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Hyperbolic Triangle Groups

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Hyperbolic Triangle Groups

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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Abstract

This paper will be on hyperbolic reflections and triangle groups. We will compare hyperbolic reflection groups to Euclidean reflection groups. The goal of this project is to give a clear exposition of the geometric, algebraic, and number theoretic properties of Euclidean and hyperbolic reflection groups.

Acknowledgements

I want to thank my wife, Lyana for being extremely supportive and patient with me while I completed my degree. I would like to thank my children, Alisa and Elijah for bringing me joy while working on my thesis. I would like to thank Dr. Jeffrey Meyer for being extremely patient and helpful during my research, this thesis would not be possible without his help. I would like to also thank my committee, Dr. Trapp and Dr. Sarli for their comments and suggestions. Finally, thank you to my parents and in-laws for their support and love. I am forever in everyone's debt.

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Chapter 1

Introduction

Hyperbolic geometry is non-Euclidean geometry that follows all postulates from Euclidean geometry except parallel postulate. In Euclidean geometry, lines are parallel if they do not intersect. Therefore, one form of the parallel postulate in Euclidean geometry states the following: *given any line and a point not on that line you can construct a unique line going through that point and parallel to the given line*, see Figure 1.1.

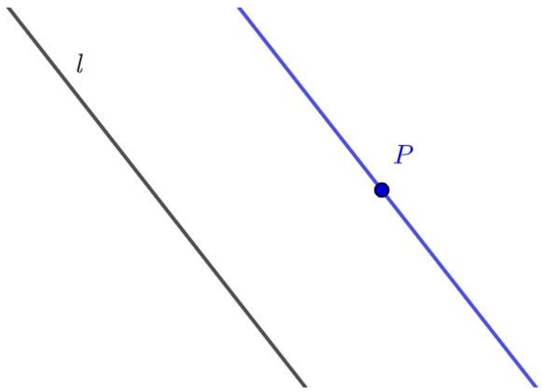


Figure 1.1: Parallel Lines

Now in Euclidean geometry there are two quantifiers that go with the fifth postulate. The first is that there exists a line that would be parallel to any given line through a point not on that line. The second is that parallel line is unique, which means there exists only one parallel line to any given line and a point not on that line. The reason the fifth postulate fails in hyperbolic geometry is because, while we do have the existence of parallel lines, they are not unique. In hyperbolic geometry given a line and a point not on that line, there exists infinitely many parallel lines through that point and not just one. It is no surprise that Euclid was unable to prove the postulate from the first four postulates. Euclid wrote a book called the ***Elements*** where he outlined what we now call Euclidean geometry. In this book he included his postulates, propositions, theorems, definitions, and mathematical proofs. The Elements was used as a textbook for geometry for many years and is still the foundation of Euclidean geometry to this day.

The story goes that Euclid was not content with assuming the parallel postulate, so much so, that he tried to avoid using it as much as possible. In fact, he only used the first four postulates for the first 28 propositions of the Elements, but was forced to invoke the parallel postulate on the 29th proposition. The reason it is not a theorem is because the fifth postulate is entirely self-consistent and only remains a fact in Euclidean geometry. Many mathematicians tried to prove this postulate but failed to do so. However, with these failures also came triumphs, where new birth of geometry was introduced to the world called "hyperbolic geometry". While the first four postulates hold, the fifth does not in hyperbolic geometry because given a line (geodesic) and a point not on that geodesic you can construct infinity many lines through that point that will be parallel to the given geodesic. Three people: Gauss, Lobachevsky and Bolyai can be credited with discovery of hyperbolic geometry, however, Gauss never published his ideas, and Lobachevsky was the first to present his views to the world mathematical community. This is why hyperbolic geometry is frequently referred to as "Lobachevskian geometry" or "Bolyai–Lobachevskian geometry" [Tha08].

Both Euclidean and hyperbolic geometries have triangles, symmetries, translations, reflections, and rotations. Furthermore, a lot of the properties and theorems hold true in both. My focus will be in exploration of hyperbolic reflection groups and more specifically hyperbolic triangle groups. I will research hyperbolic triangle groups and how they behave in the hyperbolic plane. I will explore hyperbolic reflection groups and compare them to Euclidean reflections groups.

At the core one of the most fundamental geometric symmetry is a reflection ("mirror image"). We may not recognize it but we are surrounded by this amazing concept every time your see tile on the kitchen floor or bathroom walls. Artists apply reflections to show beautiful tessellations or patterns in their paintings or sculptures. Engineers apply reflections to enhance the structure or to add life to the building they are designing. This fundamental concept obeys certain properties or rules. Analyzing reflections can allow us to find groups that keep the properties. We can then start exploring these groups in the Euclidean plane which we call reflection groups. We will look at discrete reflection groups and how the behave and which polygons create patterns, tiles, or unique images through composition of reflections. We will explore and analyze reflections through intersecting lines and parallel lines to see how each effects our polygons in each case. I will provide visual aids through use of images and interactive programs in GeoGebra. The use of technology will aid the reader to visualize clearly what happens when reflections through two intersecting lines or through two parallel lines are composed. GeoGebra gives freedom to the reader to open a program and move the objects and points and clearly see how they behave under these transformations.

The reader needs to have a concrete understanding of the importance of reflections in Euclidean geometry because this will translate into hyperbolic geometry. Our focus is going to be on composition of reflections that yield discrete groups. We are going to see that in Euclidean geometry we have a finite set of discrete groups. We will analyze the structure of these groups and see the amazing tessellations that are created by these discrete groups and their corresponding pictures. More specifically there are only three discrete groups in Euclidean geometry that make these tessellations. They are known as (3,3,3) triangle group associated to the $60° - 60° - 60°$ (regular) triangle, (2,4,4) triangle group associated to the $45° - 45° - 90°$ triangle, and (2,3,6) triangle group associated to the $30° - 60° - 90°$ triangle. Please see the following three images that display those unique tessellations.

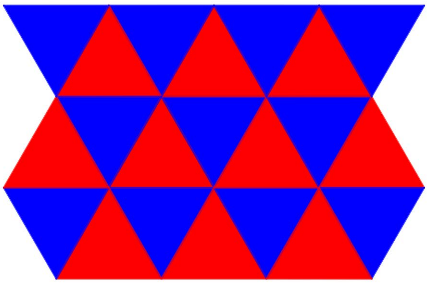


Figure 1.2: 3,3,3

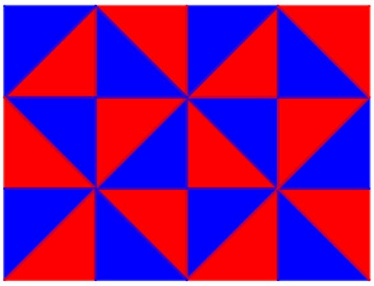


Figure 1.3: 2,4,4

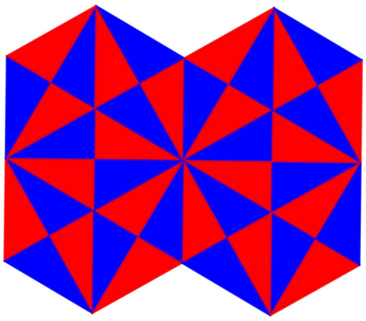


Figure 1.4: 2,3,6

We know that in Euclidean geometry the interior angles of a triangle sum to $180°$. Therefore, we are limited on how many discrete triangle groups exist in Euclidean space. This restriction does not hold in hyperbolic space. Knowing that a the sum of interior angles of a hyperbolic triangle is less than $180°$, we find other discrete groups of isometries in hyperbolic space that do not exist in Euclidean space. We will describe many examples to see which tessellations can be created by these discrete groups. Our goal will also include the importance of these groups in Euclidean and hyperbolic geometry.

Cartographers display maps on a positively curved or spherical surface in the plane which distorts the image of the map. Similarly, in hyperbolic geometry we display the negative curvature which in turn also distorts distances, areas, and shapes in two dimensional plane. To visualize this we are going to look at couple of artists that show case this cartography skill. M.C. Escher took his Angel-Devil drawing and displayed it in Poincare Disk, see Figure 1.5 and Figure 1.6. Jos Leys took M.C. Escher's famous drawing of Angel-Devil drawing in Euclidean plane and displayed it in hyperbolic upper half-plane model, see Figure 1.7.

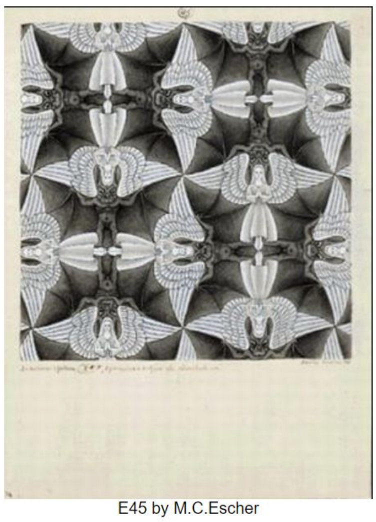


Figure 1.5: M.C. Escher Original

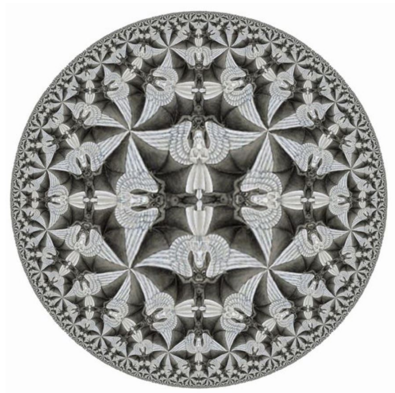


Figure 1.6: Poincare Disk

Figure 1.7: Upper Half-Plane

If you notice that all the angels and devils in Figure 1.5 are exactly the same size and shape. What is not so obvious is that all the angels and devils in Figure 1.6

and Figure 1.7 are also same size and shape. Visually it is obvious that they are not but because we are trying to put something on two-dimensional plane which belongs in hyperbolic plane the image gets distorted. By the end of this exposition we should see why or how these tessellations are different from one plane to the other. See other images on different geometric planes by Jos Leys that were inspired by M.C. Escher `http://www.josleys.com/show_gallery.php?galid=325`.

My thesis will contain images of different discrete groups and links to online resources that will allow the reader to participate in the learning of these unique groups. The reader will get a deeper understanding of each group and its structure in both Euclidean and hyperbolic geometry. We will look at special cases and what makes these special cases unique. Most importantly we will look at the geometric and algebraic properties of these groups.

Chapter 2

Euclidean Geometry

Let us define and explain notation we will use in Euclidean geometry. I will denote Euclidean plane as $\mathbb{E}^2$. We shall denote $\mathbb{R}$ the field of real numbers and denote $\mathbb{R}^2$ as the coordinate pair (x, y) in $\mathbb{E}^2$. Given any two points in $\mathbb{E}^2$, we can calculate distance between two points using the Pythagorean Theorem. The distance formula is: $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ where d represents Euclidean length in $\mathbb{E}^2$. It is also a well known fact that law of cosines is also the result of Pythagorean Theorem and as a result law of cosines gives Side-Angle-Side property (SAS). SAS states that if the corresponding sides, angles, and sides are congruent on two different triangles then the two triangles must be congruent. If two triangles are congruent, then all of the corresponding sides must be equal length and all of the corresponding angles must be equal measure.

