1995

Describing and distinguishing knots

Lisa A. Padgett

DESCRIBING AND DISTINGUISHING KNOTS

A Project

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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ABSTRACT

The theory of knots has recently become a "hot" topic in mathematics, although the study of knots began in the early 1900's. The most important question when dealing with knots is whether two knots are actually equivalent, i.e., whether one knot can be manipulated into the other knot without cutting or splicing the knot. Different fields in mathematics are used to help us distinguish knots, such as topology and algebra. I will explain the different approaches starting with the older methods involving groups up through the more modern techniques. The theory of knots deals with a vast amount of mathematics, so in some areas I will only touch on the subject and leave it for the reader to investigate further on their own.

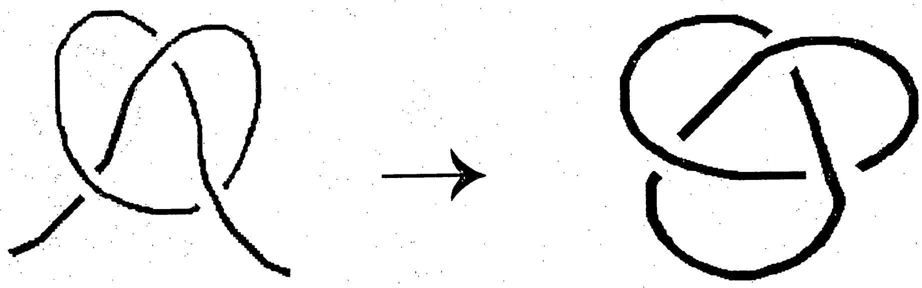
ACKNOWLEDGEMENTS

I would like to thank my husband Terry, for all his love and support throughout these past years while I have been working on my masters. I would also like to thank all my professors while here at CSUSB, especially the following who sparked my interest in several areas of mathematics: Dr. Gary Griffing, Dr. Zahid Hassan, Dr. Joel Stein, Dr. Peter Williams and Dr. Robert Stein.

TABLE OF CONTENTS

INTRODUCTION

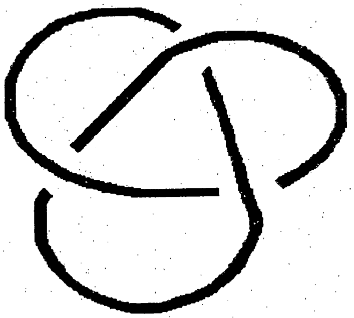
Almost everyone is familiar with knots in some form or another. For example, when tying shoe laces, the trefoil knot is used. By connecting the ends of the shoe laces after performing the initial knot when tying your shoes, you have the trefoil knot as shown below.



Initial Knot $\rightarrow$ Trefoil Knot

In mathematics, it is essential that we splice the ends together to form one continuous curve so that two knots can be compared. This leads to the following definition: A <u>knot</u> is a simple closed polygonal curve in $\mathbf{R}^3$. A knot is considered to be a subset of 3-dimensional space which is homeomorphic to the circle. Recall, two topological spaces X and Y are called <u>homeomorphic</u> if there is a continuous bijective mapping from X to Y whose inverse is continuous. We will give 2-dimensional representations of knots as shown above. Even though knots exist in $\mathbf{R}^3$, we will need to write them down so we will use 2-dimensional diagrams.

The most important question when dealing with knots is whether two knots, such as the trefoil and figure eight, are "equivalent".



Trefoil Knot

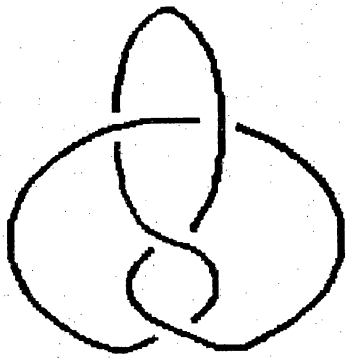


Figure-eight Knot

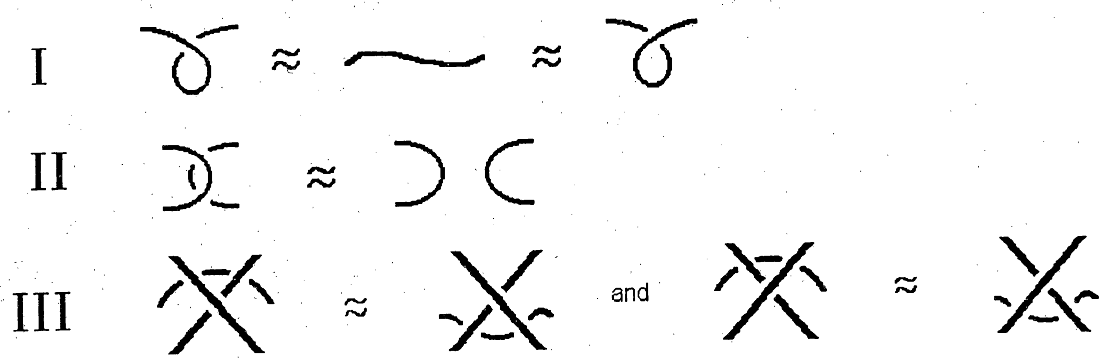
If we could manipulate one knot into the other by moving it around without cutting or retying, then the two knots are said to be <u>equivalent</u>. Formally, knots K_1 and K_2 are said to be equivalent if there exists a homeomorphism of $\mathbf{R}^3$ onto itself which maps K_1 onto K_2. If two knots are equivalent, they are said to be of the same <u>knot type</u>. A particular question of equivalence occurs when we have a knot K_1 and the unknot $O = K_2$. In this case, if K_1 is equivalent to K_2, then K_1 is said to be <u>unknotted.</u>

Algebraic objects called <u>invariants</u> are used to determine whether two knots are of the same knot type, i.e., are equivalent knots. The geometric problem of manipulating one knot into another can be very difficult, so we change it into an algebraic problem which hopefully will be easier to solve. Let I_1 be an invariant for knot K_1, and I_2 be an invariant for knot K_2. Then if K_1

and K_2 are equivalent knots (in the topological sense), then their invariants I_1 and I_2 must be equivalent (in the algebraic sense).

Therefore, using the contrapositive, if invariants I_1 and I_2 are not equivalent, then the knots K_1 and K_2 are not equivalent. But if the invariants are equivalent, nothing has been proven. It is only when the invariants are shown to be inequivalent that we can conclude that the knots must be inequivalent. Thus, the use of invariants helps us only to prove two knots are inequivalent. If the invariants of two knots K_1 and K_2 are inequivalent, then K_1 cannot be manipulated into K_2 no matter how hard we try or how clever we are.

In the late twenties/early thirties, Reidemeister showed two knots K_1 and K_2 are equivalent if and only if K_1 can be turned into K_2 by a finite sequence of "moves". These moves are called the Reidemeister moves and are shown below. (Where in each diagram, only the relevant portion of the knot is shown.)

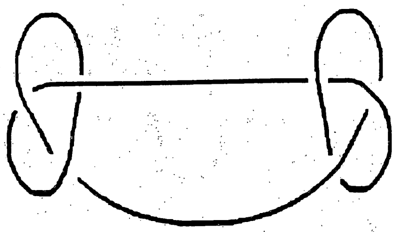


The first invariant I began studying was a certain group associated with a knot. This group is the so-called fundamental group of the complement of the knot in $\mathbf{R}^3$. It can be shown that knots of the same type have isomorphic groups. Given two knots, if one can show that their corresponding groups are not isomorphic, then the two knots are not equivalent. Groups associated with knots are given by presentations, i.e., a collection of generators and relations. Two groups are said to be of the same presentation type if they have "isomorphic presentations". (Two presentations are isomorphic if one can be obtained from the other using a finite sequence of Tietze Transformations, examples later.)

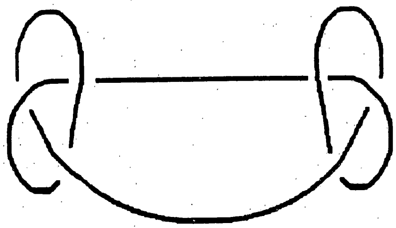
In the theory of groups, the problem of determining whether two presentations give isomorphic groups is, in general, unsolvable. So, since determining whether two groups are isomorphic can sometimes be very difficult, we must consider other invariants. One of these invariants is the sequence of elementary ideals which are defined in terms of the matrices formed using the presentation of the group. Another invariant is the sequence of Alexander knot polynomials which can be defined in terms of the elementary ideals. Since each new invariant is defined in terms of the previous, the new invariants will not give us any more information than the previous ones did; however, the new invariants may be easier to distinguish, easier to calculate, and easier to algebraically manipulate. Later, I will show the use of each invariant and how each invariant contains less information than the preceding one (knot polynomials containing the least information in the following diagram).

Knot type
↓
Presentation type
↓
Sequence of elementary ideals
↓
Sequence of knot polynomials

After examining the invariants above, I will show that these are not "strong enough" invariants to distinguish the granny knot and square knot (shown below). That is, each of the invariants in the list above are equivalent for both the granny knot and square knot.



Granny Knot



Square Knot

(Remember, invariants being equivalent does <u>not</u> necessarily imply knots are equivalent. Only if invariants are inequivalent can we conclude knots are inequivalent.)

Although these knots look very similar and the invariants above are all equivalent, we will later show it is not possible to turn one into the other no matter what we do. We will use more modern invariants to prove that the granny knot and square knot are <u>not</u> equivalent. These invariants are the Conway-

Alexander polynomial, the more general Jones polynomial, and in some sense the most general Homfly polynomial. The Homfly polynomial was so named because of its founders who all discovered it at the same time (used first letter of their names).

CHAPTER I

KNOT GROUPS AND INVARIANCE

The first invariant I studied was the fundamental group of the complement of a knot. To understand this invariant, we first need to understand the fundamental group for an arbitrary topological space X. Then we will investigate the applications of the fundamental group to knot theory. For a topological space X, a path a is a continuous mapping a: $[0, t_a] \rightarrow X$, where t_a is the stopping time, $t_a \geq 0$. A path a has initial point, $a(0)$, and terminal point, $a(t_a)$, in X. The two paths

$$a(t) = (1, t) \quad 0 \leq t \leq 2\pi \quad \text{and}$$

$$b(t) = (1, 2t) \quad 0 \leq t \leq 2\pi$$

are distinct even though they have the same stopping time, (2π), same initial point, (1, 0), same terminal point, $(1, 2\pi)$, and same set of image points. To be equal paths, a and b must have the same domain of definition, i.e., terminal points are the same, $t_a = t_b$ and for every t in that domain, $a(t) = b(t)$ (paths are the same at any point in time). Consider two paths a and b in X, where the terminal point of a coincides with the initial point of b, i.e., $a(t_a) = b(0)$. The product $a \bullet b$ is

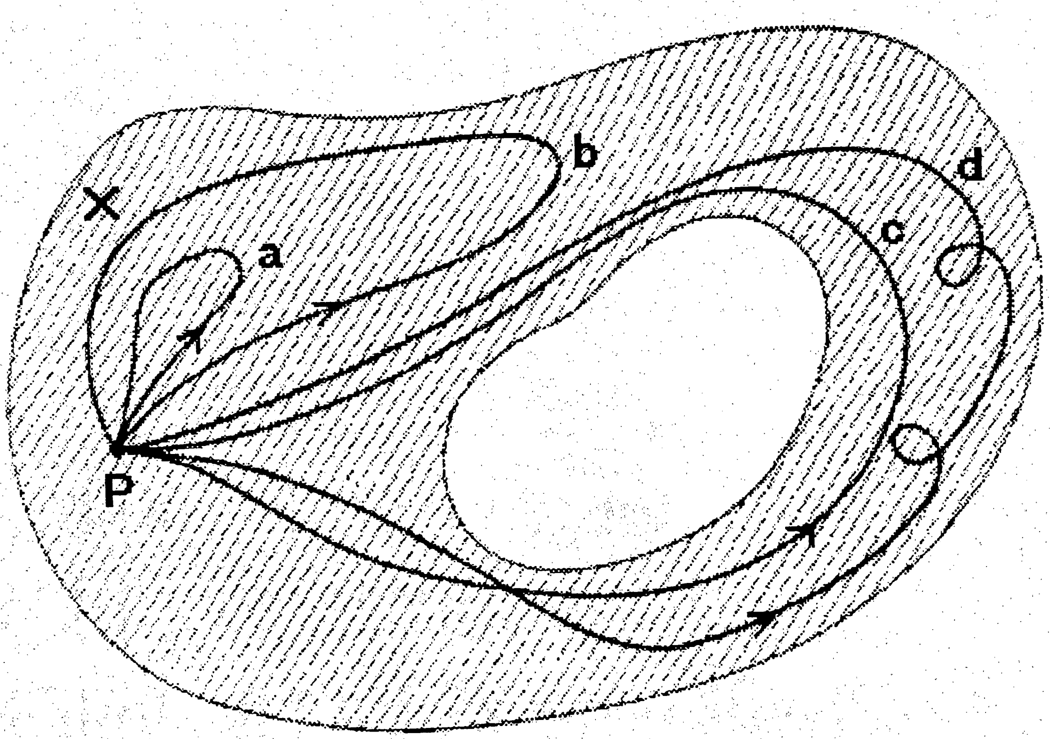
$$(a \cdot b)t = \begin{cases} a(t) & 0 \leq t \leq t_a \\ b(t - t_a) & t_a \leq t \leq t_a + t_b \end{cases}$$

