

6-2019

Geodesics on Generalized Plane Wave Manifolds

Moises Pena

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GEODESICS ON GENERALIZED PLANE WAVE MANIFOLDS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Moises Peña

June 2019

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Approved by:

Corey Dunn, Committee Chair

Zahid Hasan, Committee Member

Jeremy Aikin, Committee Member

Shawnee McMurrin, Chair, Department of Mathematics

Corey Dunn, Graduate Coordinator

ABSTRACT

A manifold is a Hausdorff topological space that is locally Euclidean. We will define the difference between a Riemannian manifold and a pseudo-Riemannian manifold. We will explore how geodesics behave on pseudo-Riemannian manifolds and what it means for manifolds to be geodesically complete. The Hopf-Rinow theorem states that, “Riemannian manifolds are geodesically complete if and only if it is complete as a metric space,” [Lee97] however, in pseudo-Riemannian geometry there is no analogous theorem since in general a pseudo-Riemannian metric does not induce a metric space structure on the manifold. Our main focus will be on a family of manifolds referred to as a generalized plane wave manifolds. We will prove that all generalized plane wave manifolds are geodesically complete.

ACKNOWLEDGEMENTS

I would like to thank Dr. Dunn for his endless support. Without his thorough knowledge and patience, the completion of this thesis would not be possible. I would also extend my gratitude to Dr. Aikin and Dr. Hasan for being part of my committee and taking their time to review this thesis. I would like to thank my friends who constantly support me and encouraged me when all felt impossible.

To my parents, I cannot express in words how supportive they have been with my academic career. They have molded me and supported me every day of my life and for that I will be forever thankful. Gracias sí se pudo.

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Chapter 1

Introduction

1.1 Introduction

Differential geometry studies geometric properties of objects by applying the techniques of calculus. We focus on a branch of differential geometry called pseudo-Riemannian geometry (sometimes referred to as semi-Riemannian geometry) which studies pseudo-Riemannian manifolds, M . These manifolds are equipped with a metric, g to “give a manifold its shape.” On these manifolds we also consider a special way of differentiating vector fields, which is sometimes called the miracle of differential geometry, called the Levi-Civita connection. We denote these pseudo-Riemannian manifolds with their metric as (M, g) .

We begin by introducing topological preliminaries in order to build up to and define what a Hausdorff space is, which will be a key component in defining a manifold. Then we transition to differential geometric properties, with our goal being to establish the definition of a smooth manifold. Thereafter, we provide definitions for tangent vectors, vector spaces, and introduce the Basis Theorem which allows us to perform calculations on a local coordinate system. It is on the tangent space in which we can begin defining a symmetric bilinear form which is a generalization of the inner product. More specifically, a symmetric bilinear form allows us to measure distances on a manifold, M , and angles of vectors on its tangent space. With added structure we will see that a symmetric bilinear form is actually the metric, g , of a manifold which we previously mentioned. Metrics will play an important role in determining if a manifold is a Riemannian manifold or

a pseudo-Riemannian manifold, the latter being the manifolds we will be interested in, depending on the signature of the metric.

We define a connection, which allows differentiation in any given direction to be measured by a tangent vector. These connections are also referred to as covariant derivatives which intuitively measure the rate of change of a vector as it slides in the direction of another vector. It is here that we introduce the Levi-Civita connection which is the only connection that satisfies certain reasonable conditions. We also define geodesics and provide supplemental definitions and theorems to properly explain what it means to be a geodesically complete manifold. We conclude this thesis with a culminating example which demonstrates the intricacies of our main theorem that a certain family of manifolds are geodesically complete.

For convenience we have provided a list of the most commonly used notation presented in this thesis:

- M : manifold
- p : point
- f : real valued function
- ϕ : mapping
- \mathcal{F} : smooth functions on M
- v : vector
- T_pM : tangent vector space to M at p
- V : vector field
- $\mathfrak{X}(M)$: set of all smooth vector fields on M
- \mathcal{B} : basis
- g : metric
- (M, g) : geometric manifold
- ∇ : connection

- Γ_{ij}^k : Christoffel symbol of the first kind
- Γ_{ijk} : Christoffel symbol of the second kind
- σ : smooth path

Chapter 2

Preliminaries

In this chapter we will provide Topological and Differential Geometric preliminaries which will arise throughout this material and serve as a foundation for the reader in order to have a working grasp of the definitions, examples, and theorems presented.

2.1 Topological Preliminaries

In order to understand what a topological space is we must first define what a topology is.

Definition 2.1. *Let X be a set and T be a set of subsets of X , then T is a topology of X if the following properties are satisfied:*

1. The empty set, \emptyset , belongs to T .
2. The set X belongs to T .
3. Any union of subsets of T is in T .
4. The finite intersection of subsets of T is in T .

When T is a topology on X , the subsets of X in T are called open.

Definition 2.2. *A topological space is an ordered pair (X, T) where X is a set and T is a topology on X . [Mun15]*

Now to better understand what a topological space is we provide the following example.

Example 2.3. Let $X = \{a, b\}$ and $T = \{\emptyset, X, \{a\}, \{b\}\}$. Then T is a topology on X since it satisfies all four conditions.

We now make the jump to open sets and open balls but first we must give a brief definition of a metric and metric spaces. A **metric** is an abstraction of the concept of distance in a space. A metric is a function that defines the distance between pairs of elements within that space.

Definition 2.4. More precisely, if $x, y \in X$ then $d(x, y)$ is said to be the distance from x to y . The distance function, d , obeys the following properties for all $x, y, z \in X$:

1. $d(x, y) \geq 0$ for all $x, y \in X$.
2. $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) = d(y, x)$.
4. $d(x, z) \leq d(x, y) + d(y, z)$.

If all properties are satisfied then d is a metric on X and (X, d) is called a metric space.

Example 2.5. Let $d(x, y) = |x - y|$, then (\mathbb{R}^n, d) is a metric space.

1. Since $|x - y| \geq 0$ then $d(x, y) \geq 0$.
2. Let $d(x, y) = 0$ then $|x - y| = 0$ thus $x = y$. The converse easily follows.
3. Also, $d(x, y) = |x - y| = |y - x| = d(y, x)$.
4. Finally by the well-known triangle inequality, $d(x, z) = |x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z| = d(x, y) + d(y, z)$.

Now that we have a concept of distance we can describe what an open ball is defined to be.

Example 2.6. Given that X is a metric space with the metric d , with $r > 0$, and $x \in X$. An open ball B of radius r around x is $B_r(x) = \{y \in \mathbb{R} \mid d(x, y) < r\}$.

We note that this ball contains all of the points y whose distance from x is less than r .

Definition 2.7. Let A be a subset of the metric space (X, d) , then A is said to be an open set if for all $x \in A$, there exists $r > 0$ such that given an arbitrary point $y \in X$ with $d(x, y) < r$, $B_r(x)$ is contained in A .

Example 2.8. Let $X = [0, 2]$ then $A = (1, 2)$ is open in X , since $(1, 2) = B_{\frac{1}{2}}(\frac{3}{2})$.

Definition 2.9. Let x be a point in X and let A be a subset of X . Then A is said to be a neighborhood of the point x if there exists an open set, B , in A such that $x \in B \subset A$.

Let us look at the following examples of a subset considered to be a neighborhood and a subset not considered to be a neighborhood.

Example 2.10. Assume we are in the standard topology of \mathbb{R} which is generated by the collection of all open intervals, $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$. Let $x = 1$ and $A = (0, 2)$, then A is a neighborhood of x since there exists an open set $B = (\frac{1}{2}, 2)$ which is a subset contained in A .

We conclude the section of Topological preliminaries with the definition of a *Hausdorff* space which will serve as a component for the definition of a manifold, which we will define later.

