rich objects. In fact, matroids share a connection between several areas of mathematics. For instance, we can understand what a matroid is from the perspective of linear alegbra as a matrix.

Some, but not all, matroids consist of a pair $(E, \mathcal{I})$, where E is a finite set of vectors over a field $\mathbb{F}$ and $\mathcal{I}$ is the collection of all subsets of E that are linearly independent. This class of matroids is called the $\mathbb{F}$-*representable matroids*. In general, it is quite difficult to determine if a given matroid M is $\mathbb{F}$-representable, for some field $\mathbb{F}$, or for any field, for that matter. One way to depict an $\mathbb{F}$-representable matroid is by a matrix A over a field $\mathbb{F}$, where the columns of A form the ground set E of M. That is, we can obtain a matroid-dependence structure by the column dependencies of matrix A. We denoted $M[A]$ as the column dependence matroid defined on matrix A. Let's take a look at the following two examples showcasing a matroid drawn from a matrix.

Example 1.5. Let A be the following matrix over $\mathbb{R}$.

$$A = \begin{array}{c} \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 3 & 5 & 0 \end{pmatrix} \end{array}$$

Consider the column vectors $a = (1,0)$, $b = (0,1)$, $c = (1,3)$, $d = (2,5)$, and $e = (0,0)$ in $\mathbb{R}^2$. In this example we are going to look at the subsets of the column vectors that are linearly independent and linearly dependent. Recall that a set of vectors is *linearly dependent* if some non-trivial *linear combination* of the vectors is the zero vector [GM12]. Moreover, the vectors are linearly dependent if the amount of vectors are greater than the dimension of your vector space. Since matrix A lives in $\mathbb{R}^2$, any subset of three or more vectors is a linear dependent set. For example, the set of vectors abd is linearly dependent since there exists a non-trivial linear combination of the vectors that results in the zero vector. A linear combination of abd is $-2(1,0) - 5(0,1) + (2,5) = (0,0)$.

As we mentioned, every subset of three or more vectors is linearly dependent. With that being said, every single vector is linearly independent except e and every pair of vectors is linearly independent except the pairs including e. We leave it to the reader to verify that there does not exist a linear combination that results in the zero vector for every single vector except e and every pair of vectors except the pairs including e. Generally, this method of finding and listing the independent and dependent sets of the column vectors of a matrix is very tedious. However, matroid theory takes on a geometric approach to describe these sets. We will now show the process of drawing a geometric figure that illustrates the column vectors dependencies, i.e. a picture of a matroid for the column dependencies of matrix A. The goal is to take the vectors on this space and project them onto a space of dimension one less. Since we are in $\mathbb{R}^2$, we are going to project the vectors into $\mathbb{R}^1$. In general, rather than looking at a picture in $\mathbb{R}^n$, we can project the picture onto $\mathbb{R}^{n-1}$ and maintain the same dependence structure. So we can still see the same information without having to draw something in a higher dimensional space.

Let's first start by drawing the five column vectors of matrix A on a plane since they exist in $\mathbb{R}^2$. See Figure 1.1. Recall that e is the zero vector so e is a point on the plane.

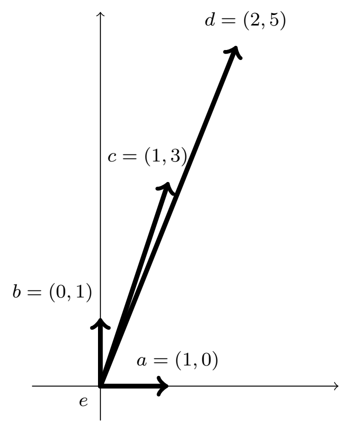


Figure 1.1: Vectors a, b, c, d, and e from matrix A.

For the next step, we need to take the vector space of dimension one less. In this case it is a line. So, we will take a line ℓ and position it in a way so that it passes through each vector. The line ℓ must be non-parallel to the set of vectors. See Figure 1.2. The intersection between the vectors and line ℓ is a picture of the resulting matroid. So, we just focus on the line with the four collinear points and ignore the original vectors. We can now stretch or shrink each non-zero vector until it meets the line we created. We should have a total four collinear points lying on line ℓ. Notice that e does not lie on ℓ since e is the zero vector and is a dependent set of size 1. That is, e is a loop. And that's it! What we have drawn, in Figure 1.3, is a geometric representation of the matroid $M[A]$ given by the matrix A. We must note that the order of the four collinear points of the resulting matroid in the goemetry does not matter.

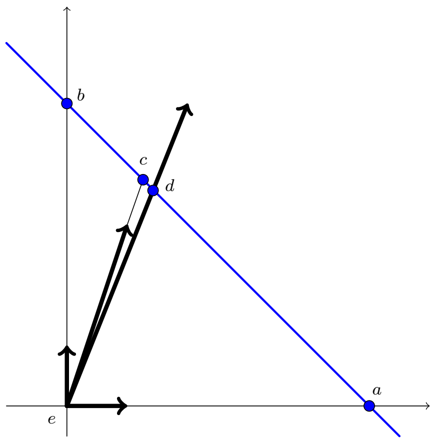


Figure 1.2: Vectors of matrix A with line ℓ.

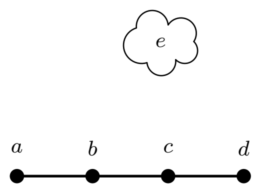


Figure 1.3: The matroid $M[A]$.

Now the question is, how do we distinguish the sets of linearly independent and linearly dependent vectors of matrix A in the geometry? Each column vector of A except e is translated into points in the geometry. Since e is the zero vector, we know that e is a dependent set of size 1 (a loop) and is translated into a cloud shape in the geometry.

Moreover, if three vectors of A are linearly dependent, then they are translated into three corresponding points that are collinear in the geometry. In addition, if two vectors of A are linearly independent, then they lie on line ℓ as distinct points and are independent in the matroid. As we mentioned before, the goal of this drawing procedure is to reduce dimension while preserving the same information. In Figure 1.1, the vectors lived in $\mathbb{R}^2$, so the rank of matrix A is 2. Notice in Figure 1.3, the matroid is drawn exactly one dimension less. It follows that the process of drawing matroids reduces the dimension by one. So we have, rank = dimension+1. Thus, the rank of matroid $M[A]$ is 2. Recall that the rank can also be found by the largest independent subset of $M[A]$.

We can also verify that $M[A]$ is indeed a matroid by the independence axioms (I1), (I2), and (I3). Since the $\emptyset \in \mathcal{I}$, axiom (I1) holds. Since every subset of an independent set of $M[A]$ is independent, axiom (I2) holds. Lastly, since we can take any two independent sets I and J of $M[A]$, where $1 = |I| < |J| = 2$, then we can find some element $x \in J - I$ such that $I \cup \{x\}$ is independent. Thus, axiom (I3) holds. Therefore $M[A]$ is a matroid.

In the next example, we'll see how this procedure works when we are dealing with a higher dimensional space.

Example 1.6. Let B be the following matrix over $\mathbb{R}$.

$$B = \begin{array}{c} \\ \end{array}\begin{matrix} a & b & c & d & e \end{matrix}$$

$$B = \begin{pmatrix} 1 & 0 & 2 & 1 & 4 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Similar to Example 1.5, we will draw the geometric figure that illustrates the column vectors dependencies of matrix B. Since the column vectors of B exist in $\mathbb{R}^3$, we draw the vectors in a three dimensional space. See Figure 1.4.

Next, we are taking the vector space of dimenson one less. In this case its a plane. So, we will take a plane P and position it so that it intersects through each vector. The plane P must be non-parallel to the set of vectors. We can now stretch or shrink each non-zero vector until it meets the plane P. The intersection between the vectors and the plane P is a picture of the resulting matroid. See Figure 1.5.

Let's distinguish the sets of linearly independent and linearly dependent vectors of matrix B in the geometry. Every single vector is linearly dependent, so each column

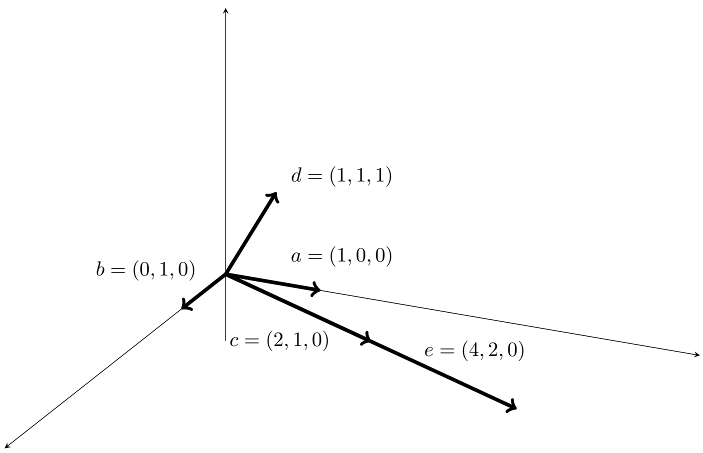


Figure 1.4: Vectors a, b, c, d, and e from matrix B.

vector of B is translated into points in the geometry. In addition, every pair of vectors is linearly independent except the pair ce. So every pair of vectors except ce lies on the plane P as distinct points and are independent in the matroid. Since ce is a pair of linearly dependent vectors of matrix B, ce is projected onto the plane P as a pair of multiple points in matroid $M[B]$. That is, ce is dependent set of size 2 and is translated into a pair of points on top of one another in the geometry. Moreover, every set of three vectors are linearly independent except the sets abc and abe. So every set of three vectors except abc and abe are translated into three corresponding points that are not collinear in the geometry. Since abc and abe are sets of three linearly dependent vectors of matrix B, abc and abe are projected onto the plane P as two three-point lines in geometry. Notice that abd, acd, ade, bcd, and bde are all sets of three vectors that are linearly independent in matrix B that span the whole vector space of B. In other words, these sets form a basis for the vector space of B. As we can see d is a vector that is contained in every basis. That is, d is a coloop and is projected onto the plane P as a distinct non-collinear point in the geometry of $M[B]$. Lastly, the rank of matroid $M[B]$ is 3 since we know the size of our basis is 3. See Figure 1.6.

