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Permutation and Monomial Progenitors

Crystal Diaz

Permutation and Monomial Progenitors

A Thesis

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Faculty of

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in

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by

Crystal Diaz

June 2020

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Approved by:

Dr. Zahid Hasan, Committee Chair

Dr. Min-Lin Lo, Committee Member

Dr. Belisario Ventura, Committee Member

Dr. David Maynard, Chair, Department of Mathematics

Dr. Corey Dunn, Graduate Coordinator

ABSTRACT

In this thesis, we searched several monomial and permutation progenitors for symmetric presentations of important images, nonabelian simple groups, their automorphism groups, or groups that have these as their factor groups. Our target non-abelian simple groups included sporadic groups, linear groups, and alternating groups. In this presentation, we have described our search for the homomorphic images through the permutation progenitor $2^{*15} : (D_5 \times 3)$ and constuction of a monomial representation through the group $2^3 : 3$. We have constructed $PGL(2,7)$ over $2^3 : 3$ on 6 letters and $L_2(11)$ over $2^2 : 3$ on 8 letters. We have also given our construction of $S_5 \times 2$ and $L_2(25)$ as homomorphic images of the monomial progenitors $3^{*3} :_m D_4$ and $S^{*6} : S_5$. In addition, we have described as to how to solve the extension problem for finite groups through the example of the group $(4 \times 2^2) :^{\cdot} A_4$. We note that the symmetric presentations and constructions given in this presentation are original, to the best of our knowledge.

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Introduction

In Chapter 1, we will discuss progenitors. We begin by listing definitions and theorems relevant to progenitors. In this chapter, we construct permutation progenitors of free products. We utilize the computing program MAGMA to very the success of a built progenitor. We will also find the homomorphic images of progenitors factored by given relations.

In Chapter 2, we will discuss a monomial progenitor. The monomial progenitor will be constructed using a process known as the "lifting process." This process grants us the ability to produce a monomial matrix to obtain a new control group on which our monomial progenitor will be constructed from.

In Chapter 3, we describe the process to solve the extension problem for finite groups through examples.

In Chapter 4, we construct the double coset enumeration of a group G over a transitive group N of finite permutations and monomial progenitors.

Chapter 1

Preliminaries

1.1 Definitions and Theorems

Definition 1.1.1. (**Permutation**) *If X is a nonempty set, a **permutation** is the bijective mapping $\alpha : X \to X$. [Rot95]*

Definition 1.1.2. (**Disjoint**) *Two permutations $\alpha, \beta \in S_X$ are **disjoint** if every x moved by one is fixed by the other. In symbols, if $\alpha(a) \neq a$, then $\beta(a) = a$, and if $\alpha(b) = b$, then $\beta(b) \neq b$. [Rot95]*

Theorem 1.1.3. *Every permutation $\alpha \in S_n$, is either a cycle or a product of disjoint cycles. [Rot95]*

Definition 1.1.4. (**Semigroup**) *A **semigroup** $(G, *)$ is a nonempty set G equipped with an associative operation $*$. [Rot95]*

Definition 1.1.5. (**Symmetric Group**) *The **symmetric group**, denoted* S_n *is the set of all permutations of the nonempty set* $X = \{1, 2, ..., n\}$. S_n *is a group of order* $n!$ *on* n *letters. [Rot95]*

Definition 1.1.6. (**Group**). *A **group** is a semigroup G containing an element e such that*
(i) $e * a = a = a * e$ *for all* $a \in G$
(ii) for every $a \in G$*, there is an element* $b \in G$ *with* $a * b = e = b * a$*. [Rot95]*

Definition 1.1.7. (**Order**) *If G is a group, then the **order** of G, dentoted* $|G|$*, is the number of elements in G. [Rot95]*

Definition 1.1.8. (**Free Group**) *If* X *is a subset of a group* F*, then* F *is a **free group** with basis* X *if, for every group G and every function* $f : X \to G$*, there exists a unique homomorphism* $\varphi : F \to G$ *extending* f*. [Rot95]*

Definition 1.1.9. (**Presentation**) *Let* X *be a set and let* Δ *be a family of words on* X*. A group G has generators* X *and relations* Δ *if* $G \cong F/R$*, where F is the free group with basis* X *and* R *is the normal subgroup of* F *generated by* Δ*. The ordered pair* $(X|\Delta)$ *is called a **presentation** of* G*. [Rot95]*

Definition 1.1.10. (**Progenitor**) *A **progenitor** is a semi-direct product of the following form: P*$\cong 2^{*n}:N = \{\pi w \mid \pi \in N$*, and w is a word in the* $t_i\}$*, where* 2^{*n} *denotes a free product of n copies of a cyclic group of order 2 generated by involutions* t_i *for i=1,...,n; and N is* *a transitive permutation group of degree n which acts on the free product by permuting the involutory generators.[Curt96]*

Lemma 1.1.11. (**Factoring Lemma**) *(Know as the Grindstaff Lemma) Factoring the progenitor m^{*n}:N by (t_i, t_j) for $1 \leq i < j \leq n$ gives the group m^n:N.[Grind15]*

1.2 Permutation Progenitor $2^{*15} : (D_5 \times 3)$

We will write the progenitor generated by $x \sim (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ and $y \sim (1,4)(2,8)(3,12)(6,9)(7,13)(11,14)$. The presentation of N is $< x, y \mid y^2, x^{-4}yxy >$. We use MAGMA to find the permutation that stabilizes 1 in N, denoted by N^1. The permutation that stabilizes 1 is $(2,5)(3,9)(4,13)(7,10)(8,14)$
$(12,15)$. We use our Schreier System to find the word that corresponds to the permutation $(2,5)(3,9)(4,13)(7,10)(8,14)(12,15)$ which is y^x. In order to complete our progenitor of G, we add the stabilizer of 1 in N to the presentation. We also add t^2 to the presentation since our $t_i's$ are of order 2. Thus the progenitor of $G =< x, y, t \mid y^2, x^{-4}yxy, t^2, (t, y^x) >$. In order to verify the progenitor, we must re-write our presentation of N in terms of the stabiliser and orbits of the stabiliser of N^1. The orbits of N^1 are $\{1\}$, $\{6\}$, $\{11\}$, $\{2,5\}$, $\{3,9\}$, $\{4,13\}$, $\{7,10\}$, $\{8,14\}$, and $\{12,15\}$. We can verify our progenitor in MAGMA by applying the Grindstaff Lemma on the following code: $G < x, y, t >:=$
$Group < x, y, t | y^2, x^{-4} * y * x * y, t^2, (t, y^x), (t, t^{x^5}), (t, t^{x^{10}}), (t, t^x), (t, t^{x^2}), (t, t^{x^3}), (t, t^{x^6}), (t, t^{x^7}),$
$(t, t^{x^{11}}) >;$
#G;
Our progenitor G is infinite. In order to find a finite presentation, we must factor G by relations.

1.2.1 Writing First Order Relations

We can compute all the possible relations of G by computing the orbits of the centralizer. In order to achieve this, we must find identify the conjugacy classes of N.

Conjugacy Classes of N

Class	Representative of Class	# of elements in the class
C_1	e	1
C_2	$x^2yx = (1,13)(3,6)(4,10)(5,14)(8,11)(9,15)$	5
C_3	$(xy)^2 = (1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15)$	1
C_4	$(yx^{-1})^2 = (1,11,6)(2,12,7)(3,13,8)(4,14,9)(5,15,10)$	1
C_5	$x^3 = (1,4,7,10,13)(2,5,8,11,14)(3,6,9,12,15)$	2
C_6	$x^2yxy = (1,7,13,4,10)(2,8,14,5,11)(3,9,15,6,12)$	2
C_7	$xy = (1,8,6,13,11,3)(2,12,7)(4,5,9,10,14,15)$	5
C_8	$yx^{-1} = (1,3,11,13,6,8)(2,7,12)(4,15,14,10,9,5)$	5
C_9	$x = (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$	2
C_{10}	$x^2 = (1,3,5,7,9,11,13,15,2,4,6,8,10,12,14)$	2
C_{11}	$yx^{-2}y = (1,8,15,7,14,6,13,5,12,4,11,3,10,2,9)$	2
C_{12}	$yx^{-1}y = (1,12,8,4,15,11,7,3,14,10,6,2,13,9,5)$	2

Table 1.1: Conjugacy Classes $2^{*15} : (D_5 \times 3)$

Now that we found the conjugacy classes and have identified the representative for each class, we proceed to find the centraliser and orbits of the centraliser of each class.

Orbits of Centraliser(N, Rep)

Class	Representative	Centraliser(N, Rep)	Orbits of Centraliser(N, Rep)
C_2	x^2yx	$<(1,13)(3,6)(4,10)(5,14)(8,11)(9,15)>$	{2, 12, 7}, {1, 13, 11, 8, 6, 3}, {4, 10, 14, 5, 9, 15}
C_3	$(xy)^2$	$<(1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15)>$	{1, 2, 4, 3, 8, 5, 12, 9, 6, 13, 10, 7, 14, 11, 15}
C_4	$(yx^{-1})^2$	$<(1,11,6)(2,12,7)(3,13,8)(4,14,9)(5,15,10)>$	{1, 2, 4, 3, 8, 5, 12, 9, 6, 13, 10, 7, 14, 11, 15}
C_5	x^3	$<(1,4,7,10,13)(2,5,8,11,14)(3,6,9,12,15)>$	{1, 4, 11, 7, 14, 6, 10, 2, 9, 13, 5, 12, 8, 15, 3}
C_6	x^2yxy	$<(1,7,13,4,10)(2,8,14,5,11)(3,9,15,6,12)>$	{1, 7, 11, 13, 2, 6, 4, 8, 12, 10, 14, 3, 5, 9, 15}
C_7	xy	$<(1,8,6,13,11,3)(2,12,7)(4,5,9,10,14,15)>$	{2, 12, 7}, {1, 13, 8, 11, 6, 3}, {4, 10, 5, 14, 9, 15}
C_8	yx^{-1}	$<(1,3,11,13,6,8)(2,7,12)(4,15,14,10,9,5)>$	{2, 7, 12}, {1, 13, 3, 6, 11, 8}, {4, 10, 15, 9, 14, 5}
C_9	x	$<(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)>$	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}
C_{10}	x^2	$<(1,3,5,7,9,11,13,15,2,4,6,8,10,12,14)>$	{1, 3, 5, 7, 9, 11, 13, 15, 2, 4, 6, 8, 10, 12, 14}
C_{11}	$yx^{-2}y$	$<(1,8,15,7,14,6,13,5,12,4,11,3,10,2,9)>$	{1, 8, 15, 7, 14, 6, 13, 5, 12, 4, 11, 3, 10, 2, 9}
C_{12}	$yx^{-1}y$	$<(1,12,8,4,15,11,7,3,14,10,6,2,13,9,5)>$	{1, 12, 8, 4, 15, 11, 7, 3, 14, 10, 6, 2, 13, 9, 5}

Table 1.2: Orbits of Centraliser $2^{*15} : (D_5 \times 3)$

From the orbits of the centraliser, we obtain the following first order relations:

First Order Relation (N, Rep)

Class	Relations
C_2	$x^2yxt^x, x^2yxt, x^2yxt^{x^3},$
C_3	$(xy)^2t,$
C_4	$(yx^{-1})^2t,$
C_5	$x^3t,$
C_6	$x^2yxyt,$
C_7	$xyt^x, xyt, xyt^{x^3},$
C_8	$yx^{-1}t^x, yx^{-1}t, yx^{-1}t^{x^3},$
C_9	$xt,$
C_{10}	$x^2t,$
C_{11}	$yx^{-2}yt,$
C_{12}	$yx^{-1}yt$

Table 1.3: First Order Relations $2^{*15} : (D_5 \times 3)$

Now we add the first order relations to the progenitor to obtain a homomorphic image of G.

$G =< x, y, t | y^2, x^{-4}yxy, t^2, (t, y^x), (t, t^{x^5}), (t, t^{x^{10}}), (t, t^x), (t, t^{x^2}),$
$(t, t^{x^3}), (t, t^{x^6}), (t, t^{x^7}), (t, t^{x^{11}}), (x^2yxt^x)^a, (x^2yxt)^b, (x^2yxt^{x^3})^c,$
$((xy)^2t)^d, ((yx^{-1})^2t)^e, (x^3t)^f, (x^2yxyt)^g, (xyt^x)^h, (xyt)^i, (xyt^{x^3})^j,$
$(yx^{-1}t^x)^k, (yx^{-1}t)^l, (yx^{-1}t^{x^3})^m, (xt)^n, (x^2t)^o, (yx^{-2}yt)^p, (yx^{-1}yt)^q >$

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 983040

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 5 160

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 6 48

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 10 320

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 12 48

0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 15 491520

1.3 Permutation Progenitor $2^{*15} : (D_3 \times 5)$

We will write the presentation for the progenitor of the Transitive Group 15. Let N be the subgroup generated by $x \sim (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$ and $y \sim (1,11)(2,7)(4,14)(5,10)(8,13)$. The presentation of the subgroup N is $< x, y \mid y^2, x^{-4}yx^{-1}y >$. We will let $t \sim t_1$, this means that t commutes with the stabiliser of 1 in N. We use MAGMA to find the permutation that stabilizes 1 in N. The permutation that stabilizes 1 is $(2,12)(3,8)(5,15)(6,11)(9,14)$. We apply the Schreier System in MAGMA. The Schreier System will produce the word corresponding to its permutation representation. The word corresponding to the permutation $(2,12)(3,8)(5,15)(6,11)(9,14)$ is y^x. We add the y^x to the presentation of N and obtain $< x, y, t | y^2, x^{-4}yx^{-1}y, y^x, t^2, (t, y^x) >$. We proceed to find the orbits of the stabilizer of 1. The orbits of the stabilizer are $\{1\}, \{4\}, \{7\}, \{10\}, \{13\}, \{2,12\}, \{3,8\}, \{5,15\}, \{6,11\}$, and $\{9,14\}$. We make t commute with the orbits of the stabilizer. We add the each to the progenitor to obtain the presentation $G =< x, y, t \mid y^2, x^{-4}yx^{-1}y, t^2, (t, y^x), (t, t^{x^3}), (t, t^{xy}), (t, t^{x^4y}), (t, t^{x^7y}), (t, t^x), (t, t^{x^2}), (t, t^{x^4}),$ $(t, t^y), (t, t^{x^3y}) >$.

The progenitor is infinite. In order to make it progenitor finite, we factor the progenitor by relations.

1.3.1 First Order Relations

We can compute all the possible first order relations by computing all the orbits of the centralizes of the conjugacy classes of N. Let's find the classes of N.

Class	Representative of Class	# of elements in the class
C_1	e	1
C_2	$y^x = (2,12)(3,8)(5,15)(6,11)(9,14)$	3
C_3	$xyx^{-1}y = (1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15)$	2
C_4	$x^3 = (1,4,7,10,13)(2,5,8,11,14)(3,6,9,12,15)$	1
C_5	$x^{-3} = (1,13,10,7,4)(2,14,11,8,5)(3,15,12,9,6)$	1
C_6	$xyx^{-2}y = (2,11,5,14,8)(3,12,6,15,9)$	1
C_7	$x^2yx^{-1}y = (1,7,13,4,10)(2,8,14,5,11)(3,9,15,6,12)$	1
C_8	$xy = (2,3,14,15,11,12,8,9,5,6)$	3
C_9	$x^{-2}y = (1,4,7,10,13)(2,15,8,6,14,12,5,3,11,9)$	3
C_{10}	$yx^2 = (1,13,10,7,4)(2,9,11,3,5,12,14,6,8,15)$	3
C_{11}	$yx^{-1} = (1,10,4,13,7)(2,6,5,9,8,12,11,15,14,3)$	3
C_{12}	$x = (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15)$	2
C_{13}	$x^2 = (1,3,5,7,9,11,13,15,2,4,6,8,10,12,14)$	2
C_{14}	$yx^{-1}y = (1,5,9,13,2,6,10,14,3,7,11,15,4,8,12)$	2
C_{15}	$x^{-2} = (1,14,12,10,8,6,4,2,15,13,11,9,7,5,3)$	2

Table 1.4: Conjugacy Classes $2^{*15} : (D_3 \times 5)$

Now that we have listed the conjugacy classes of N, we will find the centraliser of each class. In addition, we wil find the orbits of each centraliser.

Class	Representative	Centraliser(N, Rep)	Orbits of Centraliser(N, Rep)
C_2	y^x	$< (2,12)(3,8)(5,15)(6,11)(9,14) >$	{1,13,10,7,4}, {2,12,14,9,11,6,8,3,5,15}
C_3	$xyx^{-1}y$	$< (1,6,11)(2,7,12)(3,8,13) (4,9,14)(5,10,15) >$	{1,6,13,11,3,10,8, 15,7,5,12,4,2,9,14}
C_4	x^3	$< (1,4,7,10,13)(2,5,8,11,14) (3,6,9,12,15) >$	{ 1, 2, 11, 3, 7, 12, 4, 8, 13, 5, 14, 9, 6, 10, 15 }
C_5	x^{-3}	$< (1,13,10,7,4)(2,14,11,8,5) (3,15,12,9,6) >$	{ 1, 2, 11, 3, 7, 12, 4, 8, 13, 5, 14, 9, 6, 10, 15}
C_6	$xyx^{-2}y$	$< (2,11,5,14,8)(3,12,6,15,9) >$	{ 1, 2, 11, 3, 7, 12, 4, 8, 13, 5, 14, 9, 6, 10, 15 }
C_7	$x^2yx^{-1}y$	$< (1,7,13,4,10) (2,8,14,5,11)(3,9,15,6,12) >$	{1, 2, 11, 3, 7, 12, 4, 8, 13, 5, 14, 9, 6, 10, 15}
C_8	xy	$< (2,3,14,15,11,12,8,9,5,6) >$	{1, 7, 13, 4, 10 }, { 2, 12, 3, 8, 14, 9, 15, 5, 11, 6}
C_9	$x^{-2}y$	$< (1,4,7,10,13) (2,15,8,6,14,12,5,3,11,9) >$	{1, 4, 7, 10, 13 }, {2, 12, 15, 5, 8, 3, 6, 11, 14, 9 }
C_{10}	yx^2	$< (1,13,10,7,4) (2,9,11,3,5,12,14,6,8,15) >$	{ 1, 13, 10, 7, 4 }, { 2, 12, 9, 14, 11, 6, 3, 8, 5, 15 }
C_{11}	yx^{-1}	$< (1,10,4,13,7) (2,6,5,9,8,12,11,15,14,3) >$	{ 1, 10, 4, 13, 7 }, { 2, 12, 6, 11, 5, 15, 9, 14, 8, 3 }
C_{12}	x	$< (1,2,3,4,5,6,7,8, 9,10,11,12,13,14,15) >$	{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 }
C_{13}	x^2	$< (1,3,5,7,9,11,13, 15,2,4,6,8,10,12,14) >$	{ 1, 3, 5, 7, 9, 11, 13, 15, 2, 4, 6, 8, 10, 12, 14 }
C_{14}	$yx^{-1}y$	$< (1,5,9,13,2,6,10, 14,3,7,11,15,4,8,12) >$	{ 1, 5, 9, 13, 2, 6, 10, 14, 3, 7, 11, 15, 4, 8, 12 }
C_{15}	x^{-2}	$< (1,14,12,10,8,6,4,2, 15,13,11,9,7,5,3) >$	{1, 14, 12, 10, 8, 6, 4, 2, 15, 13, 11, 9, 7, 5, 3 }

Table 1.5: Orbits of Centraliser $2^{*15} : (D_3 \times 5)$

From the orbits of the centraliser, we obtain the following first order relations:

First Order Relations (N, Rep)

Class	Relations
C_2	$(y^x t)^a$
C_3	$(xyx^{-1}yt^{x^3})^b$
C_4	$(x^3 t^{xy})^c$
C_5	$(x^{-3} t^{yx^{-1}})^d$
C_6	$(xyx^{-2}yt^{yx^2})^e$
C_7	$(x^2 yx^{-1}yt^x)^f$
C_8	$(xyt^{yx})^g$
C_9	$(x^{-2}yt^{x^2})^h$
C_{10}	$(yx^2 t^{xyx})^i$
C_{11}	$(yx^{-1} t^{xyx^{-2}})^j$
C_{12}	$(xt^{x^{-1}})^k$
C_{13}	$(x^2 t^{xyx^{-1}})^l$
C_{14}	$(yx^{-1}yt^y)^m$
C_{15}	$(x^{-2} t^{yx^{-2}})^n$

Table 1.6: First Order Relations $2^{*15} : (D_3 \times 5)$

We add the first order relations to the progenitor to obtain a homomorphic image of G,

$G = < x, y, t \mid y^2, x^{-4}yx^{-1}y, y^x, t^2, (t, y^x), (y^x t)^a, (xyx^{-1}yt^{x^3})^b, (x^3 t^{xy})^c, (x^{-3} t^{yx^{-1}})^d, (xyx^{-2}yt^{yx^2})^e,$
$(x^2 yx^{-1}yt^x)^f, (xyt^{yx})^g, (x^{-2}yt^{x^2})^h, (yx^2 t^{xyx})^i, (yx^{-1} t^{xyx^{-2}})^j, (xt^{x^{-1}})^k, (x^2 t^{xyx^{-1}})^l, (yx^{-1}yt^y)^m,$

$(x^{-2}t^{yx^{-2}})^n >$

1.4 Permutation Progenitor $2^{*24} : (4 \times 2 : S_3)$

In this section we will write the presentation for the progenitor $2^{*24} : (4 \times 2) : S_3$ N is generated by $x \sim (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16)(17,23)(18,21)(20,24)$, $y \sim (1,15,17)(2,13,18)(3,11,19)(4,16,20)(5,10,21)(6,14,22)(7,9,23)(8,12,24)$, and $z \sim (1,2,4,5)(3,8,6,7)(9,16,12,15)(10,11,13,14)(17,22,20,19)(18,24,21,23)$. The presentation of $N =< x, y, z | x^2, y^3, z^4, (y^{-1}x)^2, z^{-2}y^{-1}z^2y, (yz^{-1}x)^2, z^{-1}y^{-1}z^{-1}y^{-1}zy^{-1} >$

The notation 2^{*24}, tells us we have 24 $t's$ of order 2. We will let $t \sim t_1$, this means that t will commute with the stabilizer of 1 in N. We use MAGMA to find the permutations that stabilize 1 in N. The permututation that stabilizes 1 is $(2,6)(3,5)(7,8)(9,21)$ $(10,17)(11,23)(12,18)(13,20)(14,24)(15,22)(16,19)$.
We use our Schreier System to determine the permutation is $xz^{-1}y^{-1}$. We add $xz^{-1}y^{-1}$ to our presentation of N to get the presentation of 2^{*24}. Thus our progenitor is $G =< x, y, z | x^2, y^3, z^4, (y^{-1}x)^2, z^{-2}y^{-1}z^2y, (yz^{-1}x)^2, z^{-1}y^{-1}z^{-1}y^{-1}zy^{-1}, t^2, (t, xz^{-1}y^{-1}) >$. Our progenitor G is infinite. In order to make it finite we factor by relations.

1.4.1 Writing the First Order Relations

First order relations are written in the form $(\pi_i^a)^b = 1$, where $a \in N$ and w is a work in the $t_i's$. We can compute all the possible relations by computing the orbits of the centralizers of the Conjugacy Classes of N. Let's find the classes of N.

Conjugacy Classes of N

Class	Representative of Class	# of elements in the class
C_1	e	1
C_2	$z^2 = (1,4)(2,5)(3,6)(7,8)(9,12)(10,13)(11,14)(15,16)(17,20)(18,21)(19,22)(23,24)$	1
C_3	$zxz = (1,11)(2,15)(3,12)(4,14)(5,16)(6,9)(7,13)(8,10)(17,20)(18,19)(21,22)$	12
C_4	$y = (1,15,17)(2,13,18)(3,11,19)(4,16,20)(5,10,21)(6,14,22)(7,9,23)(8,12,24)$	8
C_5	$z = (1,2,4,5)(3,8,6,7)(9,16,12,15)(10,11,13,14)(17,22,20,19)(18,24,21,23)$	6
C_6	$yz = (1,9,18,4,12,21)(2,14,20,5,11,17)(3,13,24,6,10,23)(7,16,19,8,15,22)$	8
C_7	$zx = (1,10,3,16,4,13,6,15)(2,12,7,11,5,9,8,14)(17,22,24,18,20,19,23,21)$	6
C_8	$z^{-1}x = (1,13,3,15,4,10,6,16)(2,9,7,14,5,12,8,11)(17,19,24,21,20,22,23,18)$	6

Table 1.7: Conjugacy Classes of $2^{*24} : (4 \times 2 : S_3)$

Next we must find the centraliser of each class representative. Once we have found the centraliser of each class, we must then find the orbit of each centraliser.

Orbits of Centraliser(N,Rep)

Class	Representative	Centraliser(N,Rep)	Orbits of Centraliser(N,Rep)
C_1	e	1	
C_2	z^2	<(1, 4)(2, 5)(3, 6)(7, 8)(9, 12)(10, 13) (11, 14)(15, 16)(17, 20)(18,21)(19, 22)(23, 24)>	{ 1, 9, 15, 2, 23, 16, 7, 17, 10, 13, 4, 18, 8, 20, 12, 3, 22, 21, 11, 5, 14, 24, 6, 19}
C_3	zxz	<(1, 11)(2, 15)(3, 12)(4, 14)(5, 16) (6, 9)(7, 13)(8, 10)(17, 20)(18,19)(21, 22)>	{ 17, 20 },{ 23, 24 }, { 1, 11, 4, 14 }, { 2, 15, 5, 16 }, { 3, 12, 6, 9 }, { 7, 13, 8, 10 }, { 18, 19, 21, 22 }
C_4	y	<(1, 15, 17)(2, 13, 18)(3, 11, 19) (4, 16, 20)(5, 10, 21)(6, 14,22)(7, 9, 23)(8, 12, 24)>	{1, 15, 4, 17, 16, 20 }, { 2, 13, 5, 18, 10, 21 }, { 3, 11, 6, 19, 14, 22 }, { 7, 9, 8, 23, 12, 24 }
C_5	z	<(1, 2, 4, 5)(3, 8, 6, 7) (9, 16, 12, 15)(10, 11, 13, 14) (17, 22, 20,19)(18, 24, 21, 23)>	{ 1, 2, 20, 4, 19, 5, 17, 22 }, { 3, 8, 18, 6, 24, 7, 21, 23 }, { 9, 16, 13, 12, 14, 15, 10, 11 }
C_6	yz	<(1, 9, 18, 4, 12, 21)(2, 14, 20, 5, 11, 17) (3, 13, 24, 6, 10, 23)(7,16, 19, 8, 15, 22)>	{ 1, 9, 18, 4, 12, 21 }, { 2, 14, 20, 5, 11, 17 }, { 3, 13, 24, 6, 10, 23 }, { 7, 16, 19, 8, 15, 22 }
C_7	zx	<(1, 10, 3, 16, 4, 13, 6, 15)(2, 12, 7, 11, 5, 9, 8, 14) (17, 22, 24,18, 20, 19, 23, 21)>	{ 1, 10, 3, 16, 4, 13, 6, 15 }, { 2, 12, 7, 11, 5, 9, 8, 14 }, { 17, 22, 24, 18, 20, 19, 23, 21 }
C_8	$z^{-1}x$	<(1, 13, 3, 15, 4, 10, 6, 16)(2, 9, 7, 14, 5, 12, 8, 11) (17, 19, 24,21, 20, 22, 23, 18)>	{ 1, 13, 3, 15, 4, 10, 6, 16 }, { 2, 9, 7, 14, 5, 12, 8, 11 }, { 17, 19, 24, 21, 20, 22, 23, 18 }

Table 1.8: Orbits of Centraliser $2^{*24} : (4 \times 2 : S_3)$

From the orbits of the centraliser, we obtained the first order relations as shown in the table above.

