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ASSESSING STUDENT UNDERSTANDING WHILE SOLVING LINEAR EQUATIONS USING FLOWCHARTS AND ALGEBRAIC METHODS

Edima Umanah

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A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Teaching: Mathematics

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ABSTRACT

Solving linear equations has often been taught procedurally by performing inverse operations until the variable in question is isolated. Students do not remember which operation to undo first because they often memorize operations with no understanding of the underlying meanings. The study was designed to help assess how well students are able to solve linear equations. Furthermore, the lesson is designed to help students identify solving linear equations in more than one-way. The following research questions were addressed in this study: Does the introduction of multiple ways to think about linear equations lead students to flexibly incorporate appropriate representations/strategies in solving problems involving linear equations? Which representations do students use to solve linear equations and in what context?

By using the do/undo flowchart for solving linear equations, students' learning will develop relations between concepts, and their learning will involve understanding and interpreting concepts. In this study, two methods were taught to students to collect one set of data on solving linear equations. Students completed pre and posttest, and some students were selected to participate in a 10-15 minute interview based on their responses from their assessments to clear up any ambiguity on the post-assessment. During the interview process, I took notes. The findings on the pre-post assessments were qualitatively evaluated and revealed that students from the control/comparison group struggled to recall the inverse operation strategy used for solving linear equations in one variable.

The findings from the pre-post assessments also show that the experimental/treatment group may have benefited more from using the flowchart. The pre-post assessments were examined for each group because solving linear equations in one-variable is unfortunately taught using one procedure. However, the statistical analysis showed no significant difference between the groups.

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CHAPTER ONE

INTRODUCTION

Background

Multiple representations refer to different ways of describing or symbolizing a single mathematical idea: verbally, visually, numerically or symbolically. These representations can be used to develop, communicate, and understand different aspects or properties of a mathematical solution, object, or operation. They may include a wide range of thinking tools for problem solving in mathematics including graphs, diagrams, tables, grids, formulas, symbols, words, and pictures.

The intent of teaching multiple representations is to improve students' knowledge and proficiency and flexibility in solving a variety of mathematical problems. In this paper, we will focus on solving linear equations using diagrams. In middle schools, teaching with multiple representations can support learning new ideas. Solving linear equations using multiple representations will help students to develop a deep understanding of multiple ways to see problems and their solutions. This, in turn, builds flexible thinking when solving problems or something like this in mathematics. Symbolic procedures alone are no longer adequate to meet the demands of higher education. The flowchart method is also an algebraic method that involves the use of inverse operations. Thus, various models of teaching are needed in order to support different student learning styles. In meeting the goals of the common core standards, teachers will need to embrace strategies that will support those standards. In particular the Standards for Mathematical Practice SMP 7 and SMP 1 require students to “look for and make use of structure” and suggest that students should be able to represent a problem in different ways. The use of multiple representations is one way in which this standard could be supported. This will help learners to identify and evaluate efficient strategies for a solution. In this case, the teacher might help the students identify why using a “flowchart” to solve a linear equation is just as valid as solving the linear equation “algebraically”. Moreover, students may begin to understand why flowchart representations may be more useful in certain scenarios.

Goal and Research Questions

The goal of this MAT project is to investigate student learning when solving linear equations while using flow charts and algebraic representations to effect positive changes in my teaching and in student learning. In doing so, I will have the opportunity to extend existing professional development experiences to meet my individual needs and the needs of my students.

Using multiple representations in solving algebraic linear equations should enable all students to

- Create and use representations to organize, record, and communicate mathematical ideas;

- Select, apply, and translate among mathematical representations to solve algebraic problems;
- Use representations to model and interpret physical, social, and mathematical phenomena (NCTM 2000, p. 67).

By using these representations, students will have the ability to select, apply, and translate among different representations (Fried and Amit, 2004). Students will learn how to solve algebraic equations with the use of graphs and diagrams (flowcharts).

The representations are aimed at having students not only develop proficiency in solving linear equations, but also develop a conceptual understanding of the solution process. “The ways in which mathematical ideas are represented is fundamental to how people can understand and use those ideas. When students gain access to mathematical representations and the ideas they represent, they have a set of tools that significantly expand their capacity to think mathematically” (NCTM, 2000 p.67). The goal of this research is to encourage students to express their ideas by providing tools that will allow them to show how the process for solving linear algebraic equations makes sense to them.

The following questions are addressed in this study:

- Does the introduction of multiple ways to think about linear equations lead students to flexibly incorporate appropriate representations in solving problems involving linear equations?

- Which representations do students use to solve linear equations and in what context? How do students use representations when presented with a specific task?

Significance

It is an instructional challenge for most teachers in middle and high school to implement strategies for solving linear equations. This foundational topic is frequently taught procedurally and students along with teachers believe that this is the only way. It starts out with using the order of operations and then students are taught either multiply/divide to isolate the variable. This may all seem so simple for the teachers to teach using the procedural strategy but it has created frenzy for most students to understand and apply it as a learning tool. This procedure is presented in textbooks for teachers to replicate over and over again.

In my experience while observing other teachers, they continue to use this procedure year after year in the classroom because students may be confused when they are taught more than one way. Teachers may not know how to show multiple strategies on solving equations.

The same procedures are taught over and over again to students because teachers believe that students will become confused and the textbooks teach procedures that are outlined step by step without conceptual understanding. At the elementary level, students begin solving equations in this form: $n + 6 = 10$. Students are able to understand this equation. As the students progress through the K-5 modules at the elementary level, the circles are replaced with variables and it is a shift for students to become independent learners. In previous years observations, some students struggle with when to add and subtract terms and when they need to undo the operations. They also get stuck when they come to a solution with a zero. For instance, $3x = 0$. They aren't confident about what to do afterward. Students do not understand that the sequence of learning mathematics in previous grades leads to an understanding of future topics in mathematics unless teachers support them in transferring their learning to new concepts. It is also essential for teachers to help students gain fluency with procedures such as eliminating a fraction so that the work does not become cumbersome in the end and they do not get frustrated. For instance, it helps to have students read and write out the problem first. Three multiplied by x decreased by 2 and divided by 4 or multiplied by one-fourth equals negative five. With the written words, the students can construct a flow chart and solve the above equation using Do/Undo order of operations. If students could improve their fluency in working with expressions and equations through strategies like the DO/UNDO flowcharts, it would greatly improve students' opportunities to succeed in secondary mathematics.

CHAPTER TWO

LITERATURE REVIEW

Introduction

Students often tend to stick to the standard strategy, even when they have the freedom to use other strategies. Lack of strategy freedom can prevent students from exploring alternative strategies on their own. Students that tend to stick to known methods are less likely to think flexibly and discover more efficient solution strategies for various situations. The same procedures are taught over and over again to students because teachers believe that students will become confused and the textbooks teach procedures that are outlined step by step without conceptual understanding.

Benefits of Teaching with Multiple Strategies

Star (2005) suggests that there is a possible trade-off in the initial stages of learning between the goal of the flexible use of multiple strategies and the goal of mastery of a standard algorithm. Star and Rittle-Johnson (2008) showed that prompting students to solve the same equation in different ways provides better results on items measuring students' strategic flexibility. By "student flexibility" we refer to the practice of allowing students to pursue multiple solution strategies within a given problem. (Waalkens, Aleven, and Taatgen 2013) asked the question, "But does greater freedom mean that students learn more robustly?"

They developed three versions of the same Intelligent Tutoring System (ITS) for solving linear algebraic equations that differed only in the amount of freedom given to students. The three conditions are (a) strict standard strategy, (b) flexible standard strategy and multi-strategy. The strict standard strategy adhered to a specific standard strategy, while the other two versions (flexible and multi) adhered to minor and major variations, respectively. According to Waalkens, Aleven, and Taatgen, with both the strict and flexible strategies, all equations had to be solved with a standard strategy that is widely used in American middle-school mathematics textbooks. They claimed that this standard strategy can solve almost all linear equations and is described as follows: First, use the distributive law to expand any term in parentheses. Second, combine constant terms and variable terms on each side of the equation. Third, move variable terms to one side of the equation and constant terms to the other side. And finally, divide both sides by the coefficient of the variable. The authors go on to say that students had the most freedom in the multi-strategy method because they could solve the linear equations with any strategy that progresses towards the goal of arriving at a solution. For example, in the linear equation $2(x + 1) = 4$, students are allowed to divide both sides of the equation by 2 instead of using the distributive law to expand the term in parentheses, a step that is required in the two stricter methods. With the multi-strategy method, students have the most freedom because they can solve the equations with any strategy that progresses toward the goal of solving the equation. Waalkens, Aleven, and Taatgen's study concluded that ITS helped students improve their equation-solving skills. However, allowing minor or major strategy variations did not make a difference in learning gain, motivation, or perceived strategy freedom, compared to strictly enforcing a standard strategy with which students were familiar, without allowing any variations.