2.1 Euclidean Transformations

In this section I will analyze Euclidean transformations. I will focus on reflection, translations and rotations. Why are they important? Since we are looking to achieve beautiful tessellation images we need a way to duplicate a unique image multiple times. Transformations or composition of specific transformations allows us to create patterns and images we are looking for. We will also use upper case letter R for reflection, upper case letter T for translation, and Rot for rotation. Now let us define these Euclidean transformations starting with reflection.

Definition 2.1. *A* ***reflection*** *in a line k is a transformation of $\mathbb{E}^2$, denoted R_k, such that if P is on k then P is fixed, and otherwise R_k maps P to P' such that k is the perpendicular bisector of PP'. The line k is called the mirror of the reflection.*

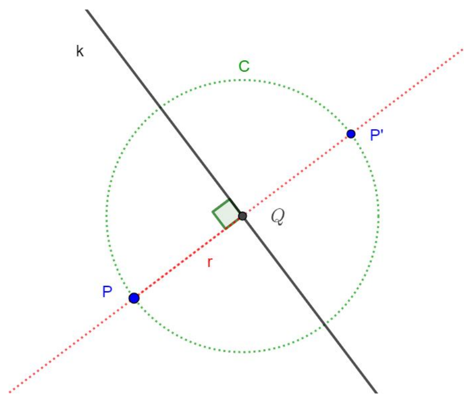
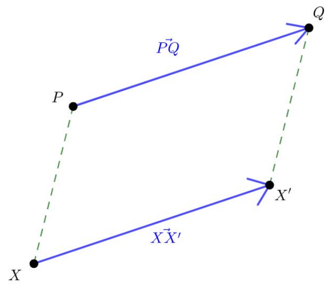


Figure 2.1: Euclidean Reflection

As seen in Figure 2.1 given line k and a point P not on that line we can reflect our point P across the line k, R_k. Draw any line k and choose any point not on that line. Then draw a perpendicular line to line k passing through point P. Draw a circle C with the radius $r = d(Q, P)$ where Q is the intersection of k and the dropped perpendicular. Our reflected point will be on the perpendicular line through points P and Q with the radius $r = d(Q, P)$ across the line k and on circle C. Therefore, when we reflect point P across line k we get point P'.

Reflection is one of the ways we can move the image on the Euclidean plane to create images of tessellations. Another transformation we use is translation.

Definition 2.2. *A* ***translation*** *through a vector PQ is a transformation of a plane, denoted $T_{\vec{PQ}}$ such that if $T_{\vec{PQ}}$ maps X to X', then the vector $XX' \cong PQ$.*



Recall that by definition a *vector* has magnitude (length) and direction and vectors are congruent when both their length and direction are equivalent.

A third transformation that can be used to move the image around the Euclidean plane is rotation.

Definition 2.3. *A* ***rotation*** *about a point C through an angle with measure θ, denoted $R_{C,\theta}$, is a transformation of a plane where C is mapped to itself and for any point X distinct from C if $R_{C,\theta}$ maps X to X', then $d(X', C) = d(X, C)$ and measure of angle $XCX' = \theta$. C is called the center of the rotation.*

2.2 Compositions of Reflections

In Euclidean geometry, as well as in hyperbolic geometry, reflections are at the core of all transformations. All of the transformations seem unique in their own way but in fact reflections are considered "atomic" transformations because from reflections you can build rotation and translation. In $\mathbb{E}^2$ composition of two reflections yields either a rotation or translation. If we are given two arbitrary lines in $\mathbb{E}^2$, then those lines will intersect or will be parallel to each other. These are the only possibilities in Euclidean plane. Let us explore each one in more detail.

Case 1: Two Intersecting Lines

Proposition 2.4. *If $l_1, l_2 \subset \mathbb{E}^2$ are two intersecting lines and the angle from l_2 to l_1 is α, then $R_{l_1} \circ R_{l_2} = Rot_{F,2\alpha}$.*

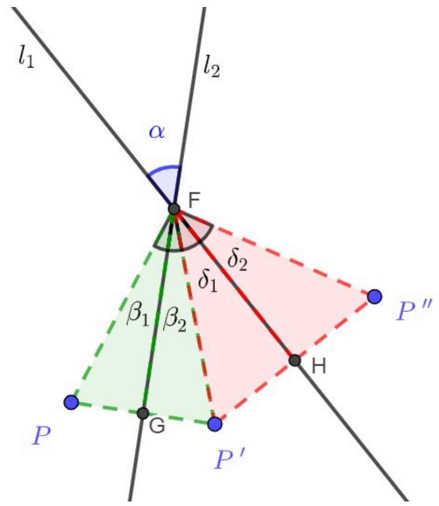


Figure 2.2: Two Intersecting Lines

Proof. Let us reflect point P across the line l_2, we get point P'. By Definition 2.1 $d(PG) = d(GP')$. Since $d(FG) = d(FG)$ and $\angle PGF$ and $\angle P'GF$ are right angles by Definition 2.1. Thus, $\triangle PGF \cong \triangle P'GF$ by SAS and therefore $\angle\beta_1 \cong \angle\beta_2$. Similarly, after reflecting point P' across line l_1 we can show that $\angle\delta_1 \cong \angle\delta_2$. It is a well known fact that in Euclidean geometry vertical angles are congruent. Note that in Figure 2.2 measure of angel $(m\angle)$ $\alpha = m\angle\beta_2 + m\angle\delta_1$ and the $m\angle PFP'' = m\angle\beta_1 + m\angle\beta_2 + m\angle\delta_1 + m\angle\delta_2 = m\angle 2\alpha$. Thus, $R_{l_1} \circ R_{l_2} = Rot_{F,2\alpha}$. □

Example 2.5. *In Figure 2.3 we observe a composition of reflections on* $\triangle GFE$ *which yields* $R_{x-axis} \circ R_{y-axis} = Rot_{origin,180}$. *Note that the two axis intersect at the origin at the right angle. The resulting composition of reflections is rotated about the origin twice the angle of intersection.*

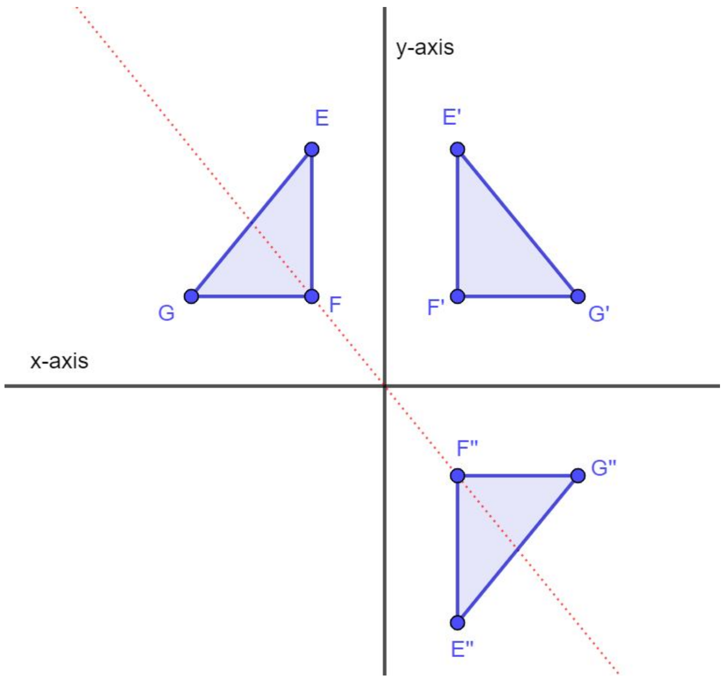


Figure 2.3: Triangle Reflected about y-axis and x-axis

Case 2: Two Parallel Lines

Proposition 2.6. *If $l_1, l_2 \subset \mathbb{E}^2$ are parallel lines and $\vec{v}$ is vector from l_1 to l_2, then $R_{l_2} \circ R_{l_1} = T_{2\vec{v}}$.*

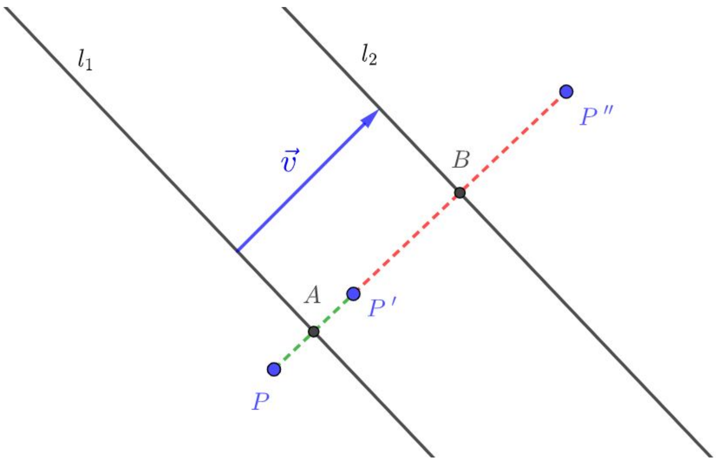


Figure 2.4: Two Parallel Lines

Proof. Let $\vec{v}$ be the distance between line l_1 and line l_2 as seen in Figure 2.4. By Definition 2.1 $d(PA) = d(AP')$. Similarly, $d(P'B) = d(BP'')$. Since the distance between two parallel lines is $d(AP') + d(P'B) \implies \vec{v} = d(AP') + d(P'B)$. Furthermore, the distance between point P and point P'' is $d(PA) + d(AP') + d(P'B) + d(BP'') \implies d(PP'') = d(PA) + d(AP') + d(P'B) + d(BP'') = 2\vec{v}$. Thus, $R_{l_2} \circ R_{l_1} = T_{2\vec{v}}$. □

2.3 Isometries

We want to be certain that regardless of the transformation we are performing on these images they do not change size or shape. Therefore we want to make sure that these transformations are isometries.

Definition 2.7. *A Euclidean* ***isometry****, or* ***isometry*** *of* $\mathbb{E}^2$*, is a function* $f : \mathbb{E}^2 \to \mathbb{E}^2$*, which preserves Euclidean distance, that is*

$$d(f(P_1), f(P_2)) = d(P_1, P_2) \text{ for all } P_1, P_2 \in \mathbb{E}^2.$$

Proposition 2.8. *All reflections are isometries.*

To prove that reflection is an isometry we need to show that it preserves distance. In the Figure 2.5 point P and point Q are reflected about the the line l.