Relations (N,Rep)

Class	Relations
C_2	z^2t
C_3	$zxzt^{y^{-1}}$, $zxzt^{yzy}$, $zxzt$, $zxzt^{z}$, $zxzt^{xy^{-1}z}$, $zxzt^{xz^{-1}x}$, $zxzt^{xyz}$,
C_4	$yt, yt^{z}, yt^{xy^{-1}z}, yt^{xy^{-1}}$
C_5	$zt, zt^{xy^{-1}z}, zt^{yz},$
C_6	$yzt^{z}, yzt^{xy^{-1}z}, yzt^{xy^{-1}},$
C_7	$zxt, zxt^{z}, zxt^{y^{-1}},$
C_8	$z^{-1}xt, z^{-1}xt^{z}, z^{-1}xt^{y^{-1}}$

Table 1.9: First Order Relations $2^{*24} : (4 \times 2 : S_3)$

Now we add our first order relations to our progenitor to obtain a homomorphic image of G,

$G =< x, y, z, t | x^2, y^3, z^4, (y^{-1}x)^2, z^{-2}y^{-1}z^2y, (yz^{-1}x)^2, z^{-1}y^{-1}z^{-1}y^{-1}zy^{-1}, (z^2t)^a, (zxzt^{y^{-1}})^b,$
$(zxzt^{yzy})^c, (zxzt)^d, (zxzt^{z})^e, (zxzt^{xy^{-1}z})^f, (zxzt^{xz^{-1}x})^g, (zxzt^{xyz})^h, (yt)^i, (yt^{z})^j, (yt^{xy^{-1}z})^k,$
$(yt^{xy^{-1}})^l, (zt)^m, (zt^{xy^{-1}z})^n, (zt^{yz})^o, (yzt)^p, (yzt^{z})^q, (yzt^{xy^{-1}z})^r, (yzt^{xy^{-1}})^s, (zxt)^t, (zxt^{z})^u,$
$(zxt^{y^{-1}})^v, (z^{-1}xt)^{a1}, (z^{-1}xt^{z})^{a2}, (z^{-1}xt^{y^{-1}})^{a3} >$

```
    if #G gt 48 then a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,a1,a2,a3, #G;
end if; end for;
```

Chapter 2

Monomial Progenitors

2.1 Preliminary

Definition 2.1.1. (**Formula for Induced Character**)

$$\varphi_{\alpha}^{G}(x) = \frac{n}{h_{\alpha}} \sum_{\omega \in C_{\alpha} \cap H} = \varphi(\omega), \alpha = 1, 2, 3, ..., m$$

. **Definition 1.1.12.** (**Character**) *Let $A(x) = (a_{ij}(x))$ be a matrix representation of G of degree m. We consider the characteristic polynomial of $A(x)$, namely*

$$det(\lambda I - A(x)) = \begin{bmatrix} \lambda - a_{11}(x) & \lambda - a_{12}(x) & ... & \lambda - a_{1m}(x) \\ \lambda - a_{21}(x) & \lambda - a_{22}(x) & ... & \lambda - a_{1m}(x) \\ ... & ... & & ... \\ \lambda - a_{m1}(x) & \lambda - a_{m2}(x) & ... & \lambda - a_{mm}(x) \end{bmatrix}$$

This is a polynomial of degree m in λ, with the coefficient of $-\lambda^{m-1}$ is

$\varphi(x) = a_{11}(x) + a_{22}(x) + .. + a_{mm}(x)$

It is customary to call the right-hand side of this equation the trace of $A(x)$, abbreviated to $tr A(x)$, so that $\varphi(x) = tr A(x)$

*We regard $\varphi(x)$ as a function on G with values in field K, and we call it the **character** of $A(x)$.[Led77]*

Theorem 2.1.2. *The number of irreducible characters of G is equal to the number of conjugacy classes of G.[Led77]*

Definition 2.1.3. (**Degree of a Character**) *The sum of squares of the degrees of the distinct irreducible characters of G is equal to $|G|$. The **degree of a character** χ is $\chi(1)$. Note that a character whose degree is 1 is called a **linear character**.[Led77]*

Definition 2.1.4. (**Lifting Process**) *Let N be a normal subgroup of G and suppose that $A_0(Nx)$ is a representation of degree m of the group G/N. Then $A(x) = A_0(Nx)$ defines a representation of G/N lifted from G/N. If $\varphi_0(Nx)$ is a character of $A_0(Nx)$, then $\varphi(x) = \varphi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = Im, \varphi(u) = m = \varphi(1)$. The lifting process preserves irreducibility.[Led77]*

Definition 2.1.5. (**Induced Character**) *The character of A(x), which is called the **induced character** of ϕ, will be dentoted by ϕ^G. Thus, $\phi^G = trA(x) = \sum_{i=1}^{n} \phi(t_i x t_i{}^{-1})$.[Led77]*

Definition 2.1.6. (**Formula for Induced Character**)

$$\varphi_\alpha^G(x) = \frac{n}{h_\alpha} \sum_{\omega \in C_\alpha \cap H} = \varphi(\omega), \alpha = 1, 2, 3, ..., m$$

.

2.2 A Monomial Progenitor for $2^3 : 3$

$G = 2^3 : 3$ is generated by $xx = (3,6)$ and $yy = (1,3,5)(2,4,6)$

The conjugacy classes of the group G are $C_1 = ID(G)$

$C_2 = (1,4)(2,5)(3,6)$

$C_3 = (1,4),(2,5),(3,6)$

$C_4 = (1,4)(3,6),(2,5)(3,6),(1,4)(2,5)$

$C_5 = (1,3,5)(2,6,4),(1,6,5)(2,4,3),(1,3,2)(4,6,5),(1,6,2)(3,5,4)$

$C_6 = (1,5,3)(2,6,4),(1,5,6)(2,3,4),(1,2,6)(3,4,5)(1,2,3)(4,5,6)$

$C_7 = (1,6,2,4,3,5),(1,3,5,4,6,2),(1,3,2,4,6,5),(1,6,5,4,3,2)$

$C_8 = (1,5,3,4,2,6),(1,5,6,4,2,3)(1,2,6,4,5,3),(1,2,3,4,5,6)$

Let us consider the subgroup H of G,

$H = Id(G),(1,4)(2,5),(2,5)(3,6),(1,4)(3,6),(2,5),(1,4),(3,6),(1,4)(2,5)(3,6)$

The conjugacy classes of H are

$D_1 = Id(G)$

$D_2 = (1,4)(2,5)$

$D_3 = (2,5)(3,6)$

$D_4 = (1,4)(3,6)$

$D_5 = (2,5)$

$D_6 = (1,4)$

$D_7 = (3,6)$

$D_8 = (1,4)(2,5)(3,6)$

Class	$D1$	$D2$	$D3$	$D4$	$D5$	$D6$	$D7$	$D8$
Size	1	1	1	1	1	1	1	1
Representative	$Id(G)$	$(1,4)(2,5)$	$(2,5)(3,6)$	$(1,4)(3,6)$	$(2,5)$	$(1,4)$	$(3,6)$	$(1,4)(2,5)(3,6)$
ϕ	1	-1	-1	1	-1	1	1	-1

In the table below, we have the characters ϕ of G corresponding to the subgroup H.

Class	$C1$	$C2$	$C3$	$C4$	$C5$	$C6$
Size	1	1	3	3	4	4
Representative	$Id(G)$	$(1,4)(2,5)(3,6)$	$(1,4)$	$(1,4)(3,6)$	$(1,3,5)(2,6,4)$	$(1,5,3)(2,6,4)$
ϕ^G	3	-3	1	-1	0	0

Class	$C7$	$C8$
Size	4	4
Representative	$(1,6,2,4,3,5)$	$(1,5,3,4,2,6)$
ϕ^G	0	0

a) We want to induce the ϕ of H up to G to obtain the character ϕ^G.

$\phi_\alpha^G = \frac{n}{h_\alpha}\sum_{w\in H\cap C_\alpha}\phi(w)$, where $n = \frac{|G|}{|H|} = \frac{24}{8} = 3$.

$\phi_1^G = \frac{n}{h_1}\sum_{w\in H\cap C_1}\phi(w)$

So, $\phi_1^G = \frac{3}{1}(\phi(1)) = 3(1) = 3$

$\phi_2^G = \frac{n}{h_2} \sum_{w \in H \cap C_2} \phi(w)$

So, $\phi_2^G = \frac{3}{1}(\phi(1,4)(2,5)(3,6)) = 3(-1) = -3$

$\phi_3^G = \frac{n}{h_3} \sum_{w \in H \cap C_3} \phi(w)$

So, $\phi_3^G = \frac{3}{3}(\phi(1,4) + \phi(2,5) + \phi(3,6)) = 1(1 + -1 + 1) = 1$

$\phi_4^G = \frac{n}{h_4} \sum_{w \in H \cap C_4} \phi(w)$

So, $\phi_4^G = \frac{3}{3}(\phi(1,4)(3,6) + \phi(2,5)(3,6) + \phi(1,4)(2,5)) = 1(1 + -1 + -1) = -1$

$\phi_5^G = \frac{n}{h_5} \sum_{w \in H \cap C_5} \phi(w)$

So, $\phi_5^G = \frac{3}{4}(\phi(0)) = \frac{3}{4}(0) = 0$

$\phi_6^G = \frac{n}{h_6} \sum_{w \in H \cap C_6} \phi(w)$

So, $\phi_6^G = \frac{3}{4}(\phi(0)) = \frac{3}{4}(0) = 0$

$\phi_7^G = \frac{n}{h_7} \sum_{w \in H \cap C_7} \phi(w)$

So, $\phi_7^G = \frac{3}{4}(\phi(0)) = \frac{3}{4}(0) = 0$

$\phi_8^G = \frac{n}{h_8} \sum_{w \in H \cap C_8} \phi(w)$

So, $\phi_8^G = \frac{3}{4}(\phi(0)) = \frac{3}{4}(0) = 0$

b) Show the monomial representation has the generators

$$A(xx) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } A(yy) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$G = HeUH(1,3,5)(2,4,6),\ H(1,5,3)(2,6,4).$

Let $t_1 = e,\ t_2 = (1,3,5)(2,4,6),\ t_3 = (1,5,3)(2,6,4)$

$$A(xx) = \begin{bmatrix} \phi(t_1 x t_1^{-1}) & \phi(t_1 x t_2^{-1}) & \phi(t_1 x t_3^{-1}) \\ \phi(t_2 x t_1^{-1}) & \phi(t_2 x t_2^{-1}) & \phi(t_2 x t_3^{-1}) \\ \phi(t_3 x t_1^{-1}) & \phi(t_3 x t_2^{-1}) & \phi(t_3 x t_3^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} \phi(exe) & \phi(ex(1,5,3)(2,6,4)) & \phi(ex(1,3,5)(2,4,6)) \\ \phi((1,3,5)(2,4,6)xe) & \phi((1,3,5)(2,4,6)x(1,5,3)(2,6,4)) & \phi((1,3,5)(2,4,6)x(1,3,5)(2,4,6)) \\ \phi((1,5,3)(2,6,4)xe) & \phi((1,5,3)(2,6,4)x(1,5,3)(2,6,4)) & \phi((1,5,3)(2,6,4)x(1,3,5)(1,3,5)(2,4,6)) \end{bmatrix}$$

$$= \begin{bmatrix} \phi((3,6) & \phi((1,5,3,4,2,6)) & \phi((1,3,2,4,6,5)) \\ \phi((1,6,2,4,3,5)) & \phi((1,4,)) & \phi((1,2,6,4,5,3)) \\ \phi((1,5,6,4,2,3)) & \phi((1,3,5,4,6,2)) & \phi((2,5)) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A(yy) = \begin{bmatrix} \phi(t_1 y t_1^{-1}) & \phi(t_1 y t_2^{-1}) & \phi(t_1 y t_3^{-1}) \\ \phi(t_2 y t_1^{-1}) & \phi(t_2 y t_2^{-1}) & \phi(t_2 y t_3^{-1}) \\ \phi(t_3 y t_1^{-1}) & \phi(t_3 y t_2^{-1}) & \phi(t_3 y t_3^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} \phi(eye) & \phi(ey(1,5,3)(2,6,4)) & \phi(ey(1,3,5)(2,4,6)) \\ \phi((1,3,5)(2,4,6)ye) & \phi((1,3,5)(2,4,6)y(1,5,3)(2,6,4)) & \phi((1,3,5)(2,4,6)y(1,3,5)(2,4,6)) \\ \phi((1,5,3)(2,6,4)ye) & \phi((1,5,3)(2,6,4)y(1,5,3)(2,6,4)) & \phi((1,5,3)(2,6,4)y(1,3,5)(1,3,5)(2,4,6)) \end{bmatrix}$$

$$= \begin{bmatrix} \phi((1,3,5)(2,4,6)) & \phi((e)) & \phi((1,5,3)(2,6,4)) \\ \phi((1,5,3)(2,6,4)) & \phi((1,3,5)(2,4,6)) & \phi((e)) \\ \phi((e)) & \phi((1,5,3)(1,5,3)(2,6,4)) & \phi((1,3,5)(2,4,6)) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) Give a permutation representation of $A(xx)$ and $A(yy)$ of the monomial representation of part (b).

$$A(xx) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ where } a_{11} = 1,\ a_{22} = 1, \text{ and } a_{33} = 2.$$

This give us,

$t_1 \to t_1$,

$t_2 \to t_2$,

and $t_3 \to t_3^2$

1	2	3	4	5	6
t_1	t_2	t_3	t_1^2	t_2^2	t_3^2
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_1	t_2	t_3^2	t_1^2	t_2^2	t_3
1	2	6	4	5	3

Therefore, $A(xx) = (3,6)$.

$A(yy) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ where $a_{12} = 1$, $a_{23} = 1$, and $a_{31} = 1$.

This gives us,

$t_1 \to t_2$,

$t_2 \to t_3$

$t_3 \to t_1$

1	2	3	4	5	6
t_1	t_2	t_3	t_1^2	t_2^2	t_3^2
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_2	t_3	t_1^2	t_2^2	t_3^2	t_1
2	3	1	5	6	4

Therefore, $A(yy) = (1,2,3)(4,5,6)$.

(d) Give the presentation of the monomial progenitor $3^{*3} :_m (2^3 : 3)$.

A presentation for S_4 is $< x, y \mid x^2, y^3, (xy)^6 >$. We need to find the Normaliser$\{t, t^2\}$.

The Stabiliser$(N, \{1,4\}) = (3,6), (2,5), (1,4)$. This means that t commutes with all 3.

Thus, our presentation is $< x, y, t \mid x^2, y^3, (xy)^6, (x, y)^2, (t, x), (t, yxy^{-1}), t^{(x^y)} = t^2 >$.

Chapter 3

Ismorphism Types

3.1 Preliminaries

Definition 3.1.1. (**Abelian**) *A pair of elements a and b in a group **commutes** if a*b=b*a. A group is **abelian** if every pair of its elements commutes. [Rot95]*

Definition 3.1.2. (**Homomorphism**) *Let (G,*) and (H,∘) be groups. A function f:G↔H is a **homomorphism** if, for all a, b ∈ G, f(a*b) = f(a) ∘ f(b). [Rot95]*

Definition 3.1.3. (**Isomorphism**) *An **isomorphism** is a homomorphism that is also a bijection. We say that G is **isomorphic** to H, denoted G≅H, if there exists an isomorphism f:G ↔ H. [Rot95]*

Theorem 3.1.4. *Let p be a prime. A group G of order p^n is cyclic if and only if it is an abelian group having a unique subgroup of order p. [Rot95]*

Definition 3.1.5. (**normal subgroup**) *A subgroup* $K \leq G$ *is a* ***normal subgroup****, denoted by* $k \trianglelefteq G$*, if* $gKg^{-1} = K$ *for every* $g \in G$*. [Rot95]*

Theorem 3.1.6. (**First Isomorphism Theorem**) *Let* $f\colon G \longrightarrow H$ *be a homomorphism with kernal K. Then K is a normal subgroup of G and* $G/K \cong im(f)$*. [Rot95]*

Theorem 3.1.7. (**Second Isomorphism Theorem**) *Let N and T be subgroups of G with N normal. Then* $N \cap T$ *is normal in T and* $T/(N \cap T) \cong NT/N$*. [Rot95]*

Theorem 3.1.8. (**Third Isomorphism Theorem**) *Let* $K \leq H \leq G$*, where both K and H are normal subgroups of G. Then H/K is a normal subgroup of G/K and* $(G/K)(H/K) \cong G/H$*. [Rot95]*

Theorem 3.1.9. (**Correspondence Theorem**) *Let* $K \trianglelefteq G$ *and let* $v\colon G \longrightarrow G/K$ *be the natural map. Then* $S \mapsto v(S) = S/K$ *is a bi-jection from the family of all those subgroups S of G which contain K to the family of all the subgroups of G/K. Moreover, if we denote S/K by* S^**, then: (i)* $T \leq S$ *if and only if* $T^* \leq S^*$*, and then* $[S:T] = [S^*:T^*]$*; and (ii)* $T \trianglelefteq S$ *if and only if* $T^* \trianglelefteq S^*$*, and then* $S/T \cong S^*/T^*$*. [Rot95]*

Definition 3.1.10. (**maximal normal subgroup**) *A subgroup* $H \leq G$ *is a* ***maximal normal subgroup*** *of G if there is no normal subgroup N of G with* $H < N < G$*. [Rot95]*

Definition 3.1.11. (**simple**) *A group* $G \neq 1$ *is* ***simple*** *if it has no normal subgroups other* *than G and itself. [Rot95]*

Definition 3.1.12. (**direct product**) *If H and K are groups, then their* ***direct product****, denoted by H×K, is the group with elements all ordered pairs (h,k), where h∈H and k∈K, and with the operation (h,k)(h',k')=(hh',kk'). [Rot95]*

Theorem 3.1.13. (**Jordan-Holder Theorem**) *Every two composition series of a group G are equivalent.*
Suppose that the finite group G has two composition series
$G = B_0 > B_1 > \ldots > B_n = \{1\}$ *and* $G = C_0 > C_1 > \ldots > C_m = \{1\}$. *Then* $n = m$ *and the lists of composition factors for the two series are identical in the sense that if* $|H| \leq |G|$ *and* $\Phi(H) = \{i \geq 1 : B_{i-1}/B_i \cong H\}$ *and* $\Psi(H) = \{i \geq 1 : C_{i-1}/C_i \cong H\}$ *then* $\Phi(H) = \Psi(H)$. *[Rot95]*

Definition 3.1.14. (**semi-direct product**) *A group G is a* ***semi-direct product*** *of K by Q, denoted by G=K×Q, if K◁G and K has a complement* $Q_1 \cong Q$*. One also says that G splits over K. [Rot95]*

Definition 3.1.15. (**Mixed-Extension**) *If G is an extension of an abelian group not equal to the center of G, then this is called a* ***mixed extension****. [Rot95]*

Definition 3.1.16.(**normal subgroup in composition series**) *A normal subgroup N of a group G is called a maximal normal subgroup of G if*
(a) N ≠ G

(b) whenever $N \leq M \lhd G$ *then either* $M = N$ *or* $M + G$.

By the Correspondence Theorem, if $N \lhd G$ *and* $N \neq G$ *then every normal subgroup of G/N corresponds to a normal subgroup of G containing N. So a normal subgroup N is maximal if and only if* G/N *is simple.*

Definition 3.1.17. (**Composition series**)*Given a group G, a **composition series** for G of length n is a sequence of subgroups* $G{=}B_0 > B_1 > \cdots > B_n = 1_G$ *such that*

(i) $B_i \lhd B_{i-1}$ *for i= 1,... ,n.*

(ii) B_{i-1}/B_i *is simple for i=1,...,n. In particular,* B_i *is a maximal normal subgroup of G and* B_{i-1} *is simple. The (isomorphism classes of the) quotient groups* B_i/B_{i-1} *are called composition factors of G.*

Example 3.1.18.

S_4 *has the following composition series of length 4, where K is the Klein group*

$\{(1),(12)(34),(13)(24),(14)(23)\}$.

$S_4 > A_4 > K > \{(12)(34)\} > \{1\}$

We know that $A_4 \lhd S_4$*; the composition factor* $S_4/A_4 \cong C_2$.

We have seen that $K \lhd A_4$*; and* $A_4/K \cong C_3$

All subgroups of K are normal in K, because K is abelian.

Both $K/\{(12)(34)\}$ *and* $\{(12)(34)\}/\{1\}$ *are isomorphic to* C_2.

So the composition factors of S_4 *are* C_2 *(three times) and* C_3 *(once).*

3.2 Isomorphism Type $4^2 : 4$

Our goal is to find the Isomorphism type of the transitive group N on 8 letters. N is a group of order 64 and is generated by $x \sim (2,6)(3,7)$, $y \sim (1,3)(4,8)(5,7)$, and $z \sim (1,2,3,8)(4,5,6,7)$. The Normal Lattice of N is given below.

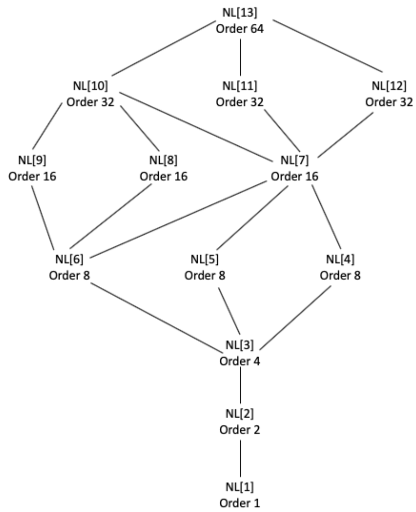

Figure 3.1: Normal Lattice of $4^2 : 4$

The largest normal abelian subgroup of N is $NL[8]$, which is of order 16. From this we can conclude that $NL[8]$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_4$ or $\mathbb{Z}_{16}$. We see that $NL[8]$ has two generators of order 4, $A \sim (2,8,6,4)$ and $B \sim (1,3,5,7)(2,8,6,4)$. Thus, $NL[8] \cong 4 \times 4 = 4^2$.

A presentation for $NL[8]$ is given by $< A, B \mid A^4, B^4, (AB) >$.

$NL[8]$ is an abelian subgroup of order 16 and N is of order 64. If N has a normal subgroup of order 4, then N is a direct-product. However, N does not have a normal subgroup of order 4. Thus we can conclude that N is a semi-direct product of $NL[8]$.

By factoring N by $NL[8]$ we obtain that N is an extension of $NL[8]$ by $N/NL[8]$. $N/NL[8]$ is generated by $C \sim (1,2,3,8)(4,5,6,7)$. So $N/NL[8] =$ $< C >$ which is of order 4.

Now we conjugate the generators of $NL[8] =< A, B >$ by C and compute A^C and B^C.

$$\begin{aligned} A^C &= (2,8,6,4)^{(1,2,3,8)(4,5,6,7)} \\ &= (1,7,5,3) \\ &= AB^3 \end{aligned} \tag{3.1}$$

$$\begin{aligned} B^C &= (1,3,5,7)(2,8,6,4)^{(1,2,3,8)(4,5,6,7)} \\ &= (1,7,5,3)(2,8,6,4) \\ &= A^2B^3 \end{aligned} \tag{3.2}$$

The presentation of N is $< a, b, c \mid a^4, b^4, (ab), c^4, a^c = ab^3, b^c = a^2b^3 >$. Therefore, N is the semi-direct product of $4^2 : 4$.

3.3 Ismorphism Type $(4 \times 2^2) : S_3$

Our goal is to find the Isomorphism type of the transitive group N on 14 letters. N is a group of order 96 and is generated by $w \sim (1,7)(3,9)(4,10)(6,12)$ $x \sim (14,7,10)(2,5,8,11)(3,6,9,12)$ $y \sim (1,5,9)(2,6,10)(3,7,11)(4,8,12)$ and $z \sim (1,5)(2,10)(4,8)(7,11)$. The Normal Lattice of N is given by The largest normal abelian subgroup of N is $NL[7]$ which is of order 16. From this we can conclude that $NL[7]$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_4 \times \mathbb{Z}_4$ or $\mathbb{Z}_{16}$. We see that $NL[7]$ has one generators of order 4 and three generators of order 2 $A \sim (1,4,7,10)(2,5,8,11)(3,6,9,12)$, $B \sim (3,9)(6,12)$, $C \sim (1,7)(3,9)(4,10)(6,12)$ and $D \sim (1,7)(2,8)(4,10)(5,11)$. However, I need to determine

Figure 3.2: Normal Lattice of $(4 \times 2^2) : S_3$

if all four generator are needed to generator $NL[7]$ which is of order 16. We conclude that $NL[7]$ is generated by ABC. Thus $NL[7] \cong 4 \times 2^2$.

A presentation for $NL[7]$ is given by $< A, B, C \mid A^4, B^2, C^2(A, B)(A, C)(B, C)) >$. $NL[7]$ is an abelian subgroup of order 16 and N is of order 96. If N has a normal subgroup of order 6, then N is a direct-product. However, N does not have a normal subgroup of order 6. Thus we can conclude that N is a semi-direct product of $NL[7]$ by S_3.

By factoring N by $NL[7]$ we obtain that N is an extension of $NL[7]$ by $N/NL[7]$. $N/NL[7]$ is generated by $D \sim (1,5,9)(42,6,10)(3,7,11)(4,8,12)$ which is of order 3 and $E \sim (1,5)(2,10)(4,8)(7,11)$ which is of order 2. So $N/NL[7] =< D, E >$ which is of order 6.

Now we conjugate the generators of $NL[7] =< A, B, C >$ by D and E and compute A^D,B^D, C^D, A^E, B^E, and C^E.

$$\begin{aligned} A^D &= (1,4,7,10)(2,5,8,11)(3,6,9,12)^{(1,5,9)(2,6,10)(3,7,11)(4,8,12)} \\ &= (1,4,7,10)(2,5,8,11)(3,6,9,12) \\ &= A \end{aligned} \tag{3.3}$$

$$\begin{aligned} B^D &= (3,9)(6,12)^{(1,5,9)(2,6,10)(3,7,11)(4,8,12)} \\ &= (7,1)(4,10) \\ &= B * C \end{aligned} \tag{3.4}$$

$$\begin{aligned} C^D &= (1,7)(3,9)(4,10)(6,12)^{(1,5,9)(2,6,10)(3,7,11)(4,8,12)} \\ &= (1,7)(2,8)(4,10)(5,11) \\ &= A^2 * B \end{aligned} \tag{3.5}$$

$$\begin{aligned} A^E &= (1,4,7,10)(2,5,8,11)(3,6,9,12)^{(1,5)(2,10)(4,8)(7,11)} \\ &= (1,4,7,10)(2,5,8,11)(3,6,9,12) \\ &= A \end{aligned} \tag{3.6}$$

$$\begin{aligned} B^E &= (3,9)(6,12)^{(1,5)(2,10)(4,8)(7,11)} \\ &= (3,9)(6,12) \\ &= B \end{aligned} \tag{3.7}$$

$$\begin{aligned} C^E &= (1,7)(3,9)(4,10)(6,12)^{(1,5)(2,10)(4,8)(7,11)} \\ &= (2,8)(3,9)(5,11)(6,12) \\ &= A^2 * B * C \end{aligned} \tag{3.8}$$

The presentation of N is $< a, b, c, d, e \mid a^4, b^2, c^2, (a, b), (a, c), (b, c), d^3, e^2, (d * e)^2, a^d = a, b^d = b * c, c^d = a^2 * b, a^e = a, b^e = b, c^e = a^2 * b * c >$. Therefore, N is the semi-direct product of $4 \times 2^2 : S_3$.

3.4 Isomorphism Type $(4 \times 2^2) : \bullet A_4$

Our goal is to find the Isomorphism type of the transitive group N on 24 letters. N is a group of order 192 and is generated by $x \sim (1,3)(2,4)(5,23)(6,24)(11,12)(13,14)(15,16)(17,18)(19,22)(20,21)$ and $y \sim (1,7,22,24,10,19)(2,8,21,23,9,20)(3,11,15,6,14,18)(4,12,16,5,13,17)$.