Benefit of Teaching Multiple Representations

Greeno and Hall mentioned

> Forms of representation need not be taught as though they are ends in themselves. Instead, they can be considered as useful tools for constructing understanding and for communicating information and understanding. If students simply complete assignments of constructing representations in forms that are already specified, they do not have opportunities to learn how to weigh the advantages and disadvantages of different forms or representations or how to use those representations as tools with which to build their conceptual understanding. (1997, p. 362)

They go on to say that representations enhance the problem-solving ability and that students often construct representations in forms that help them see patterns and perform calculations.

"The use of multiple representations with or without technology, is one of the major topics in mathematics education that has gained importance in recent decades" (Ozgun-Koca, 1998). The significance of representing the solution of linear algebraic equations in multiple ways provides the same objective of more than one form. It is necessary to see how students use these representations. It is suggested that multiple representations provide an environment for students to abstract and understand major concepts (McArthur et al.1998, Yerushalmy, 1991) while constructivist theory suggests that we need to understand students' thinking processes in order to facilitate their learning in more empowering ways (Stepphe, 1991). Understanding students' thinking and their preferences while choosing a representation type for solving algebraic linear algebraic equations helps mathematics teachers gain insight into student thinking. Representations such as the do/undo flow chart and algebraic method are tools that provide the same information in more than one form. The role of these tools in the task mentioned above is to represent solving linear algebraic equations using multiple concretizations of a concept, mitigate certain difficulties and to make mathematics more attractive and interesting (Ozgun-Koca, 98). Dienes' mentioned that conceptual learning is maximized when children are exposed to a mathematical concept through a variety of physical contexts or embodiments. In other words, we should not expect that all students would perceive the same concept from one representation. Algebraic concepts have become a study of procedures and rules instead of exploration and concepts, which should lead to generalizations that justify the rules.

The Meaning of the Equal Sign

Early elementary school children … view the equal sign as a symbol that separates a problem and its answer. (Kieran 1981, p. 324). Students will have difficulty in solving linear equations if they do not understand what the equal signs mean. It is important to build on what students might have seen in the elementary grades such as a problem like this:

$$8 + 4 = n + 5.$$

If teachers can build on the above problem to find out what students understand by it, then the difficulty in solving this linear equation, $3x - 4 = 7x + 8$ may be less.

Many studies such as (Austin & Vollrath, 1989 and Star & Ozgun-Koca, 1998) have been carried out emphasizing the use of multiple strategies in mathematics. Based on such studies, the hypothesis proposed in this paper is that exposing students to multiple representations/multiple strategies using tools such as a graphical method and a flow chart will lead to improved flexibility in problem-solving. For the sake of this action research, both methods are described below

CHAPTER THREE

METHODOLOGY

This action research will be conducted to determine the effectiveness of teaching that includes using the direct teaching model in the use of flow charts and algebra to find solutions to linear equations. I will use quantitative and qualitative methods in this study to compare the results of the two groups that learned using a flowchart and direct teaching method (experimental/treatment group) and the other group (control/comparison group) that received only the direct teaching method in solving linear algebraic equations in one variable. During the data collection period, I will conduct the interviews for certain participants. Interview participants will be chosen based on the responses, which need clarification. The pre-post assessment given to the participants is identical. However, the interview questions might vary depending on the responses of the participants.

Solving Linear Equations in One Variable using a Flowchart

This section describes the instructional methods, i.e., the flow chart and for solving algebraic linear equations. Here is an example of an application of the flowchart method below.

Solve for x: 5(x+7)/3 = 20.

Start with *x*	*x*
Add 7	*x*+7
Multiply by 5	5(*x*+7)
Divide by 3	5(*x*+7)/3
Equals 20	5(*x*+7)/3 = 20

Figure 1. The DO Strategy

Start with 20	20
Multiply by 3	20(3) = 60
Divide by 5	60/5 = 12
Subtract 7	12-7 = 5
Equals *x*	5 = *x*

Figure 2. The UNDO Strategy

Incorporating the DO/UNDO Flowchart alongside the Traditional Method

Students will learn how to solve linear algebraic equations using a DO/UNDO flow chart. They will also learn how to solve linear algebraic equations using a traditional algorithmic approach. The intent of the DO/UNDO method is to help students understand how to use inverse operations in an appropriate order and why that order matters. The "DO" part of the flowchart outlines what has been done to the variable in creating the equation. Within the "UNDO" part of the flow chart, inverse operations are applied in the reverse order so that the unknown value of the variable may be determined. In other words, the DO/UNDO flowchart may be used to help students identify what is being done to the variable so that they may, in turn, correctly use inverse operations to solve for the variable. In doing so, students should better understand the important role order of operations plays in solving equations.

When applying the algebra for solving linear algebraic equations alongside the flowchart, the learning goal for students is to conceptually understand how and why the algorithm works.

Research Lessons

Implementing conceptual exercises on solving linear equations in five Grade 8 classes were based on the Key to Algebra Book 3 for Equations. A linear equation is a topic first covered in Grade 6, thus students may have some prior knowledge regarding this concept. Solving linear equations extends the ability for students to write and solve equations that require more steps. The students will solve an equation by using the undoing method. Students will have to focus on operations being applied to the variable and the use of inverse operations to undo them. The undo method can be classified as a graphic organizer, which helps to emphasize operations in the reverse order to solve for the unknown variable in the given equation. Then the algebraic method helps the student to develop a technique that will be useful in solving other kinds of equations. I implemented various lessons for two weeks.

- Lesson 1, One-step equations were created for students to think of a number that would make a true sentence. Conceptually, students will learn that only one number will work. The flowchart was introduced to the experimental/treatment group simultaneously with the standard procedure when solving one-step and two-step equations. Each one-step/two-step equation was solved using a flowchart before solving the same equation with the standard procedure strategy side by side.
- Lesson 2, Solving Equations with variables on both sides. Students will learn quickly that not all equations are simple. Students will learn to collect like variables on the same side.
- Lesson 3, Solving Multi-Step Equations. Students will have to check to see whether one or both sides of the equation can be simplified before they use inverse operations to solve an equation.

- Lesson 4, Using Equations to Solve Problems. Students will have to make up an equation for each problem and then solve the equation to reach a solution.
- Lesson 5, Age Puzzles. Using Algebra, students will come up with an about age and follow the same procedure done in lesson 4.
- Lesson 6, The Multiplication Principle for Equations. Students will learn how to use the Multiplication Principle to solve equations by multiplying both sides of an equation by the same number.

Pre-and Post-Assessments

A pre-assessment was given to determine students' prior knowledge of solving linear equations before the lessons were implemented. Students' approach to each problem was also taken into consideration. The assessment consisted of 8 items, and students were given two class periods to complete the assessment. Having used mental math at some point during their math years to solve equations, students could think freely without the use of the rules of algebra on a few of the items. The last two problems were a good example of how students displayed their conceptual understanding without the use of algebra methodically. Reached solutions on the pre-test were analyzed to address misconceptions when solving linear equations. Between concepts and procedures, their responses were also checked for some procedural fluency and quality of explanations. The lessons were implemented an hour each every day for two weeks (10 days) with whole group instruction, differentiated instruction in small groups and mathematical discussions. Making connections between the daily given tasks and classroom discussions helped with procedural approaches to increasing student understanding through visual representation/diagrams (flowcharts). While solving linear equations, the flowchart was a tool used to help build coherence, perseverance, and reasoning abilities in students.

The post-assessment was implemented in the next part of my research. The same questions were featured on the post-assessment to illustrate and document their academic gain. The lessons were taught in a specific sequence so that they can build on the previous problem. Students were encouraged to solve each problem one step at a time. With the post-assessment, the goal was to help students understand how using the flowchart diagram shifts them to solving the linear equations algebraically to become more independent learners.

To solve linear algebraic equation problems students need help developing and making sense of the rules they are using to show them how to employ a variety of strategies. As students are taught how to develop a deeper understanding of solving equations, they are given an opportunity to solve linear equation problems in different contexts. The designed pre and post-assessments for my research engaged students to make solving equations meaningful. Assessments were collected using standard school practice since all students took the assessment as part of a regular classroom routine. The data used for this study only came from students who had parental consent.

Interviews

Interviews were conducted with students who had parental consent to determine their perceptions regarding how their thinking was affected by the introduction of various representations on the post-assessment. I chose from a variety of questions I had prepared ahead of time depending on the circumstances of the students' response to a specific item on a paper. During the interview process, I jotted down the students' responses to my question(s). I chose students based on their responses to the post-assessment using the following criteria: written solution using the algebraic method of solving linear equations and it is correct or incorrect, a written solution with limited understanding of using the algebraic method, written solution using a different strategy, written solution using conceptual understanding and written solutions with just an answer and no explanation.