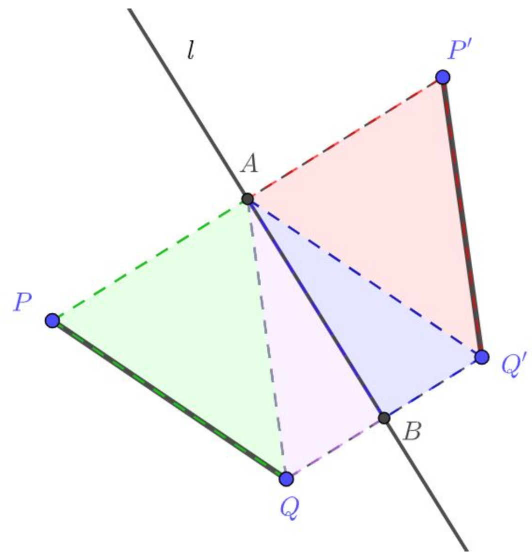


Figure 2.5: Proving Euclidean Reflection

Proof. Let points P and point Q be $\in \mathbb{E}^2$ where point P' is the reflection across the line l of P and point Q' is the reflection across the line l of Q. Need to show $d(PQ) = d(P'Q')$. By Definition 2.1 we know that $d(PA) = d(AP')$ and $d(QB) = d(BQ')$, furthermore, $\angle QBA$, $\angle Q'BA$, $\angle PAB$, and $\angle P'AB$ are right angles. Side $d(AB) = d(AB)$ by reflexive property, therefore, $\triangle QBA \cong \triangle Q'BA$ by SAS property. Since $\triangle QBA \cong \triangle Q'BA$ this implies sides $d(QA) = d(Q'A)$. Now all is left for us is to show is that $\angle QAP \cong \angle Q'AP'$. Recall that $\angle PAB$ and $\angle P'AB$ are right angles thus $m\angle PAQ = 90° - m\angle BAQ$. Similarly $m\angle P'AQ' = 90° - m\angle BAQ'$. Since $\angle BAQ \cong \angle BAQ'$, then $\angle PAQ \cong \angle P'AQ'$. Therefore, $\triangle PAQ \cong \triangle P'AQ'$ by SAS property and $\overline{PQ} \cong \overline{P'Q'}$. Thus, $d(PQ) = d(P'Q')$ and since reflection preserved the distance, therefore, it is an isometry. □

Proposition 2.9. *The composition of isometries is an isometry.*

Proof. By Definition 2.7 any isometry preserves length. Thus composition of isometries also preserves length, therefore, composition of isometries is an isometry. □

Theorem 2.10. *Reflections, Translations, and Rotations are isometries in* $\mathbb{E}^2$.

Proof. We need to show that each transformation preserves length. Reflections are isometries by Proposition 2.8. To show translations are isometries we will combine our propositions. By Proposition 2.8 all reflections are isometries. By Proposition 2.6 we proved that composition of reflections through parallel lines is a translation. By Proposition 2.9 we proved that composition of isometries are isometries, therefore, translations are isometries. Similarly we can show rotations are isometries by combining our propositions. By Proposition 2.8 all reflections are isometries. By Proposition 2.4 composition of reflections through intersecting lines is a rotation. By Proposition 2.9 we proved that composition of reflections are isometries, therefore, rotations are also isometries. □

Chapter 3

Hyperbolic Geometry

We will be modeling hyperbolic geometry using the upper half-plane. Let $\mathbb{C}$ be the complex plane. We will use well known notation for the real and imaginary parts of $z = x + iy \in \mathbb{C}, Re(z) = x, Im(z) = y$.

Definition 3.1. *The subset* $\mathbb{H}^2 = \{z \in \mathbb{C} \mid Im(z) > 0\}$ *of the complex plane* $\mathbb{C}$ *is called the* ***upper half-plane****. Similarly, the upper half-plane can be represented in plane* $\mathbb{R}^2$ *as a subset* $\mathbb{H}^2 = \{x, y \in \mathbb{R}^2 : y > 0\}$.

We will define the length between two points in $\mathbb{H}^2$ using the *hyperbolic metric*. Let $I = [0, 1]$ and $\gamma : I \to \mathbb{H}^2$ be a piecewise differentiable path $\gamma = z(t) = x(t) + iy(t)$. Then its *hyperbolic length* $h(\gamma)$ is given by

$$h(\gamma) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt = \int_0^1 \frac{\left|\frac{dz}{dt}\right|}{y(t)} dt. \tag{3.1}$$

The hyperbolic distance between two points is the infimum of the lengths of all paths between them. [Kat92]

Definition 3.2. *The upper half-plane model of the* ***hyperbolic plane*** *is the upper half-plane together with the hyperbolic length given by Equation 3.1.*

Definition 3.3. *A* ***hyperbolic line*** *in* $\mathbb{H}^2$*, also known as geodesic, is a distance minimizing path from one point to the another.*

Theorem 3.4. *The geodesics in $\mathbb{H}^2$ are semicircles and straight lines orthogonal to the real axis $\mathbb{R}$.*

See proof of this theorem in [Kat92, Theorem 1.2.1].

Let us consider a fixed line XY on the x - axis of the Euclidean plane (see Figure 3.1). The x-axis itself is *not* in $\mathbb{H}^2$, but rather "the boundary at infinity" which we will denote as $\partial\mathbb{H}^2$.

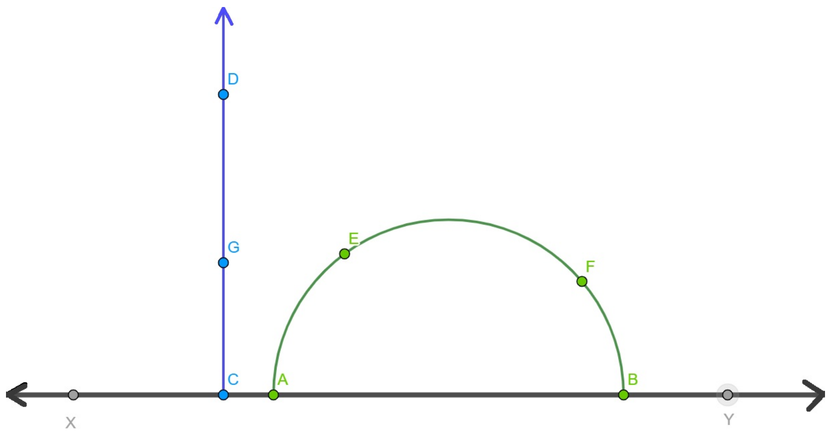


Figure 3.1: Upper Half-Plane

A line in upper half-plane is either a semicircle like EF with center located on XY or the intersecting perpendicular line DG to XY. It is important to note that in upper half-plane points C, A and B are not in $\mathbb{H}^2$, but in the boundary $\partial\mathbb{H}^2$. We will call the points where a geodesics intersects the boundary its **endpoints**.

If you recall that in Euclidean geometry the "existence" and the "uniqueness" of parallel lines make up the fifth postulate. However, in hyperbolic geometry parallel lines do exist but they are not "unique". There are two types of parallel lines in hyperbolic plane, *asymptotically parallel* and *ultraparallel*. Asymptotically parallel lines are hyperbolic lines that share an endpoint. In the Figure 3.2 we are given line l and a point P not on line in upper half-plane. The asymptotically parallel lines to line l are line n and line m. Since line n shares a point at x - axis with line l and line m shares a point at positive infinity with line l. Ultraparallel line are hyperbolic lines that do not share an endpoint. The ultraparallel lines to line l in Figure 3.2 are lines k and q. Both line are parallel to line l and go through the point P and do no share any of the endpoints.

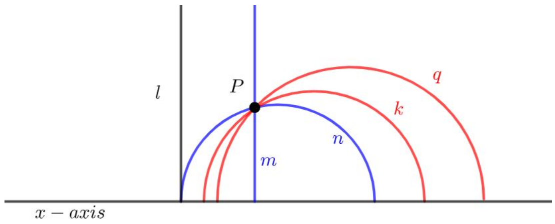


Figure 3.2: Asymptotic and Ultraparallel

Earlier in the introduction I said that one of the best ways to view hyperbolic geometry is being a cartographer. I also showed a famous Angel-Devil drawing by M.C. Escher in three different planes. It is important to note that hyperbolic disk "Poincaré disk" model and upper half-plane model can be transferred from one plane to another via a *Möbius* transformation from two inversions (reflections in the inversive plane). [Hit18]

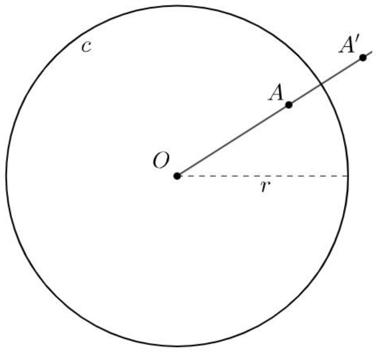
Definition 3.5. *The elements of the projective special linear group $PSL(2, \mathbb{R})$ are the Möbius transformations, rational functions from $\mathbb{C}$ to $\mathbb{C}$ of the form $z \mapsto \frac{az+b}{cz+d}$ with $ad - bc = 1$ and $a, b, c, d \in \mathbb{R}$.*

3.1 Hyperbolic Transformations

In this section I will analyze hyperbolic transformations. Just like in the Euclidean section, I will focus on reflections, translations, and rotations. Since we are looking to achieve beautiful tessellations in hyperbolic plane we want to be able to replicate isometric images in $\mathbb{H}^2$.

Definition 3.6. *If c is a Euclidean circle with radius r and center O, then a* ***circle inversion*** *through c is a transformation of the plane sending point A to the point A' such that A' lies on the ray from O to A, and satisfies $d(OA) \cdot d(OA') = r^2$, where d denotes Euclidean distance.*

Observe that if O lies on the x-axis, circle inversions restrict to a transformation of the upper half-plane.



Definition 3.7. *If l is a geodesic in $\mathbb{H}^2$, then* ***hyperbolic reflection*** *is the transformation $I(l) : \mathbb{H}^2 \longmapsto \mathbb{H}^2$ such that if*

1) l is vertical line orthogonal to the real axis, then it is equivalent to Euclidean reflection through a vertical line.

2) l is a semicircle with its center on x-axis, then it is the circle inversion in the upper half-plane.

3.2 Compositions of Reflections

In Euclidean geometry we had composition of two reflections yield rotations when the lines intersected and yielded translations when the lines were parallel. In hyperbolic geometry we will take a similar approach as we did in Euclidean geometry.

In the compositions of reflections section in Euclidean geometry we reflected about two intersecting lines or parallel lines. In hyperbolic geometry we can reflect about two intersecting hyperbolic lines, two parallel lines that meet at infinity (*asymptotically parallel*), and two parallel lines that do not share endpoints (*ultraparallel*).

Möbius transformations are classified algebraically by their trace. There are three types of elements in $PSL(2,\mathbb{R}) = \{z \to T(z) = \dfrac{az+b}{cz+d} \mid ad - bc = 1\}$ that are recognized by the value of its *trace*: $Tr(T) = |a+d|$. If $Tr(T) < 2$, T is called *elliptic*; if $Tr(T) = 2$, T is called *parabolic*; and if $Tr(T) > 2$, T is called *hyperbolic*. [Kat92, 2.1]. We will show that each of these can be realized by the composition of two hyperbolic reflections, *elliptic* case with two lines intersecting, *parabolic* case with two asymptotically parallel lines, and *hyperbolic* case with ultraparallel lines.

My goal is to compare the algebraic approach to my geometric approach and to show that results are equivalent. Now let us look at each case from algebraic and geometric perspective.

3.2.1 Elliptic

For the moment, let T be an elliptic transformation represented by the matrix A. An elliptic transformation has a fixed point $O \in \mathbb{H}^2$. By analyzing the eigenvalues, A is elliptic if and only if it is conjugate in $SL(2,\mathbb{R})$ to a matrix $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. This happens when the trace value is $Tr(T) < 2$. This *Möbius* transformation can be represented by the conjugate PBP^{-1}, where P^{-1} takes the fixed point O in $\mathbb{H}^2$ and maps it to i in $\mathbb{H}^2$. Then matrix B "rotates" it by the angle θ about the new fixed point i. Finally, matrix P maps the point O from i back to the original location in $\mathbb{H}^2$. This whole algebraic process is the same thing as reflecting through the two intersecting geodesics.