The Normal Lattice of N is

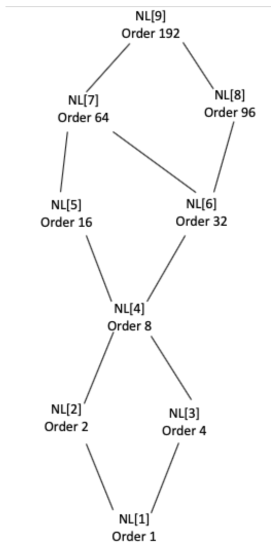

Figure 3.3: Normal Lattice of 4×2^2

The largest normal abelian subgroup of N is NL[5], which is of order 16. This implies that NL[5] can be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, or $\mathbb{Z}_4 \times \mathbb{Z}_4$, or $\mathbb{Z}_{16}$. $NL[5]$ is generated by three elements,
$< A, B, C >=< (1,5,24,4)(2,6,23,3)(7,11,10,14)(8,12,9,13)(15,22,18,19)(16,21,17,20),$
$(1,2)(3,4)(5,6)(15,17)(16,18)(19,21)(20,22)(23,24),$
$(1,23)(2,24)(3,5)(4,6)(7,8)(9,10)(11,12)(13,14) >$. The presentation for NL[5] is given by $< a, b, c \mid a^4, b^2, c^2, (a,b), (a,c), (b,c) >$. Since $NL[5]$ is an abelian subgroup of order 16 and N is of order 192, we are looking for a normal subgroup of order 12. However, N does not have a normal subgroup of order 12.

Now we factor N by $NL[5]$ and see that N is an extension of $NL[5]$ by $N/NL[5]$. Thus, $N/NL[5] \cong q =< NL[5]D, NL[5]E >$ and $N/NL[5] =< D, E >$ is of order 12.
$D \sim (1,3)(2,4)(5,23)(6,24)(11,12)(13,14)(15,16)(17,18)(19,22)(20,21)$ of order 2 and $E \sim (1,7,22,24,10,19)(2,8,21,23,9,20)(3,11,15,6,14,18)(4,12,16,5,13,17)$ of order 6. Now we conjugate every generator in $NL[5]$ by D and E.

$$\begin{aligned} A^D &= (1,5,24,4)(2,6,23,3)(7,11,10,14)(8,12,9,13)(15,22,18,19)(16,21,17,20)^D \\ &= (1,4,24,5)(2,3,23,6)(7,12,10,13)(8,11,9,14)(15,20,18,21)(16,19,17,22) \\ &= ABC \end{aligned} \tag{3.9}$$

$$\begin{aligned} B^D &= (1,2)(3,4)(5,6)(15,17)(16,18)(19,21)(20,22)(23,24)^D \\ &= (1,2)(3,4)(5,6)(15,17)(16,18)(19,21)(20,22)(23,24) \\ &= B \end{aligned} \tag{3.10}$$

$$\begin{aligned} C^D &= (1,23)(2,24)(3,5)(4,6)(7,8)(9,10)(11,12)(13,14)^D \\ &= (1,23)(2,24)(3,5)(4,6)(7,8)(9,10)(11,12)(13,14) \\ &= C \end{aligned} \tag{3.11}$$

$$\begin{aligned} A^E &= (1,5,24,4)(2,6,23,3)(7,11,10,14)(8,12,9,13)(15,22,18,19)(16,21,17,20)^E \\ &= (1,6,24,3)(2,5,23,4)(7,13,10,12)(8,14,9,11)(15,19,18,22)(16,20,17,21) \\ &= A^3C \end{aligned} \tag{3.12}$$

$$\begin{aligned} B^E &= (1,2)(3,4)(5,6)(15,17)(16,18)(19,21)(20,22)(23,24)^E \\ &= (1,23)(2,24)(3,5)(4,6)(7,8)(9,10)(11,12)(13,14) \\ &= C \end{aligned} \tag{3.13}$$

$$\begin{aligned} C^E &= (1,23)(2,24)(3,5)(4,6)(7,8)(9,10)(11,12)(13,14)^E \\ &= (7,9)(8,10)(11,13)(12,14)(15,16)(17,18)(19,20)(21,22) \\ &= A^2BC \end{aligned} \tag{3.14}$$

The presentation of N is $< a,b,c,d,e | a^4, b^2, c^2, (a,b), (a,c), (b,c), d^2, e^3, (d*e)^3, a^d = abc, b^d = b, c^d = c, a^e = a^3c, b^e = c, c^3 = a^2bc >$.

However when we verify if the presentation is isomorphic to N, MAGMA tells us that it is not.

```
MAGMA CODE:
> H<a,b,c,d,e>:=Group<a,b,c,d,e|a^4,b^2,c^2,(a,b),(a,c),(b,c),d^2,
e^3,(d*e)^3,a^d=a*b*c,b^d=b,c^d=c,a^e=a^3*c,b^e=c>;
```

```
> H<a,b,c,d,e>:=Group<a,b,c,d,e|a^4,b^2,c^2,(a,b),(a,c),(b,c),d^2,
e^3,(d*e)^3,a^d=a*b*c,b^d=b,c^d=c,a^e=a^3*c,b^e=c,c^e=a^2*b*c>;
> #H;
192

> f,H1,k:=CosetAction(H,sub<H|Id(H)>);
> IsIsomorphic(H1,N);
false
```

From this we can conclude that N is not a semi-direct product of $NL[5]$, rather it is a mixed extension. This means that some elements of $N/NL[5]$ can be written in terms of the elements in NL[5]. In order to proceed, we must check the order of the elements of $NL[5]$.

In MAGMA we compute:

```
> Order(q.1);
2
> Order(T2) eq Order(q.1);
true
> Order(q.2);
3
> Order(T3) eq Order(q.2);
false
> Order(T2*T3) eq Order(q.1*q.2);
false
> Order(T3);
6
> Order(q.2);
3
```

Given that there exists a homomorphism from N to $N/NL[5]$, we know $T[2], T[3] \in N$. However, we verified through MAGMA that $D = T[2]$ is of order 3, and $E = T[3]$ is order of E is 6. This means that $E^3 \in NL[5]$, and $(D*E)^3 = AB$.

```
> H<a,b,c,d,e>:=Group<a,b,c,d,e|a^4,b^2,c^2,(a,b),(a,c),(b,c),
```

```
d^2,e^3=a^2,(d*e)^3=a*b,a^d=a*b*c,b^d=b,
c^d=c,a^e=a^3*c,b^e=c,c^e=a^2*b*c>;
> #H;
192
> f,H1,k:=CosetAction(H,sub<H|Id(H)>);
> IsIsomorphic(H1,N);
true Mapping from: GrpPerm: H1 to GrpPerm: N
Composition of Mapping from: GrpPerm: H1 to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: N
```

Therefore, the presentation of N is $< a, b, c, d, e | a^4, b^2, c^2, (a, b), (a, c), (b, c), d^2, e^3 = a^2, (d * e)^3 = ab, a^d = abc, b^d = b, c^d = c, a^e = a^3 c, b^e = c, c^e = a^2 bc >$. Thus N is the mixed extension of $(4 \times 2^2) : \bullet A_4$

Chapter 4

Double Coset Enumeration

4.1 Preliminaries

Definition 4.1.1. (**right coset**) *If S is a subgroup of G and if t ∈ G, then a* ***right coset*** *of S ∈ G is the subset of G: St = {st:s ∈ S} (a* ***left coset*** *is tS = {ts:s∈S}). One calls t a* ***representative*** *of St (and also tS). [Rot95]*

Theorem 4.1.2. *If S≤G, then any two right (or any two left)cosets of S in G are either identical or disjoint. [Rot95]*

Theorem 4.1.3. *If S≤G, then the number right cosets of S in G is equal to the number of left cosets of S in G. [Rot95]*

Definition 4.1.4. (**index**) *If S≤G, then the* ***index*** *of S in G, denoted [G:S], is the number of right cosets of S in G. [Rot95]*

Definition 4.1.5. (**conjugate**) *If $x \in G$, then a **conjugate** of x in G is an element of the form axa^{-1} for some $a \in G$. [Rot95]*

Definition 4.1.6. (**double coset**) *If S and T are subgroups of G, then a **double coset** is a subset of G of the form SgT, where $g \in G$. [Rot95]*

Definition 4.1.7. (**G-set**) *If X is a set and G is a group, then X is a **G-set** if there is a function $\alpha\colon G\times X \longrightarrow X$ (called an **action**), denoted by $\alpha\colon (g, x) \mapsto gx$, such that:*

(i) $1x = x$ for all $x \in X$; and

(ii) $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$. [Rot95]

Definition 4.1.8. (**acts**) *G **acts** on X, if $|X| = n$, then n is called the **degree** of the G-set X. [Rot95]*

Definition 4.1.9. (**G-orbit**) *If X is a G-set anf $x \in X$, then the **G-orbit** of x is $\vartheta(x) = \{gx : g \in G\} \subset X$, ($\vartheta(x)$ denoted Gx). [Rot95]*

Definition 4.1.10. (**stabilizer**) *If X is a G-set and $x \in X$, then the **stabilizer** of x, denoted by G_x, is the subgroup $G_x = g \in G : gx = x \leq G$. [Rot95]*

Theorem 4.1.11. *If X is a G-set and $x \in X$, then $|\vartheta(x)| = [G : G_x]$. [Rot95]*

Corollary 4.1.12. *If a finite group G acts on a set X, then the number of elements in any* *orbit is a divisor of* $|G|$. *[Rot95]*

Corollary 4.1.13. *(i) If G is a finite group and* $x \in G$*, then the number of conjugates of* $x \in G$ *is* $[G : C_G(x)]$ *(*C_G*, is* ***centralizer****). (ii) If G is a finite group and* $H \le G$*, then the number of conjugates of* $H \in G$ *is* $[G : N_G(H)]$ *(*N_G*, is* ***normalizer****). [Rot95]*

Definition 4.1.14. (**transitive**) *A G-set X is* ***transitive*** *if it has only one orbit; that is for every* $x, y \in X$*, there exists* $\sigma \in G$ *with* $y = \sigma x$*. [Rot95]*

Definition 4.1.15. (**Point Stabilizer**) *A* ***point stabilizer*** *of w in N, denoted by* N^w, $N^w = \{n \in N \mid w^n = w\} \le N$*, where w is word of the* t_i'*s*

Lemma 4.1.16. *The point stabilizer* N^w *is a subgroup of N. Apply the subgroup test to* N^w.

1. $w^e = w \Rightarrow e \in N^w$

2. Let $a, b \in N^w$*, we want to show that* $ab \in N^w$

$w^{ab} = (w^a)^b$

$= w^b$

$= w$

$\Rightarrow ab \in N^w$

3. Let $a \in N^w$*. Show* $a^{-1} \in N^w$*. Given* $w^a = w$

Then $w^{aa^{-1}} = wa^{-1}$*. So* $w = w^{-a}$*. Thus* $a^{-1} = w$*, and* $a^{-1} \in N^w$*.*

Definition 4.1.17. (**Coset Stabilizing Group**) *The coset stabilizing group of the coset* Nw *is* $N^{(w)=\{n \in N | Nw^n = Nw}$ *where w is a word in the* t_i'*s.*

Lemma 4.1.18. *The coset stabilizer $N^{(w)}$ is a subgroup of N. Apply the subgroup test to N^{w}.*

1. $e \in N^{(w)}$ since $Nw^{e} = Nw$

2. Let $a, b \in N^{(w)}$, we want to show that $ab \in N^{(w)}$

$Nw^{a} = Nw$ and $Nw^{b} = Nw$, Then $Nw^{ab} = N(w^{a})^{b}$

$= (Nw^{a})^{b}$

$= N(w)^{b}$

$= Nw \Rightarrow ab \in N^{(w)}$

3. Let $a \in N^{(w)}$. Show $a^{-1} \in N^{(w)}$. Given $Nw^{a} = Nw$

$\Rightarrow (Nw^{a}) = Nw$

$\Rightarrow (Nw^{a})^{a^{-1}} = (Nw)^{a^{-1}}$

$\Rightarrow Nw^{aa^{-1}} = Nw^{a^{-1}}$

$\Rightarrow Nw = Nw^{-a}$. Thus $a^{-1} \in N^{(w)}$.

Lemma 4.1.19. $N^{w} \leq N^{(w)}$

Let $a \in N^{w}$ and $N^{(w)} = \{n \in N \mid Nw^{n} = Nw\}$

$\Rightarrow w^{a} = w$

$\Rightarrow Nw^{a} = Nw$

$\Rightarrow a \in N^{(w)}$.

Lemma 4.1.20. *The number of right cosets in the double cosets NwN is $\frac{|N|}{|N^{(w)}|}$, since $Na \neq Nb \Longleftrightarrow N^{(w)}a \neq N^{(w)}b$.*

Lemma 4.1.21.(**Equality of Right Cosets** $Nw_{1} = Nw_{2}$

$\Longleftrightarrow w_1 \in Nw_2$

$\Longleftrightarrow \exists n \in N \ni w_1 = nw_2$

Lemma 4.1.22.(**Equality of Double Cosets** *Let* $NwN = \{Nw^n \mid n \in N\} = \{mw^n \mid n, m \in N\}$ *define a double coset. Let* $Nw_1N = \{Nw_1^n \mid n \in N\}$ *be one double coset and* $w_1 \in Nw_1N$ *and let Let* $Nw_2N = \{Nw_2^n \mid n \in N\}$ *be a different double coset. Then* $Nw_1N = Nw_2N$

$w_1 \in Nw_1N = Nw_2N$

$\Longleftrightarrow w_1 \in Nw_2N$

$\Longleftrightarrow w_1 = mw_2n \textit{ wherem}, n \in N$

$\Longleftrightarrow w_1 = mn^{-1}w_2n$

$\Longleftrightarrow w_1 = mnw_2^n$

$\Longleftrightarrow w_1 = gw_2^n \textit{ whereg} = mn \in N$

Definition 4.1.23. (**Double Coset Algorithm**) *Perform the double coset enumeration of group G over transitive group N, where double cosets take the form NwN = {Nwn | n ∈ N} = {*Nw^n *| n ∈ N}.*

(i) Compute the point-stabilizer N^w *and coset stabilizer of each double coset.*

(ii) Compute the number of right cosets by using the formula $\frac{|N|}{|N^{(w)}|}$*, where* $N^{(w)}$ *= {n ∈ N |* Nw^n *= Nw} is the coset stabilizer of the right coset.*

(iii) For each double coset NwN, compute the orbits of $N^{(w)}$*. It suffices to determine the double coset of* Nwt_i *for a single representative of each orbit. Note,* $N^{(w)} \geq N^w$ *is always true.*

(iv) Determine which double coset each coset representative Nwt_i *belongs to, (repeat the process until closed by coset multiplication) .*

4.2 Double Coset Enumeration of PGL(2,7) over $2^2:3$

$G = \frac{2^6:2^2:3}{(xt^y t^x)^2,(xt^y(t^y)^x)^3}$ Consider the group,

$G < x, y, t >= Group < x, y, t | x^3, y^3, (xy)^2, t^2, (t, xy^{-1}x) >$ factored by $(xt^y t^x)^2$ and $(xt^y(t^y)^x)^3$ where $x = (1,3,5)(2,4,6)$, $y = (1,2,6)(3,4,5)$ and $t = t_1$. Now we substitute the values of x and y and expand our relation $(xt^y t^x)^2 = e$ to obtain,

$$\begin{aligned}\left((135)(246)\,t_1^{(126)(345)}\,t_1^{(135)(246)}\right)^2 &= ((135)(246)\,t_2\,t_3)^2\\ &= (135)(246)\,t_2\,t_3(135)(246)\,t_2\,t_3\\ &= (135)(246)(135)(246)(t_2\,t_3)^{(135)(246)}\,t_2\,t_3\\ &= (153)(264)\,t_4\,t_5\,t_2\,t_3\end{aligned} \tag{4.1}$$

Our first relation $(153)(264)\,t_4\,t_5\,t_2\,t_3 = e$ can be written as $(153)(264)\,t_4\,t_5 = t_3\,t_2$

Similarly, we expand our second relation to obtain,

$$\begin{aligned}\left((135)(246)\,t_1^{(126)(346)}\,(t_1^{(126)(345)})^{(135)(246)}\right)^3 &= ((135)(246)\,t_2\,t_4)^3\\ &= (135)(246)\,t_2\,t_4(135)(246)\,t_2\,t_4(135)(246)\,t_2\,t_4\\ &= (135)(246)^3(t_2\,t_4)^{(135)(246)^2}(t_2\,t_4)^{(135)(246)}\,t_2\,t_4\\ &= e\,t_6\,t_2\,t_4\,t_6\,t_2\,t_4\end{aligned} \tag{4.2}$$

Our second relation $t_6\,t_2\,t_4\,t_6\,t_2\,t_4 = e$ can be written as $t_6\,t_2\,t_4 = t_4\,t_2\,t_6$.

We conjugate the first relation $(153)(264)\,t_4\,t_5 = t_3\,t_2$ by all the elements of N to obtain twelve new relations.

$$(153)(264)\,t_4\,t_5^{(153)(264)} = t_3\,t_2^{(153)(264)} \Rightarrow (153)(264)\,t_2\,t_3 = t_1\,t_6$$
$$(153)(264)\,t_4\,t_5^{(156)(234)} = t_3\,t_2^{(156)(234)} \Rightarrow (123)(456)\,t_2\,t_6 = t_4\,t_3$$
$$(153)(264)\,t_4\,t5^{(132)(465)} = t_3\,t_2^{(132)(465)} \Rightarrow (156)(234)\,t_6\,t_4 = t_2\,t_1$$
$$(153)(264)\,t_4\,t_5^{(126)(345)} = t_3\,t_2^{(126)(345)} \Rightarrow (156)(234)\,t_5\,t_3 = t_4\,t_6$$
$$(153)(264)\,t_4\,t_5^{(162)(354)} = t_3\,t_2^{(162)(354)} \Rightarrow (123)(456)\,t_3\,t_4 = t_5\,t_1$$
$$(153)(264)\,t_4\,t_5^{(165)(243)} = t_3\,t_2^{(126)(345)} \Rightarrow (126)(345)\,t_3\,t_1 = t_2\,t_4 \tag{4.3}$$
$$(153)(264)\,t_4\,t_5^{(14)(25)} = t_3\,t_2^{(14)(25)} \Rightarrow (156)(234)\,t_1\,t_2 = t_3\,t_5$$
$$(153)(264)\,t_4\,t_5^{(14)(36)} = t_3\,t_2^{(14)(36)} \Rightarrow (123)(456)\,t_1\,t_5 = t_6\,t_2$$
$$(153)(264)\,t_4\,t_5^{(135)(246)} = t_3\,t_2^{(135)(246)} \Rightarrow (153)(264)\,t_6\,t_1 = t_5\,t_4$$
$$(153)(264)\,t_4\,t_5^{(123)(456)} = t_3\,t_2^{(123)(456)} \Rightarrow (126)(345)\,t_5\,t_6 = t_1\,t_3$$
$$(153)(264)\,t_4\,t_5^{(25)(36)} = t_3\,t_2^{(25)(36)} \Rightarrow (126)(345)\,t_4\,t_2 = t_6\,t_5$$

We conjugate the second relation $t_6\,t_2\,t_4 = t_4\,t_2\,t_6$ by all the elements of N to obtain twelve new relations.

$$\begin{aligned}
t_6 t_2 t_4^{(153)(264)} &= t_4 t_2 t_6^{(153)(264)} \Rightarrow t_4 t_6 t_2 = t_2 t_6 t_4 \\
t_6 t_2 t_4^{(156)(234)} &= t_4 t_2 t_6^{(156)(234)} \Rightarrow t_1 t_3 t_2 = t_2 t_3 t_1 \\
t_6 t_2 t_4^{(132)(465)} &= t_4 t_2 t_6^{(132)(465)} \Rightarrow t_5 t_1 t_6 = t_6 t_1 t_5 \\
t_6 t_2 t_4^{(126)(345)} &= t_4 t_2 t_6^{(126)(345)} \Rightarrow t_1 t_6 t_5 = t_5 t_6 t_1 \\
t_6 t_2 t_4^{(162)(354)} &= t_4 t_2 t_6^{(162)(354)} \Rightarrow t_2 t_1 t_3 = t_3 t_1 t_2 \\
t_6 t_2 t_4^{(165)(243)} &= t_4 t_2 t_6^{(126)(345)} \Rightarrow t_5 t_4 t_3 = t_3 t_4 t_5 \\
t_6 t_2 t_4^{(14)(25)} &= t_4 t_2 t_6^{(14)(25)} \Rightarrow t_6 t_5 t_1 = t_1 t_5 t_6 \\
t_6 t_2 t_4^{(14)(36)} &= t_4 t_2 t_6^{(14)(36)} \Rightarrow t_3 t_2 t_1 = t_1 t_2 t_3 \\
t_6 t_2 t_4^{(135)(246)} &= t_4 t_2 t_6^{(135)(246)} \Rightarrow t_2 t_4 t_6 = t_6 t_4 t_2 \\
t_6 t_2 t_4^{(123)(456)} &= t_4 t_2 t_6^{(123)(456)} \Rightarrow t_4 t_3 t_5 = t_5 t_3 t_4 \\
t_6 t_2 t_4^{(25)(36)} &= t_4 t_2 t_6^{(25)(36)} \Rightarrow t_3 t_5 t_4 = t_4 t_5 t_3
\end{aligned} \tag{4.4}$$

We will use our technique of double coset enumeration to show that $|G| \leq 336$.

<u>1st Double Coset [∗]</u>

Let [∗] represent the double coset which contains $NeN = \{N(e)^n \mid n \in N\} = \{N\}$. The coset stabilizer of NeN = N. The number of single cosets in [∗] is equal to $\frac{|N|}{|N|} = \frac{12}{12} = 1$. The orbits of N on $\{1,2,3,4,5,6\}$ are $\{1,2,3,4,5,6\}$, that is, there is one single orbit. Now select a representative from the single orbit, say 1, and find the double coset that contains Nt_1. We determine that Nt_1 belongs to a new double coset Nt_1 N denoted by [1]. There are 6 elements in the orbit $\{1,2,3,4,5,6\}$, therefore, all 6 symmetric generators will move forward.

Figure 4.1: Cayley Graph PGL(2,7) over $2^2:3$

2nd Double Coset

Let [1] represent the double coset that contains all the elements in Nt_1N.

$$\begin{aligned} Nt_1N &= \{N(t_1)^n \mid n \in N\} \\ &= \{Nt_1^e, Nt_1^{(1,5,3)(2,6,4)}, Nt_1^{(1,5,6)(2,3,4)}, Nt_1^{(1,3,2)(4,6,5)}, \\ &Nt_1^{(1,2,6)(3,4,5)}, Nt_1^{(1,6,2)(3,5,4)}, Nt_1^{(1,6,5)(2,4,3)}, Nt_1^{(1,4)(2,5)}, \\ &Nt_1^{(1,4)(3,6)}, Nt_1^{(1,3,5)(2,4,6)}, Nt_1^{(1,2,3)(4,5,6)}, Nt_1^{(2,5)(3,6)}\} \\ &= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6\} \end{aligned} \tag{4.5}$$

The point stabilizer of N^1 is e, (25)(36), since these are the only two elements that stabilize 1. The coset stabilizer of $N^{(1)} = \{e, (25)(36)\}$. The number of single cosets in [1] is equal to $\frac{|N|}{|N^{(1)}|} = \frac{12}{2} = 6$. The orbits of N on $\{1,2,3,4,5,6\}$ are $\{1\}, \{4\}, \{2,5\}$, and $\{3,6\}$. Now select a representative from each single orbit, $1 \in \{1\}$,$4 \in \{4\}$, $2 \in \{2,5\}$,$3 \in \{3,6\}$ and determine the double cosets that contains Nt_1t_1, Nt_1t_4, Nt_1t_2, and Nt_1t_3. We have four possible new double cosets. We use our first relation to determine if we will have four distinct double cosets.

$Nt_1t_1 = Ne \in [*]$, since there is one element in the orbit $\{1\}$, one symmetric generator will return to $[*]$.

Nt_1t_2 belongs to a new double coset Nt_1t_2N denoted by [12], since there are two elements in the orbit $\{2,5\}$, two symmetric generators will move forward.

Nt_1t_3 denoted by [13]= [12]. If we conjugate our first relation $(153)(264)t_4t_5 = t_3t_2$ by $(123)(456)$ we get

$$(153)(264)t_4t_5^{(123)(456)} = t_3t_2^{(123)(456)}$$

$$(126)(345)t_5t_6 = t_1t_3$$

Now, $t_1t_3 = (126)(345)t_5t_6$

So $Nt_1t_3 = Nt_5t_6 = N(t_1t_2)^{(153)(246)} \in [12]$

But $Nt_1t_3 \not\in [12]$ so $Nt_1t_3 = [12]$

Nt_1t_4 belongs to a new double coset Nt_1t_4N denoted by [14], since there is one element in the orbit $\{4\}$, one symmetric generator will move forward.

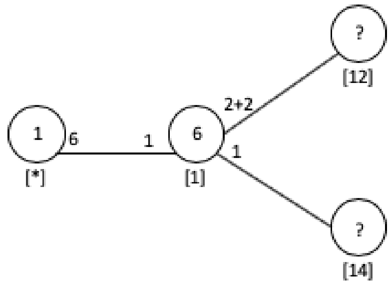

Figure 4.2: Cayley Graph PGL(2,7) over $2^2:3$

3rd Double Coset [12]

Let [12] represent the double coset containing the elements,

$$\begin{aligned} Nt_1t_2N &= \{N(t_1t_2)^n \mid n \in N\} \\ &= \{N(t_1t_2)^e, N(t_1t_2)^{(1,5,3)(2,6,4)}, N(t_1t_2)^{(1,5,6)(2,3,4)}, N(t_1t_2)^{(1,3,2)(4,6,5)}, \\ &N(t_1t_2)^{(1,2,6)(3,4,5)}, N(t_1t_2)^{(1,6,2)(3,5,4)}, N(t_1t_2)^{(1,6,5)(2,4,3)}, N(t_1t_2)^{(1,4)(2,5)}, \\ &N(t_1t_2)^{(1,4)(3,6)}, N(t_1t_2)^{(1,3,5)(2,4,6)}, N(t_1t_2)^{(1,2,3)(4,5,6)}, N(t_1t_2)^{(2,5)(3,6)}\} \\ &= \{Nt_1t_2, Nt_5t_6, Nt_5t_3, Nt_3t_1, Nt_2t_6, Nt_6t_1, \\ &Nt_6t_4, Nt_4t_5, Nt_4t_2, Nt_3t_4, Nt_2t_3, Nt_1t_5,\} \end{aligned} \tag{4.6}$$

Lemma $Nt_1t_2t_4 = Nt_2t_3t_1$

Conjugate the original relation $(153)(264)t_4t_5 = t_3t_2$ by $(165)(243)$ to get,

$$\begin{aligned} (153)(264)t_4t_5^{(165)(243)} &= t_3t_2^{(14)(25)} \\ (156)(234)t_3t_1 &= t_2t_4 \\ \text{Then } t_1\underline{t_2t_4} &= t_1(156)(234)t_3t_1 \\ &= (126)(453)t_2t_3t_1 \\ \Rightarrow Nt_1t_2t_4 &= Nt_2t_3t_1 \end{aligned} \tag{4.7}$$

Lemma $Nt_1t_2t_5 = Nt_3$

Conjugate the original relation $(153)(264)t_4t_5 = t_3t_2$ by $(14)(25)$ to get,

$$\begin{aligned}
(153)(264)t_4t_5^{(14)(25)} &= t_3t_2^{(14)(25)} \\
(156)(234)t_1t_2 &= t_3t_5 \\
\Rightarrow t_1t_2 &= (165)(243)t_3t_5 \\
\text{Then } \underline{t_1t_2}t_5 &= (165)(243)t_3t_5t_5 \\
&= (165)(243)t_3 \\
\Rightarrow Nt_1t_2t_5 &= Nt_3
\end{aligned} \tag{4.8}$$

Lemma $Nt_1t_2t_6 = Nt_3t_4t_3$

Conjugate the original relation $(153)(264)t_4t_5 = t_3t_2$ by $(156)(234)$ to get,

$$\begin{aligned}
(153)(264)t_4t_5^{(156)(234)} &= t_3t_2^{(156)(234)} \\
(123)(456)t_2t_6 &= t_4t_3 \\
\Rightarrow t_2t_6 &= (132)(465)t_4t_3 \text{Then } t_1\underline{t_2t_6} \\
= t_1(132)(465)t_4t_3 \\
&= (132)(465)t_3t_4t_3 \\
\Rightarrow Nt_1t_2t_6 &= Nt_3t_4t_3
\end{aligned} \tag{4.9}$$

The point stabilizer of N^{12} is $\{e\}$. The coset stabilizer of $N^{(12)} = \{e\}$. The number of single cosets in [12] is equal to $\frac{|N|}{|N^{(12)}|} = \frac{12}{1} = 12$. The orbits of N on $\{1,2,3,4,5,6\}$ are $\{1\},\{2\},\{3\},\{4\},\{5\}$,and $\{6\}$. Now select a representative from each single orbit, $1 \in \{1\}$, $2 \in \{2\}$, $3 \in \{3\}$, $4 \in \{4\}$,$5 \in \{5\}$, and $6 \in \{6\}$ and determine the double cosets to which they belong.