It is important that we point out that we do not draw two-point line segments if they are the only two points on that line. For example, in Figure 1.6, the points a and d form a line segment between them, however, we do not draw this line. We avoid drawing two-point lines in the picture for the sake of cleanliness, but we still recognize

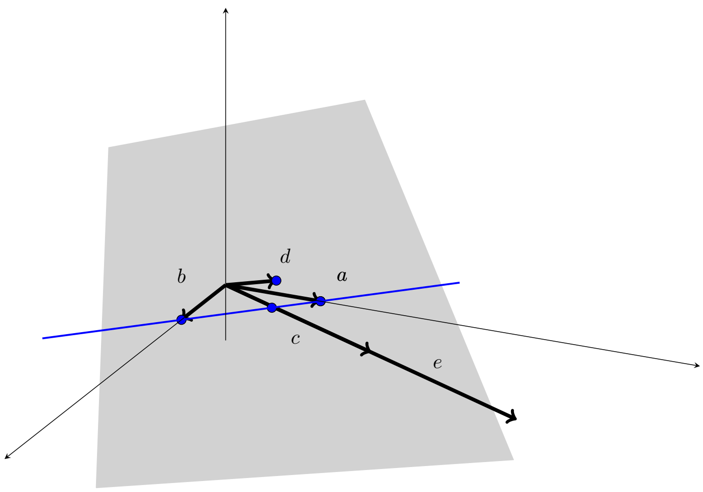


Figure 1.5: Vectors of matrix B with plane P.

the existence of the two-point lines.

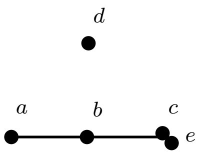


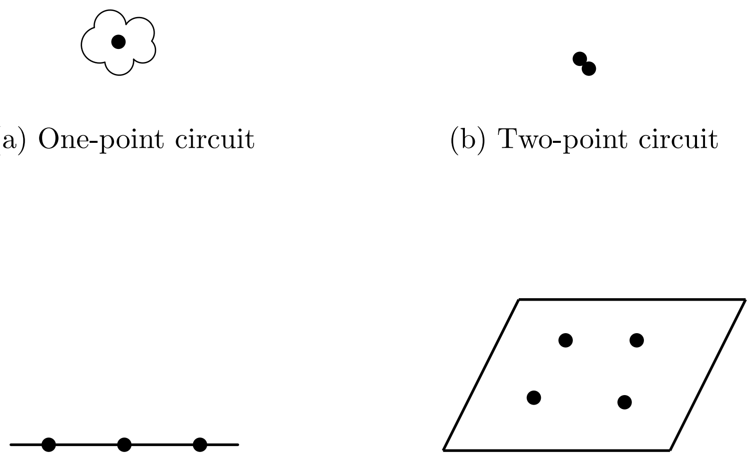
Figure 1.6: The matroid $M[B]$.

To verify that $M[B]$ is indeed a matroid, we must check that $M[B]$ satisfies the independence axioms (I1), (I2), and (I3). Since the $\emptyset \in \mathcal{I}$, axiom (I1) holds. Since every subset of an independent set of $M[B]$ is independent, axiom (I2) holds. Lastly, since we can take any two independent sets I and J of $M[B]$, where $|I| < |J|$, then we can find some element $x \in J - I$ such that $I \cup \{x\}$ is independent. Thus, axiom (I3) holds. Therefore $M[B]$ is a matroid. In fact, the notion of linear independence of vectors always satisfies axioms (I1), (I2), and (I3).

In both examples, matrix A and B are represented as matroids by drawing $M[A]$ and $M[B]$ as point-line incidence geometries. The process of drawing vectors in $\mathbb{R}^n$, then projecting them onto a space of dimension $n-1$ to arrive at a geometry is not something we will always be able to do. If $n \geq 4$, we will run into some trouble since we can't draw these vectors in $\mathbb{R}^4$ or greater. In fact, not all matroids come from vectors. There are non-representable matroids and you just can't look at them in terms of vectors. Nonetheless, any matroid can be viewed geometrically, and viewing matroids this way is often advantageous. Given any matroid M, we would like to be able to immediately see the geometry without requiring the context of vectors.

1.2 Matroid geometry

In Section 1.1, we saw how column vector dependencies of a matrix can be illustrated as a point-line incidence geometry to represent a matroid. However, it is much easier to determine the independent and dependent sets of a matroid just by looking directly at the geometry. Geometrically, the circuits of a matroid M to look like the following:



(a) One-point circuit (b) Two-point circuit

(c) Three-point circuit (d) Four-point circuit

Figure 1.7: In (a), we depict a one element dependent set as a cloud shape. In (b), we depict a two element dependent set as pair of points on top of each other. In (c), we depict a three element dependent set as three collinear points. In (d), we depict a four element dependent set as four coplanar points.

These geometric illustrations of the circuits will help us distinguish the dependent sets from the independent sets in a matroid when we refer to a geometric picture.

Example 1.7. Given the geometry of matroid $M[A]$ in Figure 1.3, where $E = \{a, b, c, d, e\}$, we determine the independent sets $\mathcal{I}$, bases $\mathcal{B}$, and circuits $\mathcal{C}$ of $M[A]$ directly from the geometry.

As we saw in Example 1.5, the element e is a dependent singleton and is therefore a loop, which is depicted by placing e inside a cloud. So, we know that $e \notin \mathcal{I}$. Since the matroid $M[A]$ has rank 2, the independent sets $\mathcal{I}$ cannot contain any subsets of E of size more than 2. The independent sets $\mathcal{I}$, of $M[A]$, can be characterized by the following:

- $\emptyset$ is independent.
- Each subset of size 1 is independent except e.
- Each subset of size 2 is independent: ab, ac, ad, bc, bd, and cd are all independent.

From here, we can easily find the bases of matroid $M[A]$ by gathering the maximal independent sets from $\mathcal{I}$. So $\mathcal{B} = \{ab, ac, ad, bc, bd, cd\}$. Lastly, the circuits of matroid $M[A]$ are the minimal subsets that we excluded from $\mathcal{I}$. Thus, $\mathcal{C} = \{e, abc, acd, abd, bcd\}$.

Now say we didn't know the geometry of a matroid, but we did know the ground set E and bases $\mathcal{B}$. Can we build the geometry of the matroid? Of course we can!

Example 1.8. We draw the geometry of matroid M, from Example 1.6, on finite set $E = \{a, b, c, d, e\}$ having bases $\mathcal{B} = \{abd, acd, ade, bcd, bde\}$.

Since we are given the bases of M, we know the maximal sets of $\mathcal{I}(M)$. We also know that for every basis $B \in \mathcal{B}$, every subset of B is independent. Now, the cardinality of any basis is equal to the rank of the matroid. In this case, for every $B \in \mathcal{B}$, $|B| = 3$, which implies $r(M) = 3$. Notice that the element d is contained in every set of $\mathcal{B}$. Recall that we call such an element a coloop. The sets that are not contained in some $B \in \mathcal{B}$ must be dependent. By observation, the set abc is not a basis, but every proper subset of abc is contained in some basis. That is, abc is not independent but every subset of abc is independent. Therefore abc is a minimally dependent set and is therefore a three-element circuit. So abc must be three collinear points in the geometry. Now, the set ce is not contained in any basis, but c is, and so is e. So every proper subset of ce is independent. Thus, ce is a a two-element circuit. So ce must be parallel class of size 2 in the geometry. We have completely determined what each element from the ground set is represented by in the point-line incidence geometry for the rank 3 matroid. See Figure 1.8.

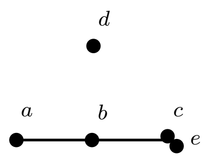


Figure 1.8: Matroid for Example 1.8.

In the next section, we introduce matroids that have certain features, and how these particular matroids that carry these features belong to a matroid class.

1.3 Classes of matroids

There are two fundamental classes of matroids we wish to consider. A *uniform matroid*, denoted $U_{r,n}$, where $r \leq n$, is a matroid where E is an n-element set and I is the collection of all subsets of E with at most r elements. Note that uniform matroids have rank r, and only circuits of size $r+1$. If we consider $r = 0$, the matroid $U_{0,n}$ has rank 0, and all circuits are of size 1. So $U_{0,n}$ is the matroid consisting of n loops. If we consider $r = 1$, the matroid $U_{1,n}$ is the rank 1 matroid in which all circuits are of size 2. So our uniform matroid is a bundle of n parallel points. Geometrically, we depict these matroids in Figure 1.9. Note that in the case of $U_{1,n}$, all the points should be thought of as occupying the same space. Also, we depict loops geometrically as points in a cloud.

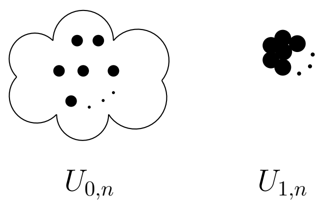


Figure 1.9: The uniform matroids $U_{0,n}$ and $U_{1,n}$.

We find the class of uniform matroids interesting to study since uniform matroids function as the building blocks of all matroids. For example, say we have a rank r matroid M. Any basis of M is $U_{r,r}$. In other words, the heart of every matroid has uniform matroids living in there. We will further study the many interesting characteristics of this class in Chapter 3.

The next class of matroids we wish to consider is the class of paving matroids. A *paving matroid* is a rank r matroid whose circuits are either of size r or $r+1$. The matroid $\mathcal{W}^3$, in Figure 1.10, is an example of paving matroid since it is rank 3 and all of its circuits are of size 3 or 4. Additionally, the class of uniform matriods is contained in the class of paving matroids since uniform matroids have $r+1$-element circuits, which is a subset of paving matroids. For example, the matroid $U_{1,2}$ is uniform, but it is also paving since $U_{1,2}$ has rank 1 with circuits of size 2. Studying the class of paving matroids is highly motivating since almost all matroids are paving. In 1970, mathematicians Henry Crapo and Gian-Carlo Rota mentioned in their Combinatorial Geometries book [CR70] that they "consider it likely that paving matroid would actually predominate in any asymptotic enumeration of geometries." We will continue exploring the class of paving matroids in Chapter 4.

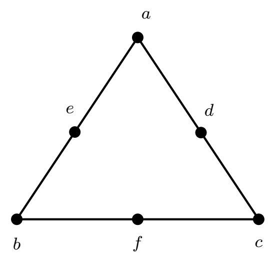


Figure 1.10: The matroid $\mathcal{W}^3$ is paving.

Chapter 2

Creating new matroids from existing matroids

In this chapter, we will discuss one way to create new matroids from existing matroids by using two important operations: *deletion* and *contraction*. These two operations remove an element from a matroid M and produces a smaller matroid called a *minor*. We will define these operations and demonstrate what happens geometrically under these operations.

2.1 Deletion and contraction

We begin by defining the deletion and contraction of an element from a matroid. Let M be a matroid on the ground set E with independent sets $\mathcal{I}$. If $e \in E$, where e is not a coloop, then the matroid obtained by *deleting* e, denoted $M \backslash e$, is the matroid with a ground set $E - \{e\}$ whose independent sets are those members of $\mathcal{I}$ that do not contain e. If $e \in E$, where e is not an loop, then the matroid obtained by *contracting* e, denoted M/e, is the matroid with ground set $E - \{e\}$ whose independent sets are formed by choosing all those members of $\mathcal{I}$ that contain e, and then removing e from each such set. An easy way to view the independents sets $\mathcal{I}$ of M under these two operations is as if the independent sets were split in two collections: independent sets that do not contain e and independent sets that contain e and then remove e from each set.