Proposition 3.8. *([Kat92, 3.3.4]) Given elliptic $T(z)$, there exists l_1, l_2 which intersect such that $I(l_2) \circ I(l_1) = T$.*

I will now prove the converse of this proposition. The composition of reflections represented by $I(l_2) \circ I(l_1)$ is not a matrix but it is a combination of *Möbius* transformations that can be represented by a matrix. I will define these transformations as T. Thus $I(l_2) \circ I(l_1)$ would represent some *Möbius* transformation $T(z) = \frac{az+b}{cz+d}$ which then can be written as a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Proposition 3.9. *Given two hyperbolic lines l_1, l_2 which intersect in $\mathbb{H}^2$, the composition $I(l_2) \circ I(l_1)$ is elliptic. Furthermore, if $l_1 = ((x_1, 0), r_1)$, $l_2 = ((x_2, 0), r_2)$, and $d = |x_2 - x_1|$, then $I(l_2) \circ I(l_1)$ can be represented by PBP^{-1}, $P^{-1} = \begin{bmatrix} 1 & -x \\ 0 & y \end{bmatrix}$ where $x = \dfrac{r_2^2 - r_1^2 + x_1^2 - x_2^2}{2x_1 - 2x_2}$ and $y = \sqrt{r_1^2 - (x - x_1)^2}$. Furthermore, $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ where $\theta = \arccos\left(\frac{r_1^2 + r_2^2 - d^2}{2 \cdot r_1 \cdot r_2}\right)$*

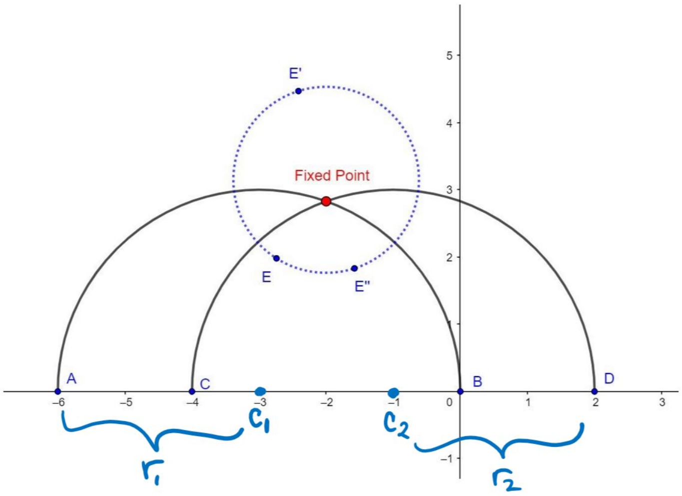


Figure 3.3: Elliptic Transformation

The elliptic transformation is a composition of reflections through two circles. We can see that point E is the initial point which was reflected through the circle of center C_1 and a radius of r_1. That reflection was the result of E'. Now if we take E' and reflect that point through circle of center C_2 and a radius of r_2 the result will be point E''. Note that regardless where we choose the point, the composition of these reflections will always be a " hyperbolic rotation" about the fixed point where the two circles intersect.

Proof. To calculate the point of intersection we will have to look at equation of the circles. Since both circles will lie on x-axis we can use the following two arbitrary circle equation: $(x - x_1)^2 + y^2 = r_1^2$ and $(x - x_2)^2 + y^2 = r_2^2$. By expanding both equations and solving for x we get the following:

$$x^2 - 2x_1x + x_1^2 + y^2 = r_1^2$$
$$x^2 - 2x_2x + x_2^2 + y^2 = r_2^2$$
$$-x^2 + 2x_1x - x_1^2 - y^2 = -r_1^2$$
$$2x_1x - 2x_2x + x_2^2 - x_1^2 = r_2^2 - r_1^2$$
$$x(2x_1 - 2x_2) = r_2^2 - r_1^2 + x_1^2 - x_2^2$$

$$x = \frac{r_2^2 - r_1^2 + x_1^2 - x_2^2}{2x_1 - 2x_2}. \tag{3.2}$$

After you find the value of x, use the following equation to find the value of y.

$$y = \pm\sqrt{r_1^2 - (x - x_1)^2} \tag{3.3}$$

Since our focus is only on the upper half-plane we can ignore the negative value of y and just use the positive value of y to find the exact fixed point (x, y).

Since we now have our fixed point now we need to find our angle of intersection by the use of Law of Cosine, see Figure 3.4.

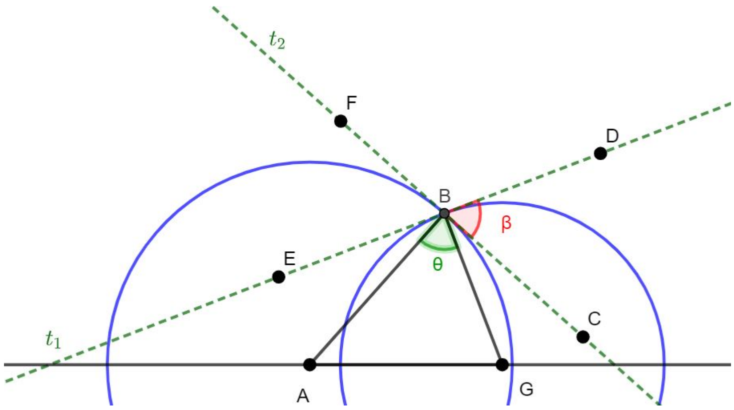


Figure 3.4: Elliptic

Note segment AB is perpendicular to tangent line t_2 and segment GB is perpendicular to tangent line t_1. Therefore, $\angle ABC$ and $\angle GBE$ are right angles. Since $m\angle\theta + m\angle GBC = 90°$ and $m\angle\beta + m\angle GBC = 90°$ then $\angle\theta \cong \angle\beta$. Law of Cosine formula, $\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2 \cdot r_1 \cdot r_2}$, gives us the exact angle of intersection between two geodesics. Finding the *Möbius* transformation that takes our fixed point to i gives the matrix needed to represent our composition of hyperbolic reflections. Thus, $I(l_2) \circ I(l_1)$ which represents our $PBP^{-1} = T(z)$ is as follows: $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 1 & -x \\ 0 & y \end{bmatrix}$.

$\square$

I created a manipulative of this proof in GeoGebra:
`https://www.geogebra.org/m/wsnwwbt2`

Example 3.10. *Given C_1 is centered at $(-3, 0)$ with radius 3 and C_2 is centered at $(-1, 0)$ with radius 3, find the matrix that will represent the composition of hyperbolic reflections through these Euclidean circles.*

First let us find the x value by referring to the equation (3.2) from above. Substitute appropriate values from our example into the equation (3.2). We get the following:

$$x = \frac{3^2 - 3^2 + (-3)^2 - (-1)^2}{2(-3) - 2(-1)}$$
$$x = -2$$

Now let us use equation (3.3) to calculate our y value. Using substitution we get the following:

$$y = \pm\sqrt{3^2 - (-2 - (-3))^2}$$
$$y = 2\sqrt{2}$$

Since we are only working in the Upper Half-Plane we can ignore the negative value of y. Therefore, your fixed point will be at $(-2, 2\sqrt{2})$.

To calculate the angle of intersection let us use Law of Cosine.

$$\cos\theta = \frac{r_1^2 + r_2^2 - d^2}{2 \cdot r_1 \cdot r_2}$$
$$\cos\theta = \frac{3^2 + 3^2 - 2^2}{2 \cdot 3 \cdot 3}$$
$$\theta = \arccos\left(\frac{3^2 + 3^2 - 2^2}{2 \cdot 3 \cdot 3}\right)$$
$$\theta = \arccos\left(\frac{7}{9}\right)$$
$$\theta \approx 38.94°$$

The *Möbius* transformation that takes the fixed point to i is $\begin{bmatrix} 1 & -x \\ 0 & y \end{bmatrix}$ which represents our P^{-1}. Now use substitution to compute PBP^{-1}. Therefore, $P^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 2\sqrt{2} \end{bmatrix}$, $B \approx \begin{bmatrix} \cos(38.94) & \sin(38.94) \\ -\sin(38.94) & \cos(38.94) \end{bmatrix}$ and $P = \begin{bmatrix} 1 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{2\sqrt{2}} \end{bmatrix}$. Thus $PBP^{-1} \approx \begin{bmatrix} 1.22 & 2.67 \\ -0.22 & 0.33 \end{bmatrix}$ which implies that the *Möbius* transformation that represents the composition of hyperbolic reflections through the given circles is represented by $T(z) \approx \frac{1.22 \cdot z + 2.67}{-0.22 \cdot z + 0.33}$.

Using GeoGebra I was able to create this example and show you that our calculations are correct (see Figure 3.5). GeoGebra link for Example 3.10
`https://www.geogebra.org/m/efpgncem`.

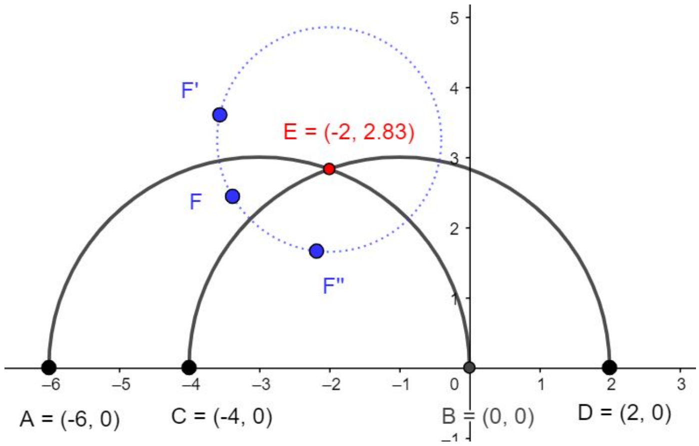


Figure 3.5: Example 3.10

3.2.2 Parabolic

Now, let T be a parabolic transformation represented by the matrix A. A parabolic transformation has one fixed point in the boundary $\partial\mathbb{H}^2$. By analyzing the eigenvalues, A is parabolic if and only if it conjugate over $\mathbb{R}^2$, to the matrix $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. This happens when the trace value is $Tr(T) = 2$. This *Möbius* transformation that can be represented by the conjugate PBP^{-1}, where P^{-1} takes the fixed point and maps it to infinity. Then matrix B translates everything one unit to the right. Finally, matrix P maps the fixed point back to its original location. This whole process is the same thing as reflecting through the two asymptotically parallel geodesics.

Proposition 3.11. *([Kat92, 3.3.4]) Given parabolic $T(z)$, there exists l_1, l_2 which are asymptotically parallel such that $I(l_2) \circ I(l_1) = T$.*

I will now prove the converse of this proposition.