$Nt_1t_2t_1 \in [121]$, which is a new double coset. Thus one symmetric generator moves forward.

$Nt_1t_2t_2 \in [1]$, since $Nt_1t_2t_2 = Nt_1e = Nt_1$. Thus, one symmetric generator goes back.

$Nt_1t_2t_3 \in [123]$, which is a new double coset. Thus one symmetric generator moves forward.

$Nt_1t_2t_4 \in [123]$. From the lemma, we know $Nt_1t_2t_4 = Nt_2t_3t_1 = N(t_1t_2t_3)^{(1,2,3)(4,5,6)}$.Then $Nt_1t_2t_4 \in [123]$. Thus, one symmetric generator goes to [123].

$Nt_1t_2t_5 \in [1]$. From the lemma $Nt_1t_2t_5 = Nt_3 = Nt_1^{(132)(465)}$. Then $Nt_1t_2t_5 \in [1]$. Thus, one symmetric generator moves goes back to [1].

$Nt_1t_2t_6 \in [121]$. From the lemma $Nt_1t_2t_6 = Nt_3t_4t_3 = N(t_1t_2t_1)^{(135)(246)}$. Then $Nt_1t_2t_6 \in [121]$. Thus, one symmetric generator moves forward.

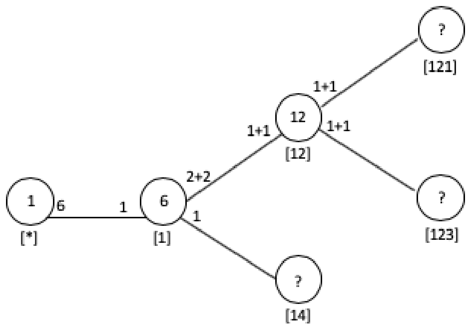

Figure 4.3: Cayley Graph PGL(2,7) over $2^2:3$

4th Double Coset [14]

Let [14] represent the double coset containing the elements,

$$
\begin{aligned}
Nt_1t_4N &= \{N(t_1t_4)^n \mid n \in N\} \\
&= \{N(t_1t_4)^e, N(t_1t_4)^{(1,5,3)(2,6,4)}, N(t_1t_4)^{(1,5,6)(2,3,4)}, N(t_1t_4)^{(1,3,2)(4,6,5)}, \\
&N(t_1t_4)^{(1,2,6)(3,4,5)}, N(t_1t_4)^{(1,6,2)(3,5,4)}, N(t_1t_4)^{(1,6,5)(2,4,3)}, N(t_1t_4)^{(1,4)(2,5)}, \\
&N(t_1t_4)^{(1,4)(3,6)}, N(t_1t_4)^{(1,3,5)(2,4,6)}, N(t_1t_4)^{(1,2,3)(4,5,6)}, N(t_1t_4)^{(2,5)(3,6)}\} \\
&= \{Nt_1t_4, Nt_5t_2, Nt_3t_6, Nt_2t_5, Nt_6t_3, Nt_4t_1,
\end{aligned} \tag{4.10}
$$

The point stabilizer of 14 is $\{e, (25)(36)\}$. In order to see the number of elements that are in the coset stabilizer, we must identify the element that stabilizes the coset.

If we conjugate our relation by (135)(246) to obtain,

$$(153)(264)\,t_6\,t_1 = t_5\,t_4.$$

$$\text{So } t_2(153)(264)\,t_1 = t_5\,t_4.$$

$$\text{Thus } (153)(264)\,t_1\,t_4\,t_5 = t_4.$$

$$\text{Therefore, } t_1\,t_4\,t_5\,t_2 = (135)(246) \in N. \tag{4.11}$$

$$\text{Then} N t_1\,t_4\,t_5\,t_2 = N$$

$$\text{Hence } N t_1\,t_4 = N t_2\,t_5.$$

$$N(t_1\,t_4)^{(126)(345)} = t_2\,t_5$$

$\Rightarrow$ (126)(345) belongs to $N^{(14)}$.

Therefore, the coset stabilizer of Nt_1t_4 is $< e, (25)(36), (126)(345) >= N$. The number of single cosets in [14] is equal to $\frac{|N|}{|N^{(14)}|} = \frac{12}{12} = 1$. The orbits of N on $\{1,2,3,4,5,6\}$ are $\{1,2,3,4,5,6\}$. Now select a representative from the single orbit, $4 \in \{1,2,3,4,5,6\}$, and determine the double cosets to which it belongs.

$Nt_1t_4t_4 \in [1]$, since $t_4t_4 = t_4^2 = e$ and $Nt_1t_4t_4 = Nt_1e \in [1]$. Thus all 6 symmetric generators go back to [1].

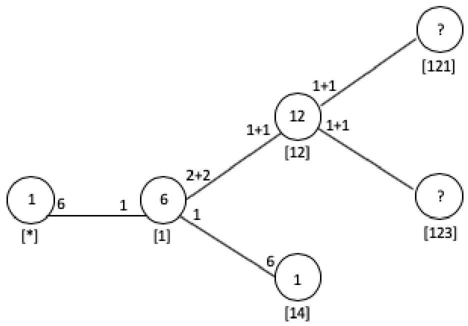

Figure 4.4: Cayley Graph PGL(2,7) over $2^2 : 3$

5th Double Coset [121]

Let [121] represent the double coset containing the elements,

$$Nt_1t_2t_1N = \{N(t_1t_2t_1)^n \mid n \in N\}$$
$$= \{N(t_1t_2t_1)^e, N(t_1t_2t_1)^{(1,5,3)(2,6,4)}, N(t_1t_2t_1)^{(1,5,6)(2,3,4)}, N(t_1t_2t_1)^{(1,3,2)(4,6,5)},$$
$$N(t_1t_2t_1)^{(1,2,6)(3,4,5)}, N(t_1t_2t_1)^{(1,6,2)(3,5,4)}, N(t_1t_2t_1)^{(1,6,5)(2,4,3)}, N(t_1t_2t_1)^{(1,4)(2,5)},$$
$$N(t_1t_2t_1)^{(1,4)(3,6)}, N(t_1t_2t_1)^{(1,3,5)(2,4,6)}, N(t_1t_2t_1)^{(1,2,3)(4,5,6)}, N(t_1t_2t_1)^{(2,5)(3,6)}\}$$
$$= \{Nt_1t_2t_1, Nt_5t_6t_5, Nt_5t_3t_5, Nt_3t_1t_3, Nt_2t_6t_2, Nt_6t_1t_6,$$
$$Nt_6t_4t_6, Nt_4t_5t_4, Nt_4t_2t_4, Nt_3t_4t_3, Nt_2t_3t_2, Nt_1t_5t_1,\} \tag{4.12}$$

The point stabilizer of 121 is $\{e\}$. However, since $Nt_1t_2t_1 = Nt_3t_1t_3$, the coset stabilizer of $N^{(121)} = \{e, (132)(465), (1,2,3)(4,5,6)\}$. The number of single cosets in [121] is equal to $\frac{|N|}{|N^{(121)}|} = \frac{12}{3} = 4$. The orbits of N on $\{1,2,3,4,5,6\}$ are $\{1,2,3\}$ and $\{4,5,6\}$. Now select a representative from each single orbit, $1 \in \{1,2,3\}$ and $4 \in \{4,5,6\}$ and determine the double cosets to which they belong.

$Nt_1t_2t_1t_1 \in [12]$. Since $t_1t_1 = t_1^2 = e$. Then $Nt_1t_2t_1t_1 = Nt_1t_2t_1^e = Nt_1t_2 \in [12]$. Thus 3 symmetric generators go back to [12].

$Nt_1t_2t_1t_4 \in [12]$. If we conjugate our original relation $(153)(264)t_4t_5 = t3t_2$ by $(132)(465)$ we get

$$\begin{aligned} ((153)(264)t_4t_5)^{(132)(465)} &= (t_3t_2)^{(132)(465)} \\ (156)(234)t_6t_4 &= t_2t_1 \end{aligned} \tag{4.13}$$

Also, if we conjugate our original relation $(153)(264)t_4t_5 = t_3t_2$ by $(123)(456)$ we get

$$\begin{aligned} ((153)(264)t_4t_5)^{(123)(456)} &= (t_3t_2)^{(132)(465)} \\ (126)(345)t_5t_6 &= t_1t_3 \end{aligned} \tag{4.14}$$

We use both relations to show,

$$\begin{aligned} t_1\underline{t_2t_1}t_4 &= t_1(156)(234)t_6t_4t_4 \\ &= (156)(234)t_5t_6e \\ &= (156)(234)t_5t_6 \end{aligned} \tag{4.15}$$

$Nt_1t_2t_1t_4 = Nt_5t_6 = N(t_1t_2)^{(153)(264)} \in [12]$. Therefore, $Nt_1t_2t_1t_4 \in [12]$ Thus 3 symmetric generators go back to [12].

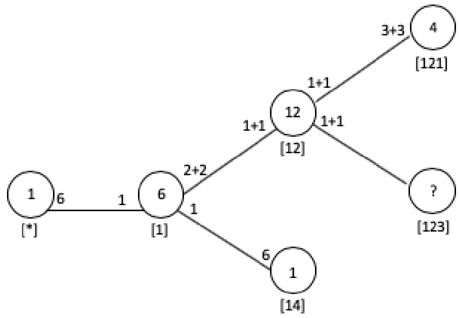

Figure 4.5: Cayley Graph PGL(2,7) over $2^2:3$

6th Double Coset [123]

Let [123] represent the double coset containing the elements,

$$
\begin{aligned}
Nt_1t_2t_3N &= \{N(t_1t_2t_3)^n \mid n \in N\} \\
&= \{N(t_1t_2t_3)^e, N(t_1t_2t_3)^{(1,5,3)(2,6,4)}, N(t_1t_2t_3)^{(1,5,6)(2,3,4)}, N(t_1t_2t_3)^{(1,3,2)(4,6,5)}, \\
&N(t_1t_2t_3)^{(1,2,6)(3,4,5)}, N(t_1t_2t_3)^{(1,6,2)(3,5,4)}, N(t_1t_2t_3)^{(1,6,5)(2,4,3)}, N(t_1t_2t_3)^{(1,4)(2,5)}, \\
&N(t_1t_2t_3)^{(1,4)(3,6)}, N(t_1t_2t_3)^{(1,3,5)(2,4,6)}, N(t_1t_2t_3)^{(1,2,3)(4,5,6)}, N(t_1t_2t_3)^{(2,5)(3,6)}\} \\
&= \{Nt_1t_2t_3, Nt_5t_6t_1, Nt_5t_3t_4, Nt_3t_1t_2, Nt_2t_6t_4, Nt_6t_1t_5, \\
&Nt_6t_4t_2, Nt_4t_5t_3, Nt_4t_2t_6, Nt_3t_4t_5, Nt_2t_3t_1, Nt_1t_5t_6,\}
\end{aligned}
\tag{4.16}
$$

The point stabilizer of N^{123} is $\{e\}$. However, since $Nt_1t_2t_3 = Nt_5t_3t_4$, the coset stabilizer of $N^{(123)} = \{e, (1,5,6)(2,3,4), (1,6,5)(2,4,3)\}$. The number of single cosets in [123] is equal to $\frac{|N|}{|N^{(12)}|} = \frac{12}{3} = 4$. The orbits of N on $\{1,2,3,4,5,6\}$ are $\{1,5,6\}$ and $\{2,3,4\}$. Now select a representative from each single orbit, $1 \in \{1,5,6\}$ and $3 \in \{2,3,4\}$ and determine the double cosets to which they belong.

$Nt_1t_2t_3t_1 \in [12]$. If we conjugate our original second relation $t_6t_2t_4 = t_4t_2t_6$ by $(14)(36)$ we get

$$t_6t_2t_4^{(14)(36)} = t_4t_2t_6^{(14)(36)}$$
$$\Rightarrow t_3t_2t_1 = t_1t_2t_3 \tag{4.17}$$

Now we show,

$$\underline{t_1t_2t_3}\,t_1 = t_3t_2t_1t_1$$
$$= t_3t_2$$
$$= (153)(264)t_4t_5 \tag{4.18}$$
$$\Rightarrow Nt_1t_2t_3t_1 = Nt_4t_5 = N(t_1t_2)^{(14)(25)} \in [12]$$

Therefore, $Nt_1t_2t_3t_1 \in [12]$. Thus, 3 symmetric generators go back to [12].

$Nt_1t_2t_3t_3 \in [12]$. Since, $t_3t_3 = t_3^2 = e$. Then $Nt_1t_2t_3t_3 = Nt_1t_2t_3^2 = Nt_1t_2$. Thus 3 symmetric generators go back to [12].

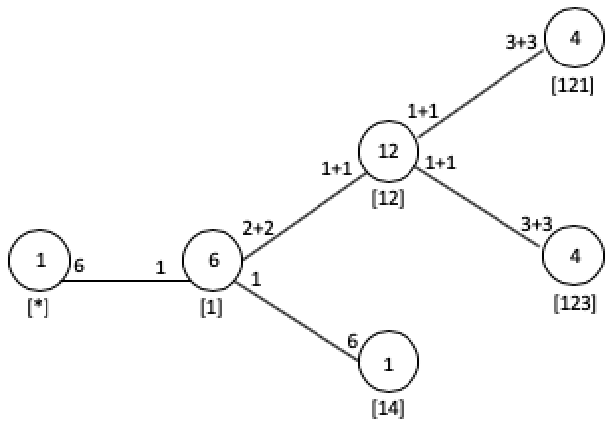

Figure 4.6: Cayley Graph PGL(2,7) over $2^2:3$

4.3 Double Coset Enumeration of PSL(2,11) over D_6

$G = \frac{2^6:N}{(xyt^y)^3,(xt^{y^2}t)^5}$ Consider the group, $G < x, y, t > := Group < x, y, t | x^3, y^3, (x*y)^2, t^2, (t, x*y^{-1}*x), t*t^y*t^{x*y}*t^y = y^2*t^{x^2}*t^{y^2} >$ factored by $(xyt^y)^3$ and $(xt^{y^2}t)^5$ where $x = (1,3,5)(2,4,6)$, $y = (1,2,6)(3,4,5)$, $t's$ are of order 2 and $t = t_1$. Next we will expand our relations.

Expanding the relation $(x*y*t^y)^3$

Let $\pi = xy = (1,4)(2,5)$

$\pi^2 = (xy)^2 = e$

and $\pi^3 = (xy)^3 = (1,4)(2,5)$ Now we will expand our relation $(x*y*t^y)^3 = e$

$$
\begin{aligned}
(x * y * t^{y})^3 &= (\pi\, t_1^{(1,2,6)(3,4,5)})^3 \\
&= (\pi\, t_2)^3 \\
&= \pi\, t_2 \cdot \pi\, t_2 \cdot \pi\, t_2 \\
&= \pi^3\, t_2^{\pi^2}\, t_2^{\pi}\, t_2 \\
&= (1,4)(2,5)\, t_2^{e}\, t_2^{(1,4)(2,5)}\, t_2 \\
&= (1,4)(2,5)\, t_2\, t_5\, t_2
\end{aligned} \tag{4.19}
$$

Our relation is $(1,4)(2,5)\,t_2\,t_5\,t_2 = e$ which can also be written as $(1,4)(2,5)\,t_2\,t_5 = t_2$. We use this relation to find other relation by conjugating by the elements in N.

$$\big((1,4)(2,5)\,t_2\,t_5\big)^{(1,3,5)(2,4,6)} = t_2^{(1,3,5)(2,4,6)} \Rightarrow (1,4)(3,6)\,t_4\,t_1 = t_4 \tag{4.20}$$

$$\big((1,4)(2,5)\,t_2\,t_5\big)^{(1,2,6)(3,4,5)} = t_2^{(1,2,6)(3,4,5)} \Rightarrow (2,5)(3,6)\,t_6\,t_3 = t_6 \tag{4.21}$$

$$\big((1,4)(2,5)\,t_2\,t_5\big)^{(1,5,3)(2,6,4)} = t_2^{(1,5,3)(2,6,4)} \Rightarrow (2,5)(3,6)\,t_6\,t_3 = t_6 \tag{4.22}$$

$$\big((1,4)(2,5)\,t_2\,t_5\big)^{(1,6,2)(3,5,4)} = t_2^{(1,6,2)(3,5,4)} \Rightarrow (1,4)(3,6)\,t_1\,t_4 = t_1 \tag{4.23}$$

$$\big((1,4)(2,5)\,t_2\,t_5\big)^{(1,4)(2,5)} = t_2^{(1,4)(2,5)} \Rightarrow (1,4)(2,5)\,t_5\,t_2 = t_5 \tag{4.24}$$

$$\big((1,4)(2,5)\,t_2\,t_5\big)^{(2,5)(3,6)} = t_2^{(2,5)(3,6)} \Rightarrow (1,4)(2,5)\,t_5\,t_2 = t_5 \tag{4.25}$$

$$\big((1,4)(2,5)\,t_2\,t_5\big)^{(1,3,2)(4,6,5)} = t_2^{(1,3,2)(4,6,5)} \Rightarrow (1,4)(3,6)\,t_1\,t_4 = t_1 \tag{4.26}$$

$$\big((1,4)(2,5)\,t_2\,t_5\big)^{(1,2,3)(4,5,6)} = t_2^{(1,2,3)(4,5,6)} \Rightarrow (2,5)(3,6)\,t_3\,t_6 = t_3 \tag{4.27}$$

$$\big((1,4)(2,5)\,t_2\,t_5\big)^{(1,5,6)(2,3,4)} = t_2^{(1,5,6)(2,3,4)} \Rightarrow (2,5)(3,6)\,t_3\,t_6 = t_3 \tag{4.28}$$

$$\big((1,4)(2,5)\,t_2\,t_5\big)^{(1,6,5)(2,4,3)} = t_2^{(1,6,5)(2,4,3)} \Rightarrow (1,4)(3,6)\,t_1\,t_4 = t_1 \tag{4.29}$$

$$\big((1,4)(2,5)\,t_2\,t_5\big)^{(1,6,5)(2,4,3))} = t_2^{(1,6,5)(2,4,3)} \Rightarrow (1,4)(3,6)\,t_4\,t_1 = t_4 \tag{4.30}$$

$$\tag{4.31}$$

Expanding our relation $(xt^{y^2}t)^5$

Let $t = t_1$, $x = (1,3,5)(2,4,6)$, $x^2 = (1,5,3)(2,6,4)$, $x^3 = e$, $x^4 = (1,3,5)(2,4,6)$, and $x^5 = (1,5,3)(2,6,4)$. Also, let $y = (1,2,6)(3,4,5)$ and $y^2 = (1,6,2)(3,5,4)$. We expand our relation as follows:

$$\begin{aligned}(xt_1^{y^2}t_1)^5 &= (xt_1^{(1,6,2)(3,5,4)}t_1)^5 \\ &= (xt_6t_1)^5 \\ &= xt_6t_1 \cdot xt_6t_1 \cdot xt_6t_1 \cdot xt_6t_1 \cdot xt_6t_1 \qquad (4.32) \\ &= x^5(t_6t_1)^{x^4}(t_6t_1)^{x^3}(t_6t_1)^{x^2}(t_6t_1)^{x}t_6t_1 \\ &= (1,5,3)(2,6,4)t_2t_3t_6t_1t_4t_5t_2t_3t_6t_1\end{aligned}$$

Our relation is $(1,5,3)(2,6,4)t_2t_3t_6t_1t_4t_5t_2t_3t_6t_1 = e$ which can also be written as $(1,5,3)(2,6,4)t_2t_3t_6t_1t_4 = t_1t_6t_3t_2t_5$

We will also use the relation $(1,6,2)(3,5,4)t_5t_6t_2 = t_1t_2t_4$. We can also conjugate this relation by all the elements of N to obtain eleven new elements.

$$(1,6,2)(3,5,4)t_5t_6t_2^{(1,5,3)(2,6,4)} = t_1t_2t_4^{(1,5,3)(2,6,4)} \Rightarrow (1,3,2)(4,6,5)t_3t_4t_6 = t_5t_6t_2$$
$$(1,6,2)(3,5,4)t_5t_6t_2^{(1,5,6)(2,3,4)} = t_1t_2t_4^{(1,5,6)(2,3,4)} \Rightarrow (1,3,5)(2,4,6)t_6t_1t_3 = t_5t_3t_2$$
$$(1,6,2)(3,5,4)t_5t_6t_2^{(1,3,2)(4,6,5)} = t_1t_2t_4^{(1,3,2)(4,6,5)} \Rightarrow (1,3,5)(2,4,6)t_4t_5t_1 = t_3t_1t_6$$
$$(1,6,2)(3,5,4)t_5t_6t_2^{(1,2,6)(3,4,5)} = t_1t_2t_4^{(1,2,6)(3,4,5)} \Rightarrow (1,6,2)(3,5,4)t_3t_1t_6 = t_2t_6t_5$$
$$(1,6,2)(3,5,4)t_5t_6t_2^{(1,6,2)(3,5,4)} = t_1t_2t_4^{(1,6,2)(3,5,4)} \Rightarrow (1,6,2)(3,5,4)t_4t_2t_1 = t_6t_1t_3$$
$$(1,6,2)(3,5,4)t_5t_6t_2^{(1,6,5)(2,4,3)} = t_1t_2t_4^{(1,6,5)(2,4,3)} \Rightarrow (1,3,2)(4,6,5)t_1t_5t_4 = t_6t_4t_3 \quad (4.33)$$
$$(1,6,2)(3,5,4)t_5t_6t_2^{(1,4)(2,5)} = t_1t_2t_4^{(1,4)(2,5)} \Rightarrow (1,3,2)(4,6,5)t_2t_6t_5 = t_4t_5t_1$$
$$(1,6,2)(3,5,4)t_5t_6t_2^{(1,4)(3,6)} = t_1t_2t_4^{(1,4)(3,6)} \Rightarrow (1,6,5)(2,4,3)t_5t_3t_2 = t_4t_2t_1$$
$$(1,6,2)(3,5,4)t_5t_6t_2^{(1,3,5)(2,4,6)} = t_1t_2t_4^{(1,3,5)(2,4,6)} \Rightarrow (1,6,5)(2,4,3)t_1t_2t_4 = t_3t_4t_6$$
$$(1,6,2)(3,5,4)t_5t_6t_2^{(1,2,3)(4,5,6)} = t_1t_2t_4^{(1,2,3)(4,5,6)} \Rightarrow (1,6,5)(2,4,3)t_6t_4t_3 = t_2t_3t_5$$
$$(1,6,2)(3,5,4)t_5t_6t_2^{(2,5)(3,6)} = t_1t_2t_4^{(2,5)(3,6)} \Rightarrow (1,3,5)(2,4,6)t_2t_3t_5 = t_1t_5t_4$$

First Double Coset[$*$]

Let [$*$] represent the double coset [$*$] = $\{NeN = N(e)^n \mid n \in N = N\}$. The coset stabilizer of Ne = N. The number of single cosets in [*] is equal to $\frac{|N|}{|N|} = \frac{12}{12} = 1$. The orbits of N on $\{1,2,3,4,5,6\}$ is $\{1,2,3,4,5,6\}$, that is, there is one single orbit. Now select a representative from the single orbit, say 1, and find the double coset that contains Nt_1. We determine that Nt_1 belongs to a new double coset Nt_1N denoted by [1]. There are 6 elements in the orbit $\{1,2,3,4,5,6\}$, therefore, all 6 symmetric generators will move forward.

Figure 4.7: Cayley Graph PSL(2,11) over D_6

Second Double Coset[1]

$$
\begin{aligned}
Nt_1N &= \{N(t_1)^n \mid n \in N\} \\
&= \{Nt_1^e, Nt_1^{(1,5,3)(2,6,4)}, Nt_1^{(1,5,6)(2,3,4)}), Nt_1^{(1,3,2)(4,6,5)}, Nt_1^{(1,2,6)(3,4,5)}, \\
&Nt_1^{(1,6,2)(3,5,4)}, Nt_1^{(1,6,5)(2,4,3)}, Nt_1^{(1,4)(2,5)}, Nt_1^{(1,4)(3,6)}, Nt_1^{(1,3,5)(2,4,6)}, \\
&Nt_1^{(1,2,3)(4,5,6)}, Nt_1^{(2,5)(3,6)}\} \\
&= \{Nt_1, Nt_2, Nt_3, Nt_4, Nt_5, Nt_6\}
\end{aligned} \tag{4.34}
$$

The point stabilizer of $N^1 = \{(2,5)(3,6)\}$. Similarly, the coset stabilizer $N^{(1)} = \{(2,5)(3,6)\}$. The number of single right cosets in $N^{(1)} = \frac{|N|}{|N^{(1)}|} = \frac{12}{2} = 6$.The orbits of $N^{(1)}$ on $X = \{1,2,3,4,5,6\}$ are $\{1\}, \{2,5\}, \{3,6\}$ and $\{4\}$. Now we select a representative from each orbit, say $1 \in \{1\}, 2 \in \{2,5\}, 3 \in \{3,6\}$ and $4 \in \{4\}$ and determine the double coset it belongs to.

$Nt_1t_1 \in [*]$ since $t_1t_1 = t_1^2 = e$. Thus, one symmetric generator will move forward.

$Nt_1t_2 \in [12]$ which is a new double coset. Thus, two symmetric generators will move forward.

$Nt_1t_3 \in [13]$ which is a new double coset. Thus, two symmetric generators will move forward.

$Nt_1t_4 \in [1]$. In order to prove that $Nt_1t_4 \in [1]$, we conjugate our original relation $(1,4)(2,5)t_2t_5 = t_2$ by $(1,6,5)(2,4,3)$ to obtain $(1,4)(3,6)t_1t_4 = t_1$.

$$
\begin{aligned}
t_1t_4 &= \underline{t_1}t_4 \\
&= (1,4)(3,6)t_1t_4t_4 \\
&= (1,4)(3,6)t_1t_4^2 \\
&= (1,4)(3,6)t_1 \text{ since } t_4^2 = e \\
\Rightarrow Nt_1t_4 &= Nt_1 \in [1]
\end{aligned}
$$

Thus, one symmtric generator loops back to [1].