Now that we know what the independent sets are for $M \backslash e$ and M/e, we can discuss what the bases and circuits are for $M\backslash e$ and M/e. Let M be a matroid and e be an element that is neither a coloop nor a loop. Then the bases of the matroid $M\backslash e$ are the bases of M that do not contain e. The bases of the matroid M/e are the bases of M that contain e, and then removing e from each such basis. For circuits, there are a few different scenarios that can happen when deleting or contracting e. For deletion: If C is a circuit of M and $e \notin C$ then C is a circuit of $M\backslash e$. That is, e is not in any circuit of M so, the deletion of element e preserves the circuits of M in $M\backslash e$. If C is a circuit of M and $e \in C$ then C is not a circuit of $M\backslash e$. Recall that a circuit is a minimal dependent set so deleting e in circuit C destroys the circuit. The remaining elements from $C - e$ become independent sets in $M\backslash e$. For contraction: If C is a circuit of M and $e \notin C$ then C is a circuit of M/e. That is, e is not in any circuit of M so, the contraction of element e preserves the circuits of M in M/e. If C is a circuit of M and $e \in C$ then $C - e$ is a circuit of M/e such that $|C - e| = |C| - 1$.

Now, let's consider deleting and contracting multiple elements in a matroid. If we delete element a first and then contract element b, will we end up with the same matroid as if we contracted element b and then deleted element a? Let's take a look at the matroid from Example 1.5. The independent sets of matroid $M[A]$ are $\{\emptyset, a, b, c, d, ab, ac, ad, bc, bd, cd\}$. Note that element e is a loop and cannot be deleted. Consider $(M[A]\backslash a)/b$. When we first delete element a, the matroid $M[A]\backslash a$ has the independent sets of $M[A]$ that do not contain a. That is, $\{\emptyset, b, c, d, bc, bd, cd\}$. Now, after contracting element b, we arrive at matroid $(M[A]\backslash a)/b$. We find the independent sets of the matroid $(M[A]\backslash a)/b$ simply by taking the independent sets of $M[A]\backslash a$ that do contain b and then removing b from each such set. So, the matroid $(M[A]\backslash a)/b$ has ground set $\{c, d, e\}$ with independent sets $\{\emptyset, c, d\}$.

Next, consider $(M[A]/b)\backslash a$. When we contract element b, we find the independent sets of the matroid $M[A]/b$ by taking the independent sets of $M[A]$ that contain b and then removing b from each such set. Therefore, the independent sets of $M[A]/b$ are $\{\emptyset, b, ab, bc, bd\}$. Then deleting element a gives us matroid $(M[A]/b)\backslash a$ with ground set $\{c, d, e\}$ and independent sets $\{\emptyset, c, d\}$.

Notice that in this example, the matroids $(M[A]\backslash a)/b$ and $(M[A]/b)\backslash a$ are equal (they have the same ground set and the same independent sets). This leads us to believe that the operations of deletion and contraction commute. Indeed, this is the case, and we state this formally in the following lemma. We omit the straightforward proof which can be found in [GM12].

Lemma 2.1. *Let $a, b \in E$, the ground set of the matroid M. Assuming everything is well-defined, we have:*

(1) $(M\backslash a)\backslash b = (M\backslash b)\backslash a$*;*

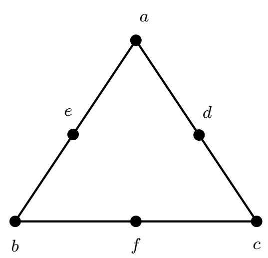
(2) $(M/a)/b = (M/b)/a$*;*

(3) $(M/a)\backslash b = (M\backslash b)/a$*.*

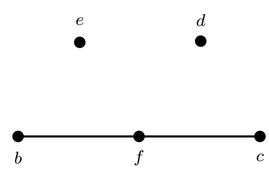
2.2 Deletion and contraction in matroid geometry

In this section, we develop techniques for determining the geometries corresponding to the matroids $M\backslash e$ and M/e. We will see how these geometries are related to the geometry of M. Geometrically, we delete e, if e is not a coloop, by simply removing the point e, and we keep in mind that in the resulting picture, we usually also remove any lines that no longer contain three distinct points. The geometry of the contraction of an element e is a bit more involved. In order to contract e, if e is not a loop, we will need a hyperplane H. Recall that a hyperplane is one dimension less than the matroid. When we contract e, we are projecting the remaining elements onto the hyperplane H along the lines between e and H.

Example 2.2. Figure 2.1 (a) shows a matroid M, where $a \in E(M)$ and a is not a coloop. Figure 2.1 (b) shows the matroid $M\backslash a$.



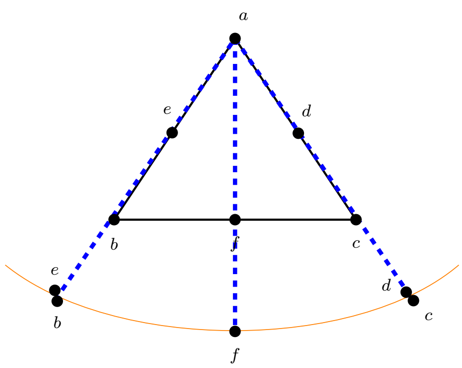
(a) The matroid M



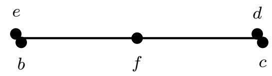
(b) The matroid $M\backslash a$

Figure 2.1: Removing point a leaves us with two-point lines be and cd.

Example 2.3. Figure 2.2 shows what happens in the matroid geometry when we contract a non-loop element. Consider the matroid M from Figure 2.1 (a). In Figure 2.2 (a), we see that when we contract the element a, we view this geometrically as projecting the remaining elements onto a hyperplane H along the lines between a and H. In fact, we can take the idea of sliding pearls down a string onto the hyperplane H. We represent the orange line as the hyperplane, the blue lines going from a to H as the strings, and the points as the pearls that are all sliding down to the hyperplane.



(a) Constructing M/a



(b) The matroid M/a

Figure 2.2: Contracting point a from the two 3-point lines aeb and adc in (a) become double points eb and dc in (b). Similarly, the two-point line af in (a) becomes a single point f in (b).

Notice that when we deleted a from M, the rank of matroid $M\backslash a$ remained the same. In fact, any element e we delete from matroid M, we have $r(M/e) = r(M)$. When we contracted a from M, the rank of matroid M/a reduced by 1. It follows that for any element e we contract from matroid M we have, $r(M/e) = r(M) - 1$.

2.3 Direct sums

An matroid operation that allows us to construct a new matroid from existing matroids is the binary operation of direct sum. Given two matroids, M_1 and M_2, we can combine them to produce a matroid $M_1 \oplus M_2$. This operation is defined as follows. Let M_1 and M_2 be matroids with disjoint ground sets E_1 and E_2. Then the *direct sum* $M_1 \oplus M_2$

is a matroid with ground set $E = E_1 \cup E_2$ and independent sets $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2)\}$. That is, an independent set of $M_1 \oplus M_2$ is formed by combining the elements from an independent set in matroid M_1 with an independent set in matroid M_2. Similarly, the bases of $M_1 \oplus M_2$ are of the form $B_1 \cup B_2$, where B_1 is a basis of matroid M_1 and B_2 is a basis of matroid M_2. It follows that the rank of $M_1 \oplus M_2$ is the sum of $r(M_1)$ and $r(M_2)$. Lastly, the circuits of $M_1 \oplus M_2$ are the circuits of M_1 combined with the circuits of M_2. That is, $\mathcal{C}(M_1 \oplus M_2) = \mathcal{C}(M_1) \cup \mathcal{C}(M_2)$.

2.4 Excluded minors

A *minor* N of a matroid M is any matroid obtained from M by performing a (possibly empty) sequence of deletions and contractions of elements of $E(M)$. Now, a collection $\mathcal{M}$ of matroids that is closed under the operations of deletion and contraction is called a *minor-closed* class . Additionally, a matroid N is an *excluded minor* for the minor-closed class $\mathcal{M}$ if $N \notin \mathcal{M}$, but every *proper* minor of N is in $\mathcal{M}$. That is, a matroid N is outside of the class and the deletions or contractions of any element of N produces a matroid that is in the class $\mathcal{M}$. The set of all excluded minors of a minor-closed class $\mathcal{M}$ of matroids is denoted $EX(\mathcal{M})$. It is possible to completely characterize a minor-closed class $\mathcal{M}$ by finding $EX(\mathcal{M})$. Such a characterization is a result of the form: A matroid M is in the class $\mathcal{M}$ if and only if M has no minor that is isomorphic to a matroid in $EX(\mathcal{M})$. We will now give some examples of minor-closed classes of matroids, together with their excluded minors.

Example 2.4. Let $\mathcal{G}$ be the class of all matroids with 6 elements or less.

This is a minor-closed class since any minor of a matroid in $\mathcal{G}$ will also have 6 or fewer elements. Now, the excluded minors for $\mathcal{G}$ is any matroid that has exactly 7 elements.

Example 2.5. Let $\mathcal{U}$ be the class of all uniform matroids.

This is a minor-closed class since all minors of any matroid in the class will be in $\mathcal{U}$. That is, any deletion or contraction of any element from any matroid in the class will also be in the class. It turns out that the matroid $U_{0,1} \oplus U_{1,1}$ is the only excluded minor for the class of uniform matroids.

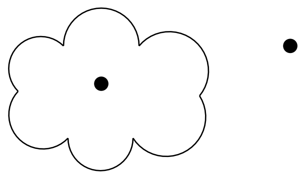


Figure 2.3: The matroid $U_{0,1} \oplus U_{1,1}$.

Example 2.6. Let $\mathcal{P}$ be the class of all paving matroids; that is, matroids M whose circuits all have size at least $r(M)$.

This is minor closed since any minor of a matroid in the class will be in the class. It turns out that the matroid $U_{0,1} \oplus U_{2,2}$ is the only excluded minor for the class of paving matroids.

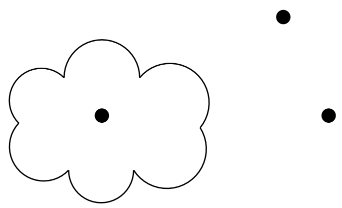


Figure 2.4: The matroid $U_{0,1} \oplus U_{2,2}$.

Chapter 3

Nearly-$\mathcal{M}$ matroids

In this chapter, we will focus on an investigation of matroids that are a deletion or contraction away from being in a minor-closed class $\mathcal{M}$. We formalize this idea as the class of *nearly*-$\mathcal{M}$ matroids. Specifically, we consider $\mathcal{M}$ to be the minor-closed class of uniform matroids and study the nearly-uniform matroids by asking the following two questions: Is the class of nearly-uniform matroids minor-closed? If so, what are the excluded minors?