Proposition 3.12. *Given two hyperbolic lines l_1, l_2, which intersect in the boundary $\partial\mathbb{H}^2$, the composition $I(l_2) \circ I(l_1)$ is parabolic. Furthermore, if $l_1 = ((x_1, 0), r_1)$, and*

$l_2 = ((x_2, 0), r_2)$*, then* $I(l_2) \circ I(l_1)$ *can be represented by* PBP^{-1}*,* $P^{-1} = \begin{bmatrix} 0 & r^2 \\ 1 & -q \end{bmatrix}$ *where* $r = \sqrt{\frac{0.5 \cdot |q - z_2| \cdot |q - z_1|}{|q - z_1| + |q - z_2|}}$*,* $q = x_2 + r_2$*,* $z_1 = x_1 + r_1$*, and* $z_2 = x_2 - r_2$*. Furthermore,* $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

In parabolic transformation the two circles will be asymptotically parallel and we will have a single fixed points on the x-axis. Let us look at what this diagram looks like in upper half-plane.

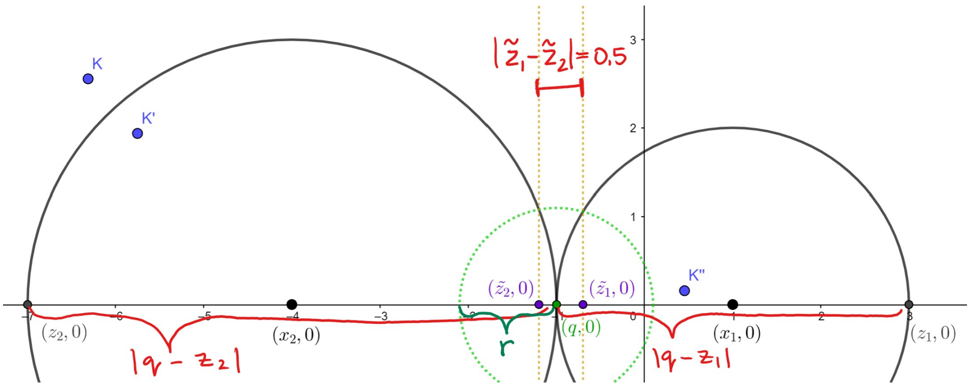


Figure 3.6: Parabolic Transformation

As seen in the Figure 3.6 the parabolic transformation is a composition of reflections through two circles. We can see that our initial point K which was reflected through the first circle, call it l_1, results in point K'. This is represented by the following notation, $I_{l_1}(K) = K'$. Now let us reflect K' through the second circle call it l_2 which will result in K'' and is represented by the following notation, $I_{l_2} \circ I_{l_1}(K) = K''$.

Proof. We can calculate the fixed point $(q, 0)$ easily. Since you are given the centers of your circles and the radii of both circles we can use that to find the fixed point. Given $l_1 = ((x_1, 0), r_1)$ and $l_2 = ((x_2, 0), r_2)$, then $q = x_2 + r_2$ and our fixed point is going to be $(q, 0)$.

To make sure that my matrix $P^{-1}AP$ transforms everything exactly one unit to the right, we need to apply our knowledge about inversions through a circle. It is a well known fact that hyperbolic reflections send circles to circles or vertical lines. As seen in

Figure 3.6 the two vertical lines that are passing through $(\tilde{z_2}, 0)$ and $(\tilde{z_1}, 0)$ are inversions through a circle centered at $(q, 0)$ and a radius of r. Now we need to make sure that these to vertical lines are exactly 0.5 units apart because as we know from Proposition 2.6, reflection through two parallel will yield to twice the distance between the two lines. Since I want to make sure that my matrix B transforms everything only one unit to the right, then I need the distance between two vertical lines to be exactly 0.5 units apart. Thus, $|\tilde{z_1} - \tilde{z_2}| = 0.5$. Using Definition 3.6 we know the following:

$$|q - \tilde{z_1}| \cdot |q - z_1| = r^2$$
$$|q - \tilde{z_2}| \cdot |q - z_2| = r^2$$

Since $|\tilde{z_1} - \tilde{z_2}| = 0.5$ this implies that we have two equations: $\tilde{z_1} - \tilde{z_2} = 0.5$ and $\tilde{z_1} - \tilde{z_2} = -0.5$. Without loss of generality, we will define the positive solution as the circle on the right side. Thus, our $\tilde{z_1} = 0.5 + \tilde{z_2}$ and using substitution we get the following systems of equations:

$$(q - 0.5) - \tilde{z_2} = \tfrac{-r^2}{|q - z_1|}$$
$$-q + \tilde{z_2} = \tfrac{-r^2}{|q - z_2|}$$

Use elimination and then solve for r to get the following equation:

$$r = \sqrt{\frac{0.5 \cdot |q - z_2| \cdot |q - z_1|}{|q - z_1| + |q - z_2|}} \tag{3.4}$$

This allows us to find the required $M\ddot{o}bius$ transformation which represents our composition of hyperbolic reflections. Thus, $I(l_2) \circ I(l_1)$ which represents our $PBP^{-1} = T(z)$ is as follows: $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 0 & r^2 \\ 1 & -q \end{bmatrix}$

□

I created a manipulative of this proof in GeoGebra:
`https://www.geogebra.org/m/a25skhfe`

Example 3.13. *Given C_1 is centered at $(-4, 0)$ with radius 3 and C_2 is centered at $(1, 0)$ with radius 2, find the matrix that will represent the composition of hyperbolic reflections through these Euclidean circles.*

First using Proposition 3.12 we need to find the following values: q, z_1, z_2, and r.

Thus,

$$q = x_2 + r_2 = -4 + 3 = -1,$$
$$z_1 = x_1 + r_1 = 1 + 2 = 3,$$
$$z_2 = x_2 - r_2 = -4 - 3 = -7,$$
$$r = \sqrt{\tfrac{0.5\cdot|q-z_2|\cdot|q-z_1|}{|q-z_1|+|q-z_2|}} = \sqrt{\tfrac{0.5\cdot|-1-(-7)|\cdot|-1-3|}{|-1-3|+|-1-(-7)|}} \approx 1.1$$

This implies that the *Möbius* transformation that takes the fixed point to infinity is $\begin{bmatrix} 0 & r^2 \\ 1 & -q \end{bmatrix}$ which represents our P^{-1}. Now use substitution to compute PBP^{-1}. Therefore, $P^{-1} \approx \begin{bmatrix} 0 & 1.2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $P \approx \begin{bmatrix} -0.83 & 1 \\ .83 & 0 \end{bmatrix}$. Thus. $PBP^{-1} \approx \begin{bmatrix} 0.17 & -0.83 \\ 0.83 & 1.83 \end{bmatrix}$ which implies that the *Möbius* transformation that represents the composition of hyperbolic reflections through the given circles is $T(z) \approx \frac{0.17\cdot z-0.83}{0.83\cdot z+1.83}$.

Using GeoGebra I was able to create this example and show that our calculations are correct (see Figure 3.6). GeoGebra link for Example 3.13
`https://www.geogebra.org/m/rtzx9zzm`.

3.2.3 Hyperbolic

Now let T be a hyperbolic transformation represented by the matrix A. A hyperbolic transformation has two fixed points in $\partial\mathbb{H}^2$, we call the two fixed points the *source* and the *sink*. By analyzing the eigenvalues, A is hyperbolic transformation if and only if it is diagonalizable over $\mathbb{R}^2$, or conjugate in $SL(2,\mathbb{R})$ to a matrix $B = \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}$, $\lambda \neq 1$. This happens when the trace value is $Tr(T) > 2$. This *Möbius* transformation can be represented by the conjugate PBP^{-1}, where P^{-1} takes the *source* and maps it to zero and takes the *sink* and maps it to infinity. Then matrix B will dilate it by a factor of λ^2 along the y-axis in the direction from zero to infinity. Finally, matrix P maps the two fixed points back to the original locations in $\mathbb{H}^2$. This again is the same thing as reflecting through the two ultraparallel geodesics.

Proposition 3.14. *([Kat92, 3.3.4]) Given hyperbolic $T(z)$, there exists l_1, l_2 which are ultraparallel such that $I(l_2) \circ I(l_1) = T$.*

I will now prove the converse of this proposition.

Proposition 3.15. *Given two ultraparallel hyperbolic lines l_1 and l_2, the composition $I(l_2) \circ I(l_1)$ is hyperbolic. Furthermore, if $l_1 = ((x_1, 0), r_1)$, $l_2 = ((x_2, 0), r_2)$, and $d = |x_2 - x_1|$, then $I(l_2) \circ I(l_1)$ can be represented by PBP^{-1}, $P = \begin{bmatrix} x_1 + ({}^+x) & x_1 + ({}^-x) \\ 1 & 1 \end{bmatrix}$ where ${}^+x = \dfrac{-(r_2^2 - r_1^2 - d^2) + \sqrt{(r_2^2 - r_1^2 - d^2)^2 - 4(d)(dr_1^2)}}{2 \cdot d}$ and ${}^-x = \dfrac{-(r_2^2 - r_1^2 - d^2) - \sqrt{(r_2^2 - r_1^2 - d^2)^2 - 4(d)(dr_1^2)}}{2 \cdot d}$. Furthermore, $B = \begin{bmatrix} \lambda & 0 \\ 0 & \dfrac{1}{\lambda} \end{bmatrix}$ where $\lambda \neq 1$ and $\lambda = e^{\cosh^{-1}\left(\frac{d^2 - r_1^2 - r_2^2}{2 \cdot r_1 \cdot r_2}\right)}$.*

The hyperbolic transformation is a composition of reflections through two parallel lines which do not share endpoints, see Figure 3.7.

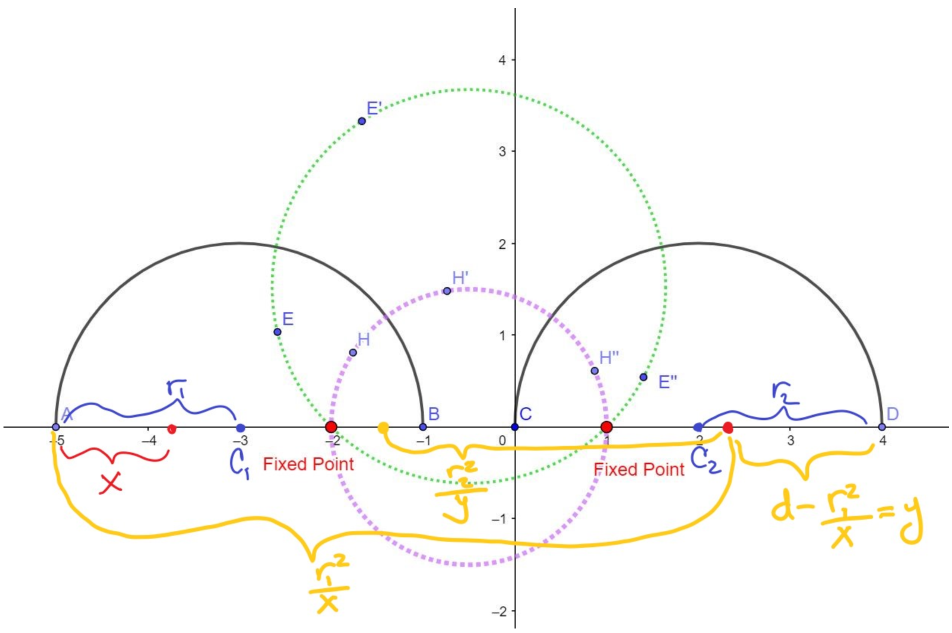


Figure 3.7: Hyperbolic Transformation

We can see that our initial point E which was reflected through the first circle, call it C_1, results in point E'. This is represented by the following notation, $I_{l_1}(E) = E'$. Now let us reflect E' through the second circle call it C_2 which will result in E'' and is represented by the following notation, $I_{l_2} \circ I_{l_1}(E) = E''$.