To determine the effectiveness of teaching, which included using the direct teaching model along with flowcharts, the student interviews (qualitative) were analyzed qualitatively. Student assessments were categorized into types of solution strategies in the charts on page 25 and 31. For example, guess and check strategy with correct solutions and no explanations, solved equation with the correct solution and no explanation, solved equation with wrong solution and no explanation, solved equation with a correct solution with minor mistakes and no explanation, solved equation with the wrong solution with major errors and no explanation, solved equation with the correct solution and an explanation, solved equation with wrong solution and an explanation and finally, a blank or an incomplete solution. Both the pre and post-assessment were compared to the quality of their solution strategies and development of their conceptual understanding from the pre to the post-assessment. Student interviews were conducted to help clear up misconceptions and reshape meaningful learning to promote conceptual understanding. The primary objective of mathematics instruction should be to improve students 'reasoning and sense-making capabilities. In mathematics learning, logic and sense making are important. Students who truly understand mathematical concepts will apply them to problem-solving and new circumstances and use them as a basis for future learning. There is ample evidence that learning is improved by teachers paying attention to the information and values that learners bring to a learning mission, using this information as a starting point for new teaching, and tracking the evolving expectations of students as teaching continues (Bautista, 2017).

CHAPTER FOUR

RESULTS

Pre-Assessment Results

The data from the first part of this study came from the pre-assessment used to evaluate the student's prior knowledge of solving linear equations and their abstract level of linear equation comprehension.

Table 1: Pre-Assessment Item 1

1. 4n + 10 = 50
<u>Written Response(s):</u> • "I was thinking to multiply 4x10 but that makes 40 and I thought n=1. I added then multiplied" (Respondent 18, 2017) • "In my thought, I already knew how to do this so it was pretty easy except #5 that was pretty tricky" (Respondent 22, 2017) • "What I did is multiply and simplify" (Respondent 16, 2017) • "I was thinking that to get 50 you would have to have 40 so 4n=40" (Respondent 24, 2017) • "I believe 4n represents 40 so if 4n is 40 and you add 10 it will give you 50" (Respondent 28, 2017) • "The answer is 10 because 4 times 10 + 10 = 50" (Respondent 9, 2017)

On item 1, most students did not explain their thoughts on how they were thinking. Four students did the guess and check method to find the value of n. Six students explained their thinking on this question. Thus, 18 out of 27 students taking the assessment gave the correct answer, *n* = 10. In analyzing several students' solutions, it was clear that they were using mental math.

Table 2: Pre-Assessment Item 2

2. $\frac{x}{3} - 4 = 2.$

Written Response(s):

- "I subtracted x from 3 and I got 2 and 4-2 = 2" (Respondent 18, 2017)
- "I was thinking that scince (since) 2 from 4 and 3 it has zero" (Respondent 24, 2017)
- "The answer is 18 because 3 goes into 18 6 times and 6 - 4 = 2" (Respondent 9, 2017)
- "X = 2 because I subtracted 4 by 2 and got 2" (Respondent 10, 2017)
- "What I did was put a 1 under the -4 and turn it into a fraction so then I can divide" (Respondent 16, 2017)
- "X = 18 because 18 3 = 6 and 6 - 4 = 2" (Respondent 13, 2017)

Of the 27 students taking the assessment, 8 students stated correctly that x = 18. Their methods involved guessing and checking and substituting their solution into the original equation. Conceptually, these students understood this problem- solving strategy helped them to come up with a solution that fits the condition. Other students had solutions such as x = 2 because they solved using inverse operations and forgot to have the fraction equal 6. Instead, they wrote

$\frac{x}{3} + \frac{6}{1}.$

Table 3: Pre-Assessment Item 3

3. -3n + 12 = -12

<u>Written Response(s):</u>

- "What I did was I put +3 on both sides then subtracted and I ended up with 12n and -9 so I put them into a fraction then divided" (Respondent 16, 2017)
- "If -3n was able to turn 12 into negative 12 I thought that it had to be lower than -12, I thought to get 12 to zero subtract 12 then subtract another 12 to get -12 which means -3n = -24 then to find n divide -24 by -3 which equals 8 so n = 8" (Respondent 15, 2017)
- "I just added the number till I got what the answer was" (Respondent 18, 2017)
- "n = 4 because 4 times -3 is -12" (Respondent 10, 2017)
- "I was thinking of how to do the problem to get an answer" (Respondent 24, 2017)
- "The answer is 0 because 0 and -0 made the 12 negatives" (Respondent 9, 2017)

Twelve students solved *n* = 8 for this question. Nine of them solved the linear equation using the algebraic method while 1 student solved using the guess and check strategy. In this item, one student wrote -3 + *n* = -4. -4 +12 = 8.

Table 4: Pre-Assessment Item 4

4. $\frac{x}{5} - (-14) = 8.$

<u>Written Response(s):</u>

- "x = -30 because -30+5=-6 and -6 + 14 = 8 (Respondent 13, 2017)

- "What I did was I turned the equation into a fraction then I multiplied finally and divided" (Respondent 16, 2017)
- "The answer is 3 because 14 - 5 = 11 and 11 - 8 = 3" (Respondent 9, 2017)

In analyzing item 4, 4 students solved x = -30. One student solved using the guess and check strategy while the other 3 students solved using the algebraic method. Four students left the question blank. Item 4 involved students at least conceptually understanding that subtracting a negative integer produces a positive integer. Even after a few students subtracted negative 14, they were still confused about how to conceptually interpret this item. One student subtracted 14 from both sides of the equation and ended up with two equivalent fractions. Another student multiplied 5 to the first term and 5 to the solution to receive *x* = 29.

Table 5: Pre-Assessment Item 5

5. $\frac{-5x+6}{2} = -22.$

Written Response(s):

- " I added the top of the fraction then I divided 2 times -11 which gave me -22" (Respondent 16, 2017)

Two students multiplied -22 times 4 to get -44 and then wrote the linear equation -5*x* + 6 = -44 to solve for a value of *x* while 3 other students used the guess and check strategy. Out of 27 students, 5 students solved for *x* correctly while 1 student made an error on the value of *x*. For further analysis of student work on this item, please see the post-assessment results on p. 29.

Table 6: Pre-Assessment Item 6

6. $\sqrt{x^2+3^2} - 5.$

<u>Written Response(s):</u>

- "What I did was multiply 3 times 3 and x times x which gave me 9x" (Respondent 16, 2017)

While some students attempted this problem, none of their strategies led them to the correct solution. A good number of students left this item blank.

Table 7: Pre-Assessment Item 7

7. "I'm thinking of a number. If you multiply it by 6 and then add 7, you will get 55. What is my number?"

<u>Written Response(s):</u>

- "Your number is 8 because 6 times 8 equals 48 and once you add 7 you get 55" (Respondent 27, 2017)

- “The number is 48” (Respondent 22, 2017)

Four students gave substantially the same answer and explanation as respondent 27 quoted above. A few students wrote the number is 8 without any explanation. Twelve other students answered this item by writing a linear equation and solved for the unknown variable using the algebraic strategy. They were successful. Five students solved this item using the guess and check strategy. Four students responded to this item by saying, “the number is 48”.

Table 8: Pre-Assessment Item 8

8. A shake at the Shack cost 80 cents and the bill for three burgers and a shake is $4.40. “How much is a burger?”

<u>Written Response(s):</u>

- “A burger is $1.20 each” (Respondent 25, 2017)
- “The burger would be 1.20$. Three burgers would be 3.60$” (Respondent 10, 2017)
- “The burgers would be $3.60 because $4.40 - .80 = $ 3.60. The answer is $3.60” (Respondent 9, 2017)
- “A burger costs $3.60” (Respondent 16, 2017)
- “The burgers are a dollar” (Respondent 7, 2017)
- “ The burger cost $2.00” (Respondent 1, 2017)

For item 8, 10 students wrote a linear algebraic equation to find their unknown. Two students wrote the correct equation and solved it correctly. One student wrote the equation $x + 3 + 80 = 4.40$; $3x + 80 = 4.40$. Then the equation was solved until $x = 1.20$. In this case, the student made an error of stating that $x + 3 = 3x$. The wrong equation was written down and the correct solution was arrived at in the end. Another student wrote the correct equation but arrived at the incorrect solution. Four students applied the guess and check strategy to arrive at the correct solution for this problem.