3.1 Definition of nearly-$\mathcal{M}$ matroids

Definition 3.1. *Let $\mathcal{M}$ be a minor-closed class of matroids. The class $\mathcal{Z}$ is called the class of nearly-$\mathcal{M}$ matroids if, for all $M \in \mathcal{Z}$ and for all $e \in E(M)$, either $M\backslash e \in \mathcal{M}$ or $M/e \in \mathcal{M}$.*

One question that arises from this definition is: Is the class of nearly-$\mathcal{M}$ matroids minor-closed? The next result states that nearly-$\mathcal{M}$ matroids are indeed a minor-closed class.

Theorem 3.2. *Let $\mathcal{M}$ be a minor-closed class of matroids. Let $\mathcal{Z}$ be the collection of matroids such that for all $M \in \mathcal{Z}$, whenever $e \in E(M)$, either $M\backslash e$ or M/e is in $\mathcal{M}$. Then $\mathcal{Z}$ is a minor-closed class of matroids that contains $\mathcal{M}$.*

Proof. Since $\mathcal{M}$ is minor-closed, it follows that every matroid in $\mathcal{M}$ is also in $\mathcal{Z}$. Hence, $\mathcal{M} \subseteq \mathcal{Z}$. Let $M \in \mathcal{Z}$ and let $y \in E(M)$. If $M \in \mathcal{M}$, then the matroid minors $M\backslash y$

and M/y are in $\mathcal{M}$ since $\mathcal{M}$ is minor closed. Thus the minor $M\backslash y \in \mathcal{Z}$ and $M/y \in \mathcal{Z}$. Suppose $M \notin \mathcal{M}$ and the minor $M\backslash y \notin \mathcal{M}$. We need to determine if the minor $M\backslash y \in \mathcal{Z}$. To be a nearly-$\mathcal{M}$ matroid, we must check that whenever $e \in E(M)$, either $(M\backslash y)\backslash e$ or $(M\backslash y)/e$ is in $\mathcal{M}$. Deleting any element e of $M\backslash y$ results in minor $(M\backslash y)\backslash e$, which is in $\mathcal{M}$, since, by Lemma 2.1, we know that the operation of deletion is commutative and if the minor $M\backslash e \in \mathcal{M}$, then $(M\backslash e)\backslash y \in \mathcal{M}$ since $\mathcal{M}$ is minor closed. Moreover, contracting any element e of $M\backslash y$ results in minor $(M\backslash y)/e$, which is in $\mathcal{M}$, since, by Lemma 2.1, we know that the operation of contraction is commutative and if the minor $M/e \in \mathcal{M}$, then $(M/e)\backslash y \in \mathcal{M}$ since $\mathcal{M}$ is minor closed. Thus the minor $M\backslash y \in \mathcal{Z}$.

Now suppose $M \notin \mathcal{M}$ and the minor $M/y \notin \mathcal{M}$. We need to determine if the minor $M/y \in \mathcal{Z}$. To be a nearly-$\mathcal{M}$ matroid, we must check that whenever $e \in E(M)$, either $(M/y)\backslash e$ or $(M/y)/e$ is in $\mathcal{M}$. Deleting any element e of M/y results in minor $(M/y)\backslash e$, which is in $\mathcal{M}$, since, by Lemma 2.1, we know that the operation of deletion is commutative and if the minor $M\backslash e \in \mathcal{M}$, then $(M\backslash e)/y \in \mathcal{M}$ since $\mathcal{M}$ is minor closed. Moreover, contracting any element e of M/y results in minor $(M/y)/e$, which is in $\mathcal{M}$, since, by Lemma 2.1, we know that the operation of contraction is commutative and if the minor $M/e \in \mathcal{M}$, then $(M/e)/y \in \mathcal{M}$ since $\mathcal{M}$ is minor closed. Thus the minor $M/y \in \mathcal{Z}$. Therefore, $\mathcal{Z}$ is a minor-closed class of matroids that contains $\mathcal{M}$. □

3.2 Nearly-uniform matroids

In the previous section, we introduced the minor-closed class of nearly-$\mathcal{M}$ matroids. In this section, we will be studying the nearly-uniform matroids and provide an excluded minor characterization of the class.

By Theorem 3.2, we know that the class of nearly-uniform matroids is minor-closed. In fact, we are able to describe the matroids that are contained in class of nearly-uniform matroids by statifying three conditions. We formalize these conditions in Theorem 3.4. Before we state and prove Theorem 3.4, we will need the following lemma.

Lemma 3.3. *If a rank r matroid M has exactly one circuit C and $|C| = r$, then M has exactly one coloop $x \in E - C$.*

Proof. We know the matroid M has exactly one circuit C. Since, $|C| = r$, we also know that $r(C) = r - 1$. Suppose there exists an element $x \in E(M) - C$, where $r(C \cup x) = r$.

Then for all $e \in C$, $((C-e) \cup x) \in \mathcal{B}$. Now suppose there exists another element $y \in E(M) - C$, where $y \neq x$. Then for all $e \in C$, $|(C-e) \cup y| = r$. That is, if we remove an element e from our circuit and then union y to the circuit, the size of the circuit will remain the same. Hence, $|(C-e) \cup x \cup y| = r+1$. That is, the matroid M contains another circuit C', where $|C'| = r+1$, which contradicts our assumption that M has exactly one circuit. So, M cannot have exactly two elements not in the circuit. It follows that M has exactly one element not in C, call it x. Since x is in no circuit, it is equivalent to say that x is in every basis of M. It follows that x must be a coloop, by definition. □

We are now ready to state and prove Theorem 3.4.

Theorem 3.4. *A rank r matroid M is nearly-uniform if and only if M satisfies the following conditions:*

(i) M is a paving matroid.

(ii) M has at most one circuit C of size r.

(iii) If M has a circuit C of size r, then C is also a hyperplane.

Proof. ($\Rightarrow$) Let M be a nearly-uniform matroid of rank r. We need to show that M satisfies the following conditions: M is paving, M has at most one circuit C of size r, and if M has a circuit C of size r, then C is a hyperplane. Suppose M is not a paving matroid. Then M has a circuit C such that $|C| \leq r-1$. Consider the matroid $M \backslash y$. For any $y \in E-C$, if we delete element y, where y is not a coloop, the matroid $M \backslash y$ has rank r and contains the circuit C with $|C| \leq r-1$, which means $M \backslash y$ is not uniform. Next, consider the matroid M/y. If we contract element y, where y is not a loop, the matroid M/y has rank $r-1$ and contains the circuit C with $|C| \leq r-1$, which means M/y is not uniform. Therefore, this contradicts our assumption that M is nearly-uniform. From this, we can conclude that M must contain only circuits of size at least r. It follows that M must be a paving matroid.

Suppose M has two distinct circuits, C_1 and C_2, of size r. Consider the matroid $M \backslash e$. For any $e \in C_2 - C_1$, if we delete element e (note that x is not a coloop since coloops are not contained in circuits), the matroid $M \backslash e$ has rank r and contains the circuit C_1, which means $M \backslash e$ is not uniform since $|C_1| = r$. Next, consider the matroid M/e. If we contract element e, where e is not a loop, the matroid M/e has rank $r-1$ and contains the circuit C_2 with $C_2 - e$ where $|C_2 - e| = r - 1$, which is not uniform. It follows that if M has at least two circuits of size r, then M is not nearly uniform, which contradicts our assumption that M is nearly-uniform. Thus, M has at most one circuit of size r.

Now let M have one circuit C of size r and suppose C is not a hyperplane. Then the closure of C is $cl(C) = C \cup X$ for some set $X \subseteq E - C$. We show the set $C \cup X$ must contain another circuit, aside from C, of size r. Since $X \subset cl(C)$, then for some $e \in C$ and $x \in X$, the rank $r(C - e) = r - 1$ where $|C - e| = r - 1$, and the rank $r((C-e) \cup x) = r-1$ where $|(C-e) \cup x| = r$. This implies $C \notin (C-e) \cup x$ and $(C-e) \cup x$ is dependent. Thus, $(C-e) \cup x$ contains a circuit of size r that is not C. This contradicts that we have at most one circuit of size r. Thus, $cl(C) = C$ and so the circuit C is a hyperplane. Therefore, M satisfies the following conditions: M is paving, M has at most one circuit C of size r, and if M has a circuit of size r, then C is a hyperplane.

($\Leftarrow$) Let M be a rank r paving matroid with at most one circuit C of size r where C is also a hyperplane. Recall that a matroid N is nearly-uniform if for all $e \in E$, either $N \backslash e$ or N/e is uniform. Suppose M has no circuits of size r. Then M is uniform, which is nearly-uniform, since uniform matroids are minor-closed.

Suppose M has one circuit of size r where C is also a hyperplane. We show that the deletion of any element in C will result in a uniform matroid and the contraction of any element not in C will also result in a uniform matroid. Consider the matroid $M \backslash x$. For all $x \in C$, if we delete element x (note that x is not a coloop since coloops are not contained in circuits), $M \backslash x$ will result in a rank r paving matroid containing either circuits of size $r+1$ or no circuits. Then the matroid $M \backslash x$ is uniform, which is nearly-uniform. Next, consider the matroid M/y, for some $y \in E$. One possible scenario is that M has exactly one circuit C of size r and no other circuits. We may choose $y \in E - C$, then since M has one circuit C of size r and y is not contained in any circuit of M, by Lemma 3.3, we know that y is a coloop. If we contract element y, the matroid M/y has rank $r-1$ and contains a circuit of size r, which is uniform, and so N/y is nearly-uniform. The other possibility is that M has exactly one circuit C of size r and has another circuit C' of size $r+1$. Let $z \in C'$ such that $z \notin C$. If we contract element z, the matroid M/z has rank $r-1$ and contains circuit C of size r, which means M/z is uniform. Thus, M/z is nearly-uniform. Therefore, the rank r matroid M is nearly-uniform. $\square$

We now have a way to describe the types of matroids that are contained in the class of nearly-uniform matroids. Since the class of nearly-uniform matroids is minor-closed, let's explore the excluded minors for this class. We begin by looking at rank 1 matroids that are not contained in the class of nearly-uniform matroids. The following lemma states that there exists exactly one excluded minor of rank 1 for this class.