Definition 2.11. Let (X, T) be a topological space. We call X a Hausdorff space if given two distinct points $x, y \in X$, there exists neighborhoods A_x and A_y in X with A_x containing x and A_y containing y such that the intersection of A_x and A_y is empty. [Mun15]

2.2 Differential Geometric Preliminaries

In this next section we will describe the Differential Geometric topics that will frequently arise. First, we describe \mathbb{R}^n with the standard topology, which is formally known as Euclidean n -space.

Definition 2.12. Euclidean n -space, \mathbb{R}^n , is the set of all ordered n -tuples of real numbers.

In particular if $p = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n$ then we define $x^i : \mathbb{R}^n \rightarrow \mathbb{R}$ to be the function that sends every point to its i th coordinate: $x^i(p_1, \dots, p_n) = p_i$. We call x^1, \dots, x^n the natural coordinate functions of \mathbb{R}^n .

Definition 2.13. Let f be a real valued function on \mathbb{R}^n then f is said to be smooth, or *infinitely differentiable*, if all partial derivatives of f of all orders exist.

Now consider a function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$, then said function is considered to be smooth given that $x^i \circ \phi$ is also smooth, with $1 \leq i \leq n$. In other words, a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is smooth if and only if all of its component functions are smooth.

Recall that a homeomorphism between two topological spaces is a continuous bijective function whose inverse function is also continuous. In Differential Geometry, more specifically manifolds, we would like to add more structure. The structure added is that of differentiability; thus if there exists a homeomorphism with the continuous function being differentiable and the the inverse function also being differentiable then we have what is called a diffeomorphism.

Definition 2.14. Let S be a topological space then, a coordinate system or chart, on S is where there exists an open subset U of S with ϕ being a homeomorphism mapping U onto $\phi(U)$, an open subset of \mathbb{R}^n .

Definition 2.15. Let S be a topological space and suppose there exists neighborhoods, U_α , which cover S . Suppose there are maps $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ for some fixed n , which form a coordinate system on S when ϕ_α are homeomorphisms that map U_α onto open subsets of \mathbb{R}^n , $\phi_\alpha(U_\alpha)$. Now suppose there exists two n -dimensional coordinate systems ϕ and ψ in a topological space S , then ϕ and ψ overlap smoothly if both composite functions, $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are smooth. The previously mentioned composite functions are also referred to as the transition functions of the coordinate system.

We call a collection of all coordinate charts an atlas. More formally, an atlas A of dimension n on a space S is a collection of n -dimensional coordinate systems such that:

1. Each point of S is located in some coordinate system of A .
2. Any n -dimensional coordinate systems in A overlap smoothly.

Definition 2.16. An atlas \mathcal{C} on S is called complete if \mathcal{C} contains all coordinate systems in S that overlap smoothly with every coordinate system in \mathcal{C} . [O'N93]

Next we provide a definition for a manifold but first we introduce a more intuitive notion of what a manifold is. For example, picture the surface of the earth as being our topological space. From afar we can see the spherical shape of this “manifold” however as we zoom in on a particular, location the local space around them now looks like \mathbb{R}^2 , omitting geographical landforms such as mountains and valleys even including them, if we look closely enough. In this local space which we are more familiar with we can do calculus. Now if we take all of the people and patch up their local spaces we can build the surface of the Earth, now composed of different local spaces or coordinate systems upon which we can do calculus.

Definition 2.17. *A manifold M is a Hausdorff topological space furnished with a complete atlas. The dimension of M is n , where any coordinate chart (ϕ, U) has $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$. Thus we call M a manifold of dimension n .*

For this text we will treat all manifolds as smooth manifolds which is defined as follows.

Definition 2.18. *A smooth manifold M is a Hausdorff topological space furnished with a complete atlas, where all of the transition functions in the atlas are smooth. [O’N93]*

Chapter 3

Tangent Space and Vector Fields

The following provides a definition of a tangent vector on a manifold which can be imagined as direction in which to differentiate. If we let M be a manifold, we denote $\mathcal{F}(M)$ as the set of smooth functions on M .

Definition 3.1. *Let M be a manifold with $p \in M$. We call v_p a tangent vector to M at p if the real valued function $v : \mathcal{F}(M) \rightarrow \mathbb{R}$ satisfies the following:*

1. \mathbb{R} -Linear: $v_p(\alpha f + \beta g) = \alpha v_p(f) + \beta v_p(g)$,
2. Leibniz Rule: $v_p(fg) = g(p)v_p(f) + f(p)v_p(g)$, for all $f, g \in \mathcal{F}(M)$ and $\alpha, \beta \in \mathbb{R}$.

Let all tangent vectors of M at p be defined as T_pM , we call this the tangent vector space to M at p . Addition and scalar multiplication in T_pM are defined as follows:

1. $(v_p + w_p)(f) = v_p(f) + w_p(f)$,
2. $(av_p)(f) = av_p(f)$, for all $v, w \in T_pM$, $f \in \mathcal{F}(M)$, and $a \in \mathbb{R}$.

Definition 3.2. *A vector field V on M is a function which assigns a tangent vector $V_p \in T_pM$ for each $p \in M$.*

We note that vector fields on a manifold are added and scaled in the following way:

1. $(fV)_p = f(p)V_p$ for all $p \in M$ and $f \in \mathcal{F}(M)$.
2. $(V + W)_p = V_p + W_p$ for all $p \in M$.

Intuitively, a vector field is a map which takes smooth functions to smooth functions. One can imagine V as a collection of arrows at each point of M . Recall we denote $\mathfrak{X}(M)$ as the set of all smooth vector fields on M .

Definition 3.3. Suppose $\phi : U \rightarrow \phi(U)$ is a coordinate chart with coordinates (x_1, \dots, x_n) . U is an open subset of a manifold M , $\phi(U)$ is an open subset of \mathbb{R}^n , and ϕ is a homeomorphism. For $f \in \mathcal{F}$, define $\tilde{f} = f \circ \phi^{-1}$. Therefore, for each $1 \leq i \leq n$ we define coordinate vector fields as $\partial_{x_i}(f) = \frac{\partial}{\partial x_i}$ which implies that at point $p \in M$ the i th coordinate vector field is $\partial_{x_i}(f)|_p = \frac{\partial}{\partial x_i}(\tilde{f})|_p$.

In the previous definition we use the notation $|_p$ to specify the vector field at point p .

Theorem 3.4 (The Basis Theorem). If $\mu = (x^1, \dots, x^n)$ is a coordinate system at $p \in M$, then $\mathcal{B} = \{\partial_{x^1}|_p, \dots, \partial_{x^n}|_p\}$ is a basis for T_pM . [O'N93]

As a direct result from the basis theorem, if (x_1, \dots, x_n) is a coordinate system on $U \subseteq M$, for any vector field V ,

$$V = \sum_{i=1}^n V(x_i)\partial_{x_i}.$$

Example 3.5. Let $U = \mathbb{R}^2$ with global coordinates (x, y) and the vector field $V = (x)\partial_y + (xy)\partial_x$. Then $V|_{(0,0)} = 0$ and $V|_{(1,2)} = \partial_y + 2\partial_x$.

We now introduce an essential operation on vector fields which we will call the bracket operation.

Definition 3.6. If V, W are vector fields on M then $[V, W]$, the bracket of V and W , is also a vector field on M , where $[V, W]_p(f) = V_p(W(f)) - W_p(V(f))$, for all $p \in M$ and $f \in \mathcal{F}(M)$.