Table 9. Summary of Pre- Assessment Analysis Overview for each Item

Item	Guess & Check strategy w/ correct solutions and no explanation T vs. C		Solved equation w/ the correct solution and no explanation T vs. C		Solved equation w/ the wrong solution and no explanation T vs. C		Solved equation w/ a correct solution and minor errors with no explanation T vs. C		Solved equation w/ a wrong solution and major errors with no explanation T vs. C		Solved equation w/ the correct solution and an explanation T vs. C		Solved equation w/ the wrong solution and an explanation T vs. C		A blank or an incomplete solution T vs. C	
1	1	3	11	4	2	3	0	0	0	0	0	0	5	1	0	0
2	3	1	1	2	11	6	0	0	0	0	0	1	4	2	0	1
3	1	0	4	4	8	4	1	0	0	0	1	0	4	2	0	2
4	1	0	2	1	11	6	0	0	0	0	0	1	1	2	2	2
5	3	0	2	0	8	6	0	0	0	0	0	1	1	0	3	3
6	0	0	0	0	11	7	0	0	1	0	0	0	1	0	3	3
7	4	1	8	4	1	2	0	1	1	0	1	0	1	0	3	2
8	2	2	7	3	3	3	2	0	0	1	1	1	3	0	3	1

T stands for the treatment group while C stands for the comparison group.

A general analysis of the above assessment is on page 31.

Post - Assessment Results

To determine if their level of conceptual understanding of solving linear equations improved, the post-assessment will be analyzed. Student flexibility practice with the use of multiple strategies will be determined on each problem. In order by the question of the assessment, the results will be discussed. Although the flow chart was taught to the treatment group alongside the standard solving equation strategy, no student used the flowchart to solve any of the equations in the post-assessment.

Table 10. Post - Assessment Item 1

1. 4n + 10 = 50
Written Response(s) • "I have to isolate the variable by using the inverse operation." (Respondent 13, 2018). • "My first step was to isolate the variable by using inverse operation than using inverse operation once again to get the final value of n."(Respondent 4, 2018).

On this item, there were 18 students who solved the equation algebraically and got it correct. Fewer students used the guess and check strategy for this item. In the pre-assessment, 4 students used the guess and check strategy while on the post-assessment, 2 students solved using the guess and check strategy. A few students solved using the algebraic strategy making minor errors. Other students need further review of this item.

Table 11. Post - Assessment Item 2

2. $\frac{x}{3} - 4 = 2.$
<u>Written Response(s)</u> • "At first I thought to use the inverse operation to get the variable alone, then I multiplied both sides by 3 to get my final answer of x = 18." (Respondent 4, 2018).

Six students solved this question algebraically and got the solution correct. Two students used the guess and check strategy. Eight students performed the inverse operation of adding 4 to both sides of the equation but failed the next procedures in this problem to arrive at the right solution. Nine students still need guidance on how to approach this problem. One student wrote a solution to be $x = 18$ without an explanation.

In general, students appeared to move from the use of guess and check strategies to the standard procedure using inverse operations to solve this equation. From the comparison group, respondent 002, used the guess and check strategy on the pre-assessment with the correct solution but on the post-assessment, respondent 2 attempted to use the standard solving equation strategy with the wrong solution. From the treatment group, on the pre-assessment respondent 19 appeared to have solved the equation comparing two fractions with the wrong solution but in the post-assessment, respondent 19, solved this equation using the standard strategy with the correct solution.

Table 12. Post - Assessment Item 3

3. -3n + 12 = -12.
<u>Written Response(s)</u> • “My first thought was to use the inverse operation to simplify the equation, then using inverse operation to isolate x and getting the final value.” (Respondent 4, 2018).

There were 7 students who applied the algebraic strategy in this item and arrived at the correct solution while doing so. Nine students incorrectly added two integers and did not include the negative sign, which led to their final answer being a negative solution instead of a positive solution. The other students failed this item because they need a review on adding and subtracting integers.

Table 13. Post - Assessment Item 4

4. $\frac{x}{5} - (-14) - 8.$
<u>Written Response(s)</u> • “At first I thought to simplify the equation by inverse operation, then I multiplied both sides by 5 to isolate x and get the final value of x.” (Respondent 4, 2018).

This item contained double negatives. Four students simplified the double negatives to a positive and applied the algebraic strategy. One student used the guess and check strategy. Other students used the inverse operation first in this problem and then solved the linear equation for *x*.

Table 14. Post - Assessment Item 5

5. $\frac{-5x+6}{2} = -22.$

For this item, 6 students solved the linear equation algebraically. Although 1 student solved the linear equation algebraically, the solution was wrong. A few students did not show any work while solving this problem and arrived at the right solution. One student in particular divided -5*x* and 6 by 2 to simplify to 2.5*x* + 3= -22, then solved the linear equation for *x* correctly.

Table 15. Post - Assessment Item 6

6. $\sqrt{x^2+3^2} = 5.$

For this item, one student left the question blank. All other students attempted the item but were unsuccessful except for two students who attempted this item using a conceptual understanding of square roots and they were partially correct.

Table 16. Post - Assessment Item 7

7. “I’m thinking of a number. If you multiply it by 6 and then add 7, you will get 55. What is my number?”
Written Response(s) • “If you multiply 12 times 3 you will get 36 then you add 15.” (Respondent 12, 2018).

Item 7 is a word problem that may need to be translated into symbols. Three students produced incorrect reasoning for this problem. One student used the guess and check strategy and arrived at the correct answer. Seventeen students created an equation and solved it using inverse operations while arriving at the correct solution. Other students created an equation close to the correct solution but fell short by either writing the wrong operation or omitting the equality symbol. The last subsequent students were able to create an equation but they could not solve it.

Table 17. Post - Assessment Item 8

8. A shake at the Shack cost 80 cents and the bill for three burgers and a shake is $4.40. “How much is a burger?”

Thirteen students were able to set up a linear equation and gave the correct answer, each burger cost $1.20. Three students subtracted 0.80 from 4.40 and then divided by 3 to get the correct answer. Six students tried setting up an equation but failed along the way. One student did set up the correct equation but was confused about the use of the order of operations. Other students were completely incorrect, leaving it blank and or incomplete.

Table 18. Summary Post- Assessment Analysis Overview for each Item

Item	Guess & Check strategy w/ correct solutions and no explanation T vs. C		Solved equation w/ the correct solution and no explanation T vs. C		Solved equation w/ the wrong solution and no explanation T vs. C		Solved equation with a correct solution and minor errors w/ no explanation T vs. C		Solved equation w/ a wrong solution and major errors w/ no explanation T vs. C		Solved equation w/ the correct solution and an explanation T vs. C		Solved equation w/ the wrong solution and an explanation T vs. C		A blank or an incomplete solution T vs. C	
1	1	1	10	6	3	3	1	0	0	0	0	0	0	0	0	0
2	2	0	4	2	9	6	0	1	0	0	0	0	0	0	0	1
3	0	0	5	2	10	3	0	4	0	0	0	0	0	0	0	1
4	1	0	2	2	11	7	1	0	0	0	0	0	0	0	0	1
5	0	0	5	1	8	9	11	0	0	0	0	0	0	0	0	0
6	0	0	0	0	13	10	0	0	0	0	0	0	0	0	0	1
7	1	0	11	6	2	4	1	0	0	0	0	0	0	0	0	0
8	0	0	7	6	7	4	1	0	0	0	0	0	0	0	1	0

In comparing the summary of the two tables, students from both groups used the guess and check strategy more during the pre-assessment than the post-assessment. In the pre-assessment, at least one student applied this strategy for all items except item 6 but in the post-assessment, students applied this strategy to items 1,2,4, and 7. They gave fewer written responses and fewer explanations in the post-assessment. Students largely moved from mental math or guess and check strategies to the standard strategy of solving linear equations. According to Hiebert, (1999, p.7) and other authors, once students are taught a procedure, they become less likely to use sense-making methods. Students provided fewer explanations on the post-assessment, but more correct answers compared to the pre-assessment, and students used the standard procedure more often on the post-assessment. In the case of solving equations with correct solutions and no explanations, students appeared to perform better on the post-assessment, as evidenced by the increased number of correct solutions from pre to post. For solved equations with the wrong solution and no explanation, students received more wrong solutions on the post-assessment than the pre-assessment.

