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ANALYSIS ON A HYPERPLANE OF THE QUATERNIONS

A Project

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Pamela Jean Whelchel

June 1995

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ABSTRACT

The purpose of this paper is to define functions on a 3-dimensional space and study the properties of these functions, which has been done for 1 and 2-dimensional space. The 3-dimensional space used is subspace of the 4-dimensional quaternionic space. Much of the work done parallels complex analysis. Because of the identities discovered, further study into quaternionic analysis is likely. Also the results allow for many applications in the area of physics.

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CHAPTER 1

Introduction

Complex matrices of the form $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ are called quaternions. The quaternions form a 4-dimensional vector space over the real numbers. I will refer to this set of matrices as \mathbf{H} and it will be written $\mathbf{H} = \{aI + b\vec{i} + c\vec{j} + d\vec{k} | a, b, c, d \in \mathbb{R}\}$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\vec{i} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ $\vec{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\vec{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. The purpose of this paper is to study the subset of \mathbf{H} where $d=0$. This subset forms a 3-dimensional vector space over the real numbers. I will refer to this subset as \mathbf{J} and it will be written $\mathbf{J} = \{a + b\vec{i} + c\vec{j} | a, b, c \in \mathbb{R}\}$. I will be discussing the algebraic and functional properties of these 3-dimensional quaternions. Much of what I have done parallels the work done in 2-dimensional complex analysis.

The first part of this paper introduces the quaternions and their algebraic properties. The set \mathbf{H} forms an algebra over the reals. Next, properties of the subset \mathbf{J} are examined. It is shown that the set \mathbf{J} is not closed under multiplication. Integral powers are closed but division in general is not. The Jordan product will be used on the set \mathbf{J} giving us a Jordan algebra. Some of the basic properties of the Jordan product will be demonstrated.

Once a Jordan algebra is established, elementary functions with domain and

codomain \mathbf{J} are defined. These functions are defined as power series with real coefficients. The exponential function is an example of an elementary function and will be used throughout this paper to demonstrate various properties or formulas of elementary functions. A discussion of the convergence of such functions will be done which parallels the complex analysis version showing the convergence of complex analytic functions. This will be followed by an explanation of why the formal derivative of an elementary function could not be defined in the same manner as is done for complex and real functions.

This leads to the next part of the paper. Since the formal derivative was not attainable, a formula relating the adjoint matrix and the Jacobian of an elementary function was created and proven. Again, the exponential is used to demonstrate the discoveries that were made in this section of the paper.

Next, some applications of these elementary functions will be explored. The divergence and curl of the exponential function as well as the general elementary function will be calculated. Stokes's theorem and the Divergence theorem will then be applied. The divergence and curl have applications in physics.

This topic was chosen because it was possible to create functions with domain and codomain \mathbf{J} , these functions being power series. This made it possible to do studies similar to real and complex analysis. It is hoped that the results of this project will find applications in areas such as elasticity and hyperbolic geometry.

CHAPTER 2

Quaternions and a 3-dimensional Hyperplane

A complex 2x2 matrix of the form $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ is called a quaternion. Quaternions are written in the form $\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

where a, b, c and d are real numbers. For the purpose of this paper, $\vec{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$\vec{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\vec{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which allows the quaternion to be written

$q = a + b\vec{i} + c\vec{j} + d\vec{k}$. The following properties are clear for I, \vec{i}, \vec{j} and \vec{k} :

$\vec{i}^2 = \vec{j}^2 = \vec{k}^2 = -I$, $\vec{i}\vec{j} = \vec{k}$, $\vec{j}\vec{i} = -\vec{k}$, $\vec{j}\vec{k} = \vec{i}$, $\vec{k}\vec{j} = -\vec{i}$, $\vec{k}\vec{i} = \vec{j}$, $\vec{i}\vec{k} = -\vec{j}$. Let α denote

the function with domain the quaternions and codomain $V_4(\mathbf{R})$ given by

$\alpha: \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} \mapsto (a, b, c, d)$. α is a bijection, and under α the structure of matrix

addition corresponds with vector addition, and the structure of scalar multiplication of

matrices corresponds to scalar multiplication. The isomorphism α means the set H of

quaternions is a 4-dimensional vector space over the real numbers.

We will now be discussing the hyperplane of the quaternions where $d = 0$. It will be shown that the subset $\mathbf{J} = \{a + b\vec{i} + c\vec{j} | a, b, c \in \mathbf{R}\}$ forms a 3-dimensional vector space over the reals.

Theorem 2.1

\mathbf{J} is a 3-dimensional subspace of the vector space $\mathbf{H} = \{a + b\vec{i} + c\vec{j} + d\vec{k} | a, b, c, d \in \mathbf{R}\}$.

Proof:

Show **J** forms a group under addition.

1. closure

Let $q_1, q_2 \in \mathbf{J}$.

$$q_1 + q_2 = (a + b_1\vec{i} + c_1\vec{j}) + (a_2 + b_2\vec{i} + c_2\vec{j}) = (a_1 + a_2) + (b_1 + b_2)\vec{i} + (c_1 + c_2)\vec{j} \in \mathbf{J}.$$

2. associative

This is inherited from **H**.

3. identity

This is also inherited from **H**.

4. inverse

Let $q_1 \in \mathbf{J}$

$$\text{so } q_1 = a_1 + b_1\vec{i} + c_1\vec{j}.$$

$$\text{Let } q_2 = -a_1 + -b_1\vec{i} + -c_1\vec{j} \in \mathbf{J}$$

$$\begin{aligned}\text{so } q_1 + q_2 &= (a_1 + b_1\vec{i} + c_1\vec{j}) + (-a_1 + -b_1\vec{i} + -c_1\vec{j}) \\ &= (a_1 + -a_1) + (b_1 + -b_1)\vec{i} + (c_1 + -c_1)\vec{j} \\ &= 0.\end{aligned}$$

Therefore $q_2 = -q_1$ is the inverse of q_1 .

Show **J** is closed under scalar multiplication.

Let $q_1 \in \mathbf{J}$ and $\alpha \in \mathbf{R}$.

$$\begin{aligned}\text{So } \alpha(q_1) &= \alpha(a_1 + b_1\vec{i} + c_1\vec{j}) \\ &= \alpha a_1 + \alpha b_1\vec{i} + \alpha c_1\vec{j}.\end{aligned}$$

Therefore $\alpha(q_1)$ is in \mathbf{J} since $\alpha a_1, \alpha b_1, \alpha c_1 \in \mathbf{R}$.

Therefore \mathbf{J} is closed under scalar multiplication, and \mathbf{J} has been proven to be a subspace of \mathbf{H} .

\mathbf{J} does not form a group under multiplication, however, because it is not closed under multiplication.

Let $q_1, q_2 \in \mathbf{J}$.

$$\begin{aligned}\text{So } q_1 q_2 &= (a_1 + b_1\vec{i} + c_1\vec{j})(a_2 + b_2\vec{i} + c_2\vec{j}) \\ &= (a_1 a_2 + a_1 b_2\vec{i} + a_1 c_2\vec{j} + a_2 b_1\vec{i} - b_1 b_2 + b_1 c_2\vec{k} + a_2 c_1\vec{j} - b_2 c_1\vec{k} - c_1 c_2) \\ &= (a_1 a_2 - b_1 b_2 - c_1 c_2) + (a_1 b_2 + a_2 b_1)\vec{i} + (a_1 c_2 + a_2 c_1)\vec{j} + (b_1 c_2 - b_2 c_1)\vec{k}.\end{aligned}$$

Since $q_1 q_2$ has a \vec{k} -component, \mathbf{J} is not closed under multiplication. While \mathbf{H} forms an algebra over the reals, \mathbf{J} is not a subalgebra because it is not closed under multiplication.

\mathbf{J} is also not commutative in general.

Let $q_1 q_2 \in \mathbf{J}$.

$$\text{So } q_1 q_2 = (a_1 a_2 - b_1 b_2 - c_1 c_2) + (a_1 b_2 + a_2 b_1)\vec{i} + (a_1 c_2 + a_2 c_1)\vec{j} + (b_1 c_2 - b_2 c_1)\vec{k},$$

$$\begin{aligned}\text{and } q_2 q_1 &= (a_2 + b_2\vec{i} + c_2\vec{j})(a_1 + b_1\vec{i} + c_1\vec{j}) \\ &= a_1 a_2 + a_2 b_1\vec{i} + a_2 c_1\vec{j} + a_1 b_2\vec{i} - b_1 b_2 + b_2 c_1\vec{k} + a_1 c_2\vec{j} - b_1 c_2\vec{k} - c_1 c_2\end{aligned}$$

$$= (a_1a_2 - b_1b_2 - c_1c_2) + (a_2b_1 + a_1b_2)\vec{i} + (a_2c_1 + a_1c_2)\vec{j} + (b_2c_1 - b_1c_2)\vec{k}$$

$\neq q_1q_2$ because the \vec{k} terms are different.

Therefore **J** is not commutative. In fact, the \vec{k} -components are opposites while the others are the same. This will be useful later. Also, since the elements of **J** do not commute, division is not well-defined, since $q_1q_2^{-1}$ and $q_2^{-1}q_1$ are not necessary equal.

Next, it will be shown that **J** is closed under integral powers, even though **J** is not closed under multiplication in general.

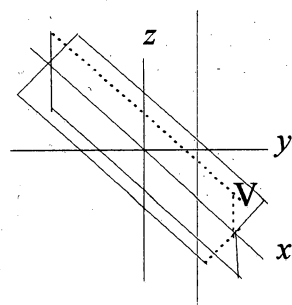
Theorem 2.2

J is closed under integral powers.

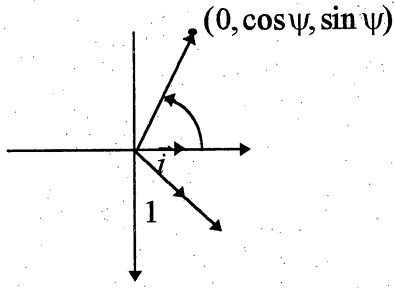
Lemma 2.3

Let **V** be any 2-dimensional subspace which contains the real axis. Then **V** is isomorphic to the complex plane as a real algebra.

Proof of Lemma:



V = a plane through the real axis.



Let Π_ψ be a plane through the real axis where $0 \leq \psi < \pi$ is the dihedral angle Π_ψ makes with the plane spanned by the real axis and the \vec{i} - axis.

The span of $(1,0,0)$ and $(0,\cos\psi,\sin\psi)$ gives the subspace V . We will refer to such a subspace V as a **C-plane**.

Let Π_ψ be a **C-plane** represented by quaternions of the form $\begin{pmatrix} t \\ s \cos \psi \\ s \sin \psi \end{pmatrix} s, t \in \mathbf{R}$.

We define $V \xrightarrow{\alpha_\psi} \mathbf{C}$ as follows:

$$\alpha_\psi: \begin{pmatrix} t \\ s \cos \psi \\ s \sin \psi \end{pmatrix} \mapsto t + si$$

Therefore $q \in V$ means $q = \begin{pmatrix} t \\ s \cos \psi \\ s \sin \psi \end{pmatrix} = t + s \cos \psi \vec{i} + s \sin \psi \vec{j}$ for some $s, t \in \mathbf{R}$.

Now that everything has been defined, we can prove that $V \cong \mathbf{C}$ by showing α_ψ is an isomorphism.

1. Show α_ψ is linear.

Let $c_1, c_2 \in \mathbf{R}$ and $q_1, q_2 \in \Pi_\psi$.

$$\begin{aligned}
\text{So } \alpha_\psi(c_1 q_1 + c_2 q_2) &= \alpha_\psi \left[c_1 (t_1, s_1 \cos \psi, s_1 \sin \psi) + c_2 (t_2, s_2 \cos \psi, s_2 \sin \psi) \right] \\
&= \alpha_\psi \left[(c_1 t_1, c_1 s_1 \cos \psi, c_1 s_1 \sin \psi) + (c_2 t_2, c_2 s_2 \cos \psi, c_2 s_2 \sin \psi) \right] \\
&= \alpha_\psi (c_1 t_1 + c_2 t_2, c_1 s_1 \cos \psi + c_2 s_2 \cos \psi, c_1 s_1 \sin \psi + c_2 s_2 \sin \psi) \\
&= \alpha_\psi \left[c_1 t_1 + c_2 t_2, (c_1 s_1 + c_2 s_2) \cos \psi, (c_1 s_1 + c_2 s_2) \sin \psi \right] \\
&= (c_1 t_1 + c_2 t_2) + (c_1 s_1 + c_2 s_2) i \\
&= c_1 t_1 + (c_1 s_1) i + c_2 t_2 + (c_2 s_2) i \\
&= c_1 (t_1 + s_1 i) + c_2 (t_2 + s_2 i) \\
&= c_1 [\alpha_\psi(t_1, s_1 \cos \psi, s_1 \sin \psi)] + c_2 [\alpha_\psi(t_2, s_2 \cos \psi, s_2 \sin \psi)] \\
&= c_1 [\alpha_\psi(q_1)] + c_2 [\alpha_\psi(q_2)] \\
&= c_1 \alpha_\psi(q_1) + c_2 \alpha_\psi(q_2).
\end{aligned}$$

Therefore α_ψ is linear.

2. Show α_ψ is a bijection.

Let $c \in \mathbf{C}$, so c can be written $c = t + si$ where $t, s \in \mathbf{R}$.

There exists $\begin{pmatrix} t \\ s \cos \psi \\ s \sin \psi \end{pmatrix} \in V$ such that $\alpha_\psi \begin{pmatrix} t \\ s \cos \psi \\ s \sin \psi \end{pmatrix} = t + si$.

Therefore α_ψ is onto.

Let $c_1, c_2 \in \mathbb{C}$ with $c_1 = c_2$.

So $c_1 = t_1 + s_1 i$ and $c_2 = t_2 + s_2 i$ where t_1, t_2, s_1 and $s_2 \in \mathbb{R}$.

Also $t_1 = t_2$ and $s_1 = s_2$ since $c_1 = c_2$.

Thus $\begin{pmatrix} t_1 \\ s_1 \cos \psi \\ s_1 \sin \psi \end{pmatrix} = \begin{pmatrix} t_2 \\ s_2 \cos \psi \\ s_2 \sin \psi \end{pmatrix}$ since each component is equal.

Therefore α_ψ is one to one.

α_ψ has now been shown to be a bijection.

3. Show α_ψ preserves multiplication.

Let $q_1, q_2 \in V$.

So $q_1 = t_1 + s_1 \cos \psi \vec{i} + s_1 \sin \psi \vec{j}$ and $q_2 = t_2 + s_2 \cos \psi \vec{i} + s_2 \sin \psi \vec{j}$.

$$\begin{aligned}
\text{Also } \alpha_\psi(q_1 q_2) &= \alpha_\psi \left[(t_1 + s_1 \cos \psi \vec{i} + s_1 \sin \psi \vec{j}) (t_2 + s_2 \cos \psi \vec{i} + s_2 \sin \psi \vec{j}) \right] \\
&= \alpha_\psi \left(\begin{aligned} &t_1 t_2 + t_1 s_2 \cos \psi \vec{i} + t_1 s_2 \sin \psi \vec{j} + s_1 t_2 \cos \psi \vec{i} - s_1 s_2 \cos^2 \psi + s_1 s_2 \cos \psi \sin \psi \vec{k} \\ &+ s_1 t_2 \sin \psi \vec{j} - s_1 s_2 \cos \psi \sin \psi \vec{k} - s_1 s_2 \sin^2 \psi \end{aligned} \right) \\
&= \alpha_\psi \left[(t_1 t_2 - s_1 s_2) + (t_1 s_2 + s_1 t_2) \cos \psi \vec{i} + (t_1 s_2 + s_1 t_2) \sin \psi \vec{j} \right] \\
&= \alpha_\psi \left[\begin{aligned} &(t_1 t_2 - s_1 s_2) \\ &(t_1 s_2 + s_1 t_2) \cos \psi \\ &(t_1 s_2 + s_1 t_2) \sin \psi \end{aligned} \right] \\
&= (t_1 t_2 - s_1 s_2) + (t_1 s_2 + s_1 t_2) i \\
&= t_1 t_2 + (t_1 s_2) i + (s_1 t_2) i - s_1 s_2 \\
&= (t_1 + s_1 i)(t_2 + s_2 i) \\
&= \alpha_\psi(q_1) \alpha_\psi(q_2).
\end{aligned}$$

Therefore α_ψ preserves multiplication.

Since α_ψ is a bijection which preserves multiplication, α_ψ is an isomorphism. An isomorphism has been constructed from an arbitrary 2-dimensional subspace containing the real axis to the complex plane. This proves the lemma. Using Lemma 2.3, we can now prove Theorem 2.2.

Proof:

Let $q \in \mathbf{J}$.

So $q = a + b\vec{i} + c\vec{j}$ for some $a, b, c \in \mathbf{R}$.

Let $a = t$, $b = s \cos \psi$ and $c = s \sin \psi$ with $s = \pm \sqrt{b^2 + c^2}$ and $\psi = \tan^{-1} \frac{c}{b}$.

So we may write $q = t + s \cos \psi \vec{i} + s \sin \psi \vec{j}$ and $q \in \Pi_\psi$.

Also $\Pi_\psi \cong \mathbb{C}$ (the complex plane).

Thus q lies on a plane isomorphic to the complex plane and therefore has the same properties a point in the complex plane has. Since the complex plane is closed under multiplication q can be multiplied with itself and the product will still be in the plane Π_ψ . Since division in the complex plane is well-defined q can be inverted and the reciprocal will still be in the plane Π_ψ . In other words, all integral powers of q remain in the plane Π_ψ . Therefore \mathbf{J} is closed under integral powers.

The Lemma also shows that the product of any two elements in a particular plane, Π_ψ , remains in that plane.

Let $q_1 = t_1 + s_1 \cos \psi \vec{i} + s_1 \sin \psi \vec{j}$ and $q_2 = t_2 + s_2 \cos \psi \vec{i} + s_2 \sin \psi \vec{j}$.

So $q_1 q_2 = (t_1 t_2 - s_1 s_2) + (t_1 s_2 + s_1 t_2) \cos \psi \vec{i} + (t_1 s_2 + s_1 t_2) \sin \psi \vec{j} \in \Pi_\psi$.

Multiplying two elements in different C-planes gives a different result, however.

Let $q_1 \in \Pi_{\psi_1}$ and $q_2 \in \Pi_{\psi_2}$.

So $q_1 = t_1 + s_1 \cos \psi_1 \vec{i} + s_1 \sin \psi_1 \vec{j}$ and $q_2 = t_2 + s_2 \cos \psi_2 \vec{i} + s_2 \sin \psi_2 \vec{j}$.

$$\begin{aligned}
\text{Thus } q_1 q_2 &= t_1 t_2 + t_1 s_2 \cos \psi_2 \vec{i} + t_1 s_2 \sin \psi_2 \vec{j} + s_1 t_2 \cos \psi_1 \vec{i} - s_1 s_2 \cos \psi_1 \cos \psi_2 + s_1 s_2 \cos \psi_1 \sin \psi_2 \vec{k} \\
&\quad + s_1 t_2 \sin \psi_1 \vec{j} - s_1 s_2 \sin \psi_1 \cos \psi_2 \vec{k} - s_1 s_2 \sin \psi_1 \sin \psi_2 \\
&= (t_1 t_2 - s_1 s_2 \cos \psi_1 \cos \psi_2 - s_1 s_2 \sin \psi_1 \sin \psi_2) + (t_1 s_2 \cos \psi_2 + s_1 t_2 \cos \psi_1) \vec{i} \\
&\quad + (t_1 s_2 \sin \psi_2 + s_1 t_2 \sin \psi_1) \vec{j} + (s_1 s_2 \cos \psi_1 \sin \psi_2 - s_1 s_2 \sin \psi_1 \cos \psi_2) \vec{k} \\
&= t_1 t_2 - s_1 s_2 \cos(\psi_1 - \psi_2) + (t_1 s_2 \cos \psi_2 + s_1 t_2 \cos \psi_1) \vec{i} + (t_1 s_2 \sin \psi_2 + s_1 t_2 \sin \psi_1) \vec{j} \\
&\quad + s_1 s_2 \sin(\psi_2 - \psi_1) \vec{k}.
\end{aligned}$$

This shows the product of two elements in different planes is not an element of **J** unless the $\sin(\psi_2 - \psi_1) = 0$. This is true when $\psi_2 - \psi_1 = n\pi$, meaning the two elements would lie on the same plane. Therefore the two elements must be in the same plane for the product to be in the same plane, or even in the subspace **J**.