Lemma 3.7 (The Bracket Operation). The following are properties of the bracket operation on $\mathfrak{X}(M)$ for all $\alpha, \beta \in \mathbb{R}$ and $V, W, X \in \mathfrak{X}(M)$:

- (1) *R-bilinearity:* $[\alpha Y + \beta Z, X] = \alpha[Y, X] + \beta[Z, X]$,
- (2) *Skew-symmetry:* $[Y, Z] = -[Z, Y]$, and
- (3) *Jacobi identity:* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

One should note that although the bracket operation on $\mathfrak{X}(M)$ is \mathbb{R} -bilinear, it is not $\mathcal{F}(M)$ -bilinear. And in two special cases the bracket operation equals zero. For any $V \in \mathfrak{X}(M)$, $[V, V] = 0$ by skew symmetry. Also, for any two coordinate vector fields of the same coordinate system, $[\partial_i, \partial_j] = 0$.

Example 3.8. Let $X = x\partial_x$ and $Y = \partial_x + \partial_y$ be vector fields on the manifold \mathbb{R}^2 , with coordinates (x, y) .

$$\begin{aligned}
 [X, Y] &= X(Y(f)) - Y(X(f)) \\
 &= x\partial_x(\partial_x(f) + \partial_y(f)) - (\partial_x + \partial_y)(x\partial_x(f)) \\
 &= x\frac{\partial^2 f}{\partial_x^2} + x\frac{\partial^2 f}{\partial_x\partial_y} - \frac{\partial f}{\partial_x} - x\frac{\partial^2 f}{\partial_x^2} - x\frac{\partial^2 f}{\partial_x\partial_y} \\
 &= -\frac{\partial f}{\partial_x} \\
 &= -\partial_x(f) \\
 &= -\partial_x.
 \end{aligned}$$

Chapter 4

Inner Products and Metrics

In this chapter we will introduce the notion of a symmetric bilinear form which is a generalization of an inner product. Let V be a vector space, a bilinear form is an \mathbb{R} -bilinear function $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$.

Definition 4.1. A symmetric bilinear form is a function $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$, satisfying the following properties for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$:

1. Symmetry: $\langle u, v \rangle = \langle v, u \rangle$.
2. Bilinearity: $\langle \alpha u, v \rangle = \langle u, \alpha v \rangle = \alpha \langle u, v \rangle$,
 $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,
 $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$.

The norm of a vector is what mathematicians refer to as its length. One usually writes this as $\|v\|$ where $\|v\|^2 = \langle v, v \rangle$. This requires $\langle v, v \rangle \geq 0$ for all v for such a name to make sense. The following example demonstrates there are symmetric bilinear forms and vectors v where $\langle v, v \rangle$ need not be nonnegative. Thus, the common notion of what the length of a vector is, must be reconsidered in this context. If $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form and $\{e_1, \dots, e_n\}$ is a basis, then $\langle \cdot, \cdot \rangle$ is determined by $\langle e_i, e_j \rangle$.

Example 4.2. Let $V = \mathbb{R}^2 = \text{span}\{e_1, e_2\}$, with a symmetric bilinear form defined as $\langle e_1, e_1 \rangle = -1$, $\langle e_2, e_2 \rangle = 1$, and $\langle e_1, e_2 \rangle = \langle e_2, e_1 \rangle = 0$. Now we will calculate

$\| xe_1 + ye_2, xe_1 + ye_2 \|^2$:

$$\begin{aligned} \| xe_1 + ye_2, xe_1 + ye_2 \|^2 &= \langle xe_1 + ye_2, xe_1 + ye_2 \rangle \\ &= \langle xe_1, xe_1 \rangle + \langle xe_1, ye_2 \rangle + \langle ye_2, xe_1 \rangle + \langle ye_2, ye_2 \rangle \\ &= x^2 \langle e_1, e_1 \rangle + xy \langle e_1, e_2 \rangle + yx \langle e_2, e_1 \rangle + y^2 \langle e_2, e_2 \rangle \\ &= -x^2 + y^2. \end{aligned}$$

Now analyzing this norm, $\| xe_1 + ye_2, xe_1 + ye_2 \|^2 = -x^2 + y^2$, we note that if:

$x > y$, then the norm squared is negative,

$x < y$, then the norm squared is positive,

$x = y$, then the norm squared is zero.

Definition 4.3. If $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form with $v \in V$, and $v \neq 0$ then:

1. If $\langle v, v \rangle \leq 0$, then v is timelike.
2. If $\langle v, v \rangle \geq 0$, then v is spacelike.
3. If $\langle v, v \rangle = 0$, then v is null.

Definition 4.4. Let V be a vector space on a manifold M , then an inner product on V is a symmetric bilinear form that satisfies the following condition, known as being nondegenerate: For all $v \in V$ with $v \neq 0$, there exists $w \in V$ such that $\langle v, w \rangle \neq 0$.

We call a vector space equipped with an inner product, an inner product space. Vectors $v, w \in V$ are orthogonal given that $\langle v, w \rangle = 0$, normally written as $v \perp w$.

Theorem 4.5 (Orthonormal Bases Exist). Given a metric g on a manifold M , there exists some orthonormal basis $\mathcal{B} = \{e_1, \dots, e_n\}$ for $T_p M$ for each $p \in M$ such that

$$\langle e_i, e_j \rangle = \pm \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ \pm 1 & \text{if } i = j. \end{cases}$$

Theorem 4.6 (Sylvester's Law of Inertia). Let $\langle \cdot, \cdot \rangle$ be a symmetric bilinear form on a finite-dimensional real vector space V . Then the number of positive diagonal entries and the number of negative entries in any diagonal matrix representation of $\langle \cdot, \cdot \rangle$ are each independent of the diagonal representation. [Fri03]

We shall order the entries of the matrix representation of a symmetric bilinear form with the quantity of $(-)$ signs first followed by the quantity of $(+)$ signs. The number of negative diagonal entries of a symmetric bilinear form is called the index. One writes the signature of a symmetric bilinear form as (p, q) where p represents the negative entries and q represents the positive entries.

Example 4.7. *Let $\langle e_1, e_2 \rangle = -1, \langle e_2, e_2 \rangle = -1$, and $\langle e_3, e_3 \rangle = +1$ on $\mathbb{R}^3 = \text{span}\{e_1, e_2, e_3\}$. Then the signature of $\langle \cdot, \cdot \rangle$ is $(2, 1)$.*

We now introduce the concept of a metric on a manifold, M . As we have discussed, a smooth manifold M is a topological space with a coordinate system near each point. A metric is an additional structure we can equip a manifold with which allows us to measure distances on the manifold and angles of vectors on the tangent spaces to the manifold.

Definition 4.8. *Let M be a smooth manifold. A metric g on M is an inner product on each tangent space of M of constant signature.*

Definition 4.9. *A geometric manifold, (M, g) , is a smooth manifold furnished with a metric g .*

If the signature of g is $(0, n)$ then the metric is positive definite and (M, g) is called a Riemannian manifold. However, if the signature of g , is (p, q) with $p, q > 0$ then the metric is called indefinite and (M, g) is called a pseudo-Riemannian manifold.

The following definitions will introduce standard notation which will be used in great quantity in the later chapters. If we have (M, g) with $p \in M$ then we denote g_p or $g(p)$ to be the metric g at point p . This is not to be confused with g_{ij} which denotes components of the metric g on an open subset U of M , which we will define with more detail.

Suppose we have (M, g) with $p \in M$ and coordinates (x_1, \dots, x_n) on an open subset U on M near p . Then by the Basis theorem, $T_p(M)$ has $\{\partial_{x_1}, \dots, \partial_{x_n}\}$ as a basis for all p in the coordinate system. Since $g(p)$ is bilinear, $g(p)$ is determined by $g(\partial_{x_i}, \partial_{x_j}) = g_{ij}(p)$. We can also denote $g_{ij} = g(\partial_{x_i}, \partial_{x_j})$. Also, since g is symmetric, $g_{ij} = g_{ji}$, for $1 \leq i, j \leq n$.