In the category of the solved equation with the correct solution with minor errors and no explanation, while students gave fewer correct pre-assessment responses (to be anticipated because the pre-assessment took place before the instruction), students who arrived at the correct answers showed more of their thought compared to their post-assessment work. For example, Respondent 002, used the guess and check strategy on the pre-assessment, for items 1-4. In the post-assessment, this respondent used the standard strategy for questions 1 and

3 only. Students had difficulty solving equations involving fractions or fractional expressions, and this was just as true in the post-assessment. Most students had difficulty applying the standard procedure of solving equations involving a numerator and denominator, which increased the intensity of the problem. Most students also moved from methods that worked for them in the pre-assessment to methods that they were trying to apply such as the standard solving equation strategy. According to Battista, “students must stay engaged in making personal sense of mathematical ideas. Furthermore, students must believe-based on their past experiences-that they are capable of making sense of mathematics” (2017). For the next three categories, students from both groups did better on the post-assessment than the pre-assessment. Finally, fewer students left blank answers during the post-assessment than the pre-assessment.

On the next few pages, the participant data is presented using 0 and 1. 0 represents the control/comparison group and 1 represents the experimental/treatment group.

Participant Data

Table 19. Pre-Test Participant Data for Items 1 - 4

Participants	Treatment	Item 1	Item 2	Item 3	Item 4
3	1	1	1	1	0
4	1	1	0	1	0
6	1	1	0	0	0
7	1	0	0	0	0
14	1	1	1	1	1
15	1	1	1	0	1
16	1	0	0	0	0
17	1	1	0	1	0
18	1	1	0	0	0
19	1	1	0	1	0
20	1	1	0	0	0
21	1	1	1	0	0
22	1	0	1	1	0
23	1	1	0	0	0
24	1	1	0	1	1
25	1	1	0	1	0
27	1	1	0	0	0
28	1	0	0	0	0
1	0	0	0	0	0
2	0	1	1	1	0.5
5	0	1	1	0	0
8	0	1	0	0	0
9	0	1	0	0	0
10	0	1	0	0	0
11	0	0	0	0	0
12	0	0	0	0	0
13	0	1	1	1	1
		Q1	Q2	Q3	Q4
		0.74	0.29	0.37	0.16

See explanation of all four tables on page 39.

Table 20. Pre-Test Participant Data for Items 5 - 8

Participants	Treatment	Item 5	Item 6	Item 7	Item 8
3	1	1	0	1	1
4	1	0	0	1	1
6	1	0	0	1	0
7	1	0	0	0	0
14	1	1	0	1	1
15	1	1	0	1	1
16	1	0	0	1	0
17	1	0	0	0	0
18	1	0	0	1	0
19	1	0	0	1	1
20	1	0	0	0	0
21	1	1	0.5	1	1
22	1	0	0	0	1
23	1	0	0	1	1
24	1	0	0	1	1
25	1	0	0	1	1
27	1	0	0	1	1
28	1	0	0	0	0
1	0	0	0	0	0
2	0	0	0	1	0
5	0	0	0	0	0
8	0	0	0	1	1
9	0	0	0	0	0
10	0	0	0	1	1
11	0	0	0	1	1
12	0	0	0	0.5	1
13	0	1	0	1	1
		Q5	Q6	Q7	Q8
		0.18	0.01	0.68	0.59

See explanation of all four tables on page 39.

Table 21. Post – Test Participant Data for Items 1 - 4

Participants	Treatment	Item 1	Item 2	Item 3	Item 4
3	1	1	0	0.5	0
4	1	1	1	1	1
6	1	0	0	0	0
7	1	0.5	0	0	0
14	1	1	1	0	1
15	1	0	1	1	0
16	1	1	0	0.5	0
17	1	1	0	0	0
18	1	1	0	0.5	0
19	1	1	1	1	1
20	1	0	0	0	0
21	1	1	1	1	1
22	1	1	0.5	0	0
23	1	1	1	1	0.5
24	1	1	0	1	0
25	1	1	0	0.5	0
27	1	0.5	0.5	1	0
28	1	1	1	0	0.5
1	0	0	0	0	0
2	0	1	0	0.5	0
5	0	1	1	0.5	0
8	0	1	0	0.5	0
9	0	0	0	0	0
10	0	1	0	0	0
11	0	1	0	0.5	0
12	0	0.5	0	0	0
13	0	1	1	1	1
		Q1	Q2	Q3	Q4
		0.75	0.37	0.44	0.22

See explanation of all four tables on page 39.

Table 22. Post-Test Participant Data for Items 5 - 8

Participants	Treatment	Item 5	Item 6	Item 7	Item 8
3	1	0	0	1	0.5
4	1	0	0	1	1
6	1	0	0	0	0
7	1	0	0	0.5	0
14	1	0.5	0	1	1
15	1	1	0.5	1	1
16	1	0	0	1	0
17	1	0	0	1	0
18	1	1	0	1	1
19	1	1	0	1	1
20	1	0	0	0	0
21	1	1	0	1	1
22	1	0	0	1	0
23	1	0	0	1	1
24	1	1	0	1	1
25	1	0	0	1	0
27	1	0.5	0	1	0
28	1	0.5	0	1	0
1	0	0	0.5	0	0
2	0	0	0	0	0
5	0	0	0	1	1
8	0	0	0	1	1
9	0	0	0	0.5	0
10	0	0	0	1	1
11	0	0	0	0.5	1
12	0	0	0	0	0
13	0	1	0	1	1
		Q5	Q6	Q7	Q8
		0.27	0.03	0.75	0.50

See explanation of all four tables on page 39.

Table 23. Change from Pre-Test to Post-Test Participant Data

Participants	Treatment	Pretest Scores	Posttest Scores	Change
3	1	6	3	-3
4	1	4	6	2
6	1	2	0	-2
7	1	0	1	1
14	1	7	5.5	-1.5
15	1	6	5.5	-0.5
16	1	1	2.5	1.5
17	1	2	2	0
18	1	2	4.5	2.5
19	1	4	7	3
20	1	1	0	-1
21	1	5.5	7	1.5
22	1	3	2.5	-0.5
23	1	3	5.5	2.5
24	1	5	5	0
25	1	4	2.5	-1.5
27	1	3	3.5	0.5
28	1	0	4	4
1	0	0	0.5	0.5
2	0	4.5	1.5	-3
5	0	2	4.5	2.5
8	0	3	3.5	0.5
9	0	1	0.5	-0.5
10	0	3	3	0
11	0	2	3	1
12	0	1.5	0.5	-1
13	0	7	7	0
		Average	Average	Average
		3.05	3.37	0.314

Mean Pre (E.G.)	3.25	Mean Post (E.G.)	3.69
St. Dev. (E.G.)	2.10	St. Dev. (E.G.)	2.17

Mean Pre (C.G.)	2.66	Mean Post (C.G.)	2.66
St. Dev. (C.G.)	2.07	St. Dev. (C.G.)	2.19

The average score of each item is at the bottom of each column for both the pre-test and post-test assessment. The pre-test average score for the whole group is 3.06 and the average group score for the post-test assessment is 3.37.The average score of the change from pre-test to post test for the experimental/treatment group is 3.72 while the average score change for the control/comparison group is 2.67. A test of statistical significance was not pursued because the results are dependent on sample size. Instead, the Cohen's D was computed as the effect size for interpreting the change between the treatment and comparison group.

Test of the effect size may help to indicate if an intervention worked and it also predicts how much of an impact to expect in scenarios such as this research. Thus, to calculate the effect size, the difference of the [mean of experimental/treatment group] and the [mean of control/comparison group] is divided by the standard deviation. The effect size of 0.19 is not large indicating a small impact on outcomes. Hence, the 0.19 effect size indicates that the difference between the gains made by two groups of students (treatment/experimental vs. control/comparison) was 0.19 standard deviations.

Interview Results

In the interview part of this study, there were 18 students who agreed to participate. Based on the responses to the post-assessment, 4 students were selected for an interview.

Respondent 019: The student expressed liking the algebraic strategy more than the flowchart strategy because getting used to what a student should know in high school was very important. Another added stress for the student was drawing the bubbles needed for the flowchart was a hassle along with writing operations and numbers verbally was a headache. This student was determined to learn how to solve equations algebraically. This student was in the experimental/treatment group. The first student was chosen because there was a score increase from a 4 on the pre-assessment and a 7 on the post-assessment. Here are the interview questions as follows:

1. Which method of solving linear algebraic equations are you most comfortable with, and why?

Verbal Response: I like solving linear equations algebraically because, with the flowchart, we are limited to the types of equations, which will work with the flowchart.

2. Before the pre-test, how many ways were you able to solve the linear equation?

Verbal Response: I knew how to use the guess and check strategy.

3. Do you think one method of solving linear equations is better than the other way?

Verbal Response: I prefer solving algebraically because it is fast and easy once I knew what to do with the equations.

4. Identify which one?

Verbal Response: I understand the flowchart and the algebraic strategy but the algebraic strategy is painless because I don’t have to draw the circles associated with solving the problem.