Lemma 3.5. *The matroid $U_{0,2} \oplus U_{1,1}$ is the only rank 1 excluded minor for the class of nearly-uniform matroids.*

Proof. The matroid $U_{0,2} \oplus U_{1,1}$ is certainly not in the class of nearly-uniform matroids since $U_{0,2} \oplus U_{1,1}$ contains two circuits of size 1, which fails the second condition of Theorem 3.4. Additionally, if one of these loops is deleted, the resulting matroid is rank 1 and still contains a loop, which is not uniform. However, deleting any non-coloop element of $U_{0,2} \oplus U_{1,1}$ results in $U_{0,1} \oplus U_{1,1}$, which is nearly-uniform. That is, for each $x \in E(U_{0,1} \oplus U_{1,1})$, either $(U_{0,1} \oplus U_{1,1})\backslash x$ or $(U_{0,1} \oplus U_{1,1})/x$ is uniform. Moreover, contracting any non-loop element of $U_{0,2} \oplus U_{1,1}$ results in $U_{0,2}$, which is nearly-uniform, since $U_{0,2}$ is a rank 0 matroid with two circuits of size 1, which is uniform. Therefore, every proper minor of $U_{0,2} \oplus U_{1,1}$ is in the class of nearly-uniform matroids. It follows that $U_{0,2} \oplus U_{1,1}$ is an excluded minor for the class of nearly-uniform matroids.

Let N be a rank 1 excluded minor for nearly-uniform matroids. We know N must contain loops since N cannot be uniform. Suppose N has exactly one loop, ℓ. Then for all $x \in E(N) - \{\ell\}$, N/x is uniform. Also, $N\backslash \ell$ is uniform. Thus, N is nearly-uniform, which contradicts our assumption that N is an excluded minor for nearly-uniform matroids. Now suppose N has at least three loops $\ell_1, \ell_2, \ell_3, ..., \ell_k$. Then the minor, $N\backslash \ell_1$, of N obtained by deleting ℓ_1 is not nearly-uniform, since $(N\backslash \ell_1)\backslash \ell_2$ contains the loop ℓ_3 and is therefore, not uniform. Note that we could not contract ℓ_2, since ℓ_2 is a loop. It follows that N must have exactly two loops, ℓ_1 and ℓ_2. Suppose N has an element x that is not a loop and is not a coloop. Then $N\backslash x$ has rank 1 and also has two loops, ℓ_1 and ℓ_2. Hence, $(N\backslash x)\backslash \ell_1$ is not a uniform matroid, which implies $N\backslash x$ is not nearly-uniform. That is, N has a minor that is not nearly-uniform, which contradicts our assumption that N is an excluded minor for nearly-paving matroids. It now follows that every element of N that is not a loop, must be a coloop. Since N is rank 1, N must have exactly one non-loop element, which must be a coloop. That is, $N \cong U_{0,2} \oplus U_{1,1}$. □

The next lemma states that there exists exactly one excluded minor of rank 2 for the class of nearly-uniform matroids. .

Lemma 3.6. *The matroid* $U_{0,1} \oplus U_{2,2}$ *is the only rank* 2 *excluded minor for the class of nearly-uniform matroids.*

Proof. The matroid $U_{0,1} \oplus U_{2,2}$ is certainly not in the class of nearly-uniform matroids since $U_{0,1} \oplus U_{2,2}$ has a circuit of size 1 and two coloops, which fails the first and second conditions of Theorem 3.4. Additionally, if one of the coloops is contracted, the resulting matroid has rank 1 with a loop, which is not uniform. However, deleting any non-coloop element of $U_{0,1} \oplus U_{2,2}$ results in $U_{2,2}$, which is uniform, and therefore nearly-uniform. Moreover, contracting any non-loop element of $U_{0,1} \oplus U_{2,2}$ results in $U_{0,1} \oplus U_{1,1}$, which is nearly-uniform. That is, for each $x \in E(U_{0,1} \oplus U_{1,1})$, either $(U_{0,1} \oplus U_{1,1})\backslash x$ or $(U_{0,1} \oplus U_{1,1})/x$ is uniform. Therefore, every proper minor of $U_{0,1} \oplus U_{2,2}$ is in the class of nearly-paving matroids. It follows that $U_{0,1} \oplus U_{2,2}$ is an excluded minor for the class of nearly-uniform matroids.

Let N be a rank 2 excluded minor for nearly-uniform matroids. We know N must contain loops since N cannot be uniform. Suppose N has at least two loops $\ell_1, \ell_2, \ell_3, ..., \ell_k$. Then the minor, $N\backslash \ell_1$, of N obtained by deleting ℓ_1 is not nearly-uniform, since for all $x \in E(N) - \{\ell_1\}$, the matroid $(N\backslash \ell_1)/x$ contains the loop ℓ_2 and is therefore, not uniform. Note that we could not contract ℓ_2, since ℓ_2 is a loop. It follows that if N contains any loops, it must have exactly one loop, ℓ. Suppose N contains a loop ℓ and a non-coloop element x. If x is not a coloop, then $N\backslash x$ has rank 2 and also has loop ℓ. Since $r(N\backslash x) > 0$, there exists an element $e \in N\backslash x$ that is distinct from ℓ. So, if we delete element e from $N\backslash x$, we obtain the rank 2 matroid $(N\backslash x)\backslash e$, which still contains loop ℓ, implying that $(N\backslash x)\backslash e$ is not uniform, and so $N\backslash x$ is not nearly-uniform. Similarly, if we contract the element e from $N\backslash x$, we obtain the rank 1 matroid $(N\backslash x)/e$, which still contains loop ℓ. Hence, $(N\backslash x)/e$ is not uniform, which implies $N\backslash x$ is not nearly-uniform. Therefore, if N has a loop ℓ and a non-coloop x, then N has minors that are not nearly-uniform, which contradicts our assumption that N is an excluded minor for nearly-uniform matroids. From this, we conclude that every element of N that is not a loop, must be a coloop. Since N is rank 2, N must have exactly two non-loop elements, which must be coloops. That is, $N \cong U_{0,1} \oplus U_{2,2}$. □

Before we start looking at excluded minors of rank 3 or higher, we must mentioned that the circuits of these matroids have certain qualities. We describe these qualities in next four lemmas.

Lemma 3.7. *Let N be a rank $r \geq 3$ excluded minor for nearly-uniform matroids. Then N has no circuit of size $k \leq r-1$.*

Proof. Suppose, to the contrary, that such a matroid N has a circuit C of size $k \leq r-1$. For any element $x \in C$, consider the minor N/x of N. We know $r(N/x) = r-1$ and circuit $C-x$ of N/x has size at most $r-2$. However, for any element $y \notin C-x$, the matroid $(N/x)\backslash y$ has rank $r-1$ and has circuit $C-x$ of size at most $r-2$, which means $(N/x)\backslash y$ is not uniform. Similary, $(N/x)/y$ has rank $r-2$ with a circuit $C-x$ of size at most $r-2$, which means $(N/x)/y$ is also not uniform. Therefore, N/x is not nearly-uniform as it contains an element y that cannot be deleted nor contracted to produce a uniform matroid. It follows that N is not an excluded minor for nearly-uniform matroids, which is a contradiction. □

Lemma 3.8. *Let N be a rank $r \geq 3$ excluded minor for nearly-uniform matroids. Then N has at least two circuits of size r.*

Proof. By Lemma 3.7, the matroid N must have at least one circuit of size r. Suppose C is the only circuit in N with $|C| = r$. Then for any element $x \in C$, the matroid $N\backslash x$ has rank r and contains no circuits, which means $N\backslash x$ is uniform. Similarly, for any element $y \notin C$, the matroid N/x has rank $r-1$ and still contains circuit C, where $|C| = r$, which means N/x is uniform. Therefore N is nearly-uniform, which contradicts our assumption that N is an excluded minor for nearly-uniform matroids. It follows that N must have at least two circuits of size r.

Lemma 3.9. *Let N be a rank $r \geq 3$ excluded minor for nearly-uniform matroids. If N has two distinct circuits, C_1 and C_2, with $|C_1| = |C_2| = r$, then $C_1 \cap C_2 = \emptyset$.*

Proof. Suppose, to the contrary, that N has two distinct circuits C_1 and C_2 with $|C_1| = |C_2| = r$, but that $C_1 \cap C_2 \neq \emptyset$. Let $x \in C_1 \cap C_2$, and consider the minor N/x of N. We know $r(N/x) = r-1$ and circuits $|C_1'| = |C_2'| = r-1$ where $C_1' = C_1 - x$ and $C_2' = C_2 - x$. However, for any element $y \in C_2'$, the matroid $(N/x)\backslash y$ has rank $r-1$ and has circuit

C_1' where $|C_1'| = r-1$, which means $(N/x)\backslash y$ is not uniform. Additionally, $(N/x)/y$ has rank $r-2$ and has circuits C_1' and C_2'', where $C_2'' = C_2' - y$, with $|C_1'| = r-1$ and $|C_2''| = r-2$. This means $(N/x)/y$ is not uniform. Therefore, N/x is not nearly-uniform as it contains an element y that cannot be deleted nor contracted to produce a uniform matroid. It follows that N is not an excluded minor for nearly-uniform matroids, which is a contradiction. □

Lemma 3.10. *Let N be a rank $r \geq 3$ excluded minor for nearly-uniform matroids. If N has two circuits C_1 and C_2, with $|C_1| = |C_2| = r$ and $C_1 \cap C_2 = \emptyset$, then $E(N) - (C_1 \cup C_2) = \emptyset$.*

Proof. Suppose, to the contrary, that N has two circuits C_1 and C_2, with $|C_1| = |C_2| = r$ where $C_1 \cap C_2 = \emptyset$ but that $E(N) - (C_1 \cup C_2) \neq \emptyset$. Let $x \in E(N) - (C_1 \cup C_2)$. Then consider the minor $N\backslash x$ of N. We know $r(N\backslash x) = r$ and $|C_1| = |C_2| = r$. However, for any element $y \in C_1$, $(N\backslash x)\backslash y$ contains the circuit C_2, where $|C_2| = r$, and $r(N\backslash x)\backslash y = r$, which is not uniform. Likewise, $(N\backslash x)/y$ has rank $r-1$ with $|C_1| = r-1$ and $|C_2| = r$, which is not uniform. Therefore, $N\backslash x$ is not nearly-uniform as it contains an element y that cannot be deleted nor contracted to produce a uniform matroid. It follows that N is not an excluded minor for nearly-uniform matroids, which is a contradiction. □

The next lemma involves a matroid operation called truncation. Ultimately, truncating a matroid M decreases the rank by one while preserving most of the circuits and independent sets of M. A matroid N is called a single-element extension of a matroid M by an element e if $N\backslash e = M$. A single-element extension of a matroid M by an element e is called a *free extension* if, for all flats F of M:

$$r(F \cup e) = \begin{cases} r(F) & \text{if } F = E(M); \\ r(F)+1 & \text{if } F \neq E(M). \end{cases}$$

We will use the notation $M+e$ to denote the free extension of M by the element e. The truncation $T(M)$ of a matroid M is obtained by performing a free extension of M by an element e, and then contracting e. That is, $T(M) = (M+e)/e$. The circuits of $T(M)$ are $\mathcal{C}(T(M)) = \{C : C \text{ is a nonspanning circuit of } M\} \cup \{B : B \text{ is a basis of } M\}$. Moreover, the basis of $T(M)$ are $\mathcal{B}(T(M)) = \{B - x : B \in \mathcal{B}(M) \text{ and } x \in B\}$.