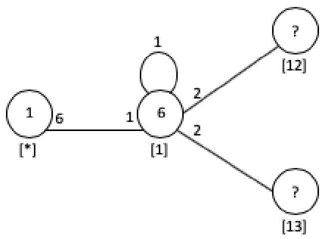

Figure 4.8: Cayley Graph PSL(2,11) over D_6

Third Double Coset[12]

$$
\begin{aligned}
Nt_1t_2N &= \{N(t_1t_2)^n \mid n \in N\}\\
&= \{N(t_1t_2)^e, N(t_1t_2)^{(1,5,3)(2,6,4)}, N(t_1t_2)^{(1,5,6)(2,3,4)}, N(t_1t_2)^{(1,3,2)(4,6,5)},\\
&N(t_1t_2)^{(1,2,6)(3,4,5)}, N(t_1t_2)^{(1,6,2)(3,5,4)}, N(t_1t_2)^{(1,6,5)(2,4,3)}, N(t_1t_2)^{(1,4)(2,5)},\\
&N(t_1t_2)^{(1,4)(3,6)}, N(t_1t_2)^{(1,3,5)(2,4,6)}, N(t_1t_2)^{(1,2,3)(4,5,6)}, N(t_1t_2)^{(2,5)(3,6)}\}\\
&= \{Nt_1t_2, Nt_5t_6, Nt_5t_3, Nt_3t_1, Nt_2t_6, Nt_6t_1, Nt_6t_4, Nt_4t_5, Nt_4t_2, Nt_3t_4, Nt_2t_3,\\
&Nt_1t_5\}
\end{aligned}
\tag{4.35}
$$

The point stabilizer of $N^{12} = \{e\}$. Similarly, the coset stabilizer $N^{(12)} = \{e\}$. The number of single right cosets in $N^{(12)} = \frac{|N|}{|N^{(12)}|} = \frac{12}{1} = 12$.The orbits of $N^{(12)}$ on $X = \{1,2,3,4,5,6\}$ are $\{1\},\{2\},\{3\},\{4\},\{5\}$ and $\{6\}$. Now we select a representative from each orbit, say $1 \in \{1\}, 2 \in \{2\}, 3 \in \{3\}$, $4 \in \{4\}, 5 \in \{5\}$ and $6 \in \{6\}$ and determine the double coset it belongs to.

$Nt_1t_2t_1 \in [121]$, which is a new double coset. Thus one symmetric generator moves forward.

$Nt_1t_2t_2 \in [1]$ since, $t_2t_2 = t_2^2 = e$ and $Nt_1e = Nt_1 \in [1]$. Thus one symmetric generator goes back to [1].

$Nt_1t_2t_3 \in [123]$ which is a new double coset. Thus, one symmetric generator moves forward.

$Nt_1t_2t_4 \in [124]$ which is a new double coset. Thus, one symmetric generator moves forward.

$Nt_1t_2t_5 \in [12]$. In order to prove that $Nt_1t_2t_5 \in [12]$, we use our original relation $(14)(25)t_2t_5 = t_2$.

$$t_1\underline{t_2t_5} = t_1(14)(25)t_2$$
$$= (14)(25)t_4t_2$$
$$\Rightarrow Nt_1t_2t_5 = Nt_4t_2 = N(t_1t_2)^{(14)(36)} \in [12].$$

$Nt_1t_2t_6 \in [13]$ which is a new double coset.

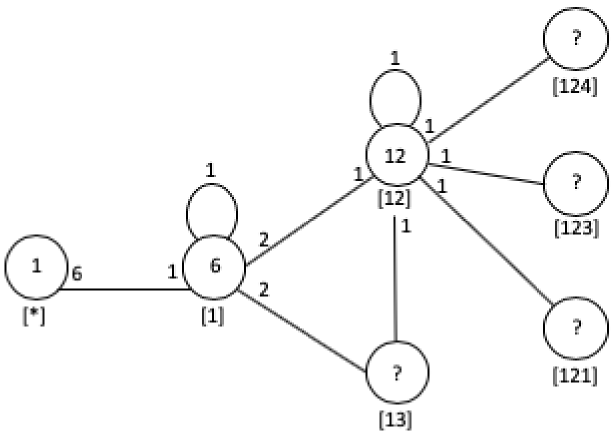

Figure 4.9: Cayley Graph PSL(2,11) over D_6

<u>Fourth Double Coset [13]</u>

$$
\begin{aligned}
Nt_1t_3N &= \{N(t_1t_3)^n \mid n \in N\} \\
&= \{N(t_1t_3)^e, N(t_1t_3)^{(1,5,3)(2,6,4)}, N(t_1t_3)^{(1,5,6)(2,3,4)}, N(t_1t_3)^{(1,3,2)(4,6,5)}, \\
&N(t_1t_3)^{(1,2,6)(3,4,5)}, N(t_1t_3)^{(1,6,2)(3,5,4)}, N(t_1t_3)^{(1,6,5)(2,4,3)}, N(t_1t_3)^{(1,4)(2,5)}, \\
&N(t_1t_3)^{(1,4)(3,6)}, N(t_1t_3)^{(1,3,5)(2,4,6)}, N(t_1t_3)^{(1,2,3)(4,5,6)}, N(t_1t_3)^{(2,5)(3,6)}\} \\
&= \{Nt_1t_3, Nt_5t_1, Nt_5t_4, Nt_3t_2, Nt_2t_4, Nt_6t_5Nt_6t_2, Nt_4t_3, Nt_4t_6, Nt_3t_5, \\
&Nt_2t_1, Nt_1t_6\}
\end{aligned}
\tag{4.36}
$$

The point stabilizer of $N^{13} = \{e\}$. Similarly, the coset stabilizer $N^{(13)} = \{e\}$. The number of single right cosets in $N^{(13)} = \frac{|N|}{|N^{(13)}|} = \frac{12}{1} = 12$. The orbits of $N^{(13)}$ on $X = \{1,2,3,4,5,6\}$ are $\{1\},\{2\},\{3\},\{4\},\{5\},\{5\}$. Now we select a representative from each orbit, say $1 \in \{1\}, 2 \in \{2\}, 3 \in \{3\}$ and $4 \in \{4\}$, $5 \in \{5\}$ and $6 \in \{6\}$ and determine the double coset it belongs to.

$Nt_1t_3t_1 \in [131]$, which is a new double coset. Thus, one symmetric generator moves forward.

$Nt_1t_3t_2 \in [123]$. In order to show this, we will use the relation $(135)(462)t_6t_1t_3 = t_5t_3t_2$ obtained by conjugating the relation $(162)(354)t_5t_6t_2 = t_1t_2t_4$ by $(156)(234)$. In addition, we will use the relation $(135)(246)t_4t_5t_1 = t_3t_1t_6$ obtained by conjugating the relation $(162)(354)t_5t_6t_2 = t_1t_2t_4$ by $(132)(465)$.

$$
\begin{aligned}
t_1 t_3 t_2 &= t_1 t_5 \underline{t_5 t_3 t_2} \\
&= t_1 t_5 (135)(246) t_6 t_1 t_3 \\
&= (135)(246) t_3 t_1 t_6 t_1 t_3 \\
&= (135)(246)(135)(246) t_4 t_5 t_1 t_1 t_3 \\
&= (153)(264) t_4 t_5 t_3 \\
\Rightarrow N t_1 t_3 t_2 &= N t_4 t_5 t_3 = N(t_1 t_2 t_3)^{(14)(25)} \in [123]
\end{aligned} \tag{4.37}
$$

$N t_1 t_3 t_3 \in [1]$. Since $t_3 t_3 = t_3^2 = e$. Therefore $N t_1 t_3 t_3 = N t_1 t_3^2 = N t_1$. Thus, one symmetric generator moves back.

We also determine

$N t_1 t_3 t_4 \in [121]$. Thus one symmetric generator goes to [121].

$N t_1 t_3 t_5 \in [12]$. Thus, one symmetric generator goes to [12].

$N t_1 t_3 t_6 \in [13]$. In order to show this, we conjugate our original relation $(14)(25) t_2 t_5 = t_2$ by (123)(456) to obtain,

$$
\begin{aligned}
(14)(25) t_2 t_5^{(123)(456)} &= t_2^{(123)(456)} \\
(25)(36) t_3 t_6 &= t_3.
\end{aligned} \tag{4.38}
$$

Now we have,

$$
\begin{aligned}
t_1 \underline{t_3 t_6} &= t_1 (25)(36) t_3 \\
&= (25)(36) t_1 t_3 \\
N t_1 t_3 t_6 &= N t_1 t_3 \in [13]
\end{aligned} \tag{4.39}
$$

Thus, one symmetric generator goes back to [13].

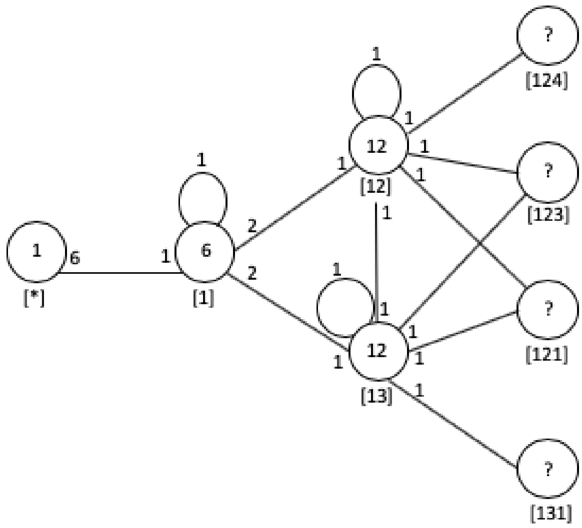

Figure 4.10: Cayley Graph PSL(2,11) over D_6

<u>Fifth Double Coset[124]</u>

$$
\begin{aligned}
Nt_1t_2t_4N &= \{N(t_1t_2t_4)^n \mid n \in N\} \\
&= \{N(t_1t_2t_4)^e, N(t_1t_2t_4)^{(1,5,3)(2,6,4)}, N(t_1t_2t_4)^{(1,5,6)(2,3,4)}, N(t_1t_2t_4)^{(1,3,2)(4,6,5)}, \\
&\quad N(t_1t_2t_4)^{(1,2,6)(3,4,5)}, N(t_1t_2t_4)^{(1,6,2)(3,5,4)}, N(t_1t_2t_4)^{(1,6,5)(2,4,3)}, N(t_1t_2t_4)^{(1,4)(2,5)}, \\
&\quad N(t_1t_2t_4)^{(1,4)(3,6)}, N(t_1t_2t_4)^{(1,3,5)(2,4,6)}, N(t_1t_2t_4)^{(1,2,3)(4,5,6)}, N(t_1t_2t_4)^{(2,5)(3,6)}\} \\
&= \{Nt_1t_2t_4, Nt_5t_6t_2, Nt_5t_3t_2, Nt_3t_1t_6, Nt_2t_6t_5, Nt_6t_1t_3, Nt_6t_4t_3, Nt_4t_5t_1, Nt_4t_2t_1, \\
&\quad Nt_3t_4t_6, Nt_2t_3t_5, Nt_4t_5t_1\}
\end{aligned}
\tag{4.40}
$$

The point stabilizer of $N^{124} = \{e\}$. However, since $Nt_1t_2t_4 = Nt_3t_4t_6$, the coset stabilizer $N^{(124)} = \{e, (1,5,3)(2,6,4), (1,3,5)(2,4,6)\}$. The number of single right cosets in

$N^{(124)} = \frac{|N|}{|N^{(124)}|} = \frac{12}{3} = 4$. The orbits of $N^{(124)}$ on $X = \{1,2,3,4,5,6\}$ are $\{1,3,5\},\{2,4,6\}$. Now we select a representative from each orbit, say $3 \in \{1,3,5\}, 2 \in \{2,4,6\}$ and determine the double coset it belongs to.

$Nt_1t_2t_4t_3 \in [121]$. In order to show this, we will use our relation, $(162)(354)t_5t_6t_2 = t_1t_2t_4$.

$$\begin{aligned}
\underline{t_1t_2t_4}t_3 &= (162)(354)t_5t_6t_2\underline{t_3} \\
&= (162)(354)t_5t_6t_2(25)(36)t_3t_6 \\
&= (162)(354)(25)(36)t_2t_3t_5t_3t_6 \\
&= (132)(465)\underline{t_2t_3t_5}t_3t_6 \\
&= (132)(465)(153)(264)t_1t_5t_4t_3t_6 \\
&= (25)(36)t_1t_5t_4t_3t_6 \\
&= (25)(36)(123)(456)t_6t_4t_3t_3t_6 \\
&= (126)(345)t_6t_4t_6
\end{aligned} \tag{4.41}$$

Thus 3 symmetric generator go back to [121].

$Nt_1t_2t_4t_2 \in [12]$. Thus 3 symmetric generators go back to [12].

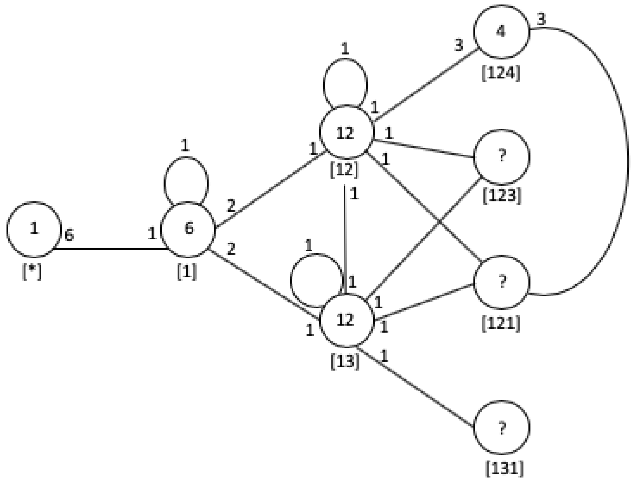

Figure 4.11: Cayley Graph PSL(2,11) over D_6

Sixth Double Coset[123]

$$
\begin{aligned}
Nt_1t_2t_3N &= \{N(t_1t_2t_3)^n \mid n \in N\} \\
&= \{N(t_1t_2t_3)^e, N(t_1t_2t_3)^{(1,5,3)(2,6,4)}, N(t_1t_2t_3)^{(1,5,6)(2,3,4)}, N(t_1t_2t_3)^{(1,3,2)(4,6,5)}, \\
&N(t_1t_2t_3)^{(1,2,6)(3,4,5)}, N(t_1t_2t_3)^{(1,6,2)(3,5,4)}, N(t_1t_2t_3)^{(1,6,5)(2,4,3)}, N(t_1t_2t_3)^{(1,4)(2,5)}, \\
&N(t_1t_2t_3)^{(1,4)(3,6)}, N(t_1t_2t_3)^{(1,3,5)(2,4,6)}, N(t_1t_2t_3)^{(1,2,3)(4,5,6)}, N(t_1t_2t_3)^{(2,5)(3,6)}\} \\
&= \{Nt_1t_2t_3, Nt_5t_6t_1, Nt_5t_3t_4, Nt_3t_1t_2, Nt_2t_6t_4, Nt_6t_1t_5, Nt_6t_4t_2, Nt_4t_5t_3, Nt_4t_2t_6, \\
&Nt_3t_4t_5, Nt_2t_3t_1, Nt_1t_5t_6\}
\end{aligned}
\tag{4.42}
$$

The point stabilizer of $N^{123} = \{e\}$. However, since $Nt_1t_2t_3 = Nt_3t_1t_2$, the coset stabilizer $N^{(123)} = \{e, (1,3,2)(4,6,5), (1,2,3)(4,5,6)\}$. The number of single right cosets in $N^{(123)} = \frac{|N|}{|N^{(123)}|} = \frac{12}{3} = 4$. The orbits of $N^{(123)}$ on $X = \{1,2,3,4,5,6\}$ are $\{1,2,3\}, \{4,5,6\}$. Now we select a representative from each orbit, say $1 \in \{1,2,3\}, 4 \in \{4,5,6\}$ and determine the double coset it belongs to.

$Nt_1t_2t_3t_1 \in [12]$,

$Nt_1t_2t_4t_4 \in [13]$.

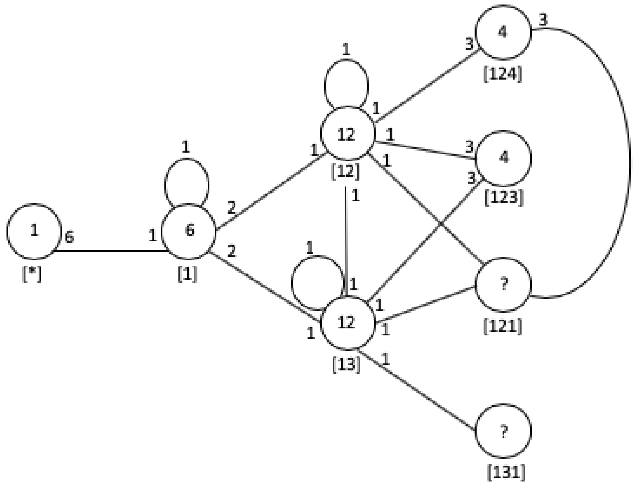

Figure 4.12: Cayley Graph PSL(2,11) over D_6

Seventh Double Coset[121]

$$Nt_1t_2t_1N = \{N(t_1t_2t_1)^n \mid n \in N\}$$
$$= \{N(t_1t_2t_1)^e, N(t_1t_2t_1)^{(1,5,3)(2,6,4)}, N(t_1t_2t_1)^{(1,5,6)(2,3,4)}, N(t_1t_2t_1)^{(1,3,2)(4,6,5)},$$
$$N(t_1t_2t_1)^{(1,2,6)(3,4,5)}, N(t_1t_2t_1)^{(1,6,2)(3,5,4)}, N(t_1t_2t_1)^{(1,6,5)(2,4,3)}, N(t_1t_2t_1)^{(1,4)(2,5)},$$
$$N(t_1t_2t_1)^{(1,4)(3,6)}, N(t_1t_2t_1)^{(1,3,5)(2,4,6)}, N(t_1t_2t_1)^{(1,2,3)(4,5,6)}, N(t_1t_2t_1)^{(2,5)(3,6)}\}$$
$$= \{Nt_1t_2t_1, Nt_5t_6t_5, Nt_5t_3t_5, Nt_3t_1t_3, Nt_2t_6t_2, Nt_6t_1t_6,$$
$$Nt_6t_4t_6, Nt_4t_5t_4, Nt_4t_2t_4, Nt_3t_4t_3, Nt_2t_3t_2, Nt_1t_5t_1\} \quad (4.43)$$

The point stabilizer of $N^{121} = \{e\}$. Similarly, the coset stabilizer $N^{(121)} = \{e\}$. The number of single right cosets in $N^{(121)} = \frac{|N|}{|N^{(12)}|} = \frac{12}{1} = 12$.The orbits of $N^{(121)}$ on $X = \{1,2,3,4,5,6\}$ are $\{1\},\{2\},\{3\},\{4\},\{5\}$ and $\{6\}$. Now we select a representative from each orbit, say $1 \in \{1\}, 2 \in \{2\}, 3 \in \{3\}$, $4 \in \{4\}, 5 \in \{5\}$ and $6 \in \{6\}$ and determine the double coset it belongs to.

We determine that,

$Nt_1t_2t_1t_1 \in [12]$, thus one symmetric generator goes to [121].

$Nt_1t_2t_1t_2 \in [121]$, thus one symmetric generator goes back to [121].

$Nt_1t_2t_1t_3 \in [13]$, thus one symmetric generator goes to [13].

$Nt_1t_2t_1t_4 \in [124]$, thus one symmetric generator goes to [124].

$Nt_1t_2t_1t_5 \in [121]$, thus one symmtric generator goes to [121].

$Nt_1t_2t_1t_6 \in [131]$, thus one symmetric generator goes to [131].

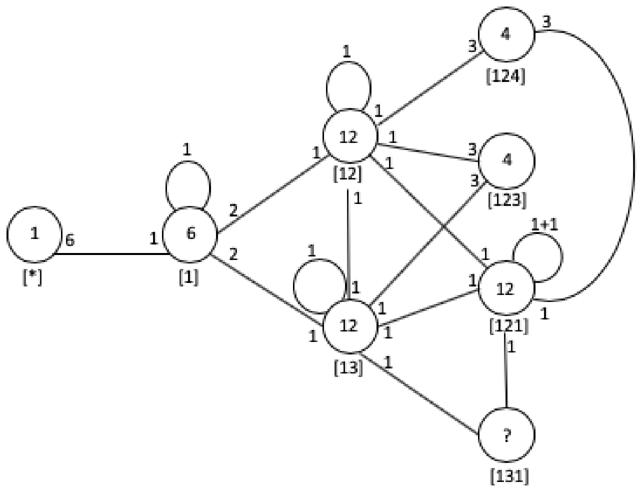

Figure 4.13: Cayley Graph PSL(2,11) over D_6

<u>Eighth Double Coset [131]</u>

$$\begin{aligned}
Nt_1t_3t_1N &= \{N(t_1t_3t_1)^n \mid n \in N\} \\
&= \{N(t_1t_3t_1)^e, N(t_1t_3t_1)^{(1,5,3)(2,6,4)}, N(t_1t_3t_1)^{(1,5,6)(2,3,4)}, N(t_1t_3t_1)^{(1,3,2)(4,6,5)}, \\
&N(t_1t_3t_1)^{(1,2,6)(3,4,5)}, N(t_1t_3t_1)^{(1,6,2)(3,5,4)}, N(t_1t_3t_1)^{(1,6,5)(2,4,3)}, N(t_1t_3t_1)^{(1,4)(2,5)}, \\
&N(t_1t_3t_1)^{(1,4)(3,6)}, N(t_1t_3t_1)^{(1,3,5)(2,4,6)}, N(t_1t_3t_1)^{(1,2,3)(4,5,6)}, N(t_1t_3t_1)^{(2,5)(3,6)}\} \\
&= \{Nt_1t_3t_1, Nt_5t_1t_5, Nt_5t_4t_5, Nt_3t_2t_3, Nt_2t_4t_2, Nt_6t_5t_6Nt_6t_2t_6, Nt_4t_3t_4, \\
&Nt_4t_6t_4, Nt_3t_5t_3, Nt_2t_1t_2, Nt_1t_6t_1\}
\end{aligned} \tag{4.44}$$

The point stabilizer of $N^{131} = \{e\}$. However, since $Nt_1t_3t_1 = Nt_2t_4t_2$, the coset stabilizer $N^{(131)} = \{e,(1,2,6)(2,3,5),(1,6,2)(3,5,4)\}$. The number of single right cosets in $N^{(131)} = \frac{|N|}{|N^{(131)}|} = \frac{12}{3} = 4$. The orbits of $N^{(131)}$ on $X = \{1,2,3,4,5,6\}$ are $\{1,2,6\},\{3,4,5\}$. Now we select a representative from each orbit, say $1 \in \{1,2,6\}, 3 \in \{3,4,5\}$ and determine the double coset it belongs to.

We determine

$Nt_1t_3t_1t_1 \in [13]$. Thus, three symmetric generatos go back to [13].

$Nt_1t_3t_1t_3 \in [121]$. Thus, three symmetric generators go to [121].

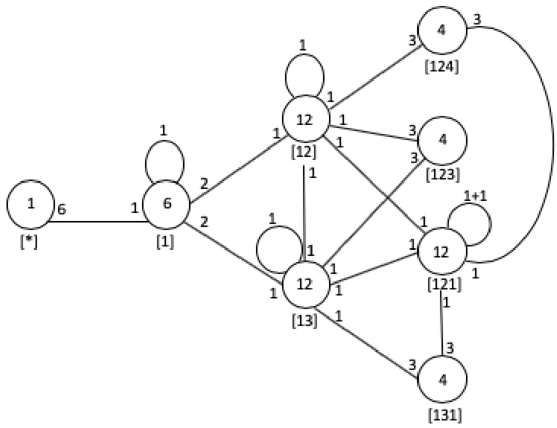

Figure 4.14: Cayley Graph PSL(2,11) over D_6

4.4 Double Coset Enumeration of $3^{*2} :_m D_4$

$$G = \frac{3^{*2}:_m D_4}{(x^2 y t t^x)^4}$$

The elements of N are $\{e, (1432), (1234), (13)(24), (12)(34), (14)(23), (13), (24)\}$, and the order of $|N| = 8$. Suppose $x = (1,2,3,4)$, $y = (2,4)$ and $t = t_1$.

Before we begin our double coset enumeration, lets expand our relation $(x^2 y t t^x)^4$.

$$\begin{aligned}
(x^2 y t t^x)^4 &= ((1,2,3,4)^2 (2,4)\, t_1\, t_1^{(1,2,3,4)})^4 \\
&= ((1,3)(2,4)(2,4)\, t_1\, t_2)^4 \\
&= ((1,3)\, t_1\, t_2)^4 \\
&= (1,3)\, t_1\, t_2 (1,3)\, t_1\, t_2 (1,3)\, t_1\, t_2 (1,3)\, t_1\, t_2 \\
&= (1,3)^4 (t_1\, t_2)^{(1,3)^3} (t_1\, t_2)^{(1,3)^2} (t_1\, t_2)^{(1,3)}\, t_1\, t_2 \\
&= t_3\, t_2\, t_1\, t_2\, t_3\, t_2\, t_1\, t_2
\end{aligned} \tag{4.45}$$

After expanding our relation we see that it is $t_3 t_2 t_1 t_2 t_3 t_2 t_1 t_2 = e$. We can simplify our relation to get $t_3 t_2 t_1 t_2 = t_4 t_3 t_4 t_1$

<u>Relations</u>

We conjugate our relation $t_3 t_2 t_1 t_2 = t_4 t_3 t_4 t_1$, by the elements of N to find new relations.

$$\begin{aligned}
t_3t_2t_1t_2^{(1,4,3,2)} = t_4t_3t_4t_1^{(1,4,3,2)} &\Rightarrow t_2t_1t_4t_1 = t_3t_2t_3t_4\\
t_3t_2t_1t_2^{(1,2,3,4)} = t_4t_3t_4t_1^{(1,2,3,4)} &\Rightarrow t_4t_3t_2t_3 = t_1t_4t_1t_2\\
t_3t_2t_1t_2^{(1,3)(2,4)} = t_4t_3t_4t_1^{(1,3)(2,4)} &\Rightarrow t_1t_4t_3t_4 = t_2t_1t_2t_3\\
t_3t_2t_1t_2^{(1,2)(3,4)} = t_4t_3t_4t_1^{(1,2)(3,4)} &\Rightarrow t_4t_1t_2t_1 = t_3t_4t_3t_2\\
t_3t_2t_1t_2^{(1,4)(2,3)} = t_4t_3t_4t_1^{(1,4)(2,3)} &\Rightarrow t_2t_3t_4t_3 = t_1t_2t_1t_4\\
t_3t_2t_1t_2^{(1,3)} = t_4t_3t_4t_1^{(1,3)} &\Rightarrow t_1t_2t_3t_2 = t_4t_1t_4t_3\\
t_3t_2t_1t_2^{(2,4)} = t_4t_3t_4t_1^{(2,4)} &\Rightarrow t_3t_4t_1t_4 = t_2t_3t_2t_1
\end{aligned} \tag{4.46}$$

We will use our technique of double coset enumeration to show that $|G| \leq 480$

Labeling			
1	2	3	4
t_1	t_2	t_1^2	t_2^2

First Double Coset

$[*] = \{NeN = N(e)^n \mid n \in N = N\}$. The coset stabilizer of Ne = N. The number of single cosets in [*] is equal to $\frac{|N|}{|N|} = \frac{8}{8} = 1$. The orbits of N on $\{1,2,3,4\}$ is $\{1,2,3,4\}$, that is, there is one single orbit. Now select a representative from the single orbit, say 1, and find the double coset that contains Nt_1. We determine that Nt_1 belongs to a new double coset Nt_1N denoted by [1]. There are 4 elements in the orbit $\{1,2,3,4\}$, therefore, all 4 symmetric generators will move forward.

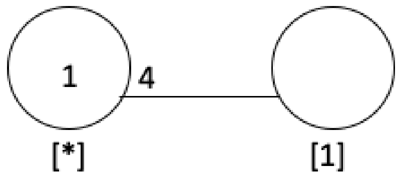

Figure 4.15: Cayley Graph of $3^{*2} :_m D_4$

<u>Second Double Coset[1]</u>

$Nt_1N = \{N(t_1)^n \mid n \in N\} = \{Nt_1, Nt_2, Nt_3, Nt_4\}$. The point stabilizer of $N^1 = \{e, (2,4)\}$. Similarly, the coset stabilizer $N^{(1)} = \{e, (2,4)\}$. The number of single right cosets in $N^{(1)} = \frac{|N|}{|N^{(1)}|} = \frac{8}{2} = 4$. The orbits of $N^{(1)}$ on $X = \{1,2,3,4\}$ are $\{1\}, \{2,4\}, \{3\}$. Now we select a representative from each orbit, say $1 \in \{1\}, 2 \in \{2,4\}$, and $3 \in \{3\}$ and determine the double coset it belongs to.