Definition 4.10. *The metric g is smooth if the functions $g_{ij} = g(\partial_{x_i}, \partial_{x_j})$ are smooth in one and hence any coordinate system (x_1, \dots, x_n) .*

Chapter 5

Connections and Geodesics

5.1 Connections

A connection is an operator that allows differentiation in any given direction to be measured by a tangent vector. More precisely, a connection is a map that takes two vector fields to produce another vector field.

Definition 5.1. A *connection*, ∇ , on a smooth manifold M is a function defined by $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ such that it satisfies the following properties.

$$(\nabla 1) \text{ (Linearity)} \quad \nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z.$$

$$(\nabla 2) \text{ (Leibniz rule)} \quad \nabla_X(fY) = X(f)Y + f\nabla_X Y.$$

$(\nabla 3)$ (\mathcal{F} -Linear) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$. For $X, Y, Z \in \mathfrak{X}(M)$ and f and g as smooth functions on M .

We call $\nabla_X Y$ the covariant derivative of Y in direction X with respect to the connection ∇ . Intuitively, this covariant derivative measures the direction of change of the X vector as it slides in the direction of the Y vector. It is also important to note that a connection, ∇ , is determined by $\nabla_{\partial_{x_i}} \partial_{x_j}$ on some coordinate system, and the following example demonstrates this result.

Example 5.2. Suppose $M = \mathbb{R}^3$ with coordinates (x, y, z) , $p \in M$ and $T_p M = \text{span}\{\partial_x, \partial_y, \partial_z\}$. If we know that $\nabla_{\partial_x} \partial_y = \nabla_{\partial_y} \partial_x = x\partial_z$, $\nabla_{\partial_z} \partial_z = y\partial_y$, and 0 elsewhere then we can compute $\nabla_X Y$ when $X = x\partial_y + z\partial_z$ and $Y = x\partial_x + z^2\partial_z$.

$$\nabla_X Y = \nabla_{(x\partial_y + z\partial_z)}(x\partial_x + z^2\partial_z) \quad (5.1)$$

$$= x\nabla_{\partial_y}(x\partial_x + z^2\partial_z) + z\nabla_{\partial_z}(x\partial_x + z^2\partial_z) \quad (5.2)$$

$$= x\nabla_{\partial_y}(x\partial_x) + x\nabla_{\partial_y}(z^2\partial_z) + z\nabla_{\partial_z}(x\partial_x) + z\nabla_{\partial_z}(z^2\partial_z) \quad (5.3)$$

$$= x(\partial_y(x) \cdot \partial_x + x\nabla_{\partial_y}\partial_x) + (\partial_y(z^2) \cdot \partial_z + z^2\nabla_{\partial_y}\partial_z) \quad (5.4)$$

$$+ z(\partial_z(x) \cdot \partial_x + x\nabla_{\partial_z}\partial_x) + z(\partial_z(z^2) \cdot \partial_z + z^2\nabla_{\partial_z}\partial_z) \quad (5.5)$$

$$= x^3\partial_z + 2z^2\partial_z + z^3y\partial_y \quad (5.6)$$

$$= yz^3\partial_y + (x^3 + 2z^2)\partial_z. \quad (5.7)$$

A quick recap of the calculations, $(\nabla 3)$ was applied on line (5.1), $(\nabla 1)$ was applied to (5.2), and $(\nabla 2)$ was applied to (5.3).

We note that if $\nabla_{\partial_{x_i}}\partial_{x_j} = \sum_{k=1}^n \Gamma_{ij}^k \partial_{x_k}$, we call Γ_{ij}^k the Christoffel Symbols of the 1st kind.

If ∇ is a connection on M , we could impose a few more restrictions that it does not necessarily have to satisfy. Those two properties are:

$$(\nabla 4) \text{ (Torsion Free) } \nabla_X Y - \nabla_Y X = [X, Y].$$

$(\nabla 5)$ (Metric Compatibility) If g is a metric on M then $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.

Theorem 5.3 (Levi-Civita connection). *Let M be a semi-Riemmanian manifold with the metric g . There exists a unique connection, ∇ , which satisfies $(\nabla 1)$ through $(\nabla 5)$ called the Levi-Civita connection. This connection is determined by the **Koszul formula**:*

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ &\quad + g(Z, [X, Y]) - g(Y, [X, Z]) - g(X, [Y, Z]). \end{aligned} \quad [O'N93]$$

Definition 5.4 (Christoffel symbols of the 2nd kind). $\Gamma_{ijk} = \frac{1}{2}(\frac{\partial}{\partial x_i}g(\partial_{x_j}, \partial_{x_k}) + \frac{\partial}{\partial x_j}g(\partial_{x_i}, \partial_{x_k}) - \frac{\partial}{\partial x_k}g(\partial_{x_i}, \partial_{x_j}))$

For notation purposes we denote $\frac{\partial}{\partial x_i}(f) = f_{/i}$. Hence we can shorten the previous definition to $\Gamma_{ijk} = \frac{1}{2}(g_{jk/i} + g_{ik/j} - g_{ij/k})$.

5.2 Geodesics

We will begin by describing the intuitive nature of geodesics on manifolds. A geodesic is meant to be a locally length minimizing curve. Given two points on a manifold, the curve of shortest length which connects both points is called a geodesic. For example, on a plane, a geodesic connecting two points is a straight line; on a sphere all geodesics are great circles. We can speak of geodesics in this intuitive sense on Riemannian manifolds because the positive definite metric makes the manifold a metric space. We will see that on pseudo-Riemannian manifolds geodesics can behave differently and in some cases can be defined as length maximizing curves, although in general a pseudo-Riemannian metric does not always make the manifold a metric space. One thing to keep in mind as we will see below is that geodesics only require a connection to be defined, which implies that geodesics can be defined on a manifold without a metric. However, most useful properties of geodesics require a metric notion which is why we are only interested in geodesics on semi-Riemannian manifolds equipped with the Levi-Civita connection.

Definition 5.5. *If $\sigma : (-\epsilon, \epsilon) \rightarrow M$ is a smooth path then it is a geodesic if $\nabla_{\sigma'} \sigma' = 0$.*

Theorem 5.6 (Geodesic Equation). *Let (x_1, \dots, x_n) be a coordinate system on $U \subset M$. A curve $\sigma \in U$ is a geodesic if and only if its coordinate functions satisfy*

$$\frac{\partial^2(x_k)}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{\partial x_i}{dt} \frac{\partial x_j}{dt} = 0$$

for $1 \leq k \leq n$.

Next we introduce the existence and uniqueness theorem for ordinary differential equations which will yield important results about geodesics.

Theorem 5.7 (Existence and Uniqueness for ODE). *Let $U \subset \mathbb{R}^n$ be an open set and I_ϵ , $\epsilon > 0$, denote the interval $-\epsilon < t < \epsilon$, $t \in \mathbb{R}$. Suppose $f^i(t, x^1, \dots, x^n)$, $i = 1, \dots, n$, to be functions of class C^r , $r \geq 1$, on $I_\epsilon \times U$. Then for each $x \in U$ there exists $\delta > 0$ and a neighborhood V of x , $V \subset U$, such that:*

1. *For each $a = (a^1, \dots, a^n) \in V$ there exists an n -tuple of C^r functions $x(t) = (x^1(t), \dots, x^n(t))$, defined on I_δ and mapping I_δ into U , which satisfy the system of first-order differential equations*

$$(*) \quad \frac{dx^i}{dt} = f^i(t, x), \quad i = 1, \dots, n,$$

and the initial conditions

$$(**) \quad x^i(0) = a^i, \quad i = 1, \dots, n.$$

For each a the functions $x(t) = (x^1(t), \dots, x^n(t))$ are uniquely determined in the sense that any other functions $\bar{x}(t), \dots, \bar{x}^n(t)$ satisfying (*) and (**) must agree with $x(t)$ on their common domain, which includes I_δ .