Proof. Let x be a distance away from the edge of the circle C_1 and now let us invert that point through the first circle. Using the equation for inversion we get the following,

$$\tilde{x} \cdot x = r_1^2 \Rightarrow \tilde{x} = \frac{r_1^2}{x}.$$

Similarly, we can calculate the second inversion using the same process and setting it to x will yield the following result,

$$d - \frac{r_2^2}{y} = x \Rightarrow d - \frac{r_2^2}{d - \frac{r_1^2}{x}} = x.$$

Now solve for x,

$$\begin{aligned}
d - \frac{r_2^2}{\frac{dx - r_1^2}{x}} &= x \\
d - \frac{r_2^2 \cdot x}{dx - r_1^2} &= x \\
d - x &= \frac{r_2^2 \cdot x}{dx - r_1^2} \\
d^2 \cdot x - d \cdot r_1^2 - d \cdot x^2 + x \cdot r_1^2 &= r_2^2 \cdot x \\
-dx^2 + d^2x + r_1^2x - r_2^2x - dr_1^2 &= 0
\end{aligned}$$

$$dx^2 + (r_2^2 - r_1^2 - d^2)x + dr_1^2 = 0. \tag{3.5}$$

$$x = \frac{-(r_2^2 - r_1^2 - d^2) \pm \sqrt{(r_2^2 - r_1^2 - d^2)^2 - 4(d)(dr_1^2)}}{2 \cdot d} \tag{3.6}$$

Since this is a quadratic, we will get two values of x using quadratic formula the ${}^{+}x$ and the ${}^{-}x$. Then $(x_1 + ({}^{+}x), 0)$ and $(x_1 + ({}^{-}x), 0)$ are your two fixed points on the x-axis. These two x values give you $P = \begin{bmatrix} x_1 + ({}^{+}x) & x_1 + ({}^{-}x) \\ 1 & 1 \end{bmatrix}$.

To calculate λ for our matrix $B = \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}$, we need to observe two ultraparallel lines and two fixed points with a geodesics going through them, see Figure 3.8.

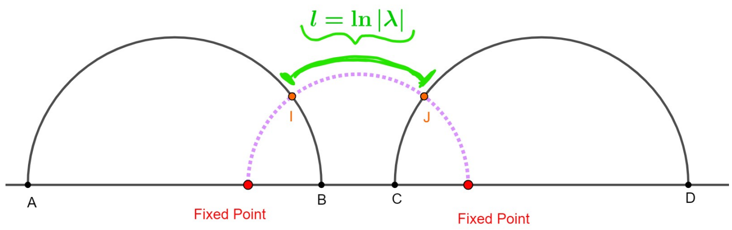


Figure 3.8: Lambda

Note that we need length between two geodesics to find λ. Using *Darboux Product* which states the following:

$$2 \cdot r_1 \cdot r_2 \cosh l = d^2 - r_1^2 - r_2^2, \tag{3.7}$$

we obtain $l = \cosh^{-1}\left(\frac{d^2 - r_1^2 - r_2^2}{2 \cdot r_1 \cdot r_2}\right)$ and therefore

$$\lambda = e^{\cosh^{-1}\left(\frac{d^2 - r_1^2 - r_2^2}{2 \cdot r_1 \cdot r_2}\right)}. \tag{3.8}$$

Since we have matrix $P = \begin{bmatrix} x_1 + (^+x) & x_1 + (^-x) \\ 1 & 1 \end{bmatrix}$ and matrix $B = \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}$ we can calculate matrix P using matrix P^{-1}. Therefore, the composition of reflections through two non intersecting circles whose centers are on x-axis can be represented by the *Möbius* transformation PBP^{-1}.

□

I created a manipulative of this proof in GeoGebra:
`https://www.geogebra.org/m/mqvdwjwb`

Example 3.16. *Given C_1 is centered at $(-3, 0)$ with radius 2 and C_2 is centered at $(2, 0)$ with radius 2, find the matrix that will represent the composition of hyperbolic reflections through these Euclidean circles.*

To start this example let us find the *source* and the *sink* using Equation 3.5 or Equation 3.6. I will be using Equation 3.5 to finding my two x values. Using Equation 3.5 we get:

$$5x^2 + (4 - 4 - 25)x + 20 = 0$$
$$5x^2 - 25x + 20 = 0$$
$$(x - 4)(x - 1) = 0$$
$$^+x = 4 \text{ and } ^-x = 1.$$

This implies that $P = \begin{bmatrix} -3+4 & -3+1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$. To find matrix B we first need to find λ using Equation 3.8. This yields the following:

$$\lambda = e^{\cosh^{-1}\left(\frac{5^2-2^2-2^2}{2\cdot 2\cdot 2}\right)}$$
$$\lambda = 4.$$

Therefore, matrix $B = \begin{bmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$ and now we can compute our algebraic calculation of PBP^{-1}. The result is $\begin{bmatrix} 1.5 & 2.5 \\ 1.25 & 2.75 \end{bmatrix}$ which implies that our $M\ddot{o}bius$ transformation is $T(z) \approx \frac{1.5\cdot z+2.5}{1.25\cdot z+2.75}$.

Using GeoGebra I was able to create this example and show you that our calculations are correct. GeoGebra link for Example 3.16.
`https://www.geogebra.org/m/macfevmw`

3.3 Isometries

In this section we want to make sure that regardless of which transformation we perform, the image does not change its shape or size.

Definition 3.17. *A transformation of $\mathbb{H}^2$ onto itself is called an **isometry** if it preserves the hyperbolic distance on $\mathbb{H}^2$.*

Just like in Euclidean geometry we showed that reflections are isometries which preserves length, we need to do the same in hyperbolic geometry as well. We want to show that hyperbolic reflection or inversions preserve length as well. This will lead into justifying that composition of reflections will preserve length.

Proposition 3.18. *Hyperbolic reflections in geodesics are isometries.*

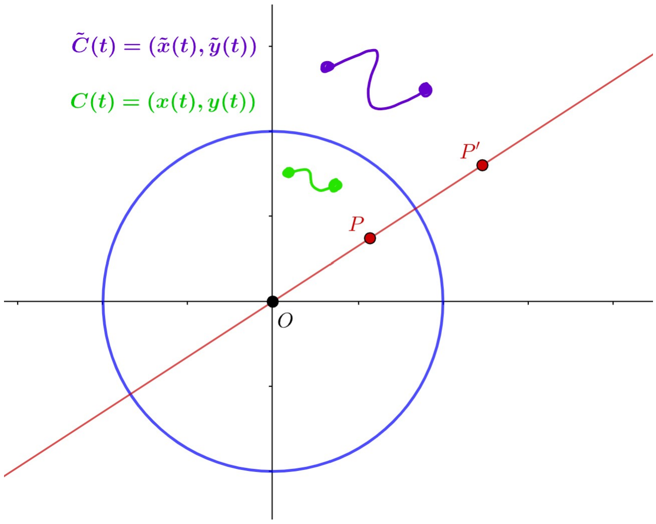


Figure 3.9: Inversion

Proof. The goal of this proof is to show that hyperbolic reflection preserve length. In upper half-plane our geodesics can be a semicircle or a vertical line. Let us assume we have some curve $C(t) = (x(t), y(t)), a < t < b$, and we reflected that function about a given geodesic and get $\tilde{C}(t) = (\tilde{x}(t), \tilde{y}(t))$, as seen in Figure 3.9. Remember that we defined our hyperbolic length to be $\int \frac{\sqrt{(x'(t))^2+(y'(t))^2}}{y(t)}dt$ in Equation 3.1 and we need to show that

$$\int_a^b \frac{\sqrt{(x'(t))^2+(y'(t))^2}}{y(t)}dt = \int_a^b \frac{\sqrt{(\tilde{x}'(t))^2+(\tilde{y}'(t))^2}}{\tilde{y}(t)}dt.$$

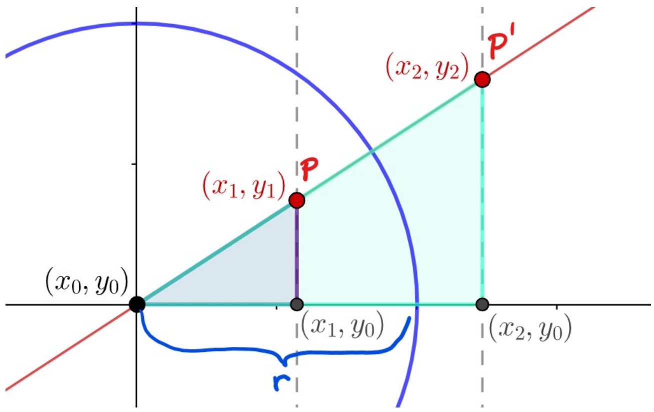


Figure 3.10: Similar Triangles

Let P be arbitrary point in $C(t)$ and P' be the reflected point in $\tilde{C}(t)$, as seen in Figure 3.10. The two points and the the center point of the circle we are reflecting about create two similar triangles. Now using Pythagorean Theorem and Definition 3.1 we get our first equation to be:

$$\sqrt{(x_1 - x_0)^2 + y_1^2} \cdot \sqrt{(x_2 - x_0)^2 + y_2^2} = r^2.$$

Knowing that P and P' have to be on the same line, we can use slope formula to get our second equation to be:

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{y_2 - y_0}{x_2 - x_0} \implies y_2 = \frac{x_2 - x_0}{x_1 - x_0} \cdot y_1.$$

Now we have two unknowns and two equations and we can solve for both unknowns, x_2 and y_2. This results in the following:

$$x_2 = \frac{r^2(x_1 - x_0)}{(x_1 - x_0)^2 + (y_1)^2} + x_0$$
$$y_2 = \frac{r^2(y_1)}{(x_1 - x_0)^2 + (y_1)^2}.$$

Let $x = (x_1 - x_0)$ using substitution. This gives us two new tilde values for our integral.

$$\tilde{x}(t) = \frac{r^2(x(t))}{(x(t))^2 + (y(t))^2} + x_0$$
$$\tilde{y}(t) = \frac{r^2(y(t))}{(x(t))^2 + (y(t))^2}$$

Now let us observe the squared derivative of $\tilde{x}(t)$ and $\tilde{y}(t)$:

$$(\tilde{x}'(t))^2 = \frac{-4r^4x(t)y(t)^3x'(t)y'(t)+4r^4x(t)^3y(t)x'(t)y'(t)+r^4x(t)^4x'(t)^2+r^4y(t)^4x'(t)^2-2r^4x(t)^2y(t)^2x'(t)^2+4r^4x(t)^2y(t)^2y'(t)^2}{(x(t)^2+y(t)^2)^4}$$

$$(\tilde{y}'(t))^2 = \frac{4r^4x(t)y(t)^3x'(t)y'(t)-4r^4x(t)^3y(t)x'(t)y'(t)+4r^4x(t)^2y(t)^2x'(t)^2+r^4x(t)^4y'(t)^2+r^4y(t)^4y'(t)^2-2r^4x(t)^2y(t)^2y'(t)^2}{(x(t)^2+y(t)^2)^4}$$

Now use substitution and then simplify to get the following:

$$\int_a^b \frac{\sqrt{(\tilde{x}'(t))^2+(\tilde{y}'(t))^2}}{\tilde{y}(t)}dt$$

$$= \int_a^b \frac{\sqrt{2x(t)^2y(t)^2x'(t)^2+2x(t)^2y(t)^2y'(t)^2+x(t)^4x'(t)^2+y(t)^4x'(t)^2+x(t)^4y'(t)^2+y(t)^4y'(t)^2}}{(x(t)^2+y(t)^2)(y(t))}$$

Continue to simplify even further, we get:

$$= \int_a^b \frac{\sqrt{(x(t)^2+y(t)^2)^2(x'(t)^2+y'(t)^2)}}{(x(t)^2+y(t)^2)(y(t))}$$

$$= \int_a^b \frac{(x(t)^2+y(t)^2)\sqrt{(x'(t)^2+y'(t)^2)}}{(x(t)^2+y(t)^2)(y(t))}$$

$$= \int_a^b \frac{\sqrt{(x'(t)^2+y'(t)^2)}}{(y(t))}$$

Thus,

$$\int_a^b \frac{\sqrt{(x'(t))^2+(y'(t))^2}}{y(t)}dt = \int_a^b \frac{\sqrt{(\tilde{x}'(t))^2+(\tilde{y}'(t))^2}}{\tilde{y}(t)}dt, \text{ as required.}$$

□

Proposition 3.19. *Elliptic, Parabolic, and Hyperbolic transformations are isometries in* $\mathbb{H}^2$.