$Nt_1t_1 = Nt_1^2 = Nt_3 \in [1]$, so one symmetric generator loops back to [1].

$Nt_1t_2 \in [12]$, which is a new double coset. This tells us two symmetric generators move forward.

$Nt_1t_3 = Nt_1t_1^2 = Nt_1^3 \in [*]$, since $Nt_1^3 = Ne$. Therefore one symmetric generator goes back to [*].

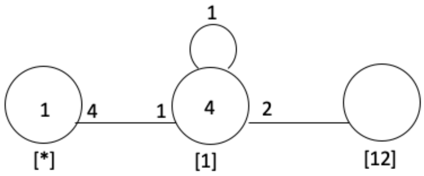

Figure 4.16: Cayley Graph of $3^{*2} :_m D_4$

Third Double Coset[12]

$$
\begin{aligned}
Nt_1t_2N &= \{N(t_1t_2)^n \mid n \in N\} \\
&= \{N(t_1t_2)^e, N(t_1t_2)^{(1,4,3,2)}, N(t_1t_2)^{(1,2,3,4)}, N(t_1t_2)^{(1,2)(3,4)}, N(t_1t_2)^{(1,4)(2,3)}, \\
&N(t_1t_2)^{(1,3)(2,4)}, N(t_1t_2)^{(1,3)}, N(t_1t_2)^{(2,4)}\} \\
&= \{Nt_1t_2, Nt_4t_2, Nt_2t_3, Nt_2t_1, Nt_4t_3, Nt_3t_4, Nt_3t_2, Nt_1t_4\}
\end{aligned}
\tag{4.47}
$$

The point stabilizer of $N^{12} = \{e\}$. Similarly, the coset stabilizer $N^{(12)} = \{e\}$. The number of single right cosets in $N^{(12)} = \frac{|N|}{|N^{(12)}|} = \frac{8}{1} = 8$. The orbits of $N^{(12)}$ on $X = \{1,2,3,4\}$ are $\{1\},\{2\},\{3\},\{4\}$. Now we select a representative from each orbit, say $1 \in \{1\}, 2 \in \{2\}, 3 \in \{3\}$ and $4 \in \{4\}$ and determine the double coset it belongs to.

$Nt_1t_2t_1 \in [121]$, which is a new double coset. This tells us one symmetric generator moves forward.

$Nt_1t_2t_2 \in [12]$. This result is obtained by evaluating our t's. $Nt_2t_2 = Nt_2^2$. From our labeling we know that $Nt_2^2 = Nt_4$. We replace Nt_2t_2 with Nt_4 to get Nt_1t_4 which is in the double coset [12]. Therefore, one symmetric generator loops back to [12].

$Nt_1t_2t_3 \in [123]$, which is a new double coset. Thus one symmetric generator moves forward.

$Nt_1t_2t_4 \in [1]$. Once again we must evaluate our t's. From our labeling $Nt_4 = Nt_2^2$. We replace Nt_4 and obtain $Nt_1t_2t_2^2$ which can be simplified to $Nt_1t_2^3$. Since our t's are of three, $t_2^3 = e$. Thus $Nt_1t_2t_2^2 = Nt_1 \in [1]$ and one symmetric generator goes back to [1].

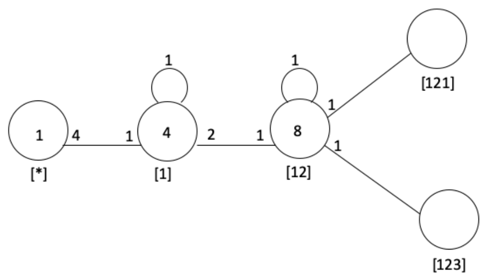

Figure 4.17: Cayley Graph of $3^{*2} :_m D_4$

<u>4th Double Coset[121]</u>

$$\begin{aligned} Nt_1t_2t_1N &= \{N(t_1t_2t_1)^n \mid n \in N\} \\ &= \{N(t_1t_2t_1)^e, N(t_1t_2t_1)^{(1,4,3,2)}, N(t_1t_2t_1)^{(1,2,3,4)}, N(t_1t_2t_1)^{(1,2)(3,4)}, \\ &N(t_1t_2t_1)^{(1,4)(2,3)}, N(t_1t_2t_1)^{(1,3)(2,4)}, N(t_1t_2t_1)^{(1,3)}, N(t_1t_2t_1)^{(2,4)}\} \\ &= \{Nt_1t_2t_1, Nt_4t_1t_4, Nt_2t_3t_2, Nt_2t_1t_2, Nt_4t_3t_4, Nt_3t_4t_3, Nt_3t_2t_3, Nt_1t_4t_1\} \end{aligned} \tag{4.48}$$

The point stabilizer of $N^{121} = \{e\}$. Similarly, the coset stabilizer $N^{(121)} = \{e\}$. The number of single right cosets in $N^{(121)} = \frac{|N|}{|N^{(121)}|} = \frac{8}{1} = 8$. The orbits of $N^{(121)}$ on $X = \{1,2,3,4\}$ are $\{1\},\{2\},\{3\},\{4\}$. Now we select a representative from each orbit, say $1 \in \{1\}, 2 \in \{2\}, 3 \in \{3\}$ and $4 \in \{4\}$ and determine the double coset it belongs to.

$Nt_1t_2t_1t_1 = Nt_1t_2t_1^2$. From our labeling we know $t_1^2 = t_3$, so we replace t_1^2 to get $Nt_1t_2t_3 \in [123]$. This tells us one symmetric generator goes to [123].

$Nt_1t_2t_1t_2 \in [1212]$, which is a new double coset. This tells us one symmetric generator moves forward.

$Nt_1t_2t_1t_3 \in [12]$. We obtain this by evaluating our t's. From our labeling we know $t_3 = t_1^2$. So we replace t_3 to get $Nt_1t_2t_1t_1^2 = Nt_1t_2t_1^3$, which simplifies to Nt_1t_2. Thus one symmetric generator goes back to [12].

$Nt_1t_2t_1t_4 \in [1214]$, which is a new double coset. This tells us one symmetric generator moves forward.

<u>5th Double Coset[123]</u>

$$\begin{aligned}
Nt_1t_2t_3N = \{&N(t_1t_2t_3)^n \mid n \in N\}\\
&= \{N(t_1t_2t_3)^e, N(t_1t_2t_3)^{(1,4,3,2)}, N(t_1t_2t_3)^{(1,2,3,4)}, N(t_1t_2t_3)^{(1,2)(3,4)},\\
&N(t_1t_2t_3)^{(1,4)(2,3)}, N(t_1t_2t_3)^{(1,3)(2,4)}, N(t_1t_2t_3)^{(1,3)}, N(t_1t_2t_3)^{(2,4)}\}\\
&= \{Nt_1t_2t_3, Nt_4t_1t_2, Nt_2t_3t_4, Nt_2t_1t_4, Nt_4t_3t_2, Nt_3t_4t_1, Nt_3t_2t_1, Nt_1t_4t_3\}
\end{aligned} \tag{4.49}$$

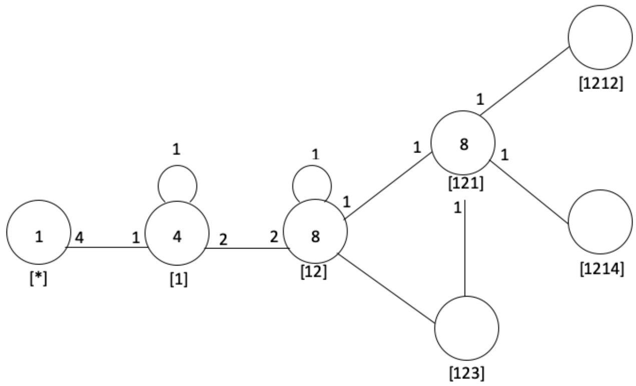

Figure 4.18: Cayley Graph of $3^{*2} :_m D_4$

The point stabilizer of $N^{123} = \{e\}$. Similarly, the coset stabilizer $N^{(123)} = \{e\}$. The number of single right cosets in $N^{(123)} = \frac{|N|}{|N^{(123)}|} = \frac{8}{1} = 8$. The orbits of $N^{(123)}$ on $X = \{1,2,3,4\}$ are $\{1\},\{2\},\{3\},\{4\}$. Now we select a representative from each orbit, say $1 \in \{1\}, 2 \in \{2\}, 3 \in \{3\}$ and $4 \in \{4\}$ and determine the double coset it belongs to.

$Nt_1t_2t_3t_1 \in [12]$. This result is obtained by using our labeling and evaluating our t's. From our labeling we know $t_3 = t_1^2$. If we replace t_3 with t_1^2, we have $Nt_1t_2t_1^2t_1 = Nt_1t_2t_1^3$. This can be simplified to Nt_1t_2, since $t_1^3 = e$. So $Nt_1t_2t_3t_1 = Nt_1t_2$, and one symmetric generator goes by to [12].

$Nt_1t_2t_3t_2 \in [1214]$, which is a new double coset. If we conjugate our original relation $(1,3)t_3t_2t_1t_2 = t_4t_3t_4t_1$ by (13) we get $(31)t_1t_2t_3t_2 = t_4t_1t_4t_3$. And $t_4t_1t_4t_3 \in [1214]$ since $Nt_1t_2t_1t_4^{(1432)} = Nt_4t_1t_4t_3$. Therefore $Nt_1t_2t_3t_2 \in [1214]$ and one symmetric generator moves forward.

$Nt_1t_2t_3t_3 \in [121]$. In order to prove this we will use our labeling.

$$\begin{aligned} Nt_1t_2t_3t_3 &= Nt_1t_2t_1^2t_1^2 \text{ since } t_3 = t_1^2 \\ &= Nt_1t_2t_1^3t_1 \\ &= Nt_1t_2t_1, \text{ since } t_1^3 = e \end{aligned} \tag{4.50}$$

so one symmetric generator goes back to [121]

$Nt_1t_2t_3t_4 \in [1234]$, which is a new double coset. Thus, one symmetric generator moves forward.

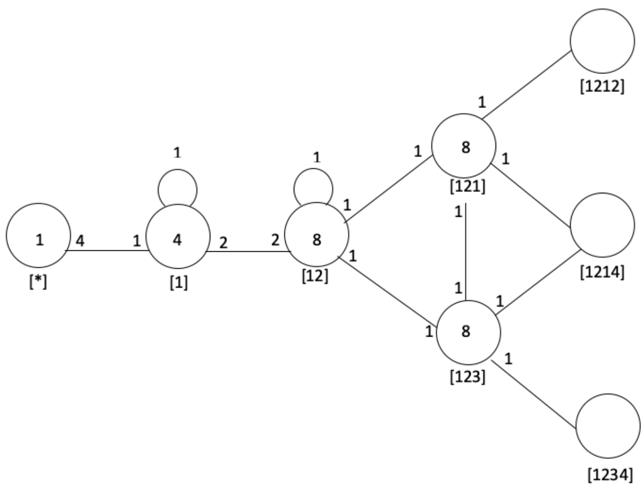

Figure 4.19: Cayley Graph of $3^{*2} :_m D_4$

<u>6th Double Coset[1212]</u>

$$
\begin{aligned}
Nt_1t_2t_1t_2N &= \{N(t_1t_2t_1t_2)^n \mid n \in N\}\\
&= \{N(t_1t_2t_1t_2)^e, N(t_1t_2t_1t_2)^{(1,4,3,2)}, N(t_1t_2t_1t_2)^{(1,2,3,4)}, N(t_1t_2t_1t_2)^{(1,2)(3,4)},\\
&= N(t_1t_2t_1t_2)^{(1,4)(2,3)}, N(t_1t_2t_1t_2)^{(1,3)(2,4)}, N(t_1t_2t_1t_2)^{(1,3)}, N(t_1t_2t_1t_2)^{(2,4)}\}\\
&= \{Nt_1t_2t_1t_2t_1, Nt_4t_1t_4t_1t_4, Nt_2t_3t_2t_3t_2, Nt_2t_1t_2t_1t_2,\\
&\quad Nt_4t_3t_4t_3t_4, Nt_3t_4t_3t_4t_3, Nt_3t_2t_3t_2t_3, Nt_1t_4t_1t_4t_1\}
\end{aligned} \tag{4.51}
$$

The point stabilizer of $N^{1212} = \{e\}$. Similarly, the coset stabilizer $N^{(1212)} = \{e\}$. The number of single right cosets in $N^{(1212)} = \frac{|N|}{|N^{(1212)}|} = \frac{8}{1} = 8$. The orbits of $N^{(1212)}$ on $X = \{1,2,3,4\}$ are $\{1\},\{2\},\{3\},\{4\}$. Now we select a representative from each orbit, say $1 \in \{1\}, 2 \in \{2\}, 3 \in \{3\}$ and $4 \in \{4\}$ and determine the double coset it belongs to.

$Nt_1t_2t_1t_2t_1 \in [12121]$ which is a new double coset. Thus, one symmetric generator moves forward.

$Nt_1t_2t_1t_2t_2 \in [1214]$, since $Nt_1t_2t_1t_2t_2 = Nt_1t_2t_1t_2^2$ and $t_2^2 = t_4$. We substitute t_4 to obtain $Nt_1t_2t_1t_4$. Thus, one symmetric generator goes to [1214].

$Nt_1t_2t_1t_2t_3 \in [1212]$. Thus, one symmetric generator goes to [1212].

$Nt_1t_2t_1t_2t_4 \in [121]$. We use our labeling to establish $Nt_1t_2t_1t_2t_4 = Nt_1t_2t_1t_2t_2^2$. We simplify $Nt_1t_2t_1t_2t_2^2$ to $Nt_1t_2t_1t_2^3 = Nt_1t_2t_1$. Thus, one symmetric generator goes back to [121].

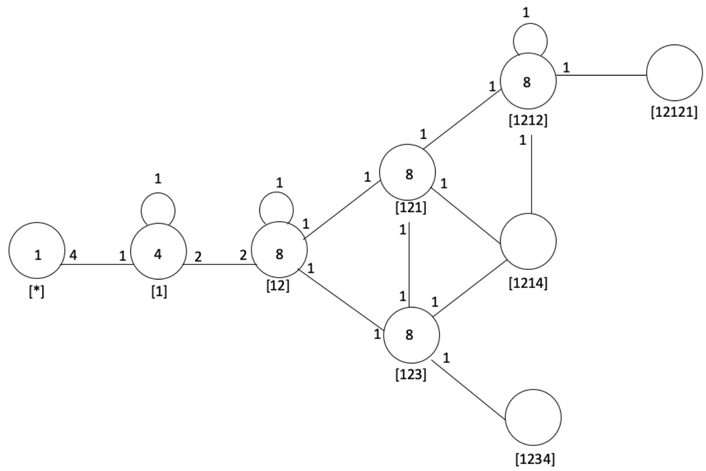

Figure 4.20: Cayley Graph of $3^{*2} :_m D_4$

7th Double Coset[1214]

$$
\begin{aligned}
Nt_1t_2t_1t_4N &= \{N(t_1t_2t_1t_4)^n \mid n \in N\} \\
&= \{N(t_1t_2t_1t_4)^e, N(t_1t_2t_1t_4)^{(1,4,3,2)}, N(t_1t_2t_1t_4)^{(1,2,3,4)}, N(t_1t_2t_1t_4)^{(1,2)(3,4)}, \\
&N(t_1t_2t_1t_4)^{(1,4)(2,3)}, N(t_1t_2t_1t_4)^{(1,3)(2,4)}, N(t_1t_2t_1t_4)^{(1,3)}, N(t_1t_2t_1t_4)^{(2,4)}\} \\
&= \{Nt_1t_2t_1t_4, Nt_4t_1t_4t_3, Nt_2t_3t_2t_1, Nt_2t_1t_2t_3, \\
&Nt_4t_3t_4t_1, Nt_3t_4t_3t_2, Nt_3t_2t_3t_4, Nt_1t_4t_1t_2\}
\end{aligned}
\tag{4.52}
$$

The point stabilizer of $N^{1214} = \{e\}$. Similarly, the coset stabilizer $N^{(1214)} = \{e\}$. The number of single right cosets in $N^{(1214)} = \frac{|N|}{|N^{(1214)}|} = \frac{8}{1} = 8$. The orbits of $N^{(1214)}$ on $X = \{1,2,3,4\}$ are $\{1\},\{2\},\{3\},\{4\}$. Now we select a representative from each orbit, say $1 \in \{1\}, 2 \in \{2\}, 3 \in \{3\}$ and $4 \in \{4\}$ and determine the double coset it belongs to.

$Nt_1t_2t_1t_4t_1 \in [123]$. If we conjugate our relation (13) $t_3t_2t_1t_3 = t_4t_3t_4t_1$ by (1,4,3,2) we get $t_2t_1t_4t_1 = t_3t_2t_3t_4$

$$
\begin{aligned}
t_1t_2t_1t_4t_1 &= t_1\underline{t_2t_1t_4t_1} \\
&= t_1t_3t_2t_3t_4 \\
&= t_1t_3t_2t_3t_4 \\
&= t_1^3t_2t_3t_4 \text{ since } t_3 = t_1^2 \\
&= t_2t_3t_4 \text{ since } t_1^3 = e
\end{aligned}
\tag{4.53}
$$

$Nt_2t_3t_4 \in [123]$ since $N(t_1t_2t_3)^{(1234)} = Nt_2t_3t_4$. Therefore, $Nt_1t_2t_1t_4t_1 = Nt_2t_3t_4 \in [123]$.

$Nt_1t_2t_1t_4t_2 \in [121]$. In order to prove this, we will use our labeling. Recall $t_4 = t_2^2$ and

$t_2^3 = e$. Therefore,

$$\begin{aligned} Nt_1t_2t_1t_4t_2 &= Nt_1t_2t_1t_2^2t_2 \\ &= Nt_1t_2t_1t_2^3 \\ &= Nt_1t_2t_1 \end{aligned} \tag{4.54}$$

Thus, one symmetric generator goes back to [121].

$Nt_1t_2t_1t_4t_3 \in [1234]$. In order to prove this, we will use the relation obtained by conjugating the original relation $t_3t_2t_1t_2 = t_4t_3t_4t_1$ by $(1,4,3,2)$ to obtain $t_2t_1t_4t_1 = t_3t_2t_3t_4$. This relation can be rewritten as $t_2t_1t_4t_1 = t_3t_2t_3t_4$.

$$\begin{aligned} t_1t_2t_1t_4t_3 &= t_1t_2t_1t_4t_3 \\ &= t_1t_2t_1t_4t_1t_1 \text{ since } t_3 = t_1^2 \\ &= t_1\underline{t_2t_1t_4t_1}t_1 \\ &= t_1t_3t_2t_3t_4t_1 \\ &= t_1t_3t_2t_3t_4t_1 \\ &= t_1t_1^2t_2t_3t_4t_1 \text{ since } t_3 = t_1^2 \\ &= t_2t_3t_4t_1 \text{ since } t_1^3 = e \end{aligned} \tag{4.55}$$

$Nt_1t_2t_1t_4t_3 = Nt_2t_3t_4t_1 = N(t_1t_2t_3t_4)^{(1,2,3,4)} \in [1234]$. Thus, one symmetric generator goes to [1234].

$Nt_1t_2t_1t_4t_4 \in [1212]$. In order to prove this we will use our labeling. If $t_4 = t_2^2$ then $t_4^2 = t_2$. Therefore $Nt_1t_2t_1t_4t_4 = Nt_1t_2t_1t_4^2$ which can be rewritten as $Nt_1t_2t_1t_2 \in [1212]$. Thus, one symmetric generator goes to [1212].

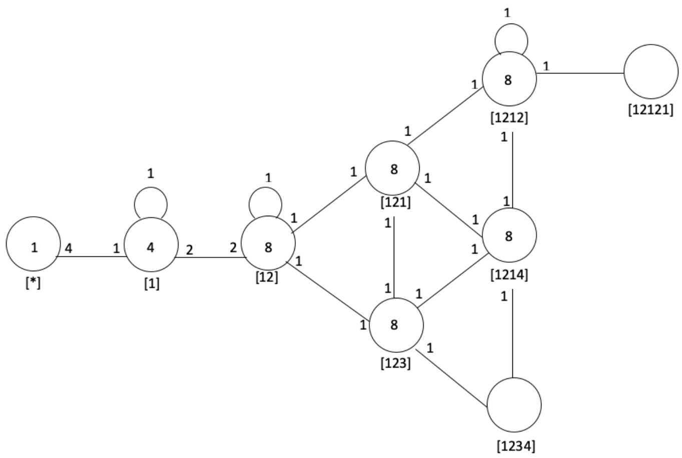

Figure 4.21: Cayley Graph of $3^{*2} :_m D_4$

<u>8th Double Coset[1234]</u>

$$
\begin{aligned}
Nt_1t_2t_3t_4N &= \{N(t_1t_2t_3t_4)^n \mid n \in N\} \\
&= \{N(t_1t_2t_3t_4)^e, N(t_1t_2t_3t_4)^{(1,4,3,2)}, N(t_1t_2t_3t_4)^{(1,2,3,4)}, N(t_1t_2t_3t_4)^{(1,2)(3,4)}, \\
&\quad N(t_1t_2t_3t_4)^{(1,4)(2,3)}, N(t_1t_2t_3t_4)^{(1,3)(2,4)}, N(t_1t_2t_3t_4)^{(1,3)}, N(t_1t_2t_3t_4)^{(2,4)}\} \\
&= \{Nt_1t_2t_3t_4, Nt_4t_1t_2t_3, Nt_2t_3t_4t_1, Nt_2t_1t_4t_3, Nt_4t_3t_2t_1, \\
&\quad Nt_3t_4t_1t_2, Nt_3t_2t_1t_4, Nt_1t_4t_3t_2\}
\end{aligned} \tag{4.56}
$$

The point stabilizer of $N^{1234} = \{e\}$. Similarly, the coset stabilizer $N^{(1234)} = \{e\}$. The number of single right cosets in $N^{(1234)} = \frac{|N|}{|N^{(1234)}|} = \frac{8}{1} = 8$. The orbits of $N^{(1234)}$ on $X = \{1,2,3,4\}$ are $\{1\},\{2\},\{3\},\{4\}$. Now we select a representative from each orbit, say $1 \in \{1\}, 2 \in \{2\}, 3 \in \{3\}$ and $4 \in \{4\}$ and determine the double coset it belongs to.

$N t_1 t_2 t_3 t_4 t_1 \in [12341]$, which is a new double coset. Thus, one symmetric generator moves forward.

$N t_1 t_2 t_3 t_4 t_2 \in [123]$. From our labeling $t_4^2 = t_2$ so we can rewrite $N t_1 t_2 t_3 t_4 t_2$ as $N t_1 t_2 t_3 t_2^2 t_2$ which is equal to $N t_1 t_2 t_3 \in [123]$ since $t_2^3 = e$. Thus $N t_1 t_2 t_3 t_4 t_2 \in [123]$. Thus, one symmetric generator goes to [123].

$N t_1 t_2 t_3 t_4 t_3 \in [1234]$. In order to prove that $N t_1 t_2 t_3 t_4 t_3 \in [1234]$ we will conjugate our original relation $t_3 t_2 t_1 t_2 = t_4 t_3 t_4 t_1$ by $(14)(23)$ we obtain $t_2 t_3 t_4 t_3 = t_1 t_2 t_1 t_4$. Now we can rewrite the relation as the following:

$$t_2 t_3 t_4 t_3 = t_1 t_2 t_1 t_4$$
$$t_2^{-1} t_2 t_3 t_4 t_3 = t_2^{-1} t_1 t_2 t_1 t_4$$
$$t_3 t_4 t_3 = t_4 t_1 t_2 t_1 t_4$$

We will use the relation to $t_3 t_4 t_3 = t_4 t_1 t_2 t_1 t_4$ to prove $N t_1 t_2 t_3 t_4 t_3 \in [1234]$.

$$\begin{aligned} t_1 t_2 \underline{t_3 t_4 t_3} &= t_1 t_2 t_2 t_4 t_1 t_2 t_1 t_4 \\ &= t_1 \underline{t_4 t_2} t_1 t_2 t_1 t_4 \\ &= \underline{t_1 t_1} t_2 t_1 t_4 \\ &= t_3 t_2 t_1 t_4 \end{aligned} \tag{4.57}$$

$N(t_1 t_2 t_3 t_4)^{(13)} = N t_3 t_2 t_1 t_4$ so $N t_3 t_2 t_1 t_4 \in [1234]$. Thus, one symmetic generator goes to [1234].

$N t_1 t_2 t_3 t_4 t_4 \in [1214]$. In order to prove this, we conjugate our original relation $t_3 t_2 t_1 t_2 = t_4 t_3 t_4 t_1$ by (13) to obtain $t_1 t_2 t_3 t_2 = t_4 t_1 t_4 t_3$. We can rewrite the relation as $t_1 t_2 t_3 t_2 =$

$t_4 t_1 t_4 t_3$.

$$\begin{aligned} t_1 t_2 t_3 \underline{t_4 t_4} &= \underline{t_1 t_2 t_3 t_2} \text{ since } t_2 = t_4^2 \\ &= t_4 t_1 t_4 t_3 \end{aligned} \tag{4.58}$$

$N t_1 t_2 t_3 t_4 t_4 = N t_4 t_1 t_4 t_3 = N(t_1 t_2 t_1 t_4)^{(1,4,3,2)} \in [1214]$. Thus, one symmetric generator goes to [1214].

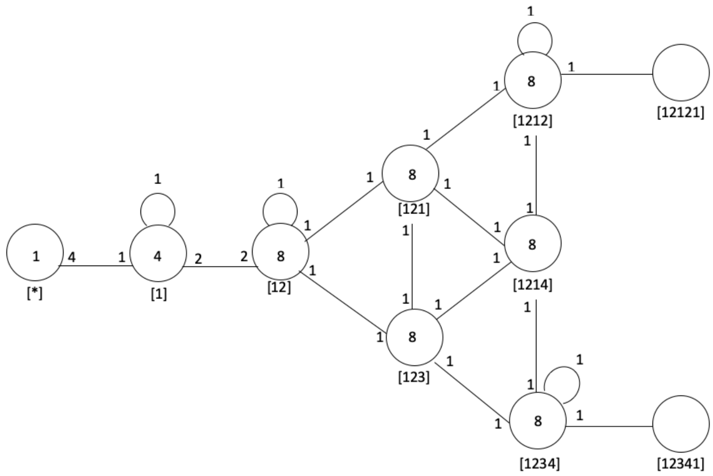

Figure 4.22: Cayley Graph of $3^{*2} :_m D_4$

9th Double Coset[12121]

$$Nt_1t_2t_1t_2t_1N = \{N(t_1t_2t_1t_2t_1)^n \mid n \in N\}$$
$$= \{N(t_1t_2t_1t_2t_1)^e, N(t_1t_2t_1t_2t_1)^{(1,4,3,2)}, N(t_1t_2t_1t_2t_1)^{(1,2,3,4)}, N(t_1t_2t_1t_2t_1)^{(1,2)(3,4)},$$
$$N(t_1t_2t_1t_2t_1)^{(1,4)(2,3)}, N(t_1t_2t_1t_2t_1)^{(1,3)(2,4)}, N(t_1t_2t_1t_2t_1)^{(1,3)}, N(t_1t_2t_1t_2t_1)^{(2,4)}\}$$
$$= \{Nt_1t_2t_1t_2t_1, Nt_4t_1t_4t_1t_4, Nt_2t_3t_2t_3t_2, Nt_2t_1t_2t_1t_2,$$
$$Nt_4t_3t_4t_3t_4, Nt_3t_4t_3t_4t_3, Nt_3t_2t_3t_2t_3, Nt_1t_4t_1t_4t_1\} \tag{4.59}$$

The point stabilizer of $N^{12121} = \{e\}$. However, since $Nt_1t_2t_1t_2t_1 = Nt_2t_1t_2t_1t_2$, then $(1,2)(3,4) \in N^{(12121)}$. Additionally, $Nt_1t_2t_1t_2t_1 = Nt_4t_3t_4t_3t_4$, then $(1,4)(2,3) \in N^{(12121)}$. The coset stabilizer $N^{(12121)} \geq N^{12121}$, since $N^{(12121)} = \{e, (1,2)(3,4), (1,4)(2,3), (1,3)(2,4)\}$. The number of single right cosets in $N^{(12121)} = \frac{|N|}{|N^{(12121)}|} = \frac{8}{4} = 2$. There is one single orbit for $N^{(12121)}$ which is $\{1,2,3,4\}$. Now we select a representative from each orbit, say $3 \in \{1,2,3,4\}$ and determine the double coset it belongs to.