Example 3.11. Let M be the matroid in Figure 3.1 with ground set $E = \{a, b, c, d, e, f\}$.

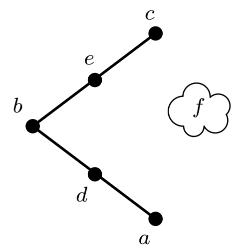


Figure 3.1: Rank 3 matroid M for Example 3.11.

To truncate M, we freely extend by element x, and then contract x. Geometrically, we can see the free extension by x in Figure 3.2 (a) and the subsequent contraction of x in Figure 3.2 (b). Since we contracted x, $T(M)$ has rank $r(M) - 1$. That is, $r(T(M)) = 2$. Now, the spanning circuits of M turn into basis in the matroid $T(M)$. Additionally, all the bases of M become spanning circuits in $T(M)$. See Figure 3.2 (b).

(a) The free extension of M by element x. (b) Rank 2 matroid $T(M)$.

Figure 3.2: The truncation $T(M)$ of a matroid M.

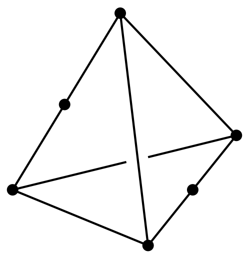
We can repeat this operation multiple times until we reach the k^{th}-truncation $T^k(M)$ of a matroid M. So, for example $T^2(M) = T(T(M))$. Finally, we have all the tools we need to establish the excluded minors of rank $r \geq 3$ for the class of nearly-uniform matroids.

Lemma 3.12. *Let N be a rank $r \geq 3$ excluded minor for nearly-uniform matroids. Then $N \cong T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r})$, the $(r-2)^{th}$ truncation of $U_{r-1,r} \oplus U_{r-1,r}$.*

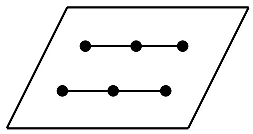
Proof. We know that the matroid $T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r})$ has rank $2r-2$ with two circuits of size r. We note that when $r \geq 3$, $2r-2$ is strictly greater than r. Now, the matroid $T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r})$ is certainly not in the class of nearly-uniform matroids since deleting any element does not decrease the rank and does not eliminate all circuits of size r. Also, contracting any element will decrease the rank to $2r-3$ but will also produce a circuit of size $r-1$ (note: when $r \geq 3$, $2r-3$ is strictly greater than $r-1$). However, deleting any element e of $T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r})$ results in the minor $(T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r}))\backslash e$, which is nearly-uniform. That is, for each $x \in E((T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r}))\backslash e)$, either $((T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r}))\backslash e)\backslash x$ or $((T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r}))\backslash e)/x$ is uniform. Moreover, contracting any element e of $T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r})$ results in the minor $(T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r}))/e$, which is nearly-uniform. That is, for each $y \in E((T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r}))/e)$, either $((T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r}))/e)\backslash y$ or $((T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r}))/e)/y$ is uniform. Therefore, every proper minor of $T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r})$ is in the class of nearly-uniform matroids. It follows that $T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r})$ is an excluded minor for the class of nearly-uniform matroids.

Let N be a rank $r \geq 3$ excluded minor for nearly-uniform matroids. We know N must contain circuits of size r or less since N cannot be uniform. By Lemma 3.8, we know we must have at least two circuits of size r. It follows by Lemma 3.9, that N has two distinct circuits, C_1 and C_2, where $C_1 \cap C_2 = \emptyset$. Furthermore, by Lemma 3.10, we know that the ground set $E(N)$ does not contain any other elements except the elements in circuits C_1 and C_2. Notice that if we have two disjoint circuits of size r, the rank of the matroid is $2r-2$. However, these circuits exist in a rank r matroid. Thus, $N \cong T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r})$. □

Consider the matroid $T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r})$ in Lemma 3.12, where $r \geq 3$. When $r = 3$, we have the matroid $T(U_{2,3} \oplus U_{2,3})$. Notice $U_{2,3} \oplus U_{2,3}$ is a rank 4 matroid with two disjoint circuits of size 3. That is, in rank 4, we will have two skew three-point lines. See Figure 3.3(a). After truncating, the two three-point lines now exist in rank 3. So, the circuits of $U_{2,3} \oplus U_{2,3}$ will become basis in $T(U_{2,3} \oplus U_{2,3})$. That is, the two skew three-point lines of $U_{2,3} \oplus U_{2,3}$ will now span a plane in the matroid $T^{(}U_{2,3} \oplus U_{2,3})$. See Figure 3.3(b).



(a) The matroid $U_{2,3} \oplus U_{2,3}$



(b) The matroid $T(U_{2,3} \oplus U_{2,3})$

Figure 3.3: The truncation of matroid $U_{2,3} \oplus U_{2,3}$.

The following theorem describes all of the excluded minors for the class of nearly-uniform matroids.

Theorem 3.13. *Let N be a rank r excluded minor for nearly-uniform matroids. Then either:*

(i) $r = 1$ *and* $N \cong U_{0,2} \oplus U_{1,1}$*; or*

(ii) $r = 2$ *and* $N \cong U_{0,1} \oplus U_{2,2}$*; or*

(iii) $r \geq 3$ *and* $N \cong T^{(r-2)}(U_{r-1,r} \oplus U_{r-1,r})$.

Proof. Statement (i) follows from Lemma 3.5. Statement (ii) follows from Lemma 3.6. Statement (iii) follows from Lemma 3.7, 3.8, 3.9, 3.10, and 3.12. □

Chapter 4

Excluded minors for nearly-paving matroids

In the last chapter, we determined the excluded minors for nearly-uniform matroids. In this chapter, we will study the class of nearly-paving matroids and determine the excluded minors for this class.

Theorem 4.1. *Let N be a rank r excluded minor for nearly-paving matroids. Then either:*

(i) $r = 2$ *and* $N \cong U_{0,2} \oplus U_{2,2}$*;*

(ii) $r = 3$ *and* $N \cong U_{0,1} \oplus U_{3,3}$*;*

(iii) $r = 3$ *and* $N \cong U_{1,1} \oplus U_{1,2} \oplus U_{1,2}$*; or*

(iv) $r \geq 4$ *and* $N \cong T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1})$*.*

4.1 Results

To prove Theorem 4.1, we will first show that each listed matroid is not contained in the class of nearly-paving matroids and then we will show, through a sequence of lemmas, that any excluded minor for nearly-paving matroids must be one of the matroids listed in Theorem 4.1.

Lemma 4.2. *The matroid* $U_{0,2} \oplus U_{2,2}$ *is the only rank* 2 *excluded minor for the class of nearly-paving matroids.*

Proof. The matroid $U_{0,2} \oplus U_{2,2}$ is certainly not in the class of nearly-paving matroids. Indeed, $U_{0,2} \oplus U_{2,2}$ contains two circuits of size 1, and if one of these loops is deleted, the resulting matroid is rank 2 and still contains a loop, which means it is not paving. However, deleting any non-coloop element of $U_{0,2} \oplus U_{2,2}$ results in $U_{0,1} \oplus U_{2,2}$, which is nearly-paving. That is, for each $x \in E(U_{0,1} \oplus U_{2,2})$, either $(U_{0,1} \oplus U_{2,2})\backslash x$ or $(U_{0,1} \oplus U_{2,2})/x$ is paving. Moreover, contracting any non-loop element of $U_{0,2} \oplus U_{2,2}$ results in $U_{0,2} \oplus U_{1,1}$, which is nearly-paving, since $U_{0,2} \oplus U_{1,1}$ is a rank 1 matroid with two circuits of size 1, which is paving. Therefore, every proper minor of $U_{0,2} \oplus U_{2,2}$ is in the class of nearly-paving matroids. It follows that $U_{0,2} \oplus U_{2,2}$ is an excluded minor for the class of nearly-paving matroids.

Let N be a rank 2 excluded minor for nearly-paving matroids. We know N must contain loops since N cannot be paving. Suppose N has exactly one loop, ℓ. Then for all $x \in E(N) - \{\ell\}$, N/x is paving. Also, $N\backslash \ell$ is paving. Thus, N is nearly-paving, which contradicts our assumption that N is an excluded minor for nearly-paving matroids. Now suppose N has at least three loops $\ell_1, \ell_2, \ell_3, ..., \ell_k$. Then the minor of N obtained by deleting ℓ_1, called $N\backslash \ell_1$, is not nearly-paving since $(N\backslash \ell_1)\backslash \ell_2$ contains the loop ℓ_3 and is therefore, not paving. Note that we could not contract ℓ_2, since ℓ_2 is a loop. It follows that N must have exactly two loops, ℓ_1 and ℓ_2. Suppose N has an element x that is not a loop and is not a coloop. Then $N\backslash x$ has rank 2 and also has two loops, ℓ_1 and ℓ_2. Hence, $(N\backslash x)\backslash \ell_1$ is not a paving matroid, which implies $N\backslash x$ is not nearly-paving. That is, N has a minor that is not nearly-paving, which contradicts our assumption that N is an excluded minor for nearly-paving matroids. It now follows that every element of N that is not a loop, must be a coloop. Since N is rank 2, N must have exactly two non-loop elements, which must both be coloops. That is, $N \cong U_{0,2} \oplus U_{2,2}$. □

We will now prove that there are exactly two rank 3 excluded minors for the class of nearly-paving matroids.

Lemma 4.3. *The matroids* $U_{0,1} \oplus U_{3,3}$ *and* $U_{1,1} \oplus U_{1,2} \oplus U_{1,2}$ *are the only two rank* 3 *excluded minors for the class of nearly-paving matroids.*

Proof. First, we show that $U_{0,1} \oplus U_{3,3}$ and $U_{1,1} \oplus U_{1,2} \oplus U_{1,2}$ are excluded minors for nearly-paving matroids. The matroid $U_{0,1} \oplus U_{3,3}$ is certainly not in the class of nearly-paving matroids since $U_{0,1} \oplus U_{3,3}$ has a circuit of size 1 and three coloops, and if one of these coloops is contracted, the resulting matroid is rank 2 with a loop, which is not paving. However, deleting any non-coloop element of $U_{0,1} \oplus U_{3,3}$ results in $U_{3,3}$, which is paving, and therefore nearly-paving. Moreover, contracting any non-loop element of $U_{0,1} \oplus U_{3,3}$ results in $U_{0,1} \oplus U_{2,2}$, which is nearly-paving. That is, for each $x \in E(U_{0,1} \oplus U_{2,2})$, either $(U_{0,1} \oplus U_{2,2})\backslash x$ or $(U_{0,1} \oplus U_{2,2})/x$ is paving. Therefore, every proper minor of $U_{0,1} \oplus U_{3,3}$ is in the class of nearly-paving matroids. It follows that $U_{0,1} \oplus U_{3,3}$ is an excluded minor for the class of nearly-paving matroids.