The following are equivalent:

1. $a \cdot b$ and $b \cdot c$ are defined
2. $a \cdot (b \cdot c)$ is defined
3. $(a \cdot b) \cdot c$ is defined

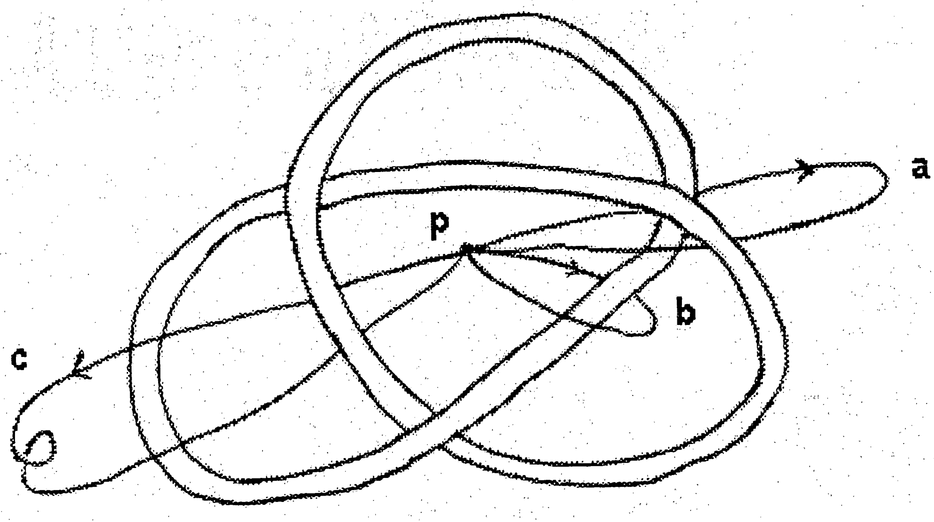
When one of them holds, the associative law $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ is valid. A path a is called an identity path if it has a stopping time $t_a = 0$. The path e is an identity if $e \cdot a = a$ and $b \cdot e = b$. a^{-1} is the inverse path formed by traversing a in the opposite direction. Thus, $a^{-1}(t) = a(t_a - t) \quad 0 \le t \le t_a$. A path whose initial and terminal points coincide is called a loop. A loop's common endpoint, p, is its basepoint. The loop with basepoint p is referred to as a p-based loop. The product of any two p-based loops is again a p-based loop. The identity path at p is a multiplicative identity. Therefore, the set of all p-based loops in X is a semi-group with identity. By adding the notion of equivalent paths, we can consider a new set whose elements are the equivalent classes of paths. The fundamental group is obtained as a combination of this construction with the idea of a loop. Path a is said to be equivalent to path b (written $a \sim b$) if and only if one can be continuously deformed into the other in the topological space X without moving the endpoints. Examples of equivalent and not equivalent loops are shown below.

$a \sim b \sim e$, since b can be shrunk to a then both a and b can be pulled into the basepoint p.
$c \sim d$, since the loops in d can be removed.
$a \nsim d$, since d cannot be pulled across the hole in the topological space X.



(If the hole were filled in, all paths would be equivalent to the identity path e.)

The application of the fundamental group to knot theory changes the focus from an arbitrary topological space to the complementary space of a knot. The <u>complementary space</u> of a knot K, consists of all of those points of $\mathbf{R}^3$ that do not belong to K, and is denoted $\mathbf{R}^3$ - K. To explain the fundamental group of the complement of a knot, I will use the tube model of the trefoil knot shown below.



Let $\mathbf{R}^3$ - K be the complement of the knot and p be a fixed base point. The set Ω is made up of loops in $\mathbf{R}^3$ - K that begin and end at p. Since Ω is infinitely large, we divide Ω into classes of equivalent loops. a and b are <u>equivalent</u> means a can be deformed into b, i.e., a can be pulled, pushed, twisted or even crossed over itself, but its beginning and ending points may not be moved and a

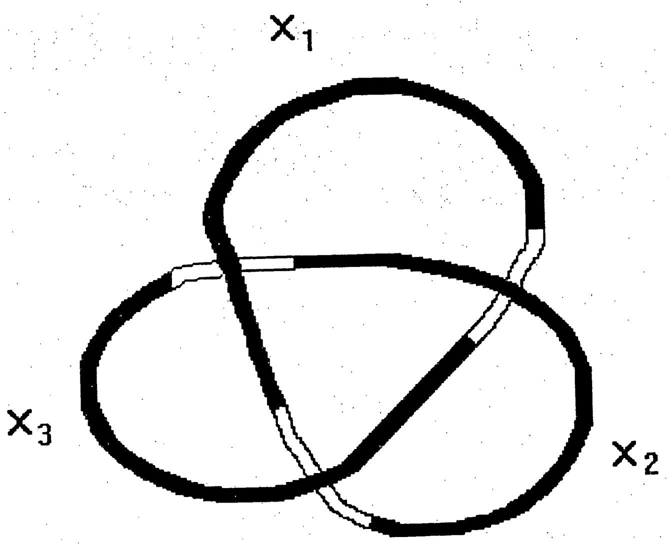
cannot come in contact with any segment of the knot. For example, in the picture above, loop a is equivalent to loop b since a can be pulled back to b. Also, c can be untwisted and shrunk back to the base point p (so c is equivalent to the identity loop e). Deformations of this type are called homotopies, and loops such as a and b that differ only by a homotopy are said to be homotopic. The class of loops homotopic to the loop a is written $[a]$. The set of loops Ω can now be regarded as a collection of homotopy classes. Multiplication of classes is defined as follows: the product $[a][b]$ is the path that begins at p, follows a back to p, and then follows b back to p. Multiplication of classes is an associative operation, so $([a][b])\,[c] = [a]([b][c])$. The class $[e]$ acts as an identity element, so $[a][e] = [e][a] = [a]$. Also, for every element $[a]$ there exists an inverse $[a]^{-1}$ such that $[a][a]^{-1} = [a]^{-1}[a] = [e]$. Therefore, Ω is a group.

The fundamental group of the complement of the knot K will be denoted by $\pi(R^3 - K)$ and called the <u>knot group of K</u> (or just the <u>knot group</u> if K is understood). This invariant can be used for distinguishing knots only if there is some way to explicitly describe it. The knot group consists of a number of equivalence classes of loops and can be calculated by constructing a finite list of objects that will completely describe the group. This list will consist of a number of group elements called <u>generators</u> and a number of equations called <u>relations</u>.

This list of generators and relations is known as a presentation of the knot group.

Next, I will explain how we get the generators and relations for the presentation of a knot group. I will need to explain about paths and corresponding notation for a knot.

A knot is divided into two classes of closed, connected segmented arcs, which are the overpasses and the underpasses. The overpasses and underpasses alternate around the knot. The overpasses are marked below in heavy lines, and labeled x_1 , x_2 , . . .

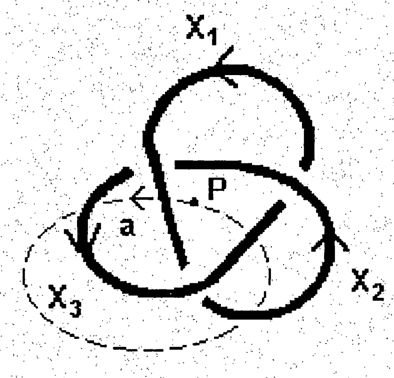


Each presentation is made with respect to an orientation of the knot K. One of the two directions along the knot is chosen as positive. We draw an arrow on K to mark the positive direction. If we fix a base point p in $\mathbf{R}^3$ - K, then each loop a (based at p) is assigned an element $a^\#$ that defines the loop. The element $a^\#$

is defined as follows:

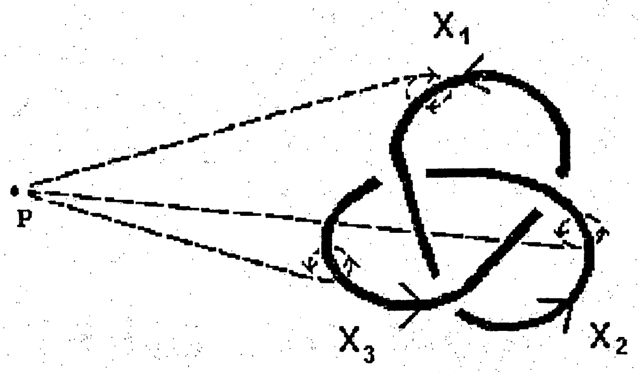
$$a^{\#} = x_{i_1}^{\epsilon_1} \cdots x_{i_k}^{\epsilon_k}$$

where overpasses crossed under by a are, in order, $x_{i_1}, \cdots, x_{i_K}$, and $\epsilon_P = 1$ or -1 depending on whether a crosses under x_{i_p} from right to left or from left to right (in other words, according as x_{i_p} and the portion of the loop under x_{i_p} form a right-handed or left-handed screw). Below is an example of a loop $a^{\#}$ that winds under the trefoil knot:



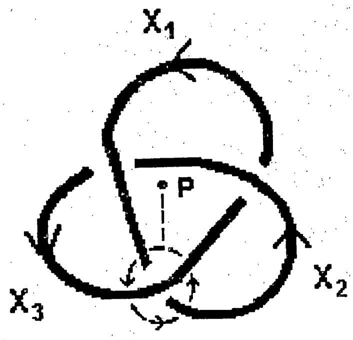
$$a^{\#} = x_1^{-1} x_3^{-1} x_2 x_3$$

It can be shown, that the loops $x_{i_1}^{\#}, \cdots, x_{i_K}^{\#}$ are the generators of the knot group. Each loop goes under and over the overcrossing of the knot, so each loop contributes one x_i. (There are no contributions when an overcrossing is passed from above, only when passed from below.)



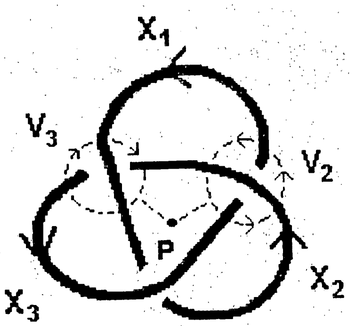
$$x_1^{\#} = x_1$$
$$x_2^{\#} = x_2$$
$$x_3^{\#} = x_3$$

From now on, we will write $x_{i_p}^{\#} = x_{i_p}$ and call $x_{i_1}, \cdots, x_{i_K}$ the <u>generators</u> for the knot group. Since the generators are in one-to-one correspondence with the overcrossings, from now on we will omit the diagram showing the loops that represent the generators. The <u>relations</u> are formed by drawing loops under and around each crossing from a base point p.



$$V_1^{\#} = x_1^{-1}x_3^{-1}x_2x_3$$

This loop V_1 that is drawn under and around one of the knots crossings can be shrunk back to the base point p, so the loop is equivalent to the identity loop e. [To see why the relationship among the generators at this crossing is given by this diagram, see C. Kosniowski's "A First Course in Algebraic Topology".] Therefore, we get the relation $x_1^{-1}x_3^{-1}x_2\,x_3 = 1$. There are two more relations using the trefoil knot as shown below.



$$V_2^{\#} = x_3^{-1}x_2^{-1}x_1x_2$$

$$V_3^{\#} = x_1^{-1}x_3^{-1}x_1x_2$$

Therefore, the presentation for the knot group of the trefoil knot is

$\pi(R^3 - K) = |x_1, x_2, x_3 : x_1^{-1}x_3^{-1}x_2x_3 = 1, x_3^{-1}x_2^{-1}x_1x_2 = 1, x_1^{-1}x_3^{-1}x_1x_2 = 1|$.

It can be shown that any <u>one</u> of the relations for a given knot is redundant and can be obtained using the other relations for that knot. Below I have shown that one of the relations for the trefoil knot is redundant. (Any one of the three can be shown to be redundant.)

$$① \quad x_1^{-1}x_3^{-1}x_2x_3 = 1$$

$$② \quad x_3^{-1}x_2^{-1}x_1x_2 = 1$$

$$③ \quad x_1^{-1}x_3^{-1}x_1x_2 = 1$$

Take ② and solve for x_1:

$$x_1 = x_2x_3x_2^{-1}.$$

Then substitute $x_1 = x_2x_3x_2^{-1}$ into ③, so ③ becomes

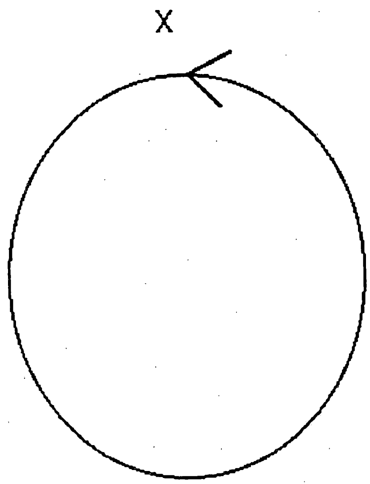
$$x_1^{-1}x_3^{-1}(x_2x_3x_2^{-1})x_2 = 1,$$

which implies equation ① $x_1^{-1}x_3^{-1}x_2x_3 = 1$.