$Nt_1t_2t_1t_2t_1t_3 = Nt_1t_2t_1t_2 \in [1212]$ since $t_1t_3 = t_1^3 = e$. Thus all four symmetric generators go back to [1212].

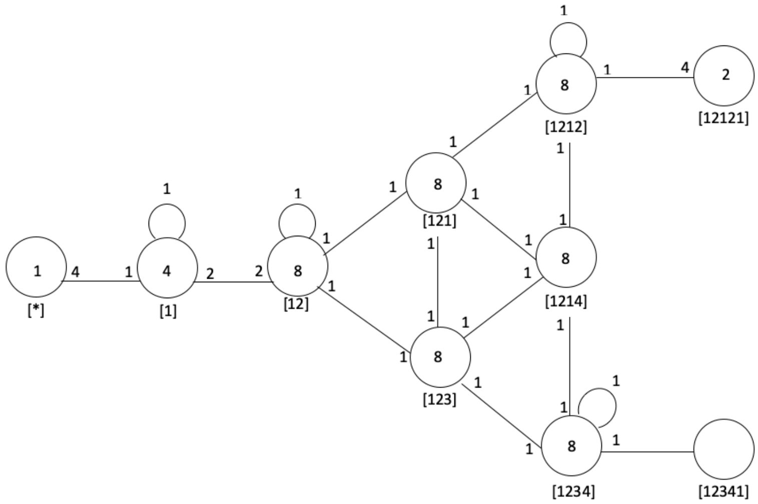

Figure 4.23: Cayley Graph of $3^{*2} :_m D_4$

<u>10th Double Coset[12341]</u>

$$
\begin{aligned}
Nt_1t_2t_3t_4t_1N &= \{N(t_1t_2t_3t_4t_1)^n \mid n \in N\} \\
&= \{N(t_1t_2t_3t_4t_1)^e, N(t_1t_2t_3t_4t_1)^{(1,4,3,2)}, N(t_1t_2t_3t_4t_1)^{(1,2,3,4)}, \\
&N(t_1t_2t_3t_4t_1)^{(1,2)(3,4)}, N(t_1t_2t_3t_4t_1)^{(1,4)(2,3)}, \\
&N(t_1t_2t_3t_4t_1)^{(1,3)(2,4)}, N(t_1t_2t_3t_4t_1)^{(1,3)}, N(t_1t_2t_3t_4t_1)^{(2,4)}\} \\
&= \{Nt_1t_2t_3t_4t_1, Nt_4t_1t_2t_3t_4, Nt_2t_3t_4t_1t_2, Nt_2t_1t_4t_3t_2, \\
&Nt_4t_3t_2t_1t_4, Nt_3t_4t_1t_2t_3, Nt_3t_2t_1t_4t_3, Nt_1t_4t_3t_2t_1\}
\end{aligned}
\tag{4.60}
$$

The point stabilizer of $N^{12341} = \{e\}$. However, since $Nt_1t_2t_3t_4t_1 = Nt_3t_2t_1t_4t_3$, then $(1,3) \in N^{(12341)}$. Thus, $N^{(12341)} \geq N^{12341}$ since $N^{(12341)} = \{e, (13)\}$. The number of single right cosets in $N^{(12341)} = \frac{|N|}{|N^{(12341)}|} = \frac{8}{2} = 4$. The orbits $N^{(12341)}$ on $X = \{1,2,3,4\}$ are $\{1,3\}$, $\{2\}$, and $\{4\}$. Now we select a representative from each orbit, say $3 \in \{1,3\}$, $2 \in \{2\}$, and $4 \in \{4\}$ and determine the double coset it belongs.

$Nt_1t_2t_3t_4t_1t_3 \in [1234]$. In order to prove this, we use our labeling $t_3 = t_1^2$ and $t_1^3 = e$

$$\begin{aligned} Nt_1t_2t_3t_4t_1t_3 &= Nt_1t_2t_3t_4t_1\underline{t_3} \\ &= Nt_1t_2t_3t_4\underline{t_1t_1^2} \\ &= Nt_1t_2t_3t_4t_1^3 \\ &= Nt_1t_2t_3t_4 \end{aligned} \tag{4.61}$$

Thus $Nt_1t_2t_3t_4t_1t_3 \in [1234]$ and two symmetric generators go to [1234].

$Nt_1t_2t_3t_4t_1t_2 \in [123412]$, which is a new double coset. Thus, one symmetric generator moves forward.

$Nt_1t_2t_3t_4t_1t_4 \in [12341]$. Thus, one symmetric generator goes to [12341].

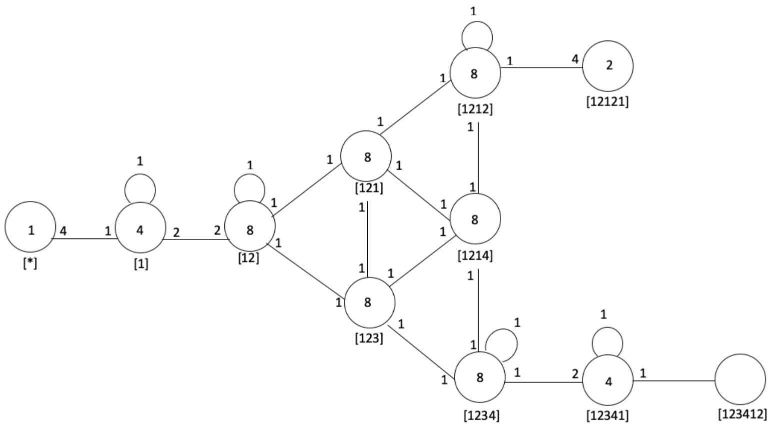

Figure 4.24: Cayley Graph of $3^{*2} :_m D_4$

11th Double Coset[123412]

$$Nt_1t_2t_3t_4t_1t_2N = \{N(t_1t_2t_3t_4t_1t_2)^n \mid n \in N\}$$
$$= \{N(t_1t_2t_3t_4t_1t_2)^e, N(t_1t_2t_3t_4t_1t_2)^{(1,4,3,2)}, N(t_1t_2t_3t_4t_1t_2)^{(1,2,3,4)},$$
$$N(t_1t_2t_3t_4t_1t_2)^{(1,2)(3,4)}, N(t_1t_2t_3t_4t_1t_2)^{(1,4)(2,3)},$$
$$N(t_1t_2t_3t_4t_1t_2)^{(1,3)(2,4)}, N(t_1t_2t_3t_4t_1t_2)^{(1,3)}, N(t_1t_2t_3t_4t_1t_2)^{(2,4)}\}$$
$$= \{Nt_1t_2t_3t_4t_1t_2, Nt_4t_1t_2t_3t_4t_1, Nt_2t_3t_4t_1t_2t_3, Nt_2t_1t_4t_3t_2t_1,$$
$$Nt_4t_3t_2t_1t_4t_3, Nt_3t_4t_1t_2t_3t_4, Nt_3t_2t_1t_4t_3t_2, Nt_1t_4t_3t_2t_1t_4\} \tag{4.62}$$

The point stabilizer of $N^{123412} = \{e\}$. However, since $Nt_1t_2t_3t_4t_1t_2 = Nt_3t_2t_1t_4t_3t_2$, then $(1,3) \in N^{(123412)}$. Additionally, $Nt_1t_2t_3t_4t_1t_2 = Nt_2t_3t_4t_1t_2t_3$, then $(1,2,3,4) \in N^{(123412)}$. Thus, $N^{(123412)} \geq N^{123412}$ since $N^{(123412)} = \{e, (1,3), (1,2,3,4), (1,3)(2,4), (1,2)(3,4)\}$. The number of single right cosets in $N^{(123412)} = \frac{|N|}{|N^{(123412)}|} = \frac{8}{4} = 2$. The orbit $N^{(123412)}$ on $X = \{1,2,3,4\}$ is $\{1,2,3,4\}$. Now we select a representative from the orbit, say 4 determine the double coset it belongs.

$Nt_1t_2t_3t_4t_1t_2t_4 \in [12341]$.

$$\begin{aligned} t_1t_2t_3t_4t_1t_2t_4 &= t_1t_2t_3t_4t_1t_2t_2^2 \\ &= t_1t_2t_3t_4t_1t_2^3 \\ &= t_1t_2t_3t_4t_1e \\ &= t_1t_2t_3t_4t_1 \end{aligned} \tag{4.63}$$

$Nt_1t_2t_3t_4t_1t_2t_4 = Nt_1t_2t_3t_4t_1 \in [12341]$. Thus, four symmetric generators go back to [12341].

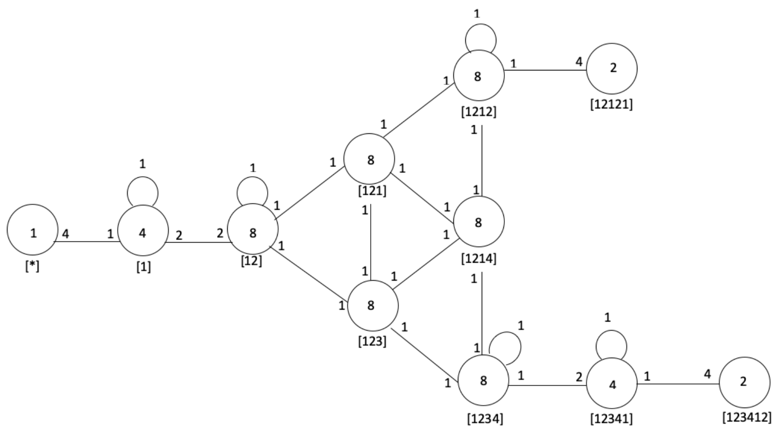

Figure 4.25: Cayley Graph of $3^{*2} :_m D_4$

Appendix A

MAGMA CODE $2^{*15}:(D_{5\times 3})$

```
N:=TransitiveGroup(15,3);
#N;
N;
Generators(N);
N.1;
N.2;
S:=Sym(15);
xx:=S!((1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15));
yy:=S!((1, 4)(2, 8)(3, 12)(6, 9)(7, 13)(11, 14));
N:=sub<S|xx,yy>;
#N;
FPGroup(N);
NN<x,y>:=Group<x,y|y^2,x^-4*y*x*y>;
#NN;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..30]];
 for i in [2..30] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
```

```
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

N1:=Stabiliser(N,1);
#N1;
N1;

Generators(N1);
for i in [1..30] do if ArrayP[i]
eq N!((2, 5)(3, 9)(4, 13)(7, 10)(8, 14)(12, 15))
then Sch[i]; end if; end for;

Orbits(Stabiliser(N,1));

G<x,y,t>:=Group<x,y,t|y^2,x^-4*y*x*y, t^2, (t,y^x),(t,t^(x^5)),
(t,t^(x^10)), (t,t^x), (t,t^(x^2)), (t,t^(x^3)), (t, t^(x^6)),
(t,t^(x^7)), (t,t^(x^11))>;

C:=Classes(N);
C;

for i in [1..48] do 1^ArrayP[i], Sch[i]; end for;

for i in [2..12] do
i, Orbits(Centraliser(N,C[i][3]));
end for;

for a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q in [0..17] do
G<x,y,t>:=Group<x,y,t|y^2,x^-4*y*x*y, t^2, (t,y^x),(t,t^(x^5)),
(t,t^(x^10)), (t,t^x), (t,t^(x^2)), (t,t^(x^3)), (t, t^(x^6)),
(t,t^(x^7)), (t,t^(x^11)),
(x^2 * y * x*t^x)^a,
(x^2 * y * x*t)^b,
(x^2 * y * x*t^(x^3))^c,
```

```
((x * y)^2*t)^d,
((y * x^-1)^2*t)^e,
(x^3*t)^f,
(x^2 * y * x * y*t)^g,
(x * y*t^x)^h,
(x * y*t)^i,
(x * y*t^(x^3))^j,
(y * x^-1*t^x)^k,
(y * x^-1*t)^l,
(y * x^-1*t^(x^3))^m,
(x*t)^n,
(x^2*t)^o,
(y * x^-2 * y*t)^p,
(y * x^-1 * y*t)^q>;

if #G gt 30 then a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,
#G;
end if; end for;
```

Appendix B

MAGMA CODE $2^{*15} : (D_3 \times 5)$

```
N:=TransitiveGroup(15,4);
#N;
N;
Generators(N);
N.1;
N.2;
S:=Sym(15);
xx:=S!((1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15));
yy:=S!((1, 11)(2, 7)(4, 14)(5, 10)(8, 13));
N:=sub<S|xx,yy>;
FPGroup(N);
NN<x,y>:=Group<x,y|y^2, x^-4*y*x^-1*y>;
#NN;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..30]];
for i in [2..30] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
end for;
```

```
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

N1:=Stabiliser(N,1);
#N1;
Generators(N1);

for i in [1..30] do if ArrayP[i]
eq N!((2, 12)(3, 8)(5, 15)(6, 11)(9, 14))
then Sch[i]; end if; end for;

G<x,y,t>:=Group<x,y,t|y^2, x^-4*y*x^-1*y, t^2, (t,y^x)>;

Orbits(Stabiliser(N,1));

xx^3;
xx*yy;
(xx^4)*yy;
(xx^7)*yy;
xx;
xx^2;
xx^4;
yy;
(xx^3)*yy;

G<x,y,t>:=Group<x,y,t|y^2, x^-4*y*x^-1*y, t^2, (t,y^x),
(t,t^(x^3)),
(t,t^(x*y)),
(t, t^((x^4)*y)),
(t,t^((x^7)*y)),
(t,t^x),
(t,t^(x^2)),
(t,t^(x^4)),
(t,t^y),
(t,t^((x^3)*y))>;
```

```
C:=Classes(N);
C;
Classes(N);

for i in [2..15] do
i, Orbits(Centraliser(N,C[i][3]));
end for;

for j in [2..15] do
C[j][3];
for i in [1..30] do
if ArrayP[i] eq C[j][3]
then Sch[i]; end if;
end for;
end for;

for a,b,c,d,e,f,g,h,i,j,k,l,m,n in [0..10] do
G<x,y,t>:=Group<x,y,t|y^2, x^-4*y*x^-1*y, y^x, t^2, (t,y^x),
(y^x*t)^a,
(x * y * x^-1 * y*t^(x^3))^b,
(x^3*t^(x*y))^c,
(x^-3*t^(y * x^-1))^d,
(x * y * x^-2 * y*t^(y * x^2))^e,
(x^2 * y * x^-1 * y*t^x)^f,
(x * y*t^(y*x))^g,
(x^-2 * y*t^(x^2))^h,
(y * x^2*t^(x * y * x))^i,
(y * x^-1*t^(x * y * x^-2))^j,
(x*t^(x^-1))^k,
(x^2*t^(x * y * x^-1))^l,
(y * x^-1 * y*t^y)^m,
(x^-2*t^(y * x^-2))^n>;

if #G gt 30 then a,b,c,d,e,f,g,h,i,j,k,l,m,n, #G;
end if; end for;
```

Appendix C

MAGMA CODE $2^{*24} : (4 \times 2 : S_3)$

```
2 * 24 : N
c = (1, 10)(2, 5)(3, 7)(4, 8)(6, 9)(11, 12)
N =
 Permutation group N acting on a set of cardinality 24
Order = 48 = 2^4 * 3
    (1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(8, 16)
    (17, 23)(18,21)(20, 24)
    (1, 15, 17)(2, 13, 18)(3, 11, 19)(4, 16, 20)(5, 10, 21)
    (6, 14, 22)(7, 9,23)(8, 12, 24)
    (1, 2, 4, 5)(3, 8, 6, 7)(9, 16, 12, 15)(10, 11, 13, 14)
    (17, 22, 20,19)(18, 24, 21, 23)
    (1, 3, 4, 6)(2, 7, 5, 8)(9, 14, 12, 11)(10, 16, 13, 15)
    (17, 24, 20,23)(18, 19, 21, 22)
    (1, 4)(2, 5)(3, 6)(7, 8)(9, 12)(10, 13)(11, 14)(15, 16)
    (17, 20)(18,21)(19, 22)(23, 24)

Stabiliser of 1 in N
 Permutation group acting on a set of cardinality 24
Order = 2
    (2, 6)(3, 5)(7, 8)(9, 21)(10, 17)(11, 23)(12, 18)
    (13, 20)(14, 24)(15,
        22)(16, 19)
*/
```

```
S:=Sym(24);
xx:=S!(1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(8, 16)
(17, 23)(18,21)(20, 24);
yy:=S!(1, 15, 17)(2, 13, 18)(3, 11, 19)(4, 16, 20)(5, 10, 21)
(6, 14, 22)(7, 9,23)(8, 12, 24);
zz:=S!(1, 2, 4, 5)(3, 8, 6, 7)(9, 16, 12, 15)(10, 11, 13, 14)
(17, 22, 20,19)(18, 24, 21, 23);
ww:=S!(1, 3, 4, 6)(2, 7, 5, 8)(9, 14, 12, 11)(10, 16, 13, 15)
(17, 24, 20,23)(18, 19, 21, 22);
pp:=S!(1, 4)(2, 5)(3, 6)(7, 8)(9, 12)(10, 13)(11, 14)(15, 16)
(17, 20)(18,21)(19, 22)(23, 24);

N:=sub<S|xx,yy,zz,ww,pp>;
#N;

#sub<S|xx,yy,zz>;
/*48*/
#sub<S|xx,yy>;
/*6*/
#sub<S|xx,zz>;
/*16*/
#sub<S|yy,zz>;
/*24*/

N:=sub<S|xx,yy,zz>;

FPGroup(N);
NN<x,y,z>:=Group<x,y,z|x^2, y^3, z^4, (y^-1 *x)^2,
z^-2 * y^-1 * z^2 * y, (y*z^-1*x)^2,
z^-1 * y^-1 * z^-1 * y^-1 *z * y^-1>;
#NN;

Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..48]];
for i in [2..48] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
```

```
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
if Eltseq(Sch[i])[j] eq 3 then P[j]:=zz; end if;
if Eltseq(Sch[i])[j] eq -3 then P[j]:=zz^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;


Orbits(Stabiliser(N,1));

N1:=Stabiliser(N,1);
N1;
N12:=Stabiliser(N,[1,2]);
C12:=Centraliser(N,N12);
C12;
/* This was to check the famous lemma.
We found out that the lemma does not apply*/

W,phi:=WordGroup(N);
rho:=InverseWordMap(N);
A:=N!(2, 6)(3, 5)(7, 8)(9, 21)(10, 17)(11, 23)(12, 18)(13, 20)
(14, 24)(15,22)(16, 19);
A;
A@rho;

AA:=function(W)
w3 := W.3^-1; w4 := W.1 * w3; w2 := W.2^-1; w5 := w4 * w2; return w5;
function> end function;

AA(NN);

Stabiliser(N,1) eq sub<N|xx * zz^-1 * yy^-1>;


G<x,y,z,t>:=Group<x,y,z,t|x^2, y^3, z^4,(y^-1 * x)^2,
```

```
z^-2 * y^-1 * z^2 * y,(y * z^-1 * x)^2,
z^-1 * y^-1 * z^-1 * y^-1 * z * y^-1, t^2, (t,x * z^-1 * y^-1)>;

Orbits(N12);

C:=Classes(N);
for j in [2..8] do
C[j][3];
for i in [1..48] do
if ArrayP[i] eq C[j][3]
then Sch[i]; end if;
end for;
end for;

#C;


C;

for i in [2..8] do i,C[i][3], Orbits(Centraliser(N,C[i][3])); end for;

for i in [1..48] do 1^ArrayP[i], Sch[i]; end for;
/*The code above will print out the names :)*/

/* MAKE t COMMUTE WITH EVERYTHING

(z^2*t),
(z*x*z*t^(y^-1)),
(z*x*z*t^(y*z*y)),
(z*x*z*t),
(z*x*z*t^z),
(z*x*z*t^(x * y^-1 * z)),
(z*x*z*t^(x * z^-1 * x)),
(z*x*z*t^( x * y * z)),
(y*t),
(y*t^z),
(y*t^(x*y^-1*z)),
(y*t^(x*y^-1)),
(z*t),
```

```
(z*t^(x * y^-1 * z)),
(z*t^(y * z)),
(y*z*t),
(y*z*t^z),
(y*z*t^(x * y^-1 * z)),
(y*z*t^(x * y^-1)),
(z*x*t),
(z*x*t^z),
(z*x*t^(y^-1)),
(z^-1 * x*t),
(z^-1 * x*t^z),
(z^-1 * x*t^(y^-1)),
*/

G<x,y,z,t>:=Group<x,y,z,t|x^2, y^3, z^4, (y^-1 *x)^2, z^-2 *
y^-1 * z^2 * y, (y*z^-1*x)^2, z^-1 * y^-1 * z^-1 * y^-1 *z *
y^-1,
(z^2*t)^a,
(z*x*z*t^(y^-1))^b,
(z*x*z*t^(y*z*y))^c,
(z*x*z*t)^d,
(z*x*z*t^z)^e,
(z*x*z*t^(x * y^-1 * z))^f,
(z*x*z*t^(x * z^-1 * x))^g,
(z*x*z*t^( x * y * z))^h,
(y*t)^i,
(y*t^z)^j,
(y*t^(x*y^-1*z))^k,
(y*t^(x*y^-1))^l,
(z*t)^m,
(z*t^(x * y^-1 * z))^n,
(z*t^(y * z))^o,
(y*z*t)^p,
(y*z*t^z)^q,
(y*z*t^(x * y^-1 * z))^r,
(y*z*t^(x * y^-1))^s,
(z*x*t)^t,
(z*x*t^z)^u,
(z*x*t^(y^-1))^v,
```

```
(z^-1 * x*t)^a.1,
(z^-1 * x*t^z)^a.2,
(z^-1 * x*t^(y^-1))^a.3>;


if #G gt 48 then
a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,a.1,a.2,a.3,
#G;
end if; end for;
```

Appendix D

MAGMA CODE $2^3 : 3$

```
S:=Sym(6);
xx:=S!(3, 6);
yy:=S!(1, 3, 5)(2, 4, 6);
G:=sub<S|xx,yy>;

Classes(G);
CT:=CharacterTable(G);
CT;

H:=sub<G|(2, 5),(2, 5)(3, 6),(1, 4)(2, 5)>;
Classes(H);
CH:=CharacterTable(H);
CH;
for i in [2..8] do for j in [7,8] do if Induction(CH[i],G) eq CT[j]
then i, j; end if; end for; end for;

T:=Transversal(G,H);
T;
G;
C:=Classes(G);
#C;
for i in[1..8] do C[i][3]; end for;
```

```
H:=sub<G|(2, 5),(2, 5)(3, 6),(1, 4)(2, 5)>;
D:=Classes(H);
#D;
 for i in[1..8] do D[i][3]; end for;

C:=CyclotomicField(2);
GG:=GL(3,C);
A:=[[C.1,0,0] : i in [1..3]];
for i ,j in [1..3] do A[i,j]:=0; end for;
for i,j in [1..3] do if T[i]*xx*T[j]^-1 in H then
A[i,j]:=CH[8](T[i]*xx*T[j]^-1); end if; end for;
B:=[[C.1,0,0] : i in [1..3]];
for i ,j in [1..3] do B[i,j]:=0; end for;
for i,j in [1..3] do if T[i]*yy*T[j]^-1 in H then
B[i,j]:=CH[8](T[i]*yy*T[j]^-1); end if; end for;
GG!A; GG!B;

Order(GG!A);
Order(GG!B);
Order(GG!A*GG!B);

H:=sub<GG|A,B>;
#H;
IsIsomorphic(H,G);

S:=Sym(6);
xx:=S!(3,6);
yy:=S!(1,2,3)(4,5,6);
N:=sub<S|xx,yy>;

#N;

IsIsomorphic(N,G);

xx*yy;

N1:=Stabiliser(N,{1,4});
N1;
```

```
#N1;

NN<a,b>:=Group<a,b|a^2,b^3,(a*b)^6, (a,b)^2>;
#NN;

(xx,yy);

NN<a,b>:=Group<a,b|a^2,b^3,(a*b)^6,(a,b)^2>;
#NN;

W:=WordGroup(N);
rho:=InverseWordMap(N);
a:=N!(3,6);
b:=N!(2,5);
c:=N!(1,4);
a@rho;
function(W)
    return W.1;
end function
xx;

b@rho;
function(W)
    w2 := W.2^-1; w3 := W.1 * w2; w4 := W.2 * w3; return w4;
end function

B:=function(W)
function>     w2 := W.2^-1; w3 := W.1 * w2; w4 := W.2 * w3;
return w4;
function> end function;

B(NN);
yy*xx*yy^-1;

c@rho;
function(W)
    w2 := W.2^-1; w6 := w2 * W.1; w7 := w6 * W.2; return w7;
end function
C:=function(W)
```

```
function>     w2 := W.2^-1; w6 := w2 * W.1; w7 := w6 * W.2;
return w7;
function> end function;
C(NN);

xx^yy;

G<x,y,t>:=Group<x,y,t|x^2,y^3,(x*y)^6,(x,y)^2, (t,x),
(t,y*x*y^-1),t^(x^y)=t^2>;
```

Appendix E

MAGMA CODE $4^2 : 4$

```
S:=Sym(8);
T:=TransitiveGroups(8);
T;
T[30];

A:=S!(2, 6)(3, 7);
B:=S!(1, 3)(4, 8)(5, 7);
C:=S!(1, 2, 3, 8)(4, 5, 6, 7);
N:=sub<S|A,B,C>;
N;

#N;
Center(N);
CompositionFactors(N);
NL:=NormalLattice(N);
NL;

for i in [1..13] do if IsAbelian(NL[i]) then i;end if;end for;
NL[8];

X:=AbelianGroup(GrpPerm,[4,4]);
IsIsomorphic(NL[8],X);
```

```
q,ff:=quo<N|NL[8]>;
q;
#q;

q eq sub<q|q.1,q.2,q.3>;
A:=N!(2, 8, 6, 4);
B:=N!(1, 3, 5, 7)(2, 8, 6, 4);

T:=Transversal(N,NL[8]);
T;

C:=N!(1, 2, 3, 8)(4, 5, 6, 7);

for i in [0..3] do for j in [0..3] do i,j, A^i*B^j; end for; end for;

A^T[3];

A*B^3;

B^T[3];

A^2*B^3;

H<a,b,c>:=Group<a,b,c|a^4,b^4,(a,b),c^4,a^c=a*b^3,b^c=a^2*b^3>;
f,H1,k:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(H1,N);
```