5. Here's what you did on this problem; please walk me through your thinking?

Verbal Response: In question number 5, I divided -22 by 2 because the left-hand side of the equations was divided by 2. Since the denominators are the same, I can get rid of the fraction. The student was told to check his/her answer with the solution he/she had. He/She then saw that the solution was wrong. I asked the student to write the problem on a separate sheet of paper. I told him/her to analyze the problem again. He/She now asked if he/she could divide the left-hand side of the equation by 2 since that was what we’re given. I told him/her yes. I handed him/her a calculator to assist with minor arithmetic calculations. The student was able to solve the equation successfully the third time around.

Respondent 013: This student was in the comparison/control group. However, this student came to the district already knowing how to solve equations algebraically. In the pre-assessment, the student had no clue how to begin Item 6 but during the post-assessment, the student used conceptual understanding to analyze the item.

This student explained Item 6 to me. The student put parentheses on the left-hand side of the equation and the student explained that since this problem equals 5, then the root on the left-hand side of this equation must equal 25. The student continued to explain that *x* would have to equal 4 since 42 + 32 = 25. I asked the student if *x* could equal anything else other than positive 4? The student thought for a moment and then mentioned that *x* can also equal negative 4.

Respondent 014: This student was in the experimental/treatment group. I do not recall this student ever wanting to use the flowchart strategy. Since I am promoting flexibility in thinking, I realized that this student was solving linear equations in a way that was suitable for the student. I noticed that this student used the guess and check strategy for the pre-assessment and the standard procedure for the post-assessment. I asked the respondent which method of solving linear algebraic equations are you most comfortable with, and why? I prefer to use the guess and check strategy because it is a lot easier for me to understand and apply it. I told the respondent to explain Item 8 in the way it was understood conceptually. For Item 8, I knew that the final cost was $4.40. I subtracted $0.80 since money can be subtracted from money. After subtracting, I ended up with $3.60 left. I knew that I had already paid for the shake and now I had to pay for the cost of three burgers. If I divide $3.60 by 3, then each burger will cost $1.20. This is the undoing method without the diagram. I was able to set up the equation but I did not solve it systematically using inverse operations.

Respondent 024: Finally, here is another student who was in the experimental/treatment group. Like the other respondents, this student did not use the flowchart strategy and started with solving linear equations algebraically. This student stayed true to using one of the strategies throughout the post-assessment. I said to the student, " Here's what you did on this problem, please walk me through your thinking"? The student replied, " I wanted to write my variable *x* means because I needed to find how much is one variable. I wrote a division problem so that I could find how much one burger would cost. After subtracting the numbers and dividing by 3, I knew I would get the cost of one burger". I asked, "Is there another way to solve this problem"? The student replied I don't know.

In comparison to the four interviews that were done, the students only used two particular strategies throughout their post-assessment. The algebraic strategy and the guess and check strategy. Respondent 14 used the reasoning of the flowchart without the graphic organizer for the burger problem. Three out of the four students used the algebraic strategy while the last student used the guess and check strategy for at least one item. Three out of the four students belonged to the treatment group while the last student belonged to the comparison group.

CHAPTER FIVE

CONCLUSION

The significance of representing the solution of linear algebraic equations in multiple ways provides the same objective of more than one form. It is necessary to see how students use these representations. It is suggested that multiple representations provide an environment for students to abstract and understand major concepts (McArthur et al.1998, Yerushalmy, 1991) while constructivist theory suggests that we need to understand students' thinking processes in order to facilitate their learning in more empowering ways (Stepphe, 1991). Understanding students' thinking and their preferences while choosing a representation type for solving algebraic linear algebraic equations help mathematics teachers gain insight into student thinking. After the last practice during the study's lesson process, the students went on Christmas break and other mini-holidays before they took the post-assessment. Representations such as the do/undo flow chart and algebraic method are tools that provide the same information in more that one form. The role of these tools in the task mentioned above is to represent solving linear algebraic equations using multiple concretizations of a concept, mitigate certain difficulties and to make mathematics more attractive and interesting (Ozgun-Koca, 98). Dienes' mentioned that conceptual learning is maximized when children are exposed to a mathematical concept through a variety of physical contexts or embodiments. In other words, we should not expect that all students would perceive the same concept from one representation.

Concluding Issues Related to Study

Does the introduction of multiple ways to think about linear equations lead students to flexibly incorporate appropriate representations/strategies in solving problems involving linear equations?

Conceptual understanding of solving linear algebraic equations at the beginning of this analysis did not prove to be absent among students because several students used the guess and check strategy and mental math strategies for sense making during the pre-assessment. Rather they lacked knowledge of the procedural steps for solving equations in one variable. Several students showed understanding by solving certain problems using mental math or the guess and checking strategies in the pre-assessment, and after instruction the students moved to more use of the standard procedure and less use of sense-making methods. Students were able to use the guess and check strategy and the algebraic strategy by the time they took the post-assessment. The treatment group displayed the above strategies during the post-assessment phase and increased from an average of 3.25 on the pre-assessment to an average of 3.69 on the post-assessment. However, the control groups' average remained the same for both assessments, which were 2.67. The need to find ways to promote comprehension of mathematics is one of the main issues that arise in mathematics education. At the beginning of this study analysis, within the "UNDO" part of the flow chart, inverse operations are applied in the reverse order so that the unknown value of the variable may be determined. In other words, the DO/UNDO flowchart may be used to help students identify what is being done to the variable so that they may, in turn, correctly use inverse operations to solve for the variable. In doing so, students should better understand the important role order of operations plays in solving equations. When applying the algebra for solving linear algebraic equations alongside the flowchart, the learning goal for students is to conceptually understand how and why the algorithm works. The treatment group showed greater average growth than the comparison group, although the effect size calculation showed the intervention had only a small effect of 0.19. A handful of students in the control setting understood how to apply the algebra associated with solving linear equations. A couple of representations on this concept gave students the ability to generate and connect flexible mathematical thinking after they attempted the pre-assessment. According to the 2012 Focus Issue on Fostering Flexible Mathematical Thinking, NCTM's Focus in High School Mathematics: Reasoning and Sense-Making (2009, pp. 9-10), students are able to adapt and expand where possible while applying previously learned principles to problems that are being presented, they seek and use connections and different representations, reconcile different approaches to solve problems including those proposed by others and they generalize a solution to a broader class of problems. Four students, Respondents 006,023,025,028 were able to display flexibility in their thinking during the post-assessment phase of this study. Although they were taught how to use the flowchart and algebraic/standard strategy, these students also displayed another strategy which is the guess and check strategy. Respondent 006 and Respondent 025 used the inverse operation strategy for problem 1 and used the guess and check strategy for problem 7. Respondent 023 displayed the inverse operation strategy for problems 1-3 and 8 while using the guess and check strategy for problem 7. Respondent 028 used the guess and check strategy for problem 2 and used the algebraic strategy for problems 1 and 7. The introduction of multiple ways to think about linear equations did not lead to greater flexibility in this research because the students did not use the multiple strategies that were introduced to them, which was using the flowchart and the algebraic strategies during the post-assessment. The guess and check strategy was used but that was not taught during the intervention of this assessment.

Which representations do students use to solve linear equations and in what context? How do students use representations when presented with a specific task?

"Students frequently use such informal approaches as guess-and-test and undoing to solve algebra word problems when they are allowed to choose a solution method"(Nathan and Koedinger, 2000). A guess and check method uses arithmetic procedures to solve algebra word problems iteratively after the unknown quantity is replaced by a number. In the undoing process, students work backward through quantitative algebra problem relationships by reversing mathematical operations and quantity order. Students use their knowledge about the environment to promote their thinking through alternate approaches to solutions. The use of informal approaches by students increases their problem-solving performance about the problems that teachers consider to be the most challenging.

After administering the pre-assessment to my experimental group, I used the DO/UNDO flowchart to help my students comprehend when a number was being added, subtracted, or multiplied in an equation. As we continued to solve linear equations using a flowchart, I also introduced solving equations algebraically simultaneously. The control group only learned how to solve linear equations algebraically without the flowchart. Students mostly used solving linear equations algebraically in both the experimental and control groups. On the post-assessment for the experimental group, most students solved the linear equations using algebra and just a handful of students used the guess and check strategy to find a solution.