Now that an isomorphism has been established between the complex plane and any plane containing the real axis, which we will refer to as a **C-plane**, similarities between complex numbers and quaternions in a plane will be discussed.

Properties of the complex plane:

1. Closed under multiplication
2. Commutative
3. Contains inverses
4. Closed under division

Properties of elements of \mathbf{J} that lie in one \mathbf{C} -plane:

1. $q_1, q_2 \in \Pi_\psi$

$q_1 q_2 \in \Pi_\psi$ (proved previously)

2. $q_1 q_2 = q_2 q_1$, since Π_ψ is isomorphic to \mathbf{C} .

Specifically

$$q_1 = t_1 + s_1 \cos \psi \vec{i} + s_1 \sin \psi \vec{j} \text{ and } q_2 = t_2 + s_2 \cos \psi \vec{i} + s_2 \sin \psi \vec{j}.$$

$$\begin{aligned} \text{So } q_1 q_2 &= (t_1 t_2 - s_1 s_2) + (t_1 s_2 + s_1 t_2) \cos \psi \vec{i} + (t_1 s_2 + s_1 t_2) \sin \psi \vec{j} \text{ (calculated previously)} \\ &= q_2 q_1 \end{aligned}$$

3. q_2 is the inverse of q_1

$$q_1 = t_1 + s_1 \cos \psi \vec{i} + s_1 \sin \psi \vec{j} \quad q_2 = \frac{t_1}{t_1^2 + s_1^2} + \left(\frac{-s_1}{t_1^2 + s_1^2} \right) \cos \psi \vec{i} + \left(\frac{-s_1}{t_1^2 + s_1^2} \right) \sin \psi \vec{j}$$

$$q_1 q_2 = (t_1 + s_1 \cos \psi \vec{i} + s_1 \sin \psi \vec{j}) \left(\frac{1}{t_1^2 + s_1^2} \right) (t_1 - s_1 \cos \psi \vec{i} - s_1 \sin \psi \vec{j})$$

$$= \frac{1}{t_1^2 + s_1^2} \left(\begin{aligned} &t_1^2 - s_1 t_1 \cos \psi \vec{i} - s_1 t_1 \sin \psi \vec{j} + s_1 t_1 \cos \psi \vec{i} + s_1^2 \cos^2 \psi - s_1^2 \cos \psi \sin \psi \vec{k} \\ &+ s_1 t_1 \sin \psi \vec{j} + s_1^2 \sin \psi \cos \psi \vec{k} + s_1^2 \sin^2 \psi \end{aligned} \right)$$

$$= \frac{1}{t_1^2 + s_1^2} (t_1^2 + s_1^2)$$

$$= 1$$

$$4. \frac{q_1}{q_2} = q_1 q_2^{-1} = q_2^{-1} q_1 \in \Pi_\psi$$

As expected, the elements of \mathbf{J} which lie in the same \mathbf{C} -plane behave the same as complex numbers.

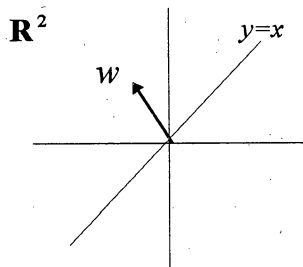
As stated previously, in general the product of two elements of \mathbf{J} is not commutative. Their products differ only in the \vec{k} -components which are negatives of each other. Geometrically $q_1 q_2$ and $q_2 q_1$ are reflections in \mathbf{J} .

Definition

Reflection in a hyperplane Suppose U is a hyperplane of the n -dimensional inner product space V , defined as the set of vectors orthogonal to a fixed unit vector w . Then we define r_w , the reflection in U , by the formula $r_w(v) = v - (2v \cdot w)w$, where v is an arbitrary vector in V .

Example:

In \mathbf{R}^2 the reflection in the line $y=x$ is given by $r_w(v) = v - (2v \cdot w)w$.



$$w = \left(\frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$$

The matrix representation for this reflection is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ relative to the standard basis.

Lemma 2.4

If $q_1, q_2 \in \mathbf{J}$ then $q_1 q_2 = r_{\vec{k}}(q_2 q_1)$.

Proof of Lemma:

Case 1:

If q_1 and q_2 are elements of the same **C**-plane then $q_1 q_2$ has no \vec{k} component, and therefore reflecting with respect to the \vec{k} -vector does nothing. $r_{\vec{k}}(q_2 q_1) = q_2 q_1$. It was shown previously that $q_2 q_1 = q_1 q_2$ when q_1 and q_2 are in the same **C**-plane. Therefore $r_{\vec{k}}(q_2 q_1) = q_2 q_1 = q_1 q_2$.

Case 2:

If q_1 and q_2 are elements of different **C**-planes, the proof is a little more complicated.

Let $q_1 \in \Pi_{\psi_1}$ and $q_2 \in \Pi_{\psi_2}$.

$$\begin{aligned} \text{So } q_1 q_2 &= (t_1 t_2 - s_1 s_2) \cos(\psi_1 - \psi_2) + (s_2 t_1 \cos \psi_2 + s_1 t_2 \cos \psi_1) \vec{i} \\ &\quad + (s_2 t_1 \sin \psi_2 + s_1 t_2 \sin \psi_1) \vec{j} + s_1 s_2 \sin(\psi_2 - \psi_1) \vec{k} \\ \text{and } q_2 q_1 &= (t_1 t_2 - s_1 s_2) \cos(\psi_1 - \psi_2) + (s_2 t_1 \cos \psi_2 + s_1 t_2 \cos \psi_1) \vec{i} \\ &\quad + (s_2 t_1 \sin \psi_2 + s_1 t_2 \sin \psi_1) \vec{j} - s_1 s_2 \sin(\psi_2 - \psi_1) \vec{k}. \end{aligned}$$

Using the reflection definition,

$$\begin{aligned}
r_{\vec{k}}(q_2 q_1) &= q_2 q_1 - (2q_2 q_1 \cdot \vec{k}) \vec{k} \\
&= q_2 q_1 - [-2s_1 s_2 \sin(\psi_2 - \psi_1)] \vec{k} \\
&= (t_1 t_2 - s_1 s_2) \cos(\psi_1 - \psi_2) + (s_2 t_1 \cos \psi_2 + s_1 t_2 \cos \psi_1) \vec{i} \\
&\quad + (s_2 t_1 \sin \psi_2 + s_1 t_2 \sin \psi_1) \vec{j} - s_1 s_2 \sin(\psi_2 - \psi_1) \vec{k} + 2s_1 s_2 \sin(\psi_2 - \psi_1) \vec{k} \\
&= (t_1 t_2 - s_1 s_2) \cos(\psi_1 - \psi_2) + (s_2 t_1 \cos \psi_2 + s_1 t_2 \cos \psi_1) \vec{i} \\
&\quad + (s_2 t_1 \sin \psi_2 + s_1 t_2 \sin \psi_1) \vec{j} + s_1 s_2 \sin(\psi_2 - \psi_1) \vec{k} \\
&= q_1 q_2.
\end{aligned}$$

This proves the lemma $r_{\vec{k}}(q_2 q_1) = q_1 q_2$.

Lemma 2.5

Let $q \in \mathbf{J}$. Then $q^{-1} = \frac{\bar{q}}{|q|^2}$.

Proof of Lemma

Let $q = a_1 + b_1 \vec{i} + c_1 \vec{j}$ and $q^{-1} = a_2 + b_2 \vec{i} + c_2 \vec{j}$.

So $qq^{-1} = a_1 a_2 + a_1 b_2 \vec{i} + a_1 c_2 \vec{j} + a_2 b_1 \vec{i} - b_1 b_2 + b_1 c_2 \vec{k} + a_2 c_1 \vec{i} - b_2 c_1 \vec{k} - c_1 c_2$.

For q^{-1} to be the inverse of q

$a_1 a_2 - b_1 b_2 - c_1 c_2 = 1$, $a_1 b_2 + a_2 b_1 = 0$, $a_1 c_2 + a_2 c_1 = 0$ and $b_1 c_2 - b_2 c_1 = 0$.

Solving for unknowns a_2, b_2 and c_2 in terms of a_1, b_1 and c_1 ,

$$a_2 = \frac{a_1}{a_1^2 + b_1^2 + c_1^2}, \quad b_2 = \frac{-b_1}{a_1^2 + b_1^2 + c_1^2} \quad \text{and} \quad c_2 = \frac{-c_1}{a_1^2 + b_1^2 + c_1^2}.$$

$$\text{So } q^{-1} = \frac{\bar{q}}{|q|^2}.$$

Checking explicitly,

$$\begin{aligned} qq^{-1} &= (a_1 + b_1\vec{i} + c_1\vec{j}) \left(\frac{1}{a_1^2 + b_1^2 + c_1^2} \right) (a_1 - b_1\vec{i} - c_1\vec{j}) \\ &= \frac{1}{a_1^2 + b_1^2 + c_1^2} (a_1^2 - a_1b_1\vec{i} - a_1c_1\vec{j} + a_1b_1\vec{i} + b_1^2 + b_1c_1\vec{k} + a_1c_1\vec{j} + b_1c_1\vec{k} + c_1^2) \\ &= \frac{1}{a_1^2 + b_1^2 + c_1^2} (a_1^2 + b_1^2 + c_1^2) \\ &= 1. \end{aligned}$$

It was determined at a later point in this study that knowing the form and relationship between $h^{-1}qh$ and hqh^{-1} , where $h, q \in \mathbf{J}$, was useful.

$$\text{Let } q = a_1 + b_1\vec{i} + c_1\vec{j} \text{ and } h = a_2 + b_2\vec{i} + c_2\vec{j}, \text{ so } h^{-1} = \frac{1}{a_2^2 + b_2^2 + c_2^2} (a_2 - b_2\vec{i} - c_2\vec{j}).$$

$$\begin{aligned}
\text{Then } h^{-1}qh &= \frac{1}{|h|^2} (a_2 - b_2 \vec{i} - c_2 \vec{j}) (a_1 + b_1 \vec{i} + c_1 \vec{j}) (a_2 + b_2 \vec{i} + c_2 \vec{j}) \\
&= \frac{1}{|h|^2} (a_1 a_2 + a_2 b_1 \vec{i} + a_2 c_1 \vec{j} - a_1 b_2 \vec{i} + b_1 b_2 - b_2 c_1 \vec{k} - a_1 c_2 \vec{j} + b_1 c_2 \vec{k} + c_1 c_2) h \\
&= \frac{1}{|h|^2} \begin{bmatrix} (a_1 a_2^2 + a_2 b_1 b_2 + a_2 c_1 c_2 - a_2 b_1 b_2 + a_1 b_2^2 - a_2 c_1 c_2 + a_1 c_2^2) \\ (a_2^2 b_1 - a_1 a_2 b_2 + a_1 a_2 b_2 + b_1 b_2^2 + b_2 c_1 c_2 + b_2 c_1 c_2 + b_1 c_2^2) \vec{i} \\ (a_2^2 c_1 - a_1 a_2 c_2 - b_2^2 c_1 + b_1 b_2 c_2 + a_1 a_2 c_2 + b_1 b_2 c_2 + c_1 c_2^2) \vec{j} \\ (-a_2 b_2 c_1 + a_2 b_1 c_2 - a_2 b_2 c_1 + a_1 b_2 c_2 + a_2 b_1 c_2 - a_1 b_2 c_2) \vec{k} \end{bmatrix} \\
&= \frac{1}{|h|^2} \begin{bmatrix} (a_1 a_2^2 + a_1 b_2^2 + a_1 c_2^2) + (a_2^2 b_1 + b_1 b_2^2 - b_1 c_2^2 + 2b_2 c_1 c_2) \vec{i} \\ + (a_2^2 c_1 - b_2^2 c_1 + c_1 c_2^2 + 2b_1 b_2 c_2) \vec{j} + (2a_2 b_1 c_2 - 2a_2 b_2 c_1) \vec{k} \end{bmatrix}
\end{aligned}$$

Similarly,

$$hqh^{-1} = \frac{1}{|h|^2} \begin{bmatrix} (a_1 a_2^2 + a_1 b_2^2 + a_1 c_2^2) + (a_2^2 b_1 + b_1 b_2^2 - b_1 c_2^2 + 2b_2 c_1 c_2) \vec{i} \\ + (a_2^2 c_1 - b_2^2 c_1 + c_1 c_2^2 + 2b_1 b_2 c_2) \vec{j} - (2a_2 b_1 c_2 - 2a_2 b_2 c_1) \vec{k} \end{bmatrix}$$

Notice their sum is in \mathbf{J} .

Since \mathbf{J} is not closed under multiplication, the Jordan product will be used, and the set of quaternions with no \vec{k} -component together with the Jordan product will be referred to as \mathbf{J}^+ .

Definition

The Jordan product is defined as follows. Let $q_1, q_2 \in \mathbf{J}$, then $[q_1, q_2] = \frac{1}{2}(q_1 q_2 + q_2 q_1)$.

With the product of two elements in \mathbf{J} defined this way, the resulting quaternion is also in \mathbf{J} .

Let $q_1 = a_1 + b_1\vec{i} + c_1\vec{j}$ and $q_2 = a_2 + b_2\vec{i} + c_2\vec{j}$.

Using the definition of the Jordan product,

$$\begin{aligned} [q_1, q_2] &= \frac{1}{2} \left[(a_1 + b_1\vec{i} + c_1\vec{j})(a_2 + b_2\vec{i} + c_2\vec{j}) + (a_2 + b_2\vec{i} + c_2\vec{j})(a_1 + b_1\vec{i} + c_1\vec{j}) \right] \\ &= \frac{1}{2} \left[\begin{aligned} &a_1a_2 + a_1b_2\vec{i} + a_1c_2\vec{j} + a_2b_1\vec{i} - b_1b_2 + b_1c_2\vec{k} + a_2c_1\vec{j} - b_2c_1\vec{k} - c_1c_2 \\ &+ a_1a_2 + a_2b_1\vec{i} + a_2c_1\vec{j} + a_1b_2\vec{i} - b_1b_2 + b_2c_1\vec{k} + a_1c_2\vec{j} - b_1c_2\vec{k} - c_1c_2 \end{aligned} \right] \\ &= (a_1a_2 - b_1b_2 - c_1c_2) + (a_1b_2 + a_2b_1)\vec{i} + (a_1c_2 + a_2c_1)\vec{j} \in \mathbf{J}. \end{aligned}$$

Thus, using this product, we now have a Jordan algebra.

The following are various properties of the Jordan product.

Let $q_1, q_2, q_3 \in \mathbf{J}^+$ and $r \in \mathbf{R}$.

$$\begin{aligned} [q_1, q_2] &= \frac{1}{2} (q_1q_2 + q_2q_1) \\ &= \frac{1}{2} (q_2q_1 + q_1q_2) \\ &= [q_2, q_1]. \end{aligned}$$

The Jordan product is commutative.

$$\begin{aligned}
[q_1, r] &= \frac{1}{2}(q_1 r + r q_1) \\
&= \frac{1}{2}(2r q_1) \\
&= r q_1.
\end{aligned}$$

The Jordan product with a scalar is the same as scalar multiplication.

$$\begin{aligned}
[q_1, q_2 + q_3] &= \frac{1}{2}[q_1(q_2 + q_3) + (q_2 + q_3)q_1] \\
&= \frac{1}{2}(q_1 q_2 + q_1 q_3 + q_2 q_1 + q_3 q_1) \\
&= \frac{1}{2}(q_1 q_2 + q_2 q_1 + q_1 q_3 + q_3 q_1) \\
&= \frac{1}{2}(q_1 q_2 + q_2 q_1) + \frac{1}{2}(q_1 q_3 + q_3 q_1) \\
&= [q_1, q_2] + [q_1, q_3].
\end{aligned}$$

The Jordan product is distributive.

$$\begin{aligned}
[[q_1, q_2], q_3] &= \left[\frac{1}{2}(q_1 q_2 + q_2 q_1), q_3 \right] \\
&= \frac{1}{2} \left[\frac{1}{2}(q_1 q_2 + q_2 q_1) q_3 + \frac{1}{2} q_3 (q_1 q_2 + q_2 q_1) \right] \\
&= \frac{1}{4}(q_1 q_2 q_3 + q_2 q_1 q_3 + q_3 q_1 q_2 + q_3 q_2 q_1).
\end{aligned}$$

$$\begin{aligned}
[q_1, [q_2, q_3]] &= \left[q_1, \frac{1}{2}(q_2 q_3 + q_3 q_2) \right] \\
&= \frac{1}{2} \left[\frac{1}{2} q_1 (q_2 q_3 + q_3 q_2) + \frac{1}{2} (q_2 q_3 + q_3 q_2) q_1 \right] \\
&= \frac{1}{4} (q_1 q_2 q_3 + q_1 q_3 q_2 + q_2 q_3 q_1 + q_3 q_2 q_1).
\end{aligned}$$

The Jordan product is not associative as the following computation shows. The associator is computed as follows.

$$\begin{aligned}
& [[q_1, q_2], q_3] - [q_1, [q_2, q_3]] \\
&= \frac{1}{4} (q_1 q_2 q_3 + q_2 q_1 q_3 + q_3 q_1 q_2 + q_3 q_2 q_1) \\
&\quad - \frac{1}{4} (q_1 q_2 q_3 + q_1 q_3 q_2 + q_2 q_3 q_1 + q_3 q_2 q_1) \\
&= \frac{1}{4} (q_2 q_1 q_3 + q_3 q_1 q_2 + q_1 q_3 q_2 + q_2 q_3 q_1).
\end{aligned}$$

$$\begin{aligned}
q_2 q_1 q_3 &= (a_2 + b_2 \vec{i} + c_2 \vec{j})(a_1 + b_1 \vec{i} + c_1 \vec{j})q_3 \\
&= \left[\begin{aligned} &(a_1 a_2 - b_1 b_2 - c_1 c_2) + (a_2 b_1 + a_1 b_2) \vec{i} \\ &+ (a_2 c_1 + a_1 c_2) \vec{j} + (b_2 c_1 - b_1 c_2) \vec{k} \end{aligned} \right] (a_3 + b_3 \vec{i} + c_3 \vec{j}) \\
&= (a_1 a_2 a_3 - a_3 b_1 b_2 - a_3 c_1 c_2 - a_2 b_1 b_3 - a_1 b_2 b_3 - a_2 c_1 c_3 - a_1 c_2 c_3) \\
&\quad + (a_2 a_3 b_1 + a_1 a_3 b_2 + a_1 a_2 b_3 - b_1 b_2 b_3 - b_3 c_1 c_2 - b_2 c_1 c_3 + b_1 c_2 c_3) \vec{i} \\
&\quad + (a_2 a_3 c_1 + a_1 a_3 c_2 + b_2 b_3 c_1 - b_1 b_3 c_2 + a_1 a_2 c_3 - b_1 b_2 c_3 - c_1 c_2 c_3) \vec{j} \\
&\quad + (a_3 b_2 c_1 - a_3 b_1 c_2 - a_2 b_3 c_1 - a_1 b_3 c_2 + a_2 b_1 c_3 + a_1 b_2 c_3) \vec{k}.
\end{aligned}$$

[illegible]

CHAPTER 3

Elementary Functions and Their Convergence

Now that it has been established that we are working with a set that is closed under the Jordan product, functions with domain and co-domain of \mathbf{J} will be considered. The convergence of such functions will be discussed as well as the derivative.

Definition

A function $f: \mathbf{J} \rightarrow \mathbf{J}$ is called elementary at a provided it can be represented as a series

$f(q) = \sum_{n=0}^{\infty} c_n (q - a)^n$, $c_n \in \mathbf{R}$, where the domain of f is the set of all q for which this series converges.

Definition

The exponential function is defined as follows: $f(q) = e^q = \sum_{n=0}^{\infty} \frac{1}{n!} q^n$.

This is an example of an elementary function at $a=0$.

The elementary functions discussed in this paper will be centered at the origin. In other words, $a=0$ and $f(q) = \sum_{n=0}^{\infty} c_n q^n$. Without loss of generality, we can let $a=0$ and use

$f(q) = \sum_{n=0}^{\infty} c_n q^n$ in forthcoming calculations.

It will be shown that $f(q) = \sum c_n q^n$ converges on a ball centered at the origin, just

as complex analytic functions converge on a disc centered at the origin. This will be done by first proving absolute convergence, then by showing the radius of convergence R is the same in all \mathbb{C} -planes.