Therefore, ① is shown to be redundant since it can be obtained using ② and ③. [For more information on why the generators generate and where the relations come from and why one relation is redundant, refer to C. Kosniowski's "A First Course in Algebraic Topology".]

Therefore, we obtain for the knot group of the trefoil knot K, the presentation $\pi(\mathbf{R}^3 - K) = |x_1, x_2, x_3\colon\ x_3^{-1}x_2^{-1}x_1x_2 = 1,\ x_1^{-1}x_3^{-1}x_1x_2 = 1|$ where $V_1^{\#}$ has been dropped. By rewriting the relations, we get the presentation $\pi(\mathbf{R}^3 - K) = |x_1, x_2, x_3\colon\ x_3 = x_2^{-1}x_1x_2,\ x_1 = x_3^{-1}x_1x_2|$. Since x_3 is expressed in terms of x_1 and x_2, we can eliminate x_3 by substituting $x_3 = x_2^{-1}x_1x_2$ into the other relation, so $\pi(\mathbf{R}^3 - K) = |x_1, x_2\colon\ x_1 = (x_2^{-1}x_1^{-1}x_2)x_1x_2| =$ $|x_1, x_2\colon\ x_1 = x_2^{-1}x_1^{-1}x_2x_1x_2|$. By multiplying the relation through on the left by $x_1\, x_2$, we obtain the following common presentation of the knot group of the trefoil knot: $|x_1, x_2\colon\ x_1x_2x_1 = x_2x_1x_2|$. This process of substitution to simplify and rewrite the presentation is formally known as Tietze transformations. [For more information, see "Introduction to Knot Theory" by Crowell and Fox.]

Now that we have a presentation for the trefoil knot, we can prove that it cannot be untied, that is, the trefoil knot is not of the same knot type as the trivial knot. The presentation for the knot group of the trivial knot is done below.



Trivial Knot

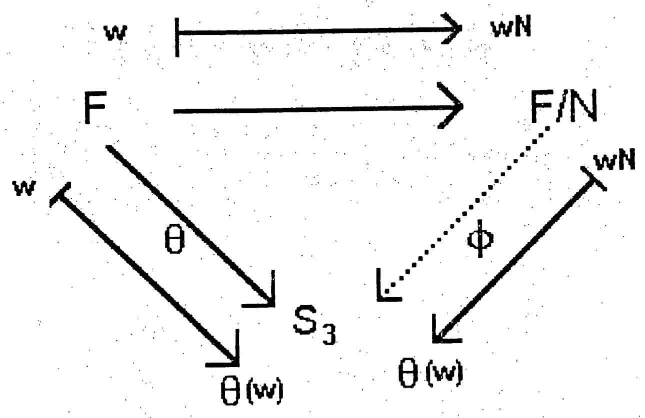
The presentation is $\pi(\mathbf{R}^3 - K) = |x : \quad|$. The trivial knot has only one generator, and therefore it has no relations. Hence, the group of the trivial knot type is infinite cyclic. To prove that the trefoil knot is not of the same knot type as the trivial knot, we must show that their knot groups are not isomophic. To prove this, I will show that the knot group for the trefoil knot $|x, y: \ xyx = yxy|$ is not infinite cyclic. To do this we must consider the symmetric group S_3 which is generated by the cycles (12) and (23). S_3 is not abelian since (12)(23) = (132) and (23)(12) = (123). The presentation G of the trefoil knot consists of a homomorphism of the free group F on x and y onto G whose kernel is N, the normal subgroup of F, generated by $xyx(yxy)^{-1}$. Then $G \cong F/N$ can be written as $|x, y: \ xyx = yxy|$. We will now show that F/N maps homomorphically onto a nonabelian group.

Consider the map:

$$\begin{aligned} \theta: \quad F &\rightarrow S_3 \\ x &\longmapsto (12) \\ y &\longmapsto (13), \end{aligned}$$

extended multiplicatively. The map θ is an onto group homomorphism.

Consider the commutative diagram:



Since the mapping ϕ is defined on cosets, we must show well defined. If $wN = w'N$, does $\phi(wN) = \phi(w'N)$ where w and w' are related by $w' = wn, n \in N$? To show $\phi(wN) = \phi(w'N)$, I first need to show $\theta(n) = e$, i.e., $N \subseteq \ker\theta$. Since $n \in N$ and θ is a group homomorphism, it will suffice to show that θ maps the generators of N to e.

$$\begin{aligned}
\theta[xyx(yxy)^{-1}] &= \theta(yxy)\theta(yxy)^{-1} \\
&= \theta(x)\theta(y)\theta(x)[\theta(y)\theta(x)\theta(y)]^{-1} \text{ since } \theta \text{ is homomorphism} \\
&= (12)(23)(12)[(23)(12)(23)]^{-1} \\
&= (13)(13)^{-1} \\
&= e.
\end{aligned}$$

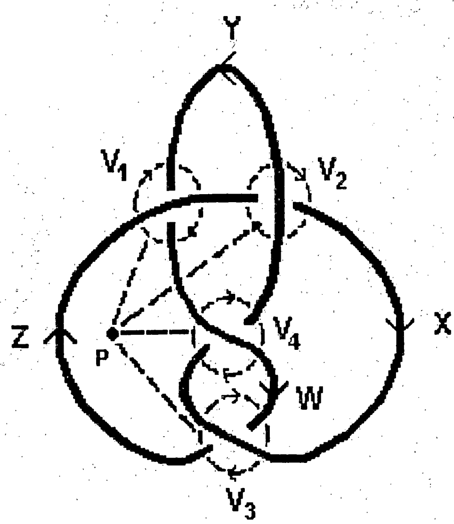
Therefore $\theta(n) = e$.

Now

$\phi(w^{/}N)$	$=$	$\phi[(wn)N]$	since $w^{/} = wn$
	$=$	$\theta(wn)$	definition of mapping ϕ
	$=$	$\theta(w)\theta(n)$	since θ is homomorphism
	$=$	$\theta(w)$	since $\theta(n) = e$ (shown above)
	$=$	$\phi(wN)$	definition of mapping.

Therefore $\phi(w^{/}N) = \phi(wN)$, so the map is well-defined. Thus, the knot group can be mapped homomorphically onto a nonabelian group. So the knot group is nonabelian (if the knot group were abelian, then its image would be abelian) and therefore is not cyclic. This shows that the knot groups for the trefoil knot and trivial knot are not isomorphic. Hence, the trefoil knot cannot be untied.

Another example of finding a presentation for a knot is done below.



$$v_1^{\#} = zyz^{-1}w^{-1}$$
$$v_2^{\#} = zy^{-1}x^{-1}y$$
$$v_3^{\#} = x^{-1}wxz^{-1}$$
$$v_4^{\#} = wy^{-1}w^{-1}x$$

Figure-eight Knot

A presentation for the knot group of the figure-eight knot K is $\pi(R^3\text{-}K) =$ $|x, y, z, w: \; z = wzy^{-1}, \; y = xyz^{-1}, \; z = x^{-1}wx|$ where $v_4^{\#}$ has been dropped. Using Tietze transformations we can substitute $z = x^{-1}wx$ in the other two relations to obtain $\pi(R^3 - K) = |x, y, w: \; x^{-1}wx = wx^{-1}wxy^{-1},$ $y = xyx^{-1}w^{-1}x|$. The second relation now gives $w = xy^{-1}xyx^{-1}$ and by substituting $w = xy^{-1}xyx^{-1}$ into the first relation we obtain $\pi(R^3 - K) =$ $|x, y: \; x^{-1}(xy^{-1}xyx^{-1})x = (xy^{-1}xyx^{-1})x^{-1}(xy^{-1}xyx^{-1})xy^{-1}|$ which can be simplified to $\pi(R^3 - K) = |x, y: \; y^{-1}xy = xy^{-1}xyx^{-1}y^{-1}x|$. By multiplying both sides on the left by $x^{-1}y$, we obtain $\pi(R^3 - K) = |x, y: \; y = x^{-1}yxy^{-1}xyx^{-1}y^{-1}x|$. Finally, by multiplying through on the right by $x^{-1}yxy^{-1}$, we obtain the common presentation of the knot group for the figure-eight knot $\pi(\mathbf{R}^3 - K) =$

$|x, y: \; yx^{-1}yxy^{-1} = x^{-1}yxy^{-1}x \,|$.

In order to prove that the figure-eight knot is distinct from the trefoil, it is sufficient to show that their groups are not isomorphic. Unfortunately, there is no easy way to determine whether or not two presentations have isomorphic groups. So what is needed are some easy to caclulate algebraic quantities which when derived from isomorphic groups, remain the same. These are the so-called group invariants. That is, since the knot group is usually too complicated as an invariant, we must pass to one that is simpler and easier to handle. One such invariant is the sequence of Alexander knot polynomials. This invariant can be used to distinguish the trefoil knot and the figure-eight knot. There is an object called the Alexander matrix which is constructed using mappings of the free group onto itself called Fox derivatives. From the Alexander matrix we can determine the sequence of elementary ideals which then gives us the sequence of Alexander knot polynomials. For the rest of the chapter, we will just write down the results of our calculations without leading reader through derivations. (For details on the calculations of fox derivatives, Alexander matrices, elementary ideals, and Alexander polynomials, the reader should consult "Introduction to Knot Theory" by Crowell and Fox.) The sequence of Alexander knot polynomials for the trefoil knot is $\Delta_1 = 1 - t + t^2$ and $\Delta_k = 1$for $k \geq 2$. The sequence of Alexander knot polynomials for the figure-eight knot is $\Delta_1 = t^2 - 3t + 1$ and $\Delta_k = 1$ for $k \geq 2$. Therefore, the trefoil and figure-eight are not equivalent knots since their Alexander knot polynomials are inequivalent. In Chapter 2, I will give a detailed description of another invariant that is easier to calculate that will also distinguish the trefoil and the figure-eight.

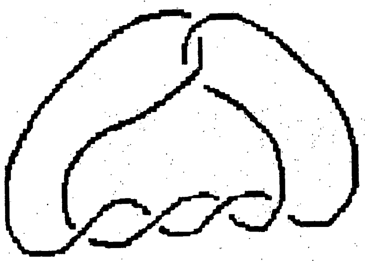


Figure 1

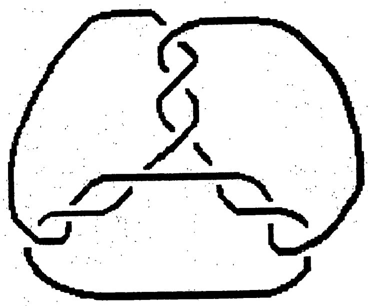


Figure 2

The sequence of Alexander knot polynomials for both figure 1 and figure 2 is $\Delta_1 = 2t^2 - 5t + 2$ and $\Delta_k = 1$ for $k \geq 2$. The sequence of Alexander knot polynomials does not distinguish these two knots. To distinguish these knots we must use another invariant. This invariant is the sequence of elementary ideals. The sequence of elementary ideals for figure 1 is $E_1 = (2t^2 - 5t + 2)$ and $E_k = (1)$ for $k \geq 2$, where (a) means the ideal generated by a in the ring $Z[t, t^{-1}]$. The sequence of elementary ideals for figure 2 is $E_1 = (2t^2 - 5t + 2)$, $E_2 = (2 - t, 1 - 2t)$ and $E_k = (1)$ for $k \geq 3$. Therefore, figure 1 and figure 2 are not equivalent knots since their sequence of elementary ideals are not equal. This example verifies that the elementary ideals are stronger invariants than the polynomials.

The following two knots cannot be distinguished using either of the two previous invariants.