Forward-Looking Guidance and Study

Before the analysis of this study, students had no clue what the alphabets (variables) were doing with numbers. They could not conceptually understand how to look at this mathematical sentence. Looking at any variable was too abstract for them. So if I were to look at this analysis in the future, I would teach students to visually represent a linear equation by transforming the mathematical problem into words. For example, $3x + 8 = 20$(a mathematical problem) to words is three times x plus eight equals twenty. Then I would have the students write the inverse operation for operation. Then we could proceed to use the DO/UNDO flowchart. I consider the advantage of the flowchart approach to be that students have a better understanding of the standard procedure process and thus by using the technique more appropriately, they would make sense of the procedure. A study involving more students could address the research question. This may help to bridge the learning from primary to secondary. It is also important for students to be able to explain their thinking process. The majority of my students did not explain their thinking. I would often say to my students that if I were to teach the way they explain their mathematical work, then they would be beyond lost and confused. If students can interpret what the equation is saying then they would be a lot more successful at problem-solving linear equations. This is simply getting the students to use mathematical academic language. I would also be very specific in my instructions on whether the students can use the flowchart as part of their strategy to solve linear equations. Based on the learning process that has been conducted I cannot conclude that students can use the DO/UNDO flowchart to solve linear equations with one variable in a formal way because every student's learning is different. For future teaching, it seems more appropriate to consider the flowchart as a stepping-stone towards the more general procedures that teachers can use to build understanding before introducing general methods.

APPENDIX A

INFORMED CONSENT

October 18, 2017

CSUSB INSTITUTIONAL REVIEW BOARD
Full Board Review
IRB# FY2017-153
Status: Approved

Ms. Edima Umanah and Prof. Madeleine Jetter
Department of Mathematics
California State University, San Bernardino
5500 University Parkway
San Bernardino, California 92407

Dear Ms. Umanah and Prof. Jetter:

Your application to use human subjects, titled "Assessing Student Understanding while Solving Linear Equations Using Flow Charts and Algebraic Methods" has been reviewed and approved by the Institutional Review Board (IRB). The informed consent document submitted with your IRB application is the official version for use in your study and cannot be changes without prior IRB approval. A change in your informed consent (no matter how minor the change) requires resubmission of your protocol as amended through the Cayuse IRB system protocol change form. **Your application is approved for one year from October 18, 2017 through October 17, 2018 . Please note the Cayuse IRB system will notify you when your protocol is due for renewal. Ensure you file your protocol renewal and continuing review form through the Cayuse IRB system to keep your protocol current and active unless you have completed your study. Please note additional conditions of IRB approval highlighted below.**

Your responsibilities as the researcher/investigator reporting to the IRB Committee include the following 4 requirements as mandated by the Code of Federal Regulations 45 CFR 46 listed below. Please note that the protocol change form and renewal form are located on the IRB website under the forms menu. Failure to notify the IRB of the above may result in disciplinary action. You are required to keep copies of the informed consent forms and data for at least three years. Please notify the IRB Research Compliance Officer for any of the following:

1) Submit a protocol modification/change in the Cayuse IRB system if any changes (no matter how minor) are proposed in your research protocol for review and approval of the IRB before implemented in your research,
2) If any unanticipated/adverse events are experienced by subjects during your research,
3) To apply for renewal and continuing review of your protocol one month prior to the protocols end date,
4) When your project has ended by emailing the IRB Research Compliance Officer.

The CSUSB IRB has not evaluated your proposal for scientific merit, except to weigh the risk to the human participants and the aspects of the proposal related to potential risk and benefit. This approval notice does not replace any departmental or additional approvals which may be required. If you have any questions regarding the

IRB decision, please contact Michael Gillespie, the IRB Compliance Officer. Mr. Michael Gillespie can be reached by phone at (909) 537-7588, by fax at (909) 537-7028, or by email at mgillesp@csusb.edu. Please include your application approval identification number (listed at the top) in all correspondence.

Best of luck with your research.

Sincerely,

Caroline Vickers

Caroline Vickers, Ph.D., IRB Chair
CSUSB Institutional Review Board

CV/MG

September 07, 2018

CSUSB INSTITUTIONAL REVIEW BOARD
Protocol Renewal
FY2017-153
Status: Approved

Ms. Edima Umanah and Prof. Madeleine Jetter

Department of Mathematics

California State University, San Bernardino
5500 University Parkway
San Bernardino, California 92407

Dear Ms. Umanah and Prof. Jetter:

Your protocol renewal to use human subjects, titled "Assessing Student Understanding while Solving Linear Equations Using Flow Charts and Algebraic Methods" has been reviewed and approved by the Chair of the Institutional Review Board (IRB). A change in your informed consent requires resubmission of your protocol's informed consent as amended for use in your study.

Your renewal is approved from **October 08, 2018** through **October 08, 2019**. Please note the Cayuse IRB system will notify you when your protocol comes up for renewal at 90, 60, and 30 days before the protocol expires. If you are no longer conducting the study you can submit a study closure through the Cayuse IRB system.

You are required to notify the IRB of the following by submitting the appropriate form (modification, unanticipated/adverse event, renewal, study closure) through the online Cayuse IRB Submission System.

1. If you need to make any changes/modifications to your protocol submit a modification form as the IRB must review all changes before implementing in your study to ensure the degree of risk has not changed.
2. If any unanticipated adverse events are experienced by subjects during your research study or project.
3. If your study has not been completed submit a study renewal to the IRB.
4. If you are no longer conducting the study or project submit a study closure.

You are required to keep copies of the informed consent forms and data for at least three years.

If you have any questions regarding the IRB decision, please contact Michael Gillespie, Research Compliance Officer. Mr. Gillespie can be reached by phone at (909) 537-7588, by fax at (909) 537-7028, or by email at mgillesp@csusb.edu. Please include your application identification number (above) in all correspondence.

August 29, 2019

CSUSB INSTITUTIONAL REVIEW BOARD
Protocol Renewal
IRB-FY2017-153
Status: Approved

Ms. Edima Umanah and Prof. Madeleine Jetter
CNS - Department of Mathematics
California State University, San Bernardino
5500 University Parkway
San Bernardino, California 92407

Dear Ms. Umanah and Prof. Jetter:

Your protocol renewal to use human subjects, titled "Assessing Student Understanding while Solving Linear Equations Using Flow Charts and Algebraic Methods" has been reviewed and approved by the Chair of the Institutional Review Board (IRB).

Your renewal is approved from September 4, 2019 through September 4, 2020. Please note the Cayuse IRB system will notify you when your protocol comes up for renewal at 90, 60, and 30 days before the protocol expires. If you are no longer conducting the study you can submit a study closure through the Cayuse IRB system.

You are required to notify the IRB of the following by submitting the appropriate form (modification, unanticipated/adverse event, renewal, study closure) through the online Cayuse IRB Submission System.

1. If you need to make any changes/modifications to your protocol submit a modification form as the IRB must review all changes before implementing in your study to ensure the degree of risk has not changed.
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If you have any questions regarding the IRB decision, please contact Michael Gillespie, Research Compliance Officer. Mr. Gillespie can be reached by phone at (909) 537-7588, by fax at (909) 537-7028, or by email at mgillesp@csusb.edu. Please include your application identification number (above) in all correspondence.

Best of luck with your research.

Sincerely,

Donna Garcia

Donna Garcia, Ph.D., IRB Chair
CSUSB Institutional Review Board

DG/MG

APPENDIX B

ASSESSMENTS

Pre/Post Test on Solving Linear Equations
Name: Date:
Period:
ID#:

Solve the following equations using any method of your choice. If you solved any of the linear equations by mental math, briefly explain your thoughts on how you were thinking. Calculators are allowed.

1. $4n + 10 = 50$

2. $\frac{x}{3} - 4 = 2$

3. $-3n + 12 = 12$

4. $\frac{x}{5} - (-14) = 8$

5. $\frac{-5x + 6}{2} = -22$

6. $\sqrt{(x^2 + 3^2)} = 5$

7. "I'm thinking of a number. If you multiply it by 6 and then add 7, you will get 55. What is my number?"

8. A shake at the Shack cost 80 cents and the bill for three burgers and a shake is $4.40. “How much is a burger?”

King, J., & Rasmussen, P. (1990). Key to Algebra Book 3 Equations.

APPENDIX C

LESSONS

Solve Linear Equations in One Variable Lessons

TABLE OF CONTENT

Here is an equation in x: $10 + x = 3$

We can **solve** this equation by substituting different numbers for x. If we find a number that works, then we have found a **solution** to the equation. The only number that will work in this equation is $^{-}7$. We can show the solution by writing

$$x = {}^{-}7$$

Solve each equation by substituting different numbers until you find a solution.

$5 + y = 9$ $y = 4$	$z + 1 = 10$	$3 + w = 4$	$a + 15 = 20$
$10 + x = 17$	$4 + t = 4$	$^{-}6 + r = {}^{-}16$	$8 + x = 6$
$c + 6 = 5$	$^{-}5 + y = {}^{-}1$	$c + {}^{-}8 = {}^{-}8$	$17 + a = 23$
$^{-}4 + a = {}^{-}9$	$^{-}3 + m = {}^{-}13$	$x + {}^{-}2 = 5$	$4 + e = {}^{-}4$

Joe picked a number and added 7 to it. The answer was 11. What was the number?