Theorem 3.1

Suppose there is some $q_1 \neq 0$ such that $\sum_{n=0}^{\infty} c_n q_1^n$ converges. Then for each q with $|q| < |q_1|$ the series $\sum_{n=0}^{\infty} c_n q^n$ is absolutely convergent.

Note:

In this proof, division of absolute values of quaternions is used. While $\frac{q_{n+1}}{q_n}$ is not well-

defined since $\frac{q_{n+1}}{q_n}$ could equal either $q_{n+1} \cdot q_n^{-1}$ or $q_n^{-1} \cdot q_{n+1}$, which are not necessarily

equal, $\frac{|q_{n+1}|}{|q_n|}$ is well-defined since each is a distance from the origin and therefore a real

number.

Proof of theorem:

Suppose the distance from q to the origin is less than or equal to some r which is strictly less than the distance from q_1 to the origin.

In other words, $|q| \leq r < |q_1|$ where $r \in \mathbb{R}$.

Since $\sum_{n=0}^{\infty} c_n q_1^n$ converges (given),

which means $\lim_{n \rightarrow \infty} c_n q_1^n = 0$,

then there exists a constant M such that $|c_n||q_1|^n \leq M$ for all n .

Therefore $|c_n||q|^n = |c_n||q_1|^n \left(\frac{|q|}{|q_1|} \right)^n \leq M\rho^n$ for all n , where $\rho = \frac{r}{|q_1|} < 1$.

Thus $\left(\frac{|q|}{|q_1|} \right)^n \leq \rho^n$ since $\frac{|q|}{|q_1|} \leq \frac{r}{|q_1|} = \rho$.

Again we are dealing with absolute values, or distances from the origin, which means we do not have to worry about division of quaternions since $|q| \in \mathbf{R}$.

Continuing the proof, we have $|c_n||q|^n \leq M\rho^n$ for all n .

But $M\rho^n$ is a geometric series which converges if $|\rho| < 1$.

Since $\rho = \frac{r}{|q_1|} < 1$, $|\rho| < 1$.

Therefore $M\rho^n$ converges absolutely, which means $|c_n||q|^n$ converges using the comparison test.

$|c_n||q|^n \leq M\rho^n$ for all n .

So $\sum_{n=0}^{\infty} c_n q^n$ is absolutely convergent, which completes the proof.

The proof of this theorem works for quaternions the same way it does for complex numbers. Next, the radius of convergence needs to be determined for

$$f(q) = \sum_{n=0}^{\infty} c_n q^n.$$

Definition:

Radius of Convergence: There are three possibilities for the radius of convergence of

$$\sum_{n=0}^{\infty} c_n q^n \text{ with } c_n \in \mathbf{R} \text{ and } q \in \mathbf{J}.$$

- 1 $\sum c_n q^n$ converges only for $q=0$.
- 2 $\sum c_n q^n$ converges for all q .
- 3 $\sum c_n q^n$ converges for some q but not all q .

If case 1 is true the radius of convergence is zero.

$$R = 0.$$

If case 2 is true the radius of convergence is infinity.

$$R = \infty.$$

If case 3 is true, the radius of convergence will be defined as follows:

Let q' be a point of convergence and q'' a point of divergence.

Then $\sum_{n=0}^{\infty} c_n (q')^n$ converges and $\sum_{n=0}^{\infty} c_n (q'')^n$ diverges.

Using the theorem just proved,

$|q| < |q'|$ means $\sum c_n q^n$ converges and $|q| > |q''|$ means $\sum c_n q^n$ diverges.

Therefore $|q'| < |q''|$.

We now define the radius of convergence to be the number R such that

$|q| < R$ means $\sum c_n q^n$ converges and $|q| > R$ means $\sum c_n q^n$ diverges.

This completes the definition of the radius of convergence. Using this definition along with Theorem 3.1, the convergence of an elementary function can be found.

Since our radius of convergence for elementary quaternionic functions is based on distances from the origin, a function will have the same radius of convergence in each \mathbb{C} -plane. In other words, the function $f(q) = \sum c_n q^n$ converges on a ball of radius R centered at the origin. f converges for all points inside the ball, diverges for all points outside the ball, and as with complex analysis f is inconclusive on the bounding sphere.

We now know that these elementary functions are absolutely convergent on an open ball centered at the origin with radius R . Next will be a theorem used to determine what the value of R is. The ratio test will be used in the next theorem so a proof of this theorem will be given.

Theorem 3.2

The Ratio Test: Let $\sum U_k$ be a series with positive terms and suppose $\lim_{k \rightarrow +\infty} \frac{U_{k+1}}{U_k} = \rho$.

- a) If $\rho < 1$ the series converges.
- b) If $\rho > 1$ the series diverges.
- c) If $\rho = 1$ the series may converge or diverge, so that another test must be tried.

Proof:

part a)

Assume $\rho < 1$ and let $r = \frac{1}{2}(1 + \rho)$.

Thus $\rho < r < 1$ since r is the midpoint between 1 and ρ .

It follows that the number $\varepsilon = r - \rho$ is positive.

Since $\rho = \lim_{k \rightarrow +\infty} \frac{U_{k+1}}{U_k}$ it follows that for k sufficiently large say $k \geq K$ the ratio $\frac{U_{k+1}}{U_k}$ are

within ε units of ρ .

Thus we will have $\frac{U_{k+1}}{U_k} < \rho + \varepsilon$ when $k \geq K$ or $\frac{U_{k+1}}{U_k} < r$ when $k \geq K$.

That is $U_{k+1} < r U_k$ when $k \geq K$.

This yields the inequalities:

$$U_{K+1} < rU_k$$

$$U_{K+2} < rU_{k+1} < r^2U_k$$

$$U_{K+3} < rU_{k+2} < r^3U_k$$

$$U_{K+4} < rU_{k+3} < r^4U_k \text{ etcetera.}$$

But $|r| < 1$.

So that $rU_k + r^2U_k + r^3U_k + \dots$ is a convergent geometric series.

From the above inequalities and the comparison test it follows that $U_{k+1} + U_{k+2} + U_{k+3} + \dots$ must also be a convergent series.

Thus $U_1 + U_2 + U_3 + \dots + U_k + \dots$ converges because convergence is unaffected by deleting a finite number of terms from the beginning of a series.

We have proven part a), the series converges if $\rho < 1$.

part b)

Assume $\rho > 1$ thus $\varepsilon = \rho - 1$ is a positive number.

Since $\rho = \lim_{k \rightarrow +\infty} \frac{U_{k+1}}{U_k}$ it follows that for k sufficiently large say $k \geq K$ the ratio $\frac{U_{k+1}}{U_k}$ is

within ε units of ρ .

Thus $\frac{U_{k+1}}{U_k} > \rho - \varepsilon$ when $k \geq K$ or $\frac{U_{k+1}}{U_k} > 1$ when $k \geq K$.

That is $U_{k+1} > U_k$ when $k \geq K$.

This yields the inequalities:

$$U_{K+1} > U_K$$

$$U_{K+2} > U_{K+1} > U_K$$

$$U_{K+3} > U_{K+2} > U_K$$

$$U_{K+4} > U_{K+3} > U_K \text{ etcetera.}$$

Since $U_k > 0$ it follows that $\lim_{k \rightarrow +\infty} U_k \neq 0$.

So $U_1 + U_2 + \dots + U_k + \dots$ diverges.

We have proven part b), the series diverges $\rho > 1$.

part c)

$\sum_{k=1}^{\infty} \frac{1}{k}$ is a divergent harmonic series.

Therefore, $\lim_{k \rightarrow \infty} \frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k \rightarrow \infty} \frac{k}{k+1} = 1$.

So $\rho = 1$.

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent ρ -series.

Therefore, $\lim_{k \rightarrow \infty} \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} = \lim_{k \rightarrow \infty} \frac{k^2}{k^2 + 2k + 1} = 1.$

So $\rho = 1.$

Therefore the ratio test does not distinguish between convergence and divergence when

$\rho = 1.$

Theorem 3.3

Suppose $\sum_{n=0}^{\infty} c_n q^n$ has a positive or infinite radius of convergence R . If $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$ exists

then $\frac{1}{R} = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|.$

Proof:

Let $L = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|.$

Then $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1} q^{n+1}}{c_n q^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1} q}{c_n} \right| = |q| \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = |q| L.$

Note: This is possible since \mathbf{J} is closed under integral powers.

Using the ratio test,

when $|q|L < 1$ the series $\sum_{n=0}^{\infty} c_n q^n$ is absolutely convergent

and when $|q|L > 1$ the series $\sum_{n=0}^{\infty} c_n q^n$ diverges.

Using the definition of R ,

$$R = \frac{1}{L} \text{ or } L = \frac{1}{R}.$$

$$\text{So } \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = \frac{1}{R}.$$

This completes the proof.

Example:

The radius of convergence for $f(x) = e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$ is infinity.

Using the ratio test for $f(q) = e^q = \sum_{n=0}^{\infty} \frac{1}{n!} q^n$,

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{q_{n+1}}{q_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{q^{n+1}}{(n+1)!} \cdot \frac{n!}{q^n} \right| = \lim_{n \rightarrow \infty} \frac{|q|}{n+1} = 0.$$

Since $\rho < 1$ for all $q \in \mathbb{J}$, the series converges absolutely and the radius of convergence is

$$R = \infty.$$

Using the previous theorem,

$$\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n!}{(n+1)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{n+1} \right| = 0.$$

$$\text{So } \frac{1}{R} = 0 \text{ and } R = \infty.$$

The exponential function has the same radius of convergence in real, complex and quaternionic space. It converges over the whole space.

It is worth pointing out that q and e^q lie in the same \mathbf{C} -plane. $e^q = \sum_{n=0}^{\infty} \frac{1}{n!} q^n$ is an infinite series in q . If q is in a particular \mathbf{C} -plane, then all powers of q are also in the \mathbf{C} -plane because the theorem stated previously showed this. Also, any finite sum of these powers is also in the same \mathbf{C} -plane. This is also true for the limit of these sums. Obviously, the same property holds for any elementary function.

Next, an attempt was made to find the derivative of an elementary function using a definition based on the calculus derivative. In calculus, when given the function $f(x) = c_n x^n$, the function $f'(x) = n c_n x^{n-1}$ is proven to be the derivative function. In complex analysis when given the function $f(z) = c_n z^n$, it is proven that the function $f'(z) = n c_n z^{n-1}$ is the derivative function. This was an important discovery in complex analysis. For both calculus and complex analysis, however, showing that f' was the derivative function required using the binomial theorem. An attempt was made to follow the complex analysis proof, but we were unable to use the binomial theorem. Rewritten for quaternions, a reasonable analogue of the binomial theorem would look as follows:

$$(q+h)^n = \sum_{j=0}^n \binom{n}{j} [q^{n-j}, h^j]. \text{ This formula turned out not to be true.}$$

Example:

$$(q+h)^3 = q^3 + qhq + hq^2 + h^2q + q^2h + qh^2 + hqh + h^3.$$

$$\begin{aligned}
\sum_{j=0}^3 \binom{3}{j} [q^{3-j}, h^j] &= \binom{3}{0} [q^3, h^0] + \binom{3}{1} [q^2, h] + \binom{3}{2} [q, h^2] + \binom{3}{3} [q^0, h^3] \\
&= q^3 + 3 \left[\frac{1}{2} (q^2 h + h q^2) \right] + 3 \left[\frac{1}{2} (q h^2 + h^2 q) \right] + h^3 \\
&= q^3 + \frac{3}{2} q^2 h + \frac{3}{2} h q^2 + \frac{3}{2} q h^2 + \frac{3}{2} h^2 q + h^3
\end{aligned}$$

which does not equal $(q + h)^3$.

Since the binomial theorem can not be used, a different approach was tried.

Letting $f(q) = \sum_{n=0}^{\infty} c_n q^n$ and $f'(q) = \sum_{n=1}^{\infty} n c_n q^{n-1}$, an attempt was made to bound the right side of the following equation.

$$[f(q+h) - f(q), h^{-1}] - \sum_{n=1}^{\infty} n c_n q^{n-1} = \left[\sum_{n=0}^{\infty} c_n (q+h)^n - \sum_{n=0}^{\infty} c_n q^n, h^{-1} \right] - \sum_{n=1}^{\infty} n c_n q^{n-1}$$

Using the right side of the above equation,

$$\begin{aligned}
&= \frac{1}{2} \left[\sum_{n=0}^{\infty} c_n (q+h)^n h^{-1} - \sum_{n=0}^{\infty} c_n q^n h^{-1} + \sum_{n=0}^{\infty} h^{-1} c_n (q+h)^n - \sum_{n=0}^{\infty} h^{-1} c_n q^n \right] - \sum_{n=1}^{\infty} n c_n q^{n-1} \\
&= \frac{1}{2} \sum_{n=2}^{\infty} \left\{ [c_n (q+h)^n - c_n q^n] h^{-1} + h^{-1} [c_n (q+h)^n - c_n q^n] \right\} \\
&\quad + \frac{1}{2} (c_1 h^{-1} - c_1 h^{-1} + h^{-1} c_1 - h^{-1} c_1 + c_1 (q+h) h^{-1} - c_1 q h^{-1} + h^{-1} c_1 (q+h) - h^{-1} c_1 q) \\
&\quad - \sum_{n=2}^{\infty} n c_n q^{n-1} - c_1
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{n=2}^{\infty} \left\{ \left[c_n (q+h)^n - c_n q^n \right] h^{-1} + h^{-1} \left[c_n (q+h)^n - c_n q^n \right] \right\} - \sum_{n=2}^{\infty} n c_n q^{n-1} \\
&= \frac{1}{2} \sum_{n=2}^{\infty} c_n \left\{ \left[(q+h)^n - q^n \right] h^{-1} + h^{-1} \left[(q+h)^n - q^n \right] - 2n q^{n-1} \right\}
\end{aligned}$$

If we had been able to bound the right side of this equation so as $h \rightarrow 0$ the whole right side went to zero, we would have had the following equation:

$$\lim_{h \rightarrow 0} \left\{ \left[f(q+h) - f(q), h^{-1} \right] - \sum_{n=1}^{\infty} n c_n q^{n-1} \right\} = 0.$$

This would mean $\lim_{h \rightarrow 0} \left[f(q+h) - f(q), h^{-1} \right] = \sum_{n=1}^{\infty} n c_n q^{n-1}$, and therefore f' would be the derivative of f .

With very little success at proving this, the cases where $n=2$ and $n=3$ were considered. While doing these, it turned out that the limit of the right side as $h \rightarrow 0$ was path dependent, and therefore we could not state that

$f'(q) = \lim_{h \rightarrow 0} \left[f(q+h) - f(q), h^{-1} \right]$ was equal to $\sum_{n=1}^{\infty} n c_n q^{n-1}$. The following shows how

the proof of $f'(q) = \sum_{n=1}^{\infty} n c_n q^{n-1}$ failed for the $n=2$ case.

$$\begin{aligned}
\left[f(q+h) - f(q), h^{-1} \right] - \sum_{n=1}^{\infty} n c_n q^{n-1} &= \frac{1}{2} \sum_{n=2}^{\infty} c_n \left[h^{-1} \left((q+h)^n - q^n \right) + \left((q+h)^n - q^n \right) h^{-1} - 2n q^{n-1} \right] \\
n &= 2 \\
&= \frac{1}{2} c_2 \left[h^{-1} \left((q+h)^2 - q^2 \right) + \left((q+h)^2 - q^2 \right) h^{-1} - 4q \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}c_2 \left(h^{-1}(qh + hq + h^2) + (qh + hq + h^2)h^{-1} - 4q \right) \\
&= \frac{1}{2}c_2 \left(h^{-1}qh + q + h + q + hqh^{-1} + h - 4q \right) \\
&= \frac{1}{2}c_2 \left(h^{-1}qh + hqh^{-1} + 2h - 2q \right).
\end{aligned}$$

As said previously in this paper $h^{-1}qh$ and hqh^{-1} were both multiplied out in component form to see if anything could be gained. The hope was that $h^{-1}qh$ and hqh^{-1} would equal $2q$. When this didn't work, an effort was made to rewrite all of the terms without using h^{-1} . This also failed. Next the terms were rewritten in components

\vec{i} , \vec{j} , and \vec{k} where $h = h_1 + h_2\vec{i} + h_3\vec{j}$ and $q = q_1 + q_2\vec{i} + q_3\vec{j}$.

Let $hq = qh + \alpha\vec{k}$ and $qh = hq - \alpha\vec{k}$ with $\alpha \in \mathbf{R}$.

These two terms differ only in the \vec{k} component which are opposites.

$$\begin{aligned}
\text{Thus } h^{-1}qh + hqh^{-1} &= (qh + \alpha\vec{k})h^{-1} + h^{-1}(hq + \alpha\vec{k}) \\
&= q + \alpha\vec{k}h^{-1} + q - \alpha h^{-1}\vec{k} \\
&= 2q + \alpha\vec{k}h^{-1} - \alpha h^{-1}\vec{k}.
\end{aligned}$$

Continuing with the $n = 2$ case, the components of h and q will be used.

$$\begin{aligned}
&\frac{1}{2}c_2(h^{-1}qh + hqh^{-1} + 2h - 2q) \\
&= \frac{1}{2}c_2(2q + \alpha\vec{k}h^{-1} - \alpha h^{-1}\vec{k}) + c_2h - c_2q \\
&= \frac{1}{2}c_2(\alpha\vec{k}h^{-1} - \alpha h^{-1}\vec{k}) + c_2h
\end{aligned}$$

$$= c_2 \left[\frac{\alpha}{2} (\vec{k} h^{-1} - h^{-1} \vec{k}) \right] + c_2 h.$$

Letting $h = h_0 + h_1 \vec{i} + h_2 \vec{j}$ and $h^{-1} = \frac{1}{|h|^2} (h_0 + h_1 \vec{i} + h_2 \vec{j})$,

$$\vec{k} h^{-1} = \frac{1}{|h|^2} (h_0 \vec{k} - h_1 \vec{j} + h_2 \vec{i}) \text{ and } h^{-1} \vec{k} = \frac{1}{|h|^2} (h_0 \vec{k} + h_1 \vec{j} - h_2 \vec{i}).$$

Substituting these values into the above equation,

$$= c_2 \left[\frac{\alpha}{2} \left(\frac{1}{|h|^2} (2h_2 \vec{i} - 2h_1 \vec{j}) \right) \right] + c_2 h$$

$$= \frac{c_2 \alpha}{|h|^2} (h_2 \vec{i} - h_1 \vec{j}) + c_2 h.$$

α = the constant of the \vec{k} component of qh which is $(q_1 h_2 - q_2 h_1) \vec{k}$.

Substituting these values into the above equation,

$$= \frac{c_2 (q_1 h_2 - q_2 h_1)}{|h|^2} (h_2 \vec{i} - h_1 \vec{j}) + c_2 h$$

$$= \frac{c_2 (q_1 h_2 - q_2 h_1) (h_2 \vec{i} - h_1 \vec{j})}{h_0^2 + h_1^2 + h_2^2} + c_2 h.$$

As $h \rightarrow 0$ the above equation has no limit because it depends on the path of h .

This proves that our attempt fails for the case where $n = 2$.

Since it was not possible to prove that the derivative of an elementary function is

$\sum_{n=1}^{\infty} n c_n q^{n-1}$, a comparison between the function defined by this series and the Jacobian will be made.

Definition

Given an elementary function, $f(q) = \sum_{n=0}^{\infty} c_n q^n$ the derived function is defined to be

$$f^*(q) = \sum_{n=1}^{\infty} n c_n q^{n-1}.$$

The next step is to find out more about the derived function and how it is related to the Jacobian.

APPENDIX

Functions with Coefficients in \mathbf{J}

Since we are dealing with a space that has the Jordan product defined on it, it is worth considering the behavior of functions with quaternionic coefficients.

Definition

A Jordan function $f: \mathbf{J} \rightarrow \mathbf{J}$ is defined as $f(q) = \sum_{n=0}^{\infty} [c_n, q^n]$ where $c_n \in \mathbf{J}$.

An attempt was made to show that these Jordan functions converge absolutely. A proof similar to the one used for elementary was tried, but did not work. The following counter-example shows that these Jordan functions do not converge absolutely.