2. These functions being uniquely determined by $a = (a^1, \dots, a^n)$ for every $a \in V$, we write them $x^i(t, a^1, \dots, a^n)$, $i = 1, \dots, n$, in which case they are of class C^r in all variables and thus determine a C^r map of $I_\delta \times V \rightarrow U$. [Boo75]

The importance of the Existence and Uniqueness for ODE theorem is that it can be applied to the geodesic equation. In summary this theorem states that given certain conditions an ordinary differential equation has a solution. The geodesic equation meets these conditions which means that we can find a solution, thus enabling us to identify geodesics on manifolds.

Next we present important facts about geodesics.

Theorem 5.8. *If $p \in M$ and $v \in T_p(M)$ then there exists an interval $(-\epsilon, \epsilon)$ about 0 and a unique geodesic $\sigma : (-\epsilon, \epsilon) \rightarrow M$ such that $\sigma'(0) = v$ and $\sigma(0) = p$.*

Lemma 5.9. *Let $\sigma, \gamma : (-\epsilon, \epsilon) \rightarrow M$ be geodesics. If there is a number $a \in (-\epsilon, \epsilon)$ such that $\sigma'(a) = \gamma'(a)$ and $\sigma(a) = \gamma(a)$ then $\sigma = \gamma$.*

The following definition will prove important in later sections.

Theorem 5.10. *Given any tangent vector $v \in T_p(M)$ there is a unique geodesic σ in M such that the following are true:*

1. Let $\sigma(0) = p$.
2. The initial velocity of σ is v ; hence $\sigma'(0) = v$.
3. The domain I of σ is the largest possible. Thus, if there exists another geodesic $\gamma : J \rightarrow M$ with initial velocity v , then $J \subset I$ and $\gamma = \sigma|_J$.

From (2) of the definition above we call the geodesic with the largest domain σ , maximal or geodesically inextendible. A semi-Riemannian manifold M for which all maximal geodesics are defined on \mathbb{R} (i.e., all geodesics are inextendible) is called geodesically complete.

Theorem 5.11 (Hopf-Rinow). *A connected Riemannian manifold is geodesically complete if and only if it is complete as a metric space. [Lee97]*

However, in pseudo-Riemannian geometry there is no corresponding theorem since geodesics do not behave the same as Riemannian manifolds since it is not a metric space induced by a metric g .

Chapter 6

Completeness of Generalized Plane Wave Manifolds

In this chapter we will be proving that all generalized plane wave manifolds are complete. We will be doing so by providing a proof of a specific generalized plane wave manifold which will show this process. This example will serve as a guideline and will allow us to better understand the proof of our main theorem. We'll begin with defining a generalized plane wave manifold.

Theorem 6.1 (Generalized Plane Wave Manifolds). *Let $x = (x_1, \dots, x_m)$ be the usual coordinates on \mathbb{R}^m . We say the manifold $(M, g) \in \mathbb{R}^m$ is a generalized plane wave manifold if:*

$$\nabla_{\partial_{x_i}} \partial_{x_j} = \sum_{k > \max(i,j)} \Gamma_{ij}^k(x_1, \dots, x_{k-1}) \partial_{x_k}.$$

[DGc05]

The following lemma will be needed for the ensuing example it will help us understand why 12 of the covariant derivatives are equal to zero.

Lemma 6.2. *Let $\{e_i\}$ be a basis for V and $\langle \cdot, \cdot \rangle$ be non-degenerate. If $\langle v, e_i \rangle = 0$ for all i , then $v = 0$.*

Proof. Suppose $v \neq 0$, since $\langle \cdot, \cdot \rangle$ is non-degenerate then there exist w such that

$\langle v, w \rangle \neq 0$. Now we express w as a linear combination, $w = \sum a_i e_i$.

$$\begin{aligned} 0 \neq \langle v, w \rangle &= \langle v, \sum a_i e_i \rangle \\ &= \sum a_i \langle v, e_i \rangle \\ &= 0, \end{aligned}$$

a contradiction, hence $v = 0$. □

In this next example we will show how as long as we know the metric and coordinate vector fields of a manifold we can calculate the covariant derivatives using the Christoffel symbols of the first and second kind. Given a metric g on a manifold M , and a point $p \in M$, we use this example to illustrate the important fact that we only need to compute the covariant derivatives of the coordinate vector fields to determine what $\nabla_X Y$ is for any vector fields X and Y on M .

For example, let $M = \mathbb{R}^4$ with coordinate vector fields $(x, y, \tilde{x}, \tilde{y})$ and a smooth function $f = f(y)$. Suppose that $g(\partial_x, \partial_x) = -2f(y)$, $g(\partial_x, \partial_{\tilde{x}}) = g(\partial_y, \partial_{\tilde{y}}) = 1$, and 0 elsewhere [DGc05]. Note that defining the metric, g , on the coordinate vector fields is sufficient to defining g on this manifold.

Now since we are dealing with four coordinate vector fields we would need to calculate 16 covariant derivatives, however we should note that for these calculations and for any future calculations we will be using the Levi-Civita connection, in which case we only need to calculate 10 covariant derivatives since the connection is torsion free. The following calculations will follow a similar method. First we will calculate the Christoffel symbols of the second kind. Second, we will use the Christoffel symbols of the second kind to derive the Christoffel symbols of the first kind. Lastly, we will use the Christoffel symbols of the first kind to calculate the covariant derivative.

Example 6.3.

1. $\nabla_{\partial_x} \partial_x = f'(y) \partial_{\tilde{y}}$,
2. $\nabla_{\partial_y} \partial_x = \nabla_{\partial_x} \partial_y = -f'(y) \partial_{\tilde{x}}$,
3. $\nabla_{\partial_y} \partial_y = 0$,
4. $\nabla_{\partial_{\tilde{x}}} \square = \nabla_{\square} \partial_{\tilde{x}} = 0$ and $\nabla_{\partial_{\tilde{y}}} \square = \nabla_{\square} \partial_{\tilde{y}} = 0$, for any coordinate vector fields. □

Proof. 1. The first covariant derivative that we will calculate is $\nabla_{\partial_x} \partial_x$.

First, we calculate the Christoffel symbols of the second kind.

$$\begin{aligned} g(\nabla_{\partial_x} \partial_x, \partial_x) &= \Gamma_{xxx} = \frac{1}{2}(g_{xx}/x + g_{xx}/x - g_{xx}/x) = 0 \\ g(\nabla_{\partial_x} \partial_x, \partial_y) &= \Gamma_{xxy} = \frac{1}{2}(g_{xy}/x + g_{xy}/x - g_{xx}/y) = \frac{1}{2}(-(-2f'(y))) = f'(y) \\ g(\nabla_{\partial_x} \partial_x, \partial_{\tilde{x}}) &= \Gamma_{xx\tilde{x}} = \frac{1}{2}(g_{x\tilde{x}}/x + g_{x\tilde{x}}/x - g_{xx}/\tilde{x}) = 0 \\ g(\nabla_{\partial_x} \partial_x, \partial_{\tilde{y}}) &= \Gamma_{xx\tilde{y}} = \frac{1}{2}(g_{x\tilde{y}}/x + g_{x\tilde{y}}/x - g_{xx}/\tilde{y}) = 0 \end{aligned}$$

Second, we calculate the Christoffel symbols of the first kind.