Equation: $x + 7 = 11$

Solution: $x = 4$

Jo picked a number and added 3 to it. The answer was 12. What was the number?

Equation:

Solution:

Chip thought of a number. He added 9 to it. The answer came out to be 14.
What was Chip's number?

Equation:

Solution:

Solve each equation.

$6x = 18$ $x = 3$	$4s = 24$	$5x = 30$	$9b = 72$
$4y = 8$	$5x = 25$	$3x = {}^{-}12$	$5s = 0$
$6t = 6$	${}^{-}4n = {}^{-}20$	$8c = {}^{-}8$	${}^{-}10e = {}^{-}30$
${}^{-}7x = {}^{-}21$	${}^{-}2x = 10$	$20x = 80$	${}^{-}7m = 7$

Robin multiplied 8 times a number. The answer was 48. What was Robin's number?

Equation:

Solution:

Jerry thought of a number. Then he multiplied it by 5. The answer came out to be 45. What was Jerry's number?

Equation:

Solution:

Jennifer started out with 8 dollars. Then she got some more money for her birthday. She ended up with 15 dollars. How much did she get for her birthday?

Equation:

Solution:

7 times some number is 28. What is the number?

Equation:

Solution:

$9x - 30 = 6$ $9(4) - 30$ | 6 $36 - 30$ 6 C The solution is 4.	$3x + 13 = 43$ The solution is ____.	$8x - 13 = 27$ The solution is ____.
$5x + 9 = 8x$ $5(3) + 9$ | $8(3)$ $15 + 9$ | 24 24 C The solution is 3.	$6x - 10 = 4x$ The solution is ____.	$4x + 15 = 9x$ The solution is ____.
$4x + 3x = 42$ $4(6) + 3(6)$ | 42 $24 + 18$ 42 C The solution is 6.	$7x + 4x = 33$ The solution is ____.	$8x - 6x = {}^{-}18$ The solution is ____.
$7x - 12 = 4x + 6$ $7(6) - 12$ | $4(6) + 6$ $42 - 12$ | $24 + 6$ 30 | 30 C The solution is 6.	$3x + 5 = 2x + 13$ The solution is ____.	$3x - 17 = 7x - 9$ The solution is ____.

Did you find solutions to all the equations? Well, don't worry if you didn't. Guessing is a fine way to solve short equations, but it doesn't work well on long ones. The rest of this booklet will show you how to solve many kinds of equations — short ones and long ones, too.

Here are some equations for you to solve. Each problem takes two steps. First simplify the equation by combining like terms. Then find the solution.

$5x + 2x = {}^{-}21$ $7x = {}^{-}21$ $x = {}^{-}3$	$2x + 3x = 30$	$3x + 3x = {}^{-}24$	$5x + 5x = {}^{-}70$
$4x + 5x = 18$	$7x - 3x = {}^{-}12$	$8x - 14x = 60$	$3x + x = 36$
$5x = 13 + 7$ $5x = 20$ $x = 4$	$3x = 10 + 2$	${}^{-}4x = 19 + 9$	$7x = 30 + 40$
$6x = 5 - 17$	$8x = 30 - 6$	$2x = 23 - 23$	$3x = 11 + 22$
$56 = 2x - 9x$ $56 = {}^{-}7x$ $x = {}^{-}8$	$77 = 6t + 5t$	$49 - 9 = 5a$ $40 = 5a$ $a = 8$	$17 + 3 = 4x$
$18 = 7s + 2s$	$15 = 3m - 8m$	$20 - 44 = 8x$	$37 - 30 = 7x$
$20 = 7y - 12y$	$18 = 9t - 3t$	$5 + 13 = {}^{-}9w$	$48 + 16 = 32y$

Solve each equation.

$5x + 3x = 19 + 5$ $8x = 24$ $x = 3$	$6x + 4x = 15 + 5$	$2x - 7x = 23 + 1$
$10x - 17x = 8 - 50$	$2x + x = 9 + 9$	$5x - 3x + 6x = 56$
$4x + 7x - 4x = 56$	$5x + 2x + 2x = 45$	$6x - 14x + 3x = 40$
$10 + 4 = 2a + 5a$	$3 + 15 = 12b - 3b$	$9 = 10r - 16r + 5r$
$12 + 36 = 13y - 7y$	$-7 = 4x + 2x + x$	$27 = 6a + 5a - 2a$
$9x - 4x - 8x = 15$	$72 = 4c + 7c - 2c$	$15z - 6z = 20 + 16$
$37 + 35 = 9s + 3s$	$6x - 10x = 28$	$6 = 11y - 4y - 5y$

Age Puzzles

Algebra makes it easy to solve certain kinds of puzzles. Here are some about age. Follow the same steps you followed on the last two pages.

Jason is 12 years older than Ted. Next year Jason will be 3 times as old as Ted. How old is Ted?

	Ted	Jason
This year:	x	$x+12$
Next year:	$x+1$	$x+13$

Equation: $x+13 = 3(x+1)$

$$x+13 = 3x+3$$
$$13 = 2x+3$$
$$10 = 2x$$
$$x = 5$$

Answer: Ted is 5 years old.

Ellen is 11 years older than Maja. Last year Ellen was twice as old as Maja. How old is Maja now?

Equation:

Answer:

Pam is 14 and her dad is 37. In how many years will Pam's dad be twice as old as she will be?

Equation:

Answer:

Sean is 20 and his brother is 12. How many years ago was Sean three times as old as his brother?

Equation:

Answer:

Minh is 16. His parents are both the same age. The three of them have lived a total of 100 years. How old are his parents?

Equation:

Answer:

Alex and Alicia are twins. Kevin is 5 years older than the twins. Their ages total 53. How old are the twins?

Equation:

Answer:

Now look at the last two problems on page 12 again. It is easy to solve these using the Addition Principle.

$$^{-}5x + 13^{-13} = {}^{-}12^{-13} \qquad {}^{-}7x - 1^{+1} = {}^{-}22^{+1}$$

$$^{-}5x = {}^{-}25 \qquad {}^{-}7x = {}^{-}21$$

$$x = 5 \qquad x = 3$$

Here are some more equations to solve using the Addition Principle:

$^{-}3x + 17 = 2$	$^{-}4x + 6 = {}^{-}10$	$^{-}5x - 18 = {}^{-}3$
$^{-}9 + 2x = {}^{-}21$	$^{-}7x + 42 = 0$	$12 + 9x = {}^{-}60$
$^{-}70 + 5x = {}^{-}20$	$^{-}2x - 98 = 2$	$36 = {}^{-}4x + 56$
$6x + 12 = 0$	$29 = 38 + x$	$0 = 12p - 48$
$18 + {}^{-}8p = {}^{-}30$	$0 = {}^{-}6x - 66$	$^{-}14 + {}^{-}3x = {}^{-}2$

Sandy and Terry got different answers when they tried to solve the same equation.

Sandy	*Terry*
$10^{-10} - 2x = 18^{-10}$	$10^{-10} - 2x = 18^{-10}$
$2x = 8$	$^{-}2x = 8$
$x = 4$	$x = {}^{-}4$

(Terry's thought: 10-2x is the same as 10+⁻2x so when I add ⁻10 I'll have ⁻2x left.)

Sandy's answer is not a solution. Terry's answer *is* a solution.

$10 - 2x$	18
$10 - 2(4)$	18
$10 - 8$	
2	
	X

$10 - 2x$	18
$10 - 2(^{-}4)$	18
$10 - {}^{-}8$	
$10 + 8$	
18	C

Below are some equations for you to solve. Remember to use Terry's method.

$23 - 4x = 27$	$12 - 5x = 27$	$6 - 8x = 30$
$32 - 4x = 0$	$40 - 3x = 34$	$15 - x = 20$
$0 = 18 - 2x$	$^{-}4 - 7x = 31$	$^{-}3 - 6x = {}^{-}21$

King, J., & Rasmussen, P. (1990). Key to Algebra Book 3 Equations.

APPENDIX D

INTERVIEW QUESTIONS

Interview Questions
Consent letters with parents agreeing to have their student interview to part in 5-10 minute interview.

1. Which method of solving linear algebraic equations are you most comfortable with, and why?
2. Before the pre-test, how many ways were you able to solve the linear equation?
3. Do you think one method of solving linear equations is better than the other way?
4. Identify which one?
5. Why do you think it is a better method?
6. Here's what you did on this problem, please walk me through your thinking?
7. Is there another way to solve this problem?
8. Why did you use this method for this problem?
9. Would that method always work?
   How do you know this/that method is easier?
10. How do you know this/that method is more efficient?

Developed by Edima Umanah

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