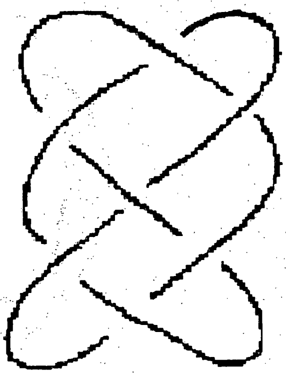


Figure 3

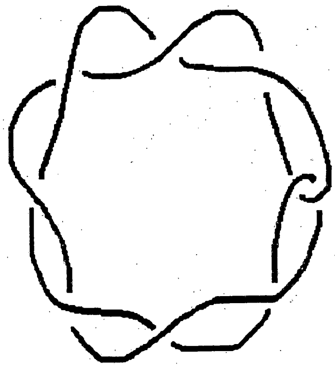
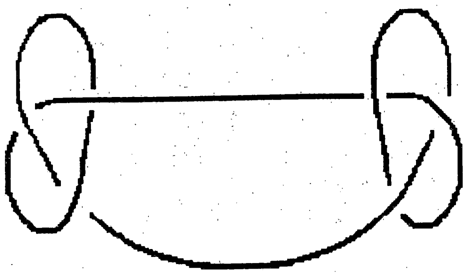


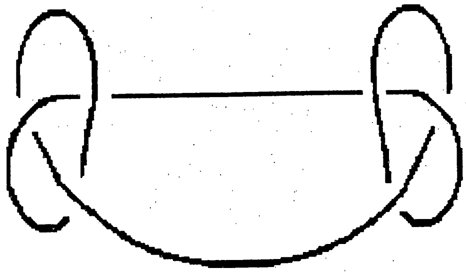
Figure 4

The Alexander matrix of each of these knots is $\| \; 4t^2 - 7t + 4 \quad 0 \; \|$. Since the sequence of elementary ideals and the sequence of Alexander knot polynomials are defined in terms of the Alexander matrix, they are equivalent for both knots. The elementary ideals and knot polynomials are not strong enough invariants to distinguish these two knots. Although it can be shown that their knot groups are nonisomorphic using other methods; therefore, the two knots are not equivalent. This shows the presentation type is a stronger invariant than either the elementary ideals or knot polynomials.

The next pair of knots not only have equivalent Alexander matrices, but they possess isomorphic groups as shown below.



Granny Knot



Square Knot

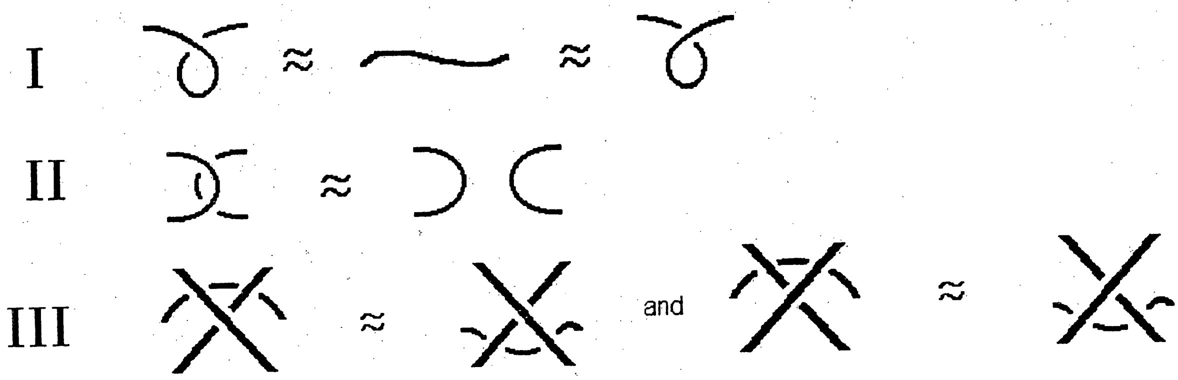
Each knot group has the presentation $\pi(\mathbf{R}^3 - K) = |\ x, y, a:\ a^{-1}xa = xax^{-1},\ a^{-1}ya = yay^{-1}\ |$. To distinguish the granny knot and square knot, we will need to use more modern techniques.

CHAPTER II

MORE MODERN TECHNIQUES

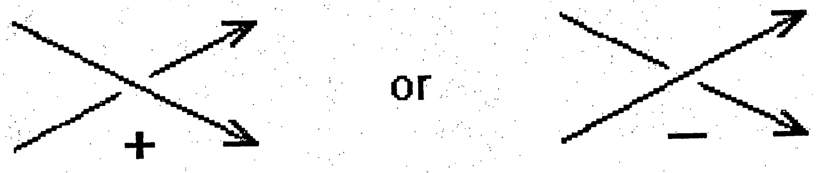
In the 70's and 80's, J. H. Conway and Louis H. Kauffman each came up with a whole new approach with which to study knots. We will focus on Kauffman's approach which uses "brackets". This new approach uses formal symbolism and a type of arithmetic with diagrams. It also uses no fundamental groups whatsoever. This more modern approach to knots not only is easier to handle but can distinguish a wider variety of knots and objects to be defined later as links.

To begin the discussion of the new approach, I must first define (or in some cases, redefine) a few terms. If we regard a knot as a single closed loop in $\mathbf{R}^3$, then a <u>link</u> will be an object consisting of one or more such loops. As referred to earlier, the following are the <u>Reidemeister Moves</u> : (Only the relevant portion of the knot or link is shown.)



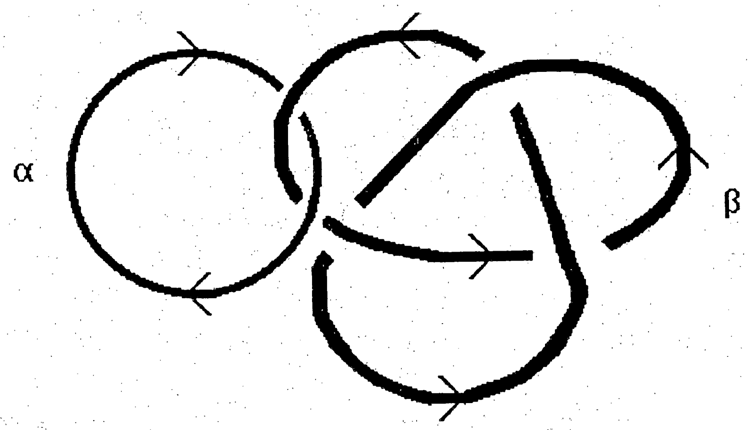
Reidemeister proved that these three moves change the structure of the diagram while leaving the topological type of the knot or link the same. That is, two knots can be manipulated one knot into the other without cutting or retying if and only if their diagrams are related by a finite sequence of Reidemeister moves. The equivalence relation generated by moves II and III is called <u>regular isotopy</u>. The equivalence relation generated by all three moves is called <u>ambient isotopy</u>.

A knot or link is said to be <u>oriented</u> if each arc in its diagram is assigned a direction (according to the right-handed screw) so that at each crossing the orientations appear either as



and have a corresponding sign of ±1.

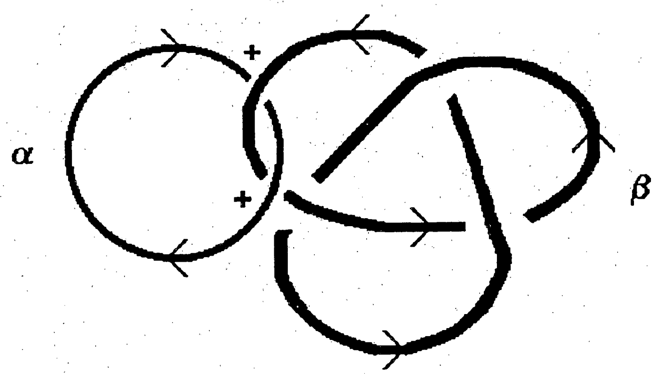
Let L = {α, β} be a link of two components α and β.



Define the <u>linking number</u> $\ell k(L) = \ell k(\alpha, \beta)$ by the formula $\ell k(\alpha, \beta) =$

$\frac{1}{2} \sum_{p \epsilon \alpha \sqcap \beta} \epsilon(p)$, where $\alpha \sqcap \beta$ denotes the set of crossings of α with β and $\epsilon(p)$ denotes the sign of the crossing.

Example:

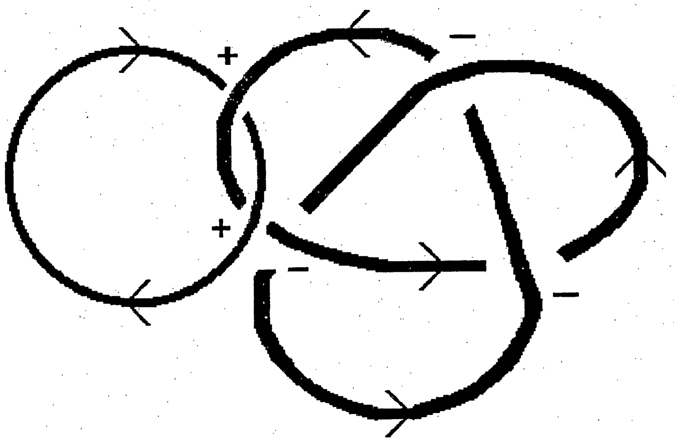


$$\ell k(\alpha, \beta) = \tfrac{1}{2}(1 + 1) = 1$$

So the linking number of the link above is 1. Notice we only consider the crossings of α with β, so where β crosses itself, there is no contribution to the linking number.

Let K be any oriented link diagram. Then the <u>writhe of K</u> (or <u>twist number of K</u>) is defined by the formula $w(K) = \sum_{p \epsilon c(K)} \epsilon(p)$, where $c(K)$ denotes the set of crossings in the diagram K.

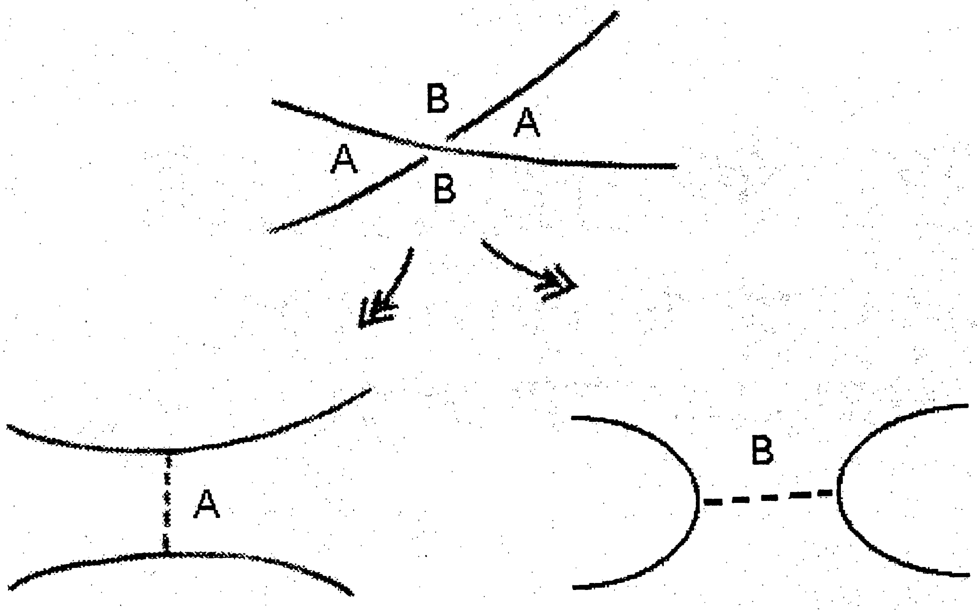
Example:



$$w(K) = 1 + 1 + {}^{-}1 + {}^{-}1 + {}^{-}1 = {}^{-}1$$

Thus, the writhe of the link above is -1. Notice all crossings were considered when calculating the writhe.

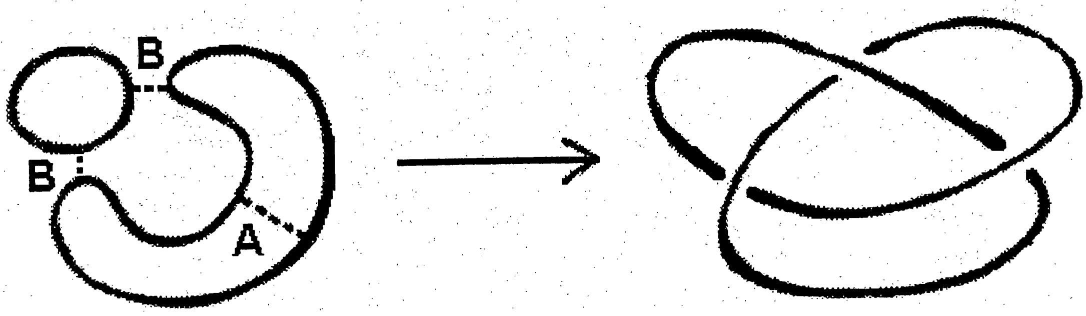
Consider a crossing in an unoriented link diagram. Two associated labelled diagrams can be obtained by <u>labelling and splicing the crossing</u> (shown below).



Type A Type B

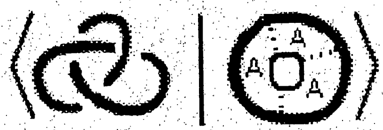
The regions labelled A (respectively B) are those that appear on the left (respectively right) to an observer walking toward the crossing along one of the undercrossing segments.

By keeping track of each splice that is performed, we can reconstruct a given knot (or link) from its descendants. A reconstruction is shown below.



final descendant given knot

The final descendants (that is, when all of the crossings having been spliced) of a knot or link K are called the <u>states of K</u>. Each state can be used to reconstruct K. Invariants of knots and links are constructed by averaging over these states. To do this, let **б** be a state of K and ⟨K|**б**⟩ denote the commutative product of the labels attached to **б**. Example shown below.