Note that, $\nabla_{\partial_x} \partial_x = \Gamma_{xx}^x \partial_x + \Gamma_{xx}^y \partial_y + \Gamma_{xx}^{\tilde{x}} \partial_{\tilde{x}} + \Gamma_{xx}^{\tilde{y}} \partial_{\tilde{y}}$.

$$\Gamma_{xxx} = 0 = g(\nabla_{\partial_x} \partial_x, \partial_x) = \Gamma_{xx}^x (-2f(y)) + \Gamma_{xx}^{\tilde{x}}$$

$$\Gamma_{xxy} = f'(y) = g(\nabla_{\partial_x} \partial_x, \partial_y) = \Gamma_{xx}^{\tilde{y}}$$

$$\Gamma_{xx\tilde{x}} = 0 = g(\nabla_{\partial_x} \partial_x, \partial_{\tilde{x}}) = \Gamma_{xx}^x = 0.$$

$$\Gamma_{xx\tilde{y}} = 0 = g(\nabla_{\partial_x} \partial_x, \partial_{\tilde{y}}) = \Gamma_{xx}^y = 0.$$

Lastly, we calculate the covariant derivative.

Now since $\Gamma_{xx}^x = 0$ then $\Gamma_{xxx} = \Gamma_{xx}^{\tilde{x}} = 0$.

Therefore, $\nabla_{\partial_x} \partial_x = f'(y) \partial_{\tilde{y}}$.

2. The second covariant derivative that we will calculate is $\nabla_{\partial_y} \partial_x$.

Calculate the Christoffel symbols of the second kind.

$$\begin{aligned} g(\nabla_{\partial_y} \partial_x, \partial_x) &= \Gamma_{yxx} = \frac{1}{2}(g_{xx}/y + g_{yx}/x - g_{yx}/x) = \frac{1}{2}(-2f'(y)) = -f'(y) \\ g(\nabla_{\partial_y} \partial_x, \partial_y) &= \Gamma_{yxy} = \frac{1}{2}(g_{xy}/y + g_{yy}/x - g_{yx}/y) = 0. \\ g(\nabla_{\partial_y} \partial_x, \partial_{\tilde{x}}) &= \Gamma_{yx\tilde{x}} = \frac{1}{2}(g_{x\tilde{x}}/y + g_{y\tilde{x}}/x - g_{yx}/\tilde{x}) = 0. \\ g(\nabla_{\partial_y} \partial_x, \partial_{\tilde{y}}) &= \Gamma_{yx\tilde{y}} = \frac{1}{2}(g_{x\tilde{y}}/y + g_{y\tilde{y}}/x - g_{yx}/\tilde{y}) = 0. \end{aligned}$$

We calculate the Christoffel symbols of the first kind.

$$\Gamma_{yxx} = -f'(y) = g(\nabla_{\partial_y} \partial_x, \partial_x) = \Gamma_{yx}^x (-2f(y)) + \Gamma_{yx}^{\tilde{x}}$$

$$\Gamma_{yxy} = 0 = g(\nabla_{\partial_y} \partial_x, \partial_y) = \Gamma_{yx}^{\tilde{y}} = 0.$$

$$\Gamma_{yx\tilde{x}} = 0 = g(\nabla_{\partial_y} \partial_x, \partial_{\tilde{x}}) = \Gamma_{yx}^x = 0.$$

$$\Gamma_{yx\tilde{y}} = 0 = g(\nabla_{\partial_y} \partial_x, \partial_{\tilde{y}}) = \Gamma_{yx}^y = 0.$$

And lastly calculate the covariant derivative.

Now since $\Gamma_{yx}^x = 0$ then $\Gamma_{yxx} = \Gamma_{yx}^{\tilde{x}} = 0$.

Therefore, $\nabla_{\partial_y} \partial_x = -f'(y) \partial_{\tilde{x}}$.

Note that since the connection is torsion free $\nabla_{\partial_y} \partial_x = \nabla_{\partial_x} \partial_y = -f'(y) \partial_{\tilde{x}}$.

Thus giving us the third covariant derivative.

3. The fourth covariant derivative that we will calculate is $\nabla_{\partial_y} \partial_y$.

Calculate the Christoffel symbols of the second kind.

$$g(\nabla_{\partial_y} \partial_y, \partial_x) = \Gamma_{yyx} = \frac{1}{2}(g_{yx}/y + g_{yx}/x - g_{yy}/x) = 0.$$

$$g(\nabla_{\partial_y} \partial_y, \partial_y) = \Gamma_{yyy} = \frac{1}{2}(g_{yy}/y + g_{yy}/y - g_{yy}/y) = 0.$$

$$g(\nabla_{\partial_y} \partial_y, \partial_{\tilde{x}}) = \Gamma_{yy\tilde{x}} = \frac{1}{2}(g_{y\tilde{x}}/y + g_{y\tilde{x}}/y - g_{yy}/\tilde{x}) = 0.$$

$$g(\nabla_{\partial_y} \partial_y, \partial_{\tilde{y}}) = \Gamma_{yy\tilde{y}} = \frac{1}{2}(g_{y\tilde{y}}/y + g_{y\tilde{y}}/y - g_{yy}/\tilde{y}) = 0.$$

We calculate then calculate the Christoffel symbols of the first kind.

$$\Gamma_{yyx} = 0 = g(\nabla_{\partial_y} \partial_y, \partial_x) = \Gamma_{yy}^x(-2f(y)) + \Gamma_{yy}^{\tilde{x}}$$

$$\Gamma_{yyy} = 0 = g(\nabla_{\partial_y} \partial_y, \partial_y) = \Gamma_{yy}^{\tilde{y}} = 0.$$

$$\Gamma_{yy\tilde{x}} = 0 = g(\nabla_{\partial_y} \partial_y, \partial_{\tilde{x}}) = \Gamma_{yy}^x = 0.$$

$$\Gamma_{yy\tilde{y}} = 0 = g(\nabla_{\partial_y} \partial_y, \partial_{\tilde{y}}) = \Gamma_{yy}^{\tilde{y}} = 0.$$

Then we calculate the covariant derivative. Now since $\Gamma_{yy}^x = 0$ then $\Gamma_{yyx} = \Gamma_{yy}^{\tilde{x}} = 0$.

Therefore, $\nabla_{\partial_y} \partial_y = 0$.

Since connection is torsion free then $\nabla_{\partial_y} \partial_x = \nabla_{\partial_x} \partial_y = -f'(y) \partial_{\tilde{x}}$.

4. Now we will show that the remaining 12 covariant derivatives are equal to zero. Instead of repeating the steps like we did for the first four covariant derivatives we will take a different approach. Note that the remaining covariant derivatives will always include at least one of the following coordinates, \tilde{x} or \tilde{y} . Thus we will show that if that coordinate is in the first slot, $\nabla_{\tilde{x}}$, or second slot, $\nabla_{\tilde{y}}$, the covariant derivative will be zero.

Let a be \tilde{x} or \tilde{y} and let b be any of the other four coordinate $x, \tilde{x}, y,$ or \tilde{y} . We will show $\Gamma_{abc} = 0$ for all c will give us $\nabla_{\partial_a} \partial_b = 0$. First, we note that the metric entry $g(\partial_a, \partial_b)$ will be constant. So when calculating the Christoffel symbols of the second kind we have $g_{a\Box/\Box} = g_{\Box a/\Box} = 0$ since the derivative of a constant is zero. Also, $g_{\Box\Box/a} = 0$ since a does not appear in any of the metric entries. Hence this implies that $\Gamma_{a\Box\Box} = \Gamma_{\Box a\Box} = \Gamma_{\Box\Box a} = 0$. Now we will use that fact to show that $\nabla_{\partial_a} \partial_b = 0$. Note that $\Gamma_{abc} = 0 = g(\nabla_{\partial_a} \partial_b, \partial_c)$. Since g is non-degenerate then by Lemma 6.1 $\nabla_{\partial_a} \partial_b = 0$. With the connection being torsion free $\nabla_{\partial_a} \partial_b = \nabla_{\partial_b} \partial_a = 0$. Thus the remaining 12 covariant derivatives are all equal to zero.