For $f(q) = \sum_{n=0}^{\infty} [c_n, q^n]$ where $c_n \in \mathbf{J}$, if $\sum_{n=0}^{\infty} [c_n, q_0^n]$ converges for some $q_0 \neq 0$

then for each q with $|q| < |q_0|$ the series $f(q) = \sum_{n=0}^{\infty} [c_n, q^n]$ is absolutely convergent. This statement can be shown to be false using a counter-example. Therefore, these Jordan functions do not converge absolutely.

Define $c_n = \cos \frac{n\pi}{2} - \sin \frac{n\pi}{2} \vec{i} + n! \sin \frac{n\pi}{2} \vec{j}$.

Let $q_0 = \alpha \vec{i}$ where $\frac{1}{2} < \alpha < 1$.

So $q_0 = \alpha e^{\frac{i\pi}{2}}$

$$\text{and } q_0^n = \alpha^n \cos \frac{n\pi}{2} + \alpha^n \sin \frac{n\pi}{2} \vec{i}.$$

To obtain values of $c_n q_0^n$, the following table will be used.

n	$\cos \frac{n\pi}{2}$	$\sin \frac{n\pi}{2}$
0	1	0
1	0	1
2	-1	0
3	0	-1

$$\begin{aligned}
 [c_n, q_0^n] &= \alpha^n \cos \frac{n\pi}{2} \cos \frac{n\pi}{2} - \alpha^n \left(-\sin \frac{n\pi}{2} \sin \frac{n\pi}{2} \right) \\
 &\quad + \left(\alpha^n \cos \frac{n\pi}{2} \cos \frac{n\pi}{2} + \alpha^n - \sin \frac{n\pi}{2} \cos \frac{n\pi}{2} \right) \vec{i} \\
 &\quad + \left(\alpha^n n! \sin \frac{n\pi}{2} \cos \frac{n\pi}{2} \right) \vec{j} \\
 &= \alpha^n + 0\vec{i} + \alpha^n n! \sin \frac{n\pi}{2} \cos \frac{n\pi}{2} \vec{j}.
 \end{aligned}$$

Therefore $\sum_{n=0}^{\infty} [c_n, q_0^n]$ converges because each component converges. The real part converges because $\alpha < 1$. The \vec{i} - component converges because it is zero. The \vec{j} - component converges because it equals zero using the table.

Now we need to find a q that is closer to the origin than q_0 such that $\sum_{n=0}^{\infty} [c_n, q^n]$

does not converge.

Let $q = \frac{1}{2} \vec{j}$.

To obtain values of $c_n q^n$, the following table will be used.

$n \quad q$

0 1

1 $\frac{1}{2} \vec{j}$

2 $-\frac{1}{4}$

3 $-\frac{1}{8} \vec{j}$

4 $\frac{1}{16}$

etc.

$$\begin{aligned}
[c_n, q^n] &= \left[\cos \frac{n\pi}{2} - \sin \frac{n\pi}{2} i + n! \sin \frac{n\pi}{2} j, \left(\frac{1}{2} j \right)^n \right] \\
&= \left(\cos \frac{n\pi}{2} \right) \left(\frac{1}{2} j \right)^n - \left(\sin \frac{n\pi}{2} \right) i \left(\frac{1}{2} j \right)^n + \left(n! \sin \frac{n\pi}{2} \right) \frac{1}{2^n} (j)^{n+1}
\end{aligned}$$

Looking at just the real part,

$$= \cos \frac{n\pi}{2} \left(\frac{1}{2} j \right)^n + \left(n! \sin \frac{n\pi}{2} \right) \frac{1}{2^n} (j)^{n+1}$$

$$n=0, 2, 4, \dots$$

$$n=1, 3, 5, \dots$$

$$= \sum_{n=0}^{\infty} a_n, \text{ where } a_n = \begin{cases} \frac{1}{2^n} & n \text{ even} \\ \frac{-n!}{2^n} & n \text{ odd} \end{cases}$$

The first piece converges, but the second part diverges.

We have just found a q such that $\sum_{n=0}^{\infty} [c_n, q^n]$ diverges. Therefore, these Jordan

functions are not absolutely convergent as hoped.

CHAPTER 4

Adjoint and Jacobians of Elementary Functions

Now that it has been established that the usual derivative is not defined as it was for real and complex analysis, a comparison between the derived function, which we will refer to as f^* , defined by $f^*(q) = \sum_{n=1}^{\infty} n c_n (q-a)^{n-1}$, and the Jacobian of f will be made.

In order to compare the Jacobian of f with the derived function, we will actually be using the adjoint of f^* , denoted A_{f^*} and defined as follows.

Definition

Let $q \in \mathbf{J}^+$. The adjoint A_q is the linear operator (a 3x3 matrix) such that $A_q(p) = [q, p]$ for all $p \in \mathbf{J}^+$.

In order to compute A_{f^*} , we must compute $[f^*, 1]$, $[f^*, \vec{i}]$, and $[f^*, \vec{j}]$.

This gives a matrix with entries as follows.

$$f^*(q) = \sum_{n=1}^{\infty} n(q-a)^{n-1}$$

1st Column

2nd Column

3rd Column

$$[n(q-a)^{n-1}, 1]$$

$$[n(q-a)^{n-1}, \vec{i}]$$

$$[n(q-a)^{n-1}, \vec{j}]$$

Letting $f^*(q) = \sum_{n=1}^{\infty} n(q-a)^{n-1} = a_{11} + a_{21}\vec{i} + a_{31}\vec{j}$ where a_{11} , a_{21} , and a_{31} are functions of

$x, y,$ and $z,$ then the columns of A_{f^*} are as follows.

$$\begin{aligned} [a_{11} + a_{21}\vec{i} + a_{31}\vec{j}, 1] & \quad [a_{11} + a_{21}\vec{i} + a_{31}\vec{j}, \vec{i}] & \quad [a_{11} + a_{21}\vec{i} + a_{31}\vec{j}, \vec{j}] \\ = a_{11} + a_{21}\vec{i} + a_{31}\vec{j} & \quad = -a_{21} + a_{11}\vec{i} & \quad = -a_{31} + a_{11}\vec{j} \end{aligned}$$

This gives the adjoint matrix $A_{f^*} = \begin{pmatrix} a_{11} & -a_{21} & -a_{31} \\ a_{21} & a_{11} & 0 \\ a_{31} & 0 & a_{11} \end{pmatrix}.$

In real and complex analysis, the Jacobian and the adjoint of the derivative function are coincidental, and the Jacobian has certain properties known as the Cauchy-Riemann equations. We will be determining what, if any, the properties of the Jacobian are for the quaternionic space \mathbf{J}^+ . We will also compare the Jacobian with the adjoint of the derived function.

Recall that $f(q) = \sum_{n=0}^{\infty} c_n q^n = u + \vec{i}v + \vec{j}w$ where $q = x + \vec{i}y + \vec{j}z$.

Thus the Jacobian $J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}.$

In order to determine the relationship between the Jacobian and the adjoint matrix, we investigate the properties of an arbitrary monomial q^n . Note that the behavior of a monomial of the form $(q - a)^n$ will have these same properties, so we can work with q^n without loss of generality. First, we work out the cases $n = 2, 3$ and 4 as examples.

Let $q = x + \vec{i}y + \vec{j}z$.

For $n = 2$,

$$\begin{aligned} f(q) = q^2 &= (x + \vec{i}y + \vec{j}z)(x + \vec{i}y + \vec{j}z) \\ &= x^2 + \vec{i}xy + \vec{j}xz - \vec{k}yz - y^2 + \vec{i}xy + \vec{j}xz + \vec{k}yz - z^2 \\ &= (x^2 - y^2 - z^2) + \vec{i}2xy + \vec{j}2xz. \end{aligned}$$

Thus $u = x^2 - y^2 - z^2$, $v = 2xy$ and $w = 2xz$.

We can now compute the entries of the Jacobian.

$$J_f = \begin{pmatrix} 2x & -2y & -2z \\ 2y & 2x & 0 \\ 2z & 0 & 2x \end{pmatrix}.$$

In order to get the entries of the adjoint matrix we need to find a_{11} , a_{21} and a_{31} as defined previously.

The derived function is $f^*(q) = 2q = 2x + \vec{i}2y + \vec{j}2z$.

So $a_{11} = 2x$, $a_{21} = 2y$ and $a_{31} = 2z$.

We can now fill in the entries of the adjoint matrix.

$$A_{f^*} = \begin{pmatrix} 2x & -2y & -2z \\ 2y & 2x & 0 \\ 2z & 0 & 2x \end{pmatrix}.$$

In this case, it appears that $J_f = A_{f^*}$.

$$\begin{aligned} \text{For } n = 3, f(q) = q^3 &= [(x^2 - y^2 - z^2) + \vec{i}2xy + \vec{j}2xz](x + \vec{i}y + \vec{j}z) \\ &= (x^3 - 3xy^2 - 3xz^2) + \vec{i}(3x^2y - y^3 - yz^2) + \vec{j}(3x^2z - y^2z - z^3). \end{aligned}$$

Thus $u = x^3 - 3xy^2 - 3xz^2$, $v = 3x^2y - y^3 - yz^2$ and $w = 3x^2z - y^2z - z^3$.

$$J_f = \begin{pmatrix} 3x^2 - 3y^2 - 3z^2 & -6xy & -6xz \\ 6xy & 3x^2 - 3y^2 - z^2 & -2yz \\ 6xz & -2yz & 3x^2 - y^2 - 3z^2 \end{pmatrix}.$$

The derived function is $f^*(q) = 3q^2 = (3x^2 - 3y^2 - 3z^2) + i\vec{6xy} + j\vec{6xz}$.

So $a_{11} = 3x^2 - 3y^2 - 3z^2$, $a_{21} = 6xy$ and $a_{31} = 6xz$.

$$A_f^* = \begin{pmatrix} 3x^2 - 3y^2 - 3z^2 & -6xy & -6xz \\ 6xy & 3x^2 - 3y^2 - 3z^2 & 0 \\ 6xz & 0 & 3x^2 - 3y^2 - 3z^2 \end{pmatrix}.$$

Now it is clear that $J_f \neq A_f^*$, although they agree in the first row and column. It is also interesting to notice some of the properties that seem to be true about J_f . Should these properties hold true in general, we would have analogues of the Cauchy-Riemann equations from complex analysis.

Theorem 4.1

Let $f(q) = \sum_{n=0}^{\infty} c_n q^n$ be an elementary function. Then the following are true about the

Jacobian matrix J_f with entries a_{ij} .

a) $a_{12} = -a_{21}$.

b) $a_{13} = -a_{31}$.

c) $a_{23} = a_{32}$.

We will continue to check this theorem for the examples worked out and will prove it

later in the chapter. Now we will check the $n = 4$ case.

For $n = 4$,

$$\begin{aligned} f(q) = q^4 &= \left[(x^3 - 3xy^2 - 3xz^2) + \vec{i}(3x^2y - y^3 - yz^2) + \vec{j}(3x^2z - y^2z - z^3) \right] (x + \vec{i}y + \vec{j}z) \\ &= (x^4 + y^4 + z^4 - 6x^2y^2 - 6x^2z^2 + 2y^2z^2) + \vec{i}(4x^3y - 4xy^3 - 4xyz^2) \\ &\quad + \vec{j}(4x^3z - 4xy^2 - 4xz^3). \end{aligned}$$

Thus $u = x^4 + y^4 + z^4 - 6x^2y^2 - 6x^2z^2 + 2y^2z^2$, $v = 4x^3y - 4xy^3 - 4xyz^2$ and

$$w = 4x^3z - 4xy^2z - 4xz^3.$$

$$J_f = \begin{pmatrix} 4x^3 - 12xy^2 - 12xz^2 & 4y^3 - 12x^2y + 4yz^2 & 4z^3 - 12x^2z + 4y^2z \\ 12x^2y - 4y^3 - 4yz^2 & 4x^3 - 12xy^2 - 4xz^2 & -8xyz \\ 12x^2z - 4y^2z - 4z^3 & -8xyz & 4x^3 - 4xy^2 - 12xz^2 \end{pmatrix}.$$

The derived function is $f^*(q) = 4q^3 = (4x^3 - 12xy^2 - 12xz^2) + \vec{i}(12x^2y - 4y^3 - 4yz^2) + \vec{j}(12x^2z - 4y^2z - 4z^3).$

$$A_f = \begin{pmatrix} 4x^3 - 12xy^2 - 12xz^2 & -12x^2y + 4y^3 + 4yz^2 & -12x^2z + 4y^2z + 4z^3 \\ 12x^2y - 4y^3 - 4yz^2 & 4x^3 - 12xy^2 - 12xz^2 & 0 \\ 12x^2z - 4y^2z - 4z^3 & 0 & 4x^3 - 12xy^2 - 12xz^2 \end{pmatrix}.$$

Once again the Jacobian and adjoint agree only in the first row and column. Also the three equations from Theorem 4.1 are true.

It would be helpful, however, to have a non-polynomial example for comparison. Going back to the exponential example again, we will find the Jacobian and adjoint matrices for this elementary function by first computing u , v and w explicitly. In order to do this we will need to refer back to the mapping used previously.

For $f(q) = e^q = \sum_{n=0}^{\infty} \frac{1}{n!} q^n$, let $q = t + s \cos \psi \vec{i} + s \sin \psi \vec{j}$.

Using the previously defined map, $\begin{pmatrix} t \\ s \cos \psi \\ s \sin \psi \end{pmatrix} \xrightarrow{\alpha_\psi} t + si$, and applying it to e^q we get the

following equations.

$$\begin{aligned} \alpha_\psi(e^q) &= \alpha_\psi\left(\sum_{n=0}^{\infty} \frac{1}{n!} q^n\right) \\ &= \sum_{n=0}^{\infty} \alpha_\psi\left(\frac{1}{n!} q^n\right) \quad (\text{since } \alpha_\psi \text{ preserves addition and convergence is absolute}) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \alpha_\psi(q^n) \quad (\text{since } \alpha_\psi \text{ preserves scalar multiplication}) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} [\alpha_\psi(q)]^n \quad (\text{since } \alpha_\psi \text{ preserves powers}). \end{aligned}$$

Therefore $\alpha_\psi(e^q) = \sum_{n=0}^{\infty} \frac{1}{n!} [\alpha_\psi(q)]^n$. This means $\alpha_\psi(e^q) = e^{\alpha_\psi(q)}$ by the definition of the exponential function. Also $e^{\alpha_\psi(q)} = e^{t+si} = e^t (\cos s + i \sin s)$. Combining the previous

results we can obtain a useable form for e^q .

$$\begin{aligned} e^q &= \alpha_\psi^{-1} \left(e^{\alpha_\psi(q)} \right) \quad (\text{since } \alpha_\psi \text{ is an isomorphism and } e^q \text{ converges absolutely}) \\ &= \alpha_\psi^{-1} (e^t \cos s + i e^t \sin s) \\ &= e^t \cos s + \vec{i} e^t \sin s \cos \psi + \vec{j} e^t \sin s \sin \psi. \end{aligned}$$

Finally, we conclude that the coordinate functions for e^q are

$$\begin{aligned} u &= e^t \cos s, \quad v = e^t \sin s \cos \psi \text{ and } w = e^t \sin s \sin \psi, \text{ where} \\ x &= t, \quad y = s \cos \psi \text{ and } z = s \sin \psi. \end{aligned}$$

Now we can begin finding the entries of the Jacobian.

$$\frac{\partial u}{\partial x} = e^t \cos s.$$

$$\frac{\partial u}{\partial y} = -e^t \sin s \frac{\partial s}{\partial y}.$$

We need to solve for $\frac{\partial s}{\partial y}$.

$$y = s \cos \psi \quad \text{and} \quad z = s \sin \psi$$

Taking the derivative of both with respect to y ,

$$1 = -s \sin \psi \frac{\partial \psi}{\partial y} + \cos \psi \frac{\partial s}{\partial y} \quad \text{and} \quad 0 = s \cos \psi \frac{\partial \psi}{\partial y} + \sin \psi \frac{\partial s}{\partial y}.$$

Solving for $\frac{\partial s}{\partial y}$,

$$\frac{\partial s}{\partial y} = \frac{-s \cos \psi \frac{\partial \psi}{\partial y}}{\sin \psi}.$$

Substituting $\frac{\partial s}{\partial y}$ into the above equation allows us to find $\frac{\partial \psi}{\partial y}$.

$$1 = -s \sin \psi \frac{\partial \psi}{\partial y} + \cos \psi \left(\frac{-s \cos \psi \frac{\partial \psi}{\partial y}}{\sin \psi} \right) \text{ which gives } 1 = \frac{-s \sin^2 \psi \frac{\partial \psi}{\partial y} - s \cos^2 \psi \frac{\partial \psi}{\partial y}}{\sin \psi}.$$

Therefore $\sin \psi = -s \frac{\partial \psi}{\partial y}$ so $\frac{\partial \psi}{\partial y} = \frac{-\sin \psi}{s}$.

We can now find the value of $\frac{\partial s}{\partial y}$ by substituting $\frac{\partial \psi}{\partial y}$ into the equation

$$\frac{\partial s}{\partial y} = \frac{-s \cos \psi \frac{\partial \psi}{\partial y}}{\sin \psi}.$$

$$\frac{\partial s}{\partial y} = \frac{-s \cos \psi \left(\frac{-\sin \psi}{s} \right)}{\sin \psi}.$$

$$\frac{\partial s}{\partial y} = -\cos \psi.$$

Substituting $\frac{\partial s}{\partial y}$ back into $\frac{\partial u}{\partial y}$ we can finish computing $\frac{\partial u}{\partial y}$.

$$\begin{aligned} \frac{\partial u}{\partial y} &= -e^t \sin s \frac{\partial s}{\partial y} \\ &= -e^t \sin s \cos \psi. \end{aligned}$$

$$\frac{\partial u}{\partial z} = -e^t \sin s \frac{\partial s}{\partial z}.$$

We need to solve for $\frac{\partial s}{\partial z}$.

$$y = s \cos \psi \quad \text{and} \quad z = s \sin \psi.$$

Taking the derivative of both with respect to z ,

$$0 = -s \sin \psi \frac{\partial \psi}{\partial z} + \cos \psi \frac{\partial s}{\partial z} \quad 1 = s \cos \psi \frac{\partial \psi}{\partial y} + \sin \psi \frac{\partial s}{\partial z}.$$

Solving for $\frac{\partial s}{\partial z}$,

$$\frac{\partial s}{\partial z} = \frac{s \sin \psi \frac{\partial \psi}{\partial z}}{\cos \psi}.$$

Substituting $\frac{\partial s}{\partial z}$ into the above equation allows us to find $\frac{\partial \psi}{\partial z}$.

$$1 = s \cos \psi \frac{\partial \psi}{\partial z} + \frac{\sin \psi s \sin \psi \frac{\partial \psi}{\partial z}}{\cos \psi}.$$

$$1 = \frac{-s \sin \psi \frac{\partial \psi}{\partial y} - s \cos^2 \psi \frac{\partial \psi}{\partial y}}{\sin \psi}.$$

$$1 = \frac{(s \cos^2 \psi + s \sin^2 \psi) \frac{\partial \psi}{\partial z}}{\cos \psi}.$$

$$1 = \frac{s}{\cos \psi} \frac{\partial \psi}{\partial z}.$$

$$\frac{\partial \psi}{\partial z} = \frac{\cos \psi}{s}.$$

We can now find the value of $\frac{\partial s}{\partial z}$ by substituting $\frac{\partial \psi}{\partial z}$ into the equation $\frac{\partial s}{\partial z} = \frac{s \sin \psi \frac{\partial \psi}{\partial z}}{\cos \psi}$.

$$\frac{\partial s}{\partial z} = \frac{s \sin \psi \cos \psi}{s \cos \psi}.$$

$$\frac{\partial s}{\partial z} = \sin \psi.$$

Substituting $\frac{\partial s}{\partial z}$ back into $\frac{\partial u}{\partial z}$ we can finish computing $\frac{\partial u}{\partial z}$.

$$\frac{\partial u}{\partial z} = -e^t \sin s \frac{ds}{dz}$$

$$= -e^t \sin s \sin \psi.$$

$$\frac{\partial v}{\partial x} = e^t \sin s \cos \psi.$$

$$\frac{\partial v}{\partial y} = e^t \left(-\sin s \sin \psi \frac{\partial \psi}{\partial y} + \cos \psi \cos s \frac{\partial s}{\partial y} \right)$$

$$= e^t \left(-\sin s \sin \psi \frac{-\sin \psi}{s} + \cos \psi \cos s \cos \psi \right)$$

$$= \frac{e^t \sin s \sin^2 \psi}{s} + e^t \cos s \cos^2 \psi.$$

$$\frac{\partial v}{\partial z} = e^t \left(-\sin s \sin \psi \frac{\partial \psi}{\partial z} + \cos \psi \cos s \frac{\partial s}{\partial z} \right)$$

$$= e^t \left(-\sin s \sin \psi \frac{\cos \psi}{s} + \cos \psi \cos s \cos \psi \right)$$

$$= \frac{-e^t \sin s \sin \psi \cos \psi}{s} + e^t \cos s \sin \psi \cos \psi.$$

$$\frac{\partial w}{\partial x} = e^t \sin s \sin \psi.$$

$$\begin{aligned}
\frac{\partial w}{\partial y} &= e^t \left(-\sin s \cos \psi \frac{\partial \psi}{\partial y} + \sin \psi \cos s \frac{\partial s}{\partial y} \right) \\
&= e^t \left(\sin s \cos \psi \frac{-\sin \psi}{s} + \sin \psi \cos s \cos \psi \right) \\
&= \frac{-e^t \sin s \sin \psi \cos \psi}{s} + e^t \cos s \sin \psi \cos \psi.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial z} &= e^t \left(\sin s \cos \psi \frac{\partial \psi}{\partial z} + \sin \psi \cos s \frac{\partial s}{\partial z} \right) \\
&= e^t \left(\sin s \cos \psi \frac{\cos \psi}{s} + \sin \psi \cos s \sin \psi \right) \\
&= \frac{e^t \sin s \cos^2 \psi}{s} + e^t \cos s \sin^2 \psi.
\end{aligned}$$

Thus, we have computed the Jacobian matrix for $f(q) = e^q$.

$$J_f = \begin{pmatrix} e^t \cos s & -e^t \sin s \cos \psi & -e^t \sin s \sin \psi \\ e^t \sin s \cos \psi & \frac{e^t \sin s \sin^2 \psi}{s} + e^t \cos s \cos^2 \psi & \frac{-e^t \sin s \sin \psi \cos \psi}{s} + e^t \cos s \sin \psi \cos \psi \\ e^t \sin s \sin \psi & \frac{-e^t \sin s \sin \psi \cos \psi}{s} + e^t \cos s \sin \psi \cos \psi & \frac{e^t \sin s \cos^2 \psi}{s} + e^t \cos s \sin^2 \psi \end{pmatrix}$$

Note that all entries are defined at $s = 0$ by taking limits as needed.