The matroid $U_{1,1} \oplus U_{1,2} \oplus U_{1,2}$ is also not in the class of nearly-paving matroids. Indeed, $U_{1,1} \oplus U_{1,2} \oplus U_{1,2}$ has two circuits of size 2, and if an element from one of these circuits is deleted, the resulting matroid is rank 3 and still contains a 2-circuit, which means that it is not paving. However, deleting any non-coloop element of $U_{1,1} \oplus U_{1,2} \oplus U_{1,2}$ results in $U_{1,2} \oplus U_{2,2}$, which is nearly-paving. That is, for each $y \in E(U_{1,2} \oplus U_{2,2})$, either $(U_{1,2} \oplus U_{2,2})\backslash y$ or $(U_{1,2} \oplus U_{2,2})/y$ is paving. Moreover, contracting any non-parallel element of $U_{1,1} \oplus U_{1,2} \oplus U_{1,2}$ results in $U_{1,2} \oplus U_{1,2}$, which is nearly-paving, since $U_{1,2} \oplus U_{1,2}$ is a paving matroid of rank 2 with two circuits of size 2. Additionally, contracting any non-coloop element of $U_{1,1} \oplus U_{1,2} \oplus U_{1,2}$ results in $U_{0,1} \oplus U_{1,1} \oplus U_{1,2}$, which is nearly-paving. That is, for each $z \in E(U_{0,1} \oplus U_{1,2} \oplus U_{1,2})$, either $(U_{0,1} \oplus U_{1,2} \oplus U_{1,2})\backslash z$ or $(U_{0,1} \oplus U_{1,2} \oplus U_{1,2})/z$ is paving. Therefore, every proper minor of $U_{1,1} \oplus U_{1,2} \oplus U_{1,2}$ is in the class of nearly-paving matroids. It follows that $U_{1,1} \oplus U_{1,2} \oplus U_{1,2}$ is an excluded minor for the class of nearly-paving matroids.

Now, we will show that if N is a rank 3 excluded minor for nearly-paving matroids, either N has loops and $N \cong U_{0,1} \oplus U_{3,3}$, or N is loopless and $N \cong U_{1,1} \oplus U_{1,2} \oplus U_{1,2}$. Let N be a rank 3 excluded minor for nearly-paving matroids. We know N must contain circuits of size at most 2 since N cannot be paving. Suppose N has at least two loops $\ell_1, \ell_2, \ell_3, ..., \ell_k$. Then the minor of N obtained by deleting ℓ_1, called $N\backslash \ell_1$, is not nearly-paving since for all $x \in E(N) - \{\ell_1\}$, the matroid $(N\backslash \ell_1)/x$ contains the loop ℓ_2 and is therefore, not paving. Note that we could not contract ℓ_2, since ℓ_2 is a loop. It follows that if N contains any loops, it must have exactly one loop, ℓ. Suppose N contains a loop ℓ and a non-coloop element x. If x is not a coloop, then $N\backslash x$ has rank 3 and also has loop ℓ. Since $r(N\backslash x) > 0$, there exists an element $e \in N\backslash x$ that is distinct from ℓ. So, if we delete element e from $N\backslash x$, we have rank 3 matroid $(N\backslash x)\backslash e$, which still contains loop ℓ, implying that $(N\backslash x)\backslash e$ is not paving, and so $N\backslash x$ is not nearly-paving. Similarly, if we contract the element e from $N\backslash x$, we have rank 2 matroid $(N\backslash x)/e$, which still contains loop ℓ. Hence, $(N\backslash x)/e$ is not paving, which implies $N\backslash x$ is not nearly-paving. Therefore, if N has a loop ℓ and a non-coloop x, then N has minors that are not nearly-paving, which contradicts our assumption that N is an excluded minor for nearly-paving matroids. From this, we conclude that every element of N that is not a loop, must be a coloop. Since N is rank 3, N must have exactly three non-loop elements, which must be coloops. That is, $N \cong U_{0,1} \oplus U_{3,3}$.

We may now assume that N is loopless. Let N be a loopless rank 3 excluded minor for nearly-paving matroids. We know such an excluded minor must have circuits of size 2. We argue by cases based on the number of distinct 2-circuits in N. First, note that if we have any parallel class of size at least three, we get a contradiction. Indeed, suppose N has a parallel class P_1 with $|P_1| \geq 3$. Let $P_1 = \{e_1, e_2, e_3, ..., e_k\}$. Consider the minor N/e_1 of N. We know $r(N/e_1) = 2$, and e_2 and e_3 are both loops in N/e_1. Since e_2 is a loop, it cannot be contracted and can only be deleted. However, $(N/e_1)\backslash e_2$ has rank 2 and also has a loop e_3. Thus, $(N/e_1)\backslash e_2$ is not paving. This implies that N has a minor, N/e_1, that is not nearly-paving. Therefore, any nontrivial parallel class in N must be a 2-circuit.

First, suppose N has exactly one non-trivial parallel class P_1 with $|P_1| = 2$. Then for all $x \in E(N)$ such that $x \notin P_1$, the matroid N/x is paving. Also, for $e_1 \in P_1$, the matroid $N\backslash e_1$ is paving. Thus, N is nearly-paving, which contradicts our assumption that N is an excluded minor for nearly-paving matroids. We conclude that N cannot have exactly one nontrivial parallel class.

Next, suppose N contains at least three non-trivial parallel classes P_1, P_2, P_3, $\ldots$, P_k. Then, for $e \in P_1$, matroids N/e and $N\backslash e$ are not paving, so N is not nearly-paving. But, $N\backslash e$ is also not nearly-paving since, for $f \in E(N\backslash e)$ such that $f \in P_2$, $(N\backslash e)\backslash f$ has rank 3 with a circuit of size 2, and $(N\backslash e)/f$ has rank 2 with a circuit of size 1. Hence, neither $(N\backslash e)\backslash f$ nor $(N\backslash e)/f$ are paving. So, $N\backslash e$ is not nearly-paving, which contradicts our assumption that N is an excluded minor for nearly-paving matroids. We conclude that N cannot have more than two nontrivial parallel classes.

It now follows that N must have exactly two non-trivial parallel classes P_1 and P_2, where $|P_1| = |P_2| = 2$. Suppose N has an element x that is not in P_1 or P_2 and is not a coloop. Then $N\backslash x$ has rank 3 and also has two 2-circuits, P_1 and P_2. For $e \in P_1$, the matroid $(N\backslash x)\backslash e$ is not a paving matroid, because $r((N\backslash x)\backslash e) = 3$ and it contains a circuit of size 2, implying that $N\backslash x$ is not nearly-paving. Similarly, $(N\backslash x)/e$ is not a paving matroid, because $r((N\backslash x)/e) = 2$ and it contains a circuit of size 1. Therefore, N has a minor that is not nearly-paving, which contradicts our assumption that N is an excluded minor for nearly-paving matroids. It now follows that every element of N that is not in P_1 or P_2, must be a coloop. Since N is rank 3, N must have exactly one such coloop. That is, $N \cong U_{1,1} \oplus U_{1,2} \oplus U_{1,2}$. □

Before we describe the excluded minors of rank $r \geq 4$, we must describe what the circuits look like when $r \geq 4$. The next four lemmas provide restrictions on circuits of rank $r \geq 4$ matroids.

Lemma 4.4. *Let N be a rank $r \geq 4$ excluded minor for nearly-paving matroids. Then N has no circuit of size $k \leq r - 2$.*

Proof. Suppose, to the contrary, that such a matroid N has a circuit C of size $k \leq r-2$. For any element $x \in C$, consider the minor N/x of N. We know $r(N/x) = r - 1$ and circuit $C - x$ of N/x has size at most $r - 3$. However, for any element $y \notin C - x$, the matroid $(N/x)\backslash y$ has rank $r-1$ and has circuit $C-x$ of size at most $r-3$, which means $(N/x)\backslash y$ is not paving. Similary, $(N/x)/y$ has rank $r - 2$ with circuit $C - x$ of size at most $r-3$, which means $(N/x)/y$ is also not paving. Therefore, N/x is not nearly-paving as it contains an element y that cannot be deleted nor contracted to produce a paving matroid. It follows that N is not an excluded minor for nearly-paving matroids, which is a contradiction. □

Lemma 4.5. *Let N be a rank $r \geq 4$ excluded minor for nearly-paving matroids. Then N has at least two circuits of size $r - 1$.*

Proof. By Lemma 4.4, N must have at least one circuit of size $r - 1$. Suppose C is the only circuit in N with $|C| = r - 1$. Then by Lemma 4.4, all other circuits of N have size at least r. In this scenario, for all $x \in C$, the matroid $N\backslash x$ has no circuits of size $r - 1$ and is therefore paving. Also, for all $y \in E - C$, the matroid N/y is a rank $r - 1$

matroid all of whose circuits have size at least $r-1$, which means N/y is also paving. Therefore, N is nearly-paving, which is a contradiction. Therefore, N must have at least two circuits of size $r-1$. □

Lemma 4.6. *Let N be a rank $r \geq 4$ excluded minor for nearly-paving matroids. If N has two distinct circuits, C_1 and C_2, with $|C_1| = |C_2| = r-1$, then $C_1 \cap C_2 = \emptyset$.*

Proof. Suppose, to the contrary, that N has two distinct circuits C_1 and C_2, with $|C_1| = |C_2| = r-1$, but that $C_1 \cap C_2 \neq \emptyset$. Let $x \in C_1 \cap C_2$, and consider the minor N/x of N. We know $r(N/x) = r-1$ and circuits $|C_1'| = |C_2'| = r-2$ where $C_1' = C_1 - x$ and $C_2' = C_2 - x$. However, for any element $y \in C_2'$, the matroid $(N/x)\backslash y$ has rank $r-1$ and has circuit C_1' where $|C_1'| = r-2$, which means $(N/x)\backslash y$ is not paving. Additionally, $(N/x)/y$ has rank $r-2$ and has circuits C_1' and C_2'', where $C_2'' = C_2' - y$, with $|C_1'| = r-2$ and $|C_2''| = r-3$. This means $(N/x)/y$ is not paving. Therefore, N/x is not nearly-paving as it contains an element y that cannot be deleted nor contracted to produce a paving matroid. It follows that N is not an excluded minor for nearly-paving matroids, which is a contradiction. □

Lemma 4.7. *Let N be a rank $r \geq 4$ excluded minor for nearly-paving matroids. If N has two circuits C_1 and C_2, with $|C_1| = |C_2| = r-1$ and $C_1 \cap C_2 = \emptyset$, then $E(N) - (C_1 \cup C_2) = \emptyset$.*

Proof. Suppose, to the contrary, that N has two circuits C_1 and C_2, with $|C_1| = |C_2| = r-1$ where $C_1 \cap C_2 = \emptyset$ but that $E(N) - (C_1 \cup C_2) \neq \emptyset$. Let $x \in E(N) - (C_1 \cup C_2)$. Then consider the minor $N\backslash x$ of N. We know $r(N\backslash x) = r$ and $|C_1| = |C_2| = r-1$. However, for any element $y \in C_1$, $(N\backslash x)\backslash y$ contains the circuit C_2, where $|C_2| = r-1$, and $r(N\backslash x)\backslash y = r$, which is not paving. Likewise, $(N\backslash x)/y$ has rank $r-1$ with $|C_1| = r-2$ and $|C_2| = r-1$, which is not paving. Therefore, $N\backslash x$ is not nearly-paving as it contains an element y that cannot be deleted nor contracted to produce a paving matroid. It follows that N is not an excluded minor for nearly-paving matroids, which is a contradiction. □

Now we can use Lemmas 4.4, 4.5, 4.6, and 4.7 to prove the excluded minors of rank $r \geq 4$ for the class of nearly-paving matroids.