□

In this following example we demonstrate using the previous manifold which we proved to be a generalized plane wave manifold, the calculations of the geodesic equation of a smooth curve on a non-Riemannian manifold. We know the manifold is non-Riemannian since $g(\partial_{\tilde{x}}, \partial_{\tilde{x}}) = 0$. If the curve satisfies the geodesic equation then it is a geodesic and we will then be able to show that it extends for infinite time.

Before we begin this next example we will utilize the following theorem from [Dan00] within the example. Also, let $\mathcal{R}[a, b]$ denote the set of Riemann integrable functions on $[a, b]$.

Theorem 6.4. *If f is continuous on $[a, b]$, then f is in $\mathcal{R}[a, b]$.*

Example 6.5. Let σ be a curve in \mathbb{R}^4 . From the previous example we will let $\sigma(t) = (x(t), y(t), \tilde{x}(t), \tilde{y}(t))$ and we determine conditions that make $\nabla_{\sigma'} \sigma' = 0$.

Let $\sigma(t) = (x(t), y(t), \tilde{x}(t), \tilde{y}(t))$ then $\sigma'(t) = x' \partial_x + y' \partial_y + \tilde{x}' \partial_{\tilde{x}} + \tilde{y}' \partial_{\tilde{y}}$ and we expand $\nabla_{\sigma'} \sigma' = \nabla_{x' \partial_x + y' \partial_y + \tilde{x}' \partial_{\tilde{x}} + \tilde{y}' \partial_{\tilde{y}}} x' \partial_x + y' \partial_y + \tilde{x}' \partial_{\tilde{x}} + \tilde{y}' \partial_{\tilde{y}}$ and set it equal to zero.

$$\begin{aligned}
\nabla_{\sigma'} \sigma' &= \nabla_{x' \partial_x + y' \partial_y + \tilde{x}' \partial_{\tilde{x}} + \tilde{y}' \partial_{\tilde{y}}} x' \partial_x + \nabla_{x' \partial_x + y' \partial_y + \tilde{x}' \partial_{\tilde{x}} + \tilde{y}' \partial_{\tilde{y}}} y' \partial_y \\
&+ \nabla_{x' \partial_x + y' \partial_y + \tilde{x}' \partial_{\tilde{x}} + \tilde{y}' \partial_{\tilde{y}}} \tilde{x}' \partial_{\tilde{x}} + \nabla_{x' \partial_x + y' \partial_y + \tilde{x}' \partial_{\tilde{x}} + \tilde{y}' \partial_{\tilde{y}}} \tilde{y}' \partial_{\tilde{y}} \\
&= x'' \partial_x + x' \nabla_{x' \partial_x + y' \partial_y + \tilde{x}' \partial_{\tilde{x}} + \tilde{y}' \partial_{\tilde{y}}} \partial_x + y'' \partial_y + y' \nabla_{x' \partial_x + y' \partial_y + \tilde{x}' \partial_{\tilde{x}} + \tilde{y}' \partial_{\tilde{y}}} \partial_y \\
&+ \tilde{x}'' \partial_{\tilde{x}} + \tilde{x}' \nabla_{x' \partial_x + y' \partial_y + \tilde{x}' \partial_{\tilde{x}} + \tilde{y}' \partial_{\tilde{y}}} \partial_{\tilde{x}} + \tilde{y}'' \partial_{\tilde{y}} + \tilde{y}' \nabla_{x' \partial_x + y' \partial_y + \tilde{x}' \partial_{\tilde{x}} + \tilde{y}' \partial_{\tilde{y}}} \partial_{\tilde{y}} \\
&= x'' \partial_x + x' \nabla_{x' \partial_x + y' \partial_y} \partial_x + y'' \partial_y + y' \nabla_{x' \partial_x + y' \partial_y} \partial_y + \tilde{x}'' \partial_{\tilde{x}} + \tilde{y}'' \partial_{\tilde{y}} \\
&= x'' \partial_x + (x')^2 \nabla_{\partial_x} \partial_x + x' y' \nabla_{\partial_y} \partial_x + y'' \partial_y + y' x' \nabla_{\partial_x} \partial_y + (y')^2 \nabla_{\partial_y} \partial_y + \tilde{x}'' \partial_{\tilde{x}} + \tilde{y}'' \partial_{\tilde{y}} \\
&= x'' \partial_x + (x')^2 f'(y) \partial_{\tilde{y}} - x' y' f'(y) \partial_{\tilde{x}} + y'' \partial_y - y' x' f'(y) \partial_{\tilde{x}} + \tilde{x}'' \partial_{\tilde{x}} + \tilde{y}'' \partial_{\tilde{y}} \\
&= x'' \partial_x + y'' \partial_y + (\tilde{x}'' - 2x' y' f'(y)) \partial_{\tilde{x}} + (\tilde{y}'' + (x')^2 f'(y)) \partial_{\tilde{y}}
\end{aligned}$$

So we have that $x'' \partial_x + y'' \partial_y + (\tilde{x}'' - 2x' y' f'(y)) \partial_{\tilde{x}} + (\tilde{y}'' + (x')^2 f'(y)) \partial_{\tilde{y}} = 0$. In order for this equation to be true we let the components of $\partial_x, \partial_y, \partial_{\tilde{x}},$ and $\partial_{\tilde{y}}$ equal to zero. Thus:

$$x'' = 0 \text{ implies that } x(t) = at + b \text{ for } a, b \in \mathbb{R},$$

$$y'' = 0 \text{ implies that } y(t) = ct + d \text{ for } c, d \in \mathbb{R},$$

$$\tilde{x}'' - 2(x')(y')(f'(y)) = 0 \text{ gives us } \tilde{x}'' = 2(x')(y')(f'(y)) = 2(a)(c)f'(c + td)$$

$$\tilde{y}'' + (x')(x')f'(y) = 0 \text{ gives us } \tilde{y}'' = -(x')(x')f'(y) = -a^2 f'(ct + d).$$

By the Fundamental Theorem of Calculus,

$$\begin{aligned}
\tilde{x}'(s) - \tilde{x}'(0) &= \int_0^s \tilde{x}''(w) dw = \int_0^s 2acf'(cw + d) dw \\
\tilde{x}'(s) &= \tilde{x}'(0) + \int_0^s 2acf'(cw + d) dw
\end{aligned}$$

By repeating this process we have,

$$\begin{aligned}
\tilde{x}(t) - \tilde{x}(0) &= \int_0^t \tilde{x}'(s) ds = \int_0^t \tilde{x}'(0) ds + \int_0^t \int_0^s (2acf'(cw + d) dw) dt \\
\tilde{x}(t) &= \tilde{x}(0) + \int_0^t \tilde{x}'(0) ds + \int_0^t \int_0^s (2acf'(cw + d) dw) dt
\end{aligned}$$

A similar process can be done to solve for $\tilde{y}(t)$. Since every continuous function is integrable we have a solution for the geodesic equation, hence our curve is a geodesic. Also, since $f : \mathbb{R} \rightarrow \mathbb{R}$ then this geodesic extends for infinite time. \square

The following theorem found in [DGc05] is the main result of this thesis and will follow a similar process exhibited in the previous example.