The Jacobian for the exponential function satisfies the equations in Theorem 4.1.

Also, since $f^*(q) = f(q)$ we have $f^*(q) = e^q = e^t \cos s + \vec{i} e^t \sin s \cos \psi + \vec{j} e^t \sin s \sin \psi$.

From which we calculate the adjoint matrix.

$$A_{f^*} = \begin{pmatrix} e^t \cos s & -e^t \sin s \cos \psi & -e^t \sin s \sin \psi \\ e^t \sin s \cos \psi & e^t \cos s & 0 \\ e^t \sin s \sin \psi & 0 & e^t \cos s \end{pmatrix}$$

We will come back to the exponential example later when a relationship between

J_f and A_{f^*} has been determined. We note, however, that the first row and column agree as in the monomial cases q^2 , q^3 and q^4 .

From looking at these examples, it was obvious that the Jacobian and the adjoint varied only in the lower right four entries. This is the reason the idea for using a derivation came about.

Definition

Let A be an algebra. A derivation on A is a linear operator D with the property

$$D(pq) = D(p)q + pD(q).$$

The set of derivations on \mathbf{H} forms a vector space and becomes a Lie algebra under the usual commutator bracket product. The following derivations were found by letting the choices for p and q vary and doing basic linear algebra. We omit the calculations.

Theorem 4.2

a) derivations on \mathbf{H}

$$\mathbf{H} = a + b\vec{i} + c\vec{j} + d\vec{k} \text{ with } p, q \in \mathbf{H}.$$

When $D(pq) = D(p)q + pD(q)$, the set of derivations consists of 4x4 matrices of

$$\text{the form } D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & 0 & -\beta \\ 0 & \alpha & \beta & \gamma \end{pmatrix} \text{ where } \alpha, \beta \text{ and } \gamma \in \mathbf{R}.$$

b) derivations on \mathbf{H}^+

$$\mathbf{H} = a + b\vec{i} + c\vec{j} + d\vec{k} \text{ with } p, q \in \mathbf{H}^+.$$

When $D([p, q]) = [D(p), q] + [p, D(q)]$, then the set of derivations consists of matrices of

$$\text{the form } D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha & -\beta \\ 0 & \alpha & 0 & -\gamma \\ 0 & \beta & \gamma & 0 \end{pmatrix} \text{ where } \alpha, \beta \text{ and } \gamma \in \mathbf{R}.$$

Corollary

$$\mathbf{J} = a + b\vec{i} + c\vec{j} \text{ with } p, q \in \mathbf{J}^+.$$

When $D([p, q]) = [D(p), q] + [p, D(q)]$, then the set of derivations consists of matrices of

$$\text{the form } D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix} \text{ where } \alpha \in \mathbf{R}.$$

Note that the vector space of derivations on \mathbf{J}^+ is 1-dimensional. Also, it worked out correctly because if a derivation matrix for \mathbf{H}^+ is restricted to \mathbf{J}^+ , it results in a

derivation matrix of the proper form. Derivations on \mathbf{J}^+ will be used to determine the relationship between J_f and A_f ..

Definition

Given $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix}$, we define D^* to be the unique derivation such that

$$DD^* = D^*D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \text{ Thus, } D^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} \\ 0 & -\frac{1}{\alpha} & 0 \end{pmatrix} \text{ relative to the standard basis.}$$

The following property of the derivation matrix proved very useful later in determining the relationship between J_f and A_{f*} .

Let A be any 3 by 3 matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \text{ and let } D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix}.$$

$$\begin{aligned} \text{Then } D^*AD &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} \\ 0 & -\frac{1}{\alpha} & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\alpha}a_{31} & \frac{1}{\alpha}a_{32} & \frac{1}{\alpha}a_{33} \\ -\frac{1}{\alpha}a_{21} & -\frac{1}{\alpha}a_{22} & -\frac{1}{\alpha}a_{23} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{33} & -a_{32} \\ 0 & -a_{23} & a_{22} \end{pmatrix}. \end{aligned}$$

The definition of D^* together with the above property are what led to the formula relating the Jacobian and adjoint.

As stated earlier, the Jacobian and adjoint differed only in the lower right four entries. By using the derivation matrix, the process for determining the relationship between the Jacobian and adjoint began to move along rapidly. Before long, a formula was stated that worked for the examples given earlier.

We begin first by examining the general monomial case.

$$\text{Let } f(q) = q^n = u + \vec{i}v + \vec{j}w.$$

$$\text{Then } f^*(q) = nq^{n-1} = a_{11} + a_{21}\vec{i} + a_{31}\vec{j}.$$

$$J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \text{ and } A_f = \begin{pmatrix} a_{11} & -a_{21} & -a_{31} \\ a_{21} & a_{11} & 0 \\ a_{31} & 0 & a_{11} \end{pmatrix}.$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix} \text{ and } D^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha} \\ 0 & -\frac{1}{\alpha} & 0 \end{pmatrix}.$$

We will also use the conjugate and the divergence of the conjugate.

$$\bar{f} = u - \vec{i}v - \vec{j}w \text{ and } \text{div } \bar{f} = \text{trace } J_{\bar{f}}.$$

Using all of these items, the main theorem of this paper can be stated and will then be proven for monomials.

Theorem 4.3

Let f be an elementary function on \mathbf{J}^+ and D any derivation on \mathbf{J}^+ . Then

$$A_{f^*} = J_f + D^* [J_f + (\text{div} \bar{f})I] D \text{ where } I \text{ is the } 3 \times 3 \text{ identity matrix.}$$

Proof:

The following is a proof for $f(q)=q^n$. After this is completed, the theorem will be proven for elementary functions.

Given $q = x + \vec{i}y + \vec{j}z$ and $f(q) = q^n$ we will make the following change of variables.

$$\text{Let } \rho = \sqrt{x^2 + y^2 + z^2} \text{ and } L = \frac{y\vec{i}}{\sqrt{y^2 + z^2}} + \frac{z\vec{j}}{\sqrt{y^2 + z^2}}. \text{ Note that } L^2 = -1.$$

$$\text{And set } \cos^2 \omega = \frac{x^2}{x^2 + y^2 + z^2} \text{ and } \sin^2 \omega = \frac{y^2 + z^2}{x^2 + y^2 + z^2}.$$

$$\text{So that } q = \rho (\cos \omega + \sin \omega L).$$

Prove $q^n = \rho^n (\cos n\omega + \sin n\omega L)$ by induction.

For $n = 2$,

$$\begin{aligned} q^2 &= \rho(\cos \omega + \sin \omega L) \rho(\cos \omega + \sin \omega L) \\ &= \rho^2 (\cos^2 \omega + 2 \cos \omega \sin \omega L + \sin^2 \omega L^2) \\ &= \rho^2 (\cos^2 \omega + \sin 2\omega L + \sin^2 \omega (-1)) \\ &= \rho^2 (\cos^2 \omega + \sin 2\omega L - \sin^2 \omega) \\ &= \rho^2 (\cos 2\omega + \sin 2\omega L). \end{aligned}$$

Assume true for $n=k$ and prove true for $n=k+1$.

$$\begin{aligned}
q^{k+1} &= q^k q \\
&= \rho^k (\cos k\omega + \sin k\omega L) \rho (\cos \omega + \sin \omega L) \\
&= \rho^{k+1} (\cos k\omega \cos \omega + \cos k\omega \sin \omega L + \cos \omega \sin k\omega L - \sin \omega \sin k\omega) \\
&= \rho^{k+1} [\cos(k+1)\omega + \sin(k+1)\omega L].
\end{aligned}$$

Therefore $q^n = \rho^n (\cos n\omega + \sin n\omega L)$ by an inductive proof.

We now have q^n written in a form that can be used in order to find the adjoint matrix and Jacobian.

$$\text{If } q^n = \rho^n (\cos n\omega + \sin n\omega L),$$

$$\text{then } q^n = \rho^n \cos n\omega + \rho^n \sin n\omega L$$

$$= \rho^n \cos n\omega + \frac{\rho^n y \sin n\omega \vec{i}}{\sqrt{y^2 + z^2}} + \frac{\rho^n z \sin n\omega \vec{j}}{\sqrt{y^2 + z^2}}.$$

$$\text{So } u = \rho^n \cos n\omega, \quad v = \frac{\rho^n y \sin n\omega}{\sqrt{y^2 + z^2}} \quad \text{and} \quad w = \frac{\rho^n z \sin n\omega}{\sqrt{y^2 + z^2}}.$$

Beginning with the adjoint matrix, we will compute its entries.

$$f^*(q) = nq^{n-1}$$

$$= n\rho^{n-1} (\cos(n-1)\omega + \sin(n-1)\omega L)$$

$$= n\rho^{n-1} \left(\cos(n-1)\omega + \sin(n-1)\omega \left[\frac{y\vec{i}}{\sqrt{y^2 + z^2}} + \frac{z\vec{j}}{\sqrt{y^2 + z^2}} \right] \right)$$

$$= \begin{pmatrix} n\rho^{n-1} \cos(n-1)\omega \\ \frac{n\rho^{n-1}y \sin(n-1)\omega}{\sqrt{y^2+z^2}} \\ \frac{n\rho^{n-1}z \sin(n-1)\omega}{\sqrt{y^2+z^2}} \end{pmatrix}.$$

$$A_{f^*} = \begin{pmatrix} n\rho^{n-1} \cos(n-1)\omega & -\frac{n\rho^{n-1}y \sin(n-1)\omega}{\sqrt{y^2+z^2}} & -\frac{n\rho^{n-1}z \sin(n-1)\omega}{\sqrt{y^2+z^2}} \\ \frac{n\rho^{n-1}y \sin(n-1)\omega}{\sqrt{y^2+z^2}} & n\rho^{n-1} \cos(n-1)\omega & 0 \\ \frac{n\rho^{n-1}z \sin(n-1)\omega}{\sqrt{y^2+z^2}} & 0 & n\rho^{n-1} \cos(n-1)\omega \end{pmatrix}.$$

Next we will find the Jacobian for $f(q)$.

$$J_f = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}.$$

Using u , v and w as defined previously we compute the Jacobian entries.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\rho^n \partial \cos n\omega}{\partial x} + \frac{\cos n\omega \partial \rho^n}{\partial x} \\
&= \rho^n \left(-n \sin n\omega \cdot \frac{\partial \omega}{\partial x} \right) + \cos n\omega \left(n\rho^{n-1} \frac{1}{2}(\rho^{-1}) \cdot 2x \right) \\
&= -n\rho^n \sin n\omega \frac{\partial \omega}{\partial x} + nx\rho^{n-2} \cos n\omega.
\end{aligned}$$

We need to find $\frac{\partial \omega}{\partial x}$ before continuing.

$$\text{We know } \cos^2 \omega = \frac{x^2}{x^2 + y^2 + z^2}.$$

We will now take the derivative with respect to x .

$$\begin{aligned}
-2 \sin \omega \cos \omega \frac{\partial \omega}{\partial x} &= \frac{(x^2 + y^2 + z^2)2x - x^2(2x)}{(x^2 + y^2 + z^2)^2}. \\
\text{So } \frac{\partial \omega}{\partial x} &= \frac{2x(y^2 + z^2)}{-\sin 2\omega \rho^4}.
\end{aligned}$$

We now substitute the value of $\frac{\partial \omega}{\partial x}$ into the above equation.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= -n\rho^n \sin n\omega \left(\frac{2x(y^2 + z^2)}{-\sin 2\omega \rho^4} \right) + nx\rho^{n-2} \cos n\omega \\
&= nx\rho^{n-2} \left(\frac{2 \sin n\omega (y^2 + z^2)}{\sin 2\omega \rho^2} + \cos n\omega \right) \\
&= nx\rho^{n-2} \left(\frac{2 \sin n\omega \rho^2 \sin^2 \omega}{\sin 2\omega \rho^2} + \cos n\omega \right)
\end{aligned}$$

$$\begin{aligned}
&= n \cos \omega \rho^{n-1} \left(\frac{2 \sin n \omega \sin^2 \omega}{\sin 2 \omega} + \cos n \omega \right) \\
&= n \rho^{n-1} \left(\frac{2 \sin n \omega \sin \omega \sin \omega \cos \omega}{\sin 2 \omega} + \cos n \omega \cos \omega \right) \\
&= n \rho^{n-1} (\sin n \omega \sin \omega + \cos n \omega \cos \omega) \\
&= n \rho^{n-1} \cos(n-1) \omega.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial v}{\partial x} &= \frac{y}{\sqrt{y^2 + z^2}} \left(\rho^n n \cos n \omega \frac{\partial w}{\partial x} + \sin n \omega n \rho^{n-1} \frac{\partial \rho}{\partial x} \right) \\
&= \frac{y}{\sqrt{y^2 + z^2}} \left(\rho^n n \cos n \omega \frac{2x(y^2 + z^2)}{-\sin 2 \omega \rho^4} + \sin n \omega n \rho^{n-1} \frac{1}{2} \rho^{-1} 2x \right) \\
&= \frac{y}{\sqrt{y^2 + z^2}} \left(\frac{n \rho^n \cos n \omega 2 \rho \cos \omega \rho^2 \sin^2 \omega}{-\sin 2 \omega \rho^4} + n \rho^{n-2} \rho \cos \omega \sin n \omega \right) \\
&= \frac{y}{\sqrt{y^2 + z^2}} \left(\frac{n \rho^n \cos n \omega \sin \omega}{-\rho} + n \rho^{n-1} \cos \omega \sin n \omega \right) \\
&= \frac{n \rho^{n-1} y}{\sqrt{y^2 + z^2}} (-\cos n \omega \sin \omega + \cos \omega \sin n \omega) \\
&= \frac{n \rho^{n-1} y \sin(n-1) \omega}{\sqrt{y^2 + z^2}}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial x} &= \frac{z}{\sqrt{y^2 + z^2}} \left(\rho^n n \cos n \omega \frac{\partial w}{\partial x} + \sin n \omega n \rho^{n-1} \frac{\partial \rho}{\partial x} \right) \\
&= \frac{z}{\sqrt{y^2 + z^2}} \left(\rho^n n \cos n \omega \frac{2x(y^2 + z^2)}{-\sin 2 \omega \rho^4} + \sin n \omega n \rho^{n-1} \frac{1}{2} \rho^{-1} 2x \right) \\
&= \frac{z}{\sqrt{y^2 + z^2}} \left(\frac{\rho^{n-4} n \cos n \omega 2 \rho \cos \omega \rho^2 \sin^2 \omega}{-\sin 2 \omega} + n \rho^{n-2} \rho \cos \omega \sin n \omega \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{z}{\sqrt{y^2 + z^2}} (n\rho^{n-1} \cos n\omega \sin \omega (-1) + n\rho^{n-1} \cos \omega \sin n\omega) \\
&= \frac{n\rho^{n-1} z}{\sqrt{y^2 + z^2}} (\cos \omega \sin n\omega - \cos n\omega \sin \omega) \\
&= \frac{n\rho^{n-1} z \sin(n-1)\omega}{\sqrt{y^2 + z^2}}.
\end{aligned}$$

$$\frac{\partial u}{\partial y} = -\rho^n n \sin n\omega \frac{\partial w}{\partial y} + \cos n\omega n \rho^{n-1} \frac{\partial \rho}{\partial y}.$$

We need to find $\frac{\partial w}{\partial y}$ before continuing.

$$\text{We know } \sin^2 \omega = \frac{y^2 + z^2}{x^2 + y^2 + z^2}.$$

We will now take the derivative with respect to y.

$$2 \sin \omega \cos \omega \frac{\partial w}{\partial y} = \frac{(x^2 + y^2 + z^2)2y - (y^2 + z^2)2y}{(x^2 + y^2 + z^2)^2}.$$

$$2 \sin \omega \cos \omega \frac{\partial w}{\partial y} = \frac{2x^2 y}{\rho^4}.$$

$$\text{So } \frac{\partial w}{\partial y} = \frac{x^2 y}{\rho^4 \sin \omega \cos \omega}.$$

We now substitute the value of $\frac{\partial w}{\partial y}$ into the above equation.