⟨K|б⟩ =  = A · A · A = A^3

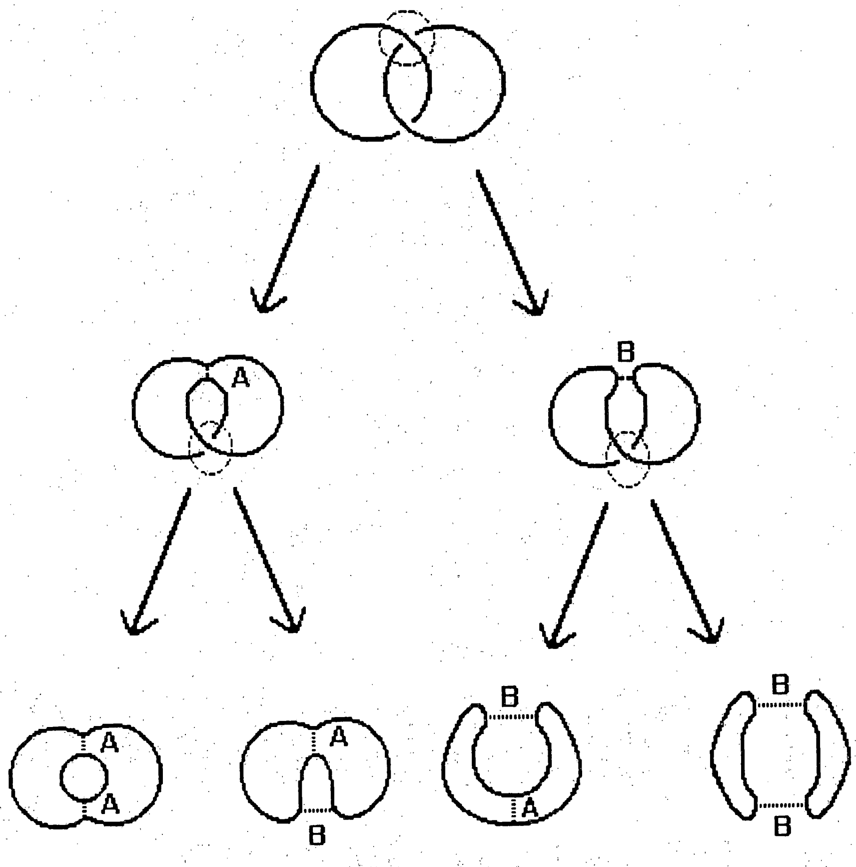
Also let ||**б**|| be one less than the number of loops in **б**.

||б|| =  = 2 - 1 = 1

Now we can define the <u>bracket</u> <u>polynomial</u>, ⟨K⟩, by the following formula:

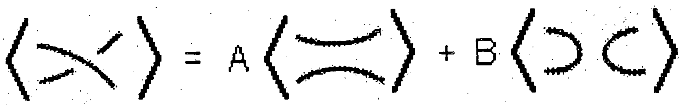
⟨K⟩ = $\sum_{б}$⟨K|**б**⟩$d^{||б||}$, where we sum over all states **б** of K.

The following is an example of the use of the bracket polynomial.

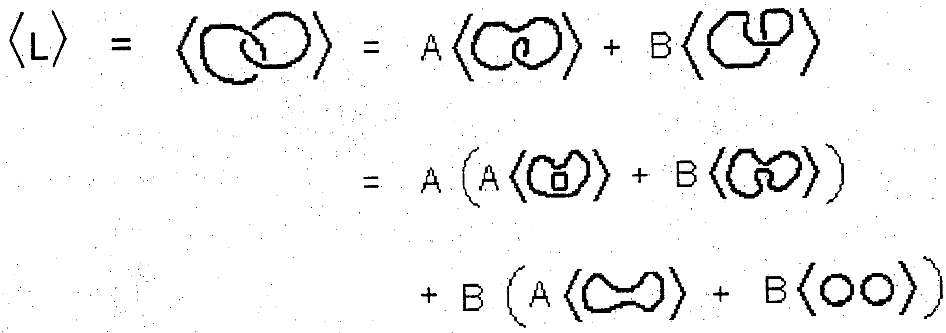


There are four states (final descendants) for this link. The bracket polynomial is calculated as follows:

$$\begin{aligned}\langle K \rangle &= A^2 d^{2-1} + ABd^{1-1} + ABd^{1-1} + B^2 d^{2-1} \\ &= A^2 d + AB + AB + B^2 d \\ &= A^2 d + 2AB + B^2 d\end{aligned}$$

Notice that at each node of the tree above, the bracket of the relevant crossing is either A times the bracket of a type A splice or B times the bracket of a type B splice, so 

holds (where only the relevant portion of the diagram is shown). An example of how this can be used to compute the bracket is done below for the link L.

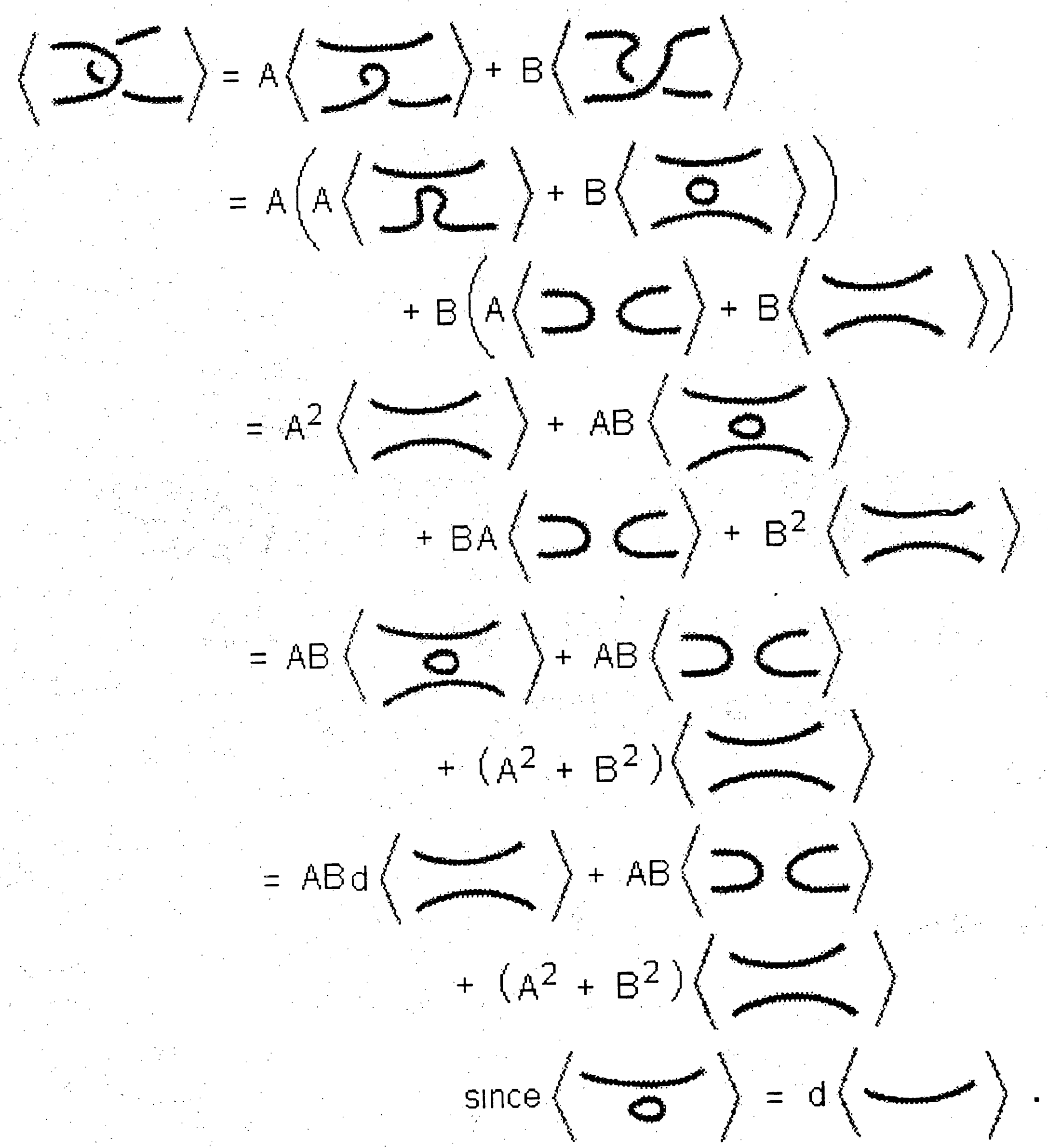


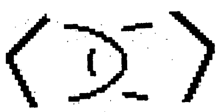
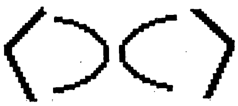
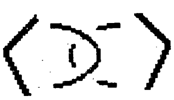
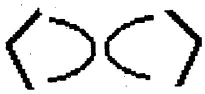
$$= A^2d^{2-1} + ABd^{1-1} + BAd^{1-1} + B^2d^{2-1}$$

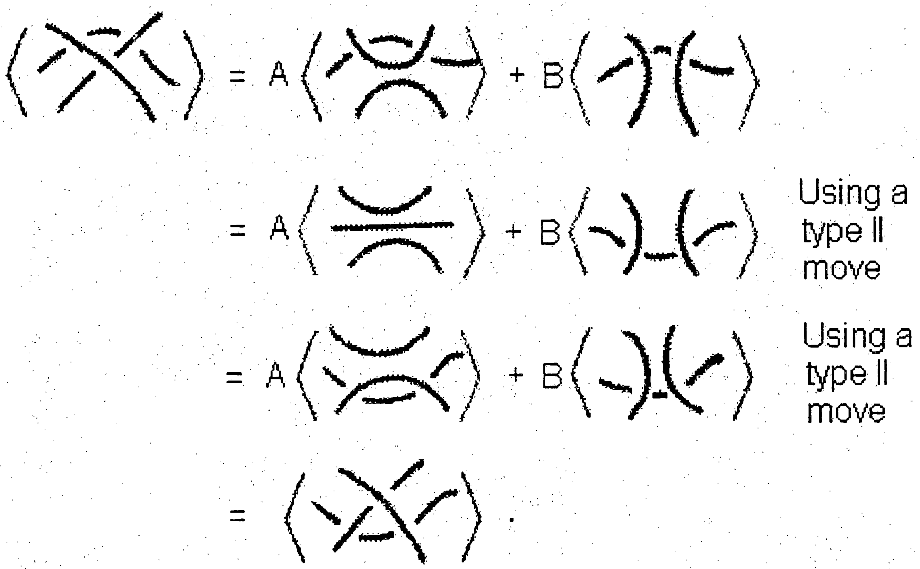
$$= A^2d + 2AB + B^2d.$$

Notice we got the same bracket polynomial as we did using the tree diagram.

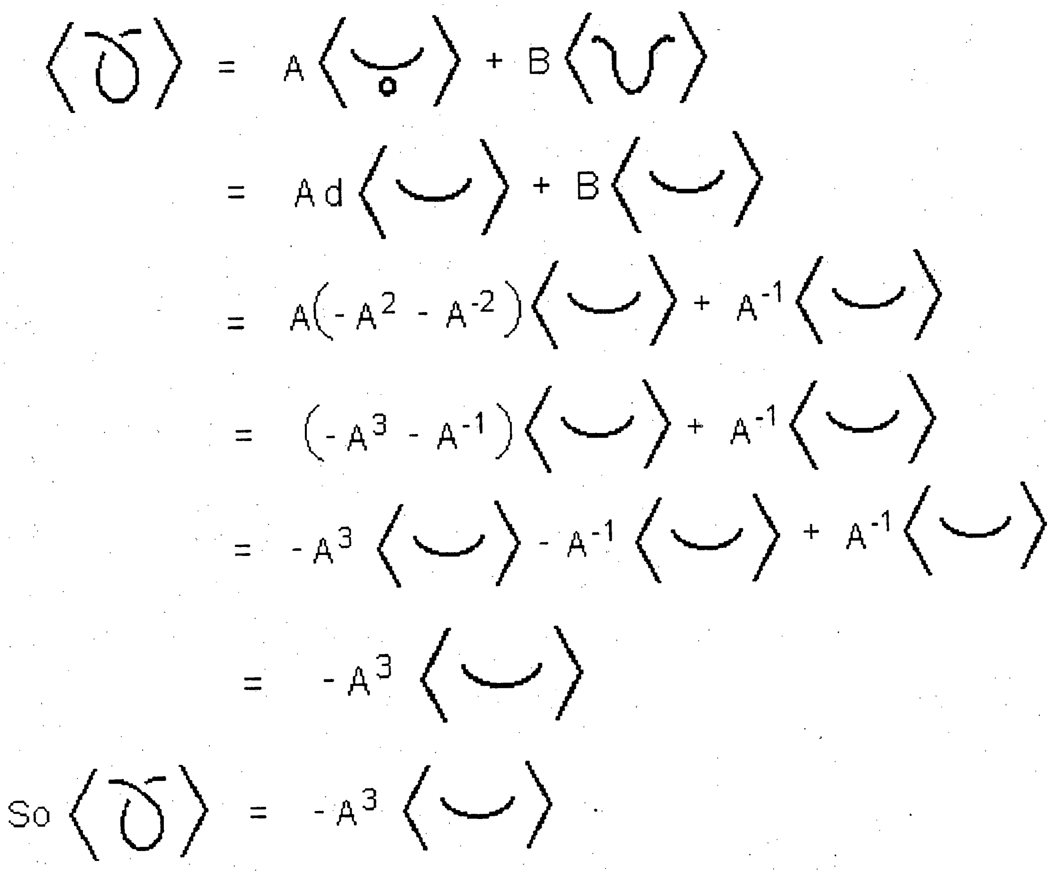
The bracket polynomial is not an invariant as it stands. We must investigate it under the Reidemeister moves and determine conditions on A, B, and d for it to become an invariant. We first investigate the bracket under type II and III moves. Consider the following:



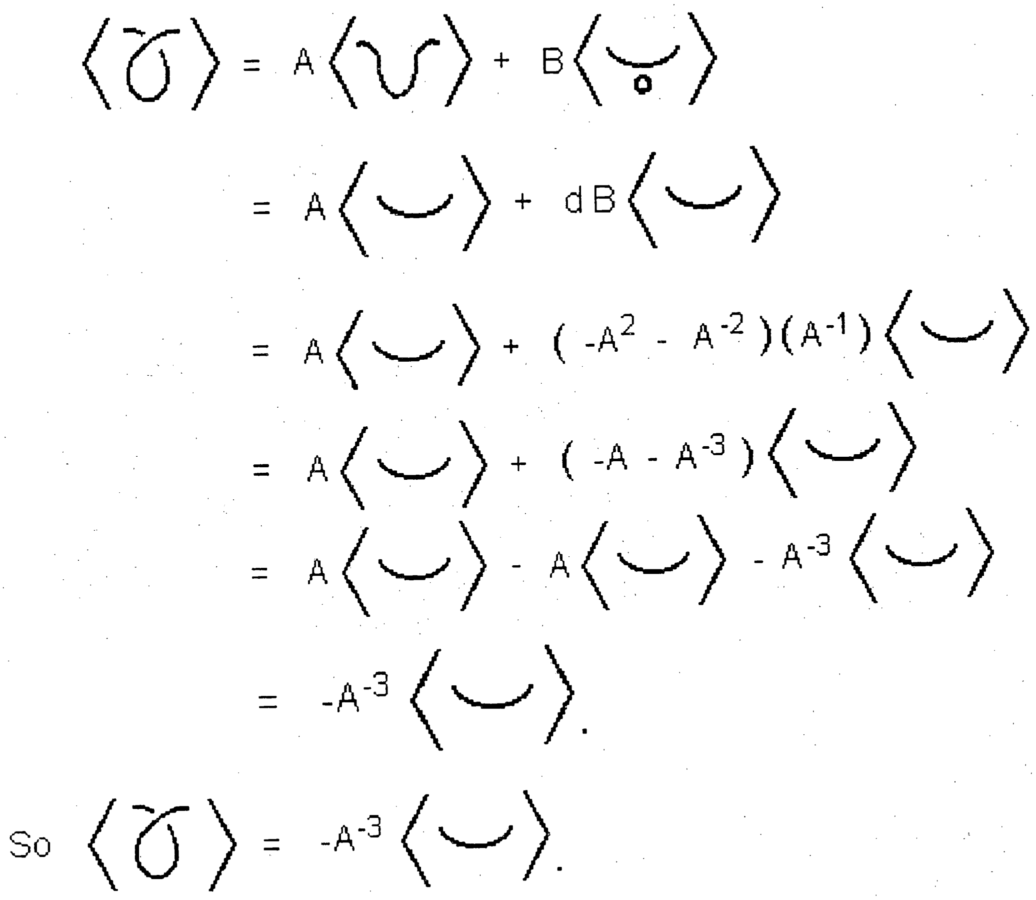
For  to equal  (type II move), it suffices to have $AB = 1$ and $d = -A^2 - A^{-2}$. Suppose $A = B^{-1}$ and $d = -A^2 - A^{-2}$, then we just showed that  = , and we now consider the bracket of a type III move.



This shows that the bracket with $B = A^{-1}$, $d = -A^{2} - A^{-2}$ is invariant under moves II and III. (That is, if two diagrams differ by a type II or type III move, their brackets are the same.) Let's now investigate how the bracket transforms under a type I move.



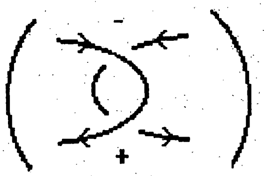
We will now calculate the bracket for the same diagram but with the "loop" having the opposite crossing.

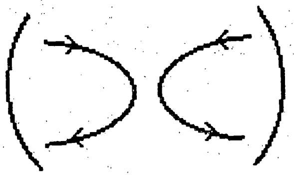


Notice that if two diagrams differ by a type I move, their brackets are not the same. Therefore, the bracket is only invariant under type II and type III moves.

To obtain an invariant of ambient isotopy (I, II and III), we must normalize the bracket. To do this, we must take a closer look at the writhe of K, w(K). Recall the $w(K) = \sum_p \epsilon(p)$ where p runs over all crossings in K, and $\epsilon(p)$ is the sign of the crossing. The writhe of K is an invariant of regular isotopy (II, III) as shown below.

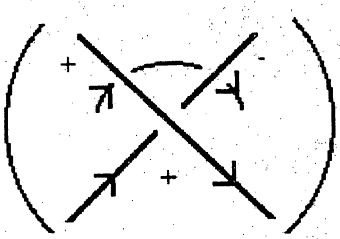
Type II move: (one possible orientation is shown)

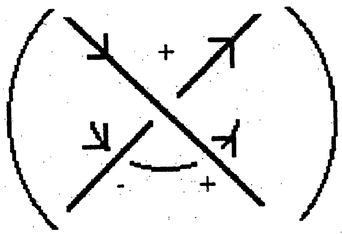
w  = -1 + 1 = 0

w  = 0

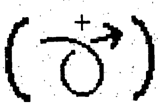
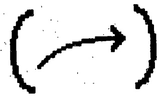
Since w  = w  (independent of orientation), the writhe is invariant under a type II move.

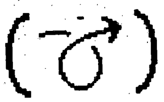
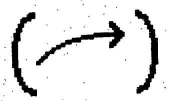
Type III move: (again, one possible orientation is shown)

w  = 1 + -1 + 1 = 1

w  = 1 + -1 + 1 = 1

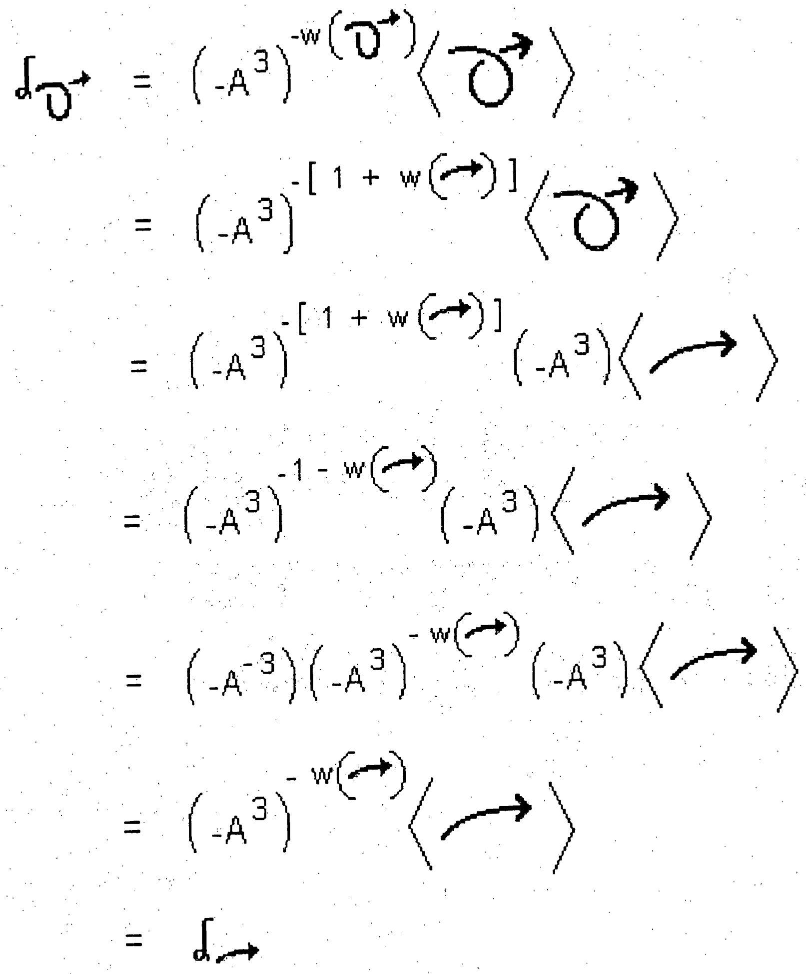
Since w  = w  (independent of orientation), the writhe is invariant under a type III move. Therefore, $w(K)$ is an invariant of regular isotopy. Also notice that (since writhe is sum on the crossings),

w  = 1 + w 

w  = -1 + w  .

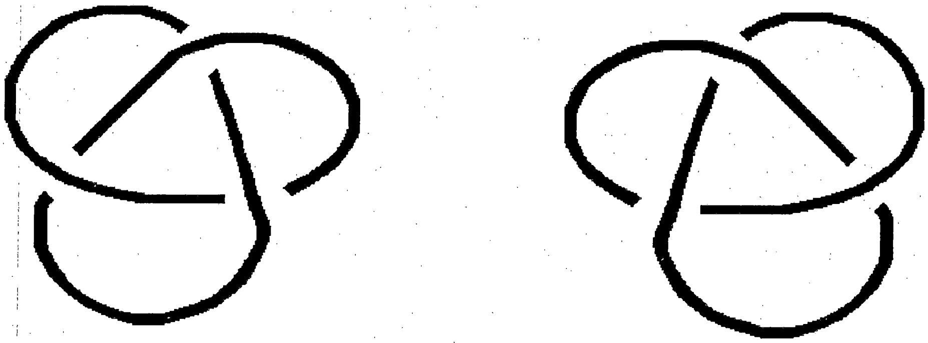
Now we can define a <u>normalized bracket</u>, f_K, for oriented links K by the formula

$f_K = (-A^3)^{-w(K)} \langle K \rangle$. I will show that the normalized bracket of f_K is an invariant of ambient isotopy. Since $w(K)$ and $\langle K \rangle$ are regular isotopy invariants, it follows that f_K is a regular isotopy invariant. Thus, we only need to check that f_K is invariant under type I moves.



This shows f_K is invariant under type I moves. Therefore, the normalized bracket polynomial f_K is an invariant of ambient isotopy.

Before I show the use of the normalized bracket polynomial f_K, I would like to define the mirror image of a knot or link. The <u>mirror image of K</u> is obtained by exchanging all overcrossings and undercrossings of K. The trefoil knot T and its mirror image T* are shown below. The trefoil and its mirror image have isomorphic knot groups (left as an easy check for the reader), so they could not be distinguished using previous methods, but using the normalized bracket polynomial they will be shown to be distinct knots.

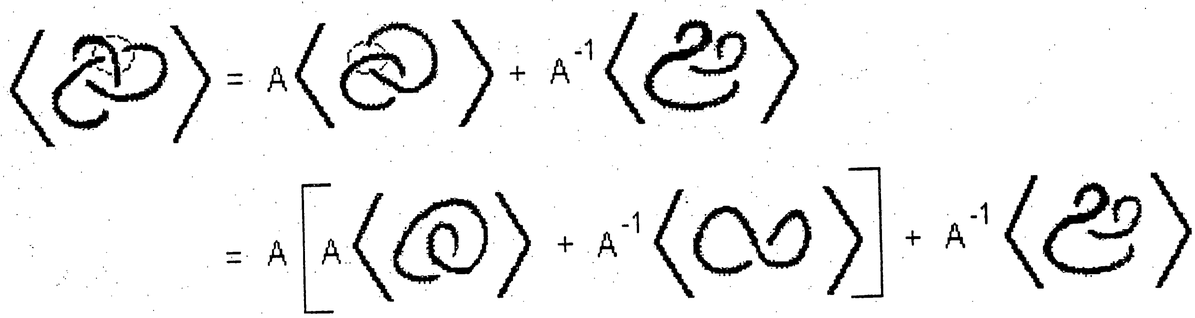


T T *

Trefoil knot and its mirror image

Using the normalized bracket polynomial f_K , I will show that the trefoil knot cannot be deformed into its mirror image T* . This will show that the trefoil knot is topologically distinct from its mirror image.

Trefoil knot T



$$= A[A\,(-A^{3}) + A^{-1}(-A^{-3})] + A^{-1}(-A^{-3})(-A^{-3})$$

$$= -A^{5} - A^{-3} + A^{-7}$$

So $\langle T \rangle = -A^{5} - A^{-3} + A^{-7}$ and $w(T) = 3$, (independent of orientation).