Proof. By Definition 3.17 any isometry preserves length. Thus composition of isometries also preserves length, therefore, composition of isometries is an isometry.

□

Theorem 3.20. *Elliptic and Hyperbolic transformations are an isometry in* $\mathbb{H}^2$.

Proof. We know that hyperbolic reflections are isometries by Proposition 3.18. Note that one reflections is orientation reversing, however, two reflections will be orientation preserving. By Proposition 3.19 we know that composition of hyperbolic reflections are isometries, therefore, elleptic and hyperbolic transformations are isometries in $\mathbb{H}^2$.

□

Chapter 4

Triangle Reflection Groups

In this section I would like to focus on triangles that when reflected will form beautiful tessellations.

Definition 4.1. *A* ***hyperbolic triangle reflection group*** *is a discrete group which is generated by the reflections through the walls of* $\triangle(p, q, r)$*.*

Let X denote a hyperbolic triangle with interior angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, and $\frac{\pi}{r}$. Since the sides of a triangle always meet (possibly at infinity), the reflection group they generate come in one of the two types. First, the (full) reflection group $\triangle(p, q, r)$ is generated by the reflections through the walls of $X(p, q, r)$. The fact that the angles of X are submultiples of π guarantees that $\triangle(p, q, r)$ acts properly discontinuously on $\mathbb{H}^2$. Second, the subset $\triangle^+(p, q, r)$ of orientation preserving transformation is an index 2 subgroup. In Figure 4.1 we can see that blue triangles are orientation preserving and the red triangles will be the filled space by orientation reversing triangles.

A triangle group is discrete if and only if the angles of the triangle are submultiples of π, where $\frac{\pi}{p}$ and $p \in \mathbb{N}$ [Hit18]. Note that X lies in $\mathbb{H}^2$ precisely when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ since the sum of the interior angles in the hyperbolic triangle is strictly less than π.

We have proven that composition of reflections through two intersecting Euclidean and hyperbolic lines give us “rotation” about the intersection point. We have also proven that composition of reflections through two Euclidean parallel lines and two asymptotically parallel lines in $\mathbb{H}^2$ are just “translation” of twice the distance between the two parallel lines.

In my introduction I mention that there are three unique triangles that will create tessellations in the Euclidean plane. The three triangles are: $(3, 3, 3)$ equiangular triangle, $(2, 4, 4)$ right isosceles triangle, and $(2, 3, 6)$ right scalene triangle. These three unique triangles are the only triangles that can be used to create tessellations, as seen in Figure 4.1.

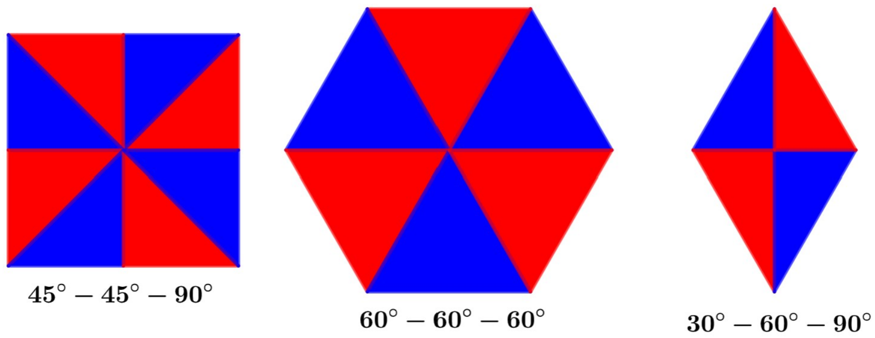


Figure 4.1: Euclidean Reflection Triangles

The reason Euclidean geometry has only three unique triangles that can make tessellations is because the sum of the Euclidean triangle must add up to 180°. Thus, there are only so many possibilities for that to happen. All three angles on the triangle have to be a quotient of π. This needs to happen because all three angles have to create not overlapping tilling for it to create discrete group. Furthermore, it has to satisfy the following equation: $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ since the sum of the interior angles of Euclidean triangle must equal to 180°.

This of course does not apply to hyperbolic geometry which allows us to find more unique triangles in that produce beautiful tessellations. My goal with hyperbolic triangles is to analyze which circles can produce these unique triangles where I can create tessellations in $\mathbb{H}^2$.

I introduced three types of elements in $PSL(2, \mathbb{R})$ in 3.2.

1. Elliptic, where we learned that composition of reflections through intersecting lines is like a rotation around a point within $\mathbb{H}^2$.

2. Parabolic, where we learned that composition of reflections through asymptotically

parallel lines is like rotation around a point on $\partial\mathbb{H}^2$.

3. Hyperbolic, where we learned that composition of reflections through ultraparallel lines is like transformation along a geodesic in $\mathbb{H}^2$.

We have also shown that all of these can be represented by a *Möbius* transformation. Therefore, I would like to observe what types of *Möbius* transformations can produce tessellations given circles and their radii.

Hyperbolic triangle groups are generated by elliptic and/or parabolic transformations. Since I would like to analyze these discrete groups and also provide a *Möbius* transformation for their generators, I will distinguish two types of triangles in $\mathbb{H}^2$. Let me call the first types of triangles *ideal triangles* and the second types of triangles I will call *elliptic triangles*.

4.1 Ideal Triangles

An ideal triangle in $\mathbb{H}^2$ is a triangle with at least one vertex on $\partial\mathbb{H}^2$, see Figure 4.2 and Figure 4.3.

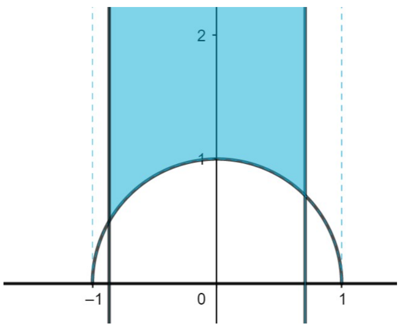


Figure 4.2: Ideal Triangle

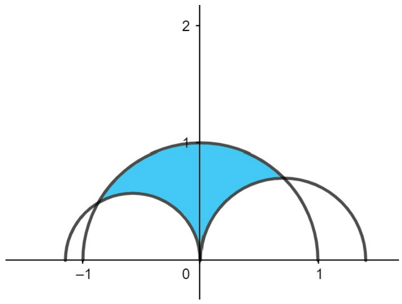


Figure 4.3: Transformed Ideal Triangle

Note that any ideal triangle in Figure 4.2 can always be transformed into a triangle in Figure 4.3. To map ideal triangle from Figure 4.2 to Figure 4.3 you invert vertical lines through the unit circle and you get what you see in Figure 4.3. The transformed ideal triangle has two angles that can be rotated using elliptic or parabolic transformation and the third angle can be translated using parabolic transformation.

Proposition 4.2. *If l is the geodesic from -1 to $+1$ and you are constructing a $\triangle^+(p,q,\infty)$, then the equation of the left vertical line of an ideal triangle is $x_L = -\cos\left(\frac{\pi}{p}\right)$ where $p \geq 2$ and p is an integer and the equation of the right vertical line of an ideal triangle is $x_R = \cos\left(\frac{\pi}{q}\right)$ where $q \geq 2$ and q is an integer, as seen in Figure 4.2.*

Proof. We are given unit circle therefore we can use trigonometry to find values of x_L and x_R, see Figure 4.4. I need to find the equations of these lines so I can later invert them through the unit circle and have three semicircles. To find equations of those vertical lines all is needed is the x-values for those points.

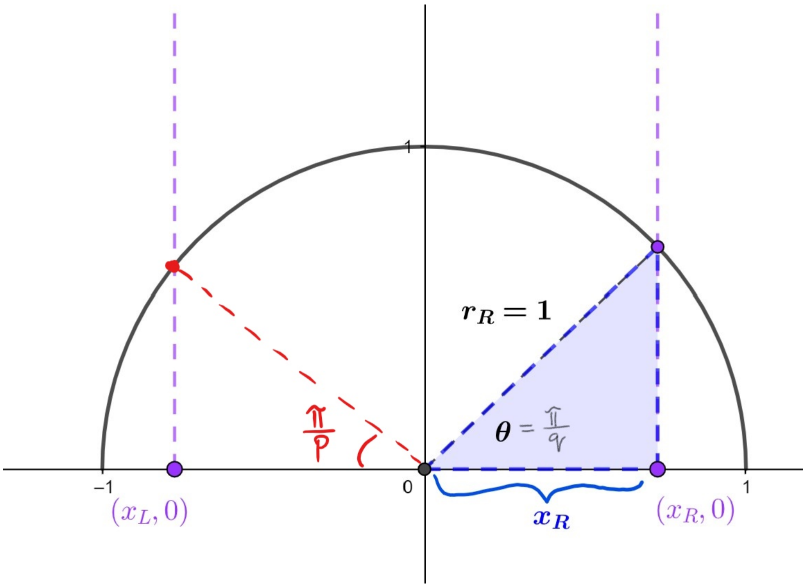


Figure 4.4: Finding x_R

Using trigonometry we know that cosine is equal to adjacent side divided by the hypotenuse, therefore, $\cos(\theta) = \left(\frac{x_R}{1}\right)$ where $\theta = \frac{\pi}{q}$ where $q \geq 2$ and q is an integer. Thus, $x_R = \cos\left(\frac{\pi}{q}\right)$. Similarly, the value of $x_L = -\cos\left(\frac{\pi}{p}\right)$ where $p \geq 2$ and p is an integer.

□

Proposition 4.3. *If given $\triangle^+(p,q,\infty)$ with vertical lines x_L and x_R from Proposition 4.2, these vertical lines can be inverted through unit circle and produce a left and a right circles. Left circle will be centered at $\left(\frac{1}{2\cdot x_L}, 0\right)$ with the radius $r_L = \left|\frac{1}{2\cdot x_L}\right|$ and the right circle will be centered at $\left(\frac{1}{2\cdot x_R}, 0\right)$ with the radius $r_R = \left|\frac{1}{2\cdot x_R}\right|$.*

Proof. Assume we have the values for x_L and x_R from Proposition 4.2. We will focus on finding the center and the radius of the right circle first and same approach can be applied to find the left circle. In the Figure 4.5 we want to know the radius and the center of the blue circle to the right of the unit circle.