$$\frac{\partial u}{\partial y} = -\rho^n n \sin n\omega \left(\frac{x^2 y}{\rho^4 \sin \omega \cos \omega} \right) + \cos n\omega n \rho^{n-1} \frac{1}{2} \rho^{-1} 2y$$

$$= n\rho^{n-1}y\left(\frac{-\rho\sin n\omega\cos^2\omega\rho^2}{\rho^4\sin\omega\cos\omega} + \cos n\omega\rho^{-1}\right)$$

$$= n\rho^{n-1}y\left(\frac{-\sin n\omega\cos\omega}{\rho\sin\omega} + \frac{\cos n\omega}{\rho}\right)$$

$$= n\rho^{n-1}y\left(\frac{-\sin n\omega\cos\omega}{\rho\sin\omega} + \frac{\cos n\omega\sin\omega}{\rho\sin\omega}\right)$$

$$= \frac{n\rho^{n-1}y}{\sqrt{y^2+z^2}}(-\sin n\omega\cos\omega + \cos n\omega\sin\omega)$$

$$= \frac{n\rho^{n-1}y\sin(\omega-n\omega)}{\sqrt{y^2+z^2}}$$

$$= \frac{n\rho^{n-1}y\sin(n-1)\omega}{\sqrt{y^2+z^2}}.$$

$$\frac{\partial v}{\partial y} = \frac{\sqrt{x^2+y^2}}{y^2+z^2}\left(\rho^n\left[yn\cos n\omega\frac{\partial w}{\partial y} + \sin n\omega\right] + y\sin n\omega n\rho^{n-1}\frac{\partial \rho}{\partial y}\right)$$

$$- \frac{\rho^n y \sin n\omega \frac{1}{2}(y^2+z^2)^{-\frac{1}{2}}zy}{y^2+z^2}$$

$$= \frac{1}{y^2+z^2}\left[\frac{\rho\sin\omega\rho^n yn\cos n\omega x^2 y}{\rho^4\sin\omega\cos\omega} + \rho\sin\omega\rho^n\sin n\omega\right. \\ \left.+ \rho\sin\omega y\sin n\omega n\rho^{n-1}y\rho - \rho^n y\sin n\omega y\frac{1}{\rho\sin\omega}\right]$$

$$= \frac{1}{y^2+z^2}\left[\frac{n\rho^{n-3}y^2\cos^2\omega\rho^2\cos n\omega}{\cos\omega} + \rho^{n+1}\sin\omega\sin n\omega\right. \\ \left.+ n\rho^{n-1}y^2\sin\omega\sin n\omega - \frac{\rho^{n-1}y^2\sin n\omega}{\sin\omega}\right]$$

$$\begin{aligned}
&= \frac{1}{y^2 + z^2} \left[n\rho^{n-1}y^2 \cos\omega \cos n\omega + n\rho^{n-1}y^2 \sin\omega \sin n\omega \right. \\
&\quad \left. + \rho^{n+1} \sin\omega \sin n\omega - \frac{\rho^{n-1}y^2 \sin n\omega}{\sin\omega} \right] \\
&= \frac{n\rho^{n-1}y^2 \cos(n-1)\omega}{y^2 + z^2} + \frac{\rho^{n+1} \sin\omega \sin n\omega}{y^2 + z^2} - \frac{\rho^{n-1}y^2 \sin n\omega}{(y^2 + z^2) \sin\omega} \\
&= \frac{n\rho^{n-1}y^2 \cos(n-1)\omega}{y^2 + z^2} + \frac{\rho^{n+1} \sin\omega \sin n\omega - \rho^{n-1}y^2 \sin n\omega}{(y^2 + z^2) \sin\omega} \\
&= \frac{n\rho^{n-1}y^2 \cos(n-1)\omega}{y^2 + z^2} + \frac{\rho^{n-1}z^2 \sin n\omega}{(y^2 + z^2) \sin\omega} \\
&= \frac{\rho^{n-1}}{y^2 + z^2} \left(ny^2 \cos(n-1)\omega + \frac{z^2 \sin n\omega}{\sin\omega} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial y} &= \frac{\sqrt{y^2 + z^2}}{y^2 + z^2} \left(\rho^n zn \cos n\omega \frac{\partial \omega}{\partial y} + \sin n\omega zn \rho^{n-1} \frac{\partial \rho}{\partial y} \right) \\
&\quad - \frac{\rho^n \sin n\omega \frac{1}{2}(y^2 + z^2)^{-\frac{1}{2}} 2y}{y^2 + z^2} \\
&= \frac{\sqrt{y^2 + z^2}}{y^2 + z^2} \left(\frac{\rho^n zn \cos n\omega x^2 y}{\rho^4 \sin\omega \cos\omega} + \sin n\omega zn \rho^{n-1} \frac{1}{2} \rho^{-1} 2y \right) \\
&\quad - \frac{\rho^n z \sin n\omega y (y^2 + z^2)^{-\frac{1}{2}}}{y^2 + z^2} \\
&= \frac{1}{y^2 + z^2} \left(\frac{\rho^{n+1} zn \sin\omega \cos n\omega \cos^2\omega \rho^2 y}{\rho^4 \sin\omega \cos\omega} + \rho^{n-1} zn \sin\omega \sin n\omega y \right. \\
&\quad \left. - \frac{\rho^n z \sin\omega y}{\rho \sin\omega} \right) \\
&= \frac{1}{y^2 + z^2} \left(n\rho^{n-1}yz \cos n\omega \cos\omega + n\rho^{n-1}yz \sin\omega \sin n\omega - \frac{\rho^{n-1}yz \sin n\omega}{\sin\omega} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{y^2 + z^2} \left(n\rho^{n-1}yz \cos(n-1)\omega - \frac{\rho^{n-1}yz \sin n\omega}{\sin \omega} \right) \\
&= \frac{\rho^{n-1}yz}{y^2 + z^2} \left(n \cos(n-1)\omega - \frac{\sin n\omega}{\sin \omega} \right).
\end{aligned}$$

$$\frac{\partial u}{\partial z} = -n\rho^n \sin n\omega \frac{\partial \omega}{\partial z} + \cos n\omega n\rho^{n-1} \frac{\partial \rho}{\partial z} \quad \text{where} \quad \frac{\partial \rho}{\partial z} = \frac{1}{2} \rho^{-1} 2z.$$

We need to find $\frac{\partial \omega}{\partial z}$ before continuing.

$$\text{We know } \sin^2 \omega = \frac{y^2 + z^2}{x^2 + y^2 + z^2}.$$

We will now take the derivative with respect to z .

$$2 \sin \omega \cos \omega \frac{\partial \omega}{\partial z} = \frac{(x^2 + y^2 + z^2)2z - (y^2 + z^2)2z}{\rho^4}.$$

$$\text{So } \frac{\partial \omega}{\partial z} = \frac{x^2 z}{\rho^4 \sin \omega \cos \omega}.$$

We now substitute the value of $\frac{\partial \omega}{\partial z}$ into the above equation.

$$\begin{aligned}
\frac{\partial u}{\partial z} &= -n\rho^n \sin n\omega \left(\frac{x^2 z}{\rho^4 \sin \omega \cos \omega} \right) + n\rho^{n-2} z \cos n\omega \\
&= \frac{-n\rho^n \sin n\omega \rho^2 \cos^2 \omega}{\rho^4 \sin \omega \cos \omega} + n\rho^{n-2} z \cos n\omega \\
&= \frac{-n\rho^{n-2} z \sin n\omega \cos \omega}{\sin \omega} + n\rho^{n-2} z \cos n\omega \\
&= \frac{n\rho^{n-2} z}{\sin \omega} (-\sin n\omega \cos \omega + \cos n\omega \sin \omega)
\end{aligned}$$

$$= \frac{n\rho^{n-2}z}{\sqrt{y^2+z^2}} (\sin(\omega - n\omega))$$

$$= \frac{-n\rho^{n-1}z \sin(n-1)\omega}{\sqrt{y^2+z^2}}.$$

$$\frac{\partial v}{\partial z} = \frac{\sqrt{x^2+y^2}}{y^2+z^2} \left(y\rho^n n \cos n\omega \frac{\partial w}{\partial z} + y \sin n\omega n\rho^{n-1} \frac{\partial \rho}{\partial z} \right)$$

$$- \frac{\rho^n y \sin n\omega \left[\frac{1}{2} (y^2+z^2)^{-\frac{1}{2}} 2z \right]}{y^2+z^2}$$

$$= \frac{\sqrt{x^2+y^2}}{y^2+z^2} (y\rho^n n \cos n\omega x^2 z + y \sin n\omega n\rho^{n-1} z\rho^{-1})$$

$$- \frac{\rho^n y \sin n\omega (y^2+z^2)^{-\frac{1}{2}}}{y^2+z^2}$$

$$= \frac{\rho^{n-1}yz}{y^2+z^2} \left(\frac{\rho \sin \omega \rho n \cos n\omega \rho^2 \cos^2 \omega}{\rho^4 \sin \omega \cos \omega} + n\rho \sin \omega \sin n\omega \rho^{-1} - \rho \sin n\omega \frac{1}{\rho \sin \omega} \right)$$

$$= \frac{\rho^{n-1}yz}{y^2+z^2} \left(n \cos n\omega \cos \omega + n \sin \omega \sin n\omega - \frac{\sin n\omega}{\sin \omega} \right)$$

$$= \frac{\rho^{n-1}yz}{y^2+z^2} \left(n \cos(n-1)\omega - \frac{\sin n\omega}{\sin \omega} \right).$$

$$\frac{\partial w}{\partial z} = \frac{\sqrt{y^2 + z^2}}{y^2 + z^2} \left[\rho^n (zn \cos n\omega \frac{\partial \omega}{\partial z} + \sin n\omega) + z \sin n\omega n \rho^{n-1} \frac{\partial \rho}{\partial z} \right] - \frac{\rho^n z \sin n\omega \frac{1}{2} (y^2 + z^2)^{-\frac{1}{2}} 2z}{y^2 + z^2}$$

$$= \frac{1}{y^2 + z^2} \left(\frac{\rho \sin \omega \rho^n zn \cos n\omega \frac{\partial \omega}{\partial z}}{\rho^4 \sin \omega \cos \omega} + \rho \sin \omega \rho^n \sin n\omega + \rho \sin \omega z \sin n\omega n \rho^{n-1} \rho^{-1} z - \rho^n z \sin n\omega z \frac{1}{\rho \sin \omega} \right)$$

$$= \frac{1}{y^2 + z^2} \left(\frac{n \rho^{n-3} z^2 \cos n\omega \rho^2 \cos^2 \omega}{\cos \omega} + \rho^{n+1} \sin \omega \sin n\omega + n \rho^{n-1} z^2 \sin \omega \sin n\omega - \frac{\rho^{n-1} z^2 \sin n\omega}{\sin \omega} \right)$$

$$= \frac{1}{y^2 + z^2} \left(n \rho^{n-1} z^2 \cos n\omega \cos \omega + n \rho^{n-1} z^2 \sin n\omega \sin \omega + \rho^{n+1} \sin \omega \sin n\omega - \frac{\rho^{n-1} z^2 \sin n\omega}{\sin \omega} \right)$$

$$= \frac{n \rho^{n-1} z^2 \cos(n-1)\omega}{y^2 + z^2} + \frac{\rho^{n+1} \sin^2 \omega \sin n\omega - \rho^{n-1} z^2 \sin n\omega}{(y^2 + z^2) \sin \omega}$$

$$= \frac{n \rho^{n-1} z^2 \cos(n-1)\omega}{y^2 + z^2} + \frac{\rho^{n+1} (y^2 + z^2) \sin n\omega - \rho^{n-1} z^2 \sin n\omega}{(y^2 + z^2) \sin \omega}$$

$$= \frac{n \rho^{n-1} z^2 \cos(n-1)\omega}{y^2 + z^2} + \frac{\rho^{n+1} y^2 \sin n\omega}{(y^2 + z^2) \sin \omega}$$

$$= \frac{\rho^{n-1}}{y^2 + z^2} \left(n z^2 \cos(n-1)\omega - \frac{y^2 \sin n\omega}{\sin \omega} \right)$$

Now that all of the entries have been found we can write the Jacobian as a matrix.

$$J_f = \begin{pmatrix} n\rho^{n-1} \cos(n-1)\omega & -\frac{n\rho^{n-1}y \sin(n-1)\omega}{\sqrt{y^2+z^2}} & -\frac{n\rho^{n-1}z \sin(n-1)\omega}{\sqrt{y^2+z^2}} \\ \frac{n\rho^{n-1}y \sin(n-1)\omega}{\sqrt{y^2+z^2}} & \frac{\rho^{n-1}}{y^2+z^2} \left[\frac{ny^2 \cos(n-1)\omega}{\sin \omega} + \frac{z^2 \sin n\omega}{\sin \omega} \right] & \frac{\rho^{n-1}yz}{y^2+z^2} \left[\frac{n \cos(n-1)\omega}{\sin \omega} - \frac{\sin n\omega}{\sin \omega} \right] \\ \frac{n\rho^{n-1}z \sin(n-1)\omega}{\sqrt{y^2+z^2}} & \frac{\rho^{n-1}yz}{y^2+z^2} \left[\frac{n \cos(n-1)\omega}{\sin \omega} - \frac{\sin n\omega}{\sin \omega} \right] & \frac{n\rho^{n-1}}{y^2+z^2} \left[\frac{nz^2 \cos(n-1)\omega}{\sin \omega} + \frac{y^2 \sin n\omega}{\sin \omega} \right] \end{pmatrix}$$

We now need to compute the divergence of the conjugate.

$$\begin{aligned} \operatorname{div} \bar{f} &= n\rho^{n-1} \cos(n-1)\omega - \frac{\rho^{n-1}}{x^2+z^2} \left[ny^2 \cos(n-1)\omega + \frac{z^2 \sin n\omega}{\sin \omega} \right] \\ &\quad - \frac{\rho^{n-1}}{x^2+z^2} \left[nz^2 \cos(n-1)\omega + \frac{y^2 \sin n\omega}{\sin \omega} \right] \\ &= n\rho^{n-1} \cos(n-1)\omega - \frac{\rho^{n-1}}{x^2+z^2} \left[\frac{ny^2 \cos(n-1)\omega + nz^2 \cos(n-1)\omega}{\sin \omega} + \frac{y^2 \sin n\omega}{\sin \omega} \right] \\ &= n\rho^{n-1} \cos(n-1)\omega - \frac{\rho^{n-1}}{x^2+z^2} \left[\frac{n \cos(n-1)\omega (y^2+z^2)}{\sin \omega} + \frac{\sin n\omega}{\sin \omega} (y^2+z^2) \right] \\ &= n\rho^{n-1} \cos(n-1)\omega - \rho^{n-1} \left[n \cos(n-1)\omega + \frac{\sin n\omega}{\sin \omega} \right] \\ &= -\rho^{n-1} \frac{\sin n\omega}{\sin \omega}. \end{aligned}$$

$$\text{So } \operatorname{div} \bar{f} = -\rho^{n-1} \frac{\sin n\omega}{\sin \omega}.$$

Next, the parts of the formula will be computed and the formula will be shown to be true.

$$J_f + \operatorname{div} \bar{f}$$

$$= \begin{pmatrix} \frac{n\rho^{n-1} \cos(n-1)\omega}{\rho^{n-1} \frac{\sin n\omega}{\sin \omega}} & -\frac{n\rho^{n-1} y \sin(n-1)\omega}{\sqrt{y^2+z^2}} & -\frac{n\rho^{n-1} z \sin(n-1)\omega}{\sqrt{y^2+z^2}} \\ \frac{n\rho^{n-1} y \sin(n-1)\omega}{\sqrt{y^2+z^2}} & \frac{\rho^{n-1}}{y^2+z^2} \left[\begin{matrix} ny^2 \cos(n-1)\omega \\ + \frac{z^2 \sin n\omega}{\sin \omega} \end{matrix} \right] & \frac{\rho^{n-1} yz}{y^2+z^2} \left[\begin{matrix} n \cos(n-1)\omega \\ - \frac{\sin n\omega}{\sin \omega} \end{matrix} \right] \\ \frac{n\rho^{n-1} y \setminus z \sin(n-1)\omega}{\sqrt{y^2+z^2}} & \frac{\rho^{n-1} yz}{y^2+z^2} \left[\begin{matrix} n \cos(n-1)\omega \\ - \frac{\sin n\omega}{\sin \omega} \end{matrix} \right] & \frac{\rho^{n-1}}{y^2+z^2} \left[\begin{matrix} nz^2 \cos(n-1)\omega \\ + \frac{y^2 \sin n\omega}{\sin \omega} \end{matrix} \right] \\ & & -\frac{\rho^{n-1} \sin n\omega}{\sin \omega} \end{pmatrix}$$

$$D^*[J_f + (\operatorname{div} \bar{f})I]D$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\rho^{n-1}}{y^2 + z^2} \begin{bmatrix} ny^2 \cos(n-1)\omega \\ + \frac{z^2 \sin n\omega}{\sin \omega} \end{bmatrix} & \frac{\rho^{n-1}yz}{y^2 + z^2} \begin{bmatrix} n \cos(n-1)\omega \\ - \frac{\sin n\omega}{\sin \omega} \end{bmatrix} \\ - \frac{\rho^{n-1} \sin n\omega}{\sin \omega} & & \\ 0 & \frac{\rho^{n-1}yz}{y^2 + z^2} \begin{bmatrix} n \cos(n-1)\omega \\ - \frac{\sin n\omega}{\sin \omega} \end{bmatrix} & \frac{\rho^{n-1}}{y^2 + z^2} \begin{bmatrix} nz^2 \cos(n-1)\omega \\ + \frac{y^2 \sin n\omega}{\sin \omega} \end{bmatrix} \\ & & - \frac{p^{n-1} \sin n\omega}{\sin \omega} \end{pmatrix}.$$

$$J_f + D^* [J_f + (\operatorname{div} \bar{f})I] D$$

$$= \begin{pmatrix} n\rho^{n-1} \cos(n-1)\omega & -\frac{n\rho^{n-1} y \sin(n-1)\omega}{\sqrt{y^2 + z^2}} & -\frac{n\rho^{n-1} z \sin(n-1)\omega}{\sqrt{y^2 + z^2}} \\ \frac{n\rho^{n-1} y \sin(n-1)\omega}{\sqrt{y^2 + z^2}} & n\rho^{n-1} \cos(n-1)\omega & 0 \\ \frac{n\rho^{n-1} y \setminus z \sin(n-1)\omega}{\sqrt{y^2 + z^2}} & 0 & n\rho^{n-1} \cos(n-1)\omega \end{pmatrix}$$

$$= A_{f^*}$$

To see how the entries a_{22} and a_{33} were shown to be equal, see below.

$$\begin{aligned} a_{22} &= \frac{\rho^{n-1}}{y^2 + z^2} \left[ny^2 \cos(n-1)\omega + \frac{z^2 \sin n\omega}{\sin \omega} \right] \\ &\quad + \frac{\rho^{n-1}}{y^2 + z^2} \left[nz^2 \cos(n-1)\omega + \frac{y^2 \sin n\omega}{\sin \omega} \right] - \frac{\rho^{n-1} \sin n\omega}{\sin \omega} \\ &= n\rho^{n-1} \cos(n-1)\omega + \frac{\rho^{n-1} \sin n\omega}{\sin \omega} - \frac{\rho^{n-1} \sin n\omega}{\sin \omega} \\ &= n\rho^{n-1} \cos(n-1)\omega. \end{aligned}$$

$$\begin{aligned}
a_{33} &= \frac{\rho^{n-1}}{y^2 + z^2} \left[nz^2 \cos(n-1)\omega + \frac{y^2 \sin n\omega}{\sin \omega} \right] \\
&\quad + \frac{\rho^{n-1}}{y^2 + z^2} \left[ny^2 \cos(n-1)\omega + \frac{z^2 \sin n\omega}{\sin \omega} \right] - \frac{\rho^{n-1} \sin n\omega}{\sin \omega} \\
&= n\rho^{n-1} \cos(n-1)\omega + \frac{\rho^{n-1} \sin n\omega}{\sin \omega} - \frac{\rho^{n-1} \sin n\omega}{\sin \omega} \\
&= n\rho^{n-1} \cos(n-1)\omega.
\end{aligned}$$

Theorem 4.3 has been proven for functions in J^+ of the form $f(q)=q^n$. The equation $A_{f^*} = J_f + D^*[J_f + (\text{div} \bar{f})I]D$ was only proven for monomials without coefficients. If a real coefficient were introduced, it would still be true because the coefficient could be pulled out in front. Since the formula contains all linear operations it follows from basic linear algebra that the formula is also true for finite sums of monomials. In other words, it is also true for polynomials.

Is the formula $A_{f^*} = J_f + D^*[J_f + (\text{div} \bar{f})I]D$ also true for convergent power series $f(q) = \sum_{n=0}^{\infty} c_n q^n$? It was shown in Chapter 3 that f converges, and therefore f^* converges on the same open ball. Since A_{f^*} is made up of the three components of f^* , all the entries in A_{f^*} converge. Thus, A_{f^*} is well-defined for a power series. Also five entries in J_f are the same as the ones in A_{f^*} so that leaves only four entries left to check. We need to check these four entries to make sure each one converges.

Let $f(q) = \sum_{n=0}^{\infty} c_n q^n$.

Using our previous change of variables,

$$\begin{aligned}
 f(q) &= \sum_{n=0}^{\infty} c_n (\rho^n \cos n\omega + \sin n\omega L) \\
 &= \sum_{n=0}^{\infty} c_n \left[\rho^n \cos n\omega + \sin n\omega \left(\frac{y\vec{i} + z\vec{j}}{\sqrt{y^2 + z^2}} \right) \right] \\
 &= \sum_{n=0}^{\infty} \left(c_n \rho^n \cos n\omega + \frac{c_n \rho^n y \sin n\omega \vec{i}}{\sqrt{y^2 + z^2}} + \frac{c_n \rho^n z \sin n\omega \vec{j}}{\sqrt{y^2 + z^2}} \right) \\
 &= \sum_{n=0}^{\infty} c_n \rho^n \cos n\omega + \sum_{n=0}^{\infty} \frac{c_n \rho^n y \sin n\omega \vec{i}}{\sqrt{y^2 + z^2}} + \sum_{n=0}^{\infty} \frac{c_n \rho^n z \sin n\omega \vec{j}}{\sqrt{y^2 + z^2}}.
 \end{aligned}$$

This gives us u , v and w .