Thus, $f_T = (-A^{3})^{-w(T)}\langle T \rangle$

$$= (-A^{3})^{-3}(-A^{5} - A^{-3} + A^{-7})$$

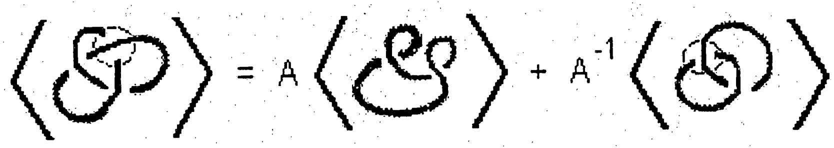
$$= -A^{-9}\,(-A^{5} - A^{-3} + A^{-7})$$

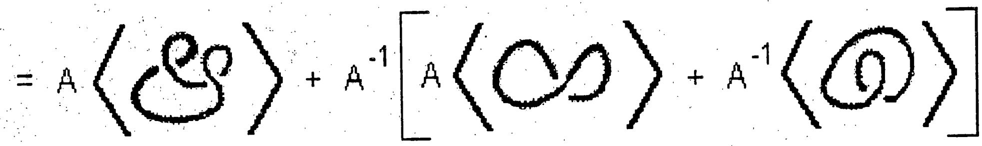
$$= A^{-4} + A^{-12} - A^{-16}$$

Therefore, the normalized bracket polynomial for the trefoil knot is

$f_T = A^{-4} + A^{-12} - A^{-16}$.

Mirror Image T*





$$= \quad A(-A^3)(-A^3) + A^{-1}\left[A(-A^3) + A^{-1}(-A^{-3})\right]$$

$$= \quad A^7 - A^3 - A^{-5}$$

So $\langle T^* \rangle = A^7 - A^3 - A^{-5}$

and $w(T^*) = -3$, (independent of orientation).

Thus, $f_{T^*} \quad = \quad (-A^3)^{-w(T^*)}\langle T^* \rangle$

$$= \quad (-A^3)^3\,(A^7 - A^3 - A^{-5})$$

$$= \quad -A^9\,(A^7 - A^3 - A^{-5})$$

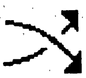
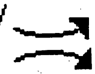
$$= \quad -A^{16} + A^{12} + A^4$$

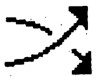
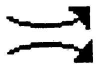
Therefore, the normalized bracket polynomial for the mirror image of the trefoil knot is $f_{T^*} = -A^{16} + A^{12} + A^4$. Since $f_T \neq f_{T^*}$, we conclude that the trefoil is not ambient isotopic to its mirror image. That is, the trefoil knot is not topologically equivalent to its mirror image. This is the first example of modern techniques being more powerful than the methods in Chapter 1.

In 1984, using representations of certain algebras, V. Jones discovered a polynomial which came to be called the <u>Jones polynomial</u>. The 1-variable Jones polynomial, $V_K(t)$, is a Laurent polynomial in the variable t (i.e., polynomial with integer powers of t). The polynomial satisfies:

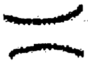
i. If K is ambient isotopic to K', then $V_K(t) = V_{K'}(t)$.

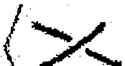
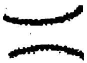
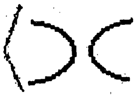
ii. V  $= 1$

iii. $t^{-1}V$  $- \; tV$  $= \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V$ 

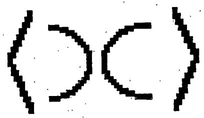
where , ,  stand for larger link diagrams that differ only by the crossing shown. Jones showed that there is a unique polynomial satisfying these identities.

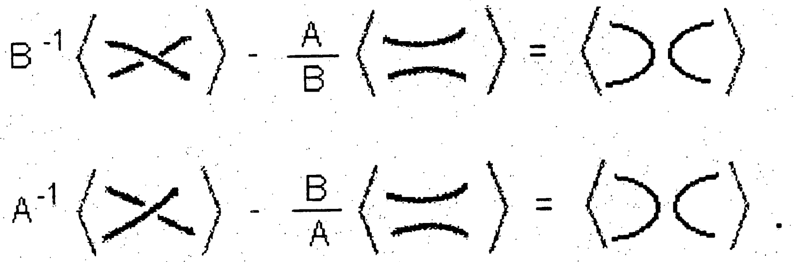
I will show that the Jones polynomial is the same as the bracket polynomial with the substitution $A = t^{-1/4}$. Recall the formulas for the bracket polynomial,

① $\langle$  $\rangle = A \langle$  $\rangle + B \langle$  $\rangle$

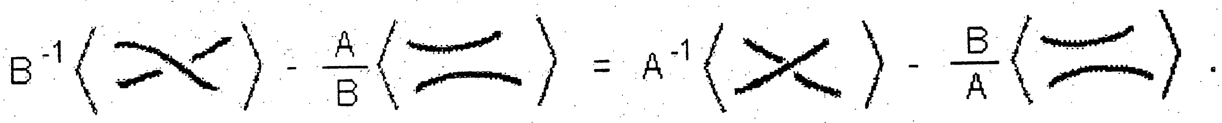
② $\langle$  $\rangle = B \langle$  $\rangle + A \langle$  $\rangle$.

By dividing the first equation by B and the second by A and solving for

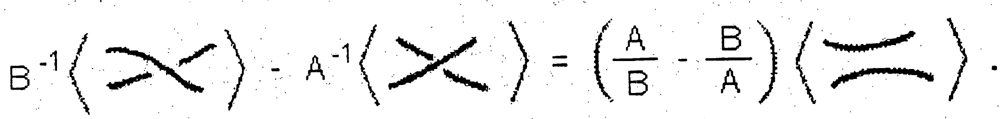
 , we obtain the following two equations



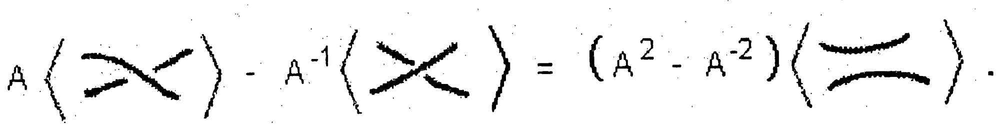
By setting them equal,



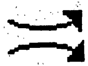
By regrouping like terms,



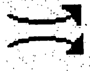
And since $B = A^{-1}$ we have,



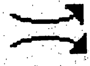
Orientating them we obtain,

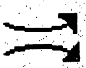
A ⟨⟩ $- A^{-1}$ ⟨⟩ $= (A^2 - A^{-2})$ ⟨⟩ .

Now let $\alpha = -A^3$ and multiply through by α^{-w} where $w = w(K)$,

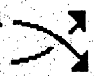
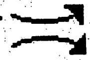
A ⟨⟩ $\alpha^{-w} - A^{-1}$ ⟨⟩ $\alpha^{-w} = (A^2 - A^{-2})$ ⟨⟩ α^{-w}.

Factoring out an α from the first term and an α^{-1} from the second term on the left,

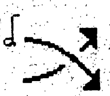
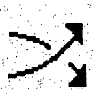
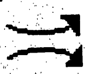
$A\alpha$ ⟨⟩ $\alpha^{-w-1} - A^{-1}\alpha^{-1}$ ⟨⟩ $\alpha^{-w+1} = (A^2 - A^{-2})$ ⟨⟩ α^{-w}.

$A\alpha$ ⟨⟩ $\alpha^{-(w+1)} - A^{-1}\alpha^{-1}$ ⟨⟩ $\alpha^{-(w-1)} = (A^2 - A^{-2})$ ⟨⟩ α^{-w}.

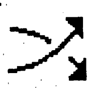
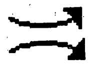
Recalling that $\int_K = (-A^3)^{-w(K)} \langle K \rangle$ allows us now to write,

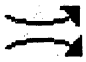
$A\alpha \int$  $- A^{-1}\alpha^{-1} \int$  $= (A^2 - A^{-2}) \int$  .

Now substituting $\alpha = -A^3$,

$-A^4 \int$  $+ A^{-4} \int$  $= (A^2 - A^{-2}) \int$  .

The final substitution $A = t^{-1/4}$ yields,

$$-t^{-1} f$$  $$+ \; t f$$  $$= \left(t^{-1/2} - t^{1/2}\right) f$$ 

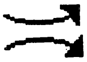
$$t^{-1} f$$  $$- \; t f$$  $$= \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) f$$  .

Therefore, with the substitution $A = t^{-1/4}$ into $f_K(A) = (-A^3)^{-w(K)} \langle K \rangle$, we notice $f_K(t^{-1/4})$ satisfies the defining identities for, $V_K(t)$, the Jones polynomial. By uniqueness of $V_K(t)$, we have $f_K(t^{-1/4}) = V_K(t)$. Thus, the normalized bracket yields the 1-variable Jones polynomial.

The Jones polynomial is structurally similar to the <u>Alexander-Conway polynomial</u> $\nabla_K(Z)$ which is a polynomial in Z with integer coefficients. This polynomial can be shown to satisfy the following properties:

i) $\nabla_K(z) = \nabla_{K'}(z)$, if the oriented links K and K' are ambient isotopic.

ii) ∇  $= 1$

iii) ∇  $-\; \nabla$  $= Z\nabla$ 

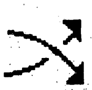
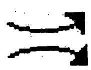
Conway showed that these properties characterize this polynomial, and that this polynomial is just a disguised and normalized form of the original Alexander polynomial.

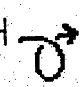
A major difference between the Jones polynomial and Conway polynomial is that the Conway polynomial does not differentiate mirror images. Hence, the Conway polynomial cannot distinguish between the trefoil and its mirror image, while the Jones polynomial can. Both the Jones and Conway-Alexander polynomials can be generalized to what is known as the Homfly polynomial, $P_K(\alpha, z)$, to be defined later. For $\alpha = t^{-1}$, $Z = \sqrt{t} - \frac{1}{\sqrt{t}}$, P_K specializes to the Jones polynomial, and for $\alpha = 1$, P_K specializes to the Conway-Alexander polynomial. Homfly is so named after its many discoverers (J. Hoste, A. Ocneanu, K. C. Millett, P. Freyd, W. B. R. Liekorish, D. Yetter).

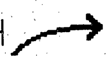
The oriented invariant $P_K(\alpha, z)$ can be regarded as the normalization of a regular isotopy invariant. The <u>regular isotopy homfly polynomial</u> $H_K(\alpha, z)$ (which we will assume exists) is defined by the following properties:

i. If the oriented links K and K' are regular isotopic, then $H_K(\alpha, z) = H_{K'}(\alpha, z)$.

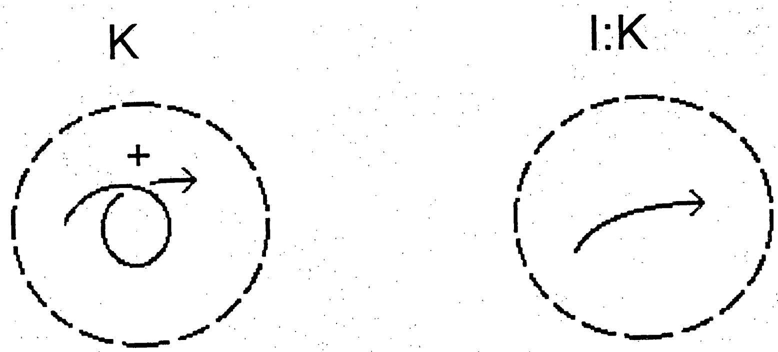
ii. H  = 1

iii. H  - H  = Z H 

iv. H  = α H 

H  = α^{-1} H 

This regular isotopy invariant can be normalized by including $\alpha^{-w(K)}$ to measure the writhe in a diagram. We then have $P_K(\alpha, z) = \alpha^{-w(K)} H_K(\alpha, z)$ which is an invariant of ambient isotopy. To prove that $P_K(\alpha, z)$ is an invariant of ambient isotopy, I will let $P_K = P_K(\alpha, z)$ so $P_K = \alpha^{-w(K)} H_K(\alpha, z)$. We know P_K is an invariant of regular isotopy since $w(K)$ and $H_K(\alpha, z)$ are invariants of regular isotopy. We only need to show P_K is an invariant under type I moves. Let $I{:}K$ represent a type I move applied to K (shown below).



So the question is, does $P_{I:K} = P_K$?

$$
\begin{aligned}
P_{I:K} &= \alpha^{-w(I:K)} H_{I:K}(\alpha, z) \\
&= \alpha^{-(w(K)-1)} \alpha^{-1} H_K(\alpha, z) \\
&= \alpha^{-w(K)} \alpha^{+1} \alpha^{-1} H_K(\alpha, z) \\
&= \alpha^{-w(K)} H_K(\alpha, z) \\
&= P_K
\end{aligned}
$$

Since $P_{I:K} = P_K$, P_K is an invariant under type I moves. (Similarly, if the type I move eliminates a negative crossing, $P_{I:K} = P_K$.) We also must show

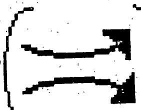
P $= 1.$

P $= a^{-w(}$$^{)} H$ 

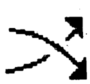
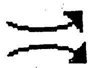
P $= a^0 H$ , since $w\,($$) = 0.$

Therefore,

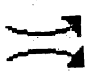
P $= 1$, since H $= 1.$

Since P_K is an invariant of ambient isotopy and P $= 1$, it only remains to find the exchange identity for P_K. Let $w = w\Big($$\Big)$,

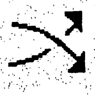
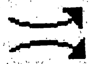
and recall that H satisfies the following identity,

H $-$ H $=$ $Z\,H$ .