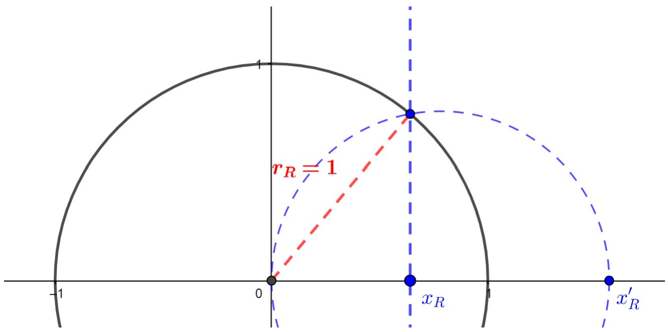


Figure 4.5: Finding Inverted Circle

Using Definition 3.6 we can calculate point x'_R, which results in $x'_R = \frac{r^2}{x_R}$. Since r will always equal 1 on a unit circle we get that $x'_R = \frac{1}{x_R}$ which is the diameter of our right circle and dividing x'_R by 2 will give us the x value of the center of our circle and the radius. Therefore, the inverted circle we are looking for will have a center at $\left(\frac{1}{2 \cdot x_R}, 0\right)$ and the radius $r_R = \left|\frac{1}{2 \cdot x_R}\right|$. $\square$

Example 4.4. *Given an ideal triangle whose $p = 6$ and $q = 3$, find all three matrices that will generate the triangle group $\triangle^+(3, 6, \infty)$.*

Since we are given $p = 6$ and $q = 3$ we know that the angles of this triangle are $0°$, $60°$, and $30°$. Apply Proposition 4.2 and Proposition 4.3 to find the inverted circles. This will give you the centers and the radius of each inverted circle through the unit circle. Now apply Proposition 3.9 to find the *Möbius* transformation that will generate triangle groups about the $30°$ angle and the $60°$ angle. Furthermore, let us apply Proposition 3.12 to find the generating triangle group about the infinity point. The three matrices that will generate the triangle group

$$\triangle^+(3, 6, \infty) = \left\langle \begin{bmatrix} \sqrt{3} & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \sqrt{3}+1 & 1 \end{bmatrix} \right\rangle.$$

GeoGebra link for additional examples.

`https://www.geogebra.org/m/wsvs3aba`

Here are some images of these types of hyperbolic triangles reflections courtesy of Dr. Meyer.

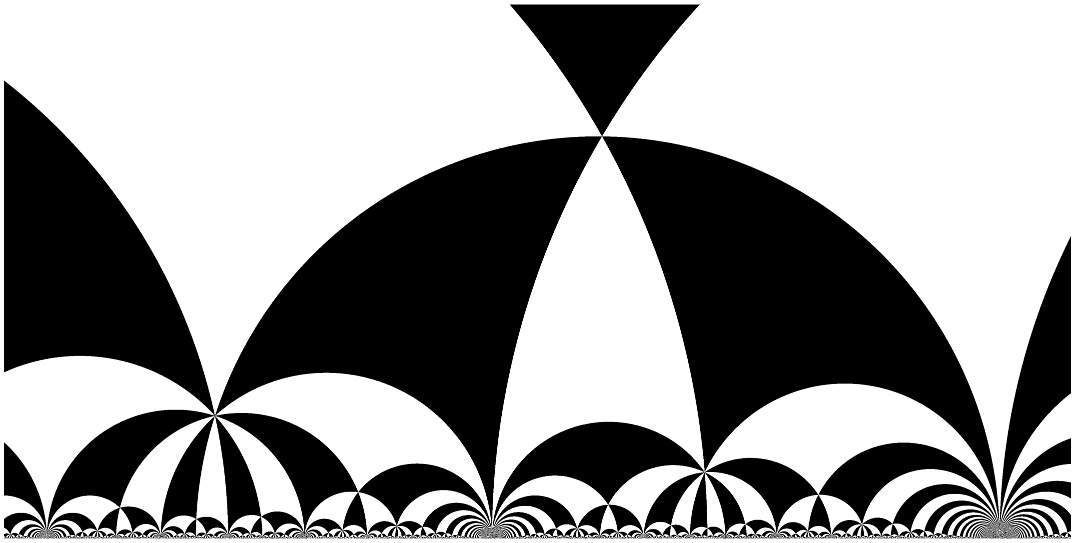


Figure 4.6: (3,6,Infinity)

4.2 Elliptic Triangles

An elliptic triangle in $\mathbb{H}^2$ is a triangle with all three angles being intersecting geodesics, see Figure 4.7.

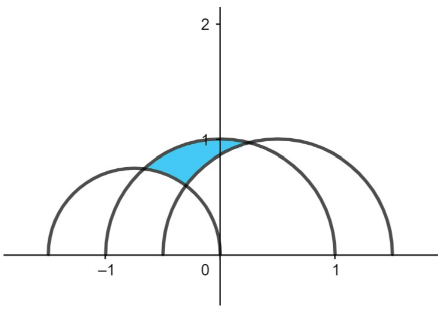


Figure 4.7: Elliptic Triangles

This implies that we can find three matrices that will generate the triangle groups $\triangle^+(p, q, r)$ given the centers and the radii of three circles. This paper will not cover different generating formulas but rather focus on one specific triangle group, which is $\triangle^{+}(2,6,6)$.

Example 4.5. *Given an elliptic triangle whose $p = 2$, $q = 6$, and $r = 6$ find all three matrices that will generate the triangle group $\triangle^{+}(2,6,6)$.*

To find this elliptic triangle we will use GeoGebra to figure out the exact centers and radii of all three circles that create it. Starting this problem I was not sure where to begin, so we will start simple. Let the first circle C_1 be centered at $(1,0)$ and a radius of $r = 1$, moreover, all three circles will have a radius of $r = 1$.

To find the centers of the other two circles let us use Proposition 3.9 and the equation $\theta = \arccos\left(\frac{r_1^2+r_2^2-d^2}{2\cdot r_1\cdot r_2}\right)$ which is equivalent to

$$\cos(\theta) = \left(\frac{r_1^2 + r_2^2 - d^2}{2 \cdot r_1 \cdot r_2}\right) \tag{4.1}$$

where $d = |x_2 - x_1|$. We know that our theta must be divisible by an integer, therefore, we know what theta is. Since we decided that all circles will have a radius of $r = 1$, then the only missing variable in the equation is d. We can calculate d given different thetas. Since we are looking for $\triangle(2,6,6)$ that means our theta will be $\theta_1 = 90°$, $\theta_2 = 30°$, and $\theta_3 = 30°$. Using special right triangles and a unit circle we can calculate that $\cos(90°) = 0$ and $\cos(30°) = \frac{\sqrt{3}}{2}$. To calculate the center of the second circle C_2, use Equation 4.1 to find d and then use $d = |x_2 - x_1|$ to find x_2. This yields that the center of C_2 is $\left(-\sqrt{2+\sqrt{3}}+1, 0\right)$. Similarly, we can calculate that the center of the third circle C_3 which is $\left(-\sqrt{2}+1, 0\right)$, as seen in Figure 4.8.

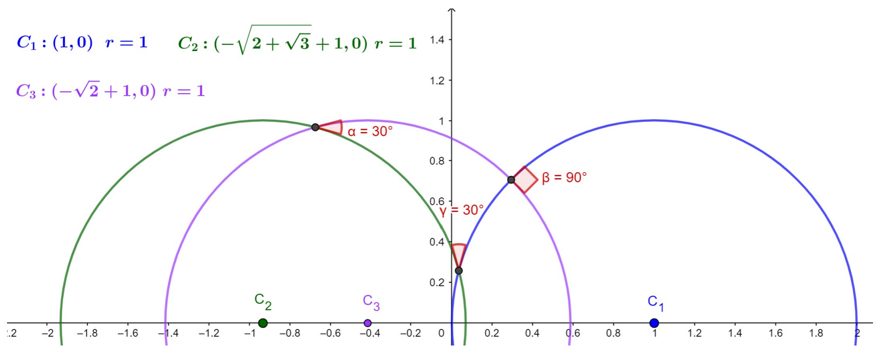


Figure 4.8: $\triangle(2,6,6)$

See GeoGebra link at,
urlhttps://www.geogebra.org/m/axshgpeg

Now that we know the exact centers and radii of our three circles we can calculate the three matrices that will generate the triangle group $\triangle^+(2,6,6)$. Apply Proposition 3.9 to find the *Möbius* transformation that will generate triangles groups about the 90° angle, 30° angle, and 30° angle. The three matrices that will generate the triangle group

$\triangle^+(2,6,6) =$

$$\left\langle \begin{bmatrix} 1-\sqrt{2} & -\sqrt{2}(-2+\sqrt{2}) \\ -\sqrt{2} & -1+\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1+\sqrt{3}-\sqrt{2+\sqrt{3}} & -2+\sqrt{6}-\sqrt{5-2\sqrt{6}} \\ -\sqrt{2+\sqrt{3}} & -1+\sqrt{2+\sqrt{3}} \end{bmatrix}, \begin{bmatrix} -1+\sqrt{3}-\sqrt{2-\sqrt{3}} & 2+\sqrt{6}-\sqrt{5+2\sqrt{6}} \\ -\sqrt{2-\sqrt{3}} & 1+\sqrt{2-\sqrt{3}} \end{bmatrix} \right\rangle.$$

Chapter 5

Conclusion

In this thesis we have analyzed reflections in Euclidean plane and hyperbolic upper half-plane. More specifically we focused on triangle reflection groups that generated tessellation. We observed some of many far reaching implications of the failure of the fifth postulate for hyperbolic geometry.

We have taken the time to show the equivalence of both algebraic and geometric approaches. One goal of this project was to show the geometric approach which is not well published as the algebraic approach. This allowed us to see the visual mathematics behind the algebraic approach in finding fundamental domain.

In this thesis we were able to find the matrix of *Möbius* transformation that would represent the composition of reflections through two hyperbolic lines. We also analyzed and proved that composition of reflections in both Euclidean and hyperbolic geometry give something like rotation or translation. In Euclidean and hyperbolic geometry if two lines intersect and you reflect an image about those two lines it will be a rotation around the intersection point. We learned that in Euclidean geometry you can only have one unique parallel line through a point outside the given line. We then proved that composition of reflections through two parallel lines is a translation in Euclidean plane. However, in hyperbolic geometry we learned that there are infinitely many parallel lines and more specifically we can have asymptotic or ultraparallel lines. We then showed that composition of reflections through asymptotically parallel lines is hyperbolic translation and composition of reflections through ultraparallel lines is hyperbolic dilation.

This project was an eyeopener for me because I have never studied hyperbolic geometry in depth. Seeing something in a different dimension allows you to see mathematics through a different lens. This project allowed my to see that composition of reflections through two geodesics are equivalent to $M\ddot{o}bius$ transformation. It connected the geometry and the algebra together in one senescence.

The next step in this project is to find exact geodesics that will generate the reflection groups we are looking for.

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