We need to check $\frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial z}$, $\frac{\partial w}{\partial y}$, and $\frac{\partial w}{\partial z}$ to make sure they converge.

$$\begin{aligned}
 \frac{\partial v}{\partial y} &= \sum_{n=0}^{\infty} \frac{c_n \rho^{n-1}}{y^2 + z^2} \left(ny^2 \cos(n-1)\omega + \frac{z^2 \sin n\omega}{\sin \omega} \right) \\
 &= \sum_{n=0}^{\infty} \frac{c_n \rho^{n-1} ny^2 \cos(n-1)\omega}{y^2 + z^2} + \sum_{n=0}^{\infty} \frac{c_n \rho^{n-1} z^2 \sin n\omega}{(y^2 + z^2) \sin \omega}.
 \end{aligned}$$

The first piece converges because $\frac{y^2}{y^2 + z^2} \leq 1$ and the rest is $\frac{\partial u}{\partial x}$ which we know

converges. The second piece is a little trickier. First, $\frac{z^2}{y^2 + z^2} \leq 1$. Next, $\frac{\sin n\omega}{\sin \omega}$ is of

indeterminate form at $\omega = 0$. So $\lim_{\omega \rightarrow 0} \frac{\sin n\omega}{\sin \omega} = \lim_{\omega \rightarrow 0} \frac{n \cos(n-1)\omega}{\cos \omega} = n$ at $\omega = 0$. Therefore,

the second piece is $\sum_{n=0}^{\infty} c_n n \rho^{n-1}$ at $\omega = 0$ which converges. So $\frac{\partial v}{\partial y}$ converges.

$$\begin{aligned} \frac{\partial v}{\partial z} &= \sum_{n=0}^{\infty} \frac{c_n \rho^{n-1} y z}{y^2 + z^2} \left(n \cos(n-1)\omega + \frac{\sin n\omega}{\sin \omega} \right) \\ &= \sum_{n=0}^{\infty} \frac{c_n \rho^{n-1} y z n \cos(n-1)\omega}{y^2 + z^2} + \sum_{n=0}^{\infty} \frac{c_n \rho^{n-1} y z \sin n\omega}{(y^2 + z^2) \sin \omega}. \end{aligned}$$

The first piece converges because $\frac{y^2}{y^2 + z^2} \leq 1$ and the rest is $\frac{\partial u}{\partial x}$ which we know

converges. The second piece also converges for the same reasons stated for $\frac{\partial v}{\partial y}$.

Therefore $\frac{\partial v}{\partial z}$ converges. The next two entries $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$ also converge because of

their similarities to the previous two entries. Now that all of the entries of J_f have been

shown to converge, we know that the formula holds for $f(q) = \sum_{n=0}^{\infty} c_n q^n$ as well. This

completes the proof of Theorem 4.3.

Corollary

Let f be a complex analytic function. Then $A_{f^*} = J_f$

Proof of Corollary.

The corollary is true because the only derivation of the complex plane is the zero

derivation. This reduces the equation in Theorem 4.3 to $A_{f^*} = J_f$.

Going back again to the exponential function which is an example of an elementary function, we can check to see if it follows our formula.

Let $f(q) = \sum_{n=0}^{\infty} \frac{1}{n!} q^n = e^q$ and show $A_{f^*} = J_f + D^*[J_f + (\text{div} \bar{f})I]D$.

$$J_f = \begin{pmatrix} e^t \cos s & -e^t \sin s \cos \psi & -e^t \sin s \sin \psi \\ e^t \sin s \cos \psi & \frac{e^t \sin s \sin^2 \psi}{s} & -\frac{e^t \sin s \sin \psi \cos \psi}{s} \\ e^t \sin s \sin \psi & -\frac{e^t \sin s \sin \psi \cos \psi}{s} & \frac{e^t \sin s \cos^2 \psi}{s} \end{pmatrix}$$

$$\text{div} \bar{f} = e^t \cos s - \frac{e^t \sin s \sin^2 \psi}{s} - e^t \cos s \cos^2 \psi - \frac{e^t \sin s \cos^2 \psi}{s} - e^t \cos s \sin^2 \psi$$

$$= e^t \cos s - \frac{e^t \sin s}{s} - e^t \cos s$$

$$= -\frac{e^t \sin s}{s}$$

$$J_f + D^*[J_f + (\operatorname{div} \bar{f})I]D$$

$$= J_f + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{e^t \sin s \sin^2 y}{s} & -\frac{e^t \sin s \sin y \cos y}{s} \\ 0 & -\frac{e^t \sin s \sin y \cos y}{s} & \frac{e^t \sin s \sin^2 y}{s} \end{pmatrix}$$

$$= \begin{pmatrix} e^t \cos s & -e^t \sin s \cos \psi & -e^t \sin s \sin \psi \\ e^t \sin s \cos \psi & e^t \cos s & 0 \\ e^t \sin s \sin \psi & 0 & e^t \cos s \end{pmatrix}$$

$$= A_{f^{**}}.$$

Our formula works for the exponential function, and yields some identities connecting the power series and closed form expressions. These will be useful later when we are working with Stokes's Theorem and the divergence theorem.

Next it is also interesting to consider what the formula gives at the point of

expansion, $q=0$. Still using $f(q)=e^q$, we will evaluate J_f and A_f at $q=0$.

If $q=0$, then $x=0$, $y=0$ and $z=0$.

So $s=0$ and $t=0$.

$$J_f = \begin{pmatrix} e^0 \cos 0 & -e^0 \sin 0 \cos \psi & -e^0 \sin 0 \sin \psi \\ e^0 \sin 0 \cos \psi & \frac{e^0 \sin 0 \sin^2 \psi}{0} & -\frac{e^0 \sin 0 \sin \psi \cos \psi}{0} \\ e^0 \sin 0 \sin \psi & -\frac{e^0 \sin 0 \sin \psi \cos \psi}{0} & \frac{e^0 \sin 0 \cos^2 \psi}{0} \end{pmatrix}.$$

$$\frac{e^0 \sin 0}{0} = 1 \text{ since } \lim_{s \rightarrow 0} \frac{\sin s}{s} = 1.$$

Therefore

$$J_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2 \psi + \cos^2 \psi & 0 \\ 0 & 0 & \cos^2 \psi + \sin^2 \psi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$D^*[J_f + (\text{div} \bar{f})I]D \text{ at } q = 0$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{e^0 \sin 0 \sin^2 \psi}{0} & \frac{e^0 \sin 0 \sin \psi \cos \psi}{0} \\ 0 & +e^0 \cos 0 \sin^2 \psi - \frac{e^0 \sin 0}{0} & -e^0 \cos 0 \sin \psi \cos \psi \\ 0 & \frac{e^0 \sin 0 \sin \psi \cos \psi}{0} & \frac{e^0 \sin 0 \sin^2 \psi}{0} \\ 0 & -e^0 \cos 0 \sin \psi \cos \psi & +e^0 \cos 0 \sin^2 \psi - \frac{e^0 \sin 0}{0} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos^2 \psi + \sin^2 \psi - 1 & \sin \psi \cos \psi - \sin \psi \cos \psi \\ 0 & \sin \psi \cos \psi - \sin \psi \cos \psi & \cos^2 \psi + \sin^2 \psi - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_f + D^*[J_f + (\text{div} \bar{f})I]D \text{ evaluated at } q = 0 \text{ is } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{This means } A_{f^*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ at } q = 0.$$

The Jacobian and the adjoint are equal when evaluated at $q = 0$ for the exponential function. Both are equal to the identity matrix. While this is true for the exponential function at the point of expansion, is this true for monomials as well?

Let $f(q) = q^2$ and $q = x + i\vec{y} + \vec{j}z$.

$$\text{So } J_f = \begin{pmatrix} 2x & -2y & -2z \\ 2y & 2x & 0 \\ 2z & 0 & 2x \end{pmatrix}$$

$$\text{At } q = 0 \quad J_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Also } A_{f^*} = \begin{pmatrix} 2x & -2y & -2z \\ 2y & 2x & 0 \\ 2z & 0 & 2x \end{pmatrix}$$

$$\text{At } q = 0 \quad A_{f^*} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It turns out that at the point of expansion, $q=0$, for the function $f(q) = q^2$ $A_{f^*} = J_f$. They are both the zero matrix. This is true for all monomials where $n > 1$. For $f(q) = q$, $J_f = A_{f^*} = I$ at $q = 0$. The same would be true for $f(q) = c_n q^n$.

Is it true for $f(q) = \sum_{n=0}^{\infty} c_n q^n$? Since all of the entries of A_{f^*} and J_f are the same as they were when using $f(q) = q^n$ except that each entry has a real coefficient and a summation now, it would still be true. The same reasons used before apply now. At $q=0$, $x=0$, $y=0$ and $z=0$. Therefore, $\rho=0$, and each entry contains a ρ so each entry equals 0. This results in the Jacobian and adjoint being equal when evaluated at $q=0$.

Theorem 4.4

For all elementary functions on \mathbf{J} , $J_f(a) = A_{f^*}(a)$. In other words, the Jacobian and adjoint matrices are equal at the point of expansion.

Finally, we formulate a conjecture which is the conjugate of Theorem 4.4.

Conjecture

Suppose we have a function $f: \mathbf{J} \rightarrow \mathbf{J}$ and J_f has c^∞ entries on an open, simply-connected set Ω . If $J_f + D^*[J_f + (\text{div} \bar{f})I]D$ satisfies our generalized Cauchy-Riemann form, is f elementary at each $a \in \Omega$.

CHAPTER 5

Applications

The last part of this paper will consider some applications of elementary functions on \mathbf{J} . We are dealing with a vector field \mathbf{F} and so the curl and divergence will be calculated. The Divergence Theorem and Stokes's Theorem will be applied to the vector field. The elementary function, $f(q) = \sum_{n=0}^{\infty} c_n q^n$, with the exponential function,

$f(q) = e^q = \sum_{n=0}^{\infty} \frac{1}{n!} q^n$, as a particular example will be considered.

We will begin by computing the divergence of an elementary function. Also, the divergence of the conjugate of f , denoted \bar{f} , will be computed.

Let $f(q) = \sum_{n=0}^{\infty} c_n q^n$, $c_n \in \mathbf{R}$.

Since $q^n = \rho^n (\cos n\omega + \sin n\omega L)$

$$\begin{aligned} &= \rho^n \left[\cos n\omega + \sin n\omega \left(\frac{y\vec{i} + z\vec{j}}{\sqrt{y^2 + z^2}} \right) \right] \\ &= \rho^n \cos n\omega + \frac{\rho^n y \sin n\omega \vec{i}}{\sqrt{y^2 + z^2}} + \frac{\rho^n z \sin n\omega \vec{j}}{\sqrt{y^2 + z^2}}, \end{aligned}$$

we have $f(q) = \sum_{n=0}^{\infty} c_n \left(\rho^n \cos n\omega + \frac{\rho^n y \sin n\omega \vec{i}}{\sqrt{y^2 + z^2}} + \frac{\rho^n z \sin n\omega \vec{j}}{\sqrt{y^2 + z^2}} \right)$

where $u = \sum_{n=0}^{\infty} c_n \rho^n \cos n\omega$, $v = \sum_{n=0}^{\infty} \frac{c_n \rho^n y \sin n\omega \vec{i}}{\sqrt{y^2 + z^2}}$ and $w = \sum_{n=0}^{\infty} \frac{c_n \rho^n z \sin n\omega \vec{j}}{\sqrt{y^2 + z^2}}$.

Since $\text{div} f = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$, these partial derivatives need to be computed.

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} c_n n \rho^{n-1} \cos(n-1)\omega.$$

$$\frac{\partial v}{\partial y} = \sum_{n=1}^{\infty} \frac{c_n \rho^{n-1}}{y^2 + z^2} \left[ny^2 \cos(n-1)\omega + \frac{z^2 \sin n\omega}{\sin \omega} \right].$$

$$\frac{\partial w}{\partial z} = \sum_{n=1}^{\infty} \frac{c_n \rho^{n-1}}{y^2 + z^2} \left[nz^2 \cos(n-1)\omega + \frac{y^2 \sin n\omega}{\sin \omega} \right].$$

$$\begin{aligned} \text{div } f &= \sum_{n=1}^{\infty} c_n \rho^{n-1} \left[n \cos(n-1)\omega + \frac{ny^2 \cos(n-1)\omega + \frac{z^2 \sin n\omega}{\sin \omega} + nz^2 \cos(n-1)\omega + \frac{y^2 \sin n\omega}{\sin \omega}}{y^2 + z^2} \right] \\ &= \sum_{n=1}^{\infty} \frac{c_n \rho^{n-1}}{y^2 + z^2} \left[(y^2 + z^2)n \cos(n-1)\omega + (y^2 + z^2)n \cos(n-1)\omega + (y^2 + z^2) \frac{\sin n\omega}{\sin \omega} \right] \\ &= \sum_{n=1}^{\infty} c_n \rho^{n-1} \left[2 \cos(n-1)\omega + \frac{\sin n\omega}{\sin \omega} \right]. \end{aligned}$$

$$\text{Similarly, } \text{div } \bar{f} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} = - \sum_{n=1}^{\infty} c_n \rho^{n-1} \frac{\sin n\omega}{\sin \omega}.$$

Now we will compute the divergence of the exponential function as well as the divergence of its conjugate. We will be using the explicit form of exponential instead of the series form.

For $f(q) = e^q = \sum_{n=0}^{\infty} \frac{1}{n!} q^n$, $e^q = e^t \cos s + \vec{i} e^t \sin s \cos \psi + \vec{j} e^t \sin s \sin \psi$.

$$\text{Again } \operatorname{div} f = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\text{where } \frac{\partial u}{\partial x} = e^t \cos s, \quad \frac{\partial v}{\partial y} = \frac{e^t}{s} \sin s \sin^2 \psi + e^t \cos s \cos^2 \psi$$

$$\text{and } \frac{\partial w}{\partial z} = \frac{e^t}{s} \sin s \cos^2 \psi + e^t \cos s \sin^2 \psi.$$

$$\begin{aligned} \text{Therefore, } \operatorname{div} f &= e^t \cos s + \frac{e^t}{s} \sin s \sin^2 \psi + e^t \cos s \cos^2 \psi + \frac{e^t}{s} \sin s \cos^2 \psi + e^t \cos s \sin^2 \psi \\ &= 2e^t \cos s + \frac{e^t}{s} \sin s \end{aligned}$$

$$\text{Similarly, } \operatorname{div} \bar{f} = \frac{-e^t \sin s}{s}.$$

Before applying the divergence theorem, we will compute the curl of

$f(q) = \sum_{n=0}^{\infty} c_n q^n$ and its conjugate, as well as the curl of the exponential function and its conjugate.

$$\text{The curl } f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right).$$

For $f(q) = \sum_{n=0}^{\infty} c_n q^n$,

$$\text{curl } f = \begin{pmatrix} \sum_{n=1}^{\infty} \frac{c_n \rho^{n-1} y z}{y^2 + z^2} \left[n \cos(n-1)\omega - \frac{\sin n\omega}{\sin \omega} \right] - \sum_{n=1}^{\infty} \frac{c_n \rho^{n-1} y z}{y^2 + z^2} \left[n \cos(n-1)\omega - \frac{\sin n\omega}{\sin \omega} \right] \\ - \sum_{n=1}^{\infty} \frac{c_n n \rho^{n-1} z \sin(n-1)\omega}{\sqrt{y^2 + z^2}} - \sum_{n=1}^{\infty} \frac{c_n n \rho^{n-1} z \sin(n-1)\omega}{\sqrt{y^2 + z^2}} \\ \sum_{n=1}^{\infty} \frac{c_n n \rho^{n-1} y \sin(n-1)\omega}{\sqrt{y^2 + z^2}} + \sum_{n=1}^{\infty} \frac{c_n n \rho^{n-1} y \sin(n-1)\omega}{\sqrt{y^2 + z^2}} \end{pmatrix}$$

$$= \left(0, -2 \sum_{n=1}^{\infty} \frac{c_n n \rho^{n-1} z \sin(n-1)\omega}{\sqrt{y^2 + z^2}}, 2 \sum_{n=1}^{\infty} \frac{c_n n \rho^{n-1} y \sin(n-1)\omega}{\sqrt{y^2 + z^2}} \right).$$

Similarly, $\text{curl } \bar{f} = (0, 0, 0)$.

Theorem 5.1

The vector field afforded by the conjugate of an elementary on \mathbf{J} is curl-free, whereas the curl of a vector field afforded by the elementary function itself lies in the \vec{ij} -plane.

Now we will compute the curl of the exponential function and its conjugate.

For $f(q) = e^q$,

$$\begin{aligned} \operatorname{curl} f &= \begin{pmatrix} -\frac{e^t}{s} \sin s \sin \psi \cos \psi + e^t \cos s \sin \psi \cos \psi \\ + \frac{e^t}{s} \sin s \sin \psi \cos \psi - e^t \cos s \sin \psi \cos \psi \\ e^t \sin s \sin \psi + e^t \sin s \sin \psi \\ -e^t \sin s \cos \psi - e^t \sin s \cos \psi \end{pmatrix} \\ &= (0, \quad 2e^t \sin s \sin \psi, \quad -2e^t \sin s \cos \psi). \end{aligned}$$

Of course, $\operatorname{curl} \bar{f} = (0, \quad 0, \quad 0)$.

In summary,

$$f(q) = \sum_{n=0}^{\infty} c_n q^n$$

$$f(q) = e^q$$

$$\operatorname{div} f = \sum_{n=1}^{\infty} c_n r^{n-1} \left[2 \cos(n-1)w + \frac{\sin nw}{\sin w} \right] \quad 2e^t \cos s + \frac{e^t}{s} \sin s$$

$$\operatorname{div} \bar{f} = \sum_{n=1}^{\infty} c_n r^{n-1} \frac{\sin nw}{\sin w} \quad \frac{-e^t \sin s}{s}$$

$$\operatorname{curl} f = \begin{pmatrix} 0, \\ -2 \sum_{n=1}^{\infty} \frac{c_n n r^{n-1} z \sin(n-1)w}{y^2 + z^2}, \\ 2 \sum_{n=1}^{\infty} \frac{c_n n r^{n-1} z \sin(n-1)w}{y^2 + z^2} \end{pmatrix}, \quad \begin{pmatrix} 0, \\ 2e^t \sin s \sin y, \\ -2e^t \sin s \cos y \end{pmatrix}$$

For comparison, we will compute the curl of the complex exponential function and its conjugate.

Let $f(z) = e^z$, where $z = x + iy$ then $f(z) = e^z = e^x (\cos y + i \sin y)$.

This gives us $u = e^x \cos y$ and $v = e^x \sin y$.

$$\begin{aligned}\text{Therefore, } \text{curl } f &= \left(0, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \\ &= (0, e^x \sin y + e^x \sin y) \\ &= (0, 2e^x \sin y).\end{aligned}$$

Similary, $\text{curl } \bar{f} = (0, 0)$.

These results coincide with the results obtained for $f(q) = e^q$ since in the xy-plane $\psi=0$ and the curl of e^q would then be $(0, 0, -2e^t \sin s)$. There is a difference in sign which is acceptable because the curl needs an orientation defined.

What about the general elementary function? It must also agree with the complex exponential function when restricted to the xy-plane. This would mean $z = 0$ and

$$\text{curl } f = \left(0, 0, 2 \sum_{n=1}^{\infty} c_n n \rho^{n-1} \sin(n-1)\omega \right). \text{ Notice that I let } \frac{y}{\sqrt{y^2}} = 1 \text{ instead of } \pm 1. \text{ It is}$$

not necessary to worry about the sign, as said previously, since it just gives the orientation. The last entries for each curl must agree. These entries are

$2e^x \sin y$ and $2 \sum_{n=1}^{\infty} c_n n \rho^{n-1} \sin(n-1)\omega$. In order to compare these, some variable changes will be made.

Let $c_n = \frac{1}{n!}$

then $2 \sum_{n=1}^{\infty} c_n n \rho^{n-1} \sin(n-1)\omega$

$$= 2 \sum_{n=1}^{\infty} \frac{n}{n!} \rho^{n-1} \sin(n-1)\omega$$

$$= 2 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left(\sqrt{x^2 + y^2} \right)^{n-1} \sin(n-1)\omega.$$

Converting from these variables to polar coordinates, the elementary function's entry is

$$2 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} r^{n-1} \sin(n-1)\omega.$$

Since the curl of the elementary function must agree with the curl of the complex exponential function, an identity results.

$$e^x \sin y = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} r^{n-1} \sin(n-1)\omega.$$

Now that the divergence and curl have been computed for both functions, the Divergence Theorem and Stokes's Theorem can be applied.