Theorem 6.6. *Let M be a generalized plane wave manifold, then M is geodesically complete.*

Proof. Let $\sigma(t) = (x_1(t), \dots, x_m(t))$ be a curve in \mathbb{R}^m . Then $\sigma(t)$ is a geodesic if and only if $\nabla_{\sigma'}\sigma' = 0$. Following a similar process as Example 6.5 we expand $\nabla_{\sigma'}\sigma'$:

$$\begin{aligned}
\nabla_{\sigma'}\sigma' &= \nabla_{\sigma'} \left(\sum_i x'_i(t) \partial_{x_i} \right) \\
&= \sum_i x''_i(t) \partial_{x_i} + \sum_i x'_i(t) \nabla_{\sigma'} (\partial_{x_i}) \\
&= \sum_i \left(x''_i(t) \partial_{x_i} + x'_i(t) \nabla_{\sum_j x'_j(t) \partial_{x_j}} \partial_{x_i} \right) \\
&= \sum_i \left(x''_i(t) \partial_{x_i} + \sum_j x'_i(t) x'_j(t) \nabla_{\partial_{x_j}} \partial_{x_i} \right) \\
&= \sum_i \left(x''_i(t) \partial_{x_i} + \sum_j x'_i(t) x'_j(t) \sum_{k > \max(i,j)} \Gamma_{ij}^k(x_1, \dots, x_{k-1}) \partial_{x_k} \right) \\
&= x''_k(t) + \sum_{k > \max(i,j)} x'_i(t) x'_j(t) \Gamma_{ij}^k(x_1, \dots, x_{k-1}) \partial_{x_k}.
\end{aligned}$$

This implies that $\sigma(t)$ is a geodesic if $x''_1(t) = 0$ and for $k > 1$ we have that $x''_k(t) + \sum_{k > \max(i,j)} x'_i(t) x'_j(t) \Gamma_{ij}^k(x_1, \dots, x_{k-1}) \partial_{x_k} = 0$.

We solve this system of equations recursively. Let $k = 1$, then $x''_1(t) = 0$ implies $x'_1(t) = a$ and $x_1(t) = at + b$ for some constants a, b .

Next consider $k = 2$:

$$\begin{aligned}
x''_2(t) + x'_1(t) x'_1(t) \Gamma_{11}^2(x_1(t)) &= 0 \\
\iff x''_2(t) + a^2 \Gamma_{11}^2(at + b) &= 0 \\
\iff x''_2(t) = -a^2 \Gamma_{11}^2(at + b).
\end{aligned}$$

Now by the Fundamental Theorem of Calculus we integrate $x''_2(t)$.

$$\int_0^s x''_2(u) du = x'_2(s) - x'_2(0) = \int_{w=0}^s -a^2 \Gamma_{11}^2(aw + b) dw$$

$$\iff x'_2(s) = x'_2(0) + \int_{w=0}^s -a^2\Gamma_{11}^2(aw + b)dw.$$

Now we integrate again:

$$\begin{aligned} \int_0^t x'_2(s)ds &= x_2(t) - x_2(0) = \int_{s=0}^t \left(x'_2(0) + \int_{w=0}^s -a^2\Gamma_{11}^2(aw + b)dw \right) ds \\ \iff x_2(t) - x_2(0) &= x'_2(0)t + \int_{s=0}^t \left(\int_{w=0}^s -a^2\Gamma_{11}^2(aw + b)dw \right) ds \\ \iff x_2(t) &= x_2(0) + x'_2(0)t + \int_{s=0}^t \int_{w=0}^s (-a^2\Gamma_{11}^2(aw + b)dw) ds. \end{aligned}$$

Note that we were able to solve for $x_2(t)$ because previously we were able to solve $x_1(t)$. Similar to Example 6.6 since every continuous function is integrable we have a solution for the geodesic equation, hence our curve is a geodesic. Also, since $f : \mathbb{R} \rightarrow \mathbb{R}$ then this geodesic extends for infinite time.

Now we consider $k = 3$:

$$\begin{aligned} x''_3(t) + (x_1(t))^2\Gamma_{11}^2(x_1(t)) + 2x'_1(t)x'_2(t)\Gamma_{12}^3(x_1(t), x_2(t)) + ((x'_2(t))^2\Gamma_{22}^3(x_1(t), x_2(t))) &= 0 \\ \iff x''_3(t) &= -[(x_1(t))^2\Gamma_{11}^2(x_1(t)) + 2x'_1(t)x'_2(t)\Gamma_{12}^3(x_1(t), x_2(t)) + ((x'_2(t))^2\Gamma_{22}^3(x_1(t), x_2(t)))] \end{aligned}$$

Now to illustrate the process and not get lost in notation, we will let we will let the right side of the above equation equal ξ . Now following the same process as above,

$$\begin{aligned} x'_3(s) - x'_3(0) &= \int_{w=0}^s (\xi)dw \\ x'_3(s) &= x'_3(0) + \int_{w=0}^s (\xi)dw \end{aligned}$$

Now we integrate again to evaluate for $x'_3(s)$.

$$\begin{aligned} x_3(t) - x_3(0) &= \int_{s=0}^t \left(x'_3(0) + \int_{w=0}^s (\xi)dw \right) ds \\ x_3(t) &= x_3(0) + x'_3(0)t + \int_{s=0}^t \int_{w=0}^s ((\xi)dw)ds \end{aligned}$$

We see that $x_3(t)$ is expressible in this way since it involves the already solved $x_1(t)$ and $x_2(t)$. We also note that this extends for infinite time for the same reason as the $k = 2$ case.

Now we consider the general case when $k = n$, supposing that all of $x_1(t), \dots, x_{n-1}(t)$ have been determined, and that these are continuous functions defined on all of \mathbb{R} :

$$\begin{aligned} x''_n(t) + \sum_{n > \max(i,j)} x'_i(t)x'_j(t)\Gamma_{ij}^n(x_1(t), \dots, x_{n-1}(t)) &= 0 \\ x''_n(t) &= - \sum_{n > \max(i,j)} x'_i(t)x'_j(t)\Gamma_{ij}^n(x_1(t), \dots, x_{n-1}(t)) = 0 \end{aligned}$$

Now following the same process as the previous cases we integrate.

$$x'_n(s) - x'_n(0) = \int_{w=0}^s \left(- \sum_{n > \max(i,j)} x'_i(t)x'_j(t)\Gamma_{ij}^n(x_1(t), \dots, x_{n-1}(t)) \right) dw$$

$$x'_n(s) = x'_n(0) + \int_{w=0}^s \left(- \sum_{n > \max(i,j)} x'_i(t)x'_j(t)\Gamma_{ij}^n(x_1(t), \dots, x_{n-1}(t)) \right) dw$$

Now we integrate to evaluate for $x_n(t)$.

$$x_n(t) - x_n(0) = \int_{s=0}^t \left(x'_n(0) + \int_{w=0}^s \left(- \sum_{n > \max(i,j)} x'_i(t)x'_j(t)\Gamma_{ij}^n(x_1(t), \dots, x_{n-1}(t)) \right) dw \right) ds$$

$$x_n(t) = x_n(0) + x'_n(0)t + \int_{s=0}^t \int_{w=0}^s \left(- \sum_{n > \max(i,j)} x'_i(t)x'_j(t)\Gamma_{ij}^n(x_1(t), \dots, x_{n-1}(t)) \right) dw ds$$

Now $x_n(t)$ can be solved for we have solved for the previous $n - 1$ cases in order to arrive at this point. And for the same reason that $k = 2, k = 3, \dots$ cases extended for infinite time so does the $k = n$ case. \square

Chapter 7

Conclusion

We began learning the basics of topology and differential geometry to establish a foundation for Riemannian geometry. Our main focus was on pseudo-Riemannian manifolds which differ from Riemannian manifolds because the former is equipped with an indefinite metric and the latter is equipped with a positive definite metric. We learned that geodesics on pseudo-Riemannian manifolds behave differently than what we are intuitively accustomed to. Using a generalized plane wave manifold of signature $(2, 2)$ we were able to demonstrate the various calculations needed to evaluate the Christoffel symbols of the first and second kind which allowed us to evaluate the covariant derivative. Through this exercise we were able to solve the geodesic equation of this manifold to show that it is geodesically complete. This example closely illustrated the more general fact that all generalized plane wave manifolds are complete.

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