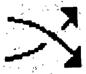
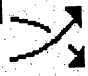
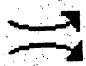
Multiplying by α^{-w},

$\alpha^{-w} H$ $-$ $\alpha^{-w} H$ $=$ $Z\,\alpha^{-w} H$ .

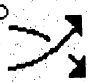
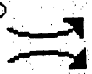
Since $\alpha\alpha^{-1} = 1$,

$\alpha\,\alpha^{-1}\,\alpha^{-w}\,H$  $-\ \alpha\,\alpha^{-1}\,\alpha^{-w}\,H$  $=\ Z\,\alpha^{-w}\,H$  .

By rewriting,

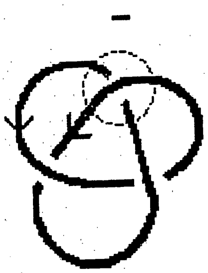
$\alpha\Big(\alpha^{-(w+1)}\,H$  $\Big) -\ \alpha^{-1}\Big(\alpha^{-(w+1)}\,H$  $\Big) =\ Z\Big(\alpha^{-w}\,H$  $\Big)$.

Therefore,

$\alpha\,P$  $-\ \alpha^{-1}\,P$  $=\ Z\,P$  .

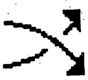
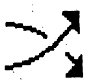
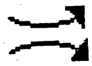
We now have the normalized polynomial which is an invariant of ambient isotopy.

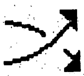
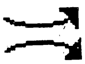
To show the use of the Homfly polynomial, I will calculate the Homfly polynomial for both the trefoil and its mirror image.

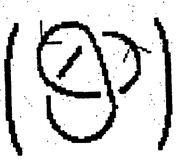
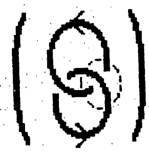


Trefoil T

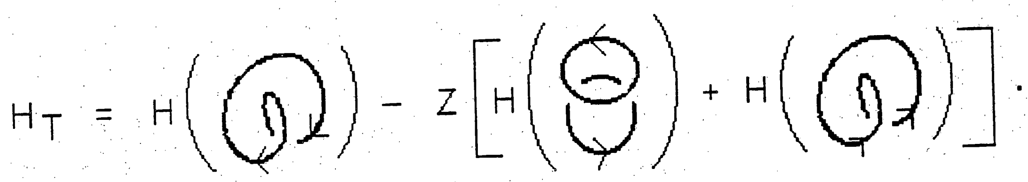
Recall, the exchange identity is as follows:

H  - H  = ZH  ; hence expanding about the negative crossing above,

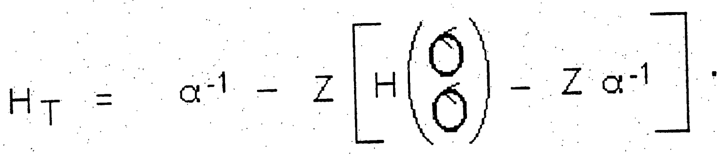
H_T = H  = H  - ZH 

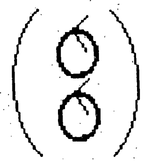
= H  - ZH  .

Using a type II move on the first term and the exchange identity on the second term, we obtain:



If we apply property (iv) twice and a type II move once, we get the following:



Below we will show H  = $\frac{\alpha - \alpha^{-1}}{Z}$. Using this,

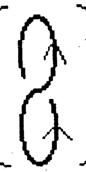
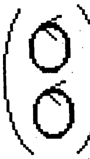
we can continue and write the above as:

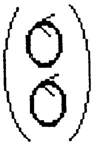
$$
\begin{aligned}
H_T &= \alpha^{-1} - Z\left(\frac{\alpha - \alpha^{-1}}{Z} - Z\alpha^{-1}\right) \\
&= \alpha^{-1} - \alpha + \alpha^{-1} + Z^2\alpha^{-1} \\
&= 2\alpha^{-1} - \alpha + Z^2\alpha^{-1}.
\end{aligned}
$$

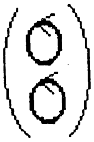
Therefore, since $w(T) = -3$,

$$
\begin{aligned}
P_T(\alpha, Z) &= \alpha^{-w(T)} H_T(\alpha, Z) \\
&= \alpha^3(2\alpha^{-1} - \alpha + Z^2\alpha^{-1}) \\
&= 2\alpha^2 - \alpha^4 + Z^2\alpha^2
\end{aligned}
$$

[Show $H\Big($  $\Big) = \dfrac{\alpha - \alpha^{-1}}{Z}$:

$H\Big($  $\Big) - H\Big($  $\Big) = Z\,H\Big($  $\Big)$ by property (iii),

$\alpha - \alpha^{-1} = Z\,H\Big($  $\Big)$ by property (iv),

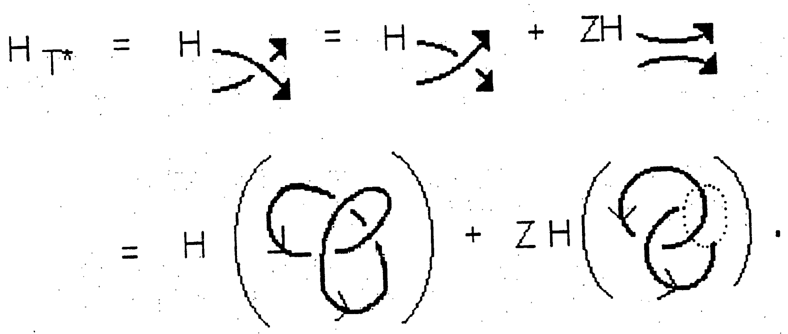
so $H\Big($  $\Big) = \dfrac{\alpha - \alpha^{-1}}{Z}$.]

Recall, the following is the diagram for the mirror image of the trefoil knot T.

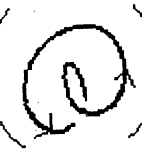
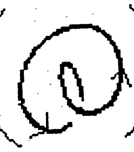
Mirror Image T*

Again, the exchange identity is

; hence expanding about the positive crossing above,



Using a type II move on the first term and the exchange identity on the second term, we obtain:

$$H_{T^*} = H\Big($$  $$\Big) + Z\Big[H\Big($$  $$\Big) + H\Big($$  $$\Big)\Big].$$

If we apply property (iv) twice and a type II move once, we get the following:

$$H_{T^*} = \alpha + Z\Big(H\Big($$  $$\Big) + Z\alpha\Big).$$

As shown earlier $H\Big($  $\Big) = \dfrac{\alpha - \alpha^{-1}}{Z}$, so we continue:

$$H_{T^*} = \alpha + Z\left(\frac{\alpha - \alpha^{-1}}{Z} + Z\alpha\right)$$

$$= \alpha + \alpha - \alpha^{-1} + Z^2\alpha$$

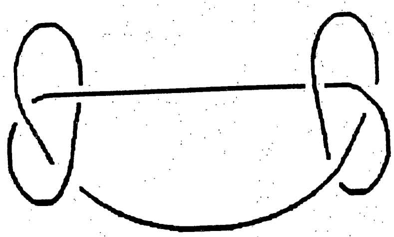
$$= 2\alpha - \alpha^{-1} + Z^2\alpha.$$

Therefore, since $w(T^*) = 3$,

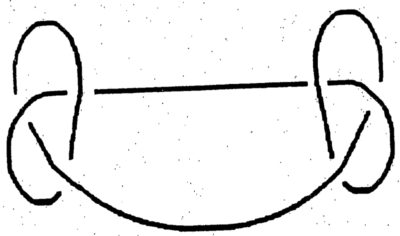
$$\begin{aligned} P_{T^*}(\alpha, z) &= \alpha^{-w(T^*)} H_{T^*}(\alpha, z) \\ &= \alpha^{-3}(2\alpha - \alpha^{-1} + z^2\alpha) \\ &= 2\alpha^{-2} - \alpha^{-4} + z^2\alpha^{-2}. \end{aligned}$$

Once again, I have shown that the trefoil knot and its mirror image are distinct, this time using the Homfly polynomial.

As mentioned earlier, the granny knot and the square knot (shown below) have isomorphic knot groups, and therefore could not be distinguished by using the methods in Chapter I.

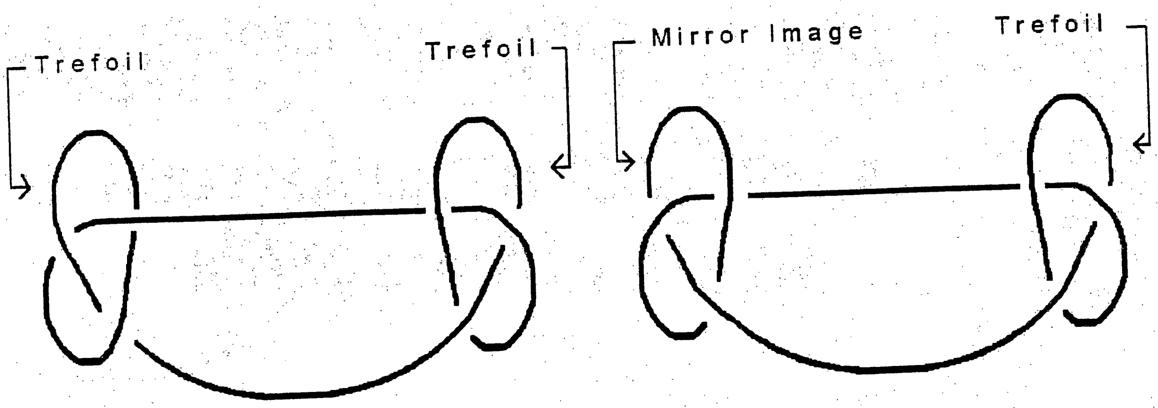


Granny Knot



Square Knot

However, the Homfly polynomial can distinguish the granny knot and square knot. But using the Homfly on these knots can be very tedious. To help with this problem, one must think of these larger knots as the "connected sum" of two smaller knots. The <u>connected sum</u> is formed by splicing two knots together so that the knots do not overlap. Therefore, the granny knot is the connected sum of two trefoil knots, and the square knot is the connected sum of a trefoil knot and its mirror image (shown below).



Granny Knot

Square Knot

It can be shown that the Homfly of the connected sum of two knots is equal to the product of their individual Homfly polynomials, i.e., $H_K = H_A \cdot H_B$ where the knot K is the connected sum of the knots A and B. Since the granny knot, G, is the connected sum of two trefoils, we have

$$
\begin{aligned}
H_G &= H_T \cdot H_T \\
&= (2\alpha^{-1} - \alpha + z^2\alpha^{-1})(2\alpha^{-1} - \alpha + z^2\alpha^{-1}) \\
&= 4\alpha^{-2} - 2 + 2z^2\alpha^{-2} - 2 + \alpha^2 - z^2 + 2z^2\alpha^{-2} - z^2 + z^4\alpha^{-2} \\
&= 4\alpha^{-2} - 4 + 4z^2\alpha^{-2} - 2z^2 + \alpha^2 + z^4\alpha^{-2}
\end{aligned}
$$

Since $w(G) = -6$, the normalized Homfly polynomial for the granny knot is

$$P_G(\alpha,z) = \alpha^6(4\alpha^{-2} - 4 + 4z^2\alpha^{-2} - 2z^2 + \alpha^2 + z^4\alpha^{-2})$$
$$= 4\alpha^4 - 4\alpha^6 + 4z^2\alpha^4 - 2z^2\alpha^6 + \alpha^8 + z^4\alpha^4 .$$

Now, since the square knot, S, is the connected sum of a trefoil and its mirror image, we have the following:

$$H_S = H_T \cdot H_{T^*}$$
$$= (2\alpha^{-1} - \alpha + z^2\alpha^{-1})(2\alpha - \alpha^{-1} + z^2\alpha)$$
$$= 4 - 2\alpha^{-2} + 2z^2 - 2\alpha^2 + 1 + z^2\alpha^2 + 2z^2 - z^2\alpha^{-2} + z^4$$
$$= 5 - 2\alpha^{-2} + 4z^2 - 2\alpha^2 + z^2\alpha^2 - z^2\alpha^{-2} + z^4 .$$

Since $w(S) = 0$, the normalized Homfly polynomial for the square knot is

$P_s(\alpha, z) = 5 - 2\alpha^{-2} + 4z^2 - 2\alpha^2 + z^2\alpha^2 - z^2\alpha^{-2} + z^4$.

The square knot does not have the same Homfly polynomial as the granny knot. Therefore, they are not equivalent knots.

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