Divergence Theorem on unit ball

Let \mathbf{F} be continuously differentiable throughout the unit ball.

Then $\int_{\text{sphere}} \mathbf{F} \cdot \bar{n} d\sigma = \int_{\text{ball}} \text{div} \mathbf{F} dV$ where \bar{n} is the outward unit normal.

In order to use the divergence theorem we will need to decide on the variables to be used and the limits of integration. Two different methods were used for doing this.

The first method is a variation on the cylindrical coordinate system.

Before integrating, the limits of integration needed to be determined. Once found, the limits were checked by integrating over the unit ball. The following variables will be used.

t - distance on the x -axis $-\sqrt{1-s^2} \leq t \leq \sqrt{1-s^2}$

s - length of vector $0 \leq s \leq 1$

ψ - angle vector makes with xy -plane $0 \leq \psi < 2\pi$

Jacobian = s

Checking the limits of integration over the unit ball, we have

$$\begin{aligned} \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} s dt ds d\psi &= \int_0^{2\pi} \int_0^1 s t \Big|_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} ds d\psi \\ &= \int_0^{2\pi} \int_0^1 2s\sqrt{1-s^2} ds d\psi. \end{aligned}$$

Letting $u = 1 - s^2$ and $du = -2s ds$,

$$= - \int_0^{2\pi} \int_0^1 u^{1/2} du d\psi$$

$$= - \int_0^{2\pi} \frac{2}{3} u^{3/2} \Big|_1^0 d\psi$$

$$= \int_0^{2\pi} \frac{2}{3} d\psi = \frac{2}{3} \psi \Big|_0^{2\pi} = \frac{4\pi}{3} \text{ which is what it should be.}$$

After checking the limits of integration, the divergence theorem was applied to the conjugate of the exponential function.

Given $f(q) = e^q$ then $\text{div} \bar{f} = \frac{-e^t}{s} \sin s$.

$$\begin{aligned}
 \text{So } \int_{\text{ball}} \int \int \text{div} \bar{f} dv &= - \int_0^1 \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \int_0^{2\pi} e^t \sin s d\psi dt ds \\
 &= - \int_0^1 \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \left[\int_0^{2\pi} e^t \sin s dt \right] ds \\
 &= - 2\pi \int_0^1 \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} e^t \sin s ds \\
 &= - 2\pi \int_0^1 e^t \sin s \left[-\sqrt{1-s^2} \right]_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} ds \\
 &= - 2\pi \int_0^1 \sin s \left(e^{\sqrt{1-s^2}} - e^{-\sqrt{1-s^2}} \right) ds \\
 &= - 4\pi \int_0^1 \sin s \sinh(\sqrt{1-s^2}) ds.
 \end{aligned}$$

No elementary antiderivative exists for this integral. Variations of this integral appear in the theory of electrodynamics. In order to evaluate it, we will try a different approach.

This time spherical coordinates will be used. We will begin by applying the divergence theorem to the conjugate of the general elementary function. The variables of integration will be ρ , ω , and ψ . They are defined below.

$\rho = \text{magnitude of } \vec{q} = \sqrt{x^2 + y^2 + z^2} \quad 0 < \rho \leq 1$

$\omega = \text{angle } \vec{q} \text{ makes with the real axis} \quad 0 < \omega \leq \pi$

$\psi = \text{angle the } \mathbf{C} \text{-plane } (\vec{q} \text{ lies in}) \text{ makes with the } xy \text{-plane} \quad 0 < \psi \leq \pi$

The Jacobian factor for spherical coordinates is $\rho^2 \sin \omega$.

Given $f(q) = \sum_{n=0}^{\infty} c_n q^n$, $c_n \in \mathbf{R}$ then $\text{div} \bar{f} = - \sum_{n=1}^{\infty} \frac{c_n \rho^{n-1} \sin n\omega}{\sin \omega}$.

$$\begin{aligned}
 \iiint_{\text{ball}} \text{div} \bar{f} dv &= - \int_0^{2\pi} \int_0^{\pi} \int_0^1 \sum_{n=1}^{\infty} c_n \rho^{n+1} \sin n\omega d\rho d\omega d\psi \\
 &= - \sum_{n=1}^{\infty} \int_0^{2\pi} \int_0^{\pi} \int_0^1 c_n \rho^{n+1} \sin n\omega d\rho d\omega d\psi \\
 &= - \sum_{n=1}^{\infty} \int_0^{2\pi} \int_0^{\pi} \left[\frac{c_n \rho^{n+2}}{n+2} \sin n\omega \right]_0^1 d\omega d\psi \\
 &= - \sum_{n=1}^{\infty} \int_0^{2\pi} \int_0^{\pi} \frac{c_n \sin n\omega}{n+2} d\omega d\psi \\
 &= - \sum_{n=1}^{\infty} \int_0^{2\pi} \left[-\frac{c_n \cos n\omega}{n(n+2)} \right]_0^{\pi} d\psi \\
 &= - \sum_{n=1}^{\infty} \int_0^{2\pi} -\frac{c_n}{n(n+2)} \left\{ \begin{array}{l} 0 \text{ when } n \text{ is even} \\ -2 \text{ when } n \text{ is odd} \end{array} \right\} d\psi \\
 &= - \sum_{n=1}^{\infty} \int_0^{2\pi} -\frac{2c_n}{n(n+2)} d\psi \text{ when } n \text{ is odd} = 0 \text{ when } n \text{ is even} \\
 &= - \sum_{n=1}^{\infty} \left[-\frac{c_n \psi}{n(n+2)} \right]_0^{2\pi} \quad n \text{ odd} \\
 &= -4\pi \sum_{n=1}^{\infty} -\frac{c_n}{n(n+2)} \quad n \text{ odd.}
 \end{aligned}$$

Letting $n = 2k + 1$ for $k = 0 \rightarrow \infty$,

we get $-4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)}$.

Therefore $\int \int \int \text{div} \bar{f} = -4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)}$.

Now the divergence theorem will be applied to the elementary function, using the same limits and Jacobian.

Given $f(q) = \sum_{n=0}^{\infty} c_n q^n$, then $\text{div} f = \sum_{n=1}^{\infty} c_n \rho^{n-1} \left[2 \cos(n-1)\omega + \frac{\sin n\omega}{\sin \omega} \right] \rho^2 \sin \omega$.

$$\begin{aligned}
 \int \int \int_{\text{ball}} \text{div} f dv &= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \sum_{n=1}^{\infty} c_n \rho^{n-1} \left[2 \cos(n-1)\omega + \frac{\sin n\omega}{\sin \omega} \right] \rho^2 \sin \omega \\
 &= \sum_{n=1}^{\infty} 2c_n \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^{n+1} \cos(n-1)\omega \sin \omega d\rho d\omega d\psi + 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)} \\
 &= \sum_{n=1}^{\infty} 2c_n \int_0^{2\pi} \int_0^{\pi} \frac{\rho^{n+2}}{n+2} \cos(n-1)\omega \sin \omega \Big|_0^1 d\omega d\psi + 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)} \\
 &= \sum_{n=1}^{\infty} \frac{2c_n}{n+2} \int_0^{2\pi} \int_0^{\pi} \cos(n-1)\omega \sin \omega d\omega d\psi + 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)} \\
 &= \sum_{n=1}^{\infty} \frac{2c_n}{n+2} \int_0^{2\pi} \int_0^{\pi} \frac{1}{2} \{ \sin[\omega - (n-1)\omega] + \sin[\omega + (n-1)\omega] \} d\omega d\psi + 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)} \\
 &= \sum_{n=1}^{\infty} \frac{2c_n}{n+2} \int_0^{2\pi} \int_0^{\pi} [\sin(2-n)\omega + \sin \omega] d\omega d\psi + 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)} \\
 &= \sum_{n=1}^{\infty} \frac{c_n}{n+2} \int_0^{2\pi} \left[\frac{-\cos(2-n)\omega}{2-n} - \frac{\cos n\omega}{n} \right]_0^{\pi} d\psi + 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)}
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{c_n}{n+2} \int_0^{2\pi} \frac{\cos(n-2)\omega}{2-n} - \frac{\cos n\omega}{n} \Bigg] d\psi + 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)} \\
&= \sum_{n=1}^{\infty} \frac{c_n}{n+2} \int_0^{2\pi} \left(-\frac{1}{n-2} + \frac{1}{n} \right) - \left(\frac{1}{n-2} - \frac{1}{n} \right) d\psi + 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)} \\
&\text{when } n \text{ is odd} \\
&n \text{ even} = 0
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{c_n}{n+2} \int_0^{2\pi} 2 \left(\frac{1}{n} - \frac{1}{n-2} \right) d\psi + 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)} \\
&n \text{ odd}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{c_n}{n+2} \left[2 \left(\frac{1}{n} - \frac{1}{n-2} \right) \psi \right]_0^{2\pi} + 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)} \\
&n \text{ odd}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{2\pi c_n}{n+2} \left[\frac{-4}{n(n-2)} \right] + 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)} \\
&n \text{ odd}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{8\pi c_n}{n(n+2)(n-2)} + \sum_{n=1}^{\infty} \frac{4\pi c_n}{n(n+2)} \\
&n \text{ odd} \qquad \qquad \qquad n \text{ odd}
\end{aligned}$$

$$\begin{aligned}
&= 4\pi \sum_{n=1}^{\infty} c_n \left[\frac{n-2}{n(n+2)(n-2)} - \frac{2}{n(n+2)(n-2)} \right] \\
&n \text{ odd}
\end{aligned}$$

$$= 4\pi \sum_{n=1}^{\infty} \frac{c_n(n-4)}{n(n+2)(n-2)} \quad n \text{ odd}$$

$$= 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}(2k-3)}{(2k+1)(2k+3)(2k-1)}$$

$$\text{Therefore } \int \int \int \text{div} f = 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}(2k-3)}{(2k+1)(2k+3)(2k-1)}.$$

Instead of integrating the divergence of the exponential function, the previous work will be applied.

$$\text{Let } f(q) = e^q = \sum_{n=0}^{\infty} \frac{1}{n!} q^n \text{ with } c_n = \frac{1}{n!} \text{ so } c_{2k+1} = \frac{1}{(2k+1)!}.$$

$$\text{From previous work, } \int \int \int \text{div} f = 4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}(2k-3)}{(2k+1)(2k+3)(2k-1)} \text{ for } f(q) = \sum_{n=0}^{\infty} c_n q^n.$$

$$\text{But for } f(q) = e^q = \sum_{n=0}^{\infty} \frac{1}{n!} q^n \text{ and } c_{2k+1} = \frac{1}{(2k+1)!},$$

$$\int \int \int \text{div} f = 4\pi \sum_{k=0}^{\infty} \frac{(2k-3)}{(2k+1)(2k+3)(2k-1)(2k+1)!}.$$

Again the previous work will be applied to find the integral of the conjugate of the exponential function.

$$\text{From previous work } \int \int \int \text{div} \bar{f} = -4\pi \sum_{k=0}^{\infty} \frac{c_{2k+1}}{(2k+1)(2k+3)} \text{ for } f(q) = \sum_{n=0}^{\infty} c_n q^n.$$

$$\text{But for } f(q) = e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!} \text{ and } c_{2k+1} = \frac{1}{(2k+1)!},$$

$$\begin{aligned} \int \int \int \text{div} \bar{f} &= -4\pi \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+3)(2k+1)!} \\ &= -4\pi \left(\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5 \cdot 3!} + \frac{1}{5 \cdot 7 \cdot 5!} + \dots + \frac{1}{(2k+1)(2k+3)(2k+1)!} + \dots \right). \end{aligned}$$

We want to rewrite this series and compare it with the integral we found previously using cylindrical coordinates. We recall the MacLauren expansion of the hyperbolic sine function from which we derive the following.

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\frac{\sinh(x)}{x} = 1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \dots$$

$$\int \frac{\sinh(x)}{x} = x + \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} + \dots$$

$$x \int \frac{\sinh(x)}{x} = x^2 + \frac{x^4}{3 \cdot 3!} + \frac{x^6}{5 \cdot 5!} + \dots$$

$$\int \left[x \int \frac{\sinh(x)}{x} \right] = \frac{x^3}{3} + \frac{x^5}{3 \cdot 5 \cdot 3!} + \frac{x^7}{5 \cdot 7 \cdot 5!} + \dots$$

$$\text{Let } g(x) = \int_0^x \frac{\sinh(t)}{t} dt = \text{shi}(x), \text{ then } \int \left[x \int \frac{\sinh(x)}{x} \right] = \int_0^1 x g(x) dx.$$

$$\text{Therefore } \int \int \int \text{div} \bar{f} = -4\pi \int_0^1 x g(x) dx$$

$$= -4\pi \int_0^1 \left[x \int_0^x \frac{\sinh(t)}{t} dt \right] dx \text{ for } f(q) = \sum_{n=0}^{\infty} c_n q^n.$$

In order to finish this, integration by parts will be used.

$$u = \int_0^x \frac{\sinh(t)}{t} dt \quad dv = x dx$$

$$du = \frac{\sinh(x)}{x} dx \quad v = \frac{1}{2} x^2$$

$$\text{So } \int \int \int \text{div} \bar{f} = -4\pi \left[\left(\int_0^x \frac{\sinh(x)}{x} \right) \left(\frac{1}{2} x^2 \right) - \int_0^1 \frac{1}{2} x^2 \frac{\sinh(x)}{x} dx \right]$$

$$= -4\pi \left[\frac{1}{2} x^2 \text{shi}(x) \Big|_0^1 - \frac{1}{2} \int_0^1 x \sinh x dx \right].$$

Again integration by parts will be used.

$$u = x \quad dv = \sinh x dx$$

$$du = dx \quad v = \cosh x$$

$$\text{So } \int \int \int \operatorname{div} \bar{f} = -4\pi \left[\frac{1}{2} x^2 \operatorname{shi}(x) \right]_0^1 - \frac{1}{2} \left\{ x \cosh x \Big|_0^1 - \int_0^1 \cosh x dx \right\}$$

$$= -2\pi x^2 \operatorname{shi}(x) \Big|_0^1 + 2\pi x \cosh x \Big|_0^1 - 2\pi \sinh x \Big|_0^1$$

$$= -2\pi \operatorname{shi}(1) + 2\pi \cosh(1) - 2\pi \sinh(1).$$

Another identity can be obtained by equating what was just found with the attempt made previously using cylindrical coordinates.

$$\text{For } f(q) = e^q$$

$$\int \int \int \operatorname{div} \bar{f} = -4\pi \int_0^1 \sin s \sinh(\sqrt{1-s^2}) ds,$$

and

$$\int \int \int \operatorname{div} \bar{f} = -2\pi \operatorname{shi}(1) + 2\pi \cosh(1) - 2\pi \sinh(1).$$

By equating these two results,

$$\begin{aligned} \int_0^1 \sin s \sinh(\sqrt{1-s^2}) ds &= \frac{1}{2} [\operatorname{shi}(1) - \cosh(1) + \sinh(1)] \\ &= \frac{1}{2} \left[\operatorname{shi}(1) - \frac{1}{2}(e^1 + e^{-1}) + \frac{1}{2}(e^1 - e^{-1}) \right] \\ &= \frac{1}{2} \left[\operatorname{shi}(1) - \frac{1}{2} \left(e - \frac{1}{e} - e - \frac{1}{e} \right) \right] = \frac{1}{2} \left[\operatorname{shi}(1) - \frac{1}{e} \right] \end{aligned}$$

An approximation for $shi(1)$ is required to estimate this integral.

$$\text{Given } shi(x) = \int_0^x \frac{\sinh t}{t} dt$$

$$\text{then } shi(1) = \int_0^1 \frac{\sinh t}{t} dt$$

$$= \int_0^1 \left(\frac{t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots}{t} \right) dt$$

$$= \int_0^1 \left(1 + \frac{t^2}{3!} + \frac{t^4}{5!} + \dots \right) dt$$

$$= \left[t + \frac{t^3}{3!3} + \frac{t^5}{5!5} + \dots \right]_0^1$$

$$= 1 + \frac{1}{3!3} + \frac{1}{5!5} + \dots > 1.$$

Therefore, $shi(1)$ is slightly greater than 1. ($shi(1) \approx 1.0573$)

Substituting this value back in,

$$\begin{aligned} \int_0^1 \sin s \sinh(\sqrt{1-s^2}) ds &= \frac{1}{2} \left[shi(1) - \frac{1}{e} \right] \\ &\approx \frac{1}{2} \left[1 - \frac{1}{e} \right]. \end{aligned}$$

Thus we have obtained an explicit evaluation of an integral form which appears in the theory of electrodynamics.

The Divergence Theorem has been applied to f and \bar{f} for $f(q) = \sum_{n=0}^{\infty} c_n q^n$ and

$f(q) = e^q$. Now some results of Stokes's Theorem will be stated.

Stokes's Theorem

If C bounds a surface S , and S lies within the region of convergence for f and \bar{f}

(respectively \bar{F}) is a vector field afforded by f (respectively \bar{f}), then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \text{curl } \mathbf{F} \cdot \bar{n} dA \text{ (and similarly for } \bar{F} \text{)}.$$

Theorem 5.2

For the conjugate of an elementary function f with

$$\text{curl } f = \left(0, \frac{-2n\rho^{n-1}z \sin(n-1)\omega}{\sqrt{y^2 + z^2}}, \frac{2n\rho^{n-1}y \sin(n-1)\omega}{\sqrt{y^2 + z^2}} \right) \text{ and } \text{curl } \bar{f} = (0, 0, 0), \text{ the line}$$

integral of \bar{F} over C is equal to zero. Whereas for f , the line integral of \mathbf{F} over any C

lying in a plane parallel to the $\vec{i}\vec{j}$ - plane is equal to zero.

CHAPTER 6

Conclusion

The purpose of this paper was to study the properties of functions defined on a distinguished hyperplane of the quaternion. We chose J to be a 3-dimensional subspace which was closed under integral powers. This permitted the construction of functions on J by power series with real coefficients.

These functions were called elementary functions and were shown to converge on a ball centered at the point of expansion which was taken without loss of generality to be the origin. The significance of these functions is that they generalize analytic functions on the complex plane to three dimensions. The studies that followed were similar to those done in real and complex analysis. Throughout the paper, the exponential function, which is an example of an elementary function, was used to demonstrate the properties of elementary functions. The exponential function, defined as a power series with real coefficients, was shown to converge on the whole space. Because the exponential function could be written in series form as well as component form, comparisons were made which led to interesting identities.

Continuing with what was done in complex analysis, an attempt was made to define a quaternionic derivative. Since elementary functions had properties similar to complex analytic functions, it was hoped that the derivative could be defined as it was done in complex analysis. This was not possible, however. Instead, the term by term

power rule was used to define a new function called the derived function, and a relationship between the adjoint of the derived function and the Jacobian was established. In real and complex space, these two matrices are identical. However, in quaternionic space, the relationship was different. An equation relating these objects was obtained using derivations on \mathbf{J}^+ .

Finally, some applications of these elementary functions were considered. The Divergence Theorem and Stokes's Theorem, which have applications in physics, were applied to the quaternionic vector field. Special emphasis was given to the exponential function as an example.

The studies done in this paper are just the beginning of what could be explored in the area of quaternionic analysis. Since the upper-half space of \mathbf{J} is a model for hyperbolic geometry, these elementary functions may have applications in this field of study. In the latter part of the paper, the curl of the vector field afforded by elementary functions and their quaternionic conjugates was computed. Since the curl of the conjugate turned out to be zero on an open ball which is simply connected, we know a scalar potential function exists for the conjugate function. What applications might be derived from the scalar potential function, and what methods could be used? Also, since complex analytic functions are useful in the theory of planar elasticity, what are the applications of these 3-dimensional elementary functions in the theory of elasticity?

Finally, since the main identity in the paper (Theorem 4.3) expresses the adjoint matrix explicitly in terms of the Jacobian matrix, we have effectively found an analogue

for the determination of coefficients of the power series expansion of a complex analytic function from the values of its derivatives. Two questions in particular occur. What properties of the Jacobian of a function $f: \mathbf{J} \rightarrow \mathbf{J}$ ensure that it has an expansion as an elementary function on some ball? Also, is there a Cauchy integral-type formula for an elementary function $f: \mathbf{J} \rightarrow \mathbf{J}$? As one can see, the topic of this paper has many applications and further areas of studies which can be explored.

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