Lemma 4.8. *Let N be a rank $r \geq 4$ excluded minor for the nearly-paving matroids. Then $N \cong T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1})$, the $(r-4)^{th}$ truncation of $U_{r-2,r-1} \oplus U_{r-2,r-1}$.*

Proof. We know that the matroid $T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1})$ has rank $2r-4$ with two circuits of size $r-1$. We note that when $r \geq 4$, $2r-4$ is strictly greater than $r-1$. Now, the matroid $T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1})$ is certainly not in the class of nearly-paving matroids since deleting any element does not decrease the rank and does not eliminate all circuits of size less than r. Also, contracting any element will decrease the rank to $2r-5$ but will also produce a circuit of size $r-2$ (Note: when $r \geq 4$, $2r-5$ is strictly greater than $r-2$). However, deleting any element e of $T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1})$ results in minor $(T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1}))\backslash e$, which is nearly-paving. This is because for each $x \in E((T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1}))\backslash e)$, either $((T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1}))\backslash e)\backslash x$ or $((T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1}))\backslash e)/x$ is paving. Moreover, contracting any element e of $T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1})$ results in minor $(T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1}))/e$, which is nearly-paving. This is because for each $y \in E((T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1}))/e)$, either $((T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1}))/e)\backslash y$ or $((T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1}))/e)/y$ is paving. Therefore, every proper minor of $T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1})$ is in the class of nearly-paving matroids. It follows that $T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1})$ is an excluded minor for the class of nearly-paving matroids.

Let N be a rank $r \geq 4$ excluded minor for nearly-paving matroids. We know N must contain circuits of size $r-1$ since N cannot be paving. By Lemma 4.5, we know we must have at least two circuits of size $r-1$. It follows by Lemma 4.6, that N has two distinct circuits, C_1 and C_2, where $C_1 \cap C_2 = \emptyset$. Furthermore, by Lemma 4.7, we know that the ground set $E(N)$ does not contain any other elements except the elements in circuits C_1 and C_2. Notice that if we have two disjoint circuits of size $r-1$, the rank of the matroid is $2r-4$. However, we need these circuits to exist in a rank r matroid so we truncate this $2r-4$ matroid $r-4$ times which produces a matroid that has rank r and still has these two circuits of size $r-1$ that are disjoint. That is, $N \cong T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1})$. □

It is a little difficult to visualize what this matroid really looks like, especially for large values of r. So to get a better understanding of the result we just proved, we will depict the matroid $T^{(r-4)}(U_{r-2,r-1} \oplus U_{r-2,r-1})$, for some small value of r. When $r=4$, we have the matroid $T^{(0)}(U_{2,3} \oplus U_{2,3})$, which means that we do not truncate. The geometry of the matroid $U_{2,3} \oplus U_{2,3}$ is shown in Figure 4.1.

When $r=5$, we have the matroid $T(U_{3,4} \oplus U_{3,4})$. Notice $U_{3,4} \oplus U_{3,4}$ is a rank

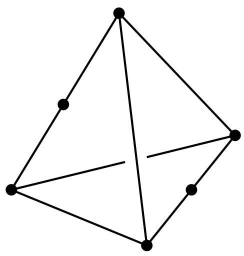
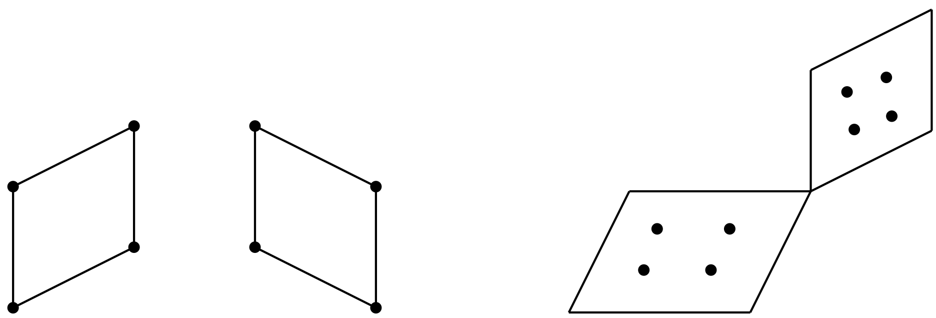


Figure 4.1: The matroid $U_{2,3} \oplus U_{2,3}$.

6 matroid with two circuits of size 4. That is, in rank 6 we have two skew planes, each with four points. See Figure 4.2 (a). However, after we truncate, these two planes share a common rank 1 space, which is how we can visualize the geometry in rank 5. See Figure 4.2 (b).



(a) The matroid $U_{3,4} \oplus U_{3,4}$ (b) The matroid $T(U_{3,4} \oplus U_{3,4})$

Figure 4.2: The truncation of matroid $U_{3,4} \oplus U_{3,4}$.

In this rank 5 example, notice that the matroid $T(U_{3,4} \oplus U_{3,4})$ is not in the class of nearly-paving matroids since deleting any element produces a matroid of rank 5 with a circuit of size 4, and the contraction of any element will result in a rank 4 matroid having a circuit of size 3. However, deleting any element e of $T(U_{3,4} \oplus U_{3,4})$ results in minor $(T(U_{3,4} \oplus U_{3,4}))\backslash e$, which is nearly-paving. This is because for each $x \in E((T(U_{3,4} \oplus U_{3,4}))\backslash e)$, either $((T(U_{3,4} \oplus U_{3,4}))\backslash e)\backslash x$ or $((T(U_{3,4} \oplus U_{3,4}))\backslash e)/x$ is paving. Moreover, contracting any element e of $T(U_{3,4} \oplus U_{3,4})$ results in the minor

$(T(U_{3,4} \oplus U_{3,4}))/e$, which is nearly-paving. This is because for each $y \in E((T(U_{3,4} \oplus U_{3,4}))/e)$, either $((T(U_{3,4} \oplus U_{3,4}))/e)\backslash y$ or $((T(U_{3,4} \oplus U_{3,4}))/e)/y$ is paving. Thus, every proper minor of $T(U_{3,4} \oplus U_{3,4})$ is in the class of nearly-paving matroids. It follows that $T^{(1)}(U_{3,4} \oplus U_{3,4})$ is an excluded minor for the class of nearly-paving matroids.

We provide the following corollary that gives us a specific excluded minor in rank 4, which is already included in the proof of Lemma 4.8. We specify the proof of this corollary to rank 4 to illustrate the excluded minor in a less abstract context.

Corollary 4.9. *The matroid $U_{2,3} \oplus U_{2,3}$ is the unique rank 4 excluded minor for the class of nearly-paving matroids.*

Proof. The matroid $U_{2,3} \oplus U_{2,3}$ is certainly not in the class of nearly-paving matroids since the deletion of any element will produce a matroid of rank 4 with circuits of size 3, and the contraction of any element will result in a rank 3 matroid having circuits of size 2. However, deleting any element e of $U_{2,3} \oplus U_{2,3}$ results in minor $(U_{2,3} \oplus U_{2,3})\backslash e$, which is nearly-paving. That is, because for each $x \in E((U_{2,3} \oplus U_{2,3})\backslash e)$, either $((U_{2,3} \oplus U_{2,3})\backslash e)\backslash x$ or $((U_{2,3} \oplus U_{2,3})\backslash e)/x$ is paving. Moreover, contracting any element of $U_{2,3} \oplus U_{2,3}$ results in a matroid isomorphic to $U_{1,2} \oplus U_{2,3}$, which is also nearly-paving, since for each $y \in E(U_{1,2} \oplus U_{2,3})$, either $(U_{1,2} \oplus U_{2,3})\backslash y$ or $(U_{1,2} \oplus U_{2,3})/y$ is paving. Therefore, every proper minor of $U_{2,3} \oplus U_{2,3}$ is in the class of nearly-paving matroids. It follows that $U_{2,3} \oplus U_{2,3}$ is an excluded minor for the class of nearly-paving matroids.

Let N be a rank 4 excluded minor for nearly-paving matroids. Then, since N is not paving, we know N must contain circuits of size 3. Suppose N has exactly one circuit C of size 3. Then, for all $x \in E(N) - \{C\}$, N/x is paving. Also, for any element $e \in C$, $N\backslash e$ is paving. Thus, N is nearly-paving, which contradicts our assumption that N is an excluded minor for nearly-paving matroids. Now suppose N has at least three circuits $C_1, C_2, C_3, ..., C_k$ of size 3. Then the minor of N obtained by deleting any element $e \in C_1$, called $N\backslash e$, is not nearly-paving since for any element $f \in C_2$, $(N\backslash e)\backslash f$ contains the circuit C_3 and is therefore, not paving. Note that if we contracted element f, matroid $(N\backslash e)/f$ still contains circuits C_2 and C_3, where C_2 now has size 2. It follows that N must have exactly two circuits C_1 and C_2 of size 3. Suppose that there exist elements in $E(N) - (C_1 \cup C_2)$. Then for all $x, y \notin C_1 \cup C_2$, the minor $N\backslash x$ is not nearly-paving since, $(N\backslash x)\backslash y$ contains circuits C_1 and C_2, and is therefore not paving. It follows that

$E(N) - (C_1 \cup C_2) = \emptyset$. Thus, N must be a rank 4 matroid with exactly two disjoint circuits of size 3. That is, $N \cong U_{2,3} \oplus U_{2,3}$. □

Theorem 4.1 now follows immediately from these lemmas.

Proof of Theorem 4.1. Statement (i) follows from Lemma 4.2. Statements (ii) and (iii) follow from Lemma 4.3. Statement (iv) follows from Lemma $4.4, 4.5, 4.6, 4.7$, and 4.8. □

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