Appendix F

MAGMA CODE $(4 \times 2^2) : S_3$

```
S:=Sym(12);
T:=TransitiveGroups(12);
T;
T[53];
A:=S!(1, 7)(3, 9)(4, 10)(6, 12);
B:=S!(1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12);
C:=S!(1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12);
D:=S!(1, 5)(2, 10)(4, 8)(7, 11);
N:=sub<S|A,B,C,D>;
N;
#N;
CompositionFactors(N);
NL:=NormalLattice(N);
NL;
for i in [1..14] do if IsAbelian(NL[i]) then i;end if;end for;
NL[7];

X:=AbelianGroup(GrpPerm,[4,2,2]);
IsIsomorphic(NL[7],X);

q,ff:=quo<N|NL[7]>;
q;
```

```
q1:=q.3;
q2:=q.4;

NL[7];

A:=N!(1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12);
B:=N!(3, 9)(6, 12);
C:=N!(1, 7)(3, 9)(4, 10)(6, 12);
D:=N!(1, 7)(2, 8)(4, 10)(5, 11);

/* I need to determine if I need all four, that is, A,B,C,D.*/

M:=sub<N|A,B,C>;
#M;

/*So I don't need D since NL[7] is order 16*/

A:=N!(1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12);
B:=N!(3, 9)(6, 12);
C:=N!(1, 7)(3, 9)(4, 10)(6, 12);

T:=Transversal(N,NL[7]);
T;

D:=N!(1, 5, 9)(2, 6, 10)(3, 7, 11)(4, 8, 12);
E:=N!(1, 5)(2, 10)(4, 8)(7, 11);

A^T[2];
B^T[2];

A^T[2] eq A^D;
> A^T[2] eq A;


for i in [0..3] do for j in [0..1] do for k in [0..1] do i,j,k,
A^i*B^j*C^k;
end for; end for; end for;

/*A^T[] eq A;
```

```
(1, 4, 7, 10)(2, 5, 8, 11)(3, 6, 9, 12)
> B^D;
(1, 7)(4, 10)

NOT NEEDED C^T[2];
(2, 8)(3, 9)(5, 11)(6, 12)

C^D;
(1, 7)(2, 8)(4, 10)(5, 11)
A^E;

A^T[3] eq A*B^2;
B^E;
B^T[3] eq B*C^2;;

C^T[3] eq C;

C^T[3];
(2, 8)(3, 9)(5, 11)(6, 12)
C^E eq C*D;

H<a,b,c,d,e>:=Group<a,b,c,d,e|a^4,b^2,c^2,(a,b),(a,c),(b,c),
d^3,e^2,(d*e)^2,a^d=a,b^d=b*c,c^d=a^2*b,a^e=a,b^e=b,c^e=a^2*b*c>;

f,H1,k:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(H1,N);
```

Appendix G

MAGMA CODE $(4 \times 2^2) \mathbin{\cdot\!\!:} A_4$

```
S:=Sym(24);
T:=TransitiveGroups(24);
T[500];

/*Permutation group acting on a set of cardinality 24
Order = 192 = 2^6 * 3
    (1, 3)(2, 4)(5, 23)(6, 24)(11, 12)(13, 14)(15, 16)(17, 18)(19, 22)
    (20, 21)
    (1, 7, 22, 24, 10, 19)(2, 8, 21, 23, 9, 20)(3, 11, 15, 6, 14, 18)
    (4, 12, 16, 5, 13, 17)*/

xx:=S!(1, 3)(2, 4)(5, 23)(6, 24)(11, 12)(13, 14)(15, 16)
(17, 18)(19, 22)(20,21);
yy:=S!(1, 7, 22, 24, 10, 19)(2, 8, 21, 23, 9, 20)(3, 11, 15, 6, 14, 18)
(4, 12, 16, 5, 13, 17);
N:=sub<S|xx,yy>;
N;

#N;

CompositionFactors(N);

NL:=NormalLattice(N);
```

```
NL;


for i in [1..9] do if IsAbelian(NL[i]) then i;end if;end for;

NL[5];
X:=AbelianGroup(GrpPerm,[4,2,2]);
IsIsomorphic(NL[5],X);

q,ff:=quo<N|NL[5]>;
q;

A:=N!(1, 5, 24, 4)(2, 6, 23, 3)(7, 11, 10, 14)(8, 12, 9, 13)
(15, 22, 18, 19)(16,21, 17, 20);
B:=N!(1, 2)(3, 4)(5, 6)(15, 17)(16, 18)(19, 21)(20, 22)(23, 24);
C:=N!(1, 23)(2, 24)(3, 5)(4, 6)(7, 8)(9, 10)(11, 12)(13, 14);
D:=N!(1, 24)(2, 23)(3, 6)(4, 5)(7, 10)(8, 9)(11, 14)(12, 13)
(15, 18)(16, 17)(19, 22)(20, 21);

/*BUT I need to determine if I need all four that is A,B,C,D.*/

A:=N!(1, 5, 24, 4)(2, 6, 23, 3)(7, 11, 10, 14)(8, 12, 9, 13)
(15, 22, 18, 19)(16,21, 17, 20);
B:=N!(1, 2)(3, 4)(5, 6)(15, 17)(16, 18)(19, 21)(20, 22)(23, 24);
C:=N!(1, 23)(2, 24)(3, 5)(4, 6)(7, 8)(9, 10)(11, 12)(13, 14);
M:=sub<N|A,B,C>;
#M;

/* 16, so I do not need D, since NL[5] is order 16
So far the presentation NL[5] is
 <a,b,c|a^{4}, b^{2}, c^{2},(a,b),(a,c),(b,c)>  */

T:=Transversal(N,NL[5]);
T;

IsIsomorphic(q,Alt(4));

FPGroup(q);
```

```
Generators(NL[5]);

A;
B;
C;

NL[5] eq sub<NL[5]|A,B,C>;

ff(T[2]) eq q.1;
ff(T[3]) eq q.2;

T2:=N!(1, 3)(2, 4)(5, 23)(6, 24)(11, 12)(13, 14)(15, 16)(17, 18)
(19, 22)(20,21);

T3:=N!(1, 7, 22, 24, 10, 19)(2, 8, 21, 23, 9, 20)
(3, 11, 15, 6, 14, 18)(4, 12,16, 5, 13,17);

for i in [0..3] do for j,k in [0..1] do
if A^T2 eq A^i*B^j*C^k then i,j,k;
end if; end for; end for;

for i in [0..3] do for j,k in [0..1] do
if B^T2 eq A^i*B^j*C^k then i,j,k;
end if; end for; end for;


for i in [0..3] do for j,k in [0..1] do
if C^T2 eq A^i*B^j*C^k then i,j,k;end if;end for; end for;

for i in [0..3] do for j,k in [0..1] do
if C^T3 eq A^i*B^j*C^k then i,j,k;end if;end for; end for;

for i in [0..3] do for j,k in [0..1] do
if A^T3 eq A^i*B^j*C^k then i,j,k;end if;end for; end for;

for i in [0..3] do for j,k in [0..1] do
if B^T3 eq A^i*B^j*C^k then i,j,k;end if;end for; end for;

H<a,b,c,d,e>:=Group<a,b,c,d,e|a^4,b^2,c^2,(a,b),(a,c),(b,c),d^2,
```

```
e^3,(d*e)^3,a^d=a*b*c,b^d=b,c^d=c,a^e=a^3*c,b^e=c>;
H<a,b,c,d,e>:=Group<a,b,c,d,e|a^4,b^2,c^2,(a,b),(a,c),(b,c),d^2,
e^3,(d*e)^3,a^d=a*b*c,b^d=b,c^d=c,a^e=a^3*c,b^e=c,c^e=a^2*b*c>;
#H;

f,H1,k:=CosetAction(H,sub<H|Id(H)>);
IsIsomorphic(H1,N);
/*false*/
#N;

Order(T2) eq Order(q.1);
true
Order(T3) eq Order(q.2);
false
Order(T2*T3) eq Order(q.1*q.2);
/*false*/
Order(T3);

Order(q.2);

for i in [0..3] do for j,k in [0..1] do
if T3^3 eq A^i*B^j*C^k then i,j,k;end if;end for; end for;

Order(T2*T3);

Order(q.1*q.2);

for i in [0..3] do for j,k in [0..1] do
if (T2*T3)^3 eq A^i*B^j*C^k then i,j,k;end if;end for; end for;

H<a,b,c,d,e>:=Group<a,b,c,d,e|a^4,b^2,c^2,(a,b),(a,c),(b,c),d^2,
e^3=a^2,(d*e)^3=a*b,a^d=a*b*c,b^d=b,c^d=c,a^e=a^3*c,b^e=c,
c^e=a^2*b*c>;
#H;

f,H1,k:=CosetAction(H,sub<H|Id(H)>);
```

Appendix H

MAGMA CODE $PSL(2,11)$

```
G<x,y,t>:=Group<x,y,t|x^3,y^3,(x*y)^2, t^2,
(t,x*y^-1*x),
t*t^y*t^(x*y)*t^y=y^2*t^(x^2)*t^(y^2),
(x*y*t^y)^3>;
#G;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
Index(G,sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);

DC:=[Id(G1),
f(t),
f(t * x * t),
f(t * y * t),
f(t * x * t * y * t),
f(t * x * t* y^-1 * t),
f(t * x * t * x^-1 * t),
f(t * y * t * y^-1 * t)];

ts:=[Id(G1) : i in [1..6]];
ts[1]:=f(t);
```

```
ts[2]:=f(t^y);
ts[3]:=f(t^x);
ts[4]:=f(t^(x*y));
ts[5]:=f(t^(x^2));
ts[6]:=f(t^(y^2));
IN:=sub<G1|f(x),f(y)>;
cst := [null : i in [1 .. 55]] where null is [Integers() |];
prodim := function(pt, Q, I)
v := pt;
 for i in I do
v := v^(Q[i]);
end for;
return v;
end function;
for i := 1 to 6 do
 cst[prodim(1, ts, [i])] := [i];
end for;
m:=0; for i in [1..55] do if cst[i] ne [] then m:=m+1;  end
if; end for; m;
/*6*/

S:=Sym(6);
xx:=S!(1, 3, 5)(2, 4, 6);
yy:=S!(1, 2, 6)(3, 4, 5);
N:=sub<S|xx,yy>;
N1:=Stabiliser(N,1);
#N1;
N1;
Orbits(N1);

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2] eq g*(DC[i])^h then i; end if; end for;
end for;/* 3 */
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[4] eq g*(DC[i])^h then i; end if; end for;
end for;/*2*/
for i in [1..8] do
```

```
for g,h in IN do
if ts[1]*ts[3] eq g*(DC[i])^h then i; break;break;
end if; end for; end for;/*4*/

/*Third Double Coset*/
S:={[1,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

N12:=Stabiliser(N,[1,2]);
N12s:=N12;
tr1:=Transversal(N,N12s);
for i := 1 to #tr1 do
   ss := [1,2]^tr1[i];
   cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne [] then m:=m+1;  end
if; end for; m;
/*18*/
Orbits(N12s);

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1] eq g*(DC[i])^h then i; end if;
end for; end for;/*7*/

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[2] eq g*(DC[i])^h then i; end if;
end for; end for;/*2*/
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[3] eq g*(DC[i])^h then i; end if;
end for; end for; /*6*/
```

```
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[4] eq g*(DC[i])^h then i; end if;
end for; end for; /*5*/
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[5] eq g*(DC[i])^h then i; end if;
end for; end for; /*3*/
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[6] eq g*(DC[i])^h then i; end if;
end for; end for; /*4*/

/*S:={[1,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
/*{[ 1, 4 ]}*/

N14:=Stabiliser(N,[1,4]);
N14s:=N14;
#N14s;
/*2*/
tr2:=Transversal(N,N14s);
for i := 1 to #tr2 do
   ss := [1,4]^tr2[i];
   cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
Orbits(N14s);
for i in [1..8] do for g,h in IN do
if ts[1]*ts[4]*ts[1] eq g*(DC[i])^h then i; end if;
end for; end for;/*1*/
for i in [1..8] do for g,h in IN do
```

```
if ts[1]*ts[4]*ts[4] eq g*(DC[i])^h then i; end if;
end for; end for;/*2*/
for i in [1..8] do for g,h in IN do
if ts[1]*ts[4]*ts[2] eq g*(DC[i])^h then i; end if;
end for; end for;/*3*/
for i in [1..8] do for g,h in IN do
if ts[1]*ts[4]*ts[3] eq g*(DC[i])^h then i; end if;
end for; end for;/*4*/ */

/*Fourth Double Coset*/
S:={[1,3]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[3]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;

N13:=Stabiliser(N,[1,3]);
N13s:=N13;
#N13s;
/*1*/
tr3:=Transversal(N,N13s);
for i := 1 to #tr3 do
   ss := [1,3]^tr3[i];
   cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
/*30*/
Orbits(N13s);

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3]*ts[1] eq g*(DC[i])^h then i; end if;
end for; end for;/*8*/
for i in [1..8] do
for g,h in IN do
```

```
if ts[1]*ts[3]*ts[2] eq g*(DC[i])^h then i; end if;
end for; end for;/*6*/
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3]*ts[3] eq g*(DC[i])^h then i; end if;
end for; end for;/*2*/
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3]*ts[4] eq g*(DC[i])^h then i; end if;
end for; end for;/*7*/
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3]*ts[5] eq g*(DC[i])^h then i; end if;
end for; end for;/*3*/
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3]*ts[6] eq g*(DC[i])^h then i; end if;
end for; end for;/*4*/

/*Fifth Double Coset*/
S:={[1,2,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[2]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;
/*
    [ 1, 2, 4 ]
    [ 3, 4, 6 ]
    [ 5, 6, 2 ]
*/

N124:=Stabiliser(N,[1,2,4]);
N124s:=N124;
tr5:=Transversal(N,N124s);
for i := 1 to #tr5 do
   ss := [1,2,4]^tr5[i];
```

```
   cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne [] then m:=m+1;  end
if; end for; m;

for n in N do if [1,2,4]^n eq [3,4,6]
then N124s:=sub<N|N124s,n>;
end if; end for;
#N124s;
Generators(N124s);
[1,2,4]^N124s;

for n in N do if [1,2,4]^n eq [5,6,2]
then N124s:=sub<N|N124s,n>;
end if; end for;
#N124s;
Generators(N124s);
[1,2,4]^N124s;

/*34*/
Orbits(N124s);

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[4] *ts[1] eq g*(DC[i])^h then i; end if;
end for; end for;/*7*/

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[4]*ts[2] eq g*(DC[i])^h then i; end if;
end for; end for;/*3*/

/*Sixth Double Coset*/
S:={[1,2,3]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[2]*ts[3]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
```

```
then print SSS[i];
end if; end for; end for;
/*
 [ 1, 2, 3 ]
    [ 3, 1, 2 ]
    [ 2, 3, 1 ]
*/
N123:=Stabiliser(N,[1,2,3]);
N123s:=N123;
tr6:=Transversal(N,N123s);
for i := 1 to #tr6 do
   ss := [1,2,3]^tr6[i];
   cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
/*38*/
for n in N do if [1,2,3]^n eq [3,1,2]
then N123s:=sub<N|N123s,n>;
end if; end for;
#N123s;
Generators(N123s);
[1,2,3]^N123s;

for n in N do if [1,2,3]^n eq [2,3,1]
then N123s:=sub<N|N123s,n>;
end if; end for;
#N123s;
Generators(N123s);
[1,2,3]^N123s;

Orbits(N123s);

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[3] *ts[1] eq g*(DC[i])^h then i;
end if; end for; end for;/*3*/

for i in [1..8] do
```

```
for g,h in IN do
if ts[1]*ts[2]*ts[3]*ts[4] eq g*(DC[i])^h then i;
end if; end for; end for;/*4*/


/*Seventh Double Coset*/
S:={[1,2,1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[2]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

N121:=Stabiliser(N,[1,2,1]);
N121s:=N121;
tr7:=Transversal(N,N121s);
for i := 1 to #tr7 do
   ss := [1,2,1]^tr7[i];
   cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne [] then m:=m+1;
end if; end for; m;
/*50*/
Orbits(N121s);

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1] *ts[1] eq g*(DC[i])^h then i;
end if; end for; end for;/*3*/
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1]*ts[2] eq g*(DC[i])^h then i;
end if; end for; end for;/*7*/
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1]*ts[3] eq g*(DC[i])^h then i;
end if; end for; end for;/*4*/
```

```
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1]*ts[4] eq g*(DC[i])^h then i;
end if; end for; end for;/*5*/
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1]*ts[5] eq g*(DC[i])^h then i;
end if; end for; end for;/*7*/
for i in [1..8] do
for g,h in IN do
if ts[1]*ts[2]*ts[1]*ts[6] eq g*(DC[i])^h then i;
end if; end for; end for;/*8*/


/*Eighth Double Coset*/
S:={[1,3,1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[1]*ts[3]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;

N131:=Stabiliser(N,[1,3,1]);
N131s:=N131;
tr8:=Transversal(N,N131s);
for i := 1 to #tr8 do
   ss := [1,3,1]^tr8[i];
   cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..55] do if cst[i] ne [] then m:=m+1;  end
if; end for; m;
/*54*/

for n in N do if [1,3,1]^n eq [2,4,2]
then N131s:=sub<N|N131s,n>;
end if; end for;
#N131s;
```

```
Generators(N131s);
[1,3,1]^N131s;

Orbits(N131s);

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3]*ts[1] *ts[1] eq g*(DC[i])^h then i;
end if; end for; end for;
/*4*/

for i in [1..8] do
for g,h in IN do
if ts[1]*ts[3]*ts[1]*ts[3] eq g*(DC[i])^h then i;
end if; end for; end for;
/*7*/
```

Appendix I

MAGMA CODE of $3^{*2} :_m D_4$

```
S:=Sym(4);
xx:=S!(1,2,3,4);
yy:=S!(2,4);
N:=sub<S|xx,yy>;
G<x,y,t>:=Group<x,y,t|x^4,y^2,(x*y)^2,t^3,(y,t),t^(x^2)=t^2,
(x^2*y*t*t^x)^4>;
#DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
f,G1,k:=CosetAction(G,sub<G|x,y>);
#G1;
CompositionFactors(G1);
H:=sub<G|x,y,
y * x^2 * t * x * t * x * t * x * t * x * t * x * t>;
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
IN:=sub<G1|f(x),f(y)>;
ts := [ Id(G1): i in [1 .. 4] ];
ts[1]:=f(t); ts[2]:=f(t^(x)); ts[3]:=ts[1]^-1;
ts[4]:=ts[2]^-1;

DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
#DoubleCosets(G,sub<G|x,y>, sub<G|x,y>);
#G1;
```

```
DC:=[ f( Id(G)),
f( t),
f( t * x * t),
f( t *x * t * x * t^-1),
f( t * x * t*x*t),
f( t * x * t * x * t^-1 * x * t^-1),
f( t * x * t * x * t * x * t^-1),
f( t * x * t * x * t * x * t),
f( t * x * t * x * t^-1 * x * t^-1 * x * t),
f(t * x * t * x * t * x * t * x * t),
f( t * x * t * x * t * x * t * x * t * x * t) ];
 Index(G1,IN);

cst := [null : i in [1 .. 60]] where null is [Integers() | ];
prodim := function(pt, Q, I)
v := pt;
for i in I do
     v := v^(Q[i]);
    end for;
return v;
end function;
for i := 1 to 4 do
      cst[prodim(1, ts, [i])] := [i];
     end for;
m:=0; for i in [1..60] do if cst[i] ne [] then m:=m+1; end if;
end for;m;
Orbits(N);
for i in [1..#DC] do for m,n in IN do if ts[1] eq m*(DC[i])^n then i;
break;end if; end for;end for;

N1:=Stabiliser(N,1);
Generators(N1);
Orbits(N1);

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[1]
eq m*(DC[i])^n then i; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]
eq m*(DC[i])^n then i; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[3]
```

```
eq m*(DC[i])^n then i; break; break;end if; end for;end for;

S:={[1,2]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
 then print SSS[i];
end if; end for; end for;


N12:=Stabiliser(N,[1,2]);
Orbits(N12);
#N12;
N12s:=sub<N|N12>;
tr1:=Transversal(N,N12s);
for i:=1 to #tr1 do
ss:=[1,2]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..60] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1]
eq m*(DC[i])^n then i; break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[2]
eq m*(DC[i])^n then i; break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]
eq m*(DC[i])^n then i; break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[4]
eq m*(DC[i])^n then i; break; break; end if; end for;end for;


S:={[1,2,1]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
```

```
for g in IN do if ts[1]*ts[2]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[1]]
 then print SSS[i];
end if; end for; end for;

N121:=Stabiliser(N,[1,2,1]);
Orbits(N121);
#N121;
N121s:=sub<N|N121>;
tr1:=Transversal(N,N121s);
for i:=1 to #tr1 do
ss:=[1,2,1]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..60] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[1]
eq m*(DC[i])^n then i; break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[2]
eq m*(DC[i])^n then i; break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[3]
eq m*(DC[i])^n then i; break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[1]*ts[4]
eq m*(DC[i])^n then i; break; break; end if; end for;end for;

S:={[1,2,3]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[3]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
 then print SSS[i];
end if; end for; end for;

N123:=Stabiliser(N,[1,2,3]);
Orbits(N123);
#N123;
```

```
N123s:=sub<N|N123>;
tr1:=Transversal(N,N123s);
for i:=1 to #tr1 do
ss:=[1,2,3]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..60] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[1]
eq m*(DC[i])^n then i; break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[2]
eq m*(DC[i])^n then i; break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[3]
eq m*(DC[i])^n then i; break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if ts[1]*ts[2]*ts[3]*ts[4]
eq m*(DC[i])^n then i; break; break; end if; end for;end for;

S:={[1,2,1,2]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[1]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
 then print SSS[i];
end if; end for; end for;

N1212:=Stabiliser(N,[1,2,1,2]);
Orbits(N1212);
#N1212;
N1212s:=sub<N|N1212>;
tr1:=Transversal(N,N1212s);
for i:=1 to #tr1 do
ss:=[1,2,1,2]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..60] do if cst[i] ne []
```

```
then m:=m+1;
end if; end for;m;

for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[1]*ts[2]*ts[1] eq m*(DC[i])^n then i; break;
break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[1]*ts[2]*ts[2] eq m*(DC[i])^n then i; break;
break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[1]*ts[2]*ts[3] eq m*(DC[i])^n then i; break;
break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[1]*ts[2]*ts[4] eq m*(DC[i])^n then i; break;
break; end if; end for;end for;

S:={[1,2,1,4]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[1]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[4]]
 then print SSS[i];
end if; end for; end for;

N1214:=Stabiliser(N,[1,2,1,4]);
Orbits(N1214);
#N1214;
N1214s:=sub<N|N1214>;
tr1:=Transversal(N,N1214s);
for i:=1 to #tr1 do
ss:=[1,2,1,4]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..60] do if cst[i] ne []
then m:=m+1;
end if; end for;m;
```

```
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[1]*ts[4]*ts[1] eq m*(DC[i])^n then i; break;
break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[1]*ts[4]*ts[2] eq m*(DC[i])^n then i; break;
break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[1]*ts[4]*ts[3] eq m*(DC[i])^n then i; break;
break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[1]*ts[4]*ts[4] eq m*(DC[i])^n then i; break;
break; end if; end for;end for;


S:={[1,2,3,4]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[3]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
*ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])[4]]
 then print SSS[i];
end if; end for; end for;

N1234:=Stabiliser(N,[1,2,3,4]);
Orbits(N1234);
#N1234;
N1234s:=sub<N|N1234>;
tr1:=Transversal(N,N1234s);
for i:=1 to #tr1 do
ss:=[1,2,3,4]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..60] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[3]*ts[4]*ts[1] eq m*(DC[i])^n then i; break;
```

```
break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[3]*ts[4]*ts[2] eq m*(DC[i])^n then i; break;
break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[3]*ts[4]*ts[3] eq m*(DC[i])^n then i; break;
break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[3]*ts[4]*ts[4] eq m*(DC[i])^n then i; break;
break; end if; end for;end for;


S:={[1,2,1,2,1]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[1]*ts[2]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[1]]
 then print SSS[i];
end if; end for; end for;

N12121:=Stabiliser(N,[1,2,1,2,1]);
#N12121;
N12121s:=N12121;

for n in N do if [1,2,1,2,1]^n eq [2,1,2,1,2]
then N12121s:=sub<N|N12121s,n>;
end if; end for;
#N12121s;
Generators(N12121s);
[1,2,1,2,1]^N12121s;

for n in N do
if [1,2,1,2,1]^n eq [4,3,4,3,4]
then N12121s:=sub<N|N12121s,n>;
end if; end for;
Generators(N12121s);
[1,2,1,2,1]^N12121s;
```

```
Orbits(N12121);
#N12121;
N12121s:=sub<N|N12121>;
tr1:=Transversal(N,N12121s);
for i:=1 to #tr1 do
ss:=[1,2,1,2,1]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..60] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[1]*ts[2]*ts[1]*ts[1]  eq m*(DC[i])^n then i;
break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[1]*ts[2]*ts[1]*ts[2]  eq m*(DC[i])^n then i;
break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[1]*ts[2]*ts[1]*ts[3]  eq m*(DC[i])^n then i;
break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[1]*ts[2]*ts[1]*ts[4]  eq m*(DC[i])^n then i;
break; break; end if; end for;end for;

S:={[1,2,3,4,1]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[3]*ts[4]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
*ts[Rep(SSS[i])[4]]*ts[Rep(SSS[i])[1]]
then print SSS[i];
end if; end for; end for;

N12341:=Stabiliser(N,[1,2,3,4,1]);
#N12341;
N12341s:=N12341;
```

```
for n in N do if [1,2,3,4,1]^n eq [3,2,1,4,3]
then N12341s:=sub<N|N12341s,n>;
end if; end for;
#N12341s;
Generators(N12341s);
[1,2,3,4,1]^N12341s;


Orbits(N12341);
#N12341;
N12341s:=sub<N|N12341>;
tr1:=Transversal(N,N12341s);
for i:=1 to #tr1 do
ss:=[1,2,3,4,1]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..60] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[3]*ts[4]*ts[1]*ts[1] eq m*(DC[i])^n then i;
break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[3]*ts[4]*ts[1]*ts[2] eq m*(DC[i])^n then i;
break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[3]*ts[4]*ts[1]*ts[3] eq m*(DC[i])^n then i;
break; break; end if; end for;end for;
for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[3]*ts[4]*ts[1]*ts[4] eq m*(DC[i])^n then i;
break; break; end if; end for;end for;

S:={[1,2,3,4,1,2]};
SS:=S^N;SS;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for g in IN do if ts[1]*ts[2]*ts[3]*ts[4]*ts[1]*ts[2]
```

```
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
*ts[Rep(SSS[i])[4]]*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
 then print SSS[i];
end if; end for; end for;

N123412:=Stabiliser(N,[1,2,3,4,1,2]);
#N123412;
N123412s:=N123412;

for n in N do if [1,2,3,4,1,2]^n eq [2,3,4,1,2,3]
then N123412s:=sub<N|N123412s,n>;
end if; end for;
#N123412s;
Generators(N123412s);
[1,2,3,4,1,2]^N123412s;

for n in N do if [1,2,3,4,1,2]^n eq [3,2,1,4,3,2]
then N123412s:=sub<N|N123412s,n>;
end if; end for;
#N123412s;
Generators(N123412s);
[1,2,3,4,1,2]^N123412s;


Orbits(N123412);
#N123412;
N123412s:=sub<N|N123412>;
tr1:=Transversal(N,N123412s);
for i:=1 to #tr1 do
ss:=[1,2,3,4,1,2]^tr1[i];
cst[prodim(1,ts,ss)]:=ss;
end for;
m:=0; for i in [1..60] do if cst[i] ne []
then m:=m+1;
end if; end for;m;

for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[3]*ts[4]*ts[1]*ts[2]*ts[1]
eq m*(DC[i])^n then i; break; break;
```

```
end if; end for;end for;

for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[3]*ts[4]*ts[1]*ts[2]*ts[2]
eq m*(DC[i])^n then i; break; break;
end if; end for;end for;

for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[3]*ts[4]*ts[1]*ts[2]*ts[3]
eq m*(DC[i])^n then i; break; break;
end if; end for;end for;

for i in [1..#DC] do for m,n in IN do if
ts[1]*ts[2]*ts[3]*ts[4]*ts[1]*ts[2]*ts[4]
eq m*(DC[i])^n then i; break; break;
end if; end for;end